Internal force analysis and load distribution for cooperative multi-robot manipulation

Sebastian Erhart, *Student Member, IEEE,*
Sandra Hirche, *Senior Member, IEEE*

***Abstract*—The load distribution strategy in cooperative manipulation tasks allocates suitable force and torque setpoints to an ensemble of manipulators in order to implement a desired action on the manipulated object. Due to the manipulator redundancy, the load distribution computed by means of a generalized inverse of the grasp matrix is not uniquely determined. Controversial results on the non-squeezing property of specific load distributions exist in literature. In this article we propose a new paradigm for the analysis of internal wrenches based on the kinematic constraints imposed to the manipulator ensemble. We unify previous results by showing that there exists no unique non-squeezing load distribution and illustrate the consequences of our findings by means of several examples. In particular, the presented results provide a new perspective on the decomposition of interaction forces into internal and external components as required for cooperative multi-manipulator control schemes.**

***Index Terms*—Cooperative Manipulators, Load distribution, Internal stress, Grasping, Kinematics.**

I. Introduction

In cooperative manipulation tasks several manipulators handle a common object. Compared to the single manipulator case the task capacity in terms of payload and dexterity is significantly increased. Application domains range from manufacturing, construction, agriculture and forestry via service robotics to search and rescue robotics. The benefits come at the cost of an increased complexity for coordinating the manipulator ensemble. In order to transfer the object from an initial to a final configuration, the manipulators need to apply suitable forces and torques to the manipulated object. Due to multiple manipulators participating in the cooperative task, in general there exists an infinite number of individual end-effector forces and torques which result in the same desired force/torque applied to the object. A similar challenge arises in the context of grasping with multi-fingered robotic hands. The intrinsic input redundancy needs to be resolved by the load distribution strategy.

The load distribution problem in robotic manipulation tasks can be classified as a control allocation problem for an over-actuated mechanical systems, see e.g. [1] for a recent survey. The load distribution in manipulation tasks, however, is a particular input allocation problem, in which the redundant degrees of freedom for choosing the input can be given a meaningful interpretation in terms of motion-inducing components and internal wrenches applied to the object. A typical control goal in robotic manipulation tasks is the decoupled control of internal and external force/torque components [2]–[4]. This topic has received quite some attention in the robotics literature. One of the first works addressing force control in a multi-manipulator setup is [5], resolving the load distribution problem by means of a linearly constrained quadratic optimization routine. A scalar weighting factor is introduced in order to balance between assigned end-effector forces and torques, resulting in a weighted pseudoinverse for the load distribution problem. The authors of [6] claim that only a specific *non-squeezing* pseudoinverse avoids internal loading of the object. This particular load distribution is subsequently used for the analysis of interaction forces, i.e. the decomposition of manipulator forces/torques into internal and external components [7].

S. Erhart and S. Hirche are with the Chair of Information-oriented control, Technische Universität München, Arcisstraße 21, D-80290 München, Germany e-mail: {erhart,hirche}@tum.de.

Recently, the authors of [8] challenged the result for the non-squeezing pseudoinverse in [6] and proposed to use the Moore-Penrose inverse instead. A common interpretation of internal loading is that the difference between two end effector forces projected onto their geometric connecting line does not vanish [7], [9]. However, it is not clear how to extend this concept in a meaningful way to describe internal torques. Beyond the scope of cooperative multi-robot manipulation, internal forces play a central role in the context of manipulation with multi-fingered robot hands [10]. A geometrically inspired definition of internal forces is presented in [11], trying to resolve inconsistencies occurring with the use of the pseudoinverse. An alternative characterization of internal forces is presented in [12] wherein the ensemble of manipulators is approximated as an articulated mechanism. Internal forces are interpreted as the actuator wrenches required to lock this mechanism. However, the influence of the applied end-effector forces on the resulting torque is neglected. In summary, the complete characterization of internal forces *and* torques is still an open issue as well as suitable load distribution strategies that avoid internal wrenches applied to the object. The solution to the problem is essential for multi-robot manipulation. The need is particularly obvious in case of heterogeneous manipulators with different payload capacities, where the freedom to select a capacity compliant load distribution is quintessential to solve the task.

In this article we present a novel approach to the load distribution in cooperative manipulation tasks and we provide a physically motivated characterization of internal wrenches. We characterize internal loading as violation of the kinematic constraints imposed to the manipulator ensemble and derive an analytical expression for all non-squeezing load distributions. Based on our approach, we can show that the proposed load distribution in [6] is actually *not* free of internal wrenches and that the load distribution strategies presented in [8] and [12] are special cases among the set of all non-squeezing load distributions. Moreover, we show that heterogeneous load distributions do not necessarily induce internal wrenches as postulated in previous works. Our results on the grasp matrix inverse apply readily to the computation of internal forces in multi-fingered manipulation and grasp force optimization.

The remainder of this article is organized as follows. Preliminaries on rigid body kinematics and statics are presented in Section II. In Section III we present our main result on load distribution schemes free of internal wrenches. The results are illustrated and discussed in Section IV. Conclusions and directions for future work are presented in Section V.

II. Preliminaries

The manipulated object is assumed to be rigid and the end-effectors are assumed to be rigidly connected to the object. The kinematic quantities are depicted in Fig. 1.

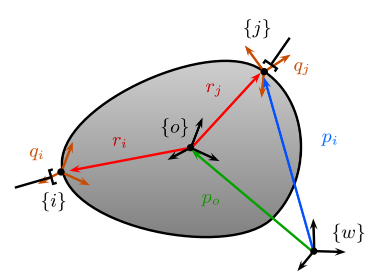

Fig. 1. Kinematic quantities relevant to the multi-robot manipulation task.

The pose of the i-th end-effector is denoted x_i and is composed of a position and orientation component, i.e. $x_i = \left(p_i^T, q_i^T\right)^T$ wherein

$p_i \in \mathbb{R}^3$ is the translation vector pointing to the i-th end-effector and $q_i \in \text{Spin}(3)$ is a unit quaternion representing the orientation of this end-effector. Throughout this paper all quantities are expressed w.r.t. the (inertial) world frame $\{w\}$ whenever the reference frame is not explicitly indicated by a leading superscript. Without loss of generality we assume in the sequel that the body-fixed object frame $\{o\}$ is located in the object's center of mass. This assumption is non-restrictive since the choice of this frame is a priori arbitrary and it can thus conveniently be translated. Note that no assumption is made about the actual object mass or inertia nor about its physical mass distribution. $S(\cdot)$ denotes the skew-symmetric matrix performing the cross-product operation, i.e. $a \times b = S(a) \cdot b$. I_3 and 0_3 are the 3×3 identity and zero matrix respectively. The quaternion product is expressed by $*$ and $R(q) \in SO(3)$ is the rotation matrix associated to the unit quaternion q. The matrix P^+ denotes a generalized inverse of the matrix P and $P^\dagger$ the Moore-Penrose inverse.

A. Rigid body kinematics

1) Translational constraint: The rigidity condition constrains the relative displacement of the object and the attached end effectors $r_i \in \mathbb{R}^3$ (cf. Fig. 1), i.e. ${}^o r_i = \text{const.}$. This means that the relative position of the manipulator with respect to the object in the body-fixed coordinate system $\{o\}$ remains constant. Using this fact one may express the position of the i-th end-effector as $p_i = p_o + {}^w R_o(q_o)\, {}^o r_i$ with the rotation matrix ${}^w R_o$ depending on the object orientation q_o. Differentiation of p_i in the inertial frame $\{w\}$ yields $\dot{p}_i = \dot{p}_o + \omega_o \times r_i$. Differentiating $\dot{p}_i$ again leads to

$$\ddot{p}_i = \ddot{p}_o + \dot{\omega}_o \times r_i + \omega_o \times (\omega_o \times r_i). \tag{1}$$

2) Rotational constraint: The relative orientation between object and manipulators is constrained, too. The use of quaternions for the parameterization of $SO(3)$ leads to the following result.

Lemma 1. *The kinematic constraint ${}^o\delta q_i = \text{const.}$, enforcing a constant relative orientation between two bodies $\{o\}$ and $\{i\}$, is equivalent to the bodies having identical angular velocities $\omega_o = \omega_i$.*

Proof. See Appendix A. □

Differentiation of ω_o w.r.t. time leads to

$$\dot{\omega}_o = \dot{\omega}_i \tag{2}$$

imposing a constraint on the admissible angular acceleration of the object and the end-effector.

B. Rigid body statics

In the sequel we refer to the general case in which each manipulator is able to apply forces *and* torques to the manipulated object. The force and torque applied by the i-th end-effector are denoted $f_i, t_i \in \mathbb{R}^3$ and concatenated to the *wrench* denoted $h_i = \left(f_i^T, t_i^T\right)^T$. The stacked vector containing all end-effector wrenches is defined as $h = (h_1^T, \ldots, h_N^T)^T$. The effective wrench acting on the origin of the object frame $\{o\}$ is denoted h_o and is unambiguously determined given the manipulator wrenches h_i according to

$$h_o = G \begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix}, \tag{3}$$

wherein the grasp matrix G [13, p. 705] depends explicitly on the kinematic parameters r_i as

$$G = \begin{bmatrix} I_3 & 0_3 & \cdots & I_3 & 0_3 \\ S(r_1) & I_3 & \cdots & S(r_N) & I_3 \end{bmatrix}. \tag{4}$$

III. Load distribution

Given a desired wrench to be applied to the object h_o^d, the set of potential desired end effector wrenches h_i^d realizing h_o^d is not unique. One approach for allocating the individual manipulator load is the use of a generalized inverse of the grasp matrix G to resolve the input redundancy inherent to (3). Thus one has

$$\begin{pmatrix} h_1^d \\ \vdots \\ h_N^d \end{pmatrix} = G^+ h_o^d. \tag{5}$$

In contrast to previous results on load allocation by means of a generalized inverse G^+ (and specifically the choice of a suitable metric in order to avoid internal wrenches), we present a more general concept of internal wrenches yielding novel degrees of freedom for the choice of a load distribution. The authors of [6] suggest that there exists a unique wrench distribution based on a particular metric which does not induce internal wrenches. Recently this result was challenged by the authors of [8] presenting the Moore-Penrose inverse $G^\dagger$ as an admissible load allocation strategy. In the sequel we show that there is *no unique* solution to the load distribution problem in (5) avoiding internal wrenches.

A. Characterization of internal wrenches

This section deals with the analysis of internal wrenches which were previously defined in [6] as the components of the wrench vector h lying in the null space of the grasp matrix G. We propose a more general formulation of internal wrenches through the following definition.

Definition 1. *Internal wrenches are end-effector wrenches for which the total virtual work is zero for any virtual displacement of the end-effectors satisfying the kinematic constraints.*

This definition has some important consequences. One immediate observation is that internal wrenches do no work to the common object. That is, internal wrenches according to Definition 1 are not *motion-inducing* and are thus in line with the nomenclature in [6]. Note that in particular any wrench belonging to the null space of the grasp matrix G yields a total virtual work of zero for an arbitrary virtual displacement compatible with the constraints. The most important difference of Definition 1 compared to previous definitions is that it is based on the kinematic constraints between the end-effectors and hence independent of the null space of G. Moreover, Definition 1 is consistent with the concept of constraining wrenches in the context of constrained multi-body systems [14]. It is well-known from Lagrangian mechanics that the total virtual work done by the constraining wrenches is zero. Internal wrenches can thus be interpreted as wrenches ensuring compliance of the manipulator motion to the imposed constraints. This idea is illustrated in Fig. 2.

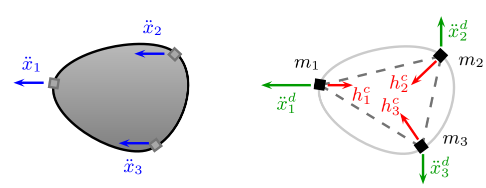

Fig. 2. Illustration of internal wrenches in a multi-robot manipulation task. The actual motion of the manipulators $\ddot{x}$ is the superposition of their desired motion $\ddot{x}^d$ and the interaction in terms of the constraining wrenches h^c.

The *desired* end-effector accelerations $\ddot{x}_i^d$ are usually available when employing common control schemes as e.g. computed torque [13, p. 143]. The set of the $\ddot{x}_i^d$ do not necessarily have to respect the kinematic constraints as depicted on the right-hand side of Fig. 2. However, the constraining wrenches h_i^c render the *actual* end-effector accelerations $\ddot{x}_i$ compatible to the imposed constraints as depicted on the left-hand side of Fig. 2. This observation links the computation of internal wrenches closely to the kinematics of the cooperative manipulator system.

In order to quantify the constraining wrenches, we reformulate the kinematic constraints in (1) and (2) in matrix form as

$$A\ddot{x} = b \tag{6}$$

by letting $\ddot{x} = (\ddot{x}_1^T, \dots, \ddot{x}_N^T)^T$, the matrix $A \in \mathbb{R}^{6(N-1)\times 6N}$

$$A = \begin{bmatrix} -I_3 & S(\Delta r_{21}) & I_3 & 0_3 & & & \\ 0_3 & -I_3 & 0_3 & I_3 & & & \\ \vdots & \vdots & & & \ddots & & \\ -I_3 & S(\Delta r_{N1}) & & & & I_3 & 0_3 \\ 0_3 & -I_3 & & & & 0_3 & I_3 \end{bmatrix}, \tag{7}$$

the vector $b \in \mathbb{R}^{6(N-1)}$ incorporating the centripetal terms $b = ([S(\omega_2)^2\Delta r_{21}]^T, 0_{1\times 3}, \dots, [S(\omega_N)^2\Delta r_{N1})]^T, 0_{1\times 3})^T$ and the relative grasp position $\Delta r_{ji} := r_j - r_i$. The constraining wrenches h^c, characterizing internal wrenches in the sense of Definition 1, result from a projection of the desired end-effector accelerations $\ddot{x}^d$ onto the kinematic constraints [14] according to

$$h^c = M^{\frac{1}{2}}(AM^{-\frac{1}{2}})^{\dagger}(b - A\ddot{x}^d) \tag{8}$$

with the matrix $M = \mathrm{diag}(m_1 I_3, J_1, \dots, m_N I_3, J_N)$ incorporating the apparent inertia of the robotic end-effectors in task-space. In fact, the kinematic error $e = b - A\ddot{x}^d$ indicates if the desired end-effector accelerations $\ddot{x}^d$ violate the imposed kinematic constraint (6). Moreover, the constraining wrench h^c vanishes whenever the acceleration of the manipulators is compatible to the kinematic constraints.

Previously, the computation of internal wrenches was performed via a decomposition of the manipulator wrenches [7] according to $h = h^{\text{ext}} + h^{\text{int}}$ into an external (motion-inducing) component h^{ext} and an internal component h^{int} without incorporating the end-effector kinematics. This approach depends implicitly on a specific load distribution for computing the generalized inverse. This becomes clear as the external/internal components are determined via

$$h^{\text{ext}} = G^+Gh \quad \text{and} \quad h^{\text{int}} = (I_{6N\times 6N} - G^+G)h, \tag{9}$$

which is based on the assumption that there exists a unique non-squeezing wrench distribution: firstly the resulting object wrench h_o is computed as presented in (3) given the manipulator wrenches h. Subsequently, the external wrenches are defined as the wrenches resulting from a particular load distribution as e.g. $G^+ = G^{\dagger}$ in [8] or $G^+ = G_{\Delta}^+$ in [6] with

$$G_{\Delta}^+ = \frac{1}{N}\begin{bmatrix} I_3 & 0_3 \\ S(r_1)^T & I_3 \\ \vdots & \vdots \\ I_3 & 0_3 \\ S(r_N)^T & I_3 \end{bmatrix}. \tag{10}$$

Remark: In continuum mechanics, internal stress is defined as the contact force between neighboring particles inside a solid body. In the scope of this article on cooperating manipulators we are not interested in the actual stress distribution *inside* the commonly manipulated object - internal stress occurs even when manipulating a rigid object with a single end-effector and can thus not be avoided.

B. Load distributions free of internal wrenches

With Definition 1 and its consequences as discussed in the previous subsection, we are ready to state our main result.

Theorem 1. *The load distribution given by*

$$G_M^+ = \begin{bmatrix} m_1^*[m_o^*]^{-1}I_3 & m_1^*[J_o^*]^{-1}S(r_1)^T \\ 0_3 & J_1^*[J_o^*]^{-1} \\ \vdots & \vdots \\ m_N^*[m_o^*]^{-1}I_3 & m_N^*[J_o^*]^{-1}S(r_N)^T \\ 0_3 & J_N^*[J_o^*]^{-1} \end{bmatrix} \tag{11}$$

for some positive-definite weighting coefficients $m_i^* \in \mathbb{R}$ and $J_i^* \in \mathbb{R}^{3\times 3}$ with

$$m_o^* = \sum_i m_i^* \tag{12}$$

$$J_o^* = \sum_i J_i^* + \sum_i S(r_i)m_i^*S(r_i)^T, \tag{13}$$

and

$$\sum_i r_i m_i^* = 0_{3\times 1} \tag{14}$$

is free of internal wrenches applied to the object.

Proof. The proof is based on a particular parameterization of the generalized inverse of the grasp matrix. This parameterization appears naturally when considering the dynamics of a *virtual* end-effector system subject to the kinematic constraints and allows to give these parameters the meaning of virtual masses and inertias. With h_o^d in hand, one readily computes the resulting virtual acceleration $\ddot{x}_o^*$ which the object would experience if it had the mass m_o^* and inertia J_o^* under the assumption that only the desired wrench h_o^d was acting on the object. This is done by inverting

$$\begin{bmatrix} m_o^*I_3 & 0_3 \\ 0_3 & J_o^* \end{bmatrix}\ddot{x}_o^* = h_o^d. \tag{15}$$

With this virtual object acceleration $\ddot{x}_o^*$ we can conclude on the (virtual) acceleration of the attached end-effectors $\ddot{x}_i^*$ by employing the kinematic constraints (1) and (2). By assigning now virtual inertias m_i^* and J_i^* to the i-th end-effector, it is straightforward to compute the required wrench h_i^d inducing the virtual end-effector acceleration $\ddot{x}_i^*$ according to

$$h_i^d = \begin{bmatrix} m_i^*I_3 & 0_3 \\ 0_3 & J_i^* \end{bmatrix}\ddot{x}_i^*. \tag{16}$$

So far, all occurring virtual inertias and thus the individual manipulator wrenches h_i^d are undetermined. However, any admissible load distribution should satisfy (3), i.e. $h_o^d = Gh^d$ being equivalent to $f_o^d = \sum_i f_i^d$ and $t_o^d = \sum_i t_i^d + \sum_i r_i \times f_i^d$. Substituting (15) and (16) for the force components and employing (1) leads to

$$m_o^*\ddot{p}_o^* = \sum_i m_i^* \left[\ddot{p}_o^* + \dot{\omega}_o^* \times r_i + \omega_o^* \times (\omega_o^* \times r_i)\right]. \tag{17}$$

Comparing the coefficients of $\ddot{p}_o^*$ immediately yields $m_o^* = \sum_i m_i^*$. Since $\dot{\omega}_o^*$ (and ω_o^*) can take arbitrary values, the virtual masses need to respect $\sum_i r_i m_i^* = 0_{3\times 1}$ in order to cancel the terms involving $\dot{\omega}_o^*$ and ω_o^* in (17). Considering the torque components in (3) and again substituting (15) and (16) combined with (1) and (2) one has

$$J_o^*\dot{\omega}_o^* = \sum_i J_i^*\dot{\omega}_o^* + \sum_i r_i \times m_i^*[\ddot{p}_o^* + \dot{\omega}_o^* \times r_i + \omega_o^* \times (\omega_o^* \times r_i)]. \tag{18}$$

Comparing coefficients yields $J_o^* = \sum_i J_i^* + \sum_i S(r_i) m_i^* S(r_i)^T$ wherein the cross-product is expressed in terms of skew-symmetric matrices. The term involving $\ddot{p}_o^*$ on the right-hand side of (18) vanishes (since $\sum_i r_i m_i^* = 0_{3\times1}$) such that only the additional term $\sum_i r_i \times m_i^* [\omega_o^* \times (\omega_o^* \times r_i)]$ remains. Recall that (15) determines solely a *virtual* object acceleration due to h_o^d at a specific time instant but no information about the object's virtual velocity is available. In order to obtain an admissible load distribution satisfying (3), a convenient choice is thus $\omega_o^* = 0_{3\times1}$ which eliminates the impact of the virtual product of inertia-like term. Note that this does not mean that the manipulated (physical) object needs to be at rest since $\ddot{x}_o^*$ is in general different from the object's *actual* acceleration (and velocity). The choice for the object's virtual velocity $\omega_o^* = 0_{3\times1}$ is arbitrary and simply ensures that at one specific time instant and a given h_o^d, an admissible set of end-effector wrenches h_i^d is computed - completely independent from the actual object dynamics. By construction, the total virtual work done by the end-effector wrenches is non-zero for any virtual displacement satisfying the constraints. The load distribution is thus free of internal wrenches according to Definition 1. □

Note that the weighting coefficients m_i^* and J_i^* (and consequently m_o^* and J_o^*) do have the meaning of inertial parameters but they are abstract parameters. They are exclusively used to parameterize the generalized inverse G_M^+ for the purpose of load distribution but they do not characterize the inertial properties of the manipulated object. A particular choice of these weighting coefficients leads to

Corollary 1. *An equal distribution of the manipulator weights according to $m_i^* = 1$ and $J_i^* = I_3$ yields*

$$G^\dagger = \frac{1}{N}\begin{bmatrix} I_3 & \bar{J}^{-1}S(r_1)^T \\ 0_3 & \bar{J}^{-1} \\ \vdots & \vdots \\ I_3 & \bar{J}^{-1}S(r_N)^T \\ 0_3 & \bar{J}^{-1} \end{bmatrix} \tag{19}$$

with $\bar{J} = I_3 + \frac{1}{N}\sum_i S(r_i)S(r_i)^T$ and (19) *being equivalent to the Moore-Penrose inverse of G.*

Proof. The Moore-Penrose inverse of a matrix might be interpreted as the solution to a quadratic programming problem with equality constraint. Thus the load distribution problem is reformulated as

$$\begin{aligned} \min_{h^d} \quad & \|h^d\|^2. \\ \text{s.t.} \quad & h_o^d = G h^d \end{aligned} \tag{20}$$

An explicit, analytical solution to this optimization problem can be obtained by computing the Schur complement

$$\bar{S} := GG^T = N\begin{bmatrix} I_3 & 0_3 \\ 0_3 & I_3 + \frac{1}{N}\sum_i S(r_i)S(r_i)^T \end{bmatrix} \tag{21}$$

which is used for computing the desired mapping

$$h^d = G^T \bar{S}^{-1} h_o^d. \tag{22}$$

By definition the Moore-Penrose inverse is equivalent to the solution of the minimization problem (20) such that

$$G^\dagger = G^T \bar{S}^{-1}. \tag{23}$$

Straightforward computation of $G^T\bar{S}^{-1}$ reveals equivalence of this expression to (19). □

IV. Illustrative Examples

The previously presented results admit some remarkable insights which will be discussed by means of the subsequent examples.

A. Non-squeezing heterogeneous load distribution

Consider the one-dimensional multi-robot manipulation example depicted in Fig. 3, wherein two manipulators cooperatively move an object in the horizontal direction.

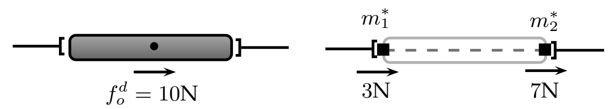

Fig. 3. Two cooperating manipulators moving a rigid object in one dimension.

The force distribution indicated at the right-hand side of Fig. 3 results obviously in the desired object force f_o^d. The relevant question in fact is if the load distribution contains internal forces applied to the object. In contrast to previous results [6]–[8] we argue that in this case there is *no* internal wrench applied to the object: Both end-effector forces contribute entirely to the desired motion of the object and thus no internal wrench is present. Existing criteria [6], [8] for the analysis of internal wrenches yield an internal force component of $\pm 2N$ for the force distribution in Fig. 3. This is due to the fact that an equal distribution of manipulator forces is assumed implicitly by using G_Δ^+ or $G^\dagger$ for the computation of internal and external wrenches in (9). By letting $m_1^* = 3$kg and $m_2^* = 7$kg the force distribution indicated in Fig. 3 is obtained. Note that this one-dimensional example is equivalent to manipulating a point mass and condition (14) is trivially met through choosing $r_1 = r_2 = 0_{3\times1}$ for any values of m_1^* and m_2^*. By considering an infinitesimal displacement of the end-effectors along the horizontal axis it becomes obvious that the total virtual work done is non-zero and the load distribution is free of internal wrenches. For the example in Fig. 3 this means that no internal wrenches are applied to the object as long as both manipulators agree and move with a common desired acceleration $\ddot{x}_o^d$ while applying the indicated end-effector forces. This observation is in contrast to the results in [6], [7] where the difference in the applied force of two manipulators projected onto their connecting line was used to conclude on internal loading.

B. Internal wrenches as constraint violation

Consider now the cooperative manipulation setup in Fig. 4.

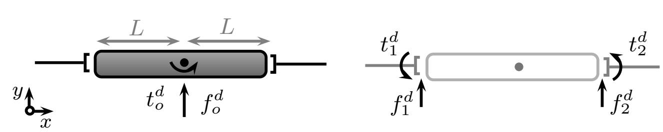

Fig. 4. Load distribution example for two cooperating manipulators.

Assume that the depicted manipulators are supposed to compensate the weight of the object such that only the y-component of the desired object force is non-zero, i.e. $f_{o,y}^d \neq 0$. The load distribution based on G_Δ^+ in (10) suggests to choose $f_{1,y}^d = f_{2,y}^d = \frac{1}{2} f_{o,y}^d$ as one might expect for an equally distributed load, but also $t_{1,z}^d = -t_{2,z}^d = -\frac{L}{2} f_{o,y}^d$, i.e. a non-zero torque for the two manipulators about the z-axis with opposite sign. We have now an illustrative explanation why the wrench distribution given by G_Δ^+ is not free of internal wrenches: the assigned manipulator torques tend to alter the orientation of each end-effector in such a way that the relative orientation does not remain constant and violates the imposed kinematic constraint.

C. Non-uniqueness of load distributions free of internal wrenches

Consider again the multi-robot manipulation setup depicted in Fig. 4. This time we would like to implement a torque about the axis perpendicular to the paper plane, i.e. only $t^d_{o,z} = \tau$ in the desired object wrench h_o. The load distribution according to the Moore-Penrose inverse $G^\dagger$ in (19) gives for $\tau = 1$Nm and $L = 1$m

$$f^d_{1,y} = -f^d_{2,y} = -\frac{1}{4}\text{N} \quad , \quad t^d_{1,z} = t^d_{2,z} = \frac{1}{4}\text{Nm}. \tag{24}$$

The choice of $m^*_i = 4$ and $J^*_i = I_3$ for the load distribution by means of G^+_M in (11) gives

$$f^d_{1,y} = -f^d_{2,y} = -\frac{4}{10}\text{N} \quad , \quad t^d_{1,z} = t^d_{2,z} = \frac{1}{10}\text{Nm}. \tag{25}$$

The load distribution obtained by the modified, non-unitary weights yields a wrench distribution which demands a smaller torque to be applied by the robotic end-effectors but leads to an equivalent object wrench. The ratio between the resulting inertial parameters m^*_o and J^*_o in (12) and (13) can be used to tune the amount of the resulting object torque t^d_o that is either produced by end-effector forces f^d_i acting over a lever r_i, or by direct application of the end-effector torques t^d_i. It is worth noticing that the wrench distribution (25) does not induce internal wrenches at the object. As a limit case for $J^*_i \to 0_3$, the desired object torque t^d_o is exclusively produced by the desired end-effector forces f^d_i and $t^d_i = 0_{3\times1}$.

D. Payload balancing for $N > 3$ manipulators

Condition (14) in Theorem 1 clearly restricts the choice of the admissible values for m^*_i. In particular, this means for $N = 3$ manipulators (and span$[r_1, r_2, r_3] = \mathbb{R}^3$) that there exists only one specific solution for the load distribution coefficients m^*_i. This follows immediately by considering (14) as a system of linear equations in m^*_i, parameterized by r_i. However, for $N > 3$ there exists an infinite number of admissible load distributions which can be used for balancing the payload between the manipulators. To this end, consider a grasp for $N = 5$ with $r_1 = (1, 0, 0)^T$m, $r_2 = (0, 1, 0)^T$m, $r_3 = (0, 0, 1)^T$m, $r_4 = (-1, -1, -1)^T$m and $r_5 = (-1, 0, 0)^T$m. One particular set of load distribution coefficients satisfying (14) with $m^*_o = 1$ is $m^*_1 = 0.368$, $m^*_2 = m^*_3 = m^*_4 = 0.132$ and $m^*_5 = 0.236$. Another set of coefficients meeting (14) and $m^*_o = 1$ is $m^*_1 = 0.256$, $m^*_2 = m^*_3 = m^*_4 = 0.244$ and $m^*_5 = 0.012$. For this second load distribution the resulting payload for the fifth manipulator is significantly lower since $0.012 \ll 0.236$.

V. Conclusions

In this article we present a novel characterization of internal wrenches applied to an object in multi-robot manipulation tasks. Based on our physically motivated approach, we are able to decouple the problem of load distribution and internal wrench analysis for cooperative manipulation tasks. On one hand, we address the load distribution problem by providing a parameterized generalized inverse of the grasp matrix which by construction is free of internal wrenches. On the other hand, we show that it is in general not possible to conclude on the presence of internal wrenches by simply analyzing the manipulator wrenches itself. A consistent analysis of internal wrenches needs to incorporate the end-effector kinematics, too. The relevance of our results is highlighted by means of several examples. The presented approach is demonstrated to be more general than former wrench decomposition approaches yielding additional degrees of freedom for the payload distribution in cooperative multi-robot manipulation tasks and simultaneously providing a fundamentally new perspective on internal forces in the context of dexterous manipulation and grasping.

Appendix
Proof of Lemma 1

The relative orientation between object and the i-th end-effector expressed in the object frame is ${}^o\delta q_i = q_o^{-1} * q_i$. After differentiation of ${}^o\delta q_i$ w.r.t. time one has [15, p. 263f.]

$$0_{4\times1} \quad = \quad (\frac{d}{dt}q_o^{-1}) * q_i + q_o^{-1} * (\frac{d}{dt}q_i) \tag{26}$$

which can be rewritten after rearranging terms as

$$0_{4\times1} = Q(q_o, q_i)\ [\omega_i - \omega_o] \tag{27}$$

with a matrix $Q(q_o, q_i) \in \mathbb{R}^{4\times3}$. The singular values of the matrix Q are $\sigma^2_k(Q) = \|q_o\|^2 \cdot \|q_i\|^2$ for $k \in \{1, 2, 3\}$ and thus the singular values of Q are non-zero for arbitrary unit quaternions q_o and q_i. This in turn means that Q has full rank for any choice of q_o and q_i. Using this fact in (27) it follows that $\omega_i - \omega_o = 0_{3\times1}$.

Acknowledgment

This work has received funding from the EU FP7/2007-2013 project n 601165 WEARHAP. The authors further would like to thank the reviewers for their constructive remarks and comments.

References

[1] T. A. Johansen and T. I. Fossen, "Control allocation a survey," *Automatica*, vol. 49, no. 5, pp. 1087 – 1103, 2013.

[2] R. Bonitz and T. Hsia, "Internal force-based impedance control for cooperating manipulators," *Robotics and Automation, IEEE Transactions on*, vol. 12, no. 1, pp. 78–89, Feb 1996.

[3] F. Caccavale, P. Chiacchio, A. Marino, and L. Villani, "Six-dof impedance control of dual-arm cooperative manipulators," *Mechatronics, IEEE/ASME Transactions on*, vol. 13, no. 5, pp. 576–586, 2008.

[4] D. Heck, D. Kostic, A. Denasi, and H. Nijmeijer, "Internal and external force-based impedance control for cooperative manipulation," in *Control Conference (ECC), 2013 European*, July 2013, pp. 2299–2304.

[5] T. Alberts and D. Soloway, "Force control of a multi-arm robot system," in *Robotics and Automation, 1988. Proceedings., 1988 IEEE International Conference on*, Apr 1988, pp. 1490–1496 vol. 3.

[6] I. D. Walker, R. A. Freeman, and S. I. Marcus, "Analysis of motion and internal loading of objects grasped by multiple cooperating manipulators," *The International Journal of Robotics Research*, vol. 10, no. 4, pp. 396–409, 1991.

[7] R. Bonitz and T. Hsia, "Force decomposition in cooperating manipulators using the theory of metric spaces and generalized inverses," in *Robotics and Automation, 1994. Proceedings., 1994 IEEE International Conference on*, May 1994, pp. 1521–1527 vol. 2.

[8] J. Chung, B.-J. Yi, and W. Kim, "Analysis of internal loading at multiple robotic systems," *Journal of Mechanical Science and Technology*, vol. 19, no. 8, pp. 1554–1567, 2005.

[9] V. Kumar and K. Waldron, "Force distribution in closed kinematic chains," in *Robotics and Automation, 1988. Proceedings., 1988 IEEE International Conference on*, Apr 1988, pp. 114–119 vol. 1.

[10] V. Lippiello, B. Siciliano, and L. Villani, "A grasping force optimization algorithm for multiarm robots with multifingered hands," *Robotics, IEEE Transactions on*, vol. 29, no. 1, pp. 55–67, Feb 2013.

[11] T. Yoshikawa and K. Nagai, "Manipulating and grasping forces in manipulation by multifingered robot hands," *Robotics and Automation, IEEE Transactions on*, vol. 7, no. 1, pp. 67–77, Feb 1991.

[12] D. Williams and O. Khatib, "The virtual linkage: a model for internal forces in multi-grasp manipulation," in *Robotics and Automation, 1993. Proceedings., 1993 IEEE International Conference on*, 1993, pp. 1025–1030 vol. 1.

[13] B. Siciliano and O. Khatib, Eds., *Springer Handbook of Robotics*. Berlin, Heidelberg: Springer, 2008.

[14] F. E. Udwadia and R. E. Kalaba, "A new perspective on constrained motion," *Proceedings: Mathematical and Physical Sciences*, vol. 439, no. 1906, pp. 407–410, 1992.

[15] J. B. Kuipers, *Quaternions and rotation sequences : A primer with applications to orbits, aerospace, and virtual reality*. Princeton, NJ: Princeton Univ. Press, 1999.