UNIVERSITY OF
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Adjoint sensitivity analysis in thermoacoustics

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with thanks to

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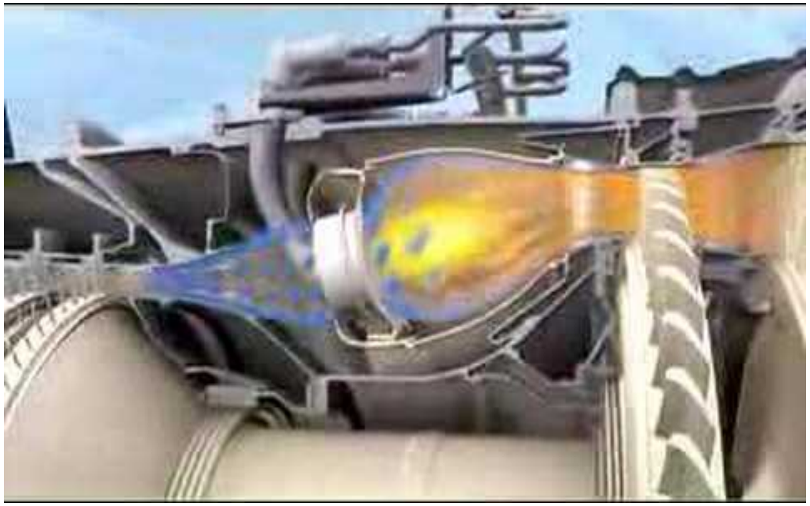
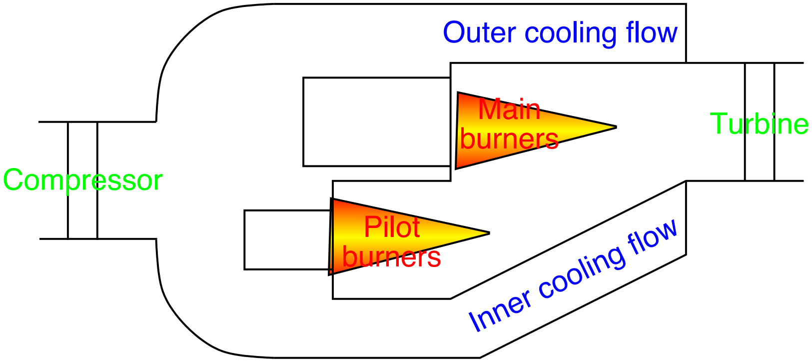
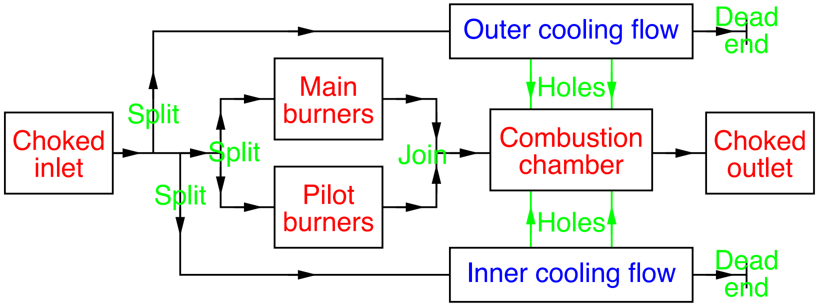
Imagine you have created a linear thermo-acoustic model of a gas turbine

Gas turbine combustion system



Acoustic network model + flame model

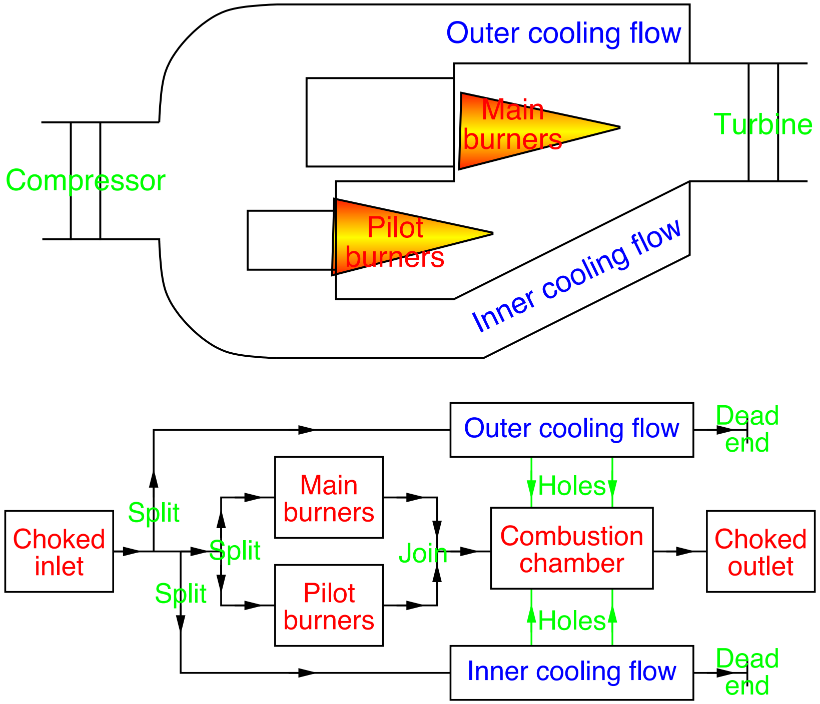




- **Many degrees of freedom**

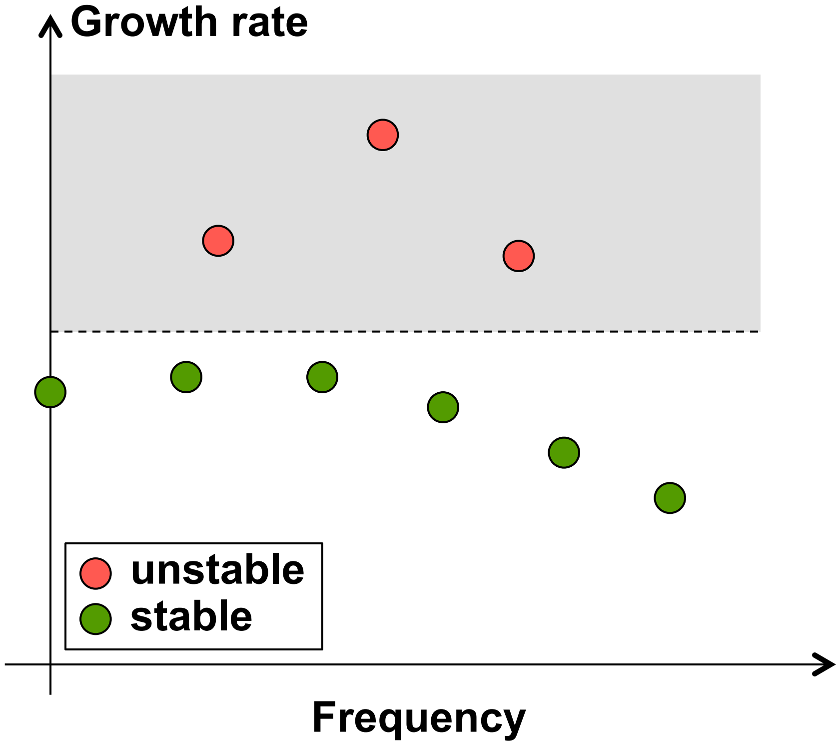
You find the growth rates and frequencies of linear modes of the model (the eigenmodes)

Acoustic network model + flame model



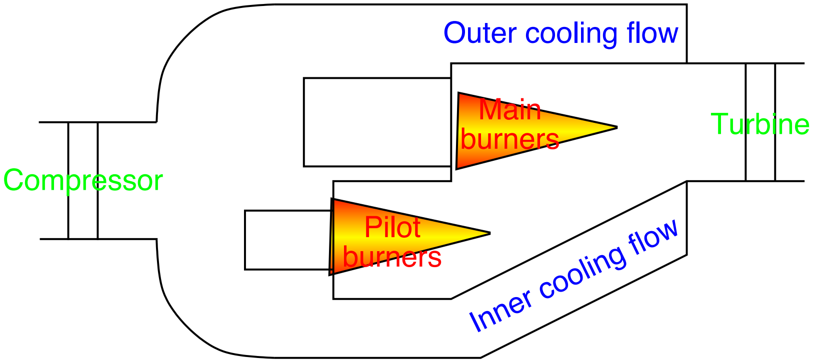
- **Many degrees of freedom**

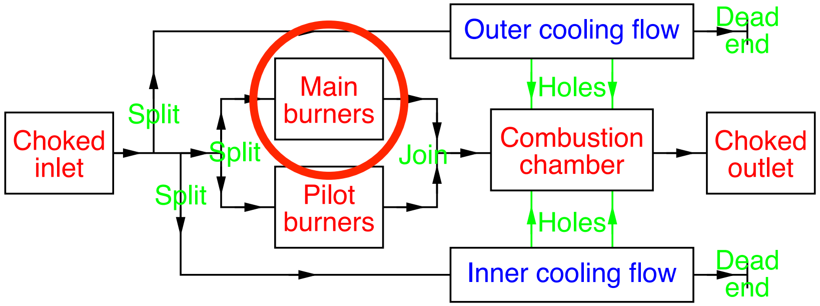
Linear modes of the model



When you change the model, the growth rates and frequencies of the modes also change. You could calculate how much they change using a finite difference method but this would take many calculations.

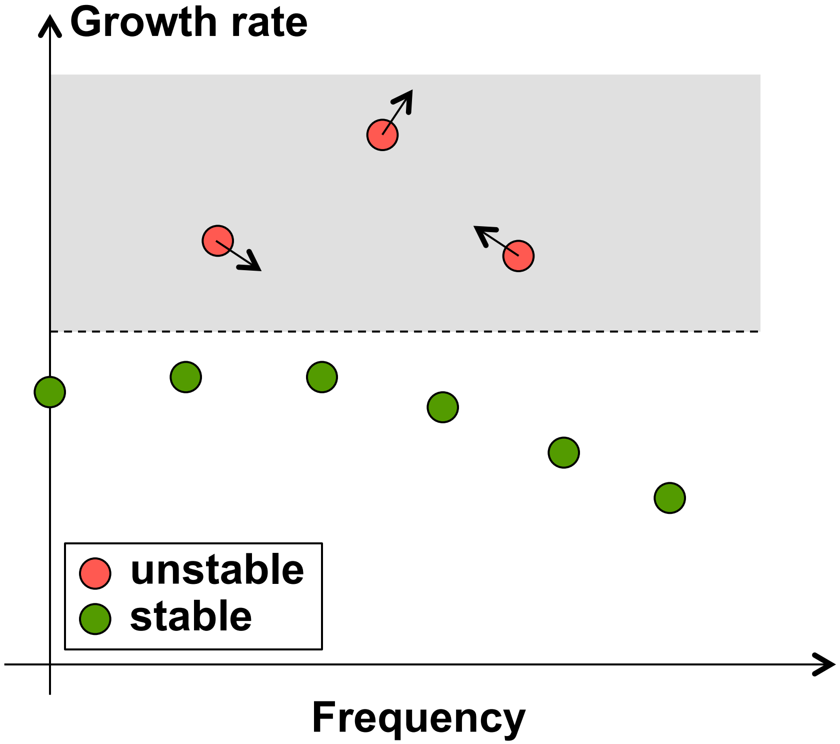
Acoustic network model + flame model





- **Many degrees of freedom**

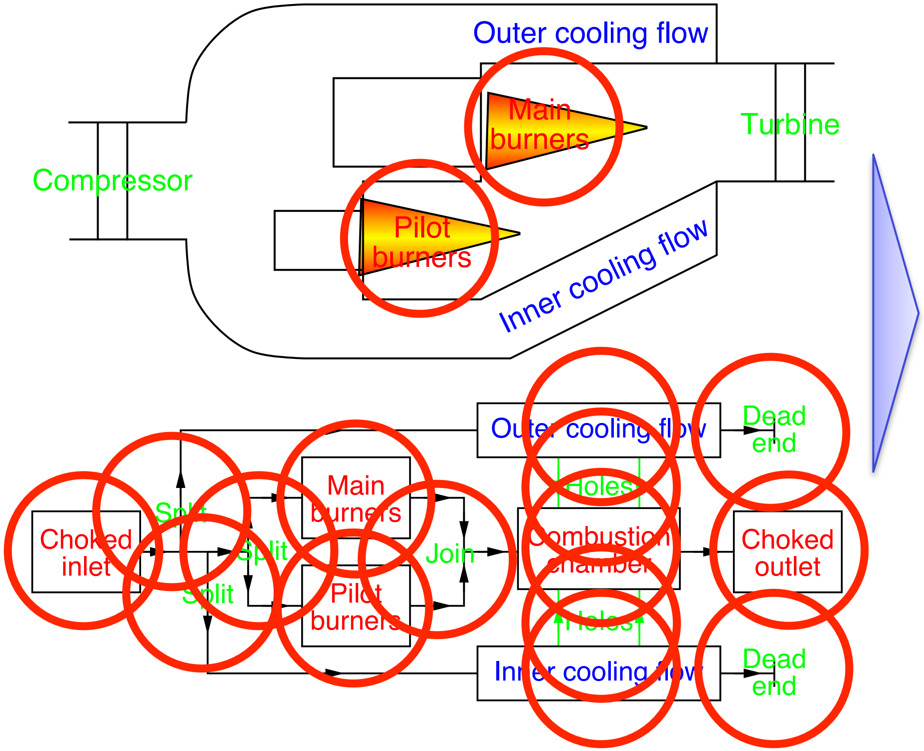
Linear modes of the model

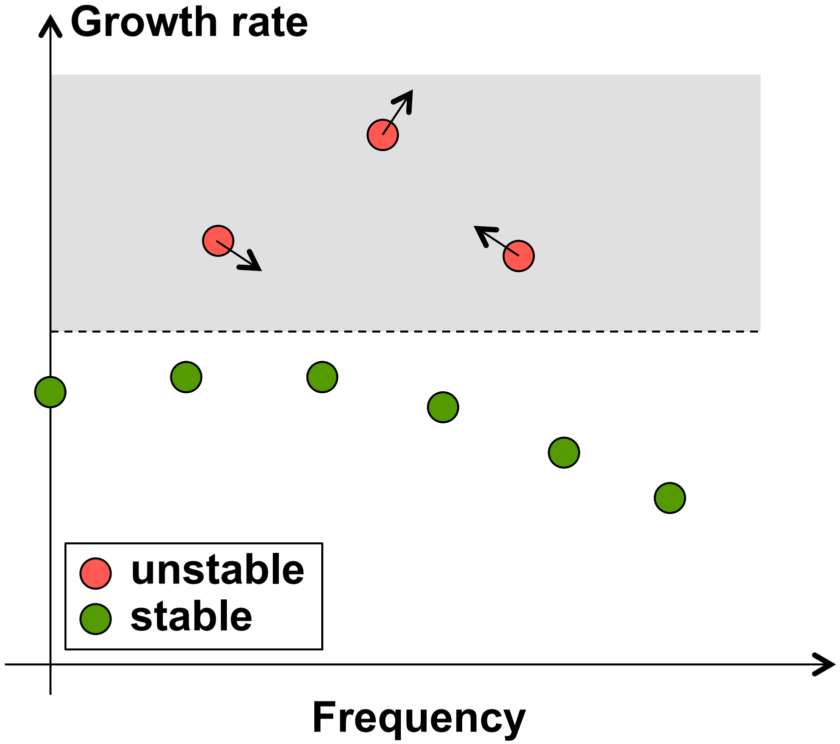


What if you could calculate the sensitivity of an eigenvalue to every single degree of freedom with just two calculations?

Acoustic network model + flame model

Linear modes of the model





- **Many degrees of freedom**


New!
Miracle Cure!
Truly amazing!
Works in minutes!
Guaranteed!

A square matrix, L , can be decomposed into a square matrix, Q , a diagonal matrix, Σ , and the inverse of Q.

$$\mathbf{L} = \mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^{-1}$$

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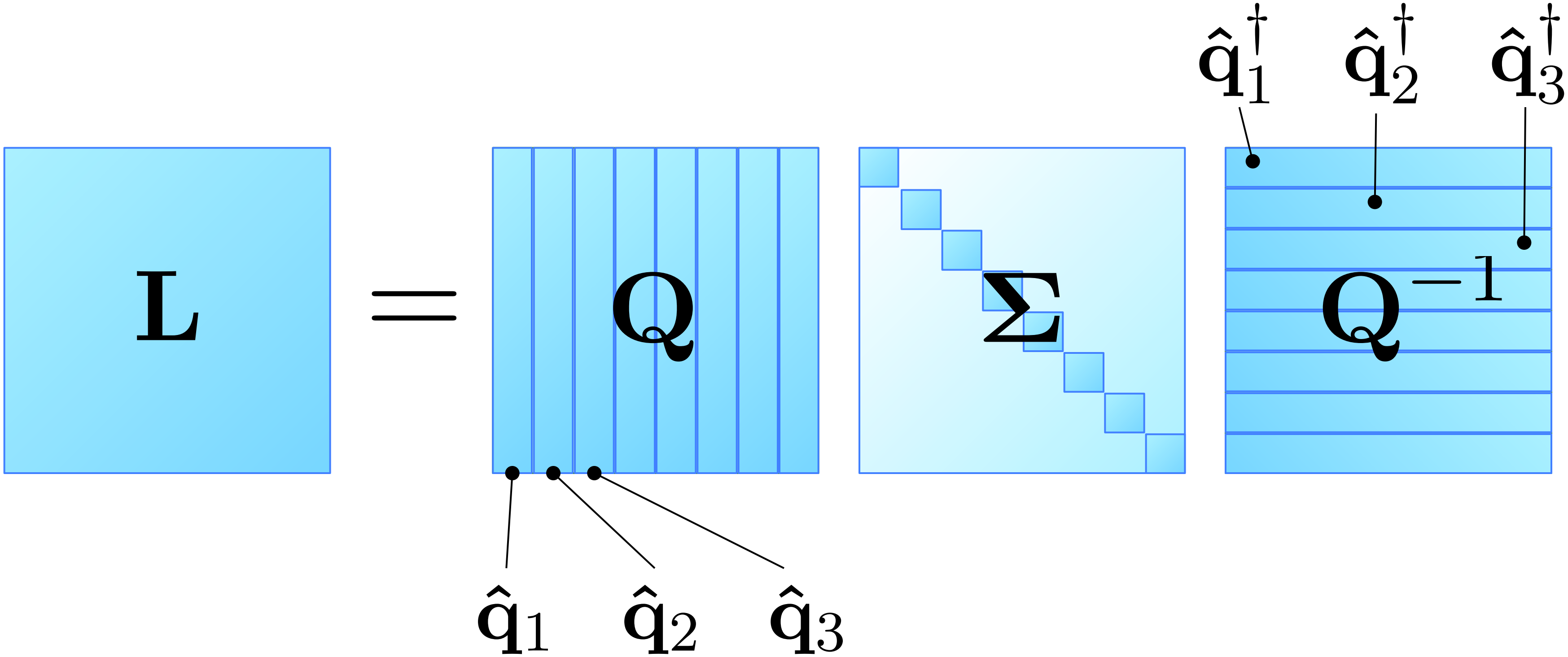

Post-multiplying by Q shows that the columns of Q are the eigenvectors of L .
(In more detail, these are the *right* eigenvectors of L)

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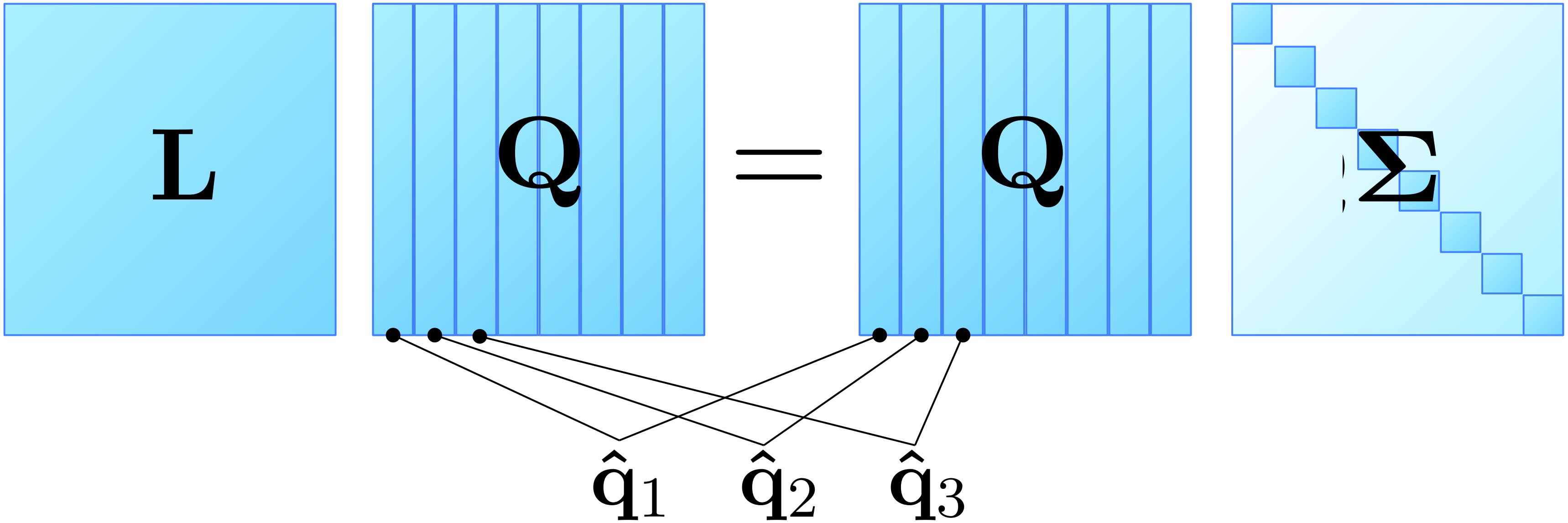
$$\mathbf{L}\mathbf{Q} = \mathbf{Q}\mathbf{\Sigma}$$



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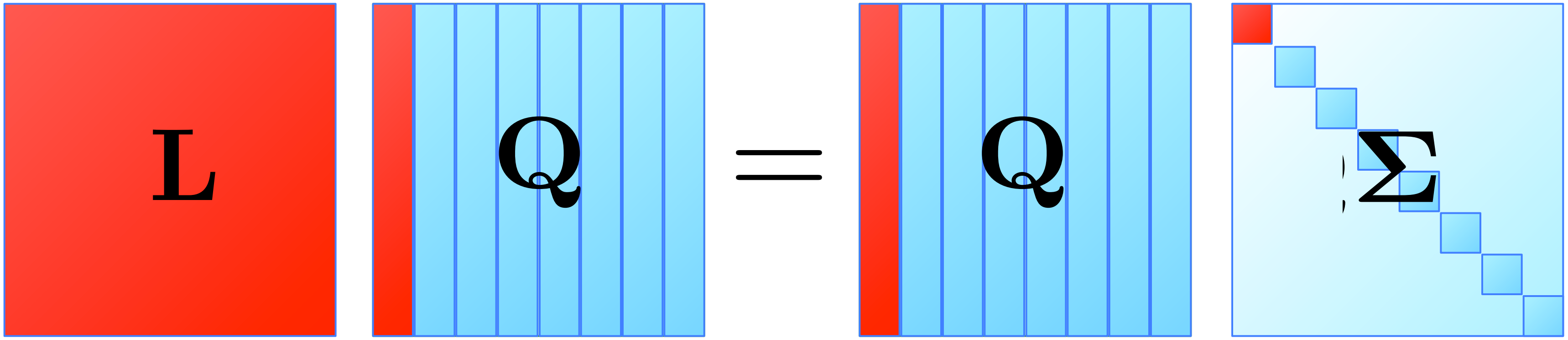

$$\mathbf{L}\hat{\mathbf{q}}_i = \sigma_i\hat{\mathbf{q}}_i$$

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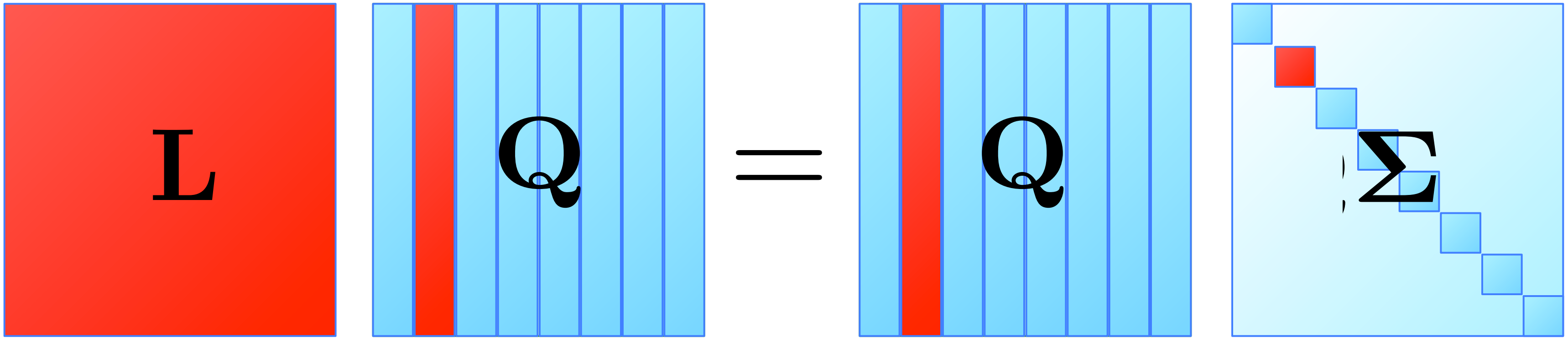
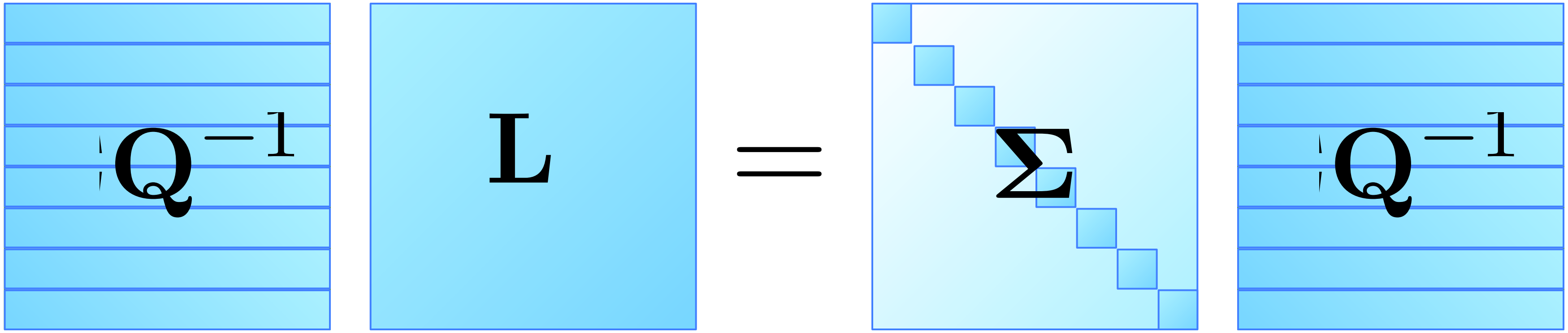


$$\mathbf{L}\hat{\mathbf{q}}_i = \sigma_i \hat{\mathbf{q}}_i$$

Pre-multiplying by Q-1 shows that the rows of Q-1 are the *left* eigenvectors of L

$$\mathbf{L} = \mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^{-1}$$

$$\mathbf{Q}^{-1}\mathbf{L} = \mathbf{\Sigma}\mathbf{Q}^{-1}$$



Pre-multiplying by Q^{-1} shows that the rows of Q^{-1} are the *left* eigenvectors of L

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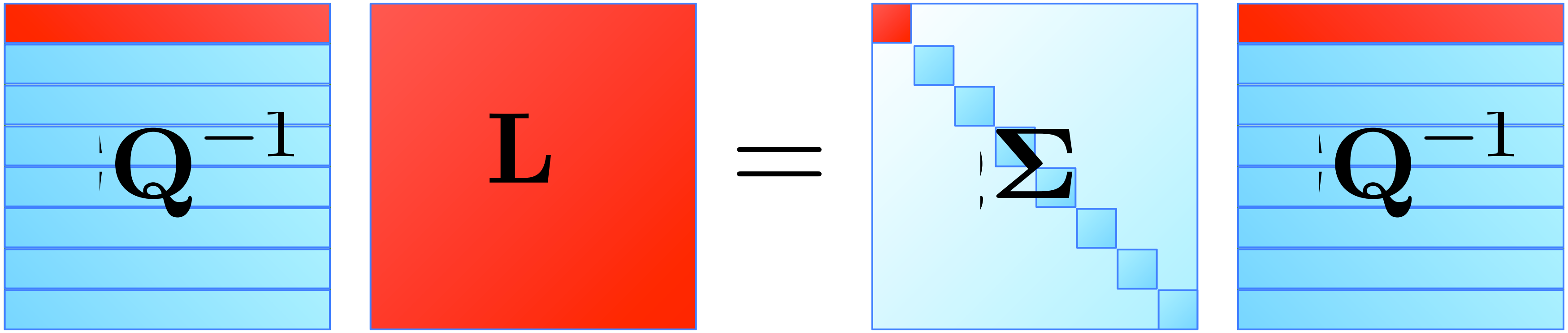

$$(\hat{\mathbf{q}}_i^\dagger)^H \mathbf{L} = \sigma_i (\hat{\mathbf{q}}_i^\dagger)^H$$

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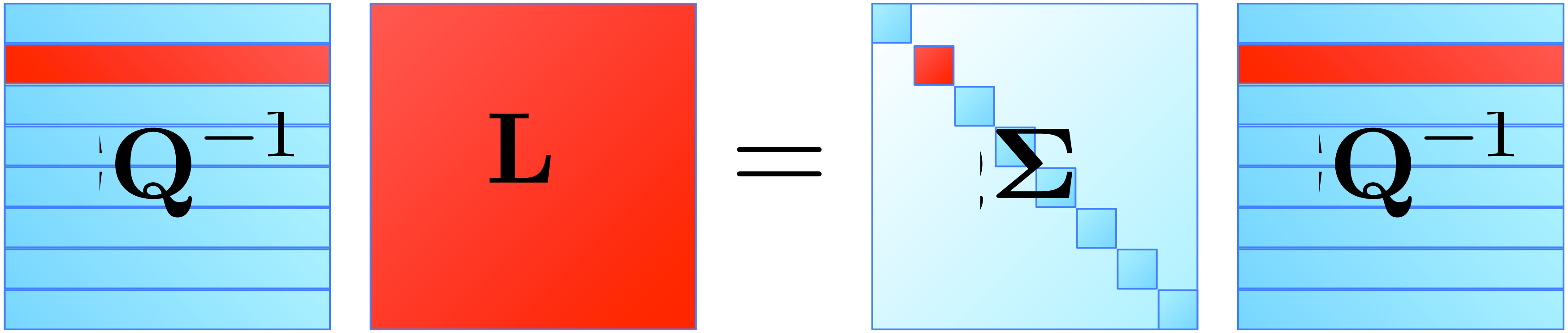
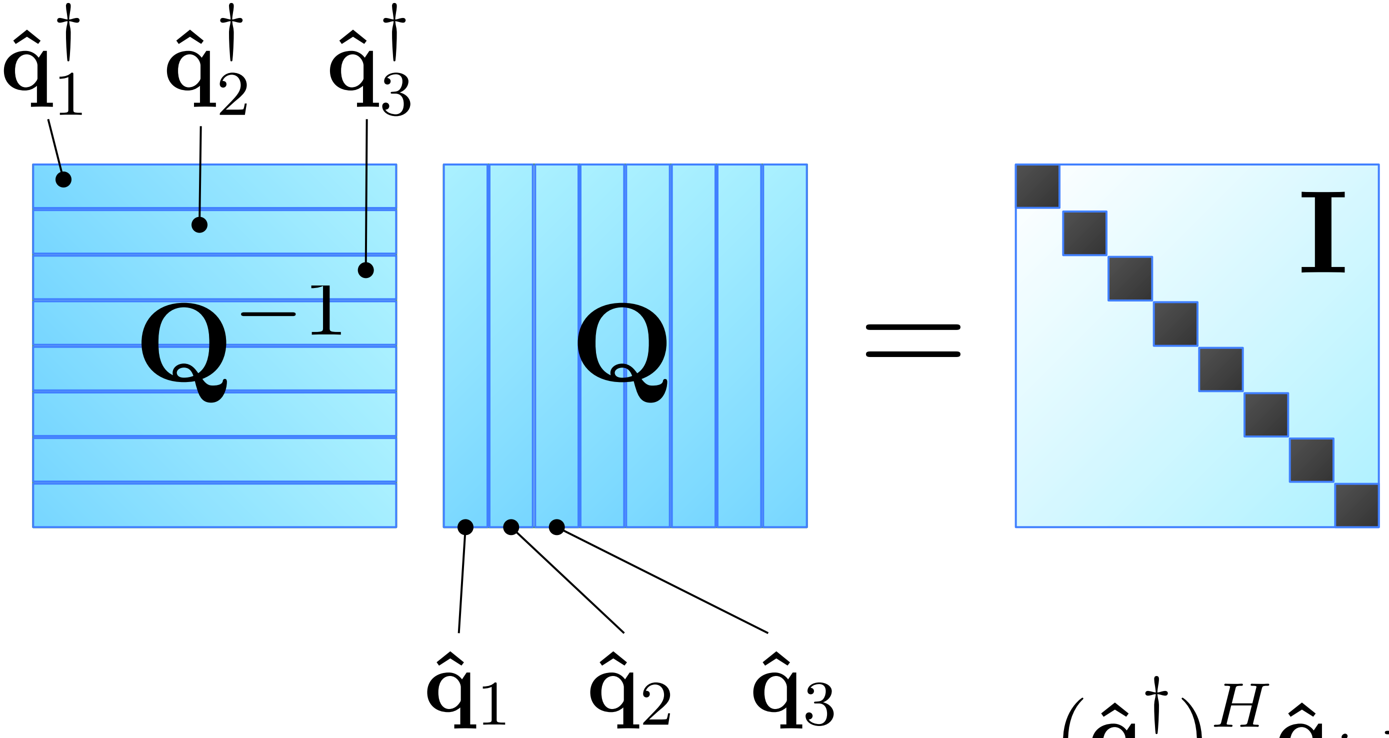


$$(\hat{\mathbf{q}}_i^{\dagger})^H \mathbf{L} = \sigma_i (\hat{\mathbf{q}}_i^{\dagger})^H$$

Because $Q^{-1}Q = I$, the rows of Q^{-1} are orthogonal to all but one of the columns of Q. In other words, the left and right eigenvectors are *bi-orthogonal*.



$$(\hat{\mathbf{q}}_j^\dagger)^H \hat{\mathbf{q}}_i = \delta_{ij}$$

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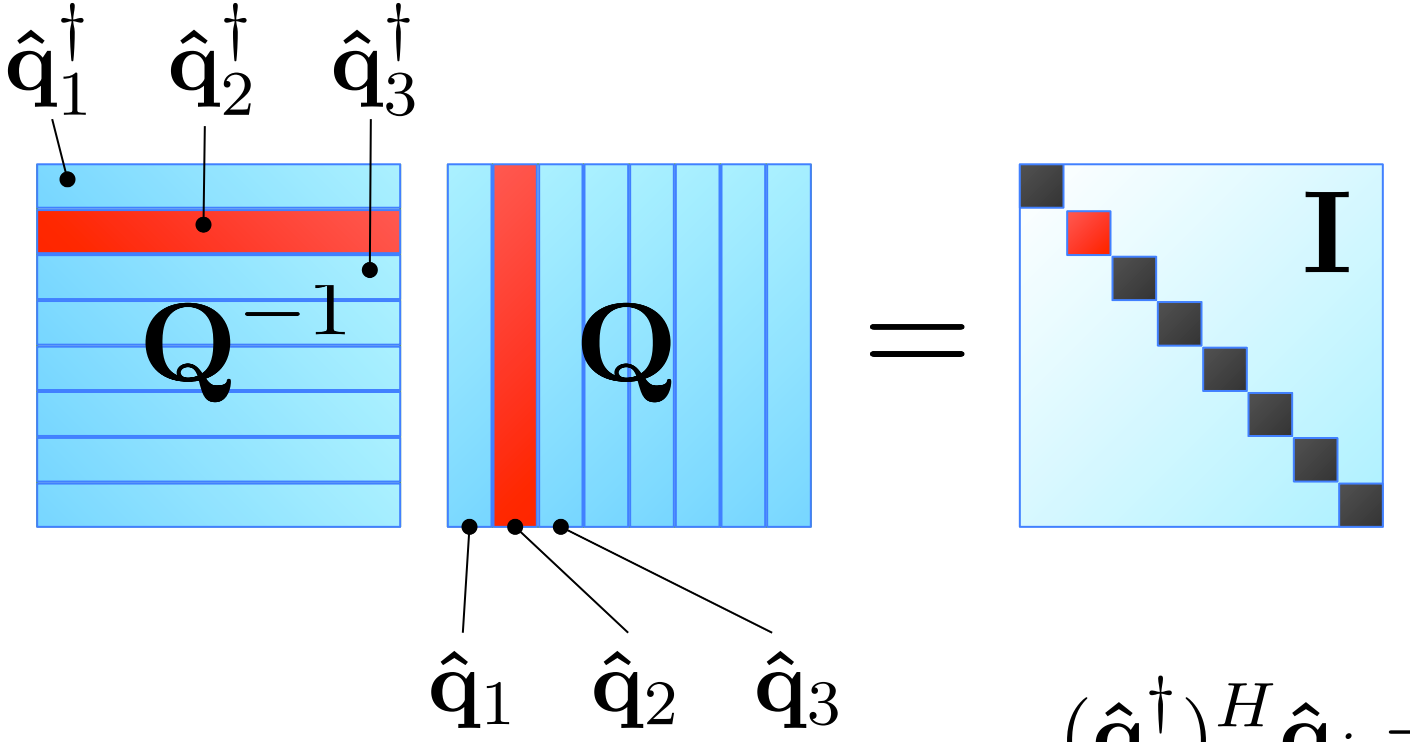

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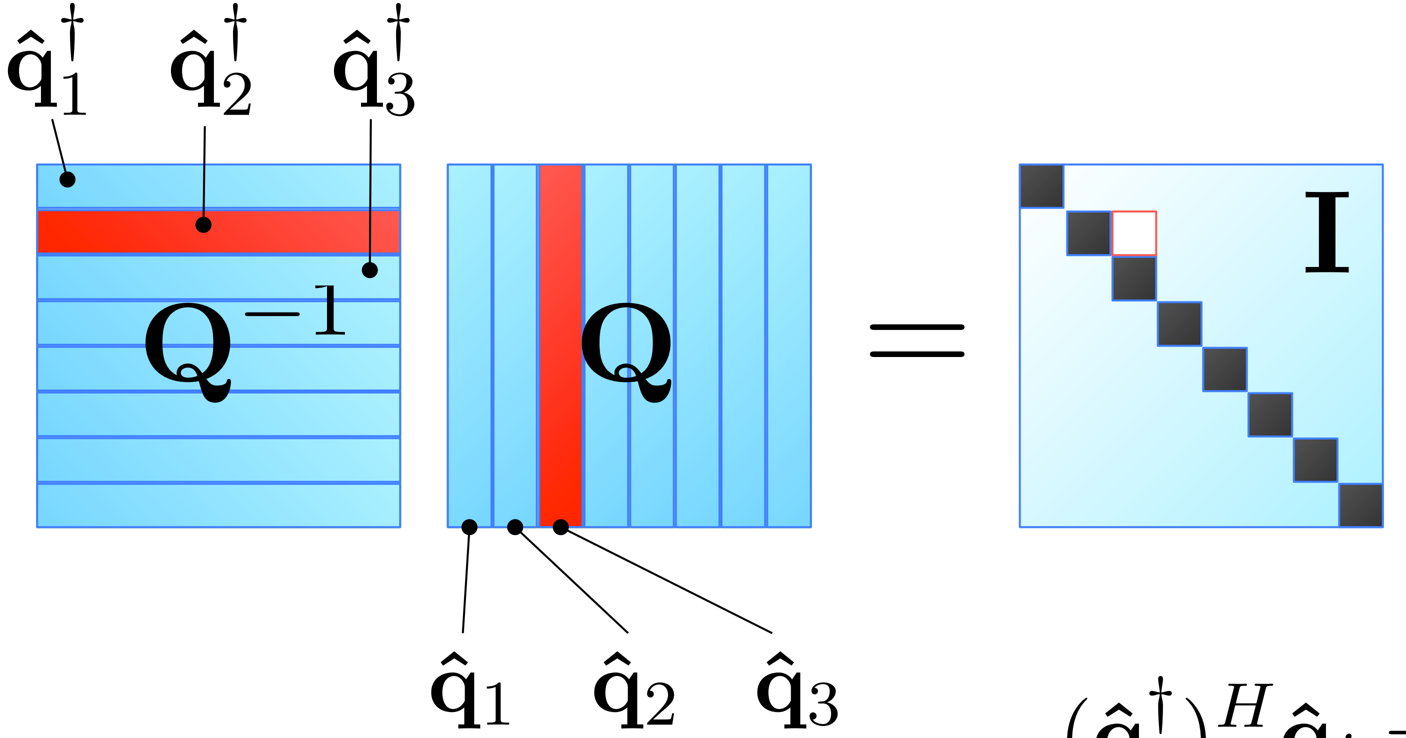


$$(\hat{\mathbf{q}}_j^\dagger)^H \hat{\mathbf{q}}_i = \delta_{ij}$$

In summary, every square matrix, L , has a set of right eigenvectors and a set of left eigenvectors, which are bi-orthogonal to each other.

$$\mathbf{L} = \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^{-1}$$

Right eigenvectors

$$\mathbf{L}\mathbf{Q} = \mathbf{Q}\boldsymbol{\Sigma}$$

Left eigenvectors

$$\mathbf{Q}^{-1}\mathbf{L} = \boldsymbol{\Sigma}\mathbf{Q}^{-1}$$

Let us consider a linearized problem in the time domain (state space formulation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q} = \mathbf{L}\mathbf{q}$$

If *t* runs from 0 to ∞ then q can be expressed as a sum of eigenmodes

$$\mathbf{q} = \sum_{i=1}^{N} \alpha_i \hat{\mathbf{q}}_i \exp(\sigma_i t)$$

each of which obeys

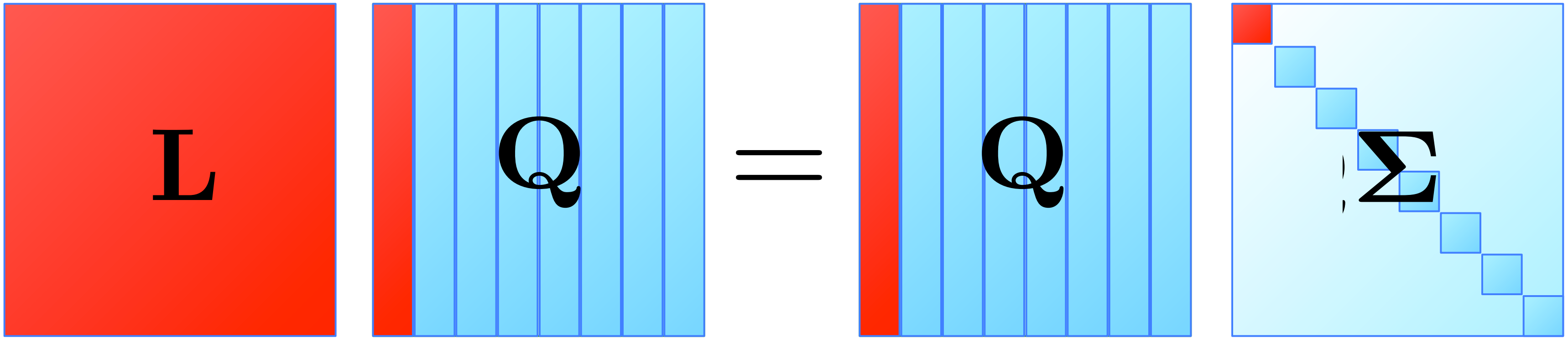
$$\sigma_i \hat{\mathbf{q}}_i = \mathbf{L}\hat{\mathbf{q}}_i$$

These are the *right* eigenfunctions of L $(\mathbf{L} - \sigma_i \mathbf{I})\hat{\mathbf{q}}_i = 0$

Reminder:

$$\mathbf{L} = \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^{-1}$$

$$\mathbf{L}\mathbf{Q} = \mathbf{Q}\boldsymbol{\Sigma}$$



$$\mathbf{L}\hat{\mathbf{q}}_i = \sigma_i \hat{\mathbf{q}}_i$$

$$(\mathbf{L} - \sigma_i \mathbf{I})\hat{\mathbf{q}}_i = 0$$

Let us consider what happens when we make a very small change to L:

$$\mathrm{L} \;\rightarrow\; \mathrm{L} + \epsilon\delta\mathrm{L}$$

The eigenvalues and the right eigenvectors change as well:

$$\sigma_i \rightarrow \sigma_i + \epsilon\delta\sigma_i$$

$$\hat{\mathbf{q}}_i \rightarrow \hat{\mathbf{q}}_i + \epsilon\delta\hat{\mathbf{q}}_i$$

and the new matrix, eigenvalues, and right eigenvectors satisfy:

$$\Big((\mathrm{L} + \epsilon\delta\mathrm{L}) - (\sigma_i + \epsilon\delta\sigma_i)\mathbf{I}\Big)(\hat{\mathbf{q}}_i + \epsilon\delta\hat{\mathbf{q}}_i) = 0$$

$$\Big((\mathrm{L} + \epsilon\delta\mathrm{L}) - (\sigma_i + \epsilon\delta\sigma_i)\mathbf{I}\Big)(\hat{\mathbf{q}}_i + \epsilon\delta\hat{\mathbf{q}}_i) = 0$$

At order ε , this is:

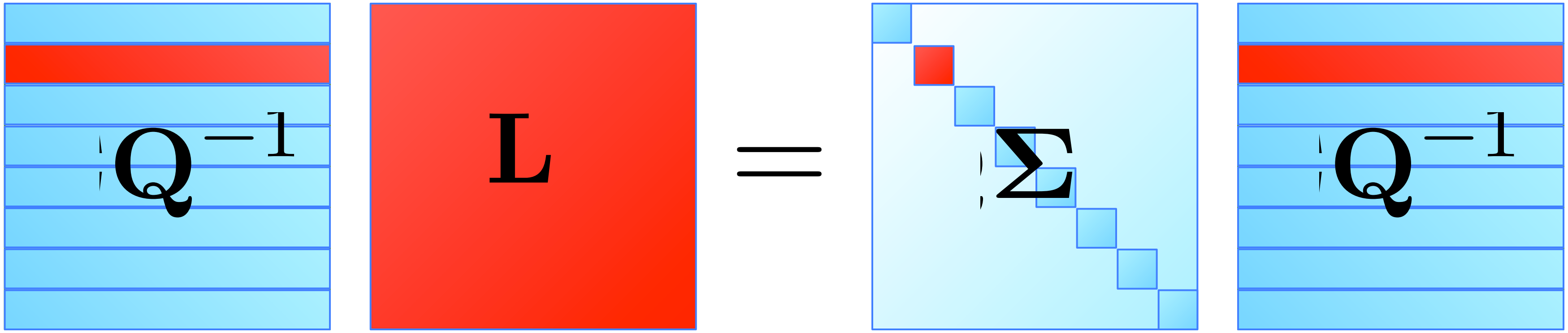
$$(\mathrm{L} - \sigma_i\mathbf{I})\epsilon\delta\hat{\mathbf{q}}_i + (\epsilon\delta\mathrm{L} - \epsilon\delta\sigma_i\mathbf{I})\hat{\mathbf{q}}_i = 0$$

We pre-multiply by the *left* eigenvector:

$$(\hat{\mathbf{q}}_i^\dagger)^H(\mathrm{L} - \sigma_i\mathbf{I})\epsilon\delta\hat{\mathbf{q}}_i + (\hat{\mathbf{q}}_i^\dagger)^H(\epsilon\delta\mathrm{L} - \epsilon\delta\sigma_i\mathbf{I})\hat{\mathbf{q}}_i = 0$$

Reminder:

$$\mathbf{Q}^{-1}\mathbf{L} = \mathbf{\Sigma}\mathbf{Q}^{-1}$$



$$(\hat{\mathbf{q}}_i^\dagger)^H \mathbf{L} = \sigma_i (\hat{\mathbf{q}}_i^\dagger)^H$$

$$(\hat{\mathbf{q}}_i^\dagger)^H (\mathrm{L} - \sigma_i \mathbf{I}) = 0$$

$$\Big((\mathrm{L} + \epsilon\delta\mathrm{L}) - (\sigma_i + \epsilon\delta\sigma_i)\mathbf{I}\Big)(\hat{\mathbf{q}}_i + \epsilon\delta\hat{\mathbf{q}}_i) = 0$$

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We pre-multiply by the *left* eigenvector:

$$(\hat{\mathbf{q}}_i^\dagger)^H(\mathrm{L} - \sigma_i\mathbf{I})\epsilon\delta\hat{\mathbf{q}}_i + (\hat{\mathbf{q}}_i^\dagger)^H(\epsilon\delta\mathrm{L} - \epsilon\delta\sigma_i\mathbf{I})\hat{\mathbf{q}}_i = 0$$

$$(\hat{\mathbf{q}}_i^\dagger)^H\delta\mathrm{L}\hat{\mathbf{q}}_i = (\hat{\mathbf{q}}_i^\dagger)^H\delta\sigma_i\hat{\mathbf{q}}_i$$

$$\big((\mathrm{L} + \epsilon\delta\mathrm{L}) - (\sigma_i + \epsilon\delta\sigma_i)\mathbf{I}\big)(\hat{\mathbf{q}}_i + \epsilon\delta\hat{\mathbf{q}}_i) = 0$$

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$$(\mathrm{L} - \sigma_i\mathbf{I})\epsilon\delta\hat{\mathbf{q}}_i + (\epsilon\delta\mathrm{L} - \epsilon\delta\sigma_i\mathbf{I})\hat{\mathbf{q}}_i = 0$$

We pre-multiply by the *left* eigenvector:

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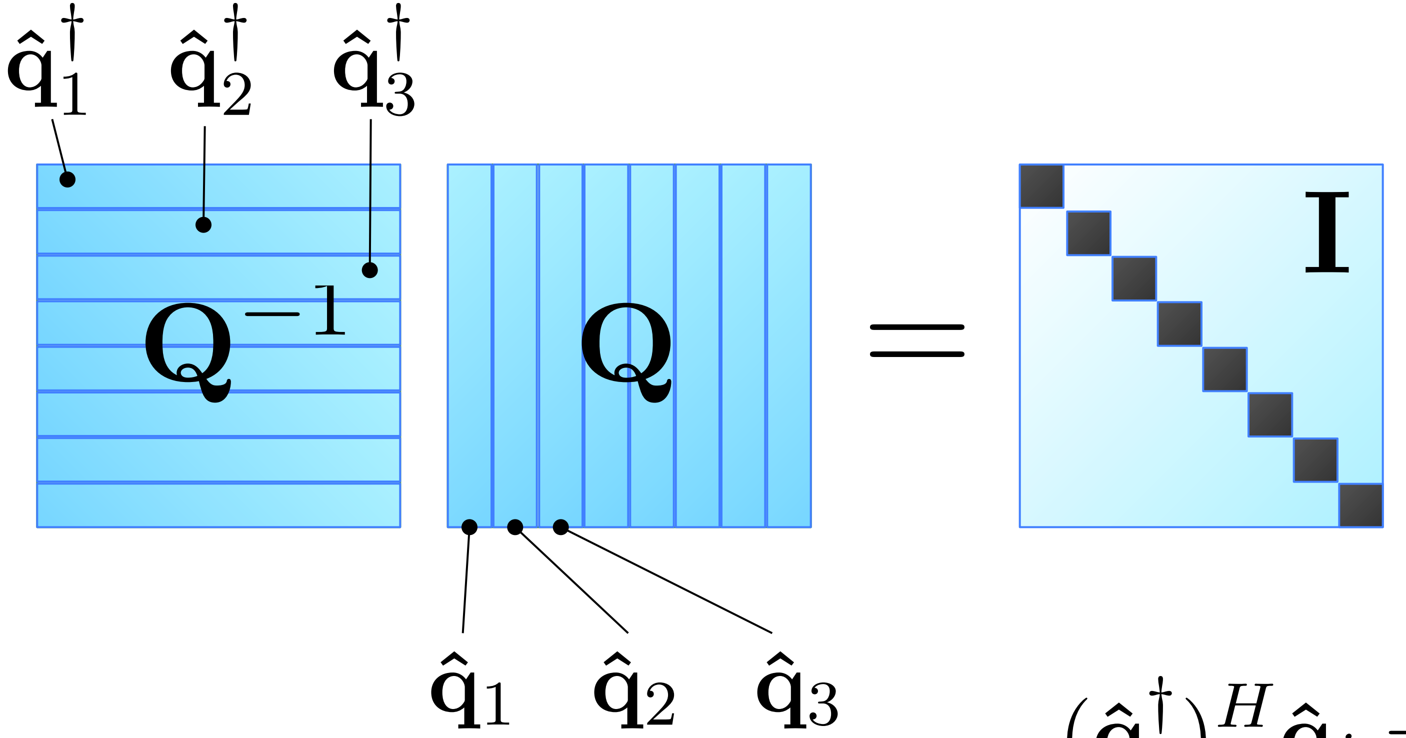
$$(\hat{\mathbf{q}}_i^\dagger)^H\delta\mathrm{L}\hat{\mathbf{q}}_i = (\hat{\mathbf{q}}_i^\dagger)^H\delta\sigma_i\hat{\mathbf{q}}_i$$

and re-arrange:

$$\boxed{\delta\sigma_i \quad = \quad \frac{(\hat{\mathbf{q}}_i^\dagger)^H\delta\mathrm{L}\hat{\mathbf{q}}_i}{(\hat{\mathbf{q}}_i^\dagger)^H\hat{\mathbf{q}}_i}}$$

We could find the left eigenvectors using the fact that Q^{-1} Q = I



$$(\hat{\mathbf{q}}_j^\dagger)^H \hat{\mathbf{q}}_i = \delta_{ij}$$

But there is an easier way to find the left eigenvectors.

Take the Hermitian transpose (the conjugate transpose) of the expression satisfied by the left eigenvector, and re-arrange:

$$\begin{aligned}
(\hat{\mathbf{q}}_i^\dagger)^H(\mathrm{L} - \sigma_i \mathbf{I}) &= 0 \\
\left((\hat{\mathbf{q}}_i^\dagger)^H(\mathrm{L} - \sigma_i \mathbf{I})\right)^H &= 0 \\
(\mathrm{L} - \sigma_i \mathbf{I})^H \hat{\mathbf{q}}_i^\dagger &= 0 \\
(\mathrm{L}^H - \sigma_i^* \mathbf{I}) \hat{\mathbf{q}}_i^\dagger &= 0
\end{aligned}$$

The *left* eigenvectors of L are the *right* eigenvectors of L^H.

In summary, here is how you evaluate the effect that any change to L has on an eigenvalue

Express your problem in state space form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q} = \mathbf{L}\mathbf{q}$$

Choose a *right* eigenmode:

$$(\mathbf{L} - \sigma_i \mathbf{I})\hat{\mathbf{q}}_i = 0$$

Find the corresponding *left* eigenvector:

$$(\mathrm{L}^H - \sigma_i^* \mathbf{I})\hat{\mathbf{q}}_i^\dagger \quad = \quad 0$$

In summary, here is how you evaluate the effect that any change to L has on an eigenvalue

Now you can work out how ANY change to L will change that eigenvalue

$$\delta\sigma_i \quad = \quad \frac{(\hat{\mathbf{q}}_i^\dagger)^H \delta \mathrm{L} \hat{\mathbf{q}}_i}{(\hat{\mathbf{q}}_i^\dagger)^H \hat{\mathbf{q}}_i}$$

δL could represent:

- **a change in the base state (base state sensitivity)**
- **the addition of a passive feedback device**
- **the addition of an active feedback device**
- **a change in one of the terms in the governing equations, to assess its influence on the instability**
- **the most influential point-wise feedback mechanism (structural sensitivity)**

For example, let us apply this to a simple linear oscillator

Here is a simple linear oscillator, which is a second order ODE:

$$\ddot{x} + b\dot{x} + c = 0$$

It can be written as two first order ODEs:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -by - cx \end{aligned}$$

And this can be expressed in state space form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q} = \mathbf{L}\mathbf{q}$$

$$\mathbf{q} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mathbf{L}\mathbf{q} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first eigenvalue, right eigenvector, and left eigenvector are (by hand):

$$\sigma_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}$$

$$\hat{\mathbf{q}}_1 = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}_1 = \begin{bmatrix} 2 \\ -b + \sqrt{b^2 - 4c} \end{bmatrix}$$

$$\hat{\mathbf{q}}_1^\dagger = \begin{bmatrix} \hat{x}^\dagger \\ \hat{y}^\dagger \end{bmatrix}_1 = \begin{bmatrix} -2c^* \\ -b^* - \sqrt{b^{*2} - 4c^*} \end{bmatrix}$$

The second eigenvalue, right eigenvector, and left eigenvector are (by hand):

$$\sigma_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

$$\hat{\mathbf{q}}_2 = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}_2 = \begin{bmatrix} 2 \\ -b - \sqrt{b^2 - 4c} \end{bmatrix}$$

$$\hat{\mathbf{q}}_2^\dagger = \begin{bmatrix} \hat{x}^\dagger \\ \hat{y}^\dagger \end{bmatrix}_2 = \begin{bmatrix} -2c^* \\ -b^* + \sqrt{b^{*2} - 4c^*} \end{bmatrix}$$

If you are feeling energetic, you can check that the eigenvectors are bi-orthogonal

$$\begin{aligned}
\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle &= (-2c^*)^* (2) + \left(-b^* - \sqrt{b^{*2} - 4c^*}\right)^* \left(-b + \sqrt{b^2 - 4c}\right) \\
&= (-4c) + \left(-b + \sqrt{b^2 - 4c}\right) \left(-b + \sqrt{b^2 - 4c}\right) \\
&= (-4c) + \left(b^2 - 2b\sqrt{b^2 - 4c} + b^2 - 4c\right) \\
&= 2b^2 - 2b\sqrt{b^2 - 4c} - 8c \\
&= 2(b^2 - 4c) - 2b\sqrt{b^2 - 4c} \\
\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_2 \rangle &= (-2c^*)^* (2) + \left(-b^* - \sqrt{b^{*2} - 4c^*}\right)^* \left(-b - \sqrt{b^2 - 4c}\right) \\
&= (-4c) + \left(-b + \sqrt{b^2 - 4c}\right) \left(-b - \sqrt{b^2 - 4c}\right) \\
&= (-4c) + (b^2 - b^2 + 4c) \\
&= 0
\end{aligned}$$

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$$\begin{aligned}
\langle \hat{\mathbf{q}}_2^\dagger, \hat{\mathbf{q}}_1 \rangle &= (-2c^*)^* (2) + \left(-b^* + \sqrt{b^{*2} - 4c^*}\right)^* \left(-b + \sqrt{b^2 - 4c}\right) \\
&= (-4c) + \left(-b - \sqrt{b^2 - 4c}\right) \left(-b + \sqrt{b^2 - 4c}\right) \\
&= (-4c) + \left(b^2 + b^2 + 4c\right) \\
&= 0 \\
\langle \hat{\mathbf{q}}_2^\dagger, \hat{\mathbf{q}}_2 \rangle &= (-2c^*)^* (2) + \left(-b^* + \sqrt{b^{*2} - 4c^*}\right)^* \left(-b - \sqrt{b^2 - 4c}\right) \\
&= (-4c) + \left(-b - \sqrt{b^2 - 4c}\right) \left(-b - \sqrt{b^2 - 4c}\right) \\
&= (-4c) + \left(b^2 + 2b\sqrt{b^2 - 4c} + b^2 - 4c\right) \\
&= 2b^2 + 2b\sqrt{b^2 - 4c} - 8c \\
&= 2(b^2 - 4c) + 2b\sqrt{b^2 - 4c}
\end{aligned}$$

Let us consider the effect of a new feedback mechanism (e.g. negative damping)

$$\begin{aligned} \dot{x} &= \epsilon x + y \\ \dot{y} &= -by - cx \end{aligned}$$

$$(\mathrm{L} + \delta\mathrm{L})\mathbf{q} = \begin{bmatrix} \epsilon & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\delta\mathrm{L} = \begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix}$$

Its influence can be worked out by hand and then compared with a new eigenvalue calculation (again, by hand)

$$\begin{aligned}
\delta\sigma_1 &= \frac{\langle \hat{\mathbf{q}}_1^\dagger, \delta \mathrm{L} \hat{\mathbf{q}}_1 \rangle}{\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle} \\
&= \frac{\hat{x}_1^{\dagger *} \epsilon \hat{x}_1}{\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle} \\
&= \epsilon \frac{\hat{x}_1^{\dagger *} \hat{x}_1}{\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle} = \epsilon \left(\frac{1}{2} + \frac{b}{2\sqrt{b^2 - 4c}} \right)
\end{aligned}$$

We can repeat this for all possible feedback mechanisms

$$\delta \mathrm{L}_{x^\dagger y} = \begin{bmatrix} 0 & \epsilon \\ 0 & 0 \end{bmatrix} \quad , \quad \delta \mathrm{L}_{y^\dagger x} = \begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix} \quad , \quad \delta \mathrm{L}_{y^\dagger y} = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix}$$

$$\delta\sigma_{1,x_1^\dagger y_1} = \frac{\hat{x}_1^{\dagger *} \epsilon \hat{y}_1}{\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle} = \epsilon \frac{-c}{\sqrt{b^2 - 4c}}$$

$$\delta\sigma_{1,y_1^\dagger x_1} = \frac{\hat{y}_1^{\dagger *} \epsilon \hat{x}_1}{\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle} = \epsilon \frac{1}{\sqrt{b^2 - 4c}}$$

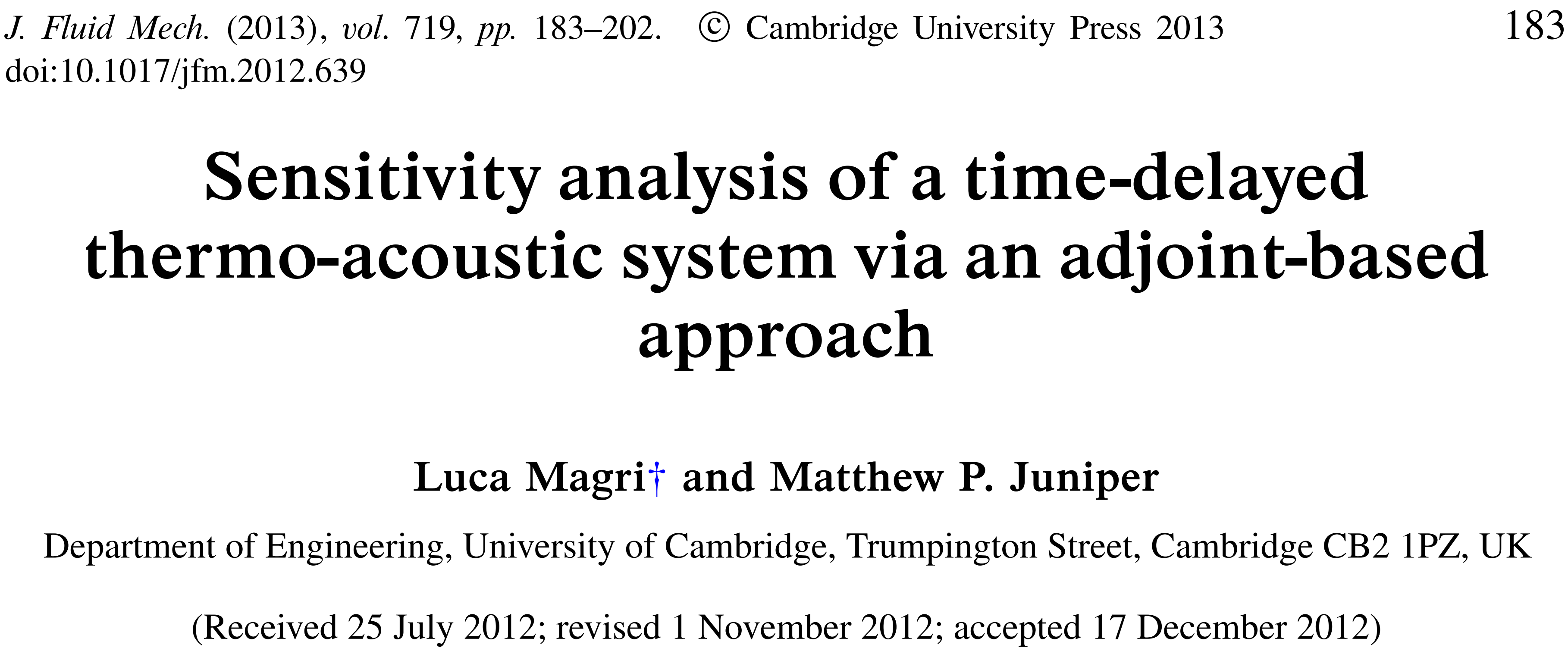
$$\delta\sigma_{1,y_1^\dagger y_1} = \frac{\hat{y}_1^{\dagger *} \epsilon \hat{y}_1}{\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle} = \epsilon \left(\frac{1}{2} - \frac{b}{2\sqrt{b^2 - 4c}} \right)$$

We can also find how the eigenvalue changes when the base state parameters, b and c, change.

$$\left.\frac{\partial \sigma_1}{\partial b}\right|_c = -\frac{\hat{y}^{\dagger *}\hat{y}}{\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle} = \frac{b}{2\sqrt{b^2 - 4c}} - \frac{1}{2}$$

$$\left.\frac{\partial \sigma_1}{\partial c}\right|_b = -\frac{\hat{y}^{\dagger *}\hat{x}}{\langle \hat{\mathbf{q}}_1^\dagger, \hat{\mathbf{q}}_1 \rangle} = \frac{-1}{\sqrt{b^2 - 4c}}$$

Now let us apply this to a simple thermoacoustic system



J. Fluid Mech. (2013), *vol.* 719, *pp.* 183–202. © Cambridge University Press 2013 183
doi:10.1017/jfm.2012.639

Sensitivity analysis of a time-delayed thermo-acoustic system via an adjoint-based approach

Luca Magri[†] **and Matthew P. Juniper**

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(Received 25 July 2012; revised 1 November 2012; accepted 17 December 2012)

The system is a Rijke tube containing a hot wire

Diagram of the Rijke tube

Non-dimensional governing equations

$$F_1 \equiv \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0$$

(note the time delay in the heat release term)

$$F_2 \equiv \underbrace{\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x}}_{\text{acoustics}} + \underbrace{\zeta p}_{\text{damping}} - \underbrace{\beta\left(\left|\frac{1}{3} + u_f(t-\tau)\right|^{1/2} - \left(\frac{1}{3}\right)^{1/2}\right)\delta(x - x_f)}_{\text{heat release at the hot wire}} = 0$$

---

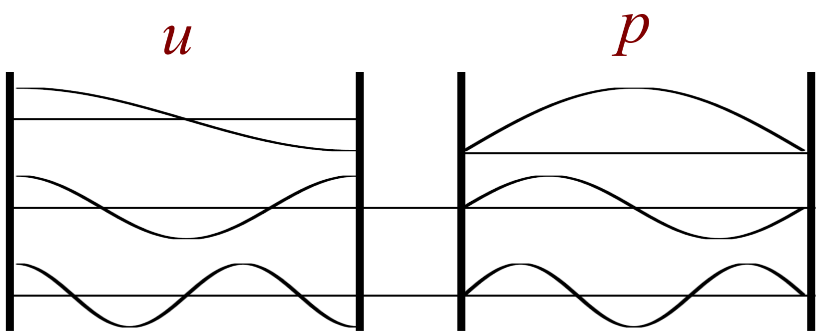
[4] Balasubramanian, K. and Sujith, R.I. "Thermoacoustic instability in a Rijke tube: nonnormality and nonlinearity" *Phys. Fluids* Vol. 20, 2008, 044103.

The governing equations are discretized by considering the fundamental 'open organ pipe' mode and its harmonics. This is a Galerkin discretization.

Discretization into basis functions

$$u = \sum_{j=1}^{\infty} \eta_j \cos(j\pi x)$$

$$p = -\sum_{j=1}^{\infty} \frac{\dot{\eta}_j}{j\pi} \sin(j\pi x)$$



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acoustics

damping

heat release at the hot wire

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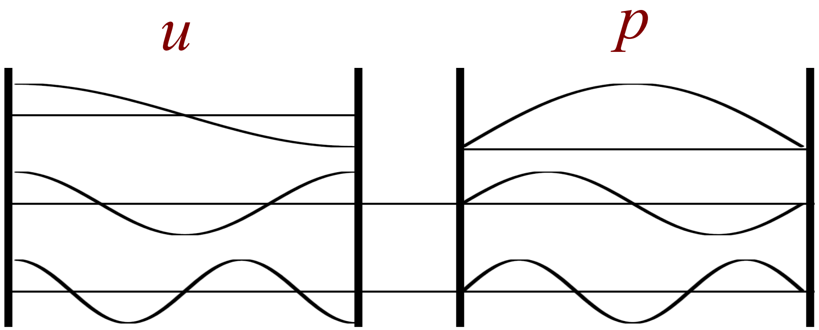


Non-dimensional discretized governing equations

$$u_f(t-\tau) = \sum_{j=1}^{\infty} \eta_j(t-\tau)\cos(j\pi x_f)$$

$$F_1^G \equiv \frac{\mathrm{d}}{\mathrm{d}t}\eta_j - \dot{\eta}_j = 0$$

$$F_2^G \equiv \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{\eta}_j}{j\pi} + j\pi\eta_j + \zeta_j \frac{\dot{\eta}_j}{j\pi} + 2\beta \sin(j\pi x_f)\left(\left|\frac{1}{3} + u_f(t-\tau)\right|^{1/2} - \left(\frac{1}{3}\right)^{1/2}\right) = 0$$

---

[4] Balasubramanian, K. and Sujith, R.I. "Thermoacoustic instability in a Rijke tube: nonnormality and nonlinearity" *Phys. Fluids* Vol. 20, 2008, 044103.

We linearize the nonlinear heat release term and the time delay (hence creating linear ODEs instead of nonlinear DDEs). This creates the state space form.

Nonlinear time-delayed term

$$u_h(t-\tau) \ll \frac{1}{3}$$

$$\tau \ll \frac{2}{N}$$

Linear with no time delay

$$\left(\left|\frac{1}{3}+u_h(t-\tau)\right|^{\frac{1}{2}} - \left(\frac{1}{3}\right)^{\frac{1}{2}}\right)$$

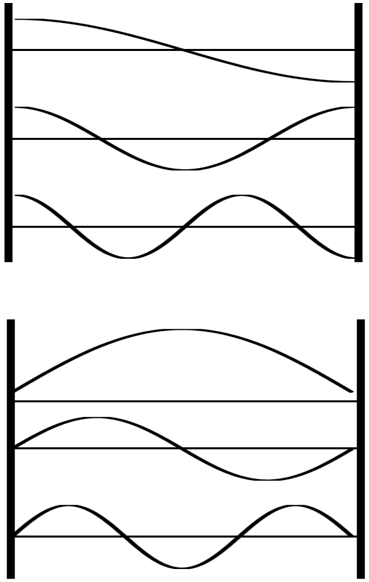
$$\approx \frac{\sqrt{3}}{2}\left(u_h(t) - \tau\frac{\partial u_h(t)}{\partial t}\right)$$

[4] Balasubramanian, K. and Sujith, R.I. “Thermoacoustic instability in a Rijke tube: nonnormality and nonlinearity” *Phys. Fluids* Vol. 20, 2008, 044103.

We linearize the nonlinear heat release term and the time delay (hence creating linear ODEs instead of nonlinear DDEs). This creates the state space form.

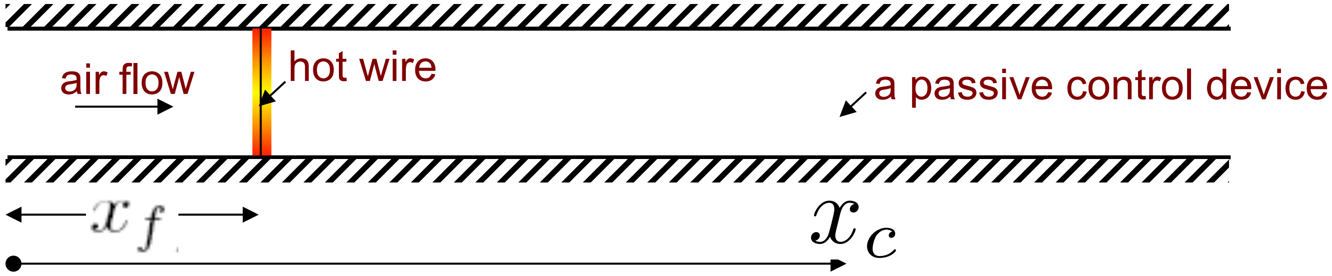
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q} = \mathbf{L}\mathbf{q}$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{u_1} \\ \vdots \\ \mathbf{u_N} \\ \mathbf{p_1} \\ \vdots \\ \mathbf{p_N} \end{bmatrix}$$



We consider a passive control device at position x_c.

Diagram of the Rijke tube



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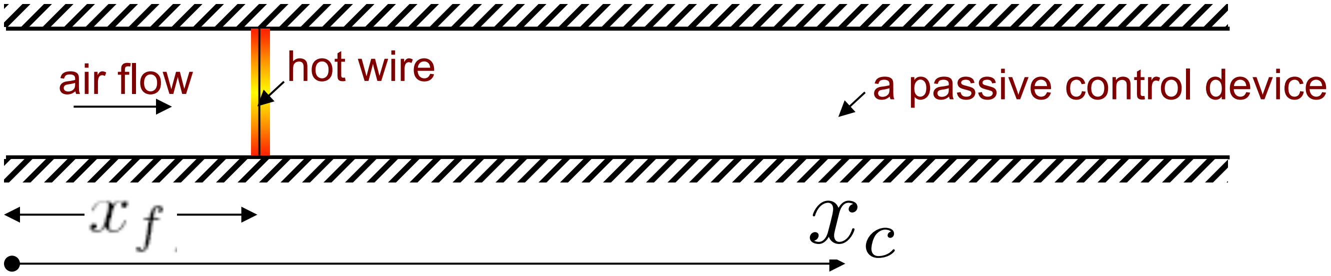
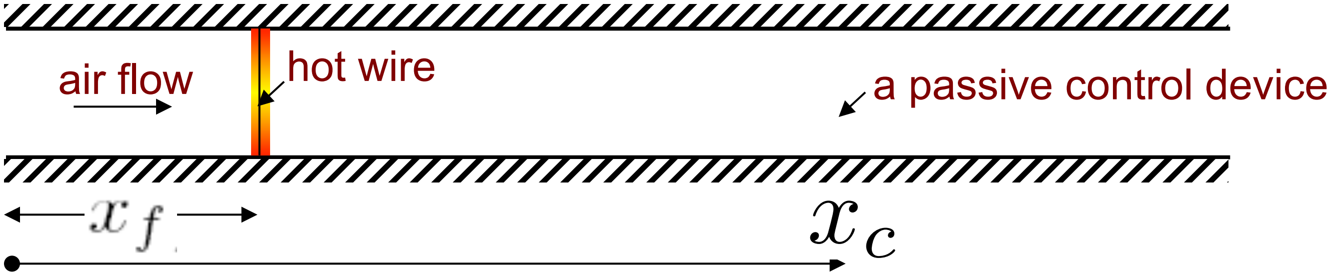


$$u(x_c)$$

We consider a passive control device at position x_c.

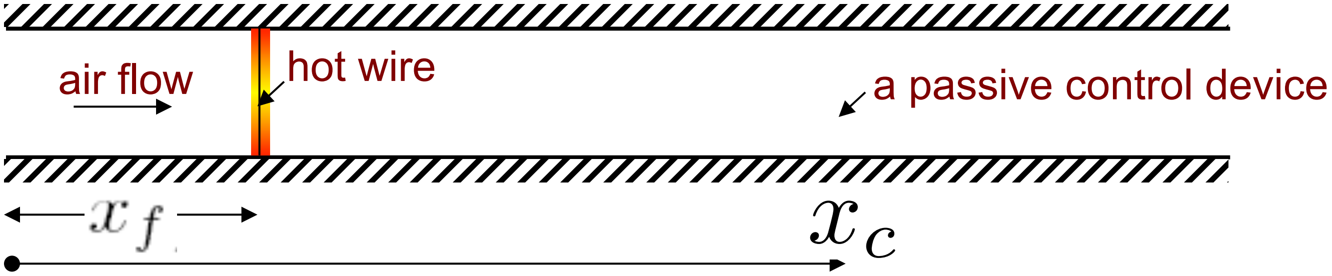
Diagram of the Rijke tube



$$\epsilon u(x_c)$$

We consider a passive control device at position x_c , which can either feed into the energy equation

Diagram of the Rijke tube



Non-dimensional governing equations

$$\epsilon u(x_c)\delta(x - x_c)$$

$$\Big\downarrow \text{ energy eq.}$$

$$F_1 \equiv \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0$$

$$F_2 \equiv \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \zeta p - \beta \left(\left| \frac{1}{3} + u_f(t - \tau) \right|^{1/2} - \left(\frac{1}{3} \right)^{1/2} \right) \delta(x - x_f) = 0$$

acoustics

damping

heat release at the hot wire

We consider a passive control device at position x_c , which can either feed into the energy equation or the momentum equation.

Diagram of the Rijke tube

Non-dimensional governing equations

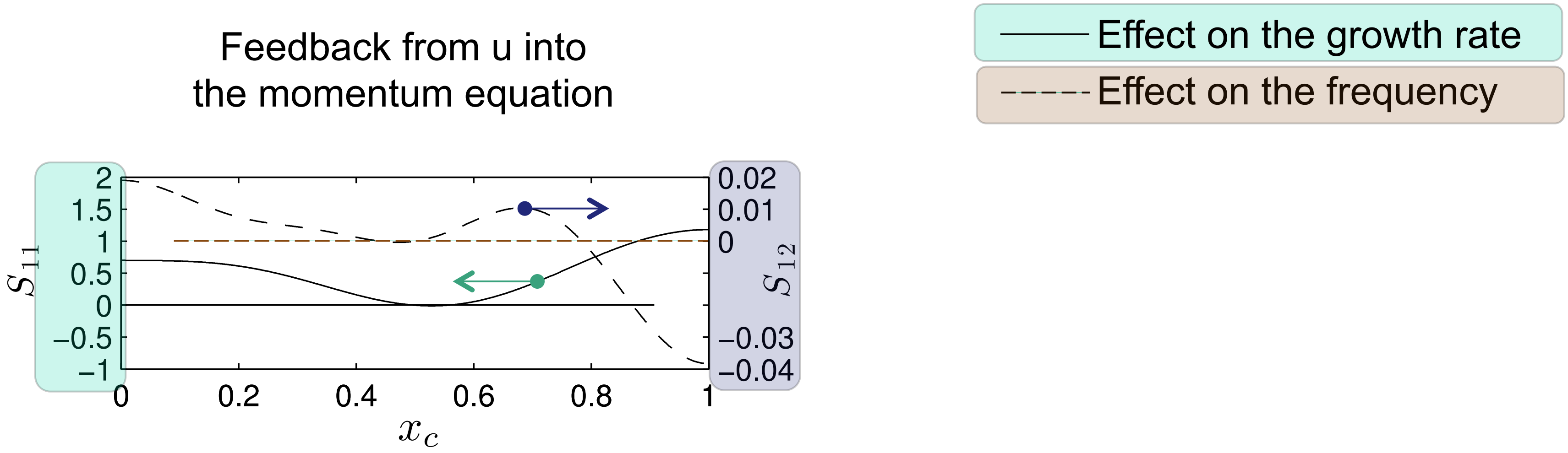
$$\epsilon u(x_c)\delta(x - x_c)$$

momentum eq.

$$F_1 \equiv \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0$$

$$F_2 \equiv \underbrace{\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x}}_{\text{acoustics}} + \underbrace{\zeta p}_{\text{damping}} - \underbrace{\beta\left(\left|\frac{1}{3} + u_f(t-\tau)\right|^{1/2} - \left(\frac{1}{3}\right)^{1/2}\right)\delta(x - x_f)}_{\text{heat release at the hot wire}} = 0$$

For example, here is the effect of a passive feedback device that, at a given point in space, produces a force proportional to the acoustic velocity. It has most influence at the downstream end of the tube.

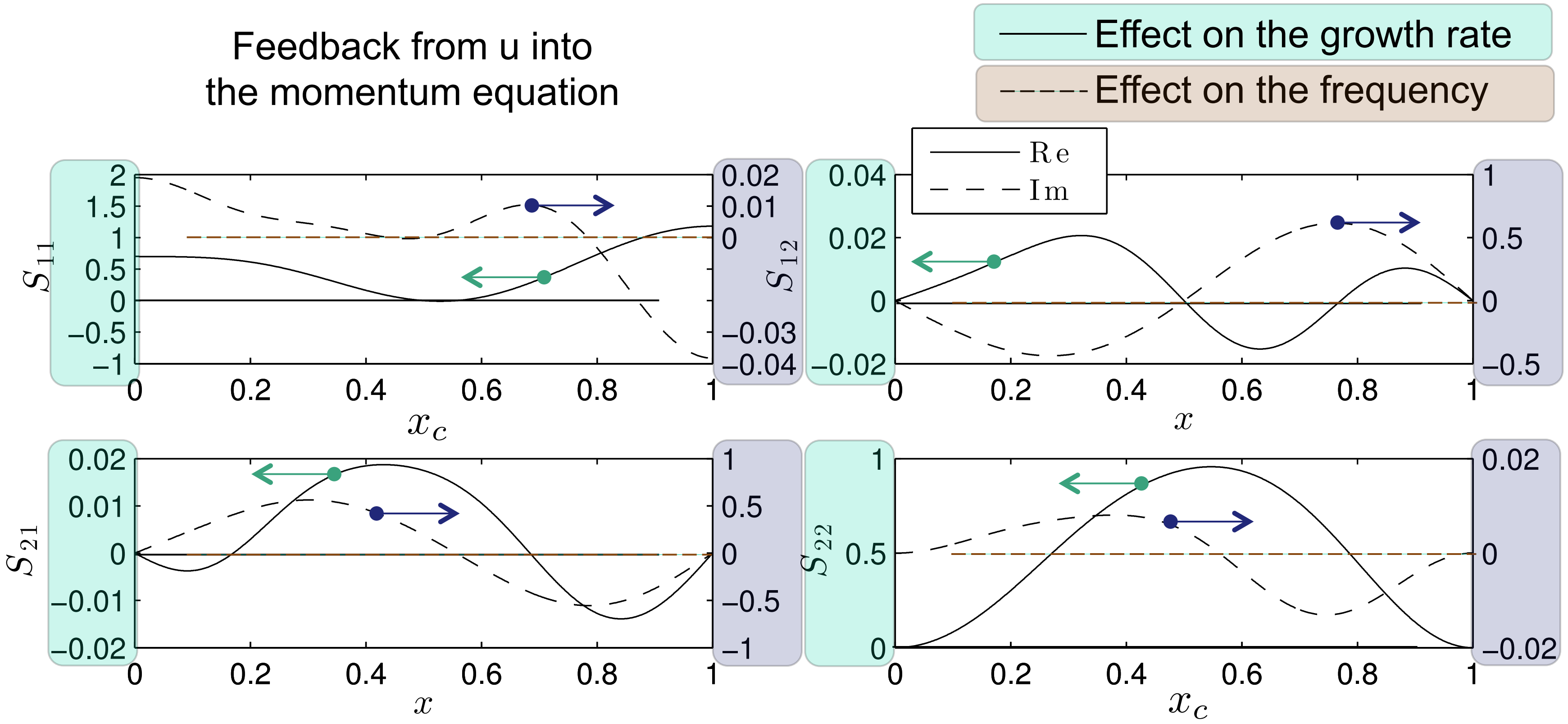


Feedback from u into the momentum equation

$$F_1 \equiv \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \epsilon u(x_c)\delta(x - x_c)$$

$$F_2 \equiv \underbrace{\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x}}_{\text{acoustics}} + \underbrace{\zeta p}_{\text{damping}} - \underbrace{\beta\left(\left|\frac{1}{3} + u_f(t-\tau)\right|^{1/2} - \left(\frac{1}{3}\right)^{1/2}\right)\delta(x - x_f)}_{\text{heat release at the hot wire}} = 0$$

The effect of all other passive devices can also be calculated.



Feedback from u into the energy equation

Feedback from p into the energy equation

For example, here is the effect of a device that increases the heat release when the acoustic pressure increases. It has most influence around the middle of the tube.

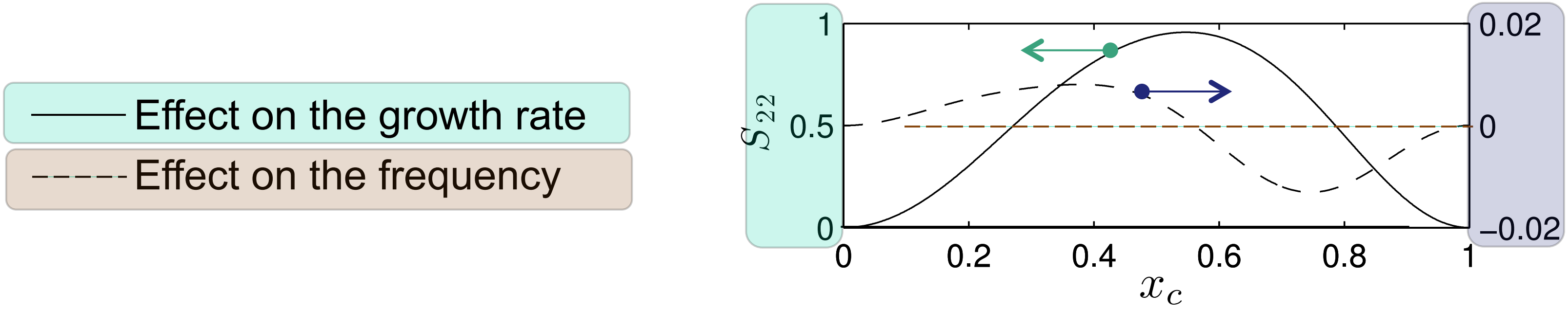
$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \zeta p - \beta \left(\left| \frac{1}{3} + u_f(t-\tau) \right|^{1/2} - \left(\frac{1}{3} \right)^{1/2} \right) \delta(x - x_f) = \epsilon p(x_c) \delta(x - x_c)$$

acoustics

damping

heat release at the hot wire



Feedback from p into the energy equation

For example, here is the effect of a device that increases the heat release when the acoustic velocity increases. It has very little influence on the growth rate, but greater influence on the frequency.

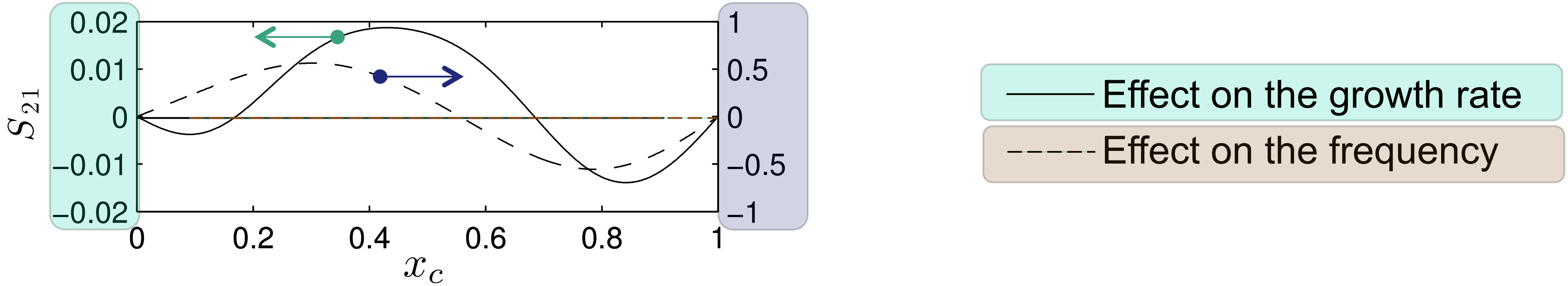
$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \zeta p - \beta\left(\left|\frac{1}{3} + u_f(t-\tau)\right|^{1/2} - \left(\frac{1}{3}\right)^{1/2}\right)\delta(x - x_f) = \epsilon u(x_c)\delta(x - x_c)$$

acoustics

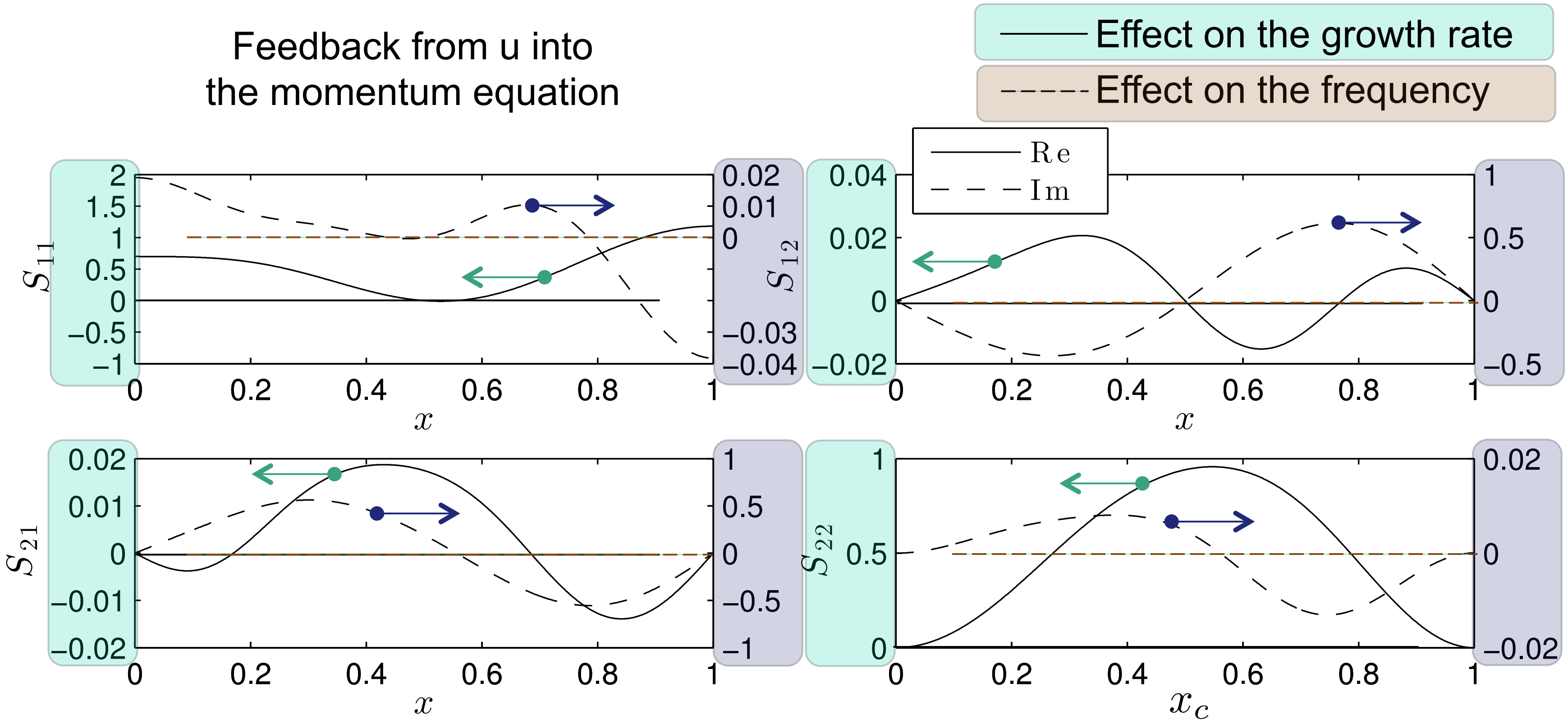
damping

heat release at the hot wire



Feedback from u into the energy equation

These building blocks can be combined in any (linear) way:

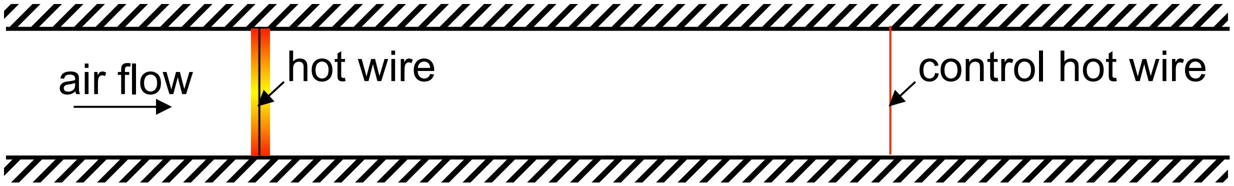


Feedback from u into the momentum equation

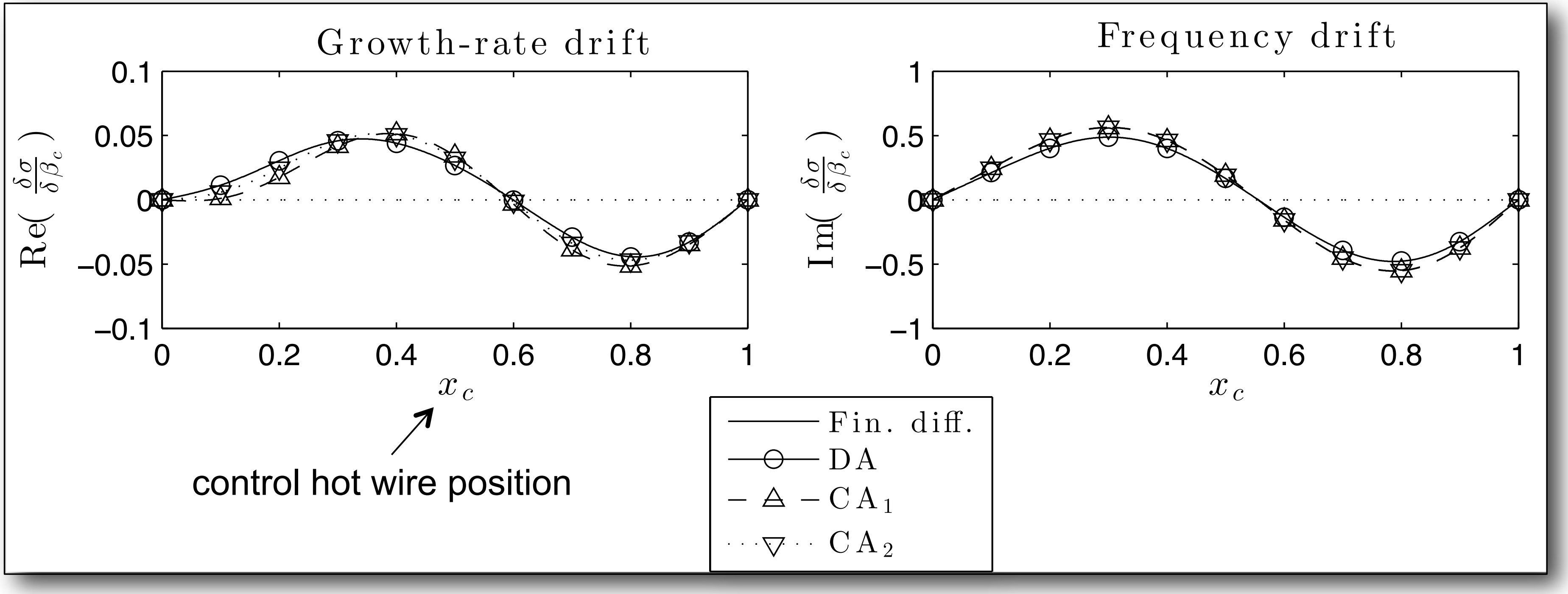
Feedback from u into the energy equation

Feedback from p into the energy equation

For example, here is the influence of another hot wire as a function of its position within the tube.



Change in the eigenvalue that would be caused by feedback from another hot wire



control hot wire position

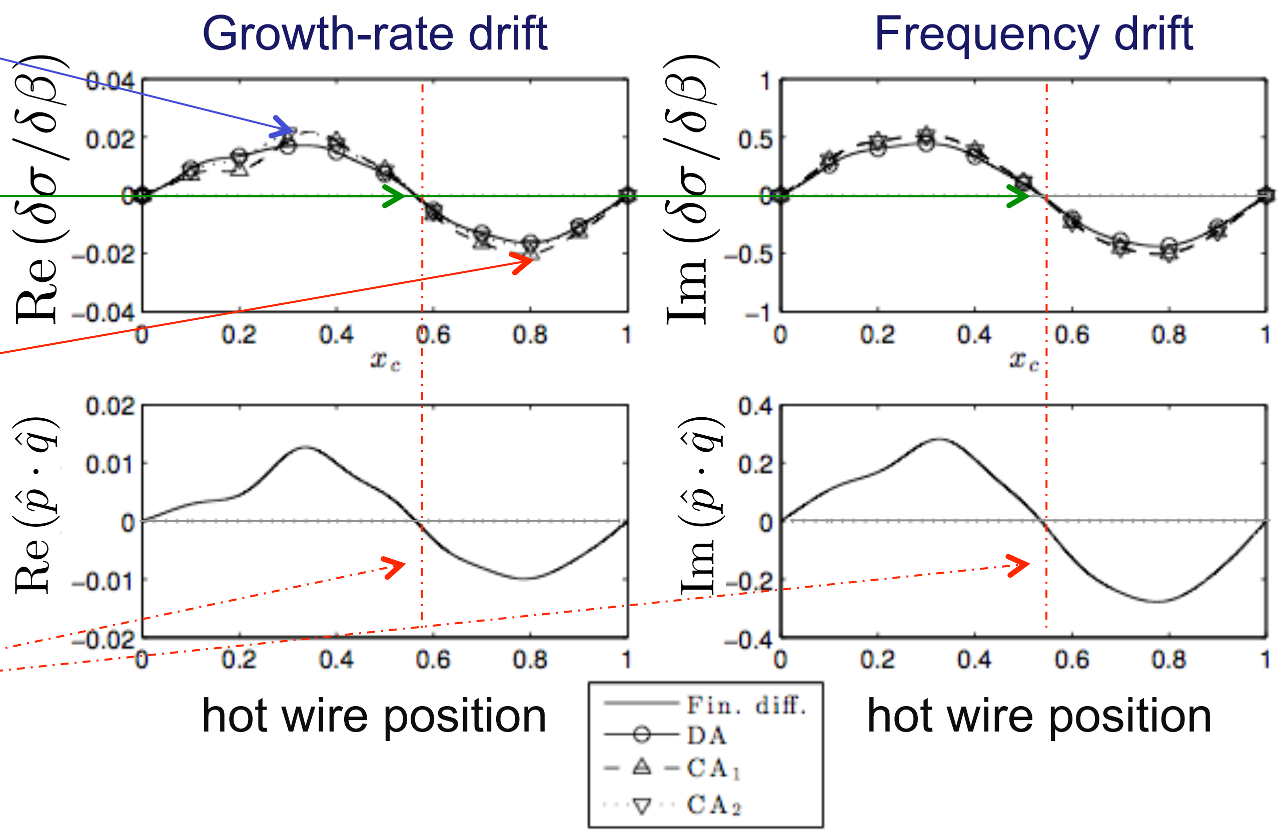
We can compare this with the Rayleigh Index for the same hot wire.

Most destabilizing

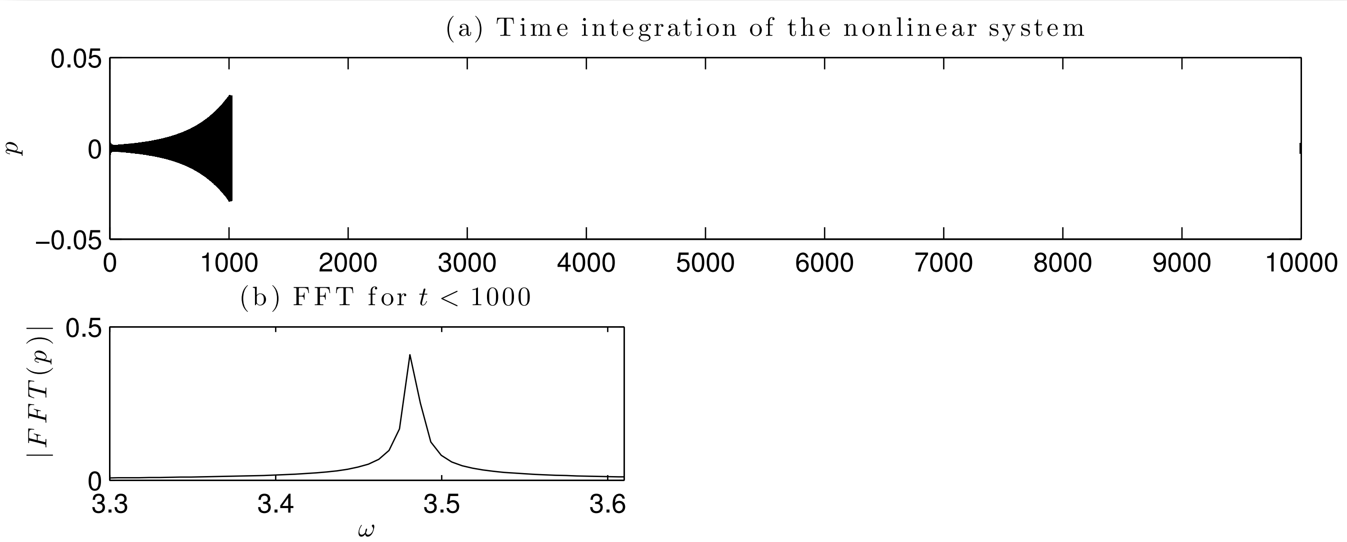
No effect

Most stabilizing

The sign matches the Rayleigh Index



The sensitivity analysis is linear so we test its predictions by applying these feedback mechanisms to the fully nonlinear system



$$\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \zeta p - \frac{2}{\sqrt{3}}\beta\left(\left|\frac{1}{3} + u(t-\tau)\right|^{\frac{1}{2}} - \left(\frac{1}{3}\right)^{\frac{1}{2}}\right)\delta(x - x_h)$$

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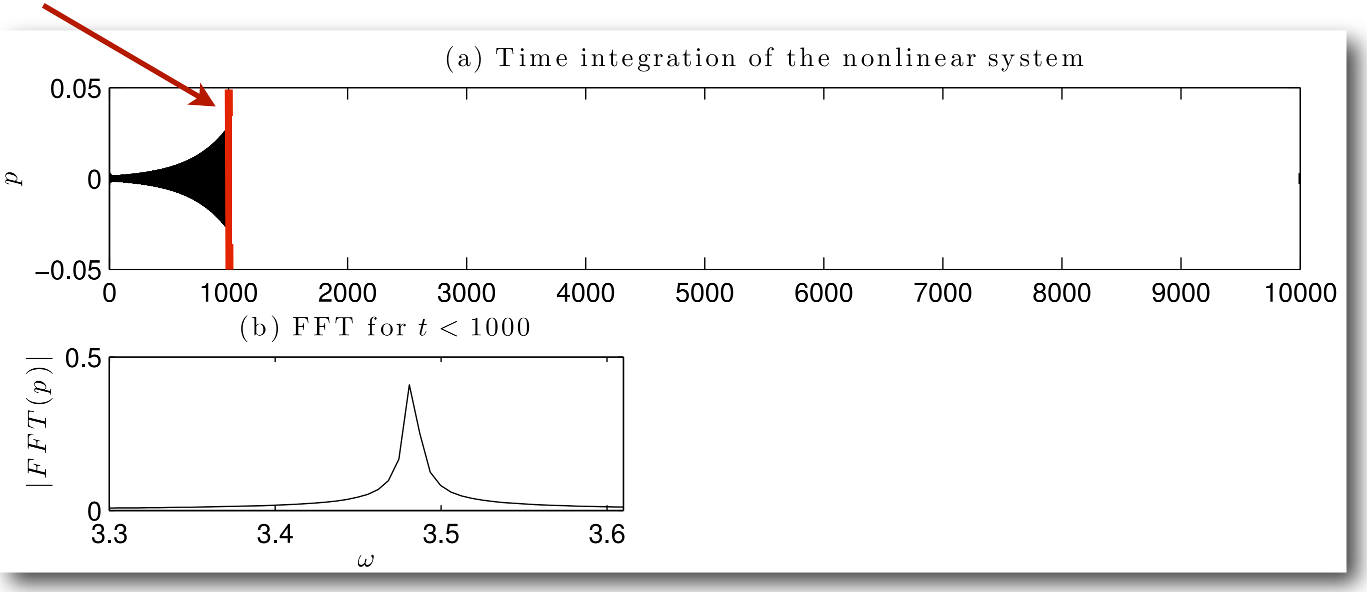
The sensitivity analysis is linear so we test its predictions by applying these feedback mechanisms to the fully nonlinear system

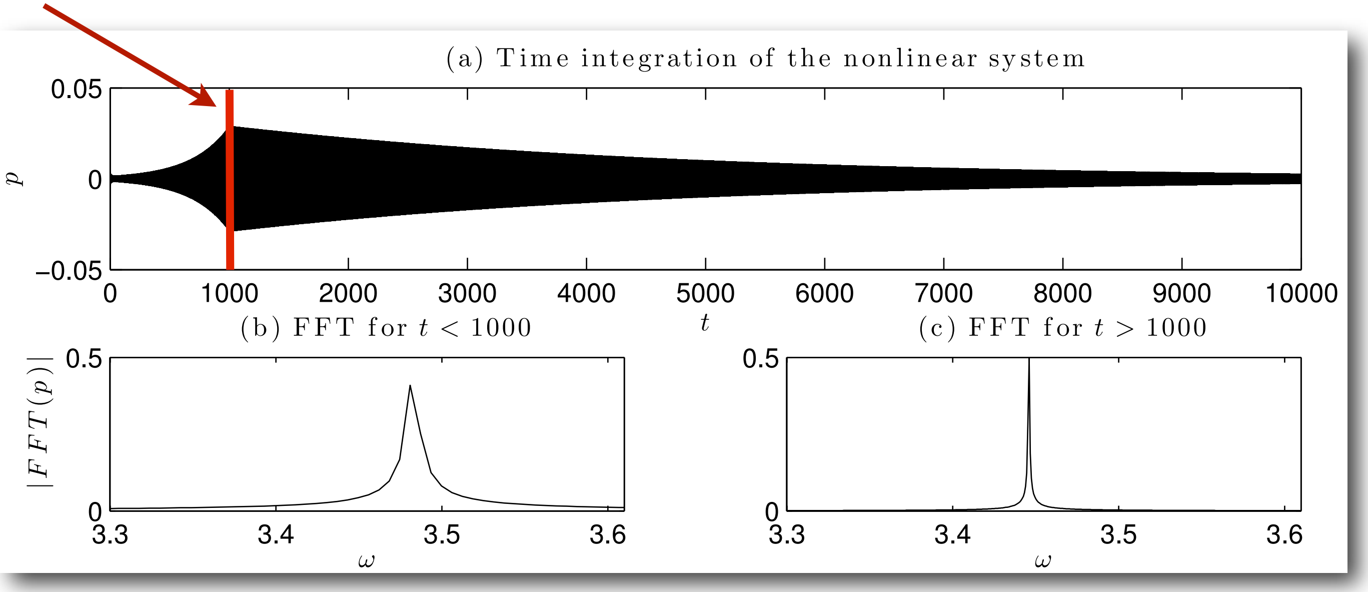
Turning the control hot wire on



$$\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \zeta p - \frac{2}{\sqrt{3}}\beta\left(\left|\frac{1}{3}+u(t-\tau)\right|^{\frac{1}{2}} - \left(\frac{1}{3}\right)^{\frac{1}{2}}\right)\delta(x-x_h)$$

$$\ldots - \frac{2}{\sqrt{3}}\beta_c\left(\left|\frac{1}{3}+u(t-\tau_c)\right|^{\frac{1}{2}} - \left(\frac{1}{3}\right)^{\frac{1}{2}}\right)\delta(x-x_c) = 0.$$

The sensitivity analysis is linear so we test its predictions by applying these feedback mechanisms to the fully nonlinear system

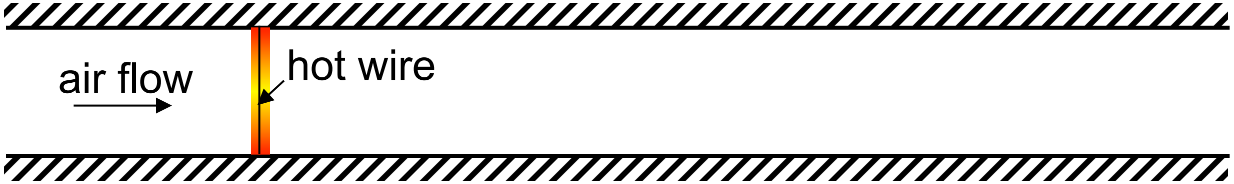


Turning the control hot wire on

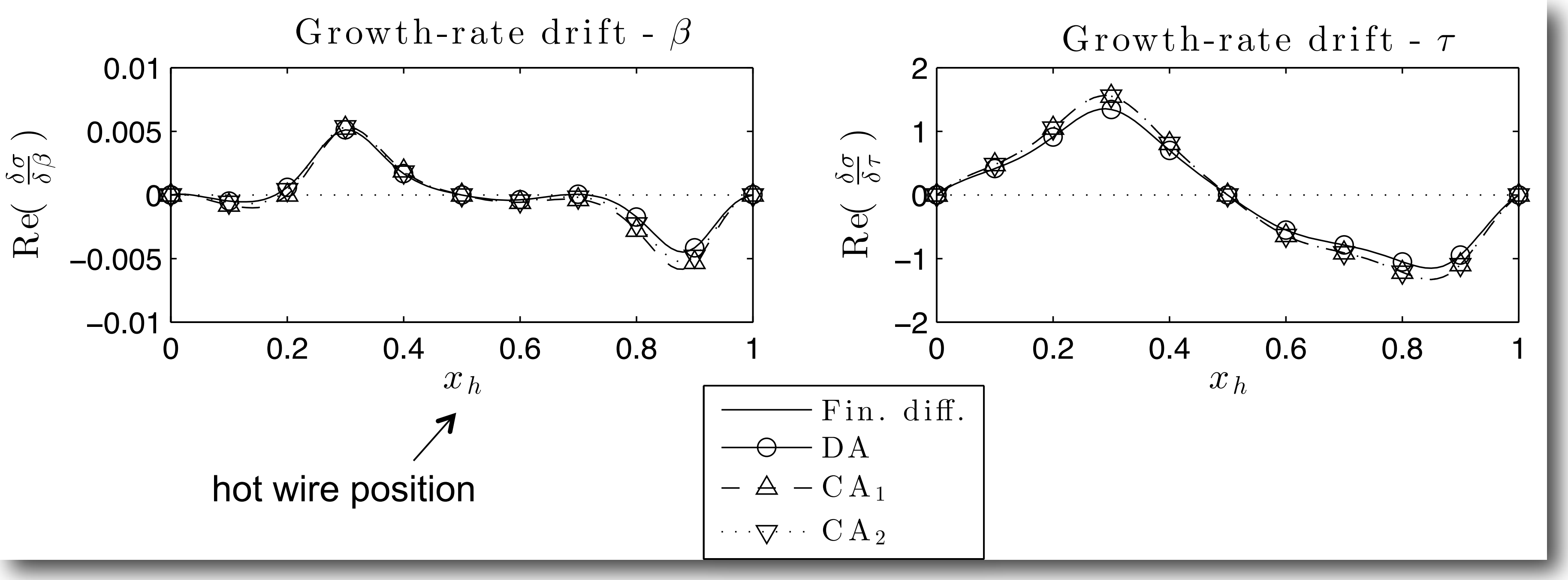
$$\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \zeta p - \frac{2}{\sqrt{3}}\beta\left(\left|\frac{1}{3} + u(t-\tau)\right|^{\frac{1}{2}} - \left(\frac{1}{3}\right)^{\frac{1}{2}}\right)\delta(x - x_h) + \dots$$

$$\dots - \frac{2}{\sqrt{3}}\beta_c\left(\left|\frac{1}{3} + u(t-\tau_c)\right|^{\frac{1}{2}} - \left(\frac{1}{3}\right)^{\frac{1}{2}}\right)\delta(x - x_c) = 0.$$

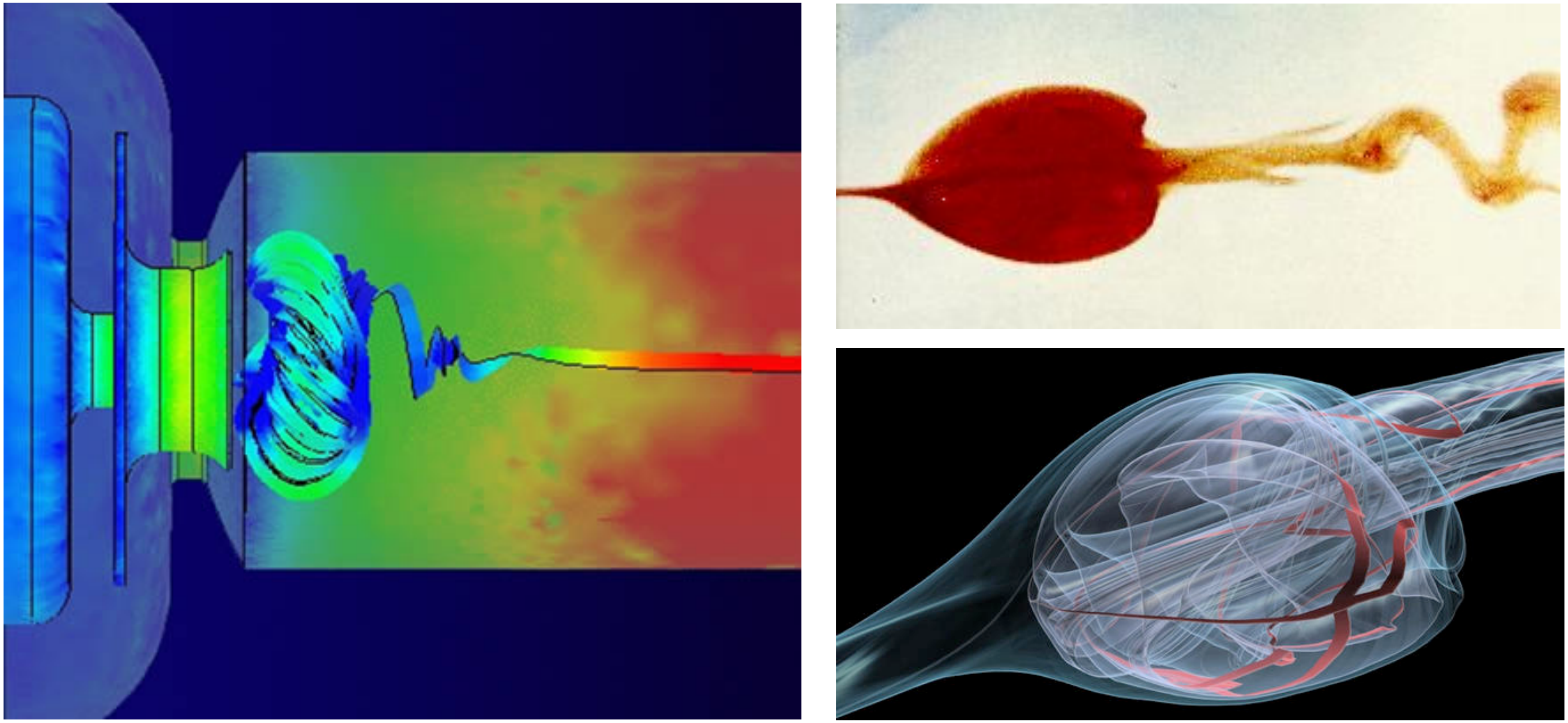
We can also calculate the change in the eigenvalue when the base state is changed.



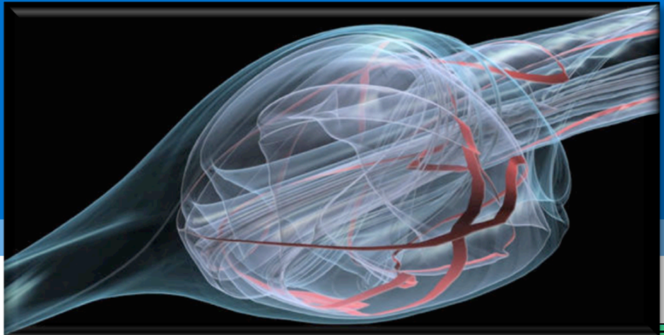
Sensitivity to changes in the base state: wire temperature (left) time delay (right)



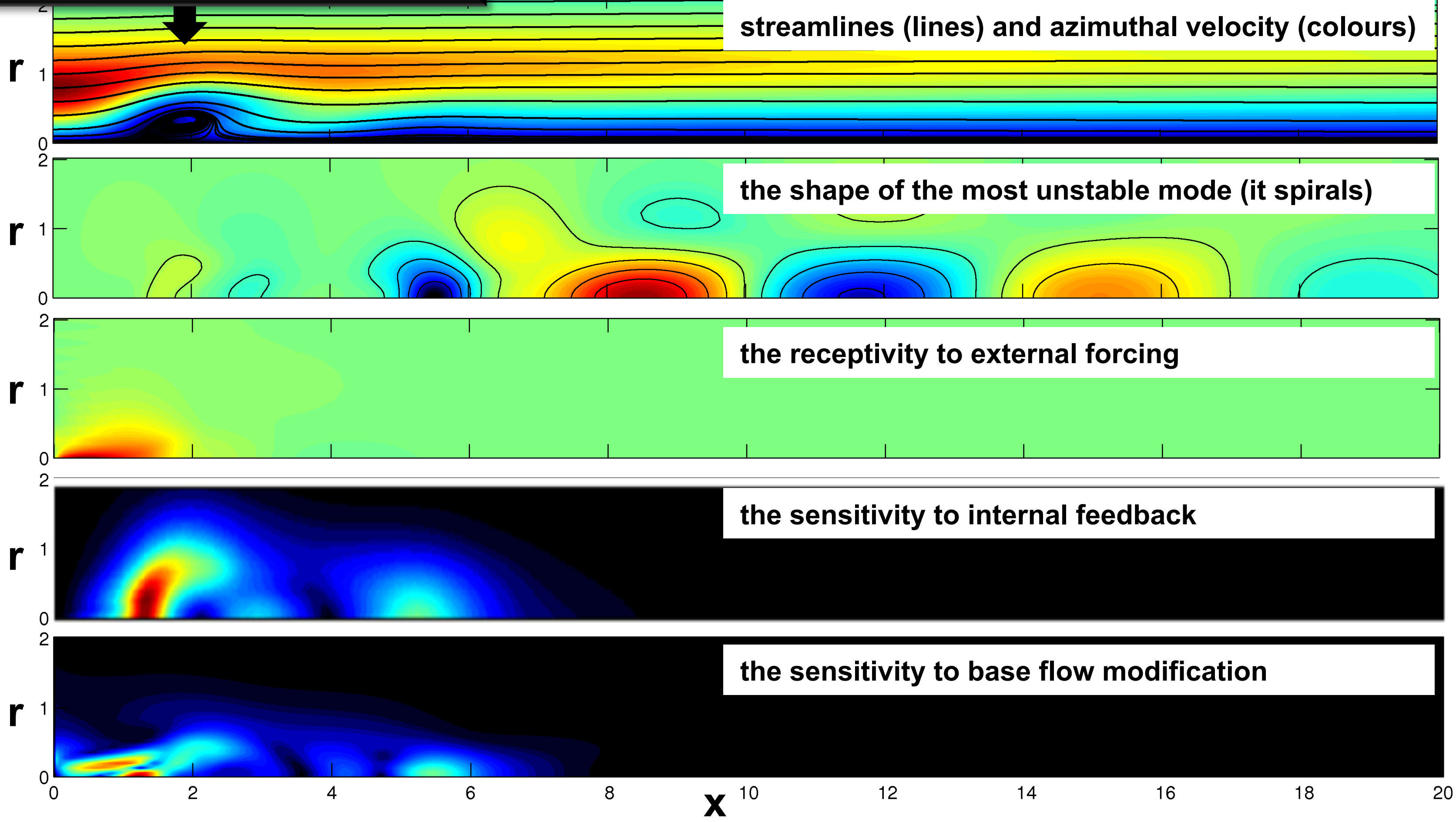
A laminar vortex breakdown bubble can be used as a toy model for the recirculating zone in a gas turbine combustion chamber.



Using adjoint methods, we find the most receptive and most sensitive regions of the flow.



Qadri & Juniper (2013)



streamlines (lines) and azimuthal velocity (colours)

the shape of the most unstable mode (it spirals)

the receptivity to external forcing

the sensitivity to internal feedback

the sensitivity to base flow modification

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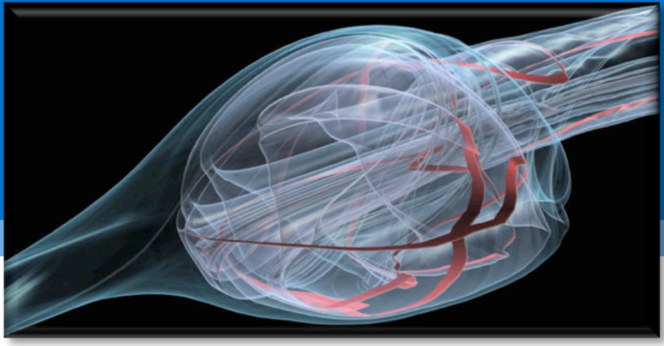
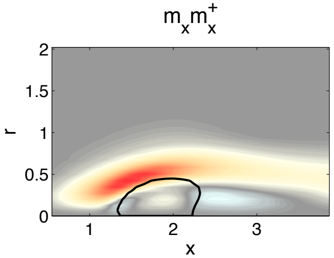
Using adjoint methods, we discover which physical feedback mechanisms drive the instability.



Qadri & Juniper (2013)

effect of the
axial velocity …



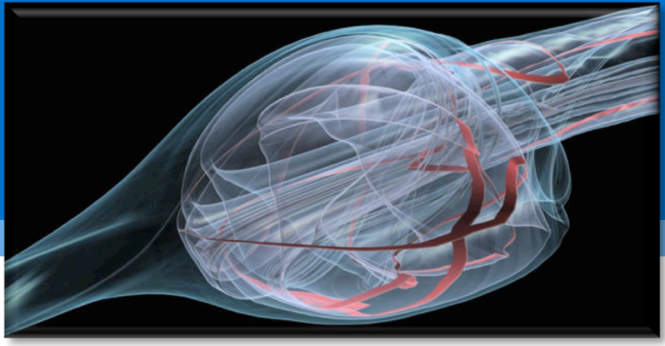
… on the axial momentum equation

destabilizing

neutral

stabilizing

Using adjoint methods, we discover which physical feedback mechanisms drive the instability.

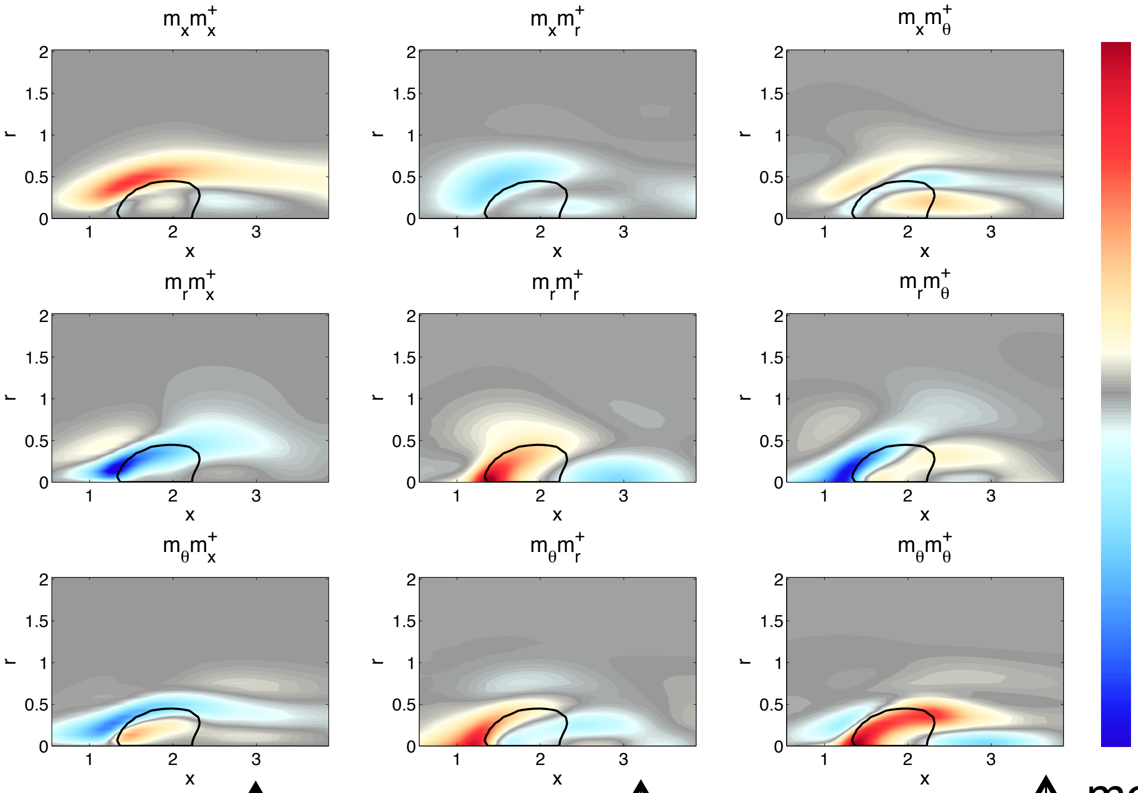


Qadri & Juniper (2013)

effect of the
axial velocity …

effect of the
radial velocity …

effect of the
azimuthal velocity …



destabilizing

neutral

stabilizing

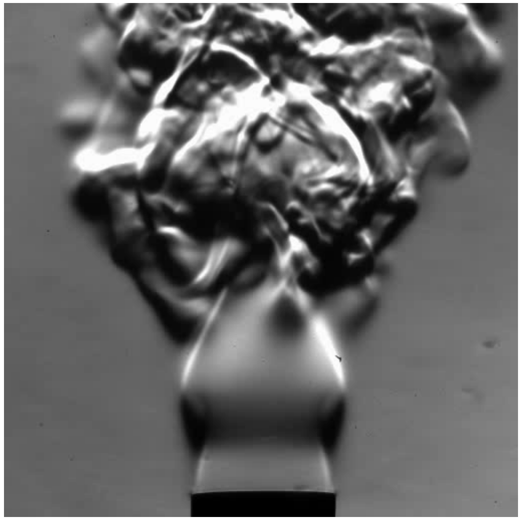
… on the … axial radial azimuthal momentum equation

I will demonstrate the base state sensitivity analysis on the varicose oscillation of a helium jet (in the absence of buoyancy)

Passive control of global instability in low-density jets

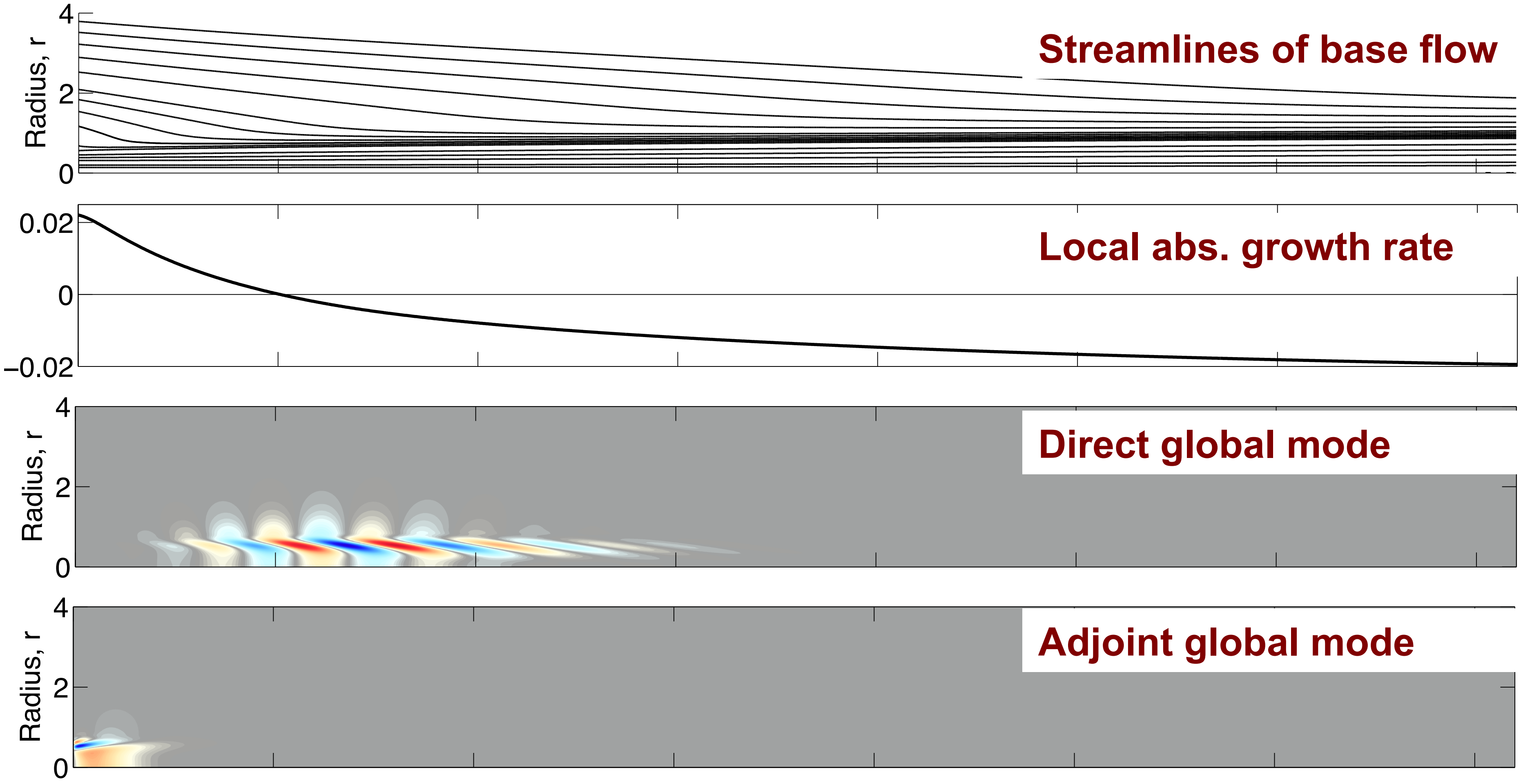
Ubaid Ali Qadri,[1] Gary J. Chandler,[1] and Matthew P. Juniper[1]

Department of Engineering, University of Cambridge, CB2 1PZ Cambridge, UK

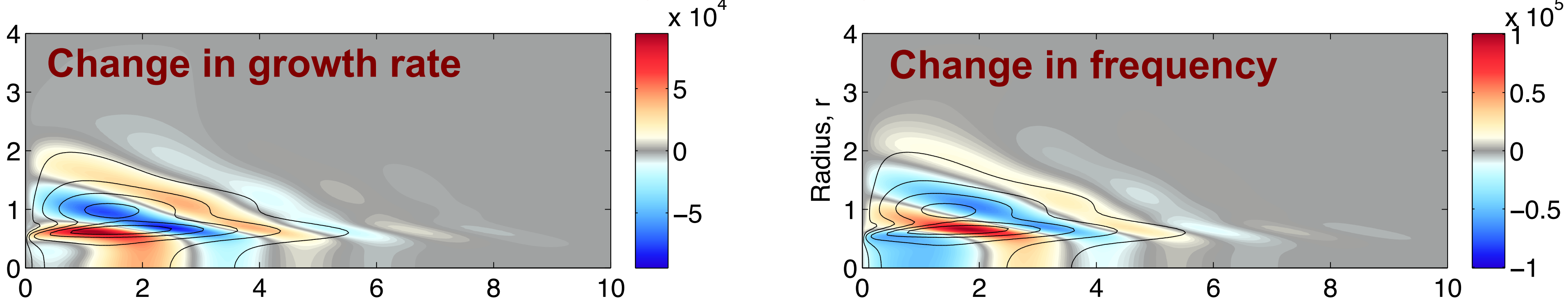


(submitted to Physics of Fluids in April 2013)

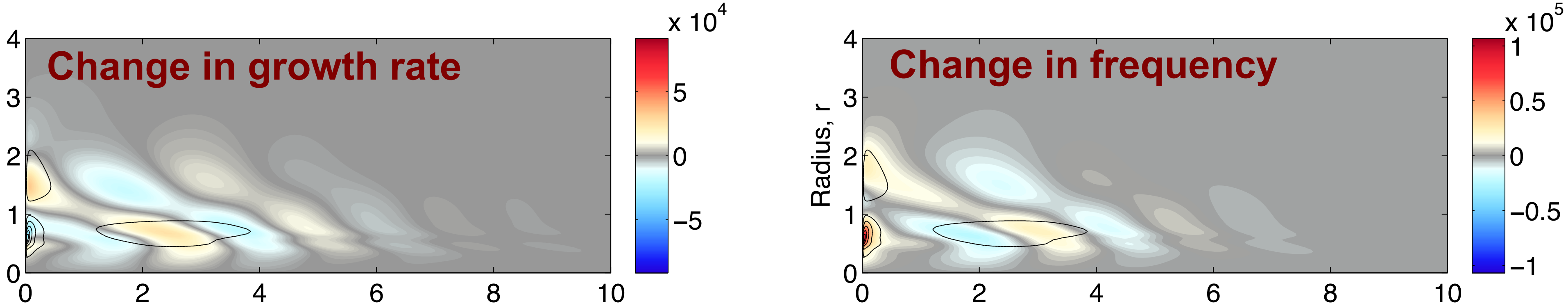
Given a base flow (top), we calculate the right eigenvector (direct global mode) and the left eigenvector (adjoint global mode).



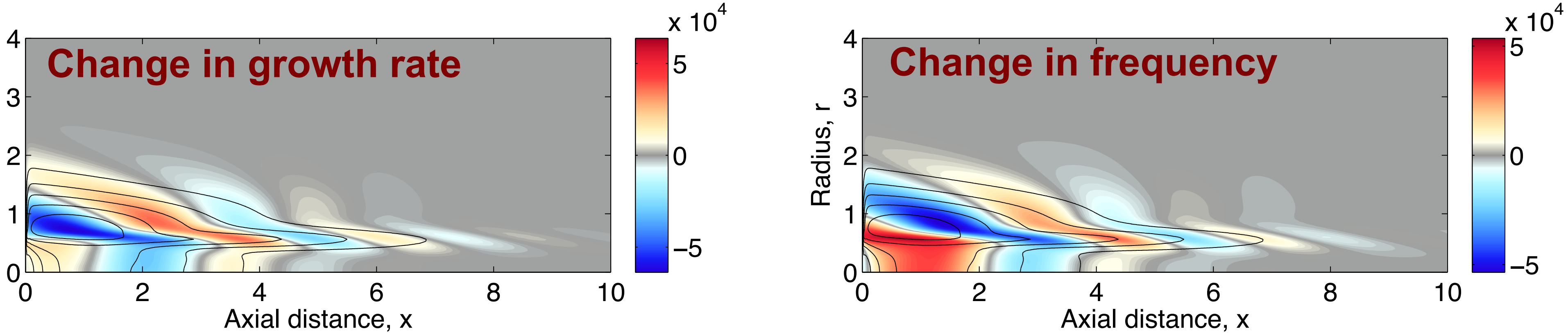
Sensitivity to a steady axial force



Sensitivity to a steady radial force

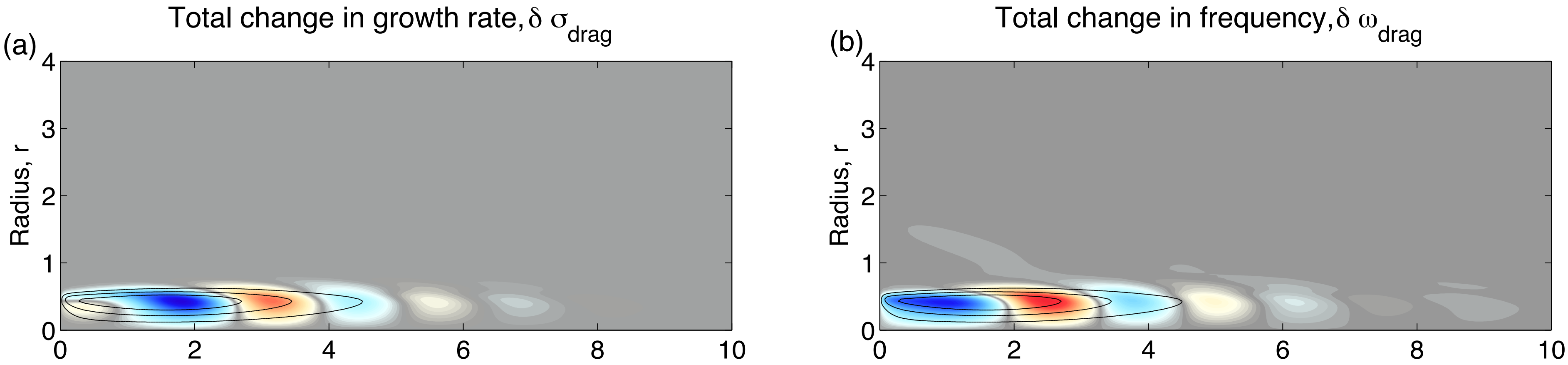


Sensitivity to a steady heat input



It is most interesting to examine the eigenvalue’s sensitivity to physical objects that can be placed in the flow.

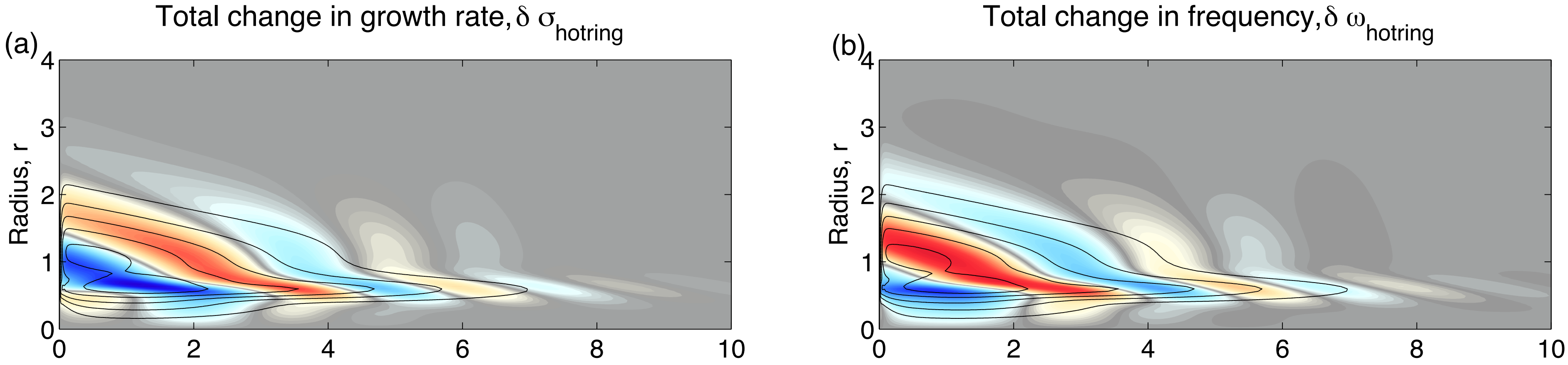
Sensitivity to a thin adiabatic ring



Total change in growth rate, $\delta\ \sigma_{drag}$

Total change in frequency, $\delta\ \omega_{drag}$

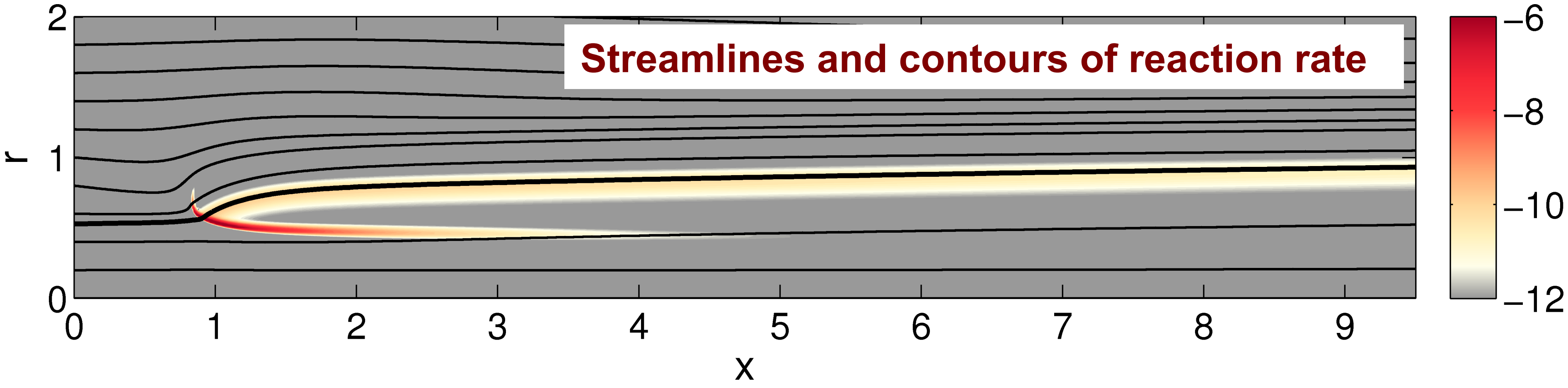
Sensitivity to a thin hot ring



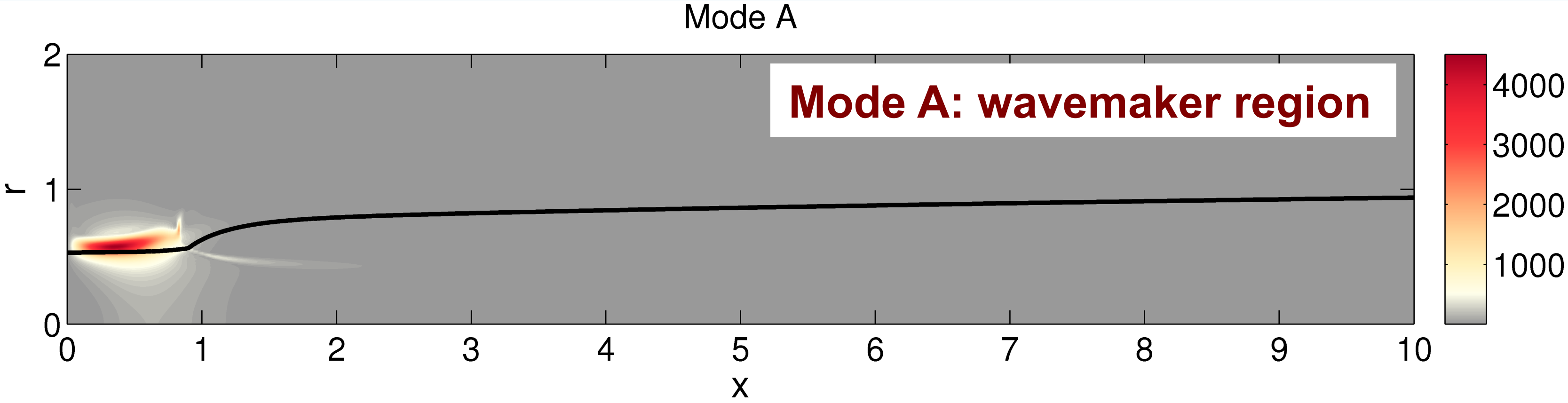
Total change in growth rate, $\delta\ \sigma_{hotring}$

Total change in frequency, $\delta\ \omega_{hotring}$

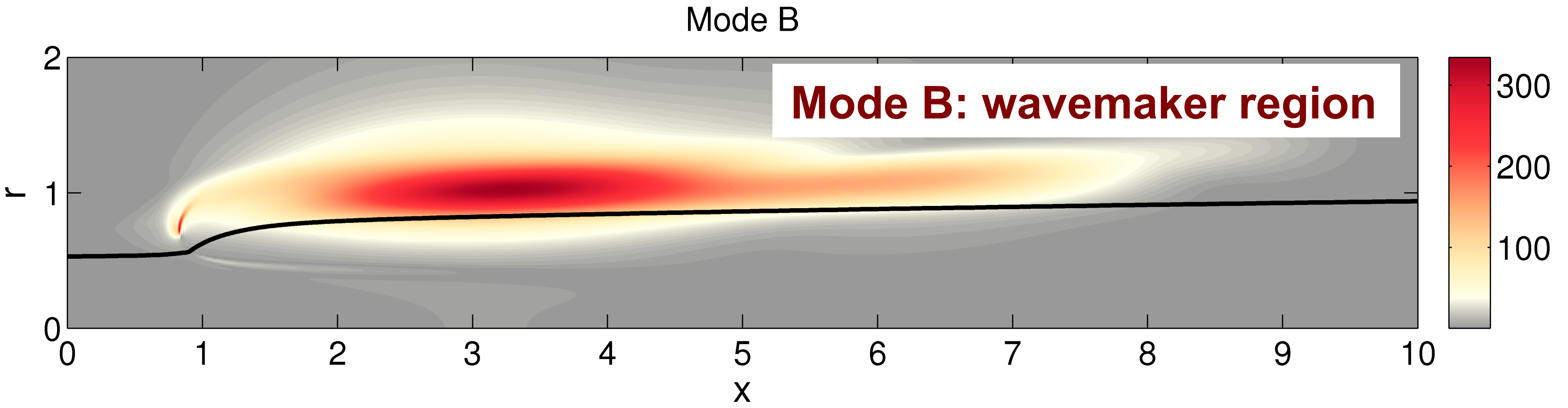
The same analysis can be performed on flames, to examine their hydrodynamic stability. Here is a lifted jet diffusion flame.



The lifted jet diffusion flame has two unstable hydrodynamic modes: one high frequency, one low frequency. They are caused by different regions of the flow.



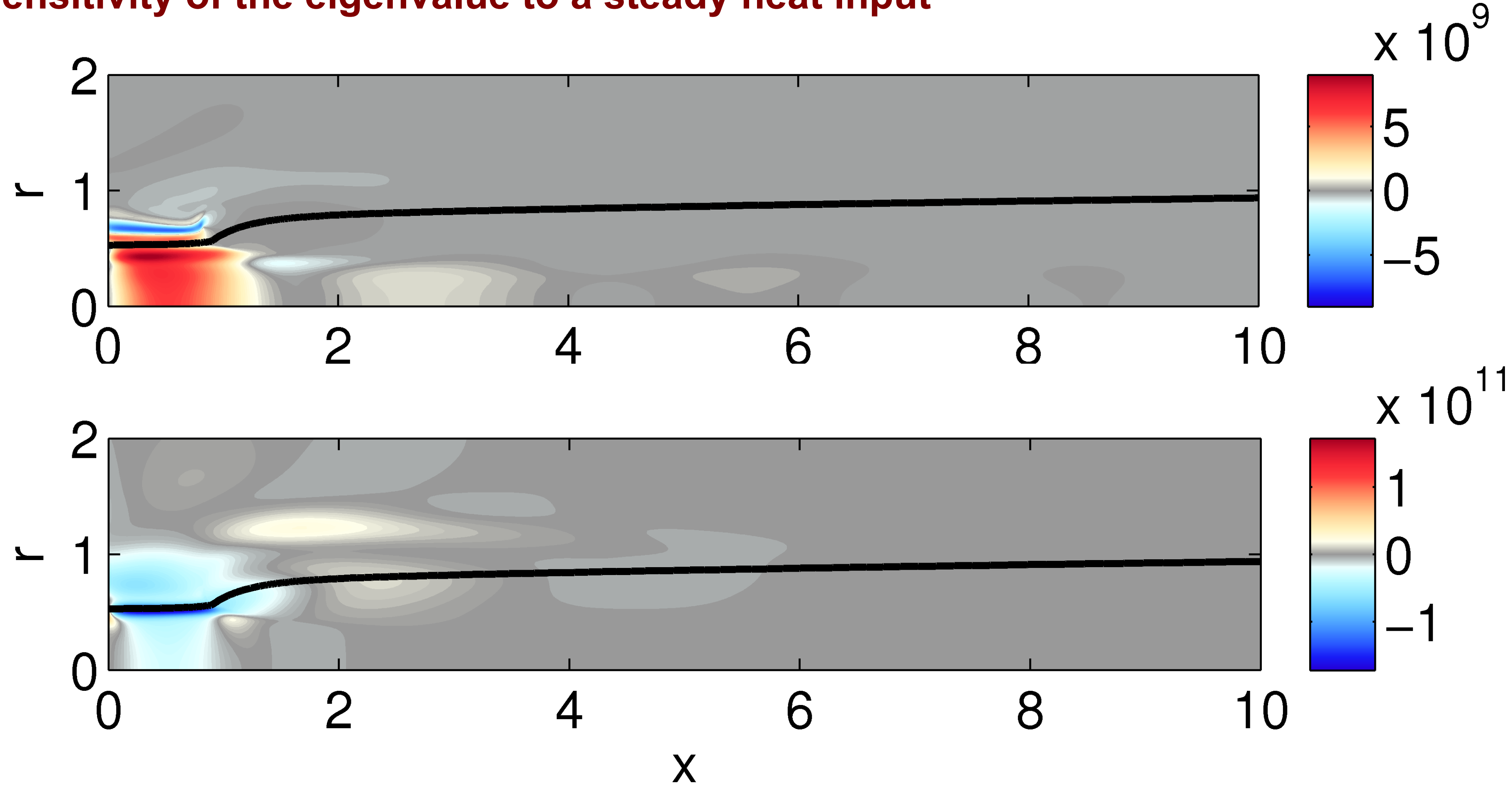
Mode A: wavemaker region



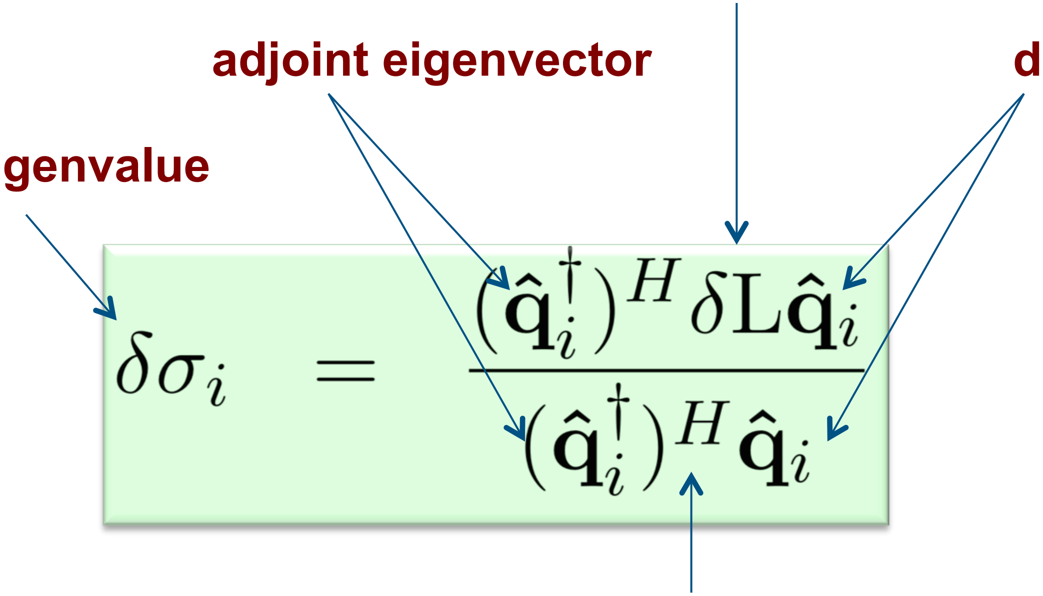
Mode B: wavemaker region

This flame turns out to be hyper-sensitive to some changes. We may find that thermoacoustic systems are also hyper-sensitive.

Sensitivity of the eigenvalue to a steady heat input



Summary



change to the linear operator

adjoint eigenvector

direct eigenvector

change to the eigenvalue

$$\delta\sigma_i = \frac{(\hat{\mathbf{q}}_i^\dagger)^H \delta \mathrm{L} \hat{\mathbf{q}}_i}{(\hat{\mathbf{q}}_i^\dagger)^H \hat{\mathbf{q}}_i}$$

conjugate transpose