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On the Stability of Prioritized Error-Based Scheduling for Resource-Constrained Networked Control Systems ⋆

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Abstract: The efficient usage of scarce communication resources is a vital necessity in networked control systems. This paper introduces a novel stochastic scheduling scheme for networked control systems with shared access through dynamic priority assignment. The overall system is assumed to consist of multiple control loops closed over a common communication network. A p-powered prioritized error-based (PEB) scheduler decides which transmission requests have the priority for channel access, and subsequently which controllers are updated with the actual states. According to the protocol, the likelihood to allocate the resource to a subsystem increases with a growing p-powered norm of its network-induced error. We show, under very mild assumptions, that the described system scheduled by the PEB protocol is stochastically stable. Moreover, numerical simulations demonstrate a significant reduction of the network-induced error variance in comparison to the other scheduling protocols such as CSMA or round robin.

Keywords: Prioritized scheduling, stochastic stability, Markov chain, drift operator.

1. INTRODUCTION

Networked Control Systems (NCS) are integrated systems composed of a multitude of small-scale elements sharing a resource-constrained communication channel. The communication constraints urge the design of modified scheduling policies to meet the real-time requirements of the control tasks while operating the network in an efficient manner. The Try-Once-Discard (TOD) protocol introduced by Walsh et al. (2002), uses only the current measurement data for transmission and immediately discards data when the transmission fails. On the other hand, the protocol hints the notion of prioritizing by choosing the measurement with the largest discrepancy between its actual value and its estimate at the controller. Therein, the Maximal Allowable Transfer Interval (MATI) discusses the stability of networked systems with deterministic communication medium by denoting the upper bound on the interval of two successive transmissions, Walsh et al. (2002); Nesic and Teel (2004). Contention-based protocols, like CSMA, are intrinsically stochastic; so they do not allow to use the notion of MATI in general, as the intervals between the transmissions usually can not be uniformly bounded with probability one. In Tabbara and Nesic (2008); Donkers et al. (2012); Antunes et al. (2009, 2012), stability of stochastic communication systems with random packet dropouts is given. Therein, the stability conditions for mean square stability and L_p stability-in-expectation are derived via Lyapunov theory.

The novelty of this work is given by considering a stochastic protocol in order to prioritize the channel access according to an error-dependent probability measure. As the errors are driven by the noise process, transmissions occur randomly in an event-based fashion. The approach uses the scarce resource of communicating more efficiently to stabilize the system and decreases the mean variance of the error in comparison with the static protocols, such as round robin and CSMA protocols. Moreover, the probabilistic nature of the protocol facilitates an approximative decentralized implementation through error-dependent back-off exponents. The system under consideration requires novel methods to analyze the asymptotic behavior of the NCS, developed in this paper. Our contribution is to analyze the stability properties of a networked system consisting of multiple loops closed over a shared communication channel by employing a stochastic scheduler with dynamically assigned priorities. Inspired by the idea of error-dependent intensity for transmission, Xu and Hespanha (2006), we introduce a stochastic variant of the TOD protocol denoted as p-powered prioritized error-based (PEB) protocol. The PEB protocol assigns to each subsystem a priority according to the individual networked-induced error. In particular, the probability of utilizing the resource increases with the p-powered norm of the estimation error. As stochastic disturbances are considered, we relax the notion of stochastic stability to ergodicity with finite second moment of the resulting Markov chain. Using drift criteria, we show that the overall system is ergodic. The key idea in the stability analysis, as is also discussed in Meyn and Tweedie (1994), lies in considering the multiple time steps drift operator to show the drift of the Lyapunov function is negative. Numerical simulations illustrate the stability of

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the system scheduled by the proposed scheme with an increased performance in terms of the average mean-squared error compared to the round robin and CSMA schemes. The simulations also show an improved performance with increasing exponent p of the PEB protocol.

The remainder of this paper is structured as follows. Problem statement is described in Section 2. Section 3 starts with some preliminaries of stochastic stability and ergodicity, and then proceeds with the stability analyses. The efficiency of the proposed approach is illustrated in Section 4 by numerical simulations.

Notation. In this paper, the Euclidean norm is denoted by $\|\cdot\|_2$. The expectation and the conditional expectation operators are denoted by $\mathsf{E}[\cdot]$ and $\mathsf{E}[\cdot|\cdot]$, respectively. The relation $\mathcal{N}(0, X)$ denotes a Gaussian random variable with zero-mean and covariance matrix X. If not otherwise stated, a state variable with superscript i indicates that it belongs to subsystem i. For constant matrices though, subscript i indicates the corresponding subsystem and superscript n denotes the matrix power.

2. PROBLEM STATEMENT

We consider a networked system composed of N independent heterogeneous subsystems which are coupled through a shared communication channel. Each individual control loop consists of a LTI stochastic plant $\mathcal{P}^i$, a stabilizing state-feedback controller $\mathcal{C}^i$, and a sensor $\mathcal{S}^i$. An event-based scheduler situated at the communication channel receives the data, in form of p-powered error norms, from all the sensors and decides if the state is an event to be scheduled for the channel utilization, as it is depicted in Fig. 1, schematically. The process $\mathcal{P}^i$ evolves by the following difference equation

$$x^i_{k+1} = A_i x^i_k + B_i u^i_k + w^i_k, \tag{1}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m_i}$, $x^i_k \in \mathbb{R}^{n_i}$, and $u^i_k \in \mathbb{R}^{m_i}$. The system noise $w^i_k \in \mathbb{R}^{n_i}$ is i.i.d. with $w^i_k \sim \mathcal{N}(0, W_i)$. For notational convenience, we assume that the system noise is unity variance Gaussian distribution, but the results hold for arbitrary positive definite W_i. Since, the stability analysis is independent of initial states, x^i_0 might have any arbitrary but symmetric distribution with bounded second moment. The scheduler output for the ith subsystem at the time-step k is described by the binary random variable $\delta^i_k \in \{0, 1\}$,

$$\delta^i_k = \begin{cases} 1 & \text{subsystem } i \text{ is updated} \\ 0 & \text{subsystem } i \text{ is blocked} \end{cases}$$

This implies for the received signal, z_k, at the controller

$$z^i_k = \begin{cases} x^i_k & \delta^i_k = 1 \\ \varnothing & \delta^i_k = 0 \end{cases}$$

It is assumed that the system data A_i, B_i, W_i, and the distribution of x_0 are known locally within each subsystem. The control law γ^i is described by the causal mappings of the past observations for each time step k, i.e.,

$$u^i_k = \gamma^i_k(Z^i_k) = -L_i \,\mathsf{E}\left[x^i_k | Z^i_k\right] \tag{2}$$

where $Z^i_k = \{z^i_0, \ldots, z^i_k\}$ represents the observation history, and L_i is the stabilizing feedback gain. This implies that each control loop is stabilized with the controller in (2) in case of the ideal communication. The controllers are updated by an estimator, in case of a blocked data transmission request, only if the closed-loop matrix $(A_i - B_i L_i)$ is Hurwitz, i.e.

$$\mathsf{E}\left[x^i_k | Z^i_k\right] = \begin{cases} x^i_k & \delta^i_k = 1 \\ (A_i - B_i L_i)\,\mathsf{E}\left[x^i_{k-1} | Z^i_k\right] & \delta^i_k = 0 \end{cases} \tag{3}$$

with the initial condition $\mathsf{E}\left[x^i_0 | Z^i_0\right] = 0$. For each subsystem i, the network-induced error state $e^i_k \in \mathbb{R}^{n_i}$ is defined as $e^i_k = x^i_k - \mathsf{E}\left[x^i_k | Z^i_{k-1}\right]$ and it evolves according to the following difference equation

$$e^i_{k+1} = \left(1 - \delta^i_k\right) A_i e^i_k + w^i_k \tag{4}$$

with the initial condition $e^i_0 = x^i_0 - \mathsf{E}\left[x^i_0\right]$. The augmented state $\left[x^i_k, e^i_k\right]$ has a triangular dynamics, according to (1)-(4), i.e. the system state x^i_k do not affect the evolution of the error state e^i_k. Hence, showing the sequence of error states e_k is stochastically stable implies the overall system's stability. The p-powered prioritized error-based (PEB) scheduling policy defines the probability that a subsystem is granted the transmission chance at time k

$$\mathsf{P}[\delta^i_k = 1 | e^j_k, j \in \{1, \ldots, N\}] = \frac{\|e^i_k\|^p_2}{\sum_{j=1}^N \|e^j_k\|^p_2} \tag{5}$$

where $p \geq 2$ is an integer. According to the PEB scheduling scheme, the highest p-powered error norm has the channel access priority, and all the other requests accompanied with the lower priorities are more likely to be dropped. As the scheduling policy is memoryless, the process is repeated in every time step k according to (5). As the resource-constrained network allows only one transmission per time step, we have the following hard constraint with probability 1 for every $k \geq 0$ as

$$\sum_{i=1}^N \delta^i_k = 1 \tag{6}$$

It is straightforward to extend the approach to a different number than one of allowed transmission per time step.

Remark 1. To implement the PEB policy approximately in a decentralized fashion, it is envisioned that every subsystem randomly determines its priority according to a probability distribution depending on its own error. In a wireless CSMA communication systems, the priority could be reflected in the error-dependent distribution of the backoff-time of each subsystem during one time step. This implies that the mean back-off time of a subsystem decreases with an increase of the error norm.

The evolution of the aggregate state $e_k \in \mathbb{R}^n$, which is defined by

$$e_k = [e^1_k, \ldots, e^N_k]^\mathsf{T} \tag{7}$$

can be regarded as a time-homogeneous Markov chain, because the scheduling policy defined in (5) is a Markov policy depending on the values of e_k. In the next section, we will investigate the stability properties of the aforementioned Markov chain.

3. STABILITY ANALYSIS

This section presents the stability of the NCS with multiple control loops coupled through a constrained resource, at which the resource utilization is scheduled by the PEB

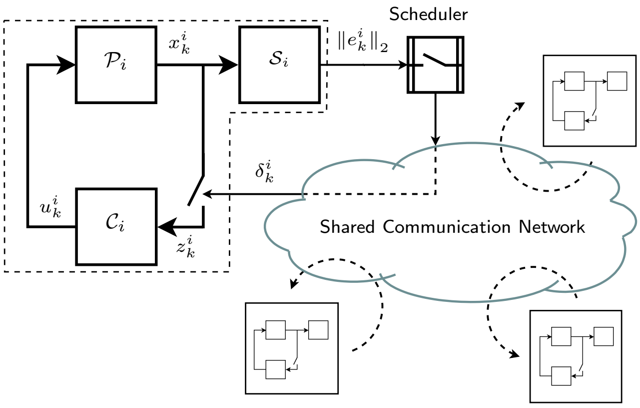

Fig. 1. A NCS with a shared communication channel

policy. We start with a two-subsystem heterogeneous network, and then extend the results for N subsystems. First, some preliminaries are introduced to facilitate the discussions.

3.1 Preliminaries

As the Markov chain defined in (7) evolves in $\mathbb{R}^n$, we have to deal with an uncountable state space. In this paper we will adopt the analysis tools for stochastic stability of Markov chains in uncountable state spaces as provided in Meyn and Tweedie (1996), Chapter 14. The notion of stability in uncountable state spaces are similar to of countable state spaces but with several generalizations, e.g. irreducibility becomes ψ-irreducibility, where ψ is a non-trivial measure of the uncountable state space, or the so-called *small sets*, which can be identified by the compact sets in our problem. These sets take the role of the finite sets in the countable Markov chains. The stability concept used in this paper is given by the f-ergodicity defined in the following.

Definition 1. (f*-ergodicity*). Let $f \geq 1$ be a real-valued function in $\mathbb{R}^n$. A Markov chain e_k is f*-ergodic*, if

(1) e_k is positive Harris recurrent with invariant probability measure π
(2) $\mathsf{E}\left[\pi(f)\right]$ is finite, where $\pi(f) = \int f(e)\pi(de)$
(3) for every initial value e_0, $\lim_{k\to\infty} \|P^k(e_0,\cdot) - \pi\|_f = 0$

The f-norm $\|\cdot\|_f$ for any signed measure ν is defined as

$$\|\nu\|_f = \sup_{|g|\leq f} |\nu(g)|$$

Remark 2. Later, we will choose f to be quadratic in order to show that the stationary distribution of the Markov chain has a finite bounded moment.

Definition 2. Let V be a real-valued function in $\mathbb{R}^n$. The drift operator Δ is defined for any non-negative measurable function V as

$$\Delta V(e_k) = \mathsf{E}[V(e_{k+1})|e_k] - V(e_k), \qquad e_k \in \mathbb{R}^n. \tag{8}$$

Theorem 1. (*Aperiodic Ergodic Theorem*). Let the Markov chain $\{e_k\}_k$ be ψ-irreducible and aperiodic and let $f(e) \geq 1$ be a real-valued function in $\mathbb{R}^n$. If a small set $\mathcal{D}$ and a non-negative real-valued function V exist such that

$$\Delta V(e) \leq -f(e), \qquad e \in \mathbb{R}^n \backslash \mathcal{D} \tag{9}$$

and $\Delta V < \infty$ for $e \in \mathcal{D}$, then $\{e_k\}_k$ is f-ergodic.

Remark 3. Since we assume Gaussian additive noise with $W_i > 0$, the transition kernel $P(e,\cdot)$ at any state e of the Markov chain e_k has a positive density function. Then, the Markov chain is ψ-irreducible and aperiodic. Analogously as in the section 5.3.5 of Meyn and Tweedie (1996), it can be concluded that all the compact sets are small.

3.2 Two-Subsystem Network

Consider two linear time-invariant control systems coupled through the communication network, with the plants and controllers given as (1) and (2). At each time k, the scheduler is provided with the error norms $\|e_k^i\|_2$. The PEB scheduler likely selects the subsystem with the higher relative error norm to access the channel and then the corresponding controller is updated. In case of dismissed transmission request, a state observer predicts the state evolution. The stability analysis is performed based on the drift conditions, but first we state the following lemma, which facilitates finding the upper bounds for the drift.

Lemma 1. (*Expected value of the ratio* $\frac{a}{b}$). Suppose a and $b \neq 0$ are two dependent random variables, then

$$\mathsf{E}\left[\frac{a}{b}|_{b\neq 0}\right] = \mathsf{E}\left[a\right]\mathsf{E}\left[\frac{1}{b}\right] + \sum_{i=1}^{\infty}(-1)^i \frac{\mathsf{E}\left[\left(a - \mathsf{E}\left[a\right]\right)\left(b - \mathsf{E}\left[b\right]\right)^i\right]}{\mathsf{E}\left[b\right]^{i+1}}$$

Proof. See Rice (2008).

The following theorem incurs the stability of the two-subsystem NCS employing the p-powered PEB allocating strategy, to

Theorem 2. Consider a NCS consists of two stochastic LTI control loops coupled through a communication channel constrained by (6), and with the stabilizing controllers γ^i as defined in (2). Then, if the channel access is scheduled as introduced in (5), then the time-homogeneous Markov chain in (7) is f-ergodic and has finite second moment.

Proof. We introduce the non-negative measurable function $V(e_k) = \sum_{i=1,2} \|e_k^i\|_2^p$ as a mapping from $\mathbb{R}^{n_1+n_2}$ to $\mathbb{R}$. It suffices to analyze the stability over the two consecutive time-steps, i.e. $[k, k+1]$. The error evolution can be written over the two time-step horizon as

$$e_{k+2}^i = \left(1-\delta_{k+1}^i\right)\left(1-\delta_k^i\right)A_i^2 e_k^i + \left(1-\delta_{k+1}^i\right)A_i w_k^i + w_{k+1}^i.$$

Hence,

$$\mathsf{E}[V(e_{k+2})|e_k] = \sum_{i=1,2} \mathsf{E}\left[\|e_{k+2}^i\|_2^p | e_k\right] =$$

$$\sum_{i=1,2} \mathsf{E}\left[\|\left(1-\delta_{k+1}^i\right)A_i\left[\left(1-\delta_k^i\right)A_i e_k^i + w_k^i\right] + w_{k+1}^i\|_2^p | e_k\right]$$

The triangle inequality incurs

$$\|e_{k+2}^i\|_2^p \leq \left(\|A_i^2 e_k^i\|_2 + \|A_i w_k^i\|_2 + \|w_{k+1}^i\|_2\right)^p$$
$$= \|A_i^2 e_k^i\|_2^p + \|A_i w_k^i\|_2^p + \|w_{k+1}^i\|_2^p + \phi^+\left(e_k^i, w_k^i, w_{k+1}^i\right)$$

where $\mathsf{E}\left[\phi^+\left(e_k^i, w_k^i, w_{k+1}^i\right)|e_k\right] \geq 0$ is bounded due to the p-moment boundedness of the Gaussian noise processes, and statistical independence of e_k, w_k, and w_{k+1}. The boundedness of ϕ^+ is preserved even if e_k^i becomes very large, since the corresponding subsystem, with the probability highly close to one, will be reseted, i.e. $\delta_{k+1}^i = 1$.

The second-order drift operator is then specified as

$$\begin{aligned}&\Delta^2 V(e_k)\\&\leq \sum_{i=1,2} \mathsf{E}\left[\| \left(1-\delta^i_{k+1}\right)\left(1-\delta^i_k\right) A_i^2 e_k^i \|_2^p | e_k\right]\\&+\sum_{i=1,2} \mathsf{E}\left[\| \left(1-\delta^i_{k+1}\right) A_i w_k^i \|_2^p | e_k\right]\\&+\sum_{i=1,2} \mathsf{E}\left[\|w^i_{k+1}\|_2^p\right]+\mathsf{E}\left[\phi^+(e_k,w_k,w_{k+1})|e_k\right]-\sum_{i=1,2}\|e_k^i\|_2^p\end{aligned}$$

Since, the system noise is unity variance Gaussian distribution, the Euclidean norm $\|w^i_{k+1}\|_2$ has a chi distribution with n_i degrees of freedom which implies that $\mathsf{E}\left[\|w^i_{k+1}\|_2^p\right]$ is bounded. At time-step k, one of the subsystems surely transmits, suppose the 1st, i.e. $\delta^1_k=1$, therefore $\delta^2_k=0$. Then, the drift operator can be simplified as follows,

$$\begin{aligned}&\Delta^2 V(e_k)\\&\leq \mathsf{E}\left[\| \left(1-\delta^2_{k+1}\right) A_2^2 e_k^2+\left(1-\delta^2_{k+1}\right) A_2 w_k^2 \|_2^p | e_k\right]\\&+\mathsf{E}\left[\| \left(1-\delta^1_{k+1}\right) A_1 w_k^1 \|_2^p | e_k\right]\\&+\sum_{i=1,2} \mathsf{E}\left[\|w^i_{k+1}\|_2^p\right]-\sum_{i=1,2}\|e_k^i\|_2^p+\mathsf{E}\left[\phi^+(e_k,w_k,w_{k+1})|e_k\right]\\&=\mathsf{E}\left[\| \left(1-\delta^2_{k+1}\right) A_2 e_{k+1}^2 \|_2^p | e_k\right]+\xi^+-\sum_{i=1,2}\|e_k^i\|_2^p\end{aligned}$$

where, $\xi^+ = \mathsf{E}\left[\| \left(1-\delta^1_{k+1}\right) A_1 w_k^1 \|_2^p | e_k\right] + \mathsf{E}\left[\phi^+|e_k\right] + \sum_{i=1,2}\mathsf{E}\left[\|w^i_{k+1}\|_2^p\right]$. We introduce the complementary binary random variable $d\in\{1,2\}$ as

$$d=\begin{cases}1 & \|e^2_{k+1}\|_2^p\leq\varepsilon_2<M\\ 2 & \|e^2_{k+1}\|_2^p>\varepsilon_2\end{cases}\tag{10}$$

where, $\varepsilon_2>0$, and d occurs with probabilities $\mathsf{P}\left[d=1|e_k\right]=\varepsilon$ and $\mathsf{P}\left[d=2|e_k\right]=1-\varepsilon$, with $\varepsilon\in[0,1]$. Employing the law of *iterated expectation*

$$\begin{aligned}&\mathsf{E}\left[\| \left(1-\delta^2_{k+1}\right) A_2 e_{k+1}^2 \|_2^p | e_k\right]\\&=\mathsf{E}\left[\mathsf{E}\left[\| \left(1-\delta^2_{k+1}\right) A_2 e_{k+1}^2 \|_2^p | e_k,d\right]|e_k\right]\\&=\mathsf{P}\left(d=1|e_k\right).\mathsf{E}\left[\| \left(1-\delta^2_{k+1}\right) A_2 e_{k+1}^2 \|_2^p | e_k,d=1\right]\\&+\mathsf{P}\left(d=2|e_k\right).\mathsf{E}\left[\| \left(1-\delta^2_{k+1}\right) A_2 e_{k+1}^2 \|_2^p | e_k,d=2\right]\\&\leq\varepsilon\varepsilon_2\|A_2\|_2^p+(1-\varepsilon)\,\mathsf{E}_2\left[\left(1-\delta^2_{k+1}\right)\|A_2e^2_{k+1}\|_2^p|e_k\right]\end{aligned}$$

where we denote $\mathsf{E}\left[\cdot|e_k,d=2\right]=\mathsf{E}_2\left[\cdot|e_k\right]$ for the sake of abbreviation. Furthermore, $\mathsf{P}\left(\|e^2_{k+1}\|_2^p>\varepsilon_2|d=2\right)=1$, so the following conservative inequality concludes

$$\mathsf{E}_2\left[\frac{1}{\|e^1_{k+1}\|_2^p+\|e^2_{k+1}\|_2^p}|e_k\right]\leq\mathsf{E}_2\left[\frac{1}{\|e^2_{k+1}\|_2^p}|e_k\right]\leq\varepsilon_2^{-1}\tag{11}$$

Then, *Lemma 1* assures

$$\begin{aligned}&\mathsf{E}_2\left[1-\delta^2_{k+1}|e_k\right]\leq\frac{2^{\frac{p}{2}}\Gamma\left(\frac{p+n_1}{2}\right)}{\varepsilon_2\Gamma\left(\frac{n_1}{2}\right)}\\&-\sum_{i_{Odd}=1}^{\infty}\frac{(i+1)!\left(2\bar{\sigma}^2\right)^{\left(\frac{i}{2}\right)}}{\left(\frac{i+1}{2}\right)!2^{i+1}\left(\mathsf{E}\left[\|w^1_k\|_2^p\right]+\mathsf{E}\left[\|e^2_{k+1}\|_2^p\right]\right)^{i+1}}=c_2\end{aligned}$$

where, Γ represents the Gamma function, and $\bar{\sigma}^2$ is $\mathrm{Var}\left[\|w^1_k\|_2^p\right]$. Superposing both upper bounds for $d\in\{1,2\}$ provides the aggregate bound on $\mathsf{E}\left[V(e_{k+2})|e_k\right]$

$$\begin{aligned}&\mathsf{E}\left[V(e_{k+2})|e_k\right]\\&\leq(1-\varepsilon)\,\mathsf{E}_2\left[\|\left(1-\delta^2_{k+1}\right)A_2e^2_{k+1}\|_2^p|e_k\right]+\varepsilon\varepsilon_2\|A_2\|_2^p+\xi^+\\&=(1-\varepsilon)\,\mathsf{E}_2\left[\left(1-\delta^2_{k+1}\right)|e_k\right].\mathsf{E}_2\left[\|A_2e^2_{k+1}\|_2^p|e_k\right]+\xi_2^+\\&\leq(1-\varepsilon)\,c_2\,\mathsf{E}_2\left[\|A_2e^2_{k+1}\|_2^p|e_k\right]+\xi_2^+\\&=(1-\varepsilon)\,c_2\|A_2\|_2^p\,\mathsf{E}_2\left[\|A_2e^2_k+w^2_k\|_2^p|e_k\right]+\xi_2^+\\&\leq(1-\varepsilon)\,c_2\left[\left(\|A_2\|_2^p\right)^2\|e^2_k\|_2^p+\|A_2\|_2^p\,\mathsf{E}_2\left[\|w^2_k\|_2^p\right]\right]+\xi_3^+\\&\leq(1-\varepsilon)\,c_2\left(\|A_2\|_2^p\right)^2\sum_{i=1,2}\|e^i_k\|_2^p+\xi_4^+\end{aligned}$$

where, the second inequality is ensured through the independence of δ^2_{k+1} and e^2_{k+1}. Moreover, $\xi_2^+=\xi^++\varepsilon\varepsilon_2\|A_2\|_2^p$, $\xi_3^+=\xi_2^++(1-\varepsilon)\,c_2\|A_2\|_2^p\,\mathsf{E}_2\left[\phi^+\left(e^2_k,w^2_k\right)|e_k\right]$ and $\xi_4^+=\xi_3^++(1-\varepsilon)\,c_2\|A_2\|_2^p\,\mathsf{E}_2\left[\|w^2_k\|_2^p\right]$.

Then the drift operator is upper bounded as

$$\Delta^2V(e_k)\leq\left[(1-\varepsilon)\,c_2\left(\|A_2\|_2^p\right)^2-1\right]\sum_{i=1,2}\|e^i_k\|_2^p+\xi_4^+$$

Define $f(e)=\epsilon\sum_{i=1,2}\|e^i_k\|_2^p-\xi_4^+$, $\epsilon>0$. Then showing $\left[(1-\varepsilon)\,c_2\left(\|A_2\|_2^p\right)^2-1\right]\leq-\epsilon$ implies $\Delta^2V(e_k)\leq-f(e)$ and the stochastic stability of the Markov chain is evident. According to the *Theorem 1*, we can find the appropriate ϵ and a compact set $\mathcal{D}$ such that for all $e\notin\mathcal{D}$

$$(1-\varepsilon)\,c_2\leq\frac{1}{\left(\|A_2\|_2^p\right)^2}\tag{12}$$

It guarantees $\mathsf{E}\left[\left(1-\delta^2_{k+1}\right)|e_k\right]\leq\frac{1}{(1-\varepsilon)\left(\|A_2\|_2^p\right)^2}$ and the proof immediately follows.

Remark 4. Although, the conservativeness of the upper bound in (11) is evident in case of large $\|e^1_{k+1}\|_2^p$, it is not troublesome, since for now we merely care about stability. For performance analysis, less conservative bounds can be employed by the available literatures, e.g. Lew (1976).

Figure 2 illustrates how the PEB scheduling policy allocates the communication resource for a two-subsystem NCS. At time k, the relative error norm of the first subsystem exceeds the second subsystem, thus the first subsystem will likely be awarded the channel access. If so, the first subsystem's error is then reseted, i.e. $\|e^1_{k+1}\|_2=\|w^1_k\|_2$. The second subsystem request for transmission is blocked, and the corresponding error magnified by $\|e^2_{k+1}\|_2=\|A_2e^2_k+w^2_k\|_2$. Clearly, if the system noise has a finite variance σ^2, then the amplified error e^2_{k+1} can be kept inside the convex set by enlarging the threshold M_2. Moreover, even if the error goes out of the safe region at a certain time-step, it will be reseted in the most forthcoming time-step and the evolution would back inside the convex set.

3.3 N-Subsystem Network

In this section the stability of the depicted NCS with arbitrary finite number of heterogeneous subsystems will be presented. We first state the following essential lemma.

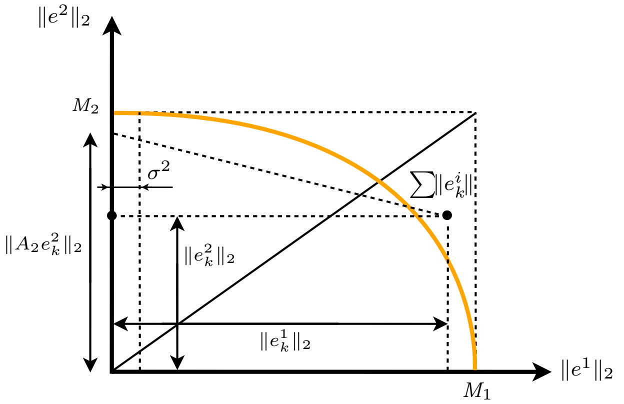

Fig. 2. PEB scheduling for a two-subsystem network.

Lemma 2. Suppose that $\delta^i_k = 1$ for some k and i. If there exists a subsystem $j \neq i$ such that $\|e^j_{k+n}\|^p_2 > M_j$, then the probability that $\delta^i_{k+n} = 1$ is upper bounded by

$$\mathsf{P}[\delta^i_{k+n} = 1 | \|e^j_{k+n}\|^p_2 > M_j, \delta^i_k = 1] \leq \frac{\sum_{k=0}^{n-1} n_j \|A^k_j\|^p_2}{M_j}$$

Proof. Taking the definition of PEB in (5) and having

$$\mathsf{P}[\delta^i_{k+n} = 1 | \|e^j_{k+n}\|^p_2 > M_j, \delta^i_k = 1]$$
$$= \mathsf{E}[\mathsf{P}[\delta^i_{k+n} = 1 | e_k] | \|e^j_{k+n}\|^p_2 > M_j, \delta^i_k = 1],$$

the result follows immediately.

Remark 5. Lemma 2 implies that the probability of two subsequent transmissions of the same subsystem within a fixed interval can be made arbitrarily small by choosing M_j, accordingly.

Theorem 3. Let a NCS consists of N arbitrary finite heterogeneous stochastic LTI control loops sharing a communication channel subject to the constraint in (6). Suppose that the stabilizing controller γ^i is given as (2). Then, the Markov chain e_k in (7), is f-ergodic and has finite second moment, if the channel access is scheduled according to the p-powered PEB scheme introduced in (5).

Proof. As in *Def. 2*, introduce the quadratic function $V(e_k) = \sum_{i=1}^N \|e^i_k\|^2_2$, and the Nth-order drift operator $\Delta^N V(e_k) = \mathsf{E}[V(e_{k+N}) | e_k] - V(e_k)$. The expectation of the drift for N time-steps later, i.e. $k+N$, will be calculated as

$$\mathsf{E}[V(e_{k+N}) | e_k] = \sum_{i=1}^N \mathsf{E} \left\| \prod_{j=0}^{N-1} \left(1 - \delta^i_{k+j}\right) A^N_i e^i_k \right\|^2_2$$
$$+ \sum_{i=1}^N \mathsf{E} \left\| \prod_{j=1}^{N-1} \left(1 - \delta^i_{k+j}\right) A^{N-1}_i w^i_k \right\|^2_2 + \dots$$
$$+ \sum_{i=1}^N \mathsf{E} \left\| \left(1 - \delta^i_{k+j}\right) A_i w^i_{k+N-2} \right\|^2_2 + \sum_{i=1}^N \mathsf{E} \left\| w^i_{k+N-1} \right\|^2_2$$

Here, we divide all the possible situations into three complementary cases in terms of transmission ordering and size of error, as follows

(1) Every subsystem $i \in [1, \dots, N]$ transmits merely once during the N time-step period, with the arbitrary size of error at each time-step.

(2) For every $i \in [1, \dots, N]$, there exists a time step $k' \in [k, k+N-1]$ s.t. $\|e^i_{k'}\|^2_2 < M_i$ for all $\delta^i_{k'} = 0$.

(3) There exists a set of subsystems m s.t. for all $i \in m$, $\|e^i_{k'}\|^2_2$ takes arbitrary values for some time-steps $k' \in [k, k+N-1]$ and for $\delta^i_{k'} = 0$.

Each of the cases can happen during the N time-step period with probability $\mathsf{P}_i \in [0,1]$, and $\sum_{i=1}^3 \mathsf{P}_i = 1$. Note that, unlike the first case at which all the subsystems eventually transmit during the N time-step period, in the second and third cases, not all the subsystems necessarily transmit, i.e. there might be some subsystems i s.t. $\delta^i_{k'} = 0$ for all $k' \in [k, k+N-1]$. On the other hand, the first case places no boundary on the error size, unlike the second case. The point in the third case is that the drift does not need to be necessarily negative since the occurrence probability can be made arbitrarily close to zero.

The first case ensures that every subsystem i, is assigned with a time-step $k+j$ with $j \in [0, \dots, N-1]$, s.t. $\delta^i_{k+j} = 1$. Therefore, the ill-effects of possibly unbounded error e^i_k would be eliminated, since the corresponding expression $\sum_{i=1}^N \mathsf{E} \left\| \prod_{j=0}^{N-1} \left(1 - \delta^i_{k+j}\right) A^N_i e^i_k \right\|^2_2$ is then vanished. Eventually, showing that the Nth order drift is negative is trivial according to the *Theorem 1*, by finding the appropriate ϵ and the compact set $\mathcal{D}$.

For the second case, we divide the subsystems into two complementary sets. The set s_1 consists of the $\bar{m}$ subsystems which are granted the resource access at least once, and the set s_2 contains the $m = N - \bar{m}$ subsystems which have not transmitted at all. Hence, the scheduler variable $\delta^{i\in s_1}_{k+j} = 1$ at least for one j, and $\delta^{i\in s_2}_{k+j} = 0$ for all j where $j \in \{0, 1, \dots, N-2\}$. Then

$$\mathsf{E}[V(e_{k+n}) | e_k]$$
$$\leq \sum_{i\in s_2} \left[\mathsf{E} \| \left(1 - \delta^i_{k+N-1}\right) A_i e^i_{k+N-1} \|^2_2 | e_k\right] + \xi^+$$

where, ξ^+ stands for the sum of the bounded terms, for abbreviation. Introduce the binary variable $d_i \in \{1, 2\}$

$$d_i = \begin{cases} 1 & \|e^i_{k+N-1}\|^2_2 \leq \varepsilon_i < M_i \\ 2 & \|e^i_{k+N-1}\|^2_2 > \varepsilon_i \end{cases} \tag{13}$$

with $\mathsf{P}(d_i = 1 | e_k) = \varepsilon$ and $\mathsf{P}(d_i = 2 | e_k) = 1 - \varepsilon$. Thus, the law of *iterated espectation* incurs

$$\mathsf{E}\left[\| \left(1 - \delta^i_{k+N-1}\right) A_i e^i_{k+N-1} \|^2_2 | e_k\right]$$
$$= \mathsf{E}\left[\mathsf{E}\left[\| \left(1 - \delta^i_{k+N-1}\right) A_i e^i_{k+N-1} \|^2_2 | e_k, d\right] | e_k\right]$$
$$= \mathsf{P}(d_i = 1 | e_k) . \mathsf{E}_1 \left[\| \left(1 - \delta^i_{k+N-1}\right) A_i e^i_{k+N-1} \|^2_2 | e_k\right]$$
$$+ \mathsf{P}(d_i = 2 | e_k) . \mathsf{E}_2 \left[\| \left(1 - \delta^i_{k+N-1}\right) A_i e^i_{k+N-1} \|^2_2 | e_k\right]$$
$$\leq \varepsilon_i \varepsilon \|A_i\|^2_2 + (1 - \varepsilon)\, \mathsf{E}_2 \left[\| \left(1 - \delta^i_{k+N-1}\right) A_i e^i_{k+N-1} \|^2_2 | e_k\right]$$

It readily follows from the policy definition in (5) that

$$\mathsf{E}_2 \left[1 - \delta^i_{k+N-1} | e_k\right] = \mathsf{E}_2 \left[\frac{\sum_{j\neq i}^N \|e^j_{k+N-1}\|^p_2}{\sum_{j=1}^N \|e^j_{k+N-1}\|^p_2} | e_k\right] \tag{14}$$

Lemma 2 ensures that, with probability arbitrarily close to one, all the subsystems in the set s_1 have bounded errors, then the following upper bound can easily be concluded for $p \geq 2$

$$\mathsf{E}_2\left[\frac{1}{\sum_{j=1}^{N}\|e^j_{k+N-1}\|_2^p}|e_k\right]\leq \mathsf{E}_2\left[\frac{1}{\sum_{j\in s_2}\|e^j_{k+N-1}\|_2^p}|e_k\right]\leq \frac{1}{\varepsilon_i}$$

The fact that a time-step k' exists s.t. $\|e^i_{k'}\|_2^p < M_i$ and further the discussions in *Lemma 2*, assure the boundedness of $\sum_{j=1}^{N\neq i}\mathsf{E}\,\|e^j_{k+N-1}\|_2^p$. Then, employing *Lemma 1* provides an upper bound for (14) as

$$\begin{aligned}&\mathsf{E}_2\left[1-\delta^i_{k+N-1}|e_k\right]\\ &\leq \frac{\sum_{j\neq i}^{N}\mathsf{E}\,\|e^j_{k+N-1}\|_2^p}{\varepsilon_i} - \sum_{i_{(Odd)}=1}^{\infty}\frac{(i+1)!\left(2\sigma'^2\right)^{\left(\frac{i+1}{2}\right)}}{2^{i+1}\left(\frac{i+1}{2}\right)E\left[b\right]^{i+1}}\\ &= c'_i\end{aligned}$$

where, σ'^2 is $\mathrm{Var}\sum_{j\neq i}^{N}\left[\|e^j_{k+N-1}\|_2^p - \mathsf{E}\,\|e^j_{k+N-1}\|_2^p\right]$, and $b = \sum_{i=1}^{N}\|e^i_{k+N-1}\|_2^p$. Summing up both bounds yields the aggregate upper bound for the drift as follows

$$\begin{aligned}&\mathsf{E}\left[V\left(e_{k+N}\right)|e_k\right]\\ &\leq (1-\varepsilon)\sum_{i\in s_2}\mathsf{E}_2\left[\|\left(1-\delta^i_{k+N-1}\right)A_i e^i_{k+N-1}\|_2^2|e_k\right]+\xi_2^+\\ &=(1-\varepsilon)\sum_{i\in s_2}\mathsf{E}_2\left[\left(1-\delta^i_{k+N-1}\right)|e_k\right]\mathsf{E}_2\left[\|A_i e^i_{k+N-1}\|_2^2|e_k\right]+\xi_2^+\\ &\leq (1-\varepsilon)\sum_{i\in s_2}c'_i\|A_i\|_2^2\,\mathsf{E}_2\left[\|e^i_{k+N-1}\|_2^2|e_k\right]+\xi_2^+\\ &=(1-\varepsilon)\sum_{i\in s_2}c'_i\|A_i\|_2^2\,\mathsf{E}_2\left[\|A_i^{N-1}e^i_k+f\left(A_i,w^i\right)\|_2^2|e_k\right]+\xi_2^+\\ &\leq (1-\varepsilon)\sum_{i\in s_2}c'_i\left(\|A_i\|_2^2\right)^N V\left(e_k\right)+\xi_3^+\end{aligned}$$

where, $\xi_2^+ = \xi^+ + \sum_{i\in s_2}\varepsilon\varepsilon_i\|A_i\|_2^2$, and $\xi_3^+ = \xi_2^+ + \mathsf{E}_2\left[\|f\left(A_i,w^i\right)\|_2^2|e_k\right]$, with $f(A_i,w^i) = A_i^{N-2}w^i_k + \ldots + A_i w^i_{k+N-3} + w^i_{k+N-2}$, which is a zero-mean Gaussian random variable. The Nth order drift operator is subsequently upper bounded as

$$\begin{aligned}&\Delta^N V\left(e_k\right)\\ &\leq\left[(1-\varepsilon)\sum_{i\in s_2}c'_i\left(\|A_i\|_2^2\right)^N-1\right]\sum_{j=1}^{N}\|e^j_k\|_2^2+\xi_3^+\end{aligned}$$

Define $f(e) = \epsilon V\left(e_k\right) - \xi_3^+$, $\epsilon > 0$. Then, $\Delta^N V\left(e_k\right) \leq -f(e)$, if $\left[(1-\varepsilon)\sum_{i\in s_2}c'_i\left(\|A_i\|_2^2\right)^N-1\right] \leq -\epsilon V\left(e_k\right)$, and the stochastic stability of the Markov chain is definitive. Based on the results in *Theorem 1*, appropriate ϵ and M_i can be found s.t.

$$(1-\varepsilon)\sum_{i\in s_2}c'_i\left(\|A_i\|_2^2\right)^N\leq 1$$

The third case declares to have multiple subsystems with unbounded errors $\|e^i_{k'}\|_2^2$ for some time-steps k'. It is sufficient to show that the drift is bounded and not necessarily negative, since according to *Lemma 2*, the probability that one subsystem does not transmit at all the time-steps, and inevitably another subsystem transmits multiple times, can be made arbitrarily small. Therefore, the drift could be positive, but it should be bounded. Considering (14), we may exclude the subsystems who transmit at least once, i.e. $j \in s_1$, because they have bounded errors. Hence, (14) can be rewritten as

$$\mathsf{E}_2\left[1-\delta^i_{k+N-1}|e_k\right] = \mathsf{E}_2\left[\frac{\sum_{j\in s_2}^{j\neq i}\|e^j_{k+N-1}\|_2^p}{\sum_{j\in s_2}\|e^j_{k+N-1}\|_2^p}|e_k\right]+\epsilon<1$$

where, $\epsilon\in(0,1)$ replaces the effect of excluded subsystems $j\in s_1$. Furthermore, e^i_{k+N-1} is a linear combination of the independent zero-mean random variables $e^i_0, w^i_0, \ldots, w^i_{k-1}$, then it has a standard normal distribution with the bounded variance $A_i^{k+N-1}\sigma^2_{e^i_0} + \sum_{l=1}^{k+N-1}A_i^{k-l}$. Thus, $\|e^i_{k+N-1}\|_2$ has a central chi distribution. Since e^i_{k+N-1}, $i\in\{1,\ldots,N\}$, are independent of each other, then the expectation $\mathsf{E}\left[\sum_{i=1}^{N}\|e^i_{k+N-1}\|_2^p|e_k\right] = \sum_{i=1}^{N}\mu_i$, and $\mathsf{E}\left[\left[\sum_{i=1}^{N}\left[\|e^i_{k+N-1}\|_2^p-\mu_i\right]\right]^{i+1}|e_k\right] = \left(2\sum_{i=1}^{N}\mu_i\right)^{\frac{i+1}{2}}i!!$, where μ_i is the pth moment of the chi distributed random variable $\|e^i_{k+N-1}\|_2$, and !! represents the odd factorial.

According to *Lemma 1*, if $a\simeq b$

$$\mathsf{E}\left[a\right].\,\mathsf{E}\left[\frac{1}{a}\right]-\sum_{i_{Odd}=1}^{\infty}\frac{\mathsf{E}\left[\left(a-\mathsf{E}\left[a\right]\right)^{i+1}\right]}{\left(\mathsf{E}\left[a\right]\right)^{i+1}}<1$$

Now, if we define $a = \sum_{j\in s_2}^{j\neq i}\|e^j_{k+N-1}\|_2^p$ and $b = \sum_{j\in s_2}\|e^j_{k+N-1}\|_2^p$, due to the unboundedness of the error values, the bound can be rewritten

$$\begin{aligned}\mathsf{E}_2\left[\frac{a}{b}\right]&\leq\frac{\mathsf{E}\left[\sum_{j\in s_2}\|e^j_{k+N-1}\|_2^p\right]}{\varepsilon_i}\\ &-\sum_{i_{Odd}=1}^{\infty}\frac{2^{\frac{i+1}{2}}\left[\sum_{i\in s_2}\mu_i\right]^{\frac{i+1}{2}}i!!}{\left[\sum_{i\in s_2}\mu_i\right]^{i+1}}<1\end{aligned}$$

Therefore, the boundedness of $\mathsf{E}\left[\sum_{j\in s_2}\|e^j_{k+N-1}\|_2^p|e_k\right]$ is immediately followed by the boundedness of the infinite summation. Finally, we can find the finite upper bound for the drift as

$$\begin{aligned}&\mathsf{P}\left(d_i=2|e_k\right).\,\mathsf{E}_2\left[\|\left(1-\delta^i_{k+N-1}\right)A_i e^i_{k+N-1}\|_2^2|e_k\right]\\ &\leq(1-\varepsilon)\sum_{i\in s_2}\|A_i\|_2^2\,\mathsf{E}_2\left[\left(1-\delta^i_{k+N-1}\right)|e_k\right]\mathsf{E}_2\left[\|e^i_{k+N-1}\|_2^2|e_k\right]\\ &\leq(1-\varepsilon)\|A_i\|_2^{2m}\,\mathsf{E}_2\left[\sum_{i\in s_2}\|e^i_{k+N-1}\|_2^2|e_k\right]\end{aligned}$$

where, m is the number of subsystems in the set s_2. Thus, the drift is bounded, and the proof is complete.

4. NUMERICAL RESULTS

We simulate a networked system comprised of two classes of subsystems - a stable and an unstable process - with system parameters $A_1 = 1.25$, $B_1 = 1$ and $A_2 = 0.75$, $B_2 = 1$, respectively. Each class of either stable or unstable systems includes finite number of homogeneous subsystems. In both classes, the state initiates with $x^1_0 = x^2_0 = 0$ and the random disturbance is given by $w^i_k\sim\mathcal{N}(0,1)$. We assume a stabilizing deadbeat control law with $L_i = A_i$ for $i\in\{1,2\}$ and a model-based observer is given by (3).

Figure 3 compares the performance of the proposed PEB protocol for different p powers with other scheduling protocols for $N \in \{2, 4, 6, 8, 10\}$ in terms of the mean variance per subsystem of the estimation error e_k^i induced by the network. The means are calculated by their empirical means through Monte Carlo simulations over a horizon of 100 000. The lower bound is determined by relaxing the initial problem to have no resource constraint, but instead restrains the total average transmission rate per time step to be 1. This can be calculated through a bilevel approach, discussed in Molin and Hirche (2013), and results in an event-triggered scheduling strategy. The round robin (RR) protocol is a time-triggered access scheme, where subsystems update their controllers periodically with a sampling period of N. The idealized carrier sense multi access protocol considered operates in the same fashion as the PEB protocol without prioritizing subsystems, i.e. the probability of updating the controller is $\frac{1}{N}$ at each time. With an increasing number of subsystems sharing the resource, the performance gap between the PEB scheduler and the other protocols becomes more evident. At the same time, the PEB scheduler deviates moderately from the lower bound, which grows slowly with increasing N. This suggests that the PEB protocol is more profitable than the round robin protocol when the resource is scarce.

By increasing the power of p, the performance of the scheduler improves, as the subsystems with higher errors get more chance to utilize the channel. In case $p \to \infty$, the transmission probability for the subsystem with the highest error tends to one. As the simulations show, the PEB scheduler is highly robust with respect to the increasing number of subsystems, compared to the other policies. Only for $N = 2$, the CSMA protocol results in an acceptable performance. For $N \geq 6$, the variance of e_k takes values of magnitude 10^{15} which suggest an unbounded variance and therefore unstable evolution. This is in accordance with theorem 2 in Molin and Hirche (2013), where the stability condition is violated for $N \geq 6$.

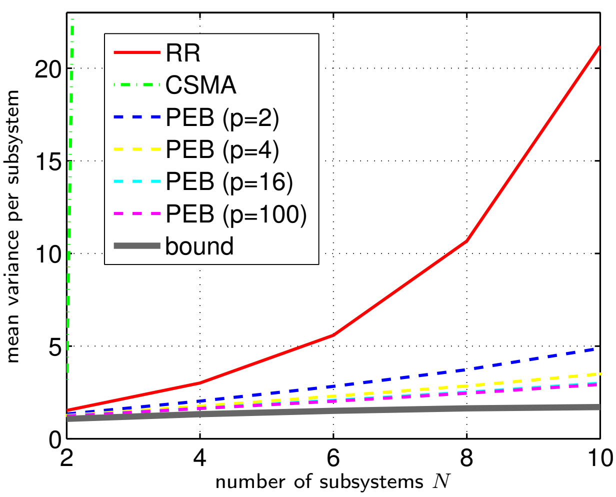

Fig. 3. Comparison of the mean steady-state variance of e_k^i for various protocols and number of subsystems.

5. CONCLUSION

We propose a new stochastic scheduling scheme, the prioritized error-based (PEB) protocol, which dynamically assigns priorities for NCS comprised of finite number of entities coupled through a scarce communication resource. The likelihood for utilizing the resource grows proportional with the p-powered norm of the networked-induced error of the subsystem. Provided with stabilizing feedback controllers, we show the stability of the overall networked system with the PEB scheduling scheme using drift criteria. In presence of disturbances, the stochastic stability is shown in terms of Markov chain ergodicity and second-order moment boundedness. Numerical results demonstrate the stability, which grows with increasing exponent p, alongside a substantial performance improvement in comparison with the other randomized protocols. Performance analysis, and considering the physically coupled systems to be scheduled with the proposed policy, are of interest as future challenges.

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