Bandwidth Efficient and Rate-Matched Low-Density Parity-Check Coded Modulation

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Outline

- Motivation: Bandwidth-Limited Communication
- Contribution: A New Coded Modulation Scheme
- Enabling Technology: Fixed-Length Distribution Matching
Bandwidth-Limited Communication
Bandwidth-Limited Communication

\[ \frac{1}{2} \log_2(1 + \text{SNR}) \]
Higher Order-Modulation

- **Equidistant** $2^m$-ASK constellation:

  $$\mathcal{X} = \{\pm 1, \pm 3, \ldots, \pm (2^m - 1)\}.$$ 

- **I/O-relation**:

  $$Y = \Delta \cdot X + Z.$$ 

  - Noise $Z$ is zero mean, variance one.
  - Input $X$ with distribution $P_X$ on $\mathcal{X}$.
  - $\Delta$ scales the constellation $\mathcal{X}$. 
Bandwidth-Limited Communication

\[ \frac{1}{2} \log_2(1 + \text{SNR}) \]

\( \{8, 16, 32, 64\} \text{ASK capacity} \)
Bandwidth-Limited Communication

\[ \frac{1}{2} \log_2(1 + \text{SNR}) \]

uniform \{8, 16, 32, 64\} ASK capacity

SNR [dB]

bits/channel use

16 18 20 22 24 26 28 30 32 34
DVB-S2 LDPC Codes, Uniform Input, FER $\leq 10^{-3}$

\[
\frac{1}{2} \log_2(1 + \text{SNR})
\]

- uniform \{8, 16, 32, 64\} ASK capacity
- uniform, \{16, 32, 64\} ASK, code rates \(\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{8}{9}, \frac{9}{10}\)
Probabilistic Amplitude Shaping
ASK Capacity

- ASK capacity:

\[ C_{\text{ask}}(P) = \max_{\Delta, P_X : E[|\Delta X|^2] \leq P} I(X; \Delta X + Z). \]

- Capacity-achieving distribution \( P_X^* \).

\[ \Rightarrow \text{Construct a codebook with entries } \sim P_X^*. \]
Capacity-Achieving Distribution $P_{X^*}$

- $P_{X^*}$ is **symmetric**:
  
  $$P_{X^*}(x) = P_{X^*}(-x).$$

- $P_{X^*}$ **factorizes**:
  
  $$P_{X^*}(x) = P_S(\text{sign}(x)) \cdot P_A(|x|)$$

  where $A := |X|$ and $S := \text{sign}(X)$.

- The sign $S$ is **uniform**:
  
  $$P_S(-1) = P_S(1) = \frac{1}{2}.$$
Shaping and Error Correction

- Binary systematic $k \times n$ generator matrix $G = [I|P]$.
- Binary information $B^k$ arbitrarily distributed.
- Uniform check bit assumption:
  
  $$R_1, R_2, \ldots \text{ iid } \sim \text{Bernoulli}(1/2).$$
Probabilistic Amplitude Shaping

- $n_c$ number of channel uses.
- Represent amplitudes by $(m - 1)$ bits: $A \mapsto b(A)$.
- Represent signs by 1 bit: $S \mapsto b(S)$.
- Multiply amplitude labels by parity matrix $P$ to get sign labels.
- Code rate $c = (m - 1)/m$.

$P_{X_i} = P_{X^*!}$
Rate Compatibility

- Control rate $H(A)$ via $P_A$.
- Control power $E[|\Delta X|^2]$ via $\Delta$. 
Enabling Technology: Distribution Matching

Emulating discrete memoryless source $P_A$:

$B^k \xrightarrow{\text{Matcher}} \tilde{A}^n$

- $B_i$ iid Bernoulli(1/2).
- $\tilde{A}_i$ approximately iid $\sim P_A$ in the sense of

$$\frac{D(P_{\tilde{A}^n} \parallel P_A^n)}{n} \xrightarrow{n \to \infty} 0, \quad \frac{k}{n} \xrightarrow{n \to \infty} H(A)$$

- invertible: $B^k$ can be recovered from $\tilde{A}^n$ with zero error.
Constant Composition Codes

The empirical distribution of an output sequence $c$ of length $n$ is

$$P_{A,c}(a) := \frac{n_a(c)}{n},$$

- $n_a(c) = |\{i : c_i = a\}|$ number of occurrences of symbol $a$ in $c$
- $P_{A,c}$ is the type of $c$
- $\mathcal{T}_{P_A}^n$ is set of all $n$-type $P_A$ sequences
- Codebook $C_{ccdm} \subseteq A^n$ is called a constant composition code if all codewords are of the same type, i.e. $C_{ccdm} \subseteq \mathcal{T}_{P_A}^n$
Constant Composition Distribution Matching

**Idea**

- Use codewords of the same type
- Choose sequence length of the matcher output $n$
- Choose $n$-type approximation $P_{\bar{A}}$ of the target distribution $P_A$
- Choose $m = \left\lfloor \log_2 |T_{P_{\bar{A}}}^n| \right\rfloor$
- Construct a unique mapping $\{0,1\}^m \rightarrow T_{P_{\bar{A}}}^n$

<table>
<thead>
<tr>
<th>Input</th>
<th>Mapping</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>1100</td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
</tr>
<tr>
<td>01</td>
<td>0110</td>
</tr>
<tr>
<td>00</td>
<td>0011</td>
</tr>
</tbody>
</table>
Bit-Metric Decoding (1)

- $\text{label}(X) = b(S)b(A) = B_1 \cdots B_m$.

- Demapper soft-decision:
  \[
  L_i = \log \frac{P_{B_i|Y}(0|Y)}{P_{B_i|Y}(1|Y)}, \quad i = 1, \ldots, m.
  \]

- No iterative demapping!
To the receiver, the channel appears as

\[ B_1 \xrightarrow{P_{L_1|B_1}} L_1 \]

\[ B_2 \xrightarrow{P_{L_2|B_2}} L_2 \]

\[ \vdots \]

\[ B_m \xrightarrow{P_{L_m|B_m}} L_m \]
Achievable Rate (2)

• Achievable rate:\(^1\)

\[ R_{\text{BMD}} = H(B) - \sum_{i=1}^{m} H(B_i|L_i) \]

\[ = \sum_{i=1}^{m} I(B_i; L_i) - D(P_B || \prod_{i} P_{B_i}). \]

\[ (\star) \]

• (\star) known as “BICM capacity”, “pragmatic capacity”,...\(^2\)

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\(^2\) Guillen i Fabregas, Martinez Bit-interleaved coded modulation with shaping, ITW 2010.
Achievable Rate (3)

\[ \frac{1}{2} \log_2 (1 + \text{SNR}) \]

- 32-ASK capacity
- \( R_{BMD} \), dependent bits
- \( R_{BMD} \), independent bits


graph showing the relationship between SNR [dB] and bits/channel use.
Probabilistic Amplitude Shaping with DVB-S2 Codes $\text{FER} \leq 10^{-3}$

\[
\frac{1}{2} \log_2(1 + \text{SNR})
\]

- uniform \(\{8, 16, 32, 64\}\) ASK capacity
- 32ASK, code rate 5/6
- 64ASK, code rate 9/10
FER $10^{-3}$ within 0.69 dB of Capacity

![Graph showing bits/channel use vs. SNR (dB)]

- $\frac{1}{2} \log_2 (1 + \text{SNR})$
- 32ASK, DVB-S2 5/6
- 64ASK, DVB-S2 9/10
- Optimized Protograph
References


- F. Steiner, G Böcherer, G. Liva *Protograph-Based LDPC Code Design for Bit-Metric Decoding*, ISIT 2015.