Energy shaping for the robust stabilization of a wheeled inverted pendulum

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Abstract: The paper deals with the robust energy-based stabilization of a wheeled inverted pendulum, which is an underactuated, unstable mechanical system subject to nonholonomic constraints. The equilibrium to be stabilized is characterized by the length of the driven path, the orientation, and the pitch angle. We use the method of Controlled Lagrangians which is applied in a systematic way, and is very intuitive, for it is physically motivated. After a detailed presentation of the model under nonholonomic constraints, we provide an elegant solution of the matching equations for kinetic and potential energy shaping for the considered systems. Simulations show the applicability and robustness of the method.

Keywords: Underactuated mechanical systems, nonholonomic systems, passivity-based control.

1. INTRODUCTION

The *wheeled inverted pendulum* (WIP) – and its commercial version, the Segway [2015, Jan] – has gained interest for human assistance and transportation in the past several years due to its high maneuverability and simple construction (see, e. g., Li et al. [2013]). A WIP – shown from the side in Figure 1 (left) – consists of a vertical body with two coaxial driven wheels mounted on the body. The actuation of both wheels in the same direction generates a forward (or backward) motion; opposite wheel velocities lead to a turning motion around the vertical axis. Mobile robotic systems based on the WIP like the intelligent two wheeled road vehicle *B2* presented by Baloh and Parent [2003], or novel and more car-like systems like the *Segway* Puma [2015, Jan] are being developed to be used as new personal urban transportation systems. Some institutes have also developed their own WIPs for research purposes, e. g., *Yamabico Kurara*, introduced by Ha and Yuta [1996], or *JOE*, presented by Grasser et al. [2002], to give only some examples. These systems can be further used as service robots like *KOBOKER* (see Lee and Jung [2011]).

The stabilization and tracking control for the WIP is challenging: The system belongs to the class of underactuated mechanical systems, since the number of control inputs is less than the number of degrees of freedom. Furthermore, the upward position of the body represents an unstable equilibrium which needs to be stabilized by feedback. In addition, the system motion is restricted by nonholonomic (nonintegrable) constraints (Bloch [2003]). These constraints do not restrict the configuration space $\tilde{\mathcal{Q}}$ on which the dynamics evolve, but the motion direction at a given point: Because of the rolling-without-slipping constraint it is not possible to move sideways, and the forward velocity of the WIP and its yaw rate are directly given by the angular velocity of the wheels. The goal of this paper is to present the design of a robust nonlinear position controller using energy shaping techniques for wheeled inverted pendulum systems.

1.1 Existing work

Several control laws have been applied to the WIP, mostly using linearized models (see Li et al. [2013], Ha and Yuta [1996], Grasser et al. [2002]). During the last decade, however, researchers have put a strong focus on the nonlinear model for control purposes: Some accessibility and controllability analysis of the WIP has been done by Pathak et al. [2005] and Nasrallah et al. [2007]. Based on the analysis of the nonlinear system, nonlinear control strategies have been developed for Segway-like systems. Pathak et al. [2005] present, e. g., two different two-level controllers based on the partially feedback linearized model for position and velocity control while maintaining stable pitch dynamics; Nasrallah et al. [2007] design in several steps a posture and velocity control for the WIP moving on an inclined plane. Many other types of modeling and control approaches have also been implemented and tested: For a very complete overview of the existing work on modeling and control of WIPs until 2012 the reader is referred to Chan et al. [2013].

Energy shaping techniques, like the method of Controlled Lagrangians, or Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC), have been successfully used for the stabilization of underactuated mechanical systems in the past, see, e. g., Ortega et al. [2002], Chang et al. [2002]. These methods are attractive since they shape the energy of the system but preserve its physical structure, and thus, appear *natural*. The idea of shaping the energy can also be expanded to mechanical systems subject to nonholonomic constraints: Maschke and Van der Schaft [1994] stabilize nonholonomic systems by shaping the potential energy. Muralidharan et al. [2009] stabilize the pitch dynamics of the WIP through IDA-PBC.

Nonholonomic systems violate one of the necessary conditions for asymptotic stabilization by smooth state feedback formulated by Brockett [1983]. Thus, for the asymptotic stabilization of a desired configuration $q \in \tilde{\mathcal{Q}}$, a discontinuous or time-varying control law is required (Astolfi [1996]). In this paper, to avoid this issue, instead of working in the WIPs six-dimensional configuration space $\tilde{\mathcal{Q}}$, we restrict our analysis to the three dimensional space $\mathcal{Q}$ with local coordinates consisting of the path length, the pitch, and the yawing angle: $\xi = [s\ \alpha\ \theta]^T \in \mathcal{Q}$. The pitch angle is physically restricted to $-\pi/2 < \alpha < \pi/2$. We design a passivity-based controller for the stabilization of an equilibrium $\xi^* \in \mathcal{Q}$. The controller is thereafter parametrized applying local linear dynamics assignment (LLDA), a method used to fix design parameters in nonlinear passivity based control by making use of the linearized model (Kotyczka [2013]). Using this approach, prescribed local dynamics (in terms of the closed-loop eigenvalues) can be achieved.

The passivity-based controller presented in this note can be systematically computed and leads to an asymptotically stable equilibrium $\xi^* \in \mathcal{Q}$ with a large domain of attraction. Since the closed-loop mechanical energy is used as Lyapunov function, the framework is remarkably intuitive for it is physically motivated. Moreover, LLDA allows for transparency concerning parameter tuning. The applicability, performance, and robustness of the developed controller is shown with a series of simulations.

Notation: For compactness of notation, the operator $\nabla_x f(x)$ is used to denote the transposed Jacobian of a vector-valued function $f(x)$. Additionally, we will use the notation $\mathrm{s}(\alpha) = \sin\alpha$, and $\mathrm{c}(\alpha) = \cos\alpha$. When obvious from the context, arguments are omitted for simplicity.

2. MODELING

In a mechanical system with nonholonomic constraints, the n-dimensional manifold $\tilde{\mathcal{Q}}$ is the configuration space, its tangent bundle $T\tilde{\mathcal{Q}}$ is the velocity phase space and a smooth (nonintegrable) distribution $\mathcal{D} \subset T\tilde{\mathcal{Q}}$ represents the constraints. The Lagrangian L is a map $L : T\tilde{\mathcal{Q}} \to \mathbb{R}$ and is defined as the kinetic energy minus the potential energy $L = T - V$. A curve $q(t)$ is said to satisfy the constraints if $\dot{q}(t) \in \mathcal{D}_q$, for all $q \in \tilde{\mathcal{Q}}$ and all times t. For k nonholonomic constraints, the admissible velocities in a point q are thus restricted to a $(n-k)$-dimensional subset ($\mathcal{D}_q \cong \mathbb{R}^{n-k}$) of the tangent space $T_q\tilde{\mathcal{Q}}$. The constraint distribution $\mathcal{D}$ is assumed to be regular, i. e., of constant rank. The widely used Lagrange-d'Alembert equations (see, e. g., Bloch [2003])

$$\frac{d}{dt}(\nabla_{\dot{q}} L) - \nabla_q L = A(q)\lambda + \sum F_{ext} \tag{1}$$

describe the dynamics of systems subject to k nonholonomic (Pfaffian) constraints of the form

$$A^T(q)\dot{q} = 0. \tag{2}$$

Assuming there are no external forces other than the input torques $\tilde{\tau}$, (1) results in

$$\tilde{M}(q)\ddot{q} + \tilde{C}(q,\dot{q})\dot{q} + \nabla_q V(q) = \tilde{\tau} + A(q)\lambda, \tag{3}$$

where $\tilde{M} = \tilde{M}^T$ is the positive definite mass matrix, and the term $\tilde{C}\dot{q}$ represents the Coriolis and centripetal forces. The constraints have been adjoined to the system using Lagrange multipliers $\lambda \in \mathbb{R}^k$ that represent the magnitude of the constraint forces which oblige the system to satisfy the constraints. The work done by these forces vanishes as can be seen by looking at the corresponding power

$$P_{constr} = \dot{q}^T A\lambda = \lambda^T A^T \dot{q} = 0. \tag{4}$$

The approach, as explained in the following, is also used, e. g., by Pathak et al. [2005] for the modeling of the WIP: Due to the nonholonomic constraints (2), the admissible velocities at $q \in \tilde{\mathcal{Q}}$ must be of the form

$$\dot{q} = S(q)\nu, \tag{5}$$

with a smooth full rank matrix S satisfying $A^T S = 0$ for all $q \in \tilde{\mathcal{Q}}$, and local coordinates of the constrained tangent space $\nu \in \mathcal{D}_q$. The admissible velocities at q lie in the subspace of $T_q\tilde{\mathcal{Q}}$ spanned by the columns of S, which is nothing but the $(n-k)$-dimensional space $\mathcal{D}_q$. Now, replace $\dot{q} = S\nu$ and $\ddot{q} = S\dot{\nu} + \dot{S}\nu$ in (3), and eliminate the constraints by pre-multiplying it by S^T

$$S^T\tilde{M}S\dot{\nu} + S^T\left(\tilde{M}\dot{S} + \tilde{C}S\right)\nu + S^T\nabla_q V = S^T\tilde{\tau}. \tag{6}$$

The dynamical system represented by (6) can also be written in the form

$$\hat{M}\dot{\nu} + \hat{C}\nu + S^T\nabla_q V = \hat{\tau} + \hat{J}\nu, \tag{7}$$

where $\hat{M} = S^T\tilde{M}S$, and $\hat{\tau} = S^T\tilde{\tau}$. Since the matrix $\hat{C}$ is solely defined by the *Christoffel symbols* of $\hat{M}$, the matching of the systems (6) and (7) requires, in general, additional gyroscopic forces $\hat{J}\nu$, where $\hat{J} = -\hat{J}^T$, which are mistakently missing in Muralidharan et al. [2009] for imposing the constraints before taking variations in the derivation of the equations of motion (see Bloch [2003]).

2.1 The wheeled inverted pendulum (WIP)

Different modeling approaches for WIPs can be found, e. g., in Pathak et al. [2005], Delgado et al. [2015], Nasrallah et al. [2007]. The dynamic parameters needed for the modeling of the WIP are listed below in Table 1 with the values used for the simulations. Figure 1 shows

m_B	body mass	1 kg
m_W	wheel mass	0.5 kg
r	wheel radius	0.05 m
b	distance from the wheel axis to the body's center of mass	0.08 m
d	half of the wheel distance	0.05 m
I_B	body's moment of inertia	
$I_{B_{xx}}$	around x-axis	1E-5 kg m^2
$I_{B_{yy}}$	around y-axis	9E-4 kg m^2
$I_{B_{zz}}$	around z-axis	4E-4 kg m^2
I_W	wheel's moment of inertia	
$I_{W_{yy}}$	around y-axis	1E-8 kg m^2
$I_{W_{zz}}$	around z-axis	1E-6 kg m^2
g	gravity constant	10 m/s^2

Table 1. System parameters

a simple scheme of the wheeled inverted pendulum. Let $\tilde{\mathcal{Q}} = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ be the configuration space and define local coordinates $q = (x,\ y,\ \theta,\ \alpha,\ \varphi_l,\ \varphi_r) \in \tilde{\mathcal{Q}}$. The coordinates φ_l and φ_r represent the absolute rotation of the left and right wheel, respectively. The equations

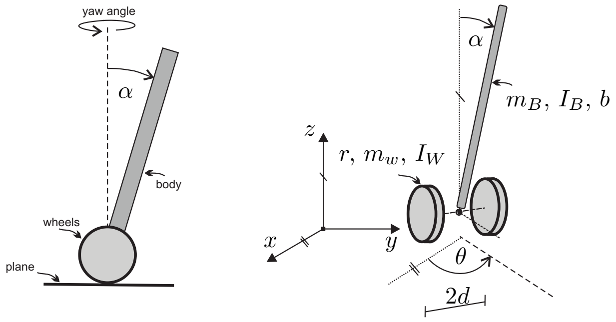

Fig. 1. The wheeled inverted pendulum

$$A^T \dot{q} = \begin{bmatrix} -\mathrm{s}(\theta) & \mathrm{c}(\theta) & 0 & 0 & 0 & 0 \\ \mathrm{c}(\theta) & \mathrm{s}(\theta) & d & 0 & -r & 0 \\ \mathrm{c}(\theta) & \mathrm{s}(\theta) & -d & 0 & 0 & -r \end{bmatrix} \dot{q} = 0 \tag{8}$$

represent the rolling-without-slipping constraints of the wheels. The velocities in a specific configuration $q \in \tilde{\mathcal{Q}}$ are thus restricted to

$$\dot{q} = S\nu = \begin{bmatrix} \mathrm{c}(\theta) & 0 & 0 \\ \mathrm{s}(\theta) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1/r & 0 & d/r \\ 1/r & 0 & -d/r \end{bmatrix} \begin{bmatrix} v \\ \dot{\alpha} \\ \dot{\theta} \end{bmatrix}, \tag{9}$$

where v is the forward velocity of the WIP. With this matrix S, the equations of motion are of the form (7) with

$$\hat{M} = \begin{bmatrix} c_1 & c_2\,\mathrm{c}(\alpha) & 0 \\ c_2\,\mathrm{c}(\alpha) & c_3 & 0 \\ 0 & 0 & I_\theta(\alpha) \end{bmatrix}, \; S^T \nabla_q V = \begin{bmatrix} 0 \\ -c_2 g\,\mathrm{s}(\alpha) \\ 0 \end{bmatrix},$$

$$\hat{\tau} = \begin{bmatrix} 1/r(\tau_r + \tau_l) \\ -\tau_r - \tau_l \\ d/r(\tau_r - \tau_l) \end{bmatrix}, \; \hat{J} = \begin{bmatrix} 0 & 0 & c_2\,\dot{\theta}\,\mathrm{s}(\alpha) \\ 0 & 0 & 0 \\ -c_2\,\dot{\theta}\,\mathrm{s}(\alpha) & 0 & 0 \end{bmatrix}, \tag{10}$$

where

$$I_\theta(\alpha) = c_4\,\mathrm{s}^2(\alpha) + c_5,$$

and

$$\begin{aligned} c_1 &= m_B + 2m_W + 2\frac{I_{W_{yy}}}{r^2}, \quad c_2 = m_B b, \\ c_3 &= m_B b^2 + I_{B_{yy}}, \quad c_4 = I_{B_{xx}} + m_B b^2 - I_{B_{zz}}, \\ c_5 &= I_{B_{zz}} + 2\frac{I_{W_{yy}} d^2}{r^2} + 2m_W d^2 + 2I_{W_{zz}}. \end{aligned}$$

The $n-k$ equations of motion (7) together with the reconstruction equation (9) describe the motion of the WIP in the space (q, ν). For a simpler analysis and control synthesis, define reduced local coordinates $\xi = [s\ \alpha\ \theta]^T \in \mathcal{Q} \subset \tilde{\mathcal{Q}}$, such that [1] $\dot{\xi} = \nu$. In the remaining of the paper we will restrict the analysis to the configuration space $\mathcal{Q}$. This is possible, since $S^T \nabla_q V = \nabla_\xi V$, for the potential forces act directly on the admissible space $\mathcal{D}$.

2.2 Input and feedback transformation

The control inputs are the motor torques on the right and on the left wheel, τ_r and τ_l, respectively. These inputs can, however, be transformed into more natural quantities for the control of the WIP. Apply the following input transformation

$$\begin{aligned} u_1 &= \tau_r + \tau_l \\ u_2 &= \frac{d}{r}(\tau_r - \tau_l), \end{aligned} \tag{11}$$

such that the new inputs u_1 and u_2 represent the resulting torque for the forward and the turning motion, respectively. The input vector can then be written as

$$\hat{\tau} = Gu = \begin{bmatrix} 1/r & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{12}$$

In order to obtain a model of the WIP suitable for energy-based control it is helpful to feedback linearize the yawing dynamics, for the terms involved in the computation get simpler. Choose

$$\begin{aligned} u_1 &= w_1, \\ u_2 &= I_\theta(\alpha) w_2 + 2c_4 \dot{\alpha}\dot{\theta}\,\mathrm{s}(\alpha)\,\mathrm{c}(\alpha) + c_2 v \dot{\theta}\,\mathrm{s}(\alpha) \end{aligned}$$

to get finally the simplified model

$$M\dot{\nu} + (C - J)\nu + \nabla_\xi V = Gw, \tag{13}$$

used for controller design. Here, the corresponding matrices and vectors are

$$M = \begin{bmatrix} c_1 & c_2\,\mathrm{c}(\alpha) & 0 \\ c_2\,\mathrm{c}(\alpha) & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \nabla_\xi V = \begin{bmatrix} 0 \\ -c_2\, g\,\mathrm{s}(\alpha) \\ 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 1/r & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, \; C - J = \begin{bmatrix} 0 & -c_2\,\dot{\alpha}\,\mathrm{s}(\alpha) & -c_2\,\dot{\theta}\,\mathrm{s}(\alpha) \\ 0 & 0 & -c_4\,\dot{\theta}\,\mathrm{s}(\alpha)\,\mathrm{c}(\alpha) \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

Note that the yawing dynamics is not completely decoupled, since a turning motion still affects the pitch and forward motion.

3. ENERGY-BASED CONTROLLER DESIGN

This section presents the methodology for the design of the energy-based controller. Since the procedures IDA-PBC and Controlled Lagrangians are equivalent (see Blankenstein et al. [2002], Chang et al. [2002]), the following controller design can be done in both frameworks analogously. We will put the focus on the Lagrangian case, for velocities are more intuitive than momenta.

As stated in the introduction, the objective is to design in a systematical way a controller which stabilizes an admissible equilibrium $\xi^* \in \mathcal{Q}$ in the reduced space [2]. An equilibrium is called admissible if $G_\perp \nabla_\xi V|_{\xi^*} = 0$, where V is the potential energy of the uncontrolled system. We formulate the conditions for the desired closed-loop equilibrium ξ^* to be (asymptotically) stable. The closed-loop system is parametrized using LLDA to achieve prescribed local dynamics in terms of the closed-loop eigenvalues.

3.1 Matching equations

The goal of the Controlled Lagrangians procedure is to transform (13) by static state feedback $w = w(\xi, \nu)$ into a Lagrangian closed-loop system. Let

$$L_c(\xi, \nu) = \frac{1}{2}\nu^T M_c(\xi)\nu - V_c(\xi) \tag{15}$$

[1] The variable s defines the path length.

[2] A given configuration $q^* \in \tilde{\mathcal{Q}}$ in the original configuration space cannot be asymptotically stabilized using the energy-based controller since Brockett's necessary condition for asymptotic stabilization is not met, in which case only convergence to a (non-intuitive) invariant set can be shown.

be the desired closed-loop Lagrangian with mass matrix $M_c(\xi) = M_c^T(\xi)$ and potential energy $V_c(\xi)$, and let us consider the Euler-Lagrange equations of motion for the target system with dissipation (and additional gyroscopic forces [3] Chang et al. [2002])

$$\frac{d}{dt}(\nabla_\nu L_c) - \nabla_\xi L_c = (J_c - R_c)\nu. \tag{16}$$

The target system dynamics evolving on $\mathcal{Q}$ can be rewritten as

$$M_c\dot{\nu} + C_c\nu + \nabla_\xi V_c = (J_c - R_c)\nu, \tag{17}$$

or equivalently

$$\dot{\nu} = -M_c^{-1}C_c\nu - M_c^{-1}\nabla V_c + M_c^{-1}(J_c - R_c)\nu, \tag{18}$$

where the matrix $J_c = J_c(\xi,\nu)$ (linear in ν) is skew symmetric, and the closed-loop damping matrix $R_c(\xi)$ is symmetric.

Proposition 1. The equilibrium $(\xi^*, 0)$ of the system (17) is stable if $M_c(\xi) > 0$ in a neighborhood Ω of ξ^*, $V_c(\xi)$ has a strict minimum at ξ^*, and $R_c \geq 0$ in Ω. The equilibrium is asymptotically stable if the system is pervasively damped.

Proof. Consider the closed-loop mechanical-type energy as Lyapunov function

$$E_c = \frac{1}{2}\nu^T M_c\nu + V_c. \tag{19}$$

The time derivative of E_c along the trajectories of (17) is

$$\dot{E}_c = -\nu^T R_c\nu, \tag{20}$$

where it has been used the fact that $\dot{M}_c = C_c + C_c^T$. Stability of the equilibrium is shown for $R_c \geq 0$. If the damping is pervasive, the largest invariant set under the closed-loop dynamics (17) contained in

$$\Big\{(\xi,\nu) \in \mathcal{Q}\times\mathbb{R}^3 \mid \dot{E}_c = 0\Big\} \tag{21}$$

equals the equilibrium $(\xi^*, 0)$. Asymptotic stability follows from La Salle's invariance principle. An estimate of the domain of attraction is given by the largest bounded level set of E_c in Ω. □

In order to formulate conditions, under which it is possible to match both, the system (13) and the desired Euler-Lagrange system (17), first replace the target dynamics (18) in the systems equations of motion (13) to get

$$\begin{aligned}&-MM_c^{-1}C_c\nu - MM_c^{-1}\nabla_\xi V_c + MM_c^{-1}(J_c - R_c)\nu\\ &+ (C - J)\nu + \nabla_\xi V = Gw.\end{aligned} \tag{22}$$

We require to find an input w, which solves (22), for the closed-loop system to take the desired form (17). Splitting the equations in terms of the dependency on the velocities ν leads to the matching equations of the potential (independent from ν) and kinetic (quadratic in ν) energy, and of the dissipation, which consists of the terms linear in ν. The resulting set of equations

$$-MM_c^{-1}(C_c - J_c)\nu + (C - J)\nu = Gw_{ke}, \tag{23a}$$

$$-MM_c^{-1}\nabla_\xi V_c + \nabla_\xi V = Gw_{pe}, \tag{23b}$$

$$-MM_c^{-1}R_c\nu = Gw_{di}, \tag{23c}$$

determines the components of the control law

$$w = w_{ke} + w_{pe} + w_{di} \tag{24}$$

related to the shaping of the kinetic and potential energy, and to damping injection. In the following we demonstrate how to solve the three matching equations independently, which is sufficient to satisfy (22).

[3] The matrix J_c serves as additional design parameter.

3.2 Shaping the kinetic energy

One can show that the upward equilibrium of the WIP cannot be stabilized simply by shaping the potential energy, i. e., solving (23b) for an appropriate potential energy $V_c(\xi)$ and keeping $M_c = M$. It is thus necessary to also shape the kinetic energy by solving (23a) for a positive definite closed-loop matrix $M_c \neq M$. The skew-symmetric matrix J_c is free, and thus, a further design parameter. Assuming that J_c and C_c are linear in the velocities, and that the kinetic shaping input is of the form

$$w_{ke} = F^T(\xi,\nu)\nu, \tag{25}$$

with $F(\xi,\nu) \in \mathbb{R}^{3\times 2}$ also linear in the velocities, more modest sufficient conditions for matching are obtained. We require

$$\left(-MM_c^{-1}(C_c - J_c) + C - J - GF^T\right)\nu = 0, \tag{26}$$

or equivalently, since (26) has to be satisfied for all $\nu \in \mathbb{R}^3$:

$$J_c = C_c + M_cM^{-1}(J - C + GF^T). \tag{27}$$

Recalling that $\dot{M}_c = C_c^T + C_c$, the skew symmetry of (27), $J_c + J_c^T = 0$, can be rewritten as

$$M_cM^{-1}(J-C+GF^T)+(J-C+GF^T)^T M^{-1}M_c+\dot{M}_c = 0. \tag{28}$$

The matrix F in (28) is the velocity feedback matrix in the kinetic energy shaping control law. To extract the conditions that have to be satisfied independently from control, we pre-multiply (28) by $G_\perp MM_c^{-1}$ and post-multiply it by $M_c^{-1}MG_\perp^T$, where $G_\perp = [r\ 1\ 0]$ is a full rank left annihilator of G, i. e., $G_\perp G = 0$. Note that the matrix equation (28) is symmetric, so is the projected equation

$$G_\perp\big((J-C)\bar{M}_cM + M\bar{M}_c(J-C)^T\big)G_\perp^T = G_\perp M\dot{\bar{M}}_cMG_\perp^T. \tag{29}$$

Thus, shaping the kinetic energy only requires the solution $\bar{M}_c$ of this differential equation for $\bar{M}_c = M_c^{-1}$. One possible solution for (29) is

$$M_c^{-1} = \begin{bmatrix} k_1 & -\dfrac{\gamma k_3\phi_1(\alpha)+g}{\gamma\phi_2(\alpha)} & 0 \\ -\dfrac{\gamma k_3\phi_1(\alpha)+g}{\gamma\phi_2(\alpha)} & \dfrac{k_3\phi_1^2(\alpha)}{\phi_2^2(\alpha)} & 0 \\ 0 & 0 & k_2 \end{bmatrix}, \tag{30}$$

with $\phi_1(\alpha) = c_1 r + c_2\,\mathrm{c}(\alpha)$, $\phi_2(\alpha) = c_3 + c_2 r\,\mathrm{c}(\alpha)$, and constant positive parameters k_1, k_2, k_3, and γ, which are chosen such that $M_c > 0$ in $-\pi/2 < \alpha < \pi/2$. The kinetic energy shaping control (25) can be now derived by pre-multiplying (27) by $G^T MM_c^{-1}$

$$w_{ke} = (G^TG)^{-1}G^T\left(MM_c^{-1}(J_c - C_c) + (C - J)\right)\nu. \tag{31}$$

The matrices C_c and J_c can be easily calculated from the matrix M_c, and premultiplying (27) by $G_\perp MM_c^{-1}$, respectively. The matrix J_c takes the form

$$J_c = \begin{bmatrix} 0 & -f_2 v - f_3\dot{\alpha} & 0 \\ f_2 v + f_3\dot{\alpha} & 0 & -f_1\dot{\theta} \\ 0 & f_1\dot{\theta} & 0 \end{bmatrix} \tag{32}$$

for some functions [4] $f_i(\xi)$.

[4] The explicit form of the functions $f_i(\xi)$ is omitted for brevity.

3.3 Shaping the potential energy

With the new mass matrix of the closed-loop system M_c, we can proceed to shape the potential energy by solving (23b). The corresponding projected matching equation is

$$G_\perp(\nabla_\xi V - MM_c^{-1}\nabla_\xi V_c) = 0, \tag{33}$$

which represents a set of linear first order PDEs and can be easily solved using a computer algebra system. The closed-loop potential energy takes the form

$$\begin{aligned} V_c(\xi) = \gamma\, \big(\ln(\phi_1(\alpha))(r^2 c_1 - c_3) - r c_2\, \mathrm{c}(\alpha)\big) \\ + \Pi_1(\Phi(s,\alpha)) + \Pi_2(\theta), \end{aligned} \tag{34}$$

where $\Pi_1(\Phi(s,\alpha))$ is a free function of the homogeneous solution

$$\begin{aligned} \Phi(s,\alpha) = s - r\alpha + \frac{\gamma}{g}(k_1 - k_3)(c_3\alpha + c_2 r\, \mathrm{s}(\alpha)) \\ + 2\frac{c_3 - c_1 r^2}{\sqrt{c_1^2 r^2 - c_2^2}} \arctan\left(\frac{(c_2 - c_1 r)(1 - \mathrm{c}(\alpha))}{\sqrt{c_1^2 r^2 - c_2^2}\ \mathrm{s}(\alpha)}\right), \end{aligned} \tag{35}$$

and $\Pi_2(\theta)$ is a free function of θ. Both, Π_1 and Π_2, need to be chosen such that $V_c(\xi)$ has an isolated minimum at $\xi = \xi^*$. The potential energy shaping control is

$$w_{pe} = (G^T G)^{-1} G^T (\nabla_\xi V - MM_c^{-1}\nabla_\xi V_c). \tag{36}$$

3.4 Damping injection and control law

To achieve asymptotic stability of the equilibrium $(\xi,\, \nu) = (\xi^*,\, 0)$, it is necessary to add (pervasive) damping according to Proposition 1, for which we need the solution of (23c) for a dissipation matrix $R_c \geq 0$, such that any possible system motions elicit energy dissipation. First, define $R_c = M_c M^{-1}\breve{R}M^{-1}M_c$, such that (23c) becomes

$$-\breve{R}M^{-1}M_c\nu = Gw_{di}. \tag{37}$$

Choose the damping matrix as $\breve{R} = GK_{di}G^T$, for $K_{di} = \mathrm{diag}(k_{d,1}, k_{d,2}) > 0$. We add damping by choosing

$$w_{di} = -K_{di}G^T M^{-1} M_c \nu. \tag{38}$$

Proposition 2. Consider the equations of motion (13). Assume there is a matrix $M_c(\xi) > 0$ and a scalar function $V_c(\xi)$ which verify (29) and (33), where the function V_c is such that $\xi^* = \arg\min V_c$. Then, the closed-loop system (13) with input w according to (24), with (31), (36), and (38), has an (asymptotically) stable equilibrium $(\xi, \nu) = (\xi^*, 0)$ for $K_{di} > 0$.

Proof. The solution of (23) is sufficient to meet the requirement for matching (22). Since G is not an invertible matrix, the equations (23) cannot be trivially solved. Premultiplying (23) by the full rank matrix

$$\begin{bmatrix} G_\perp \\ G^T \end{bmatrix} \tag{39}$$

splits the matching equations (23) into non-actuated and fully actuated parts. The fully actuated part leads straightforwardly to (24) according to (31), (36), and (38), respectively. The non-actuated part of (23c) is trivially solved for a damping matrix of the form $\breve{R} = GK_{di}G^T$. The PDE (33) represents the part of the potential matching equation which is not dependent on the input. It is clear from (27) and (28) that the non-actuated part of (23a) is equivalent to (29). The control law (24) composed of the parts corresponding to the potential and kinetic energy shaping, and the damping injection, renders (13) the closed-loop Lagrangian system (17). Stability of the desired equilibrium $(\xi^*, 0)$ follows from Proposition 1, since $M_c(\xi) > 0$, $\xi^* = \arg\min V_c$, and $R_c \geq 0$. Asymptotic stability can be shown invoking La Salle's invariance principle. □

3.5 Some remarks on the parameter choice

Since the yawing dynamics of the closed-loop system (17) is fully actuated, it can be parametrized independently and arbitrarily. Choose, e. g., the function

$$\Pi_2(\theta) = \frac{1}{2}k_p(\theta - \theta^*)^2, \quad k_p > 0. \tag{40}$$

The resulting closed-loop yawing dynamics are of the form

$$k_2\ddot{\theta} = -k_p(\theta - \theta^*) - k_{d,2}\dot{\theta} + f_1(\xi)\dot{\theta}\dot{\alpha}, \tag{41}$$

where the term quadratic in the velocities arises from (32). The closed-loop yawing dynamics (41) can be parametrized similar to a PD-controller by the choice of k_2, k_p, and $k_{d,2}$ to achieve desired local behavior. For the parametrization of the remaining dynamics we apply LLDA: The 4 free parameters k_1, k_3, γ, and $k_{d,1}$, and the free function $\Pi_1(s, \alpha)$ are chosen such that the linearized closed-loop system has desired eigenvalues at the equilibrium $(\xi^*, 0)$. The procedure results in an asymptotically stable closed-loop system with desired local dynamics and a large domain of attraction.

3.6 Robustness

In order to check the robustness of the controller, let us consider the plant

$$(M + \Delta M)\dot{\nu} + (C - J + \Delta C - \Delta J)\nu + \nabla_\xi(V + \Delta V) = Gw, \tag{42}$$

where the model uncertainties are denoted by Δ. Using the controller (24) results in a closed-loop system

$$(M_c + \Delta M)\dot{\nu} + (C_c - J_c + \Delta C - \Delta J)\nu + \nabla_\xi(V_c + \Delta V) = -R_c\nu. \tag{43}$$

Since the *real* system is of mechanical nature, the matrix ΔJ is skew symmetric, and $(\dot{\Delta M}) = \Delta C + \Delta C^T$ holds. According to Proposition 1, the closed-loop system has an (asymptotically) stable equilibrium $(\xi^*, 0)$ if $M_c + \Delta M > 0$ and $V_c + \Delta V$ has a strict minimum at ξ^*.

4. SIMULATIONS

The yawing dynamics has been parametrized by the choice of k_2, k_p, and $k_{d,2}$, such that, locally, it has closed-loop eigenvalues $\{-1, -6.2\}$. The remaining parameters k_1, k_3, γ, and $k_{d,1}$, and the function

$$\Pi_1(\Phi(s,\alpha)) = \frac{1}{2}\mu\left(\Phi(s - s^*, \alpha)\right)^2$$

are chosen such that the linearized closed-loop system has eigenvalues $\{-1, -2, -3, -6\}$. Figure 2 shows two level sets of the Lyapunov function E_c in the plane

$$\left\{(\xi,\, \nu) \in \mathcal{Q} \times \mathbb{R}^3 \mid \nu = 0,\ \theta = 0\right\}. \tag{44}$$

The level set of interest is limited by the pitch angle $|\alpha| < \pi/2$. The simulations have been run for the initial condition $\alpha_0 = 1.5\,\mathrm{rad}$, $\theta_0 = 2\,\mathrm{rad}$ and $s_0 = \dot{s}_0 = \dot{\alpha}_0 =$

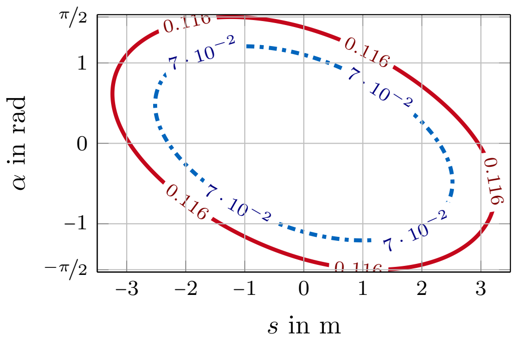

Fig. 2. Level sets of E_c for $\theta = 0$, $\nu = 0$.

$\dot{\theta}_0 = 0$, and for $s^* = 1\,\mathrm{m}$. For the disturbed model we have chosen matrices $\hat{M}$ and $\hat{C}$ in (10) to be 1.5 times the nominal value. The simulation results are shown below in Figure 3 and Figure 4.

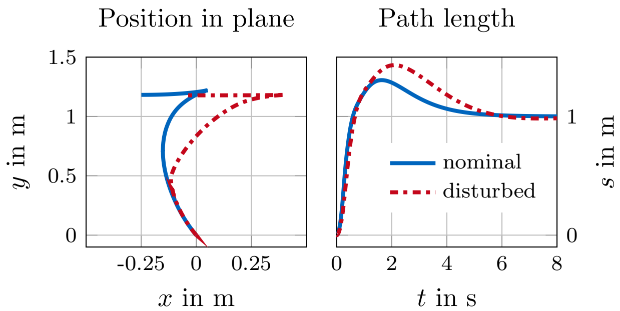

Fig. 3. Response of the path length s for the nominal (solid) and the disturbed model (dashdotted).

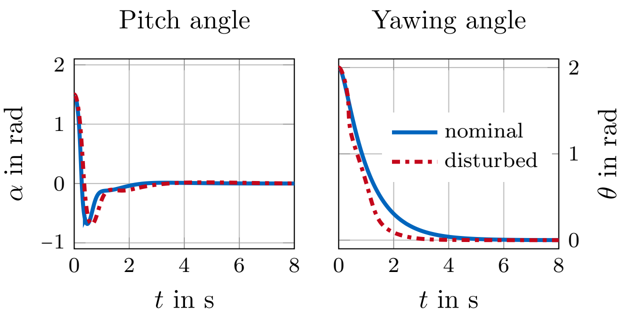

Fig. 4. Response of the pitch and yawing angle for the nominal (solid) and the disturbed model (dashdotted).

5. FURTHER WORK

The methodology presented here can be extended to the speed control without any further computations as shown in an extended version of this paper submitted to Automatica. Further, the stabilization of the position given by the cartesian coordinates x and y has been solved by other authors applying a coordinate transformation. We plan to redesign our approach to achieve that task.

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