Bounds for randomly shared risk of heavy-tailed loss factors

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Abstract

For a risk vector V, whose components are shared among agents by some random mechanism, we obtain asymptotic lower and upper bounds for the individual agents' exposure risk and the aggregated risk in the market. Risk is measured by Value-at-Risk or Conditional Tail Expectation. We assume Pareto tails for the components of V and arbitrary dependence structure in a multivariate regular variation setting. Upper and lower bounds are given by asymptotically independent and fully dependent components of V with respect to the tail index α being smaller or larger than 1. Counterexamples, where for non-linear aggregation functions no bounds are available, complete the picture.

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1 Introduction

Let V_j for $j = 1, \dots, d$ be risk variables having Pareto-tails, so that, for possibly different $K_j > 0$ and tail index $\alpha > 0$,

$$P(V_j > t) \sim K_j t^{-\alpha}, \quad t \to \infty. \tag{1.1}$$

(For two functions f and g we write $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t\to\infty} f(t)/g(t) = 1$.) We summarize all risk variables in a vector $V = (V_1, \dots, V_d)^\top$. The tail index α is assumed to be the same for all $j = 1, \dots, d$ since, when aggregating risk factors with different tail indexes, always the smallest α wins as a famous result of [5] states. That said, assuming the same α practically means to choose the subset with the smallest α out of a set of risk factors resulting in a dimension reduction.

The d risks in V are shared among q agents by some random mechanism. Let F_i denote the *exposure of agent* i and $F = (F_1, \dots, F_q)^\top$ the exposure vector. The risk sharing is governed by a random $q \times d$ matrix $A = (A_{ij})_{i,j=1}^{q,d}$ (independent of V) in such a way that $F_i = \sum_{j=1}^d A_{ij} V_j$ for $i = 1, \dots, q$ or, eqivalently, in matrix notation

$$F = AV. \tag{1.2}$$

Whether A is deterministic or stochastic may depend on the quality of available information. An internal analyst or regulator with sufficient knowledge may consider A as deterministic, whereas

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an external analyst (working for an institutional investor for instance) may consider it random due to lack of insight. This note is motivated by [14] and [13], where the risk variables V_j model large insurance claims and agents represent reinsurance companies. The claims can for instance be shared randomly with a mechanism given by a bipartite graph structure, resulting in

$$A_{ij} = \frac{\mathbb{1}(i \sim j)}{\deg(j)}, \tag{1.3}$$

where $\mathbb{1}(i \sim j)$ indicates whether agent i takes a (proportional) share of risk j or not, and $\deg(j)$ denotes the total number of agents who have chosen to insure risk j. Further examples include operational risk, modelling event types (risk variables) and business lines (agents), where Pareto tails are natural (cf. [4]), and also overlapping portfolios (common asset holding) as described in [7].

In all these applications it is of interest to quantify not only the risk of single agents, but also the market risk—saying that we mean the aggregated risk in the market—which is of high relevance to the regulator. Following ideas in [8] we assess the market risk by a risk measure on the r-norm ($r \geq 1$) or $r-$quasinorm ($0 < r < 1$) $\|F\| := \|F\|_r = \left(\sum_{i=1}^q F_i^r\right)^{1/r}$ of the exposure vector F. These aggregation functions satisfy most of the required axioms in [8] and are continuous as well as convex or concave, respectively. Our market risk measures do not necessarily satisfy the normalization condition required there: Assuming a total unit loss split to equal parts among the agents, we have $\|(1/q,\ldots,1/q)\|_r = q^{1/r-1} < 1$ if $r > 1$ and $\|(1/q,\ldots,1/q)\|_r = q^{1/r-1} > 1$ for $0 < r < 1$, hence, the normalization condition of [8] is satisfied only if $r = 1$. In the first case we see that the $r-$norm underestimates the additive risk by convexity. In particular, if the number of agents increases, the loss measured by the $r-$norm decreases. As a consequence, norms with $r > 1$ are not suitable for systemic risk assessment as they imply hereby the possibility of regulatory arbitrage. We argue, however, that this underestimation may be realistic in some applications as a larger market may be less risky due to a balance of risk as is well-known for insurance portfolios. Moreover, norms of that type can be useful in portfolio analysis, see [6]. In the second case $0 < r < 1$ the $r-$quasinorm will overestimate the additive risk. This situation in turn may be realistic whenever amplification mechanisms come into play as it happens with systemic risk. In addition, [11] employ $r-$quasinorms for portfolio construction. Our framework allows for great variability concerning the choice of the aggregation function: convex, linear or concave. The decision which aggregation function to employ is in the end application-driven and mostly a decision based on economic reasoning.

We investigate risk based on the Value-at-Risk (VaR) and Conditional Tail Expectation (CoTE), which we assess by asymptotic approximations.

Let $V_{\text{ind}}, V, V_{\text{dep}}$ be risk vectors as above with different dependence structures among the risk variables. Here V_{ind} corresponds to asymptotically independent variables and V_{dep} to asymptotically fully dependent variables in the framework of multivariate regular variation as in [13].

As in the copula world (see [3, 10]) it is possible to assess the two extreme dependence structures; i.e., $V_{\text{ind}}, V_{\text{dep}}$, and it is of high relevance to understand, if or under which conditions these extreme dependences lead to upper and lower bounds of risk for arbitrary dependence structures. Remarkably, [10] show that the comonotonic copula does not lead to an upper bound, and a procedure is provided there to find the best possible upper VaR bound. In [3], information on variance is added. This reduces the set of feasible copulas. Then the upper bound can be lower than the comonotic VaR. The related problem of sub- and super-addtivity has been investigated in [9]. We extend the setting and scope in [9] significantly: first, by allowing for diversity in the tails as in (1.1) and, second, by incorporating a stochastic market structure as in (1.2) allowing for risk assesment in a much wider way. Moreover, the results in [9] are also formulated for general aggregation functions but the effect of non-linearity—hence the possible breakdown of general bounds—is not considered there. In that sense, our results add new important aspects to the existing literature.

This note is organised as follows. In Section 2 we present V as a regularly varying vector with different dependence structures. Here we also define the risk measures VaR and CoTE for arbitrary random variables, and summarize their asymptotic behaviour in our framework. In Section 3 we derive bounds for single agent and market risk based on asymptotically independent and fully dependent random variables. We also give counter examples to present the limitations of the bounds.

2 Preliminaries

2.1 Multivariate regular variation

We recall from [17], Ch. 6 that the positive random vector $V \in \mathbb{R}_+^d$ is *multivariate regularly varying* if there is a Radon measure $\nu \not\equiv 0$ on the Borel σ-algebra $\mathcal{B} = \mathcal{B}(\mathbb{R}_+^d \setminus \{0\})$, where 0 denotes the zero vector in $\mathbb{R}^d$, such that

$$n\mathbb{P}\left[n^{-1/\alpha}V \in \cdot\right] \xrightarrow{v} \nu(\cdot), \quad n \to \infty. \tag{2.1}$$

The symbol $\xrightarrow{v}$ stands for vague convergence. Moreover, the measure ν is homogeneous of some order $-\alpha$ with $\alpha > 0$ and is called the *exponent measure of* V.

We fix a norm $\|\cdot\|$ on $\mathbb{R}^d$ in such a way that for all canonical unit vectors $\|e_j\| = 1,\ j = 1, \ldots, d$. This actually entails a slight abuse of notation as we also write $\|\cdot\|$ for the aggregation function of the vector F of agent exposures on $\mathbb{R}^q$. Denoting by $\mathbb{S}_+^{d-1} = \{x \in \mathbb{R}_+^d : \ \|x\| = 1\}$ the positive sphere in $\mathbb{R}^d$, the existence of the exponent measure ν is equivalent to the existence of a Radon measure $\rho \not\equiv 0$ on the Borel σ-algebra $\mathcal{B}(\mathbb{S}_+^{d-1})$ in such a way that for all $u > 0$

$$\frac{\mathbb{P}\left[\|V\| > ut, V\|V\|^{-1} \in \cdot\right]}{\mathbb{P}\left[\|V\| > t\right]} \xrightarrow{v} u^{-\alpha}\rho(\cdot), \quad t \to \infty, \tag{2.2}$$

holds. The measure ρ is called the *spectral measure of* V. The precise relation between ν and ρ can be found in [17], Ch. 6.

Finally, we note that convergence in (2.1) also implies

$$\frac{\mathbb{P}\left[t^{-1}V \in \cdot\right]}{\mathbb{P}\left[\|V\| > t\right]} \xrightarrow{v} \frac{\nu(\cdot)}{\nu(\{x : \|x\| > 1\})}, \quad t \to \infty. \tag{2.3}$$

The *tail index* $\alpha > 0$ is also called the index of regular variation of V, and we write $V \in \mathcal{R}(-\alpha)$.

We shall often work with the so-called *canonical exponent measure* ν^* of V, which is defined as the image measure $\nu^* = \nu \circ T$ under the transformation mapping $T : \mathbb{R}_+^d \to \mathbb{R}_+^d$, given by

$$T(x) = (\nu(\{x_1 > 1\})^{1/\alpha}x_1^{1/\alpha}, \ldots, \nu(\{x_d > 1\})^{1/\alpha}x_d^{1/\alpha})^\top.$$

Then ν^* has standardized margins and a tail index 1, corresponding to $P(V_j > x) \sim x^{-1}$ as $x \to \infty$.

The corresponding spectral measure ρ^* is called the *canonical spectral measure* and is characterized by

$$\int_{\mathbb{S}_+^{d-1}} s_j \rho^*(ds) = 1, \quad j = 1, \dots, d, \tag{2.4}$$

see [2], p. 259.

For the matrix A and a given norm $\|\cdot\|$, which gives rise to an operator norm

$$\|A\|_{\mathrm{op}} = \sup_{\|x\|=1} \|Ax\|,$$

we require througout the following:

- A satisfies the moment condition $\mathbb{E}\|A\|_{\mathrm{op}}^{\alpha+\delta} < \infty$ for some $\delta > 0$ and α as in (1.1);
- the vector V is independent of the random matrix A, while $V_1, \dots, V_d$ may not be independent of each other.

If both conditions hold, then the vector $F = AV$ is again regularly varying with exponent measure $\mathbb{E}\nu \circ A^{-1}$ (cf. [1], Proposition A.1).

2.2 Risk measures

We also recall the following risk measures.

Definition 2.1. The *Value-at-Risk (*VaR*)* of a random variable X at confidence level $1 - \gamma$ is defined as

$$\mathrm{VaR}_{1-\gamma}(X) := \inf\{t \geq 0 : \mathbb{P}[X > t] \leq \gamma\}, \quad \gamma \in (0,1),$$

and the *Conditional Tail Expectation (*CoTE*)* at confidence level $1-\gamma$, based on the corresponding VaR, as

$$\mathrm{CoTE}_{1-\gamma}(X) := \mathbb{E}[X \mid X > \mathrm{VaR}_{1-\gamma}(X)], \quad \gamma \in (0,1).$$

□

Throughout the following constants will be relevant

$$C_{\mathrm{ind}}^i = \sum_{j=1}^d K_j \mathbb{E} A_{ij}^\alpha, \quad i = 1, \dots, q, \qquad \text{and} \qquad C_{\mathrm{ind}}^S = \sum_{j=1}^d K_j \mathbb{E}\|Ae_j\|^\alpha, \tag{2.5}$$

$$C_{\mathrm{dep}}^i = \mathbb{E}(AK^{1/\alpha}\mathbb{1})_i^\alpha, \quad i = 1, \dots, q, \qquad \text{and} \qquad C_{\mathrm{dep}}^S = \mathbb{E}\|AK^{1/\alpha}\mathbb{1}\|^\alpha, \tag{2.6}$$

where we summarize the constants K_j for $j = 1, \dots, d$ from (1.1) in a diagonal matrix

$$K^{1/\alpha} := \mathrm{diag}(K_1^{1/\alpha}, \dots, K_d^{1/\alpha}). \tag{2.7}$$

Lemma 2.2 ([13], Corollaries 3.7 and 3.8)**.** *Let $F = AV = (F_1, \dots, F_q)^\top$.*
(a) Individual risk measures:
For $\alpha > 0$ the individual Value–at–Risk of agent $i \in \{1, \dots, q\}$ satisfies

$$\mathrm{VaR}_{1-\gamma}(F_i) \sim C^{1/\alpha}\gamma^{-1/\alpha}, \quad \gamma \to 0. \tag{2.8}$$

For $\alpha > 1$ the individual Conditional Tail Expectation of agent $i \in \{1, \dots, n\}$ satisfies

$$\mathrm{CoTE}_{1-\gamma}(F_i) \sim \frac{\alpha}{\alpha-1}\mathrm{VaR}_{1-\gamma}(F_i) \sim \frac{\alpha}{\alpha-1}C^{1/\alpha}\gamma^{-1/\alpha}, \quad \gamma \to 0.$$

The individual constants are either $C = C^i_{\mathrm{ind}}$ or $C = C^i_{\mathrm{dep}}$ for $V_1, \dots, V_d$ asymptotically independent or asymptotically fully dependent, respectively.

(b) Market risk measures:

The market Value–at–Risk of the aggregated vector $\|F\|$ satisfies

$$\mathrm{VaR}_{1-\gamma}(\|F\|) \sim C^{1/\alpha}\gamma^{-1/\alpha}, \quad \gamma \to 0. \tag{2.9}$$

If $\alpha > 1$ the market Conditional Tail Expectation of the aggregated vector $\|F\|$ satisfies

$$\mathrm{CoTE}_{1-\gamma}(\|F\|) \sim \frac{\alpha}{\alpha-1}\mathrm{VaR}_{1-\gamma}(\|F\|) \sim \frac{\alpha}{\alpha-1}C^{1/\alpha}\gamma^{-1/\alpha}, \quad \gamma \to 0.$$

The market constants referring to the system setting are either $C = C^S_{\mathrm{ind}}$ or $C = C^S_{\mathrm{dep}}$ for $V_1, \dots, V_d$ asymptotically independent or asymptotically fully dependent, respectively.

3 Bounds for general dependence structure

Recall from (3.12) and (3.14) of [13] that the constants (2.5) can be expressed in terms of the exponent measure via

$$C^i_{\mathrm{ind}} = \mathbb{E}\nu_{\mathrm{ind}} \circ A^{-1}(\{x : x_i > 1\}),\ i = 1, \dots, q, \quad \text{and} \quad C^S_{\mathrm{ind}} = \mathbb{E}\nu_{\mathrm{ind}} \circ A^{-1}(\{x : \|x\| > 1\}) \tag{3.1}$$

$$C^i_{\mathrm{dep}} = \mathbb{E}\nu_{\mathrm{dep}} \circ A^{-1}(\{x : x_i > 1\}),\ i = 1, \dots, q, \quad \text{and} \quad C^S_{\mathrm{dep}} = \mathbb{E}\nu_{\mathrm{dep}} \circ A^{-1}(\{x : \|x\| > 1|\}) \tag{3.2}$$

with (cf. Lemma 2.2 of [13])

$$\nu_{\mathrm{ind}}([0,x]^c) = \sum_{j=1}^d K_j x_j^{-\alpha} \quad \text{and} \quad \nu_{\mathrm{dep}}([0,x]^c) = \max_{j=1,\dots,d}\{K_j x_j^{-\alpha}\}. \tag{3.3}$$

The analogues of the constants C^i_{ind}, C^i_{dep} as well as C^S_{ind} and C^S_{dep} in the case of an arbitrary extremal dependence structure of the vector V, represented by some exponent measure ν with $\nu_{\mathrm{ind}} \neq \nu \neq \nu_{\mathrm{dep}}$, are then

$$C^i_\nu = \mathbb{E}\nu \circ A^{-1}(\{x : x_i > t\}) \ \text{ and } \ C^S_\nu = \mathbb{E}\nu \circ A^{-1}(\{x : \|x\| > t\}). \tag{3.4}$$

In the light of Lemma 2.2 it suffices to determine bounds for the constants C^i_ν and C^S_ν in order to obtain asymptotic bounds for VaR or CoTE in the respective cases.

With $K^{1/\alpha}$ from (2.7), for the exponent measure ν of the vector V with any dependence structure, we obtain

$$C^S_\nu = \mathbb{E}\nu \circ K^{1/\alpha} \circ (AK^{1/\alpha})^{-1}(\{\|x\| > 1\}) \quad \text{and} \quad C^i_\nu = \mathbb{E}\nu \circ K^{1/\alpha} \circ (AK^{1/\alpha})^{-1}(\{x_i > 1\}).$$

Note that the measure $\nu \circ K^{1/\alpha}$ has balanced tails; i.e., $\nu \circ K^{1/\alpha}(\{x_j > 1\}) = 1, j = 1, \dots, d$. Since all marginal random variables are as in (1.1), regardless of the dependence structure of the vector V, for the proofs of all theorems below we can and do assume that margins are standardized; e.g. $K_j = 1$ for $j = 1, \dots, d$. Moreover, for establishing inequalities between $C^i_{\mathrm{ind}}, C^i_{\mathrm{dep}}$ and C^i_ν or $C^S_{\mathrm{ind}}, C^S_{\mathrm{dep}}$ and C^S_ν, respectively, it is sufficient to prove the corresponding inequalities for all realizations of the random matrix A. We obtain the following bounds for the constants defining the individual risk measures.

Theorem 3.1. *Let the three d-dimensional vectors V_{ind}, V and V_{dep} be given with equal margins $V_1, \ldots, V_d$ with $\mathbb{P}[V_j > t] \sim K_j t^{-\alpha}$, but different exponent measures $\nu_{\text{ind}}, \nu, \nu_{\text{dep}}$. Then for the constants C^i referring to agent i the following inequalities hold:*

$$C^i_{\text{ind}} \le C^i_\nu \le C^i_{\text{dep}} \qquad \text{for } \alpha \ge 1, \tag{3.5}$$
$$C^i_{\text{dep}} \le C^i_\nu \le C^i_{\text{ind}} \qquad \text{for } \alpha < 1. \tag{3.6}$$

Proof. Let $a_i := A_{i\cdot}$ be the i-th row of the matrix A and $V_{\text{ind}}, V, V_{\text{dep}}$ be as above the risk vectors with different dependence structures. Corollary 3.8 in [16] provides for $\alpha \ge 1$ the inequalities

$$\limsup_{t\to\infty} \frac{\mathbb{P}[a_i V_{\text{ind}} > t]}{\mathbb{P}[a_i V]} \le 1 \;\text{ and }\; \limsup_{t\to\infty} \frac{\mathbb{P}[a_i V > t]}{\mathbb{P}\left[a_i V_{\text{dep}} > t\right]} \le 1 \tag{3.7}$$

and for $0 < \alpha < 1$ the inequalities

$$\limsup_{t\to\infty} \frac{\mathbb{P}\left[a_i V_{\text{dep}} > t\right]}{\mathbb{P}[a_i V > t]} \le 1 \;\text{ and }\; \limsup_{t\to\infty} \frac{\mathbb{P}[a_i V > t]}{\mathbb{P}[a_i V_{\text{ind}} > t]} \le 1.$$

Regarding the left inequality in (3.7), we have

$$\begin{aligned}\limsup_{t\to\infty} \frac{\mathbb{P}[a_i V_{\text{ind}} > t]}{\mathbb{P}[a_i V > t]} &= \limsup_{t\to\infty} \frac{\mathbb{P}[a_i V_{\text{ind}} > t]}{\mathbb{P}[\|V_{\text{ind}}\| > t]} \frac{\mathbb{P}[\|V_{\text{ind}}\| > t]}{\mathbb{P}\left[V_{\text{ind},i} > t\right]} \Big/ \frac{\mathbb{P}[a_i V > t]}{\mathbb{P}[\|V\| > t]} \frac{\mathbb{P}[\|V\| > t]}{\mathbb{P}[V_i] > t} \\ &= \frac{\nu_{\text{ind}} \circ A^{-1}(\{x_i > t\})\nu(\{x_i > 1\})}{\nu \circ A^{-1}(\{x_i > t\})\nu_{\text{ind}}(\{x_i > 1\})} = \frac{\nu_{\text{ind}} \circ A^{-1}(\{x_i > 1\})}{\nu \circ A^{-1}(\{x_i > 1\})} \le 1, \end{aligned} \tag{3.8}$$

since w.l.o.g all marginals are the same. The other inequalities in (3.5) as well as in (3.6) are treated analogously. □

For bounds on the market risk measures we invoke ideas from [16]. Below we sometimes write $C^i_\nu(A)$ and $C^S_\nu(A)$ instead of C^i_ν and C^S_ν, if we want to emphasize that the constants depend on a particular matrix A.

Theorem 3.2. *Let the three d-dimensional vectors V_{ind}, V and V_{dep} be given with equal margins $V_1, \ldots, V_d$ with $\mathbb{P}[V_j > t] \sim K_j t^{-\alpha}$, but different exponent measures $\nu_{\text{ind}}, \nu, \nu_{\text{dep}}$. Denote the aggregated vector $\|F\|$ for some r-norm for $r \ge 1$ or r-quasinorm for $0 < r < 1$, representing the risk in the market.*
(a) If $r \ge 1$, for the constants C^S referring to the system setting risk the following inequalities hold:

$$C^S_\nu \;\ge\; C^S_{\text{ind}} \quad \text{for } \alpha \ge r, \tag{3.9}$$
$$C^S_\nu \;\le\; C^S_{\text{ind}} \quad \text{for } 0 < \alpha \le 1. \tag{3.10}$$

(b) If $0 < r < 1$, for the constants C^S referring to the system setting the following inequalities hold:

$$C^S_\nu \;\ge\; C^S_{\text{ind}} \quad \text{for } \alpha \ge 1, \tag{3.11}$$
$$C^S_\nu \;\le\; C^S_{\text{ind}} \quad \text{for } 0 < \alpha \le r. \tag{3.12}$$

(c) However, there are matrices A_1, A_2 and an exponent measure ν_0 such that

$$C^S_{\text{ind}}(A_1) \;>\; C^S_{\nu_0}(A_1) \quad \text{for } 1 < \alpha < r, \tag{3.13}$$

$$C^S_{\nu_0}(A_2) \quad > \quad C^S_{\text{ind}}(A_2) \quad \text{for } 1 < \alpha < r, \tag{3.14}$$
$$C^S_{\text{ind}}(A_1) \quad < \quad C^S_{\nu_0}(A_1) \quad \text{for } r < \alpha < 1, \tag{3.15}$$
$$C^S_{\nu_0}(A_2) \quad < \quad C^S_{\text{ind}}(A_2) \quad \text{for } r < \alpha < 1, \tag{3.16}$$

Proof. (a) In analogy to [16] we define for $s^{1/\alpha} := (s_1^{1/\alpha}, \dots, s_d^{1/\alpha})$

$$g_{A,\alpha}(s) := \|As^{1/\alpha}\|^\alpha \quad \text{and} \quad \rho^* g_{A,\alpha} := \int_{\mathbb{S}_+^{d-1}} g_{A,\alpha}(s)\rho^*(ds)$$

for some canonical spectral measure ρ^*. Similar to (3.8), we note that

$$\frac{\nu_{\text{ind}} \circ A^{-1}(\{\|x\| > 1\})}{\nu \circ A^{-1}(\{\|x\| > 1\})} = \lim_{t\to\infty} \frac{\mathbb{P}[\|AV_{\text{ind}}\| > t]}{\mathbb{P}[\|AV\| > t]}. \tag{3.17}$$

Furthermore, we get from Propositions 3.2 and 3.3 in [16] that

$$\lim_{t\to\infty} \frac{\mathbb{P}[\|AV_{\text{ind}}\| > t]}{\mathbb{P}[\|AV\| > t]} = \frac{\rho^*_{\text{ind}} g_{A,\alpha}}{\rho^* g_{A,\alpha}} \tag{3.18}$$

holds. Hence, in order to prove (3.9) and (3.10) it is sufficient to show that $\rho^*_{\text{ind}}(g_{A,\alpha}) \le \rho^*(g_{A,\alpha})$ and $\rho^*_{\text{ind}}(g_{A,\alpha}) \ge \rho^*(g_{A,\alpha})$, respectively.

We first show (3.9). Note that for nonnegative real numbers $a_1, \dots, a_n$ and $\beta \ge 1$ the inequality

$$a_1^\beta + \dots + a_n^\beta \le (a_1 + \dots + a_n)^\beta \tag{3.19}$$

is valid. Since $\rho^*_{\text{ind}} g_{A,\alpha} = \sum_{j=1}^d \|Ae_j\|^\alpha$, and using (2.4), we write as in the proof of Theorem 3.7 of [16]

$$\rho^*_{\text{ind}} g_{A,\alpha} = \int_{\mathbb{S}_+^{d-1}} \sum_{j=1}^d \|Ae_j\|^\alpha s_j \rho^*(ds) = \int_{\mathbb{S}_+^{d-1}} \frac{\sum_{j=1}^d \|As_j^{1/\alpha} e_j\|^\alpha}{\|\sum_{j=1}^d As_j^{1/\alpha} e_j\|^\alpha} \|As^{1/\alpha}\|^\alpha \rho^*(ds).$$

In order to establish $\rho^*_{\text{ind}} g_{A,\alpha} \le \rho^* g_{A,\alpha}$ it is sufficient to bound the fraction under the right hand integral by one. For this, we recall that all the entries in A are nonnegative and that $\frac{\alpha}{r} \ge 1$. We compute

$$\sum_{j=1}^d \|As_j^{1/\alpha} e_j\|^\alpha = \sum_{j=1}^d \Big(\sum_{i=1}^q (a_{ij} s_j^{1/\alpha})^r \Big)^{\frac{\alpha}{r}} \le \Big(\sum_{j=1}^d \sum_{i=1}^q (a_{ij} s_j^{1/\alpha})^r \Big)^{\frac{\alpha}{r}} \tag{3.20}$$
$$\le \Big(\sum_{i=1}^q \Big(\sum_{j=1}^d a_{ij} s_j^{1/\alpha} \Big)^r \Big)^{\frac{\alpha}{r}} \tag{3.21}$$
$$= \|\sum_{j=1}^d As_j^{1/\alpha} e_j\|^\alpha$$

where we have applied inequality (3.19) twice.

For the bound (3.10) we use the c_r–inequality, see e.g. [15], p. 157, leading to

$$\|\sum_{i=1}^n x_i\|^\alpha \le \Big(\sum_{i=1}^n \|x_i\| \Big)^\alpha \le \sum_{i=1}^n \|x_i\|^\alpha \tag{3.22}$$

for $x_1, \dots, x_n \in \mathbb{R}^d$. In particular,

$$\rho^*_{\text{ind}} g_{A,\alpha} \quad = \quad \sum_{j=1}^d \|Ae_j\|^\alpha = \int_{\mathbb{S}_+^{d-1}} \sum_{j=1}^d \|Ae_j\|^\alpha s_j d\rho^*(s)$$

$$= \int_{\mathbb{S}_+^{d-1}} g_{A,\alpha}(s) \frac{\sum_{j=1}^d \|A s_j^{1/\alpha} e_j\|^\alpha}{\|A s^{1/\alpha}\|^\alpha} d\rho^*(s) \;\geq\; \rho^* g_{A,\alpha}$$

leading to

$$C_\nu^S = \nu \circ A^{-1}(\{\|x\| > 1\}) \leq \nu_{\text{ind}} \circ A^{-1}(\{\|x\| > 1\}) = C_{\text{ind}}^S$$

as expressed in (3.10).

(b) We can proceed analogously to the proof of part (a), simply reversing inequalities. In order to establish (3.12), we note that inequality (3.19) holds in a reverse way for $\beta \leq 1$. Consequently, also both inequalities (3.20) and (3.21) hold analogously in the opposite way. For (3.11), note that in the case of $\alpha \geq 1$ and $0 < r < 1$ both inequalities in (3.22) obviously hold in a reverse way.

(c) Concerning examples for (3.13) and (3.14), we choose ν_0 to be the image measure $\nu_0 := \nu_{\text{ind}} \circ B^{-1}$ with standard exponent measure ν_{ind} on $\mathbb{R}_+^3$ given as usual by $\nu_{\text{ind}}([0,x]^c) = \sum_{j=1}^3 x_j^{-\alpha}$ and a matrix

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Furthermore, we define the function $T : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ as

$$T(x) = ((\nu_0(\{y \in \mathbb{R}_+^2 : |y_1| > 1\})x_1)^{1/\alpha}, (\nu_0(\{y \in \mathbb{R}_+^2 : |y_2| > 1\})x_2)^{1/\alpha})^\top.$$

The measure $\nu_0^* = \nu_0 \circ T$ is then canonical; i.e., it is homogeneous of order -1 and $\nu_0^*(\{y \in \mathbb{R}_+^2 : |y_i| > 1\}) = 1$ for $i = 1, 2$. To get the canonical spectral measure, we conduct the transformation to polar coordinates by setting $\tau(x) = (\|x\|, \frac{x}{\|x\|})$. Denoting by ρ_0^* the spectral measure and defining the measure π by $d\pi(x) = x^{-2}dx$, the relation $\nu_0^* = \pi \otimes \rho_1^*$ holds. We can now calculate ρ_0^* as follows. We first note that by construction ν_0 and hence ν_0^* only have positive mass on the axes as well as on the diagonal $\{t\mathbb{1} : \; t > 0\}$. Therefore, the canonical spectral measure, living on the sphere $\mathbb{S}_+^d$, only attains mass at the points $(1,0)^\top, (0,1)^\top, \mathbb{1}/\|\mathbb{1}\|$. We first observe that $\nu_0 \circ B^{-1}(\{x : |x_i| > 1\}) = 2$ for $i = 1, 2$. This yields

$$\begin{aligned}
\rho_0^*(\{(1,0)^\top\}) &= \nu_0 \circ T(\{te_1 |\; t > 1\}) \\
&= \nu_{\text{ind}} \circ B^{-1}(\{2^{1/\alpha} te_j |\; t > 1\}) \\
&= \nu_{\text{ind}}(x \in \mathbb{R}_+^3 |\; Bx \in \{2^{1/\alpha} te_1 \in \mathbb{R}_+^2 |\; t > 1\}) \\
&= \nu_{\text{ind}}(se_1 \in \mathbb{R}_+^3 |\; sBe_2 \in \{2^{1/\alpha} te_1 \in \mathbb{R}_+^2 |\; t > 1\}) \\
&= \nu_{\text{ind}}(se_1 \in \mathbb{R}_+^3 |\; s \in [2^{1/\alpha}, \infty)) \;=\; \frac{1}{2} \;=\; \rho_0^*(\{(0,1)^\top\})
\end{aligned}$$

by symmetry. For the third atom we calculate

$$\begin{aligned}
\rho_0^*(\{\mathbb{1}/\|\mathbb{1}\|\}) &= \nu_0 \circ T(\{t\mathbb{1}/\|1\| |\; t > 1\}) \\
&= \nu_{\text{ind}} \circ B^{-1}(\{2^{1/\alpha} t\mathbb{1}/\|1\|^{1/\alpha} |\; t > 1\}) \\
&= \nu_{\text{ind}}(x \in \mathbb{R}_+^3 |\; Bx \in \{2^{1/\alpha} t\mathbb{1}/\|1\|^{\mathbb{1}/\alpha} \in \mathbb{R}_+^2 |\; t > 1\}) \\
&= \nu_{\text{ind}}(se_1 \in \mathbb{R}_+^3 | sBe_1 \in \{(2//\|1\|)^{1/\alpha} t\mathbb{1} \in \mathbb{R}_+^2 |\; t > 1\}) \\
&= \nu_{\text{ind}}(se_2 |\; s \in [(2//\|\mathbb{1}\|)^{1/\alpha}, \infty)) = \frac{\|\mathbb{1}\|}{2}.
\end{aligned}$$

Consequently, we have

$$\rho_0^* = \frac{1}{2}\delta_{(1,0)^\top} + \frac{1}{2}\delta_{(0,1)^\top} + \frac{\|1\|}{2}\delta_{\mathbb{1}/\|\mathbb{1}\|}.$$

Furthermore, the canonical spectral measures for the case of asymptotical independence and full dependence are

$$\rho^*_{\text{ind}} = \delta_{(1,0)^\top} + \delta_{(0,1)^\top} \;\text{ and }\; \rho^*_{\text{dep}} = \|\mathbb{1}\|\delta_{\mathbb{1}/\|\mathbb{1}\|}$$

In order to construct counterexamples we choose $d = q = 2$ and the function $g_{A_1,\alpha}$ with $A_1 = I_2$ the identity matrix. Then

$$\begin{aligned}
\rho^*_0 g_{A_1,\alpha} &= \int_{\mathbb{S}^1_+} \|As^{1/\alpha}\|^\alpha d\rho^*_0 \\
&= \|A(1,0)^\top\|^\alpha \rho^*_1(\{(1,0)^\top\}) + \|A(0,1)^\top\|^\alpha \rho^*_1(\{(0,1)^\top\}) + \|I_2(\mathbb{1}/\|\mathbb{1}\|)^{1/\alpha}\|^\alpha \rho^*_1(\{\mathbb{1}/\|\mathbb{1}\|\}) \\
&= 2^{-1} + 2^{-1} + \|\mathbb{1}\|^{-1}\|(1,1)^\top\|^\alpha \frac{\|\mathbb{1}\|}{2} \\
&= 1 + 2^{\frac{\alpha}{r}-1},
\end{aligned}$$

while $\rho^*_{\text{ind}} g_{A_1,\alpha} = 2$. This leads to the equivalences

$$\rho^*_0 g_{A_1,\alpha} < \rho_{\text{ind}} g_{A_1,\alpha} \;\Leftrightarrow\; 2 > 1 + 2^{\frac{\alpha}{r}-1} \;\Leftrightarrow\; 1 > 2^{\frac{\alpha}{r}-1} \;\Leftrightarrow\; r > \alpha. \tag{3.23}$$

In particular, we have for $1 < \alpha < r$,

$$C^S_{\nu_0}(A_1) < C^S_{\text{ind}}(A_1).$$

Next, we choose $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and calculate

$$\rho^*_{\text{ind}} g_{A_2,\alpha} = \|\mathbb{1}\|^\alpha + \|\mathbb{1}\|^\alpha = 2^{\frac{\alpha}{r}+1}$$

as well as

$$\rho^*_0 g_{A_2,\alpha} = \frac{1}{2}\|\mathbb{1}\|^\alpha + \frac{1}{2}\|\mathbb{1}\|^\alpha + \frac{\|\mathbb{1}\|}{2}\left\| \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{\mathbb{1}}{\|\mathbb{1}\|^{1/\alpha}} \right\|^\alpha.$$

Consequently,

$$\rho^*_{\text{ind}} g_{A_2,\alpha} < \rho^*_0 g_{A_2,\alpha} \;\Leftrightarrow\; 2 < 2^\alpha \tag{3.24}$$

Therefore, for $\alpha > 1$, $C^S_{\text{ind}}(A_2) < C^S_{\nu_0}(A_2)$. Inequalities (3.15) and (3.16) follow then from (3.23) and (3.24), respectively. □

Theorem 3.3. *Let the three d-dimensional vectors V_{ind}, V and V_{dep} be given with equal margins $V_1, \dots, V_d$ with $\mathbb{P}[V_j > t] \sim K_j t^{-\alpha}$, but different exponent measures $\nu_{\text{ind}}, \nu, \nu_{\text{dep}}$. Denote the aggregated vector $\|F\|$ for some r-norm for $r > 1$ or some r-quasinorm for $0 < r < 1$, representing the risk in the market.*
(a) If $r \geq 1$, for the constants C^S referring to the system setting the following inequalities hold:

$$C^S_\nu \leq C^S_{\text{dep}} \quad \text{for } \alpha \geq r \tag{3.25}$$
$$C^S_\nu \geq C^S_{\text{dep}} \quad \text{for } 0 < \alpha \leq 1 \tag{3.26}$$

(b) If $0 < r < 1$, for the constants C^S referring to the system setting the following inequalities hold:

$$C^S_\nu \leq C^S_{\text{dep}} \quad \text{for } \alpha \geq 1 \tag{3.27}$$

$$C_\nu^S \quad \geq \quad C_{\text{dep}}^S \quad \text{for } 0 < \alpha \leq r \tag{3.28}$$

(c) However, there are matrices A_1, A_2 and an exponent measure ν_0 such that

$$\begin{aligned}
C_{\nu_0}^S(A_1) &> C_{\text{dep}}^S(A_1) \quad \text{for } 1 < \alpha < r, & (3.29)\\
C_{\text{dep}}^S(A_2) &> C_{\nu_0}^S(A_2) \quad \text{for } 1 < \alpha < r, & (3.30)\\
C_{\nu_0}^S(A_1) &< C_{\text{dep}}^S(A_1) \quad \text{for } r < \alpha < 1, & (3.31)\\
C_{\text{dep}}^S(A_2) &< C_{\nu_0}^S(A_2) \quad \text{for } r < \alpha < 1. & (3.32)
\end{aligned}$$

Proof. We need the following inequalities, which are generalizations of Theorem 202 in [12], where such inequalities are proved for integrals with respect to Lebesgue measures. The general versions below are natural extensions using Fubini's theorem and the Hölder inequality for σ-finite measures. Suppose $(S_1, \mu_1), (S_2, \mu_2)$ are two σ-finite measure spaces and $F : S_1 \times S_2 \to \mathbb{R}$ is a product-measurable mapping. Then for $p \geq 1$ the inequality

$$\int_{S_2} \left| \int_{S_1} F(x,y) d\mu_1(x) \right|^p d\mu_2(y) \leq \left(\int_{S_1} \left(\int_{S_2} |F(x,y)|^p d\mu_2(x) \right)^{\frac{1}{p}} d\mu_1 \right)^p \tag{3.33}$$

and for $0 < p \leq 1$ the inequality

$$\int_{S_2} \left| \int_{S_1} F(x,y) d\mu_1(x) \right|^p d\mu_2(y) \geq \left(\int_{S_1} \left(\int_{S_2} |F(x,y)|^p d\mu_2(x) \right)^{\frac{1}{p}} d\mu_1 \right)^p \tag{3.34}$$

hold true.

(a) In the case $1 < r < \alpha$ we want to show (3.25); more precisely,

$$\int_{\mathbb{S}_+^{d-1}} \|A s^{1/\alpha}\|^\alpha d\rho^*(s) \leq \int_{\mathbb{S}_+^{d-1}} \|A s^{1/\alpha}\|^\alpha d\rho_{\text{dep}}^*(s) = \|A \mathbb{1}\|^\alpha. \tag{3.35}$$

To this end, we will apply (3.33) twice. In a first step, take $S_2 = \mathbb{S}_+^{d-1}$ with $\mu_2 = \rho$ and $S_1 = \{1, \ldots, q\}$ with μ_1 the counting measure, as well as $F(i,s) = \big(\sum_{j=1}^d A_{ij} s_j^{1/\alpha} \big)^r$ and $p = \frac{\alpha}{r}$. Then

$$\begin{aligned}
\int_{\mathbb{S}_+^{d-1}} \|A s^{1/\alpha}\|^\alpha d\rho^*(s) &= \int_{S_2} \left| \int_{S_1} F(x,y) d\mu_1(x) \right|^p d\mu_2(y) \\
&\leq \Big(\int_{S_1} \Big(\int_{S_2} |F(x,y)|^p d\mu_2(x) \Big)^{\frac{1}{p}} d\mu_1 \Big)^p \\
&= \Big(\sum_{i=1}^q \Big(\int_{\mathbb{S}_+^{d-1}} \Big(\sum_{j=1}^d A_{ij} s_j^{1/\alpha} \Big)^{r \frac{\alpha}{r}} d\rho^*(s) \Big)^{\frac{r}{\alpha}} \Big)^{\frac{\alpha}{r}} \qquad (3.36)
\end{aligned}$$

In the second step, take $S_2 = \mathbb{S}_+^{d-1}$ with $\mu_2 = \rho^*$ and $S_1 = \{1, \ldots, d\}$ with the weighted counting measure $\mu_1^i = \sum_{j=1}^d A_{ij} \delta_j$ for $i = 1, \ldots, q$. Further, let $F(j,s) = s_j^{1/\alpha}$ and $p = \alpha$. Then

$$\int_{\mathbb{S}_+^{d-1}} \Big(\sum_{j=1}^d A_{ij} s_j^{1/\alpha} \Big)^\alpha d\rho^*(s) \leq \Big(\sum_{j=1}^d A_{ij} \Big(\int_{\mathbb{S}_+^{d-1}} (s_j^{1/\alpha})^\alpha d\rho^*(s) \Big)^{1/\alpha} \Big)^\alpha = \big(\sum_{j=1}^d A_{ij} \big)^\alpha, \ i = 1, \ldots, q.$$

We continue with (3.36) and find that

$$\Big(\sum_{i=1}^q \Big(\int_{\mathbb{S}_+^{d-1}} \Big(\sum_{j=1}^d A_{ij} s_j^{1/\alpha} \Big)^{r \frac{\alpha}{r}} d\rho^*(s) \Big)^{\frac{r}{\alpha}} \Big)^{\frac{\alpha}{r}} \leq \Big(\sum_{i=1}^q \Big(\big(\sum_{j=1}^d A_{ij} \big)^\alpha \Big)^{\frac{r}{\alpha}} \Big)^{\frac{\alpha}{r}} = \|A \mathbb{1}\|^\alpha.$$

Relation (3.26) is shown analogously using (3.34).

(b) Inequalities (3.27) and (3.28) can be shown similar to part (a) by using the respective reverse inequalities. (c) Finally, we can use ρ_0^* in order to show (3.29) and (3.30). Taking again $A_1 = I_2$, we obtain

$$\rho_0^* g_{A_1,\alpha} = 1 + 2^{\frac{\alpha}{r}-1} \quad \text{and} \quad \rho_{\text{dep}}^* g_{A_1,\alpha} = 2^{\frac{\alpha}{r}}$$

and, consequently,

$$\rho_0^* g_{A_1,\alpha} \ > \ \rho_{\text{dep}}^* g_{A_1,\alpha} \ \Leftrightarrow \ 1 + 2^{\frac{\alpha}{r}-1} > 2^{\frac{\alpha}{r}} \ \Leftrightarrow \ 2 > 2^{\frac{\alpha}{r}} \Leftrightarrow \alpha < r. \tag{3.37}$$

Therefore, we have $C_{\nu_0}^S(A_1) > C_{\text{dep}}^S(A_1)$ for $1 < \alpha < r$.

Next, we choose $A_2 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and compute

$$\rho_1^* g_{A_2,\alpha} = 2^{\frac{\alpha}{r}} + 2^{-1} 2^{\alpha(1+\frac{1}{r})} \quad \text{and} \quad \rho_{\text{dep}}^* g_{A_2,\alpha} = 2^{\alpha(1+\frac{1}{r})}.$$

As a matter of fact,

$$\rho_{\text{dep}}^* g_{A_2,\alpha} > \rho_1^* g_{A_2,\alpha} \ \Leftrightarrow \ 2^{\alpha} > 2 \Leftrightarrow \alpha > 1; \tag{3.38}$$

i.e., we have $C_{\nu_0}^S(A_2) < C_{\text{dep}}^S$ for $1 < \alpha < r$. Relations (3.31) and (3.32) then follow from (3.37) and (3.38), respectively. □

Corollary 3.4. *Given the assumptions of Theorems 3.2 and 3.3, respectively, in the particular case* $\alpha = r = 1$ *the equalities*

$$C_{\text{ind}}^S = C_{\text{dep}}^S = C_{\nu}^S$$

hold true; i.e., the Value-at-Risk asymptotics are not influenced by the dependence structure between the risk factors.

Remark 3.5. Bounds for tail risk measures have also been studied in [3] in a non-asymptotic setting. From these bounds, one can derive asymptotic versions. As an example consider the upper bounds, while setting $q = d$ and $A = Id$ (the identity matrix, which is in particular deterministic) as well as $\alpha \in (1, \infty)$ and $\|\cdot\|_1$ as the norm. Then Theorem 3.3 implies for an arbitrary risk vector V

$$\text{VaR}_{1-\gamma}(V) \preceq \text{VaR}_{1-\gamma}(V_{\text{dep}}) \tag{3.39}$$

(where $f \preceq g$ is defined as $\limsup_{\gamma\to 0} \frac{f(1-\gamma)}{g(1-\gamma)} \leq 1$). If the distribution of $\|V\|_1$ has a density, then the Conditional Tail Expectation $\text{CoTE}_{1-\gamma}(\|V\|_1)$ from Definition 2.1 coincides with what is introduced as *Tail Value-at-Risk* $\text{TVaR}_{1\text{-}\gamma}(\|V\|_1)$ in [3].

If $\|V\|_1$ has infinite variance corresponding to $\alpha \in (1, 2]$, we get from Theorem 2.1 in [3] the inequality $\text{VaR}_{1-\gamma}(\|V\|_1) \leq \text{CoTE}_{1-\gamma}(\|V\|_1)$, which obviously implies also

$$\text{VaR}_{1-\gamma}(\|V\|_1) \preceq \text{CoTE}_{1-\gamma}(\|V_{\text{dep}}\|_1).$$

If we combine this with the Karamata asymptotic

$$\text{CoTE}_{1-\gamma}(\|V_{\text{dep}}\|_1) \sim \frac{\alpha}{\alpha-1} \text{VaR}_{1-\gamma}(\|V_{\text{dep}}\|),$$

this gives

$$\mathrm{VaR}_{1-\gamma}(V) \le \frac{\alpha}{\alpha-1}\mathrm{VaR}_{1-\gamma}(V_{\mathrm{dep}}),$$

which is in the light of (3.39) not an optimal bound as $\frac{\alpha}{\alpha-1} > 1$ for $\alpha > 1$.

If $\|V\|_1$ has finite variance, corresponding to $\alpha > 2$, Theorem 3.2 of [3] gives the inequality

$$\mathrm{VaR}_{1-\gamma}(\|V\|_1) \le \min\left\{\mu + s\sqrt{1-\gamma/\gamma}, \mathrm{CoTE}_{1-\gamma}(\|V_{\mathrm{dep}}\|_1)\right\}$$

with $\mu = \mathbb{E}\|V\|_1$ and $s^2 = \mathrm{Var}\|V\|_1$. Since $\gamma^{-1/\alpha} \le \sqrt{(1-\gamma)/\gamma}$ as $\gamma \to 0$, this also is no optimal bound compared to (3.39).

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