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Error estimates for finite element methods in shape optimization

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Abstract

This thesis is devoted to the analysis of two shape optimization problems that are discretized using finite elements. First we consider a tracking-type functional subject to an elliptic partial differential equation, then we maximize the difference between two eigenvalues of an elliptic partial differential operator corresponding to the transmission problem. Both problems are transformed on a reference domain and the existence of an optimal solution is shown. The main result for both problems is an a-priori error estimate for the error between the optimal control and a sequence of optimal controls to the fully discretized problem.

Zusammenfassung

In dieser Arbeit werde zwei Formoptimierungsprobleme betrachtet, welche mit finiten Elementen diskretisiert werden. Im ersten Beispiel wird ein tracking-type Funktional mit einer elliptischen partiellen Differentialgleichung als Nebenbedingung betrachtet, im zweiten Beispiel wird der Abstand zweier Eigenwerte des zum Transmission-Problem gehörigen elliptischen partiellen Differentialoperators maximiert. Beide Probleme werden auf ein Referenzgebiet transformiert und es wird die Existenz einer optimalen Lösung gezeigt. Das Hauptresultat für beide Beispiele ist eine a-priori Fehlerabschätzung für den Fehler zwischen der optimalen Kontrolle und einer Folge optimaler Kontrollen für das vollständig diskretisierte Problem.

Contents

1. Introduction

This thesis is devoted to a-priori error estimates for finite element discretizations of shape optimization problems. For this purpose we consider shape optimization problems, general optimal control problems as well as discretization methods including error estimates.

In optimal control of partial differential equations one is interested in minimizing a cost functional depending on a control and a state, where the control and the state are coupled via a partial differential equation, there may be additional constraints on the state and the control. In many cases, the control is given as a function or functional on the right hand side of a partial differential equation on a fixed domain, whereas the state is the solution to that equation. Within shape optimization, the control is no longer a functional but the shape of a domain. Some typical shape optimization problems are finding the shape of an airfoil that minimizes air resistance or finding the shape of a structural component that minimizes deformations induced by stress. Another practical application of shape optimization is called electrochemical machining and presented in Lu et al. [84]. As the problem of finding an optimal shape cannot, in general, be solved exactly one has to discretize the whole problem, i.e. the shape of the domain as well as the partial differential equation. The corresponding discrete solutions only yield approximations to the original solution, hence one is interested in estimating the induced error.

The main issue of this thesis is to embed shape optimization problems into the standard control theoretic framework and to rigorously prove existence and regularity results. These results provide the theoretical background for the a-priori error estimates, which are proven at the end of Chapter 2 and Chapter 3. The cost functionals under consideration are a classical L^2-tracking-type functional and a functional including the difference of the two smallest eigenvalues of a partial differential operator corresponding to the transmission problem.

Optimal control of partial differential equations is an active area of research, see, e.g., the monographs by Lions [83] and Tröltzsch [103], for the numerical treatment we refer to Hinze et al. [60].

For an introduction to shape optimization we refer to the monographs by Sokołowski & Zolésio [101], Delfour & Zolésio [37] and Ito & Kunisch [65], Chapter 11. Eppler [38] gives a short overview on how to compute shape derivatives. For the existence of optimal shapes in a general setting we mention Bucur & Buttazzo [21] and Henrot & Pierre [57].

Within this thesis, the unknown part of the boundary is parametrized as the graph of a function. This approach has already been used in various publications, see, e.g., Haslinger & Mäkinen [52], Haslinger & Neittaanmäki [53], Kunisch & Peichl [75] and Slawig [100].

Within Chapter 3, a cost functional including eigenvalues of a partial differential operator is being investigated. The relation between the eigenvalues of a partial differential operator and the underlying domain has been studied for a long time, cf. the monograph by Henrot [56] and the references cited therein. For an overview on stability estimates and sensitivity analysis we would like to mention Burenkov & Feleqi [22], Barbatis et al. [10] and Dambrine & Kateb [34].

For an introduction to the transmission problem and related regularity results we refer to Escauriaza & Mitrea [41], Escauriaza et al. [40], Xiong & Bao [109] and Caloz et. al [23]. Hintermüller & Laurain [58] showed how a transmission problem can be used within shape optimization in order to detect unknown shapes.

Braess [16] and Großman & Roos [50] give a general introduction into the numerical treatment of partial differential equations including the finite element method, an introduction to finite elements in the context of shape optimization can be found in Haslinger & Neittaanmäki [53].

Finally, an introduction to a-priori error estimates for general optimal control problems can be found in Falk [42], Hinze [59], Adar et al. [7], Casas et al. [26] and Casas & Tröltzsch [29]. There are just a few publications on error estimation and convergence analysis in shape optimization, cf. Eppler et al. [39] and Hiptmair & Paganini [61]. Vanmaele & Ženíšek [104] and the survey article by Boffi [15] give an overview on error estimates for finite elements in the context of eigenvalue problems.

This thesis is organized as follows.

Chapter 2: A model problem
In this chapter we formulate an abstract shape optimization problem with a tracking-type cost functional. We transform the problem on a reference domain and prove existence of an optimal solution of the transformed problem. The transformation approach has also been considered by Brandenburg et al. [19], Eppler & Harbrecht [39] and Ito et al. [65, 66]. Fumagalli et al. [45] applied this approach on more difficult equations. This approach leads to a problem on a fixed domain where the coefficients of the partial differential operator vary. In a more abstract setting, such types of problems have also been considered by Casas [25]. Another approach on how to deal with shape optimization problems is the so-called level set method, where the domain under consideration is defined as the zero level set of a given function. This approach has been considered by Allaire et al. [2] and Ito [64].

Afterward we derive an optimality system and use the optimality condition of first order to prove higher regularity of the optimal control. Results of this type are common within optimal control, in the context of shape optimization this idea has already been used by Carlier & Lachand-Robert [24] and Lamboley et al. [77, 78].

We then discretize the transformed problem using finite elements, and using a mild assumption on the coercivity of the second derivative of the reduced cost functional we show a-priori error estimates of optimal order for the error between the optimal control to the continuous problem and its fully discretized counterpart. For a possible numerical implementation using Newton's method we refer the reader to Laumen [80]. In Laumen [79] a general overview on numerical methods in shape optimization is given.

Some of the results of Chapter 2 are included from the author's Master's thesis [72], a detailed enumeration of those previously published results is given at the beginning of that chapter on page 5.

Chapter 3: Optimization of eigenvalues
In this chapter we formulate another abstract shape optimization problem where the cost functional is given as the difference of the first two Dirichlet eigenvalues for a transmission problem; for an overview on Neumann and Robin boundary conditions we refer to Girouardo & Polterovich [46].

As the eigenfunctions and eigenvalues of a partial differential operator generally depend on the shape of the underlying domain, it is possible to optimize functionals depending on the eigenvalues with respect to the shape of the domain. Marc Kac once asked whether it is possible to hear the shape of a drum, cf. [68]: Given all the eigenvalues of the Laplacian with homogeneous Dirichlet boundary conditions over some domain, is it possible to reconstruct the domain? Although the set of eigenvalues contains information such as the area or the diameter of the domain, the original question itself has a negative answer as has been shown by Gordon et al. [47].

Some other interesting questions concerning the behavior of the eigenvalues are the question which domain minimizes the n-th smallest eigenvector among all domains in $\mathbb{R}^2$ with a given volume. The well-known Faber-Krahn inequality states that the unique minimizer (up to sets of capacity zero) for $n = 1$ is the ball. As proven by Krahn and Szegö, cf. [74] and [90], for $n = 2$ the solution consists of two balls of the same volume. For general $n \geq 3$, the optimal domain is not known so far, cf. Henrot [55]. Furthermore, if additional constraints like connectedness or even convexity are imposed on the admissible domains, then little is known so far. Numerical approximations to some of these domains can be found in Antunes & Freitas [6] and Oudet [89].

Using the same approach as in Chapter 2 we transform the problem onto a reference domain, show the existence of an optimal control and prove higher regularity.

Again we use finite elements to discretize the problem and then show a-priori error estimates for the error between the optimal control to the continuous problem and its fully discretized counterpart. Although the obtained result is very similar to the corresponding result of Chapter 2, the methods used for its proof differ significantly.

Beside the Laplacian there is ongoing research concerning the eigenvalues of Schrödinger's operator, where the eigenfunctions have a physical interpretation as energy levels of quantum particles; but beside this physical meaning there are also some mathematical questions interesting on its own. An overview, including further references, can be found in Henrot [56]. In the context of nonlinear equations we would like to mention the p-Laplacian, cf. Lindqvist [82].

Chapter 4: Conclusion and perspectives
In this chapter we summarize the results from Chapter 2, Chapter 3 and the Appendix and discuss possible extensions and future work.

Chapter A: Appendix
The first part of the appendix contains a collection of various supplementary results like regularity results for partial differential equations and generalizations of the Bramble-Hilbert lemma and inverse estimates, which will be needed throughout this thesis.

The second part contains a generalization of a result obtained by Bramble & King [17] regarding finite element error estimates for a partial differential equation posed on a non-polygonal domain, which will be needed for the error estimation in the context of the discretization of the state and the transformation within Chapter 2 and Chapter 3.

Chapter B: Nomenclature
The last chapter contains an overview on the notation used throughout in this thesis. For a more detailed introduction into the topic of Sobolev-, and Hölder spaces we refer to Adams & Fournier [1] and Grisvard [48].

2. A model problem

The aim of this chapter is to introduce a general framework suited for the numerical analysis of shape optimization problems, apply this framework to a model problem and derive error estimates in the context of a finite element discretization of that shape optimization problem.

This chapter is organized as follows. In Section 2.1 we formulate an abstract shape optimization problem with cost functional of tracking type. We introduce a transformation to reformulate the whole problem on a fixed reference domain and prove existence of an optimal control with higher regularity. In Section 2.2 we use finite elements to discretize the control, the state and the transformation. Within Section 2.3 we first state some general stability results and then prove a-priori error estimates for the error between the optimal control to the continuous problem and its fully discretized counterpart.

As mentioned in the introduction, some of the results of this chapter, i.e. Section 2.1 without Subsubsection 2.1.3.2, Subsection 2.3.1, Subsubsection 2.3.2.1 and the beginning of Subsection 2.3.4 until and inclusive Lemma 2.3.59, have already been published within the author's Master's thesis [72] and are included in order to make this thesis more self-contained.

2.1. The problem

Within this section we are going to investigate a model shape optimization problem where the underlying shapes will be star-shaped with respect to the origin. The control variable q is an element of the control space $Q = H^2_{\text{per}}(I)$ with $I = (0, 2\pi)$ and

$$H^2_{\text{per}}(I) = \overline{C^\infty_{\text{per}}(I)}^{\|\cdot\|_{H^2(I)}}, \tag{2.1}$$

equipped with the usual H^2-norm, where

$$C^\infty_{\text{per}}(I) = \left\{ v \in C^\infty(\bar{I}) \,\middle|\, v^{(n)}(0) = v^{(n)}(2\pi)\, \forall n \in \mathbb{N}_0 \right\}.$$

The control q characterizes the domain Ω_q through

$$\Omega_q = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, r < 1 + q(\varphi),\, r = \sqrt{x^2 + y^2},\, \varphi = \arg(x + iy) \right\},$$

cf. Figure 2.1. In order to exclude a possible degeneracy of the domain Ω_q, we fix $\bar{\varepsilon} > 0$ and define the set

$$\overline{Q}^{\text{ad}} = \{ q \in Q |\, q(\varphi) \geq -1 + \bar{\varepsilon} \text{ for all } \varphi \in I \}. \tag{2.2}$$

Because of $H^2(I) \hookrightarrow C^{1,1/2}(\bar{I})$, (2.2) is well-defined. For each $q \in \overline{Q}^{\text{ad}}$ the domain Ω_q is Lipschitz, which allows for the definition of the state variable $u^q \in H^1_0(\Omega_q)$ as the weak solution to the state equation

$$\begin{cases} -\Delta u^q + u^q = f^q & \text{in } \Omega_q, \\ u^q = 0 & \text{on } \Gamma_q = \partial\Omega_q. \end{cases} \tag{2.3}$$

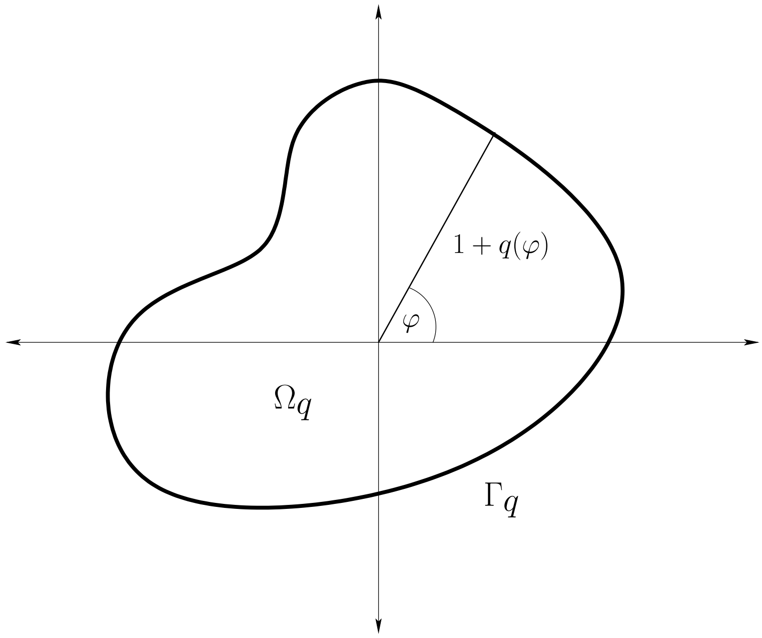

Figure 2.1.: The domain Ω_q

The shape optimization problem is then given as:

$$\text{Minimize } \tilde{J}(q, u^q) = \frac{1}{2} \left\| u^q - u_d^q \right\|_{L^2(\Omega_q)}^2 + \frac{\alpha}{2} \left\| q \right\|_{H^2(I)}^2, \quad \text{where } q \in \overline{Q}^{\text{ad}} \text{ and } u^q \in H_0^1(\Omega_q), \tag{2.4}$$

subject to (2.3) with f^q and u_d^q being sufficiently regular functions, cf. (2.7), and $\alpha > 0$ is a fixed constant.

We define the solution operator $S^q \colon \overline{Q}^{\text{ad}} \to H_0^1(\Omega_q)$, which assigns to each $q \in \overline{Q}^{\text{ad}}$ the unique solution $u^q = S^q(q)$ of (2.3). This allows for the introduction of the reduced cost functional $j \colon \overline{Q}^{\text{ad}} \to \mathbb{R}$ by

$$j(q) = \tilde{J}(q, S^q(q)). \tag{2.5}$$

In order to prove the existence of an optimal solution to (2.4) we bound $\overline{Q}^{\text{ad}}$ in $H^2(I)$.

Lemma 2.1.1. *There exists a constant $\tilde{C} = \tilde{C}(\alpha) > 0$ such that the search for a solution to* (2.4) *can be restricted to the set*

$$Q^{\text{ad}} = \left\{ q \in \overline{Q}^{\text{ad}} \,\middle|\, \|q\|_{H^2(I)} \le \tilde{C} \right\}. \tag{2.6}$$

Furthermore it holds that

$$\lim_{\alpha \to \infty} \tilde{C}(\alpha) = 0.$$

Proof. Let $q_0 = 0 \in \overline{Q}^{\text{ad}}$, a necessary condition for $q \in \overline{Q}^{\text{ad}}$ to be a solution to (2.4) is

$$j(q) \le j(q_0),$$

which reads as

$$\frac{1}{2}\left\|S^q(q)-u_d^q\right\|_{L^2(\Omega_q)}^2+\frac{\alpha}{2}\left\|q\right\|_{H^2(I)}^2\le j(q_0),$$

or equivalently

$$\|q\|_{H^2(I)}^2\le\frac{2}{\alpha}\left(j(q_0)-\frac{1}{2}\left\|S^q(q)-u_d^q\right\|_{L^2(\Omega_q)}^2\right)\le\frac{2}{\alpha}j(q_0).$$

Setting $\tilde{C}(\alpha)=\sqrt{\frac{2}{\alpha}j(q_0)}$ finishes this proof. □

Due to the boundedness of Q^{ad} in $C^1(\overline{I})$ it follows that there exists a bounded so-called holding-all domain $\tilde{\Omega}\subset\mathbb{R}^2$, such that $\Omega_q\subset\subset\tilde{\Omega}$ for all $q\in Q^{\mathrm{ad}}$. Throughout this chapter we assume for the data

$$u_d^q=u_d|_{\Omega_q},\ f^q=f|_{\Omega_q},\ \text{with } u_d,f\in C^{2,1}(\tilde{\Omega}). \tag{2.7}$$

We will therefore just write f and u_d instead of f^q and u_d^q, respectively.

2.1.1. Transformation of the problem

The aim of the following subsection is to reformulate the original problem (2.4) on a fixed reference domain Ω_0. This method is called the method of mapping, a short overview can be found in [37] and [101]. We define Ω_0 to be the unit circle and then compute a transformation T_q such that the domain Ω_q is the image of Ω_0 under that transformation, $\Omega_q=T_q(\Omega_0)$. All the results remain true if Ω_0 is replaced by any other sufficiently smooth domain sufficiently close to Ω_q in the sense of Assumption 2.1.11. In order to compute T_q it is often necessary to solve an additional partial differential equation like the equations of linear elasticity or the Laplace equation. Within this thesis we will focus on the Laplace equation. Our results remain true as long as Theorem A.1.28 holds for the chosen equation.

If one worked locally near the optimal shape instead of transforming the whole domain, then one would have to remesh the working domain every step, which is costly. As already mentioned, we bypass this remeshing at the cost of two additional Laplace equations. The reason why we choose this approach is the fact that it allows for comparing states corresponding to different shapes, which is important in the context of error estimation. Furthermore, from a practical point of view, adding some Laplace equations to the numerical solver is often less complicated than including a remeshing step.

Let $F=(F_1,F_2)^T$ be the weak solution to the following boundary value problem,

$$\begin{cases}-\Delta F=0 & \text{in } \Omega_0,\\ \quad F=q\,n & \text{on } \Gamma_0=\partial\Omega_0,\end{cases} \tag{2.8}$$

where n is the outer unit normal to Ω_0 and the Laplacian shall act on each component separately. If $F=F(q)$ solves (2.8) for a given $q\in Q$, then define $T_q=T_{F(q)}=\mathrm{Id}+F(q)$.

Lemma 2.1.2. *For $q\in Q$ it holds that $F=F(q)$ as the weak solution to (2.8) possesses the regularity $F\in H^{5/2}(\Omega_0)\hookrightarrow C^{1,1/2}(\overline{\Omega_0})$. More generally, for $s>1$ there holds the estimate*

$$\|F\|_{H^{s+1/2}(\Omega_0)}\le c_s\,\|q\|_{H^s(I)}\,.$$

Proof. As the outer unit normal n of the unit circle is uniformly bounded in $C^k(\Gamma_0)$ for every fixed $k \in \mathbb{N}$, we get

$$\|q\, n\|_{H^s(\Gamma_0)} \leq c\, \|n\|_{C^{\lceil s \rceil}(\Gamma_0)}\, \|q\|_{H^s(I)} \leq c_s\, \|q\|_{H^s(I)}\,,$$

cf. [48], Theorem 1.4.1.1, and the result follows with Theorem A.1.28. □

Let

$$\mathcal{F} = \left\{ F \in H^{5/2}(\Omega_0) \middle|\, \exists q \in Q \text{ such that } F = F(q) \text{ solves (2.8)} \right\}, \tag{2.9}$$

$$\mathcal{F}^{\mathrm{ad}} = \left\{ F \in H^{5/2}(\Omega_0) \middle|\, \exists q \in Q^{\mathrm{ad}} \text{ such that } F = F(q) \text{ solves (2.8)} \right\}, \tag{2.10}$$

be the spaces of (admissible) transformations. Note that $\mathcal{F}^{\mathrm{ad}}$ is a bounded set in $H^{5/2}(\Omega_0)$ due to Lemma 2.1.2 and (2.6). In addition, $\mathcal{F}^{\mathrm{ad}}$ need not be closed in $H^{5/2}(\Omega_0)$ for the trace operator is not surjective from $H^{k+1/2}(\Omega_0)$ to $H^k(\Gamma_0)$ for $k \in \mathbb{N}_0$ as mentioned in Theorem A.1.3.

The following functions derived from the transformation

$$T_F = \mathrm{Id} + F \tag{2.11}$$

will be used in the sequel. Some existence and regularity results concerning these functions will be shown below, some stability results concerning these functions can be found in Section 2.3.

$$\mathrm{D}T_F(x,y) = \mathrm{I} + \mathrm{D}F(x,y) = \begin{pmatrix} 1 + \partial_x F_1(x,y) & \partial_y F_1(x,y) \\ \partial_x F_2(x,y) & 1 + \partial_y F_2(x,y) \end{pmatrix}, \tag{2.12}$$

$$\gamma_F(x,y) = \det\left(\mathrm{D}T_F(x,y)\right), \tag{2.13}$$

$$\gamma'_{F,\delta F} = \frac{\mathrm{d}}{\mathrm{d}t} \gamma_{F+t\cdot\delta F}(x,y)\bigg|_{t=0}, \tag{2.14}$$

$$\gamma''_{F,\delta F,\tau F} = \frac{\mathrm{d}}{\mathrm{d}t} \gamma'_{F+t\cdot\tau F,\delta F}(x,y)\bigg|_{t=0}, \tag{2.15}$$

$$A_F(x,y) = \left(\gamma_F\, \mathrm{D}T_F^{-1} \cdot \mathrm{D}T_F^{-T}\right)(x,y), \tag{2.16}$$

where $\mathrm{D}T_F^{-1} = (\mathrm{D}T_F)^{-1}$,

$$A'_{F,\delta F}(x,y) = \frac{\mathrm{d}}{\mathrm{d}t} A_{F+t\cdot\delta F}(x,y)\bigg|_{t=0}, \tag{2.17}$$

$$A''_{F,\delta F,\tau F}(x,y) = \frac{\mathrm{d}}{\mathrm{d}t} A'_{F+t\cdot\tau F,\delta F}(x,y)\bigg|_{t=0}. \tag{2.18}$$

The following two lemmata will be needed to prove some regularity results concerning these functions.

Lemma 2.1.3. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and Lipschitz. Let $v \in H^1(\Omega)$ and $k \in \mathbb{N}$. If there exists $c_0 > 0$ such that $v(x) \geq c_0$ for almost every $x \in \Omega$, then $v^{-k} \in H^1(\Omega)$.*

Proof. We only have to show that

$$\nabla\left(v^{-k}\right) = \left(-k v^{-k-1} \nabla v\right) \in L^2(\Omega),$$

which follows from the generalized Hölder inequality and

$$v^{-k-1} \in L^\infty(\Omega),\ \nabla v \in L^2(\Omega).$$ □

Lemma 2.1.4. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{1,2\}$, be open, bounded and Lipschitz. Let $s \geq 0$ and $v \in H^s(\Omega)$. If there exists $c_0 > 0$ such that $v(x) \geq c_0$ for almost every $x \in \Omega$, then $v^{-1} \in H^s(\Omega)$.*

Proof. Let $s = k+\sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in [0,1)$. We start with the case $k=0$. If in addition $\sigma = 0$, then the result is clear. If $\sigma \in (0,1)$, then

$$\begin{aligned}
\left|v^{-1}\right|^2_{H^s(\Omega)} &= \int_\Omega \int_\Omega \frac{\left|v(x)^{-1} - v(y)^{-1}\right|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_\Omega \int_\Omega \left|\frac{1}{v(x)\,v(y)}\right|^2 \frac{|v(x)-v(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{1}{c_0^4} \left|v\right|^2_{H^s(\Omega)}.
\end{aligned}$$

Next we consider the case $k=1$, $v \in H^{1+\sigma}(\Omega)$. Due to Lemma 2.1.3 it remains to consider the case $\sigma \in (0,1)$. As $\nabla v \in H^\sigma(\Omega)$ and $v^{-3} \in H^1(\Omega)$ due to Lemma 2.1.3, it follows with Theorem A.1.5 that $-2v^{-3}\nabla v = \nabla\left(v^{-2}\right) \in H^\varepsilon(\Omega)$ for $\varepsilon < \sigma$. Hence, $v^{-2} \in H^{1+\varepsilon}(\Omega)$, and again with Theorem A.1.5 it follows that $-v^{-2}\nabla v = \nabla\left(v^{-1}\right) \in H^\sigma(\Omega)$, and $v^{-1} \in H^{1+\sigma}(\Omega)$.

The next case is $s = 2$. Due to Lemma 2.1.3 we only have to show that

$$\nabla^2\left(v^{-1}\right) = \left(2v^{-3}\nabla v \cdot \nabla v^T - v^{-2}\nabla^2 v\right) \in L^2(\Omega),$$

which follows from $v^{-2}, v^{-3} \in L^\infty(\Omega)$, $\nabla v \in H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $\nabla^2 v \in L^2(\Omega)$.

We finish the proof with induction. Assume that the statement has been shown for all $s < k$ for some $k \in \mathbb{N}$ with $k \geq 2$. Let $s = k+\sigma$, $\sigma \in [0,1)$ and $v \in H^{k+\sigma}(\Omega)$. We can further assume that $\sigma > 0$ if $k = 2$. As $v^{-1} \in H^{(k-1)+\sigma}(\Omega)$ by induction hypothesis, we get $v^{-2} \in H^{(k-1)+\sigma}(\Omega)$ with Theorem A.1.5. Furthermore, $\nabla v \in H^{(k-1)+\sigma}(\Omega)$, and again with Theorem A.1.5 we end up with $-v^{-2}\nabla v = \nabla\left(v^{-1}\right) \in H^{(k-1)+\sigma}(\Omega)$, which leads to $v^{-1} \in H^{k+\sigma}(\Omega)$. □

Lemma 2.1.5. *Let $s > 3/2$ and $q \in H^s(I)$. Then it holds that $\gamma_{F(q)}, A_{F(q)} \in H^{s-1/2}(\Omega_0)$.*

Proof. The regularity result for $\gamma_{F(q)}$ follows from (2.13), Lemma 2.1.2 and Theorem A.1.5. Because of

$$\mathrm{D}T^{-1}_{F(q)} = \frac{1}{\gamma_{F(q)}} \begin{pmatrix} 1+\partial_y F_2(q) & -\partial_y F_1(q) \\ -\partial_x F_2(q) & 1+\partial_x F_1(q) \end{pmatrix}, \tag{2.19}$$

and $\mathrm{D}F \in H^{s-1/2}(\Omega_0)$, Lemma 2.1.4 yields $\mathrm{D}T^{-1}_{F(q)} \in H^{s-1/2}(\Omega_0)$, and the regularity for $A_{F(q)}$ follows with (2.16) and the regularity result for $\gamma_{F(q)}$. □

Lemma 2.1.6. *Let $F \in \mathcal{F}^{\mathrm{ad}}$ and $\delta F, \tau F \in \mathcal{F}$. Then the following operators are Fréchet-differentiable with respect to F.*

- $\gamma_F \colon H^{5/2}(\Omega_0) \to H^{3/2}(\Omega_0)$ *with derivative*

$$\gamma'_{F,\delta F} = \gamma_F \operatorname{trace}\left(\mathrm{D}T_F^{-1} \cdot \mathrm{D}\delta F\right) = \operatorname{div}\left(\gamma_F\, \mathrm{D}T_F^{-1} \cdot \delta F\right).$$

- $\mathrm{D}T_F^{-1} \colon H^{5/2}(\Omega_0) \to H^{3/2}(\Omega_0)$ *with derivative*

$$\left(\mathrm{D}T_F^{-1}\right)'_{\delta F} = -\,\mathrm{D}T_F^{-1} \cdot \mathrm{D}\delta F \cdot \mathrm{D}T_F^{-1}.$$

- $\gamma'_{F,\delta F}\colon H^{5/2}(\Omega_0)\times H^{5/2}(\Omega_0)\to H^{3/2}(\Omega_0)$ *with derivative*

$$\begin{aligned}\gamma''_{F,\delta F,\tau F} &= \gamma_F\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\right)\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\right)-\gamma_F\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\cdot \mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\right)\\ &= \operatorname{trace}\left(\mathrm{D}\delta F\right)\operatorname{trace}\left(\mathrm{D}\tau F\right)-\operatorname{trace}\left(\mathrm{D}\delta F\cdot \mathrm{D}\tau F\right).\end{aligned}$$

- $A_F\colon H^{5/2}(\Omega_0)\to H^{3/2}(\Omega_0)$ *with derivative*

$$A'_{F,\delta F} = \operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\right)A_F-\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\cdot A_F - A_F\cdot \mathrm{D}\delta F^T\cdot \mathrm{D}T_F^{-T}. \tag{2.20}$$

- $A'_{F,\delta F}\colon H^{5/2}(\Omega_0)\times H^{5/2}(\Omega_0)\to H^{3/2}(\Omega_0)$ *with derivative*

$$\begin{aligned}&A''_{F,\delta F,\tau F}\\ &= -\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\cdot \mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\right)A_F+\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\right)\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\right)A_F\\ &-\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\right)\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\cdot A_F-\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\right)\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\cdot A_F\\ &-\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\right)A_F\cdot \mathrm{D}\tau F^T\cdot \mathrm{D}T_F^{-T}-\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\right)A_F\cdot \mathrm{D}\delta F^T\cdot \mathrm{D}T_F^{-T}\\ &+\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\cdot \mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\cdot A_F+\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\cdot \mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\cdot A_F\\ &+\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\cdot A_F\cdot \mathrm{D}\tau F^T\cdot \mathrm{D}T_F^{-T}+\mathrm{D}T_F^{-1}\cdot \mathrm{D}\tau F\cdot A_F\cdot \mathrm{D}\delta F^T\cdot \mathrm{D}T_F^{-T}\\ &+A_F\cdot \mathrm{D}\delta F^T\cdot \mathrm{D}T_F^{-T}\cdot \mathrm{D}\tau F^T\cdot \mathrm{D}T_F^{-T}+A_F\cdot \mathrm{D}\tau F^T\cdot \mathrm{D}T_F^{-T}\cdot \mathrm{D}\delta F^T\cdot \mathrm{D}T_F^{-T}.\end{aligned} \tag{2.21}$$

Proof. By a direct calculation it follows that

$$\begin{aligned}&\lim_{\|\delta F\|_{H^{5/2}(\Omega_0)}\to 0}\frac{\left\|\gamma_{F+\delta F}-\gamma_F-\gamma_F\operatorname{trace}\left(\mathrm{D}T_F^{-1}\cdot \mathrm{D}\delta F\right)\right\|_{H^{3/2}(\Omega_0)}}{\|\delta F\|_{H^{5/2}(\Omega_0)}}\\ &=\lim_{\|\delta F\|_{H^{5/2}(\Omega_0)}\to 0}\frac{\|\partial_x\delta F_1\,\partial_y\delta F_2-\partial_y\delta F_1\,\partial_x\delta F_2\|_{H^{3/2}(\Omega_0)}}{\|\delta F\|_{H^{5/2}(\Omega_0)}}\\ &\le c\lim_{\|\delta F\|_{H^{5/2}(\Omega_0)}\to 0}\frac{\|\delta F\|^2_{H^{5/2}(\Omega_0)}}{\|\delta F\|_{H^{5/2}(\Omega_0)}}=0,\end{aligned}$$

where in the second step we used Theorem A.1.5 and Theorem A.1.14. The result for $\mathrm{D}T_F^{-1}$ follows from a direct calculation, and the result for $\gamma'_{F,\delta F}$ follows from the fact that the trace is linear, Theorem A.1.11 as well as the first two parts of this lemma. The fourth part follows from the previous two parts, Theorem A.1.11 and Theorem A.1.5, and the result for $A'_{F,\delta F}$ follows from a direct calculation and the previous parts. □

Remark 2.1.7. For $A,B\in\mathbb{R}^{n\times n}$ it holds that $\operatorname{trace}(A\cdot B)=\operatorname{trace}(B\cdot A)$, hence the second derivatives $A''_{F,\delta F,\tau F}$ and $\gamma''_{F,\delta F,\tau F}$ within Lemma 2.1.6 are symmetric with respect to the directions.

Lemma 2.1.8. *For $\|q\|_{H^2(I)}\to 0$ it holds that*

- $T_{F(q)}\to \mathrm{Id}$ *in* $H^{5/2}(\Omega_0)\hookrightarrow C^{1,1/2}(\overline{\Omega_0})$,
- $\gamma_{F(q)}\to 1$ *in* $H^{3/2}(\Omega_0)\hookrightarrow C^{0,1/2}(\overline{\Omega_0})$,
- $\mathrm{D}T_{F(q)}^{-1}\to \mathrm{I}$ *in* $H^{3/2}(\Omega_0)$,

- $A_{F(q)} \to \mathrm{I}$ *in* $H^{3/2}(\Omega_0)$.

Proof. The first part follows from (2.11) and Lemma 2.1.2, the second part follows from (2.13), Lemma 2.1.5 and the first part of this lemma. The third part follows from (2.19), Theorem A.1.5 and the first two parts of this lemma, and the last part follows from (2.16), the second and the third part and again Theorem A.1.5. □

Lemma 2.1.9. *There exist* $c_0 > 0$, $0 < c_1 < c_2$ *and* $0 < c_3 < c_4$ *such that for* $\|q\|_{H^2(I)} < c_0$ *it holds that* $\gamma_{F(q)} \in [c_1, c_2]$ *and the two eigenvalues of* $A_{F(q)}$ *are elements of the interval* $[c_3, c_4]$. *Furthermore, for* $i \in \{1, 2, 3, 4\}$ *it holds that*

$$\lim_{\|q\|_{H^2(I)} \to 0} c_i = 1.$$

Proof. This lemma follows with Lemma 2.1.8 and the fact that the eigenvalues of a matrix continuously depend on its entries. □

As we use the transformation $T_{F(q)}$ to map Ω_0 onto Ω_q it is desirable that this transformation is one-to-one.

Lemma 2.1.10. *For* $\|q\|_{H^2(I)}$ *sufficiently small, the transformation* $T_{F(q)} \colon \Omega_0 \to \Omega_q$ *is bijective.*

Proof. It holds that $\Gamma_q = T_{F(q)}(\Gamma_0)$ by definition of $F(q)$. The fact that $T_{F(q)}(\Omega_0) \subset \Omega_q$ follows with the maximum principle for harmonic functions, cf. [16], Chapter I, Theorem 2.2, surjectivity follows by continuity and injectivity follows from Lemma 2.1.8 and [3], Theorem 3.8. □

Assumption 2.1.11. We assume that the constant $\tilde{C}$ in (2.6) is chosen sufficiently small such that Lemma 2.1.9 and Lemma 2.1.10 hold for all $q \in Q^{\mathrm{ad}}$.

Remark 2.1.12. With Lemma 2.1.1 it follows that Assumption 2.1.11 holds if α is sufficiently large. Furthermore, within practical applications like computing the optimal shape of an airfoil, a good approximation of the optimal shape is very often already known a-priori. For these reasons we think that this assumption is reasonable.

For the ease of notation, for $F \in \mathcal{F}^{\mathrm{ad}}$ and $u, v \in H^1(\Omega_0)$ we will use the following (bi)linear forms.

$$a(F)(u,v) = \int_{\Omega_0} \nabla u^T \cdot A_F \cdot \nabla v + uv\gamma_F \,\mathrm{d}x, \tag{2.22}$$

$$l(F)(v) = \int_{\Omega_0} (f \circ T_F) v \gamma_F \,\mathrm{d}x. \tag{2.23}$$

Lemma 2.1.13. *The bilinear form* $a(F)(\cdot,\cdot)$ *is uniformly continuous and coercive in* $H^1(\Omega_0)$, *i.e. there exist* $c_1, c_2 > 0$, *independent of* $F \in \mathcal{F}^{\mathrm{ad}}$, *such that for all* $u, v \in H^1(\Omega_0)$ *it holds that*

$$|a(F)(u,v)| \le c_1 \|u\|_{H^1(\Omega_0)} \|v\|_{H^1(\Omega_0)},$$
$$a(F)(u,u) \ge c_2 \|u\|^2_{H^1(\Omega_0)}.$$

Furthermore, there exists $c_3 > 0$, *independent of* $F \in \mathcal{F}^{\mathrm{ad}}$ *and* $p \in [1, \infty]$, *such that for* $u \in W^{1,p}(\Omega_0)$ *and* $v \in W^{1,q}(\Omega_0)$ *with* $1/p + 1/q = 1$ *the following Hölder-like inequality holds:*

$$|a(F)(u,v)| \le c_3 \|u\|_{W^{1,p}(\Omega_0)} \|v\|_{W^{1,q}(\Omega_0)}.$$

Proof. As the matrix A_F is symmetric, this lemma follows with Lemma 2.1.9 and Theorem A.1.7. □

Lemma 2.1.14. *Let $F \in \mathcal{F}^{\mathrm{ad}}$. Then there exists a unique $u \in H_0^1(\Omega_0)$ such that*

$$a(F)(u,v) = l(F)(v) \qquad \forall v \in H_0^1(\Omega_0), \tag{2.24}$$

and $\|u\|_{H_0^1(\Omega_0)} \leq c \|f \circ T_F\|_{L^2(\Omega_0)}$.

Proof. This lemma is a direct conclusion of the Lax-Milgram theorem, cf. [3], Theorem 4.2, and Lemma 2.1.13. □

Remark 2.1.15. For $q_1, q_2 \in Q^{\mathrm{ad}}$ and $F_3 \in \mathcal{F}^{\mathrm{ad}}$, let $u(q_1)$, $u(F_2)$ and $u(F_3)$ denote the unique solutions to (2.24) for $F = F(q_1)$, $F = F_2 = F(q_2)$ and $F = F_3$, respectively.

Lemma 2.1.14 motivates the introduction of another solution operator $S\colon Q^{\mathrm{ad}} \to H_0^1(\Omega_0)$, which assigns to each control $q \in Q^{\mathrm{ad}}$ the "transported" solution, i.e. let $S(q) = u(q) \in H_0^1(\Omega_0)$ be the solution of (2.24) for $F = F(q)$.

Lemma 2.1.16. *Let $q \in Q^{\mathrm{ad}}$, $F = F(q) \in \mathcal{F}^{\mathrm{ad}}$ and $v \in L^2(\Omega_q)$. Then it holds that $v \in H^1(\Omega_q)$ if and only if $v \circ T_F \in H^1(\Omega_0)$. Furthermore, the two norms $\|v\|_{H^1(\Omega_q)}$ and $\|v \circ T_F\|_{H^1(\Omega_0)}$ are equivalent for $v \in H^1(\Omega_q)$.*

Proof. Let $v \in H^1(\Omega_q)$. We have

$$\begin{aligned}
\|v\|_{H^1(\Omega_q)}^2 &= \int_{\Omega_q} v^2 + |\nabla v|^2 \,\mathrm{d}x = \int_{\Omega_0} (v \circ T_F)^2 \gamma_F + |\nabla v \circ T_F|^2 \gamma_F \,\mathrm{d}x \\
&\leq c \int_{\Omega_0} (v \circ T_F)^2 + \left|\mathrm{D}T_F^T \cdot \nabla v \circ T_F\right|^2 \,\mathrm{d}x \\
&= c \|v \circ T_F\|_{H^1(\Omega_0)}^2 \\
&\leq c \int_{\Omega_0} (v \circ T_F)^2 + |\nabla v \circ T_F|^2 \,\mathrm{d}x \leq c \int_{\Omega_0} (v \circ T_F)^2 \gamma_F + |\nabla v \circ T_F|^2 \gamma_F \,\mathrm{d}x \\
&= c \int_{\Omega_q} v^2 + |\nabla v|^2 \,\mathrm{d}x = c \|v\|_{H^1(\Omega_q)}^2,
\end{aligned}$$

where we also used Assumption 2.1.11. □

Lemma 2.1.17. *Let $F \in \mathcal{F}^{\mathrm{ad}}$, $u^q \in H_0^1(\Omega_q)$ and $u = u^q \circ T_F \in H_0^1(\Omega_0)$. Then the following two variational formulations are equivalent.*

$$\int_{\Omega_q} \left((\nabla u^q)^T \cdot \nabla v^q + u^q v^q\right) \mathrm{d}x = \int_{\Omega_q} f v^q \,\mathrm{d}x \qquad \forall v^q \in H_0^1(\Omega_q), \tag{2.25}$$

$$\int_{\Omega_0} \left(\nabla u^T \cdot A_F \cdot \nabla v + u v \gamma_F\right) \mathrm{d}x = \int_{\Omega_0} (f \circ T_F) v \gamma_F \,\mathrm{d}x \qquad \forall v \in H_0^1(\Omega_0). \tag{2.26}$$

Proof. This lemma can be shown using integration by substitution and Lemma 2.1.16. □

We are now able to reformulate problem (2.4) on the reference domain.

$$\min_{q \in Q^{\mathrm{ad}}, u \in H_0^1(\Omega_0), F \in \mathcal{F}^{\mathrm{ad}}} J(q,u,F) = \frac{1}{2} \int_{\Omega_0} (u - u_d \circ T_F)^2 \gamma_F \,\mathrm{d}x + \frac{\alpha}{2} \|q\|_{H^2(I)}^2, \tag{2.27}$$

subject to

$$\begin{cases} -\Delta F = 0 & \text{in } \Omega_0, \\ F = q\,n & \text{on } \Gamma_0, \end{cases} \qquad \text{and} \qquad \begin{cases} -\operatorname{div}(A_F \cdot \nabla u) + u\gamma_F = f \circ T_F \gamma_F & \text{in } \Omega_0, \\ u = 0 & \text{on } \Gamma_0. \end{cases}$$

Theorem 2.1.18. *Let $q \in Q^{\mathrm{ad}}$. Then q is an optimal solution to* (2.4) *if and only if it is an optimal solution to* (2.27).

Proof. This theorem follows with Lemma 2.1.17 and the fact that $\tilde{J}(q, \tilde{S}(q)) = J(q, S(q), F(q))$ holds for all $q \in Q^{\mathrm{ad}}$. □

2.1.2. Existence of a solution

Within this subsection we are going to prove that there exists a solution to problem (2.4). The following proof relies on Assumption 2.1.11 which can be partially omitted as mentioned in Remark 2.1.22. Due to Theorem 2.1.18 it is sufficient to show that (2.27) has a global solution.

Theorem 2.1.19. *If the constant $\tilde{C}$ from* (2.6) *is chosen sufficiently small in the sense of Assumption 2.1.11, then the problem* (2.27) *has a global solution.*

Proof. Let $j(q) = J(q, u(q), F(q)) \geq 0$ be the reduced cost functional. There exists a minimizing sequence $(q_n, u_n = u(q_n), F_n = F(q_n))_{n\in\mathbb{N}}$ with

$$j = \inf_{q \in Q^{\mathrm{ad}}} j(q) = \lim_{n\to\infty} j(q_n) = \lim_{n\to\infty} J(q_n, u_n, F_n).$$

As Q^{ad} is a convex, closed and bounded subset of the Hilbert space $H^2(I)$ it is weakly sequentially compact, hence there exists $\overline{q} \in Q^{\mathrm{ad}}$ such that, up to extracting a subsequence, it holds that

$$\begin{aligned} q_n &\rightharpoonup \overline{q} && \text{in } H^2(I), \\ q_n &\to \overline{q} && \text{in } H^{2-\varepsilon}(I), && \text{for } n \to \infty, \end{aligned}$$

where the second convergence is due to the compact embedding of $H^2(I)$ into $H^{2-\varepsilon}(I)$, cf. Theorem A.1.4. Let $\overline{F} = F(\overline{q}) \in \mathcal{F}^{\mathrm{ad}}$ and $\overline{u} = u(\overline{F})$. Due to Lemma 2.1.2 it follows that $F_n \to \overline{F}$ in $H^{5/2-\varepsilon}(\Omega_0)$, hence $u(F_n) \to u(\overline{F}) = \overline{u}$ in $H^1(\Omega_0)$ by Lemma 2.3.21. In addition, $\gamma_{F_n} \to \gamma_{\overline{F}}$ and $u_d \circ T_{F_n} \to u_d \circ T_{\overline{F}}$ in $L^\infty(\Omega_0)$ due to Lemma 2.3.15, which leads to

$$\lim_{n\to\infty} \left(\int_{\Omega_0} (u_n - u_d \circ T_{F_n})^2 \, \gamma_{F_n} \, \mathrm{d}x \right) = \int_{\Omega_0} \left(\overline{u} - u_d \circ T_{\overline{F}} \right)^2 \gamma_{\overline{F}} \, \mathrm{d}x. \tag{2.28}$$

As the squared H^2-norm is continuous and convex it is lower semicontinuous,

$$\liminf_{n\to\infty} \|q_n\|^2_{H^2(I)} \geq \|\overline{q}\|^2_{H^2(I)}, \tag{2.29}$$

and by adding (2.28) and (2.29) we arrive at

$$J(\overline{q}, \overline{u}, \overline{F}) \leq \liminf_{n\to\infty} J(q_n, u_n, F_n) = j, \tag{2.30}$$

and conclude that $J(\overline{q}, \overline{u}, \overline{F}) = j$. Hence $\left(\overline{q}, \overline{u}, \overline{F}\right)$ is a global solution to (2.27). □

Corollary 2.1.20. *Every minimizing sequence* $(q_n)_{n\in\mathbb{N}} \subset Q^{\mathrm{ad}}$ *contains a subsequence* $(q_{n_k})_{k\in\mathbb{N}}$ *such that* $q_{n_k} \to \overline{q}$ *in* $H^2(I)$ *for* $k \to \infty$*, where* $\overline{q}$ *is an optimal solution to* (2.27)*.*

Proof. In the proof of Theorem 2.1.19 we have already proven the existence of such a subsequence with $q_{n_k} \rightharpoonup \overline{q}$ in $H^2(I)$. As $J(q_{n_k}, u(q_{n_k}), F(q_{n_k})) \to J(\overline{q}, \overline{u}, \overline{F})$ it follows that $\|q_{n_k}\|_{H^2(I)} \to \|\overline{q}\|_{H^2(I)}$. As within Hilbert spaces weak convergence plus convergence of the norm implies strong convergence, cf. [3], U6.5, the result follows. □

Remark 2.1.21. Although the state equation (2.24) is linear, the mapping $q \mapsto u(q)$ is highly nonlinear and one cannot expect the reduced cost functional j to be convex. Therefore uniqueness of an optimal solution cannot be shown.

Remark 2.1.22. Although the proof of Theorem 2.1.19 depends on Assumption 2.1.11, this assumption can be omitted. In [52], Theorem 2.8, the authors show existence of an optimal solution for a similar shape optimization problem where they just need some sort of compactness of Q^{ad} in Q. As the proof mentioned in the source cited above is more involved than the one presented here, and as Assumption 2.1.11 is needed throughout this chapter, we decided to include the proof as stated.

2.1.3. The optimality system

Within this subsection it will be shown that the mapping $q \mapsto u(q)$ is at least twice continuously Fréchet-differentiable. We will use this fact to derive a boundary expression for the first derivative of the cost functional, and then use this representation to show higher regularity of the optimal control. This differentiability results as well as the regularity results will also be used within Section 2.3 in order to derive a-priori error estimates.

2.1.3.1. Differentiability of the control-to-state mapping and first-order optimality conditions

At first we investigate the differentiability of the control-to-state mapping $q \mapsto u(q)$, which will be shown using the implicit function theorem.

Lemma 2.1.23. *The mapping* $Q \ni q \mapsto F(q) \in \mathcal{F}$ *is at least twice continuously Fréchet-differentiable.*

Proof. As the mapping $q \mapsto F(q)$ is linear, the result follows with Lemma 2.1.2. □

Lemma 2.1.24. *The mapping* $\operatorname{int}\left(Q^{\mathrm{ad}}\right) \ni q \mapsto u(q) \in H_0^1(\Omega_0)$ *is at least twice continuously Fréchet-differentiable.*

Proof. We set $X = Q$, $X^{\mathrm{ad}} = \operatorname{int}\left(Q^{\mathrm{ad}}\right)$, $Y = Y^{\mathrm{ad}} = H_0^1(\Omega_0)$ and $Z = H^{-1}(\Omega_0)$. Furthermore, let

$$\begin{aligned} &B\colon Q \times H_0^1(\Omega_0) \to H^{-1}(\Omega_0), \\ &B(q,u) = a(F(q))(u,\cdot) - l(F(q))(\cdot). \end{aligned}$$

Then B is affine linear in u and at least twice continuously differentiable with respect to q, as follows from Lemma 2.1.6, Lemma 2.1.23 and (2.7). This lemma now follows with Theorem A.1.6. □

In order to be able to use Lemma 2.1.24 to derive optimality conditions, we make the following assumption.

Assumption 2.1.25. We assume that the optimal control $\overline{q}$ under consideration is an element of the interior of the admissible set, $\overline{q} \in \operatorname{int}(Q^{\mathrm{ad}})$.

As a result of Lemma 2.1.24, the operator S as well as the reduced cost functional j are at least twice continuously Fréchet-differentiable. The definition of the corresponding derivatives as well as some stability results can be found in Section 2.3. Due to Assumption 2.1.25 and the differentiability of j there holds a first-order optimality condition in $\overline{q}$, which reads as

$$j'(\overline{q})(\delta q) = 0 \qquad \forall \delta q \in Q. \tag{2.31}$$

Our goal is to use (2.31) to show higher regularity of the optimal control $\overline{q}$. To do so, we first have to reformulate the transformation equation.

2.1.3.2. A variational formulation for Dirichlet control problems

As the control q enters the equation for the transformation F on the boundary, (2.8) is a Dirichlet control problem which is known not to be of variational type. In [76] various possibilities on how to deal with such problems are presented. First we take a closer look at the weak solution u to the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = g & \text{on } \Gamma, \end{cases} \tag{2.32}$$

for an arbitrary Lipschitz domains $\Omega \subset \mathbb{R}^n$. If $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\Gamma)$, then one can proceed in a standard way as follows. Let $T\colon H^{1/2}(\Gamma) \to H^1(\Omega)$ be an arbitrary right inverse to the trace operator. The existence of such a T is ensured by Theorem A.1.3. The solution to (2.32) is now given as $u = u_\Omega + u_\Gamma$, where $u_\Gamma = Tg$ and $u_\Omega \in H^1_0(\Omega)$ solves

$$(\nabla u_\Omega, \nabla v) = -(\nabla u_\Gamma, \nabla v) + (f, v) \qquad \forall v \in H^1_0(\Omega). \tag{2.33}$$

The drawback of formulation (2.33) is the fact that one has to split u into the sum of the two functions u_Γ and u_Ω. This makes it more difficult to take the derivative of u with respect to g, which is crucial in order to derive an optimality system, where we have to take the derivative of (2.8) with respect to q. One possibility to overcome this difficulty is the use of the very weak formulation, which can be obtained from the weak formulation of (2.32) by partial integration once more,

$$-(u, \Delta v) + \langle g, \partial_n v\rangle = (f, v) \qquad \forall v \in H^1_0(\Omega) \cap H^2(\Omega), \tag{2.34}$$

which even allows for solutions $u \in L^2(\Omega)$ of the boundary value problem. This approach is used in [27], [36] and [85], a general overview can be found in [48] and [49]. However, it is not clear how to define a discrete approximation to (2.34), for piecewise linear ansatz functions v_h are in general no element of $H^2(\Omega)$. We will therefore stick to the approach presented in [13], which includes a discrete formulation and also coincides with the classical formulation if the input data is sufficiently regular. In what follows we will present that approach. Let $\Omega \subset \mathbb{R}^2$ be a $H^{3/2+\varepsilon}$-regular domain, which is fulfilled if Ω is either polygonal or convex. A precise formulation of this statement can be found in [49], Theorem 2.4.3 and Corollary 2.6.7. Let $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

Definition 2.1.26. Let $\mathcal{G}\colon L^2(\Omega) \to \mathbb{R}$ be defined as follows. For arbitrary $v \in L^2(\Omega)$, let $z \in H^1_0(\Omega)$ be the solution to

$$(\nabla z, \nabla w) = (v, w) \qquad \forall w \in H^1_0(\Omega).$$

Now find $\lambda \in L^2(\Gamma)$ such that

$$(\nabla z, \nabla \varphi) = (v, \varphi) + \langle \lambda, \varphi\rangle \qquad \forall \varphi \in H^1(\Omega),$$

and set

$$\mathcal{G}(v) = (f, z) - \langle g, \lambda \rangle.$$

Lemma 2.1.27. *There exists a unique $u \in L^2(\Omega)$ as the solutions to*

$$(u, v) = \mathcal{G}(v) \qquad \forall v \in L^2(\Omega). \tag{2.35}$$

Proof. This lemma follows from [13], Theorem 4.1. □

Definition 2.1.28. In order to trace the dependency on the boundary values, for $\Omega = \Omega_0$ we define, using the same notation as in Definition 2.1.26,

$$\mathcal{G}(g, v) = -\langle g\, n, \lambda \rangle,$$

where n shall denote the outer unit normal to Γ_0.

Lemma 2.1.29. *Let $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\Gamma)$. Then $u \in L^2(\Omega)$ is the unique solution of* (2.35) *if and only if it is the weak solution to* (2.32)*.*

Proof. Let $\varepsilon > 0$ be fixed such that Ω is $H^{3/2+\varepsilon}$-regular, and let

$$Z = \left\{ z \in H_0^1(\Omega) \cap H^{3/2+\varepsilon}(\Omega) \middle| \; \Delta z \in L^2(\Omega) \right\}.$$

From the definition of $H^{3/2+\varepsilon}$-regularity it follows that $-\Delta \colon Z \to L^2(\Omega)$ is bijective. Now let $u \in H^1(\Omega)$ be the weak solution to (2.32), i.e.

$$\begin{cases} (\nabla u, \nabla v) = (f, v) & \forall v \in H_0^1(\Omega), \\ \qquad u|_\Gamma = g. \end{cases}$$

As $Z \subset H_0^1(\Omega)$ it follows that

$$(\nabla u, \nabla z) = (f, z) \qquad \forall z \in Z,$$

and partial integration yields

$$(u, -\Delta z) + \langle u, \partial_n z \rangle = (f, z) \qquad \forall z \in Z.$$

Now let $v = -\Delta z$. From Definition 2.1.26 and Theorem A.1.3 it follows that $\lambda(v) = \partial_n z$, and as $u|_\Gamma = g$ we end up with

$$(u, v) = (f, z) - \langle g, \lambda \rangle \qquad \forall v \in L^2(\Omega),$$

which shows that u also solves (2.35). As the weak solution as well as the solution to (2.35) are unique, the result follows. □

2.1.3.3. The Lagrangian

Now we introduce the Lagrangian for problem (2.27) via

$$\begin{aligned}&\mathcal{L}\colon Q^{\mathrm{ad}} \times H_0^1(\Omega_0) \times H_0^1(\Omega_0) \times \mathcal{F} \times L^2(\Omega_0) \to \mathbb{R},\\ &\mathcal{L}(q,u,z,F,G) = J(q,u,F) + l(F)(z) - a(F)(u,z) + (F,G) - \mathcal{G}(q,G).\end{aligned} \tag{2.36}$$

If $u = u(q)$ and $F = F(q)$, then $\mathcal{L}(q,u,z,F,G) = j(q)$ for all $z \in H_0^1(\Omega_0)$ and $G \in L^2(\Omega_0)$. This fact is well-known and often exploited in order to obtain an optimality system. In general, one is looking for a stationary point of $\mathcal{L}$, but in order to ensure that every local minima of (2.27) is also a stationary point of $\mathcal{L}$ one needs some additional regularity which does not hold in general.

Lemma 2.1.30. *Let $q \in Q^{\mathrm{ad}}$, then $F(q) \in \mathcal{F}^{\mathrm{ad}}$ is the unique solution to*

$$\mathcal{L}'_G(q,u,z,F,G)(\delta G) = 0 \qquad \forall \delta G \in L^2(\Omega_0). \tag{2.37}$$

Proof. As $\mathcal{L}$ is linear in G, it follows that (2.37) just reads as

$$(F,\delta G) = \mathcal{G}(q,\delta G) \qquad \forall \delta G \in L^2(\Omega_0).$$

This lemma follows with Lemma 2.1.27 and Lemma 2.1.29. □

Lemma 2.1.31. *Let $q \in Q^{\mathrm{ad}}$ and $F = F(q)$. Then it holds that $u(q) \in H_0^1(\Omega_0)$ is the unique solution to*

$$\mathcal{L}'_z(q,u,z,F,G)(\delta z) = 0 \qquad \forall \delta z \in H_0^1(\Omega_0). \tag{2.38}$$

Proof. As $l(F)(\cdot)$ as well as $a(F)(u,\cdot)$ are linear, it immediately follows that $\mathcal{L}'_z$ exists. As (2.38) just reads as

$$a(F)(u,\delta z) = l(F)(\delta z) \qquad \forall \delta z \in H_0^1(\Omega_0),$$

this lemma follows with Lemma 2.1.14. □

Lemma 2.1.32. *Let $q \in Q^{\mathrm{ad}}$, $F = F(q)$ and $u = u(q)$. Then there exists a unique $z \in H_0^1(\Omega_0)$ such that*

$$\mathcal{L}'_u(q,u,z,F,G)(\delta u) = 0 \qquad \forall \delta u \in H_0^1(\Omega_0). \tag{2.39}$$

Proof. First, equation (2.39) can be written as

$$a(F)(\delta u,z) = J'_u(q,u,F)(\delta u) \qquad \forall \delta u \in H_0^1(\Omega_0),$$

which reads as

$$a(F)(\delta u,z) = ((u - u_d \circ T_F)\gamma_F, \delta u) \qquad \forall \delta u \in H_0^1(\Omega_0). \tag{2.40}$$

As the right hand side in (2.40) is a continuous functional on $L^2(\Omega_0)$, existence and uniqueness again follow with Lemma 2.1.13 and the Lax-Milgram theorem. □

Remark 2.1.33. With $z(q)$, $z(F)$ or $z(u)$ we will denote the adjoint state z as the solution to (2.39) for given q, F or u, cf. Remark 2.1.15.

To follow the standard procedure, we are now going to compute the derivative of $\mathcal{L}$ with respect to F, which exists due to Lemma 2.1.6. The goal is to prove the existence of an adjoint transformation $G \in L^2(\Omega_0)$ such that $\mathcal{L}'_F(q,u,z,F,G)(\delta F) = 0$ for all $\delta F \in \mathcal{F}$. As the transformation F enters $\mathcal{L}$ in a highly nonlinear way, we split the computation. First, it holds that

$$\begin{aligned} J'_F(q,u,F)(\delta F) &= \frac{1}{2}\int_{\Omega_0} (u - u_d \circ T_F)^2 \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big) \, \mathrm{d}x \\ &\quad - \int_{\Omega_0} (u - u_d \circ T_F)\, (\nabla u_d \circ T_F)^T \cdot \delta F \gamma_F \, \mathrm{d}x \\ &= \int_{\Gamma_0} \frac{1}{2} (u - u_d \circ T_F)^2 \, \gamma_F \delta F^T \cdot \mathrm{D}T_F^{-T} \cdot n \, \mathrm{d}s \\ &\quad - \int_{\Omega_0} (u - u_d \circ T_F)\, \nabla u^T \cdot \mathrm{D}T_F^{-1} \cdot \delta F \gamma_F \, \mathrm{d}x, \end{aligned} \tag{2.41}$$

$$\begin{aligned} l'_F(F)(\delta F, z) &= \int_{\Omega_0} f \circ T_F \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\, z + (\nabla f \circ T_F)^T \cdot \delta F \gamma_F z \, \mathrm{d}x \\ &= -\int_{\Omega_0} f \circ T_F \gamma_F \nabla z^T \cdot \mathrm{D}T_F^{-1} \cdot \delta F \, \mathrm{d}x, \end{aligned} \tag{2.42}$$

$$\begin{aligned} a'_F(F)(\delta F, u, z) &= \int_{\Omega_0} \nabla u^T \cdot A'_{F,\delta F} \cdot \nabla z + uz \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big) \, \mathrm{d}x \\ &= \int_{\Omega_0} \nabla u^T \cdot A'_{F,\delta F} \cdot \nabla z - (u\nabla z + z \nabla u)^T \cdot \mathrm{D}T_F^{-1} \cdot \delta F \gamma_F \, \mathrm{d}x, \end{aligned} \tag{2.43}$$

where we used the divergence theorem, $\nabla f \circ T_F = \mathrm{D}T_F^{-T} \cdot \nabla (f \circ T_F)$ and the analog formula for u_d as well as the fact that $uz \in W_0^{1,p}(\Omega_0)$ for $p < 2$ due to Theorem A.1.5. By combining (2.41), (2.42) and (2.43) with the definition of the Lagrangian, (2.36), we get

$$\begin{aligned} \mathcal{L}'_F(q,u,z,F,G)(\delta F) &= \int_{\Gamma_0} \frac{1}{2} (u - u_d \circ T_F)^2 \, \gamma_F \delta F^T \cdot \mathrm{D}T_F^{-T} \cdot n \, \mathrm{d}s \\ &\quad - \int_{\Omega_0} (u - u_d \circ T_F)\, \nabla u^T \cdot \mathrm{D}T_F^{-1} \cdot \delta F \gamma_F \, \mathrm{d}x \\ &\quad - \int_{\Omega_0} f \circ T_F \gamma_F \nabla z^T \cdot \mathrm{D}T_F^{-1} \cdot \delta F \, \mathrm{d}x \\ &\quad - \int_{\Omega_0} \nabla u^T \cdot A'_{F,\delta F} \cdot \nabla z + (u\nabla z + z \nabla u)^T \cdot \mathrm{D}T_F^{-1} \cdot \delta F \gamma_F \, \mathrm{d}x \\ &\quad + \int_{\Omega_0} \delta F \, G \, \mathrm{d}x. \end{aligned} \tag{2.44}$$

Lemma 2.1.34. *Let $q \in Q^{\mathrm{ad}}$, $u = u(q)$, $z = z(q)$, $F = F(q)$ and $G \in L^2(\Omega_0)$. Then there exists $d \in H^1(\Omega_0)$ such that*

$$\mathcal{L}'_F(q,u,z,F,G)(\delta F) = (-d, \delta F)_{H^1(\Omega_0)} + (\delta F, G) \qquad \forall \delta F \in H^1(\Omega_0),$$

i.e. the derivative $\mathcal{L}'_F(q,u,z,F,G)$ is a continuous linear functional on $H^1(\Omega_0)$.

Proof. Linearity of $\mathcal{L}'_F(q,u,z,F,G)(\cdot)$ follows from (2.44) and Lemma 2.1.6. With Theorem A.1.31 we get the improved regularity $u(q), z(q) \in W_0^{1,4}(\Omega_0)$, and the boundedness of $\mathcal{L}'_F$ in $H^1(\Omega_0)$ with respect to δF follows with Lemma 2.3.18 and Theorem A.1.3. The Riesz representation theorem, [3], Theorem 4.1, now ensures the existence of such an element $d \in H^1(\Omega_0)$. □

Remark 2.1.35. As $u, z \in H^{3/2-\varepsilon}(\Omega_0)$ but in general $u, z \notin H^{3/2}(\Omega_0)$ due to Theorem A.1.30, Theorem A.1.14 and the definition of $A'_{F,\delta F}$, (2.20), it follows that $\mathcal{L}'_F(q, u, z, F, G)$ is in general not a continuous linear functional on $L^2(\Omega_0)$. It follows that the equation

$$\mathcal{L}'_F(q, u(q), z(q), F(q), G)(\delta F) = 0 \qquad \forall \delta F \in H^1(\Omega_0),$$

need not have a solution $G \in L^2(\Omega_0)$ for general $q \in Q^{\mathrm{ad}}$.

With Remark 2.1.35 it follows that, in order to show the existence of an adjoint transformation G, it is necessary that the (adjoint) state as well as the transformation have a higher regularity. This is the case if the corresponding control is more regular.

2.1.3.4. Higher regularity of the optimal solution

Now we are going to prove higher regularity of the optimal control, namely $\overline{q} \in H^{9/2}(I)$. In order to do so we exploit the first-order optimality condition (2.31), which relies on Assumption 2.1.25.

$$\begin{aligned} j'(\overline{q})(\delta q) &= \frac{\mathrm{d}}{\mathrm{d}t} J\left(\overline{q} + t\,\delta q, u(\overline{q} + t\,\delta q), F(\overline{q} + t\,\delta q)\right)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}\left(\overline{q} + t\,\delta q, u(\overline{q} + t\,\delta q), z, F(\overline{q} + t\,\delta q), G\right)\Big|_{t=0} \quad \forall z \in H_0^1(\Omega_0),\, G \in L^2(\Omega_0). \end{aligned}$$

We now choose $z = z(\overline{q})$ and $G = 0$. With (2.31), Definition 2.3.2 and Lemma 2.1.32 we get

$$\begin{aligned} j'(\overline{q})(\delta q) &= \mathcal{L}'_q(\overline{q}, u(\overline{q}), z(\overline{q}), F(\overline{q}), 0)(\delta q) \\ &\quad + \mathcal{L}'_F(\overline{q}, u(\overline{q}), z(\overline{q}), F(\overline{q}), 0)(\delta F) = 0 \quad \forall \delta q \in Q,\, \delta F = F'(\overline{q})(\delta q) \in \mathcal{F}. \end{aligned} \tag{2.45}$$

With Lemma 2.1.34 we can rewrite (2.45) as

$$\alpha\left(\overline{q}, \delta q\right)_{H^2(I)} - \left(d, \delta F\right)_{H^1(\Omega_0)} = 0 \qquad \forall \delta q \in Q,\, \delta F = F'(\overline{q})(\delta q) \in \mathcal{F}, \tag{2.46}$$

with some $d \in H^1(\Omega_0)$. Using the Cauchy-Schwarz inequality it follows that

$$\left|\left(d, \delta F\right)_{H^1(\Omega_0)}\right| \leq \|d\|_{H^1(\Omega_0)} \|\delta F\|_{H^1(\Omega_0)}.$$

Furthermore the mapping $\delta q \mapsto \delta F$ is linear and $\|\delta F\|_{H^1(\Omega_0)} \leq c\, \|\delta q\|_{H^1(I)}$, which proves the existence of $d_1 \in H^1(I)$ with

$$\left(d, \delta F\right)_{H^1(\Omega_0)} = \left(d_1, \delta q\right)_{H^1(I)} \qquad \forall \delta q \in Q. \tag{2.47}$$

Inserting (2.47) into (2.46) yields

$$\alpha\left(\overline{q}, \delta q\right)_{H^2(I)} = \left(d_1, \delta q\right)_{H^1(I)} \qquad \forall \delta q \in Q. \tag{2.48}$$

To proceed, we need the following lemma.

Lemma 2.1.36. *Let $\lambda \in H^2_{\mathrm{per}}(I)$ and $\psi \in H^1_{\mathrm{per}}(I)$ such that*

$$\left(\lambda, \varphi\right)_{H^2(I)} = \left(\psi, \varphi\right)_{H^1(I)} \qquad \forall \varphi \in C^\infty_{\mathrm{per}}(I). \tag{2.49}$$

Then it holds that $\lambda \in H^3_{\mathrm{per}}(I)$.

Proof. Equation (2.49) just reads as

$$\int_0^{2\pi} \lambda''\varphi'' + \lambda'\varphi' + \lambda\varphi \,\mathrm{d}x = \int_0^{2\pi} \psi'\varphi' + \psi\varphi \,\mathrm{d}x \qquad \forall \varphi \in C^\infty_{\text{per}}(I).$$

Partial integration yields

$$\int_0^{2\pi} \big(\lambda'' - \lambda + \psi\big)\,\varphi'' \,\mathrm{d}x = \int_0^{2\pi} (-\lambda + \psi)\,\varphi \,\mathrm{d}x \qquad \forall \varphi \in C^\infty_{\text{per}}(I). \tag{2.50}$$

As (2.50) is just the definition of the second weak derivative and $(-\lambda + \psi) \in H^1_{\text{per}}(I)$, this yields

$$\big(\lambda'' - \lambda + \psi\big) \in H^3_{\text{per}}(I) \subset H^1_{\text{per}}(I),$$

and because of $\lambda, \psi \in H^1_{\text{per}}(I)$ we end up with $\lambda'' \in H^1_{\text{per}}(I)$, hence $\lambda \in H^3_{\text{per}}(I)$. □

Applying Lemma 2.1.36 to (2.48) yields $\overline{q} \in H^3(I)$, and Theorem A.1.28 proves $\overline{F} \in H^{7/2}(\Omega_0)$ and $\mathrm{D}\overline{F} \in H^{5/2}(\Omega_0) \hookrightarrow C^{1,1/2}(\overline{\Omega_0})$. As a result it holds that $A_{\overline{F}}, \gamma_{\overline{F}} \in C^{1,1/2}(\overline{\Omega_0})$, and Theorem A.1.38 yields $\overline{u}, \overline{z} \in W^{2,p}(\Omega_0)$ for $p < \infty$. Theorem A.1.3 now implies $\nabla\overline{u}|_{\Gamma_0}, \nabla\overline{z}|_{\Gamma_0} \in W^{1-1/p,p}(\Gamma_0)$. Due to this improved regularity we can further simplify some of the expressions within (2.44). First recall that

$$A'_{\overline{F},\delta F} = \operatorname{trace}\Big(\mathrm{D}T^{-1}_{\overline{F}} \cdot \mathrm{D}\delta F\Big) A_{\overline{F}} - \mathrm{D}T^{-1}_{\overline{F}} \cdot \mathrm{D}\delta F \cdot A_{\overline{F}} - A_{\overline{F}} \cdot \mathrm{D}\delta F^T \cdot \mathrm{D}T^{-T}_{\overline{F}}.$$

For the first part it holds that

$$\begin{aligned}
&- \int_{\Omega_0} \nabla\overline{u}^T \cdot A_{\overline{F}} \cdot \nabla\overline{z} \operatorname{trace}\Big(\mathrm{D}T^{-1}_{\overline{F}} \cdot \mathrm{D}\delta F\Big) \,\mathrm{d}x \\
&= - \int_{\Omega_0} \nabla\overline{u}^T \cdot \mathrm{D}T^{-1}_{\overline{F}} \cdot \mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{z} \operatorname{div}\Big(\gamma_{\overline{F}}\, \mathrm{D}T^{-1}_{\overline{F}} \cdot \delta F\Big) \,\mathrm{d}x \\
&= - \int_{\Gamma_0} \Big(\mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{u}\Big)^T \cdot \Big(\mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{z}\Big)\, \gamma_{\overline{F}} \delta F^T \cdot \mathrm{D}T^{-T}_{\overline{F}} \cdot n \,\mathrm{d}s \\
&\quad + \int_{\Omega_0} \nabla\left(\Big(\mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{u}\Big)^T \cdot \Big(\mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{z}\Big)\right)^T \cdot \mathrm{D}T^{-1}_{\overline{F}} \cdot \delta F \gamma_{\overline{F}} \,\mathrm{d}x.
\end{aligned} \tag{2.51}$$

It also holds that

$$\begin{aligned}
&\int_{\Omega_0} \nabla\left(\Big(\mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{u}\Big)^T \cdot \Big(\mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{z}\Big)\right)^T \cdot \mathrm{D}T^{-1}_{\overline{F}} \cdot \delta F \gamma_{\overline{F}} \,\mathrm{d}x \\
&+ \int_{\Omega_0} \nabla\overline{u}^T \cdot \Big(\mathrm{D}T^{-1}_{\overline{F}} \cdot \mathrm{D}\delta F \cdot A_{\overline{F}} + A_{\overline{F}} \cdot \mathrm{D}\delta F^T \cdot \mathrm{D}T^{-1}_{\overline{F}}\Big) \cdot \nabla\overline{z} \,\mathrm{d}x \\
&= \int_{\Omega_0} \nabla\overline{u}^T \cdot A_{\overline{F}} \cdot \nabla\Big(\nabla\overline{z}^T \cdot \mathrm{D}T^{-1}_{\overline{F}} \cdot \delta F\Big) + \nabla\overline{z}^T \cdot A_{\overline{F}} \cdot \nabla\Big(\nabla\overline{u}^T \cdot \mathrm{D}T^{-1}_{\overline{F}} \cdot \delta F\Big) \,\mathrm{d}x \\
&= - \int_{\Omega_0} \operatorname{div}\big(A_{\overline{F}} \cdot \nabla\overline{u}\big)\, \nabla\overline{z}^T \cdot \mathrm{D}T^{-1}_{\overline{F}} \cdot \delta F - \operatorname{div}\big(A_{\overline{F}} \cdot \nabla\overline{z}\big)\, \nabla\overline{u}^T \cdot \mathrm{D}T^{-1}_{\overline{F}} \cdot \delta F \,\mathrm{d}x \\
&\quad + 2 \int_{\Gamma_0} \Big(\mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{u}\Big)^T \cdot \Big(\mathrm{D}T^{-T}_{\overline{F}} \cdot \nabla\overline{z}\Big)\, \gamma_{\overline{F}} \delta F^T \cdot \mathrm{D}T^{-T}_{\overline{F}} \cdot n \,\mathrm{d}s.
\end{aligned} \tag{2.52}$$

If we insert (2.51) and (2.52) into (2.44) we finally arrive at

$$
\begin{aligned}
&\mathcal{L}'_F(\overline{q},\overline{u},\overline{z},\overline{F},G)(\delta F)\\
&= \int_{\Gamma_0} \frac{1}{2}\left(\overline{u}-u_d\circ T_{\overline{F}}\right)^2 \gamma_{\overline{F}}\delta F^T\cdot \mathrm{D}T_{\overline{F}}^{-1}\cdot n\,\mathrm{d}s\\
&+ \int_{\Gamma_0} \left(\mathrm{D}T_{\overline{F}}^{-T}\cdot\nabla\overline{u}\right)^T\cdot\left(\mathrm{D}T_{\overline{F}}^{-T}\cdot\nabla\overline{z}\right)\gamma_{\overline{F}}\delta F^T\cdot \mathrm{D}T_{\overline{F}}^{-1}\cdot n\,\mathrm{d}s\\
&+ \int_{\Omega_0} \underbrace{\left(-\operatorname{div}(A_{\overline{F}}\cdot\nabla\overline{u})+\overline{u}\gamma_{\overline{F}}-f\circ T_{\overline{F}}\gamma_{\overline{F}}\right)}_{=0}\left(\nabla\overline{z}^T\cdot \mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\mathrm{d}x\\
&+ \int_{\Omega_0} \underbrace{\left(-\operatorname{div}(A_{\overline{F}}\cdot\nabla\overline{z})+\overline{z}\gamma_{\overline{F}}-(\overline{u}-u_d\circ T_{\overline{F}})\gamma_{\overline{F}}\right)}_{=0}\left(\nabla\overline{u}^T\cdot \mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\mathrm{d}x\\
&+ \int_{\Omega_0}\delta F\, G\,\mathrm{d}x\\
&= \int_{\Gamma_0}\left(\frac{1}{2}\left(\overline{u}-u_d\circ T_{\overline{F}}\right)^2+\left(\mathrm{D}T_{\overline{F}}^{-T}\cdot\nabla\overline{u}\right)^T\cdot\left(\mathrm{D}T_{\overline{F}}^{-T}\cdot\nabla\overline{z}\right)\right)\gamma_{\overline{F}}\delta F^T\cdot \mathrm{D}T_{\overline{F}}^{-1}\cdot n\,\mathrm{d}s\\
&+ \int_{\Omega_0}\delta F\, G\,\mathrm{d}x,
\end{aligned}
$$

where we used the strong formulation of the (adjoint) state equation, (2.38) and (2.39), which hold due to the improved regularity of $\overline{u}$ and $\overline{z}$. As in the proof of Lemma 2.1.34 it is now possible to show that there exists $d_2 \in H^{1/2}(\Gamma_0)$ such that

$$
\mathcal{L}'_F(\overline{q},\overline{u},\overline{z},\overline{F},G)(\delta F) = -\langle d_2,\delta F\rangle + (\delta F, G) \qquad \forall \delta F\in\mathcal{F}. \tag{2.53}
$$

We now want to choose $\overline{G}\in L^2(\Omega_0)$ such that (2.53) vanishes. Due to Theorem A.1.3, there exists $H\in H^1_0(\Omega_0)\cap H^2(\Omega_0)$ with $\partial_n H = d_2$. Now define $\overline{G}=\Delta H\in L^2(\Omega_0)$, it follows that

$$
\begin{aligned}
\left(\delta F,\overline{G}\right) &= (\delta F,\Delta H)\\
&= -\left(\nabla\delta F,\nabla H\right)+\langle\delta F,\partial_n H\rangle = \langle\delta F, d_2\rangle,
\end{aligned} \tag{2.54}
$$

where the first term vanishes due to the fact that δF is weakly harmonic and $H\in H^1_0(\Omega_0)$. Inserting (2.54) into (2.53) yields

$$
\mathcal{L}'_F(\overline{q},\overline{u},\overline{z},\overline{F},\overline{G})(\delta F) = 0 \qquad \forall\delta F\in\mathcal{F},
$$

and we arrive at

$$
\begin{aligned}
j'(\overline{q})(\delta q) &= \mathcal{L}'_q(\overline{q},\overline{u},\overline{z},\overline{F},\overline{G})(\delta q)\\
&= \alpha\left(\overline{q},\delta q\right)_{H^2(I)} - \mathcal{G}(\delta q,\overline{G}) \qquad \forall\delta q\in Q.
\end{aligned}
$$

Using the definition of H and $\overline{G}$ as well as Definition 2.1.28, it finally follows that

$$
\begin{aligned}
j'(\overline{q})(\delta q) = &\int_{\Gamma_0}\left(\frac{1}{2}\left(\overline{u}-u_d\circ T_{\overline{F}}\right)^2+\left(\mathrm{D}T_{\overline{F}}^{-T}\cdot\nabla\overline{u}\right)^T\cdot\left(\mathrm{D}T_{\overline{F}}^{-T}\cdot\nabla\overline{z}\right)\right)\gamma_{\overline{F}}\delta F^T\cdot \mathrm{D}T_{\overline{F}}^{-1}\cdot n\,\mathrm{d}s\\
&+\alpha\left(\overline{q},\delta q\right)_{H^2(I)}.
\end{aligned} \tag{2.55}
$$

Remark 2.1.37. One may note that (2.55) looks similar to shape derivatives obtained by different methods, as done for example in [65], [73] and [101]. The fact that we end up with a boundary integral is due to the well-known Hadamard-Zolesio theorem, cf. [101], Theorem 2.27, which states that the derivative of a cost functional with respect to domain perturbations can always be represented as an integral over the moving part of the boundary, given sufficient smoothness of all the involved functions.

Remark 2.1.38. As $H \in H_0^1(\Omega_0) \cap H^2(\Omega_0)$ is only defined modulo $H_0^2(\Omega_0)$, it follows that $\overline{G}$ need not be unique. However, it easily follows from Definition 2.1.26 that for $\tilde{H} \in H_0^2(\Omega_0)$ and $\tilde{G} = \Delta\tilde{H}$ it holds that

$$\mathcal{G}(\delta q, \tilde{G}) = 0 \qquad\qquad \forall \delta q \in Q.$$

This observation is supported by (2.55), which is independent of H and $\overline{G}$.

The following lemma can be proven similar to Lemma 2.1.36.

Lemma 2.1.39. *Let $\lambda \in H^3_{\text{per}}(I)$ and $\psi \in H^{1/2}_{\text{per}}(I)$ such that*

$$(\lambda, \varphi)_{H^2(I)} = (\psi, \varphi)_{L^2(I)} \qquad\qquad \forall \varphi \in C^\infty_{\text{per}}(I).$$

Then it holds that $\lambda \in H^{9/2}_{\text{per}}(I)$.

Lemma 2.1.39 now proves the following theorem.

Theorem 2.1.40. *Let $\overline{q} \in Q^{\text{ad}}$ be an optimal solution to (2.27) in which Assumption 2.1.25 holds. Then we have the improved regularity $\overline{q} \in H^{9/2}(I)$.*

Remark 2.1.41. By using a bootstrap argument it is possible to show an even higher regularity of $\overline{q}$. Using Theorem A.1.28, one can show that $\overline{q} \in H^7(I)$, $\overline{F} \in H^{15/2}(\Omega_0)$, $\overline{u}, \overline{z} \in H^5(\Omega_0)$ and $\overline{G} \in H^{9/2-\varepsilon}(\Omega_0)$. A further improvement is not possible in general due to the regularity of f and u_d, cf. (2.7). Any further improvement in the regularity of f and u_d would result in a further improved regularity of $\overline{q}, \overline{u}, \overline{z}, \overline{F}$ and $\overline{G}$. For $f, u_d \in C^\infty(\hat{\Omega})$ we get $\overline{q} \in C^\infty_{\text{per}}(I)$ and $\overline{u}, \overline{z}, \overline{F}, \overline{G} \in C^\infty(\Omega_0)$.

2.2. Discretization

Within this section we are going to discretize problem (2.27) using finite elements for the control, the state and the transformation.

2.2.1. Discretization of the control

We start by discretizing the control. Let $N \in \mathbb{N}$, let $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 2\pi$ be a partition of the interval I and let $I_j = (x_j, x_{j+1})$ for $j \in \{0, \cdots, N-1\}$ be the associated subintervals. The discretization parameter will be denoted with

$$\sigma = \max_{0 \le j \le N-1} |I_j| .$$

Now we define the space of (admissible) discretized controls via

$$Q_\sigma = \left\{ q_\sigma \in Q |\ q_\sigma|_{I_j} \in \mathcal{P}^3(I_j) \, \forall j \in \{0, \cdots, N-1\} \right\}, \tag{2.56}$$

$$Q_\sigma^{\text{ad}} = Q^{\text{ad}} \cap Q_\sigma, \tag{2.57}$$

i.e. the discretized controls are piecewise polynomials of degree at most three and globally continuously differentiable. Note that it also follows that $Q_\sigma \subset W^{2,\infty}(I)$.

Definition 2.2.1. Let $i_\sigma \colon Q \to Q_\sigma$, $q \mapsto i_\sigma q$ be an interpolation operator which is defined as

$$\begin{aligned} (i_\sigma q)(x_j) &= q(x_j), \\ (i_\sigma q)'(x_j) &= q'(x_j), \end{aligned} \tag{2.58}$$

for all $j \in \{0, \cdots, N-1\}$. As the elements in Q_σ are piecewise polynomials of degree three and (2.58) gives four conditions on each subinterval I_j, it follows that $i_\sigma q$ is well-defined.

The following lemma can be shown using the Bramble-Hilbert lemma.

Lemma 2.2.2. *Let $s \in [2,4]$ and $q \in Q \cap H^s(I)$. Then it holds that*

$$\begin{aligned} \|i_\sigma q\|_{H^2(I)} &\le c\,\|q\|_{H^2(I)}\,, \\ \|q - i_\sigma q\|_{H^2(I)} &\le c_s \sigma^{s-2}\,|q|_{H^s(I)}\,. \end{aligned}$$

The first partially discretized problem now reads as

$$\min_{q_\sigma \in Q_\sigma^{\mathrm{ad}}, u \in H_0^1(\Omega_0), F \in \mathcal{F}^{\mathrm{ad}}} J(q_\sigma, u, F) \tag{2.59}$$

subject to

$$\begin{aligned} (F, G) &= \mathcal{G}(q_\sigma, G) && \forall G \in L^2(\Omega_0), \\ a(F)(u, v) &= l(F)(v) && \forall v \in H_0^1(\Omega_0). \end{aligned}$$

Theorem 2.2.3. *For $\sigma > 0$, problem* (2.59) *has an optimal solution $\overline{q}_\sigma$.*

Proof. In order to prove this theorem one may proceed as in the proof of Theorem 2.1.19 or use the fact that $\dim(Q_\sigma) < \infty$. □

2.2.2. Discretization of the state

In order to discretize the state and the transformation, we first have to construct a polygonal approximation to the domain Ω_0. For $h > 0$ sufficiently small, let $x_0, x_1, \cdots, x_{M(h)}$ denote $M(h)$ points on Γ_0 such that the distance between any two consecutive points (modulo $M(h)$) is bounded from above by h (possibly up to a given constant factor), and let $\Omega_{0,h} \subset \Omega_0$ be the polygonal and convex domain with vertices $x_0, x_1, \cdots, x_{M(h)}$. Hence, $\{\Omega_{0,h}\}_{h>0}$ is a family of domains approximating Ω_0. For fixed $h > 0$, let π_h be an admissible partition of $\Omega_{0,h}$ into triangles and/or quadrilaterals in the sense of Definition 2.2.4, where the maximal edge length of each triangle or quadrilateral shall be bounded from above by h. For each $K \in \pi_h$, let h_K denote the maximal length of its sides, and let ρ_K denote the radius of the biggest inscribed circle.

Definition 2.2.4. Let $h > 0$ be fixed. The triangulation π_h of $\Omega_{0,h}$ is called admissible if the following three conditions are satisfied.

- It holds that
$$\overline{\Omega_{0,h}} = \bigcup_{K \in \pi_h} \overline{K}.$$

- If $K_1, K_2 \in \pi_h$ and $\overline{K_1} \cap \overline{K_2} = \{x\}$ with $x \in \overline{\Omega_{0,h}}$, then x is a vertex of both K_1 and K_2.

- If $K_1, K_2 \in \pi_h$, $K_1 \neq K_2$ and $\emptyset \neq \overline{K_1} \cap \overline{K_2} \neq \{x\}$, then it holds that $\overline{K_1} \cap \overline{K_2} = \{E\}$, where $E \subset (\partial K_1 \cap \partial K_2)$ is an edge of both K_1 and K_2.

Definition 2.2.5 (Shape-regularity)**.** The family of triangulations $\{\pi_h\}_{h>0}$ is called shape-regular if there exists $\kappa > 0$ such that

$$\frac{h_K}{\rho_K} \leq \kappa,$$

holds uniformly for all $h > 0$ and $K \in \pi_h$.

Definition 2.2.6 (Quasi-uniformity)**.** The family of triangulations $\{\pi_h\}_{h>0}$ is called quasi-uniform if there exists $c_0 > 0$ such that

$$h \leq c_0 h_K,$$

holds uniformly for all $h > 0$ and $K \in \pi_h$.

Remark 2.2.7 (Usual regularity assumptions)*.* We will say that a family of admissible triangulations $\{\pi_h\}_{h>0}$ fulfills the usual regularity assumptions if it is both shape-regular and quasi-uniform in the sense of Definition 2.2.5 and Definition 2.2.6.

Remark 2.2.8*.* If $K \in \pi_h$ is a triangle, then it can be shown that shape-regularity is equivalent to the fact that the minimal inner angle of K is uniformly bounded from below. Quasi-uniformity is equivalent to the fact that the ratio of the area of the smallest triangle or quadrilateral to the area of the biggest triangle or quadrilateral within one fixed triangulation π_h is uniformly bounded from below for all $h > 0$.

If not stated otherwise we will always assume that $\{\pi_h\}_{h>0}$ fulfills the usual regularity assumptions in the sense of Remark 2.2.7. The finite element ansatz spaces are now defined as

$$V_h = \left\{ v_h \in H^1(\Omega_{0,h}) \middle| \, v_h|_K \in \mathcal{R}^1(K) \, \forall K \in \pi_h \right\}, \tag{2.60}$$

$$V_{h,0} = V_h \cap H_0^1(\Omega_{0,h}), \tag{2.61}$$

where we used the abbreviation

$$\mathcal{R}^1(K) = \begin{cases} \mathcal{P}^1(K) & \text{if } K \text{ is a triangle,} \\ \mathcal{Q}^1(K) & \text{if } K \text{ is a quadrilateral,} \end{cases} \tag{2.62}$$

where $\mathcal{Q}^1(K)$ is the space of bilinear polynomials over K. We also have to define approximations to the (bi)linear forms and the cost functional. As every function $v \in V_{h,0}$ can be extended by zero onto the whole domain Ω_0, we can regard $V_{h,0}$ as a subspace of $H_0^1(\Omega_0)$.

$$J_h(q, u, F) = \int_{\Omega_{0,h}} (u - u_d \circ T_F)^2 \gamma_F \, \mathrm{d}x + \frac{\alpha}{2} \|q\|_{H^2(I)}^2, \tag{2.63}$$

$$a_h(F)(u, v) = \int_{\Omega_{0,h}} \nabla u^T \cdot A_F \cdot \nabla v + uv\gamma_F \, \mathrm{d}x, \tag{2.64}$$

$$l_h(F)(v) = \int_{\Omega_{0,h}} f \circ T_F v \gamma_F \, \mathrm{d}x. \tag{2.65}$$

Now define the next partially discretized problem as

$$\min_{q_\sigma \in Q_\sigma^{\mathrm{ad}}, u_h \in V_{h,0}, F \in \mathcal{F}^{\mathrm{ad}}} J_h(q_\sigma, u_h, F) \tag{2.66}$$

subject to

$$\begin{aligned} (F, G) &= \mathcal{G}(q_\sigma, G) && \forall G \in L^2(\Omega_0), \\ a_h(F)(u_h, v_h) &= l_h(F)(v_h) && \forall v_h \in V_{h,0}. \end{aligned}$$

The following theorem can be proven similar to Theorem 2.2.3.

Theorem 2.2.9. *For $\sigma, h > 0$, problem (2.66) has an optimal solution $\overline{q}_{\sigma,h}$.*

2.2.3. Discretization of the transformation

The discretization of the transformation is analog to the discretization of the state, as in Subsection 2.2.2, let $\{\Omega_{0,k}\}_{k>0}$ be a sequence of polygonal approximation to Ω_0 and let $\{\pi_k\}_{k>0}$ be a corresponding family of triangulations fulfilling the usual regularity assumptions. Now let

$$V_k = \left\{ v_k \in H^1(\Omega_{0,k}) \middle| \, v_k|_K \in \mathcal{R}^1(K) \, \forall K \in \pi_k \right\}, \tag{2.67}$$

$$V_{k,0} = V_k \cap H_0^1(\Omega_{0,k}), \tag{2.68}$$

$$V_{k,\Gamma_{0,k}} = \left\{ g_k \in L^2(\Gamma_{0,k}) \middle| \, \exists v_k \in V_k \text{ such that } g_k = v_k|_{\Gamma_{0,k}} \right\}, \tag{2.69}$$

$$M_k = \left\{ v_k \in V_k | \, v_k(x_i) = 0 \text{ for all interior nodes } x_i \text{ of } \pi_k \right\}, \tag{2.70}$$

where $\mathcal{R}^1(K)$ is defined as in (2.62). Furthermore, using the extension as presented at the beginning of Subsection A.2.3 it is possible to regard V_k as subspace of $H^1(\Omega_0)$. With these definitions at hand we can now formulate the discrete transformation equation, cf. Subsubsection 2.1.3.2.

Definition 2.2.10. For $f \in L^2(\Omega_{0,k})$ and $g \in L^2(\Gamma_{0,k})$, let $\mathcal{G}_k \colon L^2(\Omega_{0,k}) \to \mathbb{R}$ be defined as follows, cf. Definition 2.1.26. For $v \in L^2(\Omega_{0,k})$, let $z_k \in V_{k,0}$ be the solution to

$$(\nabla z_k, \nabla w_k)_k = (v, w_k)_k \qquad \forall w_k \in V_{k,0}.$$

Now find $\lambda_k \in V_{k,\Gamma_{0,k}}$ such that

$$(\nabla z_k, \nabla \varphi_k)_k = (v, \varphi_k)_k + \langle \lambda_k, \varphi_k \rangle_k \qquad \forall \varphi_k \in M_k,$$

and set

$$\mathcal{G}_k(v) = (f, z_k)_k - \langle g, \lambda_k \rangle_k.$$

Lemma 2.2.11. *There exist a unique $u_k \in V_k$ as the solutions to*

$$(u_k, v_k)_k = \mathcal{G}_k(v_k) \qquad \forall v_k \in V_k.$$

Moreover it holds that

$$\begin{aligned} (\nabla u_k, \nabla v_k)_k &= (f, v_k)_k && \forall v_k \in V_{k,0}, \\ u_k|_{\Gamma_k} &= P_k g, \end{aligned}$$

where $P_k g$ is the L^2-projection of g onto the space $V_{k,\Gamma_{0,k}}$.

Proof. This lemma follows from [13], Lemma 5.1 and Theorem 5.2. □

Definition 2.2.12. In order to trace the dependency on the boundary values we define, using the same notation as in Definition 2.2.10,

$$\mathcal{G}_k(q,v) = -\langle \Pi_k(q\,n), \lambda_k \rangle_k,$$

where n shall denote the outer normal to Γ and $\Pi_k\colon L^2(\Gamma) \to V_{k,\Gamma_k}$ shall denote the orthogonal projection from Γ onto Γ_k, followed by a L^2-projection on the space of discrete boundary values, i.e. $\Pi_k(\cdot) = Q_h(\tilde{\cdot})$ in the sense of Definition A.2.12.

We also define the spaces for the discrete transformations.

$$\mathcal{F}_k = \left\{ F_k \in V_k \middle|\ \exists q_\sigma \in Q_\sigma \text{ such that } F_k = F_k(q) \text{ solves (2.74)} \right\}, \tag{2.71}$$

$$\mathcal{F}_k^{\text{ad}} = \left\{ F_k \in V_k \middle|\ \exists q_\sigma \in Q_\sigma^{\text{ad}} \text{ such that } F_k = F_k(q) \text{ solves (2.74)} \right\}. \tag{2.72}$$

Now it is possible to state the fully discretized problem.

$$\min_{q_\sigma \in Q_\sigma^{\text{ad}}, u_h \in V_{h,0}, F_k \in \mathcal{F}_k^{\text{ad}}} J_h(q_\sigma, u_h, F_k) \tag{2.73}$$

subject to

$$\begin{aligned} (F_k, G_k)_k &= \mathcal{G}_k(q_\sigma, G_k) && \forall G_k \in V_k, \\ a_h(F_k)(u_h, v_h) &= l_h(F_k)(v_h) && \forall v_h \in V_{h,0}. \end{aligned} \tag{2.74}$$

The following theorem can be proven similar to Theorem 2.2.3 and Theorem 2.2.9.

Theorem 2.2.13. *For $\sigma, h, k > 0$, problem* (2.73) *has an optimal solution $\overline{q}_{\sigma,h,k}$.*

As shown in [52] it is possible to prove that every sequence of optimal controls $\left(\overline{q}_{\sigma,h,k}\right)_{\sigma,h,k>0}$ to (2.73) contains a subsequence which converges to an optimal control to (2.27). Within Subsection 2.3.5 we will prove kind of an inverse statement, that for every local optimal control to (2.27) there exists a converging sequence of local optimal controls to (2.73).

2.3. A-priori error estimates

The aim of this section is to derive error estimates for the error between the optimal control to the continuous problem (2.27) and the optimal control to the fully discretized problem (2.73). The main result of this section will be the following, the proof can be found on page 63.

Theorem 2.3.1. *Let $\overline{q}$ be an optimal control to* (2.27)*. Then there exists a sequence $\left(\overline{q}_{\sigma,h,k}\right)_{\sigma,h,k>0}$ of local optimal controls to* (2.73) *and $c > 0$ such that*

$$\left\| \overline{q} - \overline{q}_{\sigma,h,k} \right\|_{H^2(I)} \le c \left(\sigma^2 + h^2 + k^2 \right),$$

for $\sigma, h, k \to 0$.

As already indicated in the previous theorem, we will, if not explicitily stated otherwise, always assume that σ, h and k are chosen sufficiently small. We start by recalling the definition of the state, its (partially) discretized counterparts as well as their derivatives with respect to domain perturbations.

Definition 2.3.2 (The continuous state).

- $u = S(q) \in H_0^1(\Omega_0)$ is the solution of
$$a(F)(u,v) = l(F)(v) \qquad \forall v \in H_0^1(\Omega_0),$$
where $F = F(q)$.
- $\delta u = S'(q)(\delta q) \in H_0^1(\Omega_0)$ is the solution of
$$\begin{aligned} a(F)(\delta u, v) &= \big(f \circ T_F, v \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\big) + \Big((\nabla f \circ T_F)^T \cdot \delta F, v\gamma_F\Big) \\ &\quad - \big(\nabla u, A'_{F,\delta F} \cdot \nabla v\big) - \big(u, v \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\big) \qquad \forall v \in H_0^1(\Omega_0), \end{aligned}$$
where $\delta F = F'(q)(\delta q)$.
- $\delta\tau u = S''(q)(\delta q, \tau q) \in H_0^1(\Omega_0)$ is the solution of
$$\begin{aligned} a(F)(\delta\tau u, v) &= (f \circ T_F, v\,(\operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F) - \operatorname{trace}(\mathrm{D}\delta F \cdot \mathrm{D}\tau F))) \\ &\quad + \big(\tau F^T \cdot \nabla^2 f \circ T_F \cdot \delta F, v\gamma_F\big) - \big(\nabla u, A''_{F,\delta F,\tau F} \cdot \nabla v\big) \\ &\quad + \Big((\nabla f \circ T_F)^T \cdot \delta F, v \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \tau F\big)\Big) \\ &\quad + \Big((\nabla f \circ T_F)^T \cdot \tau F, v \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\Big) \\ &\quad - \big(\nabla\tau u, A'_{F,\delta F} \cdot \nabla v\big) - \big(\nabla\delta u, A'_{F,\tau F} \cdot \nabla v\big) \\ &\quad - \big(\tau u, v \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\big) - \big(\delta u, v \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \tau F\big)\big) \\ &\quad - (uv, \operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F) - \operatorname{trace}(\mathrm{D}\delta F \cdot \mathrm{D}\tau F)) \qquad \forall v \in H_0^1(\Omega_0), \end{aligned}$$
where $\tau u = S'(q)(\tau q)$ and $\tau F = F'(q)(\tau q)$.

Definition 2.3.3 (The partially discretized state).

- $u_h = S_h(q) \in V_{h,0}$ is the solution of
$$a_h(F)(u_h, v_h) = l_h(F)(v_h) \qquad \forall v_h \in V_{h,0},$$
where $F = F(q)$.
- $\delta u_h = S'_h(q)(\delta q) \in V_{h,0}$ is the solution of
$$\begin{aligned} a_h(F)(\delta u_h, v_h) &= \big(f \circ T_F, v_h \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\big)_h + \Big((\nabla f \circ T_F)^T \cdot \delta F, v_h\gamma_F\Big)_h \\ &\quad - \big(\nabla u_h, A'_{F,\delta F} \cdot \nabla v_h\big)_h - \big(u_h, v_h \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\big)_h \qquad \forall v_h \in V_{h,0}, \end{aligned}$$
where $\delta F = F'(q)(\delta q)$.
- $\delta\tau u_h = S''_h(q)(\delta q, \tau q) \in V_{h,0}$ is the solution of
$$\begin{aligned} a_h(F)(\delta\tau u_h, v_h) &= (f \circ T_F, v_h\,(\operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F) - \operatorname{trace}(\mathrm{D}\delta F \cdot \mathrm{D}\tau F)))_h \\ &\quad + \big(\tau F^T \cdot \nabla^2 f \circ T_F \cdot \delta F, v_h\gamma_F\big)_h - \big(\nabla u_h, A''_{F,\delta F,\tau F} \cdot \nabla v_h\big)_h \\ &\quad + \Big((\nabla f \circ T_F)^T \cdot \delta F, v_h \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \tau F\big)\Big)_h \\ &\quad + \Big((\nabla f \circ T_F)^T \cdot \tau F, v_h \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\Big)_h \\ &\quad - \big(\nabla\tau u_h, A'_{F,\delta F} \cdot \nabla v_h\big)_h - \big(\nabla\delta u_h, A'_{F,\tau F} \cdot \nabla v_h\big)_h \\ &\quad - \big(\tau u_h, v_h \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big)\big)_h - \big(\delta u_h, v_h \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \tau F\big)\big)_h \\ &\quad - (u_h v_h, \operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F) - \operatorname{trace}(\mathrm{D}\delta F \cdot \mathrm{D}\tau F))_h \qquad \forall v_h \in V_{h,0}, \end{aligned}$$

where $\tau u_h = S_h'(q)(\tau q)$ and $\tau F = F'(q)(\tau q)$.

Definition 2.3.4 (The fully discretized state).

- $u_{h,k} = S_{h,k}(q) \in V_{h,0}$ is the solution of

$$a_h(F_k)(u_{h,k}, v_h) = l_h(F_k)(v_h) \qquad \forall v_h \in V_{h,0},$$

where $F_k = F_k(q)$.

- $\delta u_{h,k} = S_{h,k}'(q)(\delta q) \in V_{h,0}$ is the solution of

$$\begin{aligned} a_h(F_k)(\delta u_{h,k}, v_h) &= \left(f \circ T_{F_k}, v_h \operatorname{div}\left(\gamma_{F_k} \mathrm{D}T_{F_k}^{-1} \cdot \delta F_k\right)\right)_h + \left((\nabla f \circ T_{F_k})^T \cdot \delta F_k, v_h \gamma_{F_k}\right)_h \\ &\quad - \left(\nabla u_{h,k}, A_{F_k,\delta F_k}' \cdot \nabla v_h\right)_h - \left(u_{h,k}, v_h \operatorname{div}\left(\gamma_{F_k} \mathrm{D}T_{F_k}^{-1} \cdot \delta F_k\right)\right)_h \qquad \forall v_h \in V_{h,0}, \end{aligned}$$

where $\delta F_k = F_k'(q)(\delta q)$.

- $\delta\tau u_{h,k} = S_{h,k}''(q)(\delta q, \tau q) \in V_{h,0}$ is the solution of

$$\begin{aligned} a_h(F_k)(\delta\tau u_{h,k}, v_h) &= \left(f \circ T_{F_k}, v_h \left(\operatorname{trace}(\mathrm{D}\delta F_k) \operatorname{trace}(\mathrm{D}\tau F_k) - \operatorname{trace}(\mathrm{D}\delta F_k \cdot \mathrm{D}\tau F_k)\right)\right)_h \\ &\quad + \left(\tau F_k^T \cdot \nabla^2 f \circ T_{F_k} \cdot \delta F_k, v_h \gamma_{F_k}\right)_h - \left(\nabla u_{h,k}, A_{F_k,\delta F_k,\tau F_k}'' \cdot \nabla v_h\right)_h \\ &\quad + \left((\nabla f \circ T_{F_k})^T \cdot \delta F_k, v_h \operatorname{div}\left(\gamma_{F_k} \mathrm{D}T_{F_k}^{-1} \cdot \tau F_k\right)\right)_h \\ &\quad + \left((\nabla f \circ T_{F_k})^T \cdot \tau F_k, v_h \operatorname{div}\left(\gamma_{F_k} \mathrm{D}T_{F_k}^{-1} \cdot \delta F_k\right)\right)_h \\ &\quad - \left(\nabla \tau u_{h,k}, A_{F_k,\delta F_k}' \cdot \nabla v_h\right)_h - \left(\nabla \delta u_{h,k}, A_{F_k,\tau F_k}' \cdot \nabla v_h\right)_h \\ &\quad - \left(\tau u_{h,k}, v_h \operatorname{div}\left(\gamma_{F_k} \mathrm{D}T_{F_k}^{-1} \cdot \delta F_k\right)\right)_h - \left(\delta u_{h,k}, v_h \operatorname{div}\left(\gamma_{F_k} \mathrm{D}T_{F_k}^{-1} \cdot \tau F_k\right)\right)_h \\ &\quad - \left(u_{h,k} v_h, \operatorname{trace}(\mathrm{D}\delta F_k) \operatorname{trace}(\mathrm{D}\tau F_k) - \operatorname{trace}(\mathrm{D}\delta F_k \cdot \mathrm{D}\tau F_k)\right)_h \qquad \forall v_h \in V_{h,0}, \end{aligned}$$

where $\tau u_{h,k} = S_{h,k}'(q)(\tau q)$ and $\tau F_k = F_k'(q)(\tau q)$.

For the ease of notation we introduce the following abbreviations for some of the right hand sides within the previous definitions.

Definition 2.3.5. Let

$$\begin{aligned} l^\delta(F, \delta F, u)(v) &= \left(f \circ T_F, v \operatorname{div}\left(\gamma_F \mathrm{D}T_F^{-1} \cdot \delta F\right)\right) + \left((\nabla f \circ T_F)^T \cdot \delta F, v\gamma_F\right) - \left(\nabla u, A_{F,\delta F}' \cdot \nabla v\right) \\ &\quad - \left(u, v \operatorname{div}\left(\gamma_F \mathrm{D}T_F^{-1} \cdot \delta F\right)\right), \\ l_h^\delta(F, \delta F, u)(v) &= \left(f \circ T_F, v \operatorname{div}\left(\gamma_F \mathrm{D}T_F^{-1} \cdot \delta F\right)\right)_h + \left((\nabla f \circ T_F)^T \cdot \delta F, v\gamma_F\right)_h - \left(\nabla u, A_{F,\delta F}' \cdot \nabla v\right)_h \\ &\quad - \left(u, v \operatorname{div}\left(\gamma_F \mathrm{D}T_F^{-1} \cdot \delta F\right)\right)_h, \end{aligned}$$

and

$$
\begin{aligned}
l^{\delta,\tau}(F,\delta F,\tau F,u,\delta u,\tau u)(v) &= \left(f\circ T_F, v\left(\operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F)-\operatorname{trace}(\mathrm{D}\delta F\cdot \mathrm{D}\tau F)\right)\right)\\
&\quad+\left(\tau F^T\cdot\nabla^2 f\circ T_F\cdot\delta F, v\gamma_F\right)-\left(\nabla u, A''_{F,\delta F,\tau F}\cdot\nabla v\right)\\
&\quad+\Big(\left(\nabla f\circ T_F\right)^T\cdot\delta F, v\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\tau F\big)\Big)\\
&\quad+\Big(\left(\nabla f\circ T_F\right)^T\cdot\tau F, v\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\delta F\big)\Big)\\
&\quad-\left(\nabla\tau u, A'_{F,\delta F}\cdot\nabla v\right)-\left(\nabla\delta u, A'_{F,\tau F}\cdot\nabla v\right)\\
&\quad-\left(\tau u, v\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\delta F\big)\right)-\left(\delta u, v\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\tau F\big)\right)\\
&\quad-\left(uv,\operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F)-\operatorname{trace}(\mathrm{D}\delta F\cdot\mathrm{D}\tau F)\right),\\
l_h^{\delta,\tau}(F,\delta F,\tau F,u,\delta u,\tau u)(v) &= \left(f\circ T_F, v\left(\operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F)-\operatorname{trace}(\mathrm{D}\delta F\cdot \mathrm{D}\tau F)\right)\right)_h\\
&\quad+\left(\tau F^T\cdot\nabla^2 f\circ T_F\cdot\delta F, v\gamma_F\right)_h-\left(\nabla u, A''_{F,\delta F,\tau F}\cdot\nabla v\right)_h\\
&\quad+\Big(\left(\nabla f\circ T_F\right)^T\cdot\delta F, v\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\tau F\big)\Big)_h\\
&\quad+\Big(\left(\nabla f\circ T_F\right)^T\cdot\tau F, v\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\delta F\big)\Big)_h\\
&\quad-\left(\nabla\tau u, A'_{F,\delta F}\cdot\nabla v\right)_h-\left(\nabla\delta u, A'_{F,\tau F}\cdot\nabla v\right)_h\\
&\quad-\left(\tau u, v\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\delta F\big)\right)_h-\left(\delta u, v\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\tau F\big)\right)_h\\
&\quad-\left(uv,\operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F)-\operatorname{trace}(\mathrm{D}\delta F\cdot\mathrm{D}\tau F)\right)_h.
\end{aligned}
$$

We also recall the definitions of the cost functionals and their derivatives. Within the following three definitions let $q\in Q^{\mathrm{ad}}$, $\delta q,\tau q\in Q$, $F=F(q)$, $F_k=F_k(q)$, $\delta F=F'(q)(\delta q)$, $\tau F=F'(q)(\tau q)$, $\delta F_k=F_k'(q)(\delta q)$ and $\tau F_k=F_k'(q)(\tau q)$.

Definition 2.3.6 (The continuous cost functional)**.**

$$
\begin{aligned}
j(q) &= \frac{1}{2}\left(S(q)-u_d\circ T_F,\left(S(q)-u_d\circ T_F\right)\gamma_F\right)+\frac{\alpha}{2}\|q\|^2_{H^2(I)},\\
j'(q)(\delta q) &= \frac{1}{2}\left(S(q)-u_d\circ T_F,\left(S(q)-u_d\circ T_F\right)\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\delta F\big)\right)\\
&\quad+\Big(S'(q)(\delta q)-\left(\nabla u_d\circ T_F\right)^T\cdot\delta F,\left(S(q)-u_d\circ T_F\right)\gamma_F\Big)+\alpha\left(q,\delta q\right)_{H^2(I)},\\
j''(q)(\delta q,\tau q) &= \frac{1}{2}\left(S(q)-u_d\circ T_F,\left(S(q)-u_d\circ T_F\right)\left(\operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F)-\operatorname{trace}(\mathrm{D}\delta F\cdot\mathrm{D}\tau F)\right)\right)\\
&\quad+\Big(S'(q)(\delta q)-\left(\nabla u_d\circ T_F\right)^T\cdot\delta F,\Big(S'(q)(\tau q)-\left(\nabla u_d\circ T_F\right)^T\cdot\tau F\Big)\gamma_F\Big)\\
&\quad+\Big(S(q)-u_d\circ T_F,\Big(S'(q)(\delta q)-\left(\nabla u_d\circ T_F\right)^T\cdot\delta F\Big)\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\tau F\big)\Big)\\
&\quad+\Big(S(q)-u_d\circ T_F,\Big(S'(q)(\tau q)-\left(\nabla u_d\circ T_F\right)^T\cdot\tau F\Big)\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\delta F\big)\Big)\\
&\quad+\left(S(q)-u_d\circ T_F,\left(S''(q)(\delta q,\tau q)-\tau F^T\cdot\nabla^2 u_d\circ T_F\cdot\delta F\right)\gamma_F\right)+\alpha\left(\delta q,\tau q\right)_{H^2(I)}.
\end{aligned}
$$

Definition 2.3.7 (The partially discretized cost functional)**.**

$$
\begin{aligned}
j_h(q) &= \frac{1}{2}\left(S_h(q)-u_d\circ T_F,\left(S_h(q)-u_d\circ T_F\right)\gamma_F\right)_h+\frac{\alpha}{2}\|q\|^2_{H^2(I)},\\
j_h'(q)(\delta q) &= \frac{1}{2}\left(S_h(q)-u_d\circ T_F,\left(S_h(q)-u_d\circ T_F\right)\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\delta F\big)\right)_h\\
&\quad+\Big(S_h'(q)(\delta q)-\left(\nabla u_d\circ T_F\right)^T\cdot\delta F,\left(S_h(q)-u_d\circ T_F\right)\gamma_F\Big)_h+\alpha\left(q,\delta q\right)_{H^2(I)},
\end{aligned}
$$

$$
\begin{aligned}
&j_h''(q)(\delta q,\tau q)\\
&= \frac{1}{2}\left(S_h(q)-u_d\circ T_F, (S_h(q)-u_d\circ T_F)\,(\operatorname{trace}(\mathrm{D}\delta F)\operatorname{trace}(\mathrm{D}\tau F)-\operatorname{trace}(\mathrm{D}\delta F\cdot\mathrm{D}\tau F))\right)_h\\
&+\Big(S_h'(q)(\delta q)-(\nabla u_d\circ T_F)^T\cdot\delta F, \Big(S_h'(q)(\tau q)-(\nabla u_d\circ T_F)^T\cdot\tau F\Big)\,\gamma_F\Big)_h\\
&+\Big(S_h(q)-u_d\circ T_F, \Big(S_h'(q)(\delta q)-(\nabla u_d\circ T_F)^T\cdot\delta F\Big)\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\tau F\big)\Big)_h\\
&+\Big(S_h(q)-u_d\circ T_F, \Big(S_h'(q)(\tau q)-(\nabla u_d\circ T_F)^T\cdot\tau F\Big)\operatorname{div}\big(\gamma_F\,\mathrm{D}T_F^{-1}\cdot\delta F\big)\Big)_h\\
&+\big(S_h(q)-u_d\circ T_F, \big(S_h''(q)(\delta q,\tau q)-\tau F^T\cdot\nabla^2 u_d\circ T_F\cdot\delta F\big)\,\gamma_F\big)_h+\alpha\,(\delta q,\tau q)_{H^2(I)}\,.
\end{aligned}
$$

Definition 2.3.8 (The fully discretized cost functional)**.**

$$
\begin{aligned}
j_{h,k}(q) &= \frac{1}{2}\left(S_{h,k}(q)-u_d\circ T_{F_k}, (S_{h,k}(q)-u_d\circ T_{F_k})\,\gamma_{F_k}\right)_h+\frac{\alpha}{2}\|q\|_{H^2(I)}^2\,,\\
j_{h,k}'(q)(\delta q) &= \frac{1}{2}\Big(S_{h,k}(q)-u_d\circ T_{F_k}, (S_{h,k}(q)-u_d\circ T_{F_k})\operatorname{div}\Big(\gamma_{F_k}\,\mathrm{D}T_{F_k}^{-1}\cdot\delta F_k\Big)\Big)_h\\
&\quad+\Big(S_{h,k}'(q)(\delta q)-(\nabla u_d\circ T_{F_k})^T\cdot\delta F_k, (S_{h,k}(q)-u_d\circ T_{F_k})\,\gamma_{F_k}\Big)_h+\alpha\,(q,\delta q)_{H^2(I)}\,,
\end{aligned}
$$

$$
\begin{aligned}
&j_{h,k}''(q)(\delta q,\tau q)\\
&= \frac{1}{2}\left(S_{h,k}(q)-u_d\circ T_{F_k}, (S_{h,k}(q)-u_d\circ T_{F_k})\,(\operatorname{trace}(\mathrm{D}\delta F_k)\operatorname{trace}(\mathrm{D}\tau F_k)-\operatorname{trace}(\mathrm{D}\delta F_k\cdot\mathrm{D}\tau F_k))\right)_h\\
&+\Big(S_{h,k}'(q)(\delta q)-(\nabla u_d\circ T_{F_k})^T\cdot\delta F_k, \Big(S_{h,k}'(q)(\tau q)-(\nabla u_d\circ T_{F_k})^T\cdot\tau F_k\Big)\,\gamma_{F_k}\Big)_h\\
&+\Big(S_{h,k}(q)-u_d\circ T_{F_k}, \Big(S_{h,k}'(q)(\delta q)-(\nabla u_d\circ T_{F_k})^T\cdot\delta F_k\Big)\operatorname{div}\Big(\gamma_{F_k}\,\mathrm{D}T_{F_k}^{-1}\cdot\tau F_k\Big)\Big)_h\\
&+\Big(S_{h,k}(q)-u_d\circ T_{F_k}, \Big(S_{h,k}'(q)(\tau q)-(\nabla u_d\circ T_{F_k})^T\cdot\tau F_k\Big)\operatorname{div}\Big(\gamma_{F_k}\,\mathrm{D}T_{F_k}^{-1}\cdot\delta F_k\Big)\Big)_h\\
&+\big(S_{h,k}(q)-u_d\circ T_{F_k}, \big(S_{h,k}''(q)(\delta q,\tau q)-\tau F_k^T\cdot\nabla^2 u_d\circ T_{F_k}\cdot\delta F_k\big)\,\gamma_{F_k}\big)_h+\alpha\,(\delta q,\tau q)_{H^2(I)}\,.
\end{aligned}
$$

2.3.1. General stability estimates

This subsection is devoted to some general stability results which will be needed throughout in this section.

Lemma 2.3.9. *For $F, E\in\mathcal{F}^{\mathrm{ad}}$ it holds that*

$$
\begin{aligned}
|a(F)(u,v)-a(E)(u,v)| &\le c_\varepsilon\,\|F-E\|_{H^{2+\varepsilon}(\Omega_0)}\,\|u\|_{H^1(\Omega_0)}\,\|v\|_{H^1(\Omega_0)}\,,\\
|a(F)(u,v)-a(E)(u,v)| &\le c_\varepsilon\,\|F-E\|_{H^{3/2+\varepsilon}(\Omega_0)}\,\|u\|_{W^{1,4-\varepsilon}(\Omega_0)}\,\|v\|_{H^1(\Omega_0)}\,,\\
|a(F)(u,v)-a(E)(u,v)| &\le c_\varepsilon\,\|F-E\|_{H^{3/2+\varepsilon}(\Omega_0)}\,\|u\|_{H^{3/2-\varepsilon}(\Omega_0)}\,\|v\|_{H^1(\Omega_0)}\,.
\end{aligned}
$$

Proof. For the first part it holds that

$$
\begin{aligned}
|a(F)(u,v)-a(E)(u,v)| &\le \|u\|_{H^1(\Omega_0)}\,\|v\|_{H^1(\Omega_0)}\Big(\|A_F-A_E\|_{L^\infty(\Omega_0)}+\|\gamma_F-\gamma_E\|_{L^\infty(\Omega_0)}\Big)\\
&\le c\,\|u\|_{H^1(\Omega_0)}\,\|v\|_{H^1(\Omega_0)}\,\|F-E\|_{W^{1,\infty}(\Omega_0)}\\
&\le c_\varepsilon\,\|u\|_{H^1(\Omega_0)}\,\|v\|_{H^1(\Omega_0)}\,\|F-E\|_{H^{2+\varepsilon}(\Omega_0)}\,.
\end{aligned}
$$

Using the same approach we get

$$\begin{aligned}|a(F)(u,v)-a(E)(u,v)| &\le \|u\|_{W^{1,4-\varepsilon}(\Omega_0)}\|v\|_{H^1(\Omega_0)}\left(\|A_F-A_E\|_{L^{\frac{8-2\varepsilon}{2-\varepsilon}}(\Omega_0)}+\|\gamma_F-\gamma_E\|_{L^{\frac{8-2\varepsilon}{2-\varepsilon}}(\Omega_0)}\right)\\ &\le c\,\|u\|_{W^{1,4-\varepsilon}(\Omega_0)}\|v\|_{H^1(\Omega_0)}\|F-E\|_{W^{1,\frac{8-2\varepsilon}{2-\varepsilon}}(\Omega_0)}\\ &\le c_\varepsilon\,\|u\|_{W^{1,4-\varepsilon}(\Omega_0)}\|v\|_{H^1(\Omega_0)}\|F-E\|_{H^{3/2+\varepsilon}(\Omega_0)}\,,\end{aligned}$$

where we used the fact that $\lim_{\varepsilon\to 0^+}\frac{8-2\varepsilon}{2-\varepsilon}=4^+$. The last assertion follows from the continuous embedding $H^{3/2-\varepsilon_1}(\Omega)\hookrightarrow W^{1,4-\varepsilon_2}(\Omega)$, where $\varepsilon_2=\varepsilon_2(\varepsilon_1)>0$ can be made arbitrarily small, depending on the choice of $\varepsilon_1>0$. □

Remark 2.3.10. We would like to mention that, with a slight abuse of notation, the ε within the second and third line of the statement of Lemma 2.3.9, need not be the same (even within one line), but both can be made arbitrarily small.

Lemma 2.3.11. *For $F\in\mathcal{F}^{\mathrm{ad}}$, $F_k\in\mathcal{F}_k^{\mathrm{ad}}$ and $p\in[2,4]$ it holds that*

$$\begin{aligned}|a_h(F)(u,v)-a_h(F_k)(u,v)| &\le c\,\|F-F_k\|_{W^{1,p}(\Omega_0)}\|u\|_{W^{1,\frac{2p}{p-2}}(\Omega_{0,h})}\|v\|_{H^1(\Omega_{0,h})}\,,\\ |a_h(F)(u,v)-a_h(F_k)(u,v)| &\le c_\varepsilon\,\|F-F_k\|_{W^{1,4+\varepsilon}(\Omega_0)}\|u\|_{H^{3/2-\varepsilon}(\Omega_{0,h})}\|v\|_{H^1(\Omega_{0,h})}\,,\\ |a_h(F)(u,v)-a_h(F_k)(u,v)| &\le c_\varepsilon\,\|F-F_k\|_{W^{1,4+\varepsilon}(\Omega_0)}\|u\|_{W^{1,4-\varepsilon}(\Omega_{0,h})}\|v\|_{H^1(\Omega_{0,h})}\,.\end{aligned}$$

Proof. The proof is similar to the proof of Lemma 2.3.9. □

Lemma 2.3.12. *For $F,E\in\mathcal{F}^{\mathrm{ad}}$ and $v\in L^2(\Omega_0)$ it holds that*

$$|l(F)(v)-l(E)(v)|\le c\,\|F-E\|_{H^1(\Omega_0)}\|v\|_{L^2(\Omega_0)}\,.$$

Proof. It holds that

$$\begin{aligned}&|l(F)(v)-l(E)(v)|\\ &\le c\,\|v\|_{L^2(\Omega_0)}\Big(\|f\circ T_F\|_{L^\infty(\Omega_0)}\|\gamma_F-\gamma_E\|_{L^2(\Omega_0)}+\|\gamma_E\|_{L^\infty(\Omega_0)}\|f\circ T_F-f\circ T_E\|_{L^2(\Omega_0)}\Big)\\ &\le c\,\|v\|_{L^2(\Omega_0)}\|F-E\|_{H^1(\Omega_0)}\,,\end{aligned}$$

where in the last step we used the Lipschitz continuity of f and the boundedness of $\mathcal{F}^{\mathrm{ad}}$ in $H^{5/2}(\Omega_0)$. □

Lemma 2.3.13. *For $F\in\mathcal{F}^{\mathrm{ad}}$ and $F_k\in\mathcal{F}_k^{\mathrm{ad}}$ it holds that*

$$|l_h(F)(v)-l_h(F_k)(v)|\le c\,\|F-F_k\|_{H^1(\Omega_0)}\|v\|_{L^2(\Omega_{0,h})}\,.$$

Proof. The proof is similar to the proof of Lemma 2.3.12. □

Lemma 2.3.14. *For $F\in\mathcal{F}^{\mathrm{ad}}$ it holds that*

$$\begin{aligned}\|f\circ T_F\|_{C^{1,1/2}(\overline{\Omega_0})}&\le c, &\|\gamma_F\|_{H^{3/2}(\Omega_0)}&\le c,\\ \|u_d\circ T_F\|_{C^{1,1/2}(\overline{\Omega_0})}&\le c, &\|\mathrm{D}T_F^{-1}\|_{H^{3/2}(\Omega_0)}&\le c,\\ \|T_F\|_{H^{5/2}(\Omega_0)}&\le c.\end{aligned}$$

Proof. The estimates for f and u_d follow from the fact that $f, u_d \in C^2(\hat{\Omega})$ and the fact that $\mathcal{F}^{\mathrm{ad}}$ is bounded in $H^{5/2}(\Omega_0) \hookrightarrow C^{1,1/2}(\overline{\Omega_0})$. This last fact also proves the estimate for T_F, which, in combination with Theorem A.1.5 and Lemma 2.1.4, also proves the remaining two estimates. □

Lemma 2.3.15. *For $F, E \in \mathcal{F}^{\mathrm{ad}}$ it holds that*

$$
\begin{aligned}
\|f \circ T_F - f \circ T_E\|_{L^p(\Omega_0)} &\leq c\,\|F - E\|_{L^p(\Omega_0)} && \text{for } p \in [1,\infty],\\
\|\nabla f \circ T_F - \nabla f \circ T_E\|_{L^p(\Omega_0)} &\leq c\,\|F - E\|_{L^p(\Omega_0)} && \text{for } p \in [1,\infty],\\
\left\|\nabla^2 f \circ T_F - \nabla^2 f \circ T_E\right\|_{L^p(\Omega_0)} &\leq c\,\|F - E\|_{L^p(\Omega_0)} && \text{for } p \in [1,\infty],\\
\|u_d \circ T_F - u_d \circ T_E\|_{L^p(\Omega_0)} &\leq c\,\|F - E\|_{L^p(\Omega_0)} && \text{for } p \in [1,\infty],\\
\|\nabla u_d \circ T_F - \nabla u_d \circ T_E\|_{L^p(\Omega_0)} &\leq c\,\|F - E\|_{L^p(\Omega_0)} && \text{for } p \in [1,\infty],\\
\left\|\nabla^2 u_d \circ T_F - \nabla^2 u_d \circ T_E\right\|_{L^p(\Omega_0)} &\leq c\,\|F - E\|_{L^p(\Omega_0)} && \text{for } p \in [1,\infty],\\
\|T_F - T_E\|_{H^s(\Omega_0)} &= \|F - E\|_{H^s(\Omega_0)} && \text{for } s \geq 0,\\
\|\mathrm{D}T_F - \mathrm{D}T_E\|_{H^s(\Omega_0)} &\leq \|F - E\|_{H^{s+1}(\Omega_0)} && \text{for } s \geq 0,\\
\|\gamma_F - \gamma_E\|_{H^s(\Omega_0)} &\leq c_s\,\|F - E\|_{H^{s+1}(\Omega_0)} && \text{for } s \in [0, 3/2],\\
\left\|\mathrm{D}T_F^{-1} - \mathrm{D}T_E^{-1}\right\|_{H^s(\Omega_0)} &\leq c_s\,\|F - E\|_{H^{s+1}(\Omega_0)} && \text{for } s \in [0, 3/2].
\end{aligned}
$$

Proof. As f is Lipschitz continuous, we get

$$
\begin{aligned}
\|f \circ T_F - f \circ T_E\|_{L^p(\Omega_0)}^p &= \int_{\Omega_0} |f \circ T_F - f \circ T_E|^p \,\mathrm{d}x\\
&\leq c \int_{\Omega_0} |T_F - T_E|^p \,\mathrm{d}x\\
&= c\,\|F - E\|_{L^p(\Omega_0)}^p\,,
\end{aligned}
$$

for $p < \infty$, and it is clear that this statement also holds for $p = \infty$. The proof for the estimates of the higher derivatives as well as the proof for the estimates related to u_d follow in an analog way. What is left follows from Lemma 2.3.14 and Theorem A.1.5. □

Lemma 2.3.16. *For $F \in \mathcal{F}^{\mathrm{ad}}$ and $\delta F, \tau F \in \mathcal{F}$ it holds that*

$$
\begin{aligned}
\|\gamma_F\|_{H^{3/2}(\Omega_0)} &\leq c,\\
\left\|\gamma'_{F,\delta F}\right\|_{H^s(\Omega_0)} &\leq c_s\,\|\delta F\|_{H^{s+1}(\Omega_0)} && \text{for } s \in [0, 3/2]\,,\\
\left\|\gamma''_{F,\delta F,\tau F}\right\|_{H^s(\Omega_0)} &\leq c_s\,\|\delta F\|_{H^{s+1}(\Omega_0)}\,\|\tau F\|_{H^{s+1}(\Omega_0)} && \text{for } s > 1.
\end{aligned}
$$

Proof. The first part follows from Lemma 2.3.14, the second and the third part follow from the representations obtained in Lemma 2.1.6 and Theorem A.1.5. □

Lemma 2.3.17. *For $F, E \in \mathcal{F}^{\mathrm{ad}}$ and $\delta F, \tau F \in \mathcal{F}$ it holds that*

$$
\begin{aligned}
\|\gamma_F - \gamma_E\|_{H^s(\Omega_0)} &\leq c_s\,\|F - E\|_{H^{s+1}(\Omega_0)} && \text{for } s \in [0, 3/2]\,,\\
\left\|\gamma'_{F,\delta F} - \gamma'_{E,\delta F}\right\|_{H^s(\Omega_0)} &\leq c_s\,\|F - E\|_{H^{s+1}(\Omega_0)}\,\|\delta F\|_{H^{s+1}(\Omega_0)} && \text{for } s > 1,\\
\gamma''_{F,\delta F,\tau F} &= \gamma''_{E,\delta F,\tau F}.
\end{aligned}
$$

Proof. The first two parts of this lemma follow from the boundedness of $\mathcal{F}^{\text{ad}}$ in $H^{5/2}(\Omega_0)$ and Theorem A.1.5, the last part follows from the fact that $F \mapsto \gamma_F$ is a quadratic function. □

Lemma 2.3.18. *For $F \in \mathcal{F}^{\text{ad}}$ and $\delta F, \tau F \in \mathcal{F}$ it holds that*

$$\begin{aligned} \|A_F\|_{H^{3/2}(\Omega_0)} &\leq c, \\ \left\|A'_{F,\delta F}\right\|_{H^s(\Omega_0)} &\leq c_s \|\delta F\|_{H^{s+1}(\Omega_0)} && \text{for } s \in [0,3/2]\,, \\ \left\|A''_{F,\delta F,\tau F}\right\|_{H^s(\Omega_0)} &\leq c_s \|\delta F\|_{H^{s+1}(\Omega_0)} \|\tau F\|_{H^{s+1}(\Omega_0)} && \text{for } s \in (1,3/2]\,. \end{aligned}$$

Proof. The first part follows from Lemma 2.3.14 and Theorem A.1.5, the second and the third part of this lemma follow from the first part of this lemma and Lemma 2.1.6. □

Lemma 2.3.19. *For $F, E \in \mathcal{F}^{\text{ad}}$ and $\delta F, \tau F \in \mathcal{F}$ it holds that*

$$\begin{aligned} \|A_F - A_E\|_{H^s(\Omega_0)} &\leq c_s \|F - E\|_{H^{s+1}(\Omega_0)} && \text{for } s \in [0,3/2]\,, \\ \left\|A'_{F,\delta F} - A'_{E,\delta F}\right\|_{H^s(\Omega_0)} &\leq c_s \|F - E\|_{H^{s+1}(\Omega_0)} \|\delta F\|_{H^{s+1}(\Omega_0)} && \text{for } s \in (1,3/2]\,, \\ \left\|A''_{F,\delta F,\tau F} - A''_{E,\delta F,\tau F}\right\|_{H^s(\Omega_0)} &\leq c_s \|F - E\|_{H^{s+1}(\Omega_0)} \|\delta F\|_{H^{s+1}(\Omega_0)} \|\tau F\|_{H^{s+1}(\Omega_0)} && \text{for } s \in (1,3/2]\,. \end{aligned}$$

Proof. The first part of this lemma follows from Lemma 2.3.14, Lemma 2.3.15, Lemma 2.3.18 and Theorem A.1.5. The second and the third part follow from the first part and the references cited therein. □

2.3.2. A-priori error estimates for a general control

Within this subsection we are going to estimate the error between the continuous state and its (partially) discretized counterparts. Due to the low regularity of the matrix A_F for general $F \in \mathcal{F}^{\text{ad}}$, these estimates do not yield convergence of optimal order with respect to h and k. However, these estimates are crucial in order to prove optimality conditions of second order and the existence of the converging subsequences as stated in Theorem 2.3.1, cf. Subsection 2.3.4 and Subsection 2.3.5.

2.3.2.1. Estimates within the purely continuous case

Lemma 2.3.20. *For $q \in Q^{\text{ad}}$, $\delta q \in Q$ and $p \in (1,\infty)$ it holds that*

$$\begin{aligned} \|S(q)\|_{W^{1,p}(\Omega_0)} &\leq c_p, & \|S(q)\|_{H^{3/2-\varepsilon}(\Omega_0)} &\leq c_\varepsilon, \\ \left\|S'(q)(\delta q)\right\|_{W^{1,p}(\Omega_0)} &\leq c_{\varepsilon,p} \|\delta q\|_{H^{3/2+\varepsilon}(I)}\,, & \left\|S'(q)(\delta q)\right\|_{H^1_0(\Omega_0)} &\leq c_\varepsilon \|\delta q\|_{H^{1+\varepsilon}(I)}\,, \\ \left\|S''(q)(\delta q, \delta q)\right\|_{W^{1,p}(\Omega_0)} &\leq c_{\varepsilon,p} \|\delta q\|^2_{H^{3/2+\varepsilon}(I)}\,. \end{aligned}$$

Proof. The estimate for $S(q)$ in the $W^{1,p}$-norm follows from Theorem A.1.31. Let $F = F(q)$, in order to estimate $S'(q)(\delta q)$ in $W^{1,p}$ note that

$$\begin{aligned} \left|\left(\nabla S(q), A'_{F,\delta F} \cdot \nabla v\right)\right| &\leq \|\nabla S(q)\|_{W^{1,p}(\Omega_0)} \left\|A'_{F,\delta F}\right\|_{L^\infty(\Omega_0)} \|\nabla v\|_{L^q(\Omega_0)} \\ &\leq c_\varepsilon \|S(q)\|_{W^{1,p}(\Omega_0)} \left\|A'_{F,\delta F}\right\|_{H^{1+\varepsilon}(\Omega_0)} \|v\|_{W^{1,q}(\Omega_0)} \\ &\leq c_{\varepsilon,p} \|\delta F\|_{H^{2+\varepsilon}(\Omega_0)} \|v\|_{W^{1,q}(\Omega_0)} \\ &\leq c_{\varepsilon,p} \|\delta q\|_{H^{3/2+\varepsilon}(I)} \|v\|_{W^{1,q}(\Omega_0)}\,, \end{aligned}$$

where we used Lemma 2.3.18 and Theorem A.1.28. In the same way it holds that

$$\left|\left(\nabla S'(q)(\delta q), A'_{F,\delta F} \cdot \nabla v\right)\right| \le c_{\varepsilon,p} \left\|\delta q\right\|^2_{H^{3/2+\varepsilon}(I)} \left\|v\right\|_{W^{1,q}(\Omega_0)},$$

and

$$\left|\left(\nabla S(q), A''_{F,\delta F,\delta F} \cdot \nabla v\right)\right| \le c_{\varepsilon,p} \left\|\delta q\right\|^2_{H^{3/2+\varepsilon}(I)} \left\|v\right\|_{W^{1,q}(\Omega_0)},$$

which proves the estimate for $S''(q)(\delta q, \delta q)$.

As $A_F \in C^{0,1/2}(\overline{\Omega_0})$ we can apply Theorem A.1.30, which yields

$$\left\|S(q)\right\|_{H^{3/2-\varepsilon}(\Omega_0)} \le c_\varepsilon \left\|f \circ T_F \gamma_F\right\|_{L^2(\Omega_0)} \le c_\varepsilon.$$

Using the same theorem, Definition 2.3.2 and Theorem A.1.28 it holds that

$$\begin{aligned}
c \left\|S'(q)(\delta q)\right\|^2_{H^1_0(\Omega_0)} &\le a(F)(S'(q)(\delta q), S'(q)(\delta q)) \\
&\le c \left\|f \circ T_F\right\|_{L^\infty(\Omega_0)} \left\|S'(q)(\delta q)\right\|_{H^1_0(\Omega_0)} \left|\gamma_F \, \mathrm{D}T_{F_q}^{-1} \cdot \delta F\right|_{H^1(\Omega_0)} \\
&\quad + c \left\|\nabla f \circ T_F\right\|_{L^\infty(\Omega_0)} \left\|\delta F\right\|_{L^2(\Omega_0)} \left\|S'(q)(\delta q)\right\|_{H^1_0(\Omega_0)} \left\|\gamma_F\right\|_{L^\infty(\Omega_0)} \\
&\quad + c \left\|S(q)\right\|_{W^{1,4}(\Omega_0)} \left\|A'_{F,\delta F}\right\|_{L^4(\Omega_0)} \left\|S'(q)(\delta q)\right\|_{H^1_0(\Omega_0)} \\
&\quad + c \left\|S(q)\right\|_{L^\infty(\Omega_0)} \left\|S'(q)(\delta q)\right\|_{H^1_0(\Omega_0)} \left|\gamma_F \, \mathrm{D}T_{F_q}^{-1} \cdot \delta F\right|_{H^1(\Omega_0)} \\
&\le c \left\|S'(q)(\delta q)\right\|_{H^1_0(\Omega_0)} \left\|\delta F\right\|_{H^{3/2}(\Omega_0)} \\
&\le c_\varepsilon \left\|S'(q)(\delta q)\right\|_{H^1_0(\Omega_0)} \left\|\delta q\right\|_{H^{1+\varepsilon}(I)}.
\end{aligned}$$

□

Lemma 2.3.21. *For $q, p \in Q^{\mathrm{ad}}$ it holds that*

$$\begin{aligned}
\left\|S(q) - S(p)\right\|_{H^1_0(\Omega_0)} &\le c_\varepsilon \left\|q - p\right\|_{H^{1+\varepsilon}(I)}, \\
\left\|S(q) - S(p)\right\|_{L^2(\Omega_0)} &\le c \left\|q - p\right\|_{H^{1/2}(I)}.
\end{aligned}$$

Proof. Let $e = (S(q) - S(p))$, $F = F(q)$ and $E = F(p)$, then it holds that

$$\begin{aligned}
c \left\|e\right\|^2_{H^1_0(\Omega_0)} &\le a(F)(e, e) \\
&= a(F)(e, S(q)) - a(E)(e, S(p)) + a(E)(e, S(p)) - a(F)(e, S(p)) \\
&\le \left|l(F)(e) - l(E)(e)\right| + \left|a(E)(e, S(p)) - a(F)(e, S(p))\right| \\
&\le c \left\|F - E\right\|_{H^1(\Omega_0)} \left\|e\right\|_{L^2(\Omega_0)} + c_\varepsilon \left\|F - E\right\|_{H^{3/2+\varepsilon}(\Omega_0)} \left\|S(p)\right\|_{H^{3/2-\varepsilon}(\Omega_0)} \left\|e\right\|_{H^1_0(\Omega_0)} \\
&\le c_\varepsilon \left\|F - E\right\|_{H^{3/2+\varepsilon}(\Omega_0)} \left\|e\right\|_{H^1_0(\Omega_0)} \\
&\le c_\varepsilon \left\|q - p\right\|_{H^{1+\varepsilon}(\Omega_0)} \left\|e\right\|_{H^1_0(\Omega_0)},
\end{aligned}$$

where we used Lemma 2.3.9, Lemma 2.3.12, Lemma 2.3.20 and Lemma 2.1.2. Now let $z \in H^1_0(\Omega_0)$ solve

$$a(F)(v, z) = (e, v) \qquad \forall v \in H^1_0(\Omega_0).$$

It holds that $\|z\|_{H^{3/2-\varepsilon}(\Omega_0)} \leq c_\varepsilon \|e\|_{L^2(\Omega_0)}$, and we get

$$\begin{aligned} \|e\|^2_{L^2(\Omega_0)} &= a(F)(e,z) \\ &\leq |l(F)(z) - l(E)(z)| + |a(E)(z,S(p)) - a(F)(z,S(p))| \,. \end{aligned} \tag{2.75}$$

We are now going to estimate each of the two terms on the right hand side of (2.75) separately. First we get

$$\begin{aligned} |l(F)(z) - l(E)(z)| &\leq c\,\|F-E\|_{H^1(\Omega_0)} \|z\|_{L^2(\Omega_0)} \\ &\leq c\,\|q-p\|_{H^{1/2}(I)} \|e\|_{L^2(\Omega_0)} \,, \end{aligned}$$

and for the second term it holds that

$$|a(E)(z,S(p)) - a(F)(z,S(p))| \leq c_\varepsilon \|\nabla z\|_{L^3(\Omega_0)} \|\nabla S(p)\|_{L^6(\Omega_0)} \Big(\|A_F - A_E\|_{L^2(\Omega_0)} + \|\gamma_F - \gamma_E\|_{L^2(\Omega_0)} \Big) \,.$$

With Lemma 2.3.17 and Lemma 2.3.19 it follows that

$$\begin{aligned} \|A_F - A_E\|_{L^2(\Omega_0)} + \|\gamma_F - \gamma_E\|_{L^2(\Omega_0)} &\leq c\,\|F-E\|_{H^1(\Omega_0)} \\ &\leq c\,\|q-p\|_{H^{1/2}(I)} \,, \end{aligned}$$

and we finish the proof with Lemma 2.3.20 and the continuous embedding $H^{3/2-\varepsilon}(\Omega_0) \hookrightarrow W^{1,3}(\Omega_0)$ for $\varepsilon \leq 1/6$. □

The following technical lemma will be needed in the proof of Lemma 2.3.23.

Lemma 2.3.22. *Let $q, p \in Q^{\mathrm{ad}}$ with corresponding transformations F and E, respectively. Let $\delta q, \tau q \in Q$, $\delta F = F'(q)(\delta q)$, $\tau F = F'(q)(\tau q)$ and $v \in H^1_0(\Omega_0)$. Then it holds that*

$$\Big| l^\delta\left(F, \delta F, S(q)\right)(v) - l^\delta\left(E, \delta F, S(p)\right)(v) \Big| \leq c_\varepsilon \|q-p\|_{H^{3/2+\varepsilon}(I)} \|\delta q\|_{H^{3/2+\varepsilon}(\Omega_0)} \|v\|_{H^1_0(\Omega_0)} \,,$$

and

$$\begin{aligned} &\Big| l^{\delta,\tau}\big(F, \delta F, \tau F, S(q), S'(q)(\delta q), S'(q)(\tau q)\big)(v) - l^{\delta,\tau}\big(E, \delta F, \tau F, S(p), S'(p)(\delta q), S'(p)(\tau q)\big)(v) \Big| \\ &\leq c_\varepsilon \|q-p\|_{H^{3/2+\varepsilon}(I)} \|\delta q\|_{H^{3/2+\varepsilon}(I)} \|\tau q\|_{H^{3/2+\varepsilon}(I)} \|v\|_{H^1_0(\Omega_0)} \,. \end{aligned}$$

Proof. With Definition 2.3.5 it follows that

$$\begin{aligned} &\Big| l^\delta(F, \delta F, S(q))(v) - l^\delta(E, \delta F, S(p))(v) \Big| \\ &\leq \Big| \big(f \circ T_F, v \operatorname{div}\big(\gamma_F \,\mathrm{D}T_F^{-1} \cdot \delta F\big)\big) - \big(f \circ T_E, v \operatorname{div}\big(\gamma_E \,\mathrm{D}T_E^{-1} \cdot \delta F\big)\big) \Big| \\ &+ \Big| \Big((\nabla f \circ T_F)^T \cdot \delta F, v\gamma_F \Big) - \Big((\nabla f \circ T_E)^T \cdot \delta F, v\gamma_E \Big) \Big| \\ &+ \Big| \big(\nabla S(q), A'_{F,\delta F} \cdot \nabla v\big) - \big(\nabla S(p), A'_{E,\delta F} \cdot \nabla v\big) \Big| \\ &+ \Big| \big(S(q), v \operatorname{div}\big(\gamma_F \,\mathrm{D}T_F^{-1} \cdot \delta F\big)\big) - \big(S(p), v \operatorname{div}\big(\gamma_E \,\mathrm{D}T_E^{-1} \cdot \delta F\big)\big) \Big| \,, \end{aligned}$$

and the first estimate follows using the estimates within Subsection 2.3.1, Lemma 2.3.20 and Lemma 2.3.21. The second part can be proven using the same references as well as the first part of Lemma 2.3.23. □

Lemma 2.3.23. *For $q, p \in Q^{\text{ad}}$ and $\delta q \in Q$ it holds that*

$$\left\|S'(q)(\delta q) - S'(p)(\delta q)\right\|_{H_0^1(\Omega_0)} \le c_\varepsilon \|q-p\|_{H^{3/2+\varepsilon}(I)} \|\delta q\|_{H^{3/2+\varepsilon}(I)},$$
$$\left\|S''(q)(\delta q, \delta q) - S''(p)(\delta q, \delta q)\right\|_{H_0^1(\Omega_0)} \le c_\varepsilon \|q-p\|_{H^{3/2+\varepsilon}(I)} \|\delta q\|^2_{H^{3/2+\varepsilon}(I)}.$$

Proof. Let F and E be the respective transformations for q and p and let $e = (S'(q)(\delta q) - S'(p)(\delta q))$. It then holds that

$$\begin{aligned} c\|e\|^2_{H_0^1(\Omega_0)} &\le a(F)(e,e) \\ &\le \left|a(F)(S'(q)(\delta q), e) - a(E)(S'(p)(\delta q), e)\right| + \left|a(E)(S'(p)(\delta q), e) - a(F)(S'(p)(\delta q), e)\right|. \end{aligned}$$

With Lemma 2.3.9 and Lemma 2.3.20 we get

$$\begin{aligned} &\left|a(E)(S'(p)(\delta q), e) - a(F)(S'(p)(\delta q), e)\right| \\ &\le c_\varepsilon \|F-E\|_{H^{3/2+\varepsilon}(I)} \left\|S'(p)(\delta q)\right\|_{W^{1,4-\varepsilon}(\Omega_0)} \|e\|_{H_0^1(\Omega_0)} \\ &\le c_\varepsilon \|q-p\|_{H^{1+\varepsilon}(I)} \|\delta q\|_{H^{3/2+\varepsilon}(I)} \|e\|_{H_0^1(\Omega_0)}. \end{aligned} \tag{2.76}$$

Due to Definition 2.3.2 and Definition 2.3.5 it also holds that

$$\left|a(F)(S'(q)(\delta q), e) - a(E)(S'(p)(\delta q), e)\right| = \left|l^\delta(F, \delta F, S(q))(e) - l^\delta(E, \delta F, S(p))(e)\right|,$$

and the first part of this lemma follows with the first part of Lemma 2.3.22.

For the second part we define $d = (S''(q)(\delta q, \delta q) - S''(p)(\delta q, \delta q))$ and get

$$\begin{aligned} c\|d\|^2_{H_0^1(\Omega_0)} &\le a(F)(d,d) \\ &\le \left|a(F)(S''(q)(\delta q, \delta q), d) - a(E)(S''(p)(\delta q, \delta q), d)\right| \\ &\quad + \left|a(E)(S''(p)(\delta q, \delta q), d) - a(F)(S''(p)(\delta q, \delta q), d)\right|. \end{aligned} \tag{2.77}$$

The second part on the right hand side of (2.77) can be estimates similar to (2.76), and for the first part we can use the second part of Lemma 2.3.22. □

Lemma 2.3.24. *For $q, p \in Q^{\text{ad}}$ and $\delta q \in Q$ it holds that*

$$|j(q) - j(p)| \le c\|q-p\|_{H^2(I)},$$
$$\left|j'(q)(\delta q) - j'(p)(\delta q)\right| \le c\|q-p\|_{H^2(I)} \|\delta q\|_{H^2(I)},$$
$$\left|j''(q)(\delta q, \delta q) - j''(p)(\delta q, \delta q)\right| \le c\|q-p\|_{H^2(I)} \|\delta q\|^2_{H^2(I)}.$$

Proof. This lemma follows from Definition 2.3.6, Lemma 2.3.20, Lemma 2.3.21 and Lemma 2.3.23. □

2.3.2.2. Estimates between the continuous case and the state-discretized case

Within the following subsubsection we are going to estimate the error induced by the discretization of the state. If not stated otherwise, we will always assume that $h > 0$ is chosen sufficiently small.

Lemma 2.3.25. *For $q \in Q^{\text{ad}}$ and $\delta q \in Q$ it holds that*

$$\|S_h(q)\|_{H^{3/2-\varepsilon}(\Omega_{0,h})} \le c_\varepsilon,$$
$$\left\|S_h'(q)(\delta q)\right\|_{H^{3/2-\varepsilon}(\Omega_{0,h})} \le c_\varepsilon \|\delta q\|_{H^{3/2+\varepsilon}(I)},$$
$$\left\|S_h''(q)(\delta q, \delta q)\right\|_{H^{3/2-\varepsilon}(\Omega_{0,h})} \le c_\varepsilon \|\delta q\|^2_{H^{3/2+\varepsilon}(I)}.$$

Proof. The first part of this lemma follows from Corollary A.1.25 and Lemma 2.3.20. Let $F = F(q)$ and $\delta F = F'(q)(\delta q)$, in order to prove the second part let $\tilde{S}(q,\delta q) \in H_0^1(\Omega_0)$ be the solution to

$$a(F)(\tilde{S}(q,\delta q), v) = l^\delta(F, \delta F, S_h(q))(v) \qquad \forall v \in H_0^1(\Omega_0). \tag{2.78}$$

With Corollary A.1.25 it follows that

$$\left\| S_h'(q)(\delta q) \right\|_{H^{3/2-\varepsilon}(\Omega_{0,h})} \le c_\varepsilon \left\| \tilde{S}(q,\delta q) \right\|_{H^{3/2-\varepsilon}(\Omega_0)}, \tag{2.79}$$

and it remains to estimate the right hand side within (2.79). As $H_0^{1/2-\varepsilon}(\Omega_0) = H^{1/2-\varepsilon}(\Omega_0)$ due to [48], Theorem 1.4.2.4, it holds that

$$\begin{aligned} \left| \left(\nabla S_h(q), A'_{F,\delta F} \cdot \nabla v \right) \right| &\le \left\| A'_{F,\delta F} \cdot \nabla S_h(q) \right\|_{H^{1/2-\varepsilon}(\Omega_0)} \|\nabla v\|_{H^{-1/2+\varepsilon}(\Omega_0)} \\ &\le c_\varepsilon \|\nabla S_h(q)\|_{H^{1/2-\varepsilon}(\Omega_0)} \left\| A'_{F,\delta F} \right\|_{H^{1+\varepsilon}(\Omega_0)} \|v\|_{H^{1/2+\varepsilon}(\Omega_0)} \\ &\le c_\varepsilon \|\delta q\|_{H^{3/2+\varepsilon}(I)} \|v\|_{H^{1/2+\varepsilon}(\Omega_0)}. \end{aligned} \tag{2.80}$$

Hence the right hand side in (2.78) is an element of $H^{-1/2-\varepsilon}(\Omega_0)$, and the desired estimate for $\left\| \tilde{S}(q,\delta q) \right\|_{H^{3/2-\varepsilon}(\Omega_0)}$ follows with Definition 2.3.5 and Theorem A.1.30. In order to estimate the second derivative one defines $\tilde{S}(q,\delta q,\delta q) \in H_0^1(\Omega_0)$ as the solution to

$$a(F)(\tilde{S}(q,\delta q,\delta q), v) = l^{\delta,\tau}(F, \delta F, \delta F, S_h(q), S_h'(q)(\delta q), S_h'(q)(\delta q))(v) \qquad \forall v \in H_0^1(\Omega_0),$$

and then proceeds as in the proof of the second part of this lemma. □

Lemma 2.3.26. *For $q \in Q^{\text{ad}}$ it holds that*

$$\begin{aligned} \|S(q) - S_h(q)\|_{H^1(\Omega_{0,h})} &\le c_\varepsilon h^{1/2-\varepsilon}, \\ \|S(q) - S_h(q)\|_{L^2(\Omega_{0,h})} &\le c_\varepsilon h^{1-\varepsilon}. \end{aligned}$$

Proof. This lemma is a direct consequence of Theorem A.2.23. □

Lemma 2.3.27. *For $q \in Q^{\text{ad}}$ and $\delta q \in Q$ it holds that*

$$\begin{aligned} \left\| S'(q)(\delta q) - S_h'(q)(\delta q) \right\|_{H^1(\Omega_{0,h})} &\le c_\varepsilon h^{1/2-\varepsilon} \|\delta q\|_{H^{3/2+\varepsilon}(I)}, \\ \left\| S''(q)(\delta q,\delta q) - S_h''(q)(\delta q,\delta q) \right\|_{H^1(\Omega_{0,h})} &\le c_\varepsilon h^{1/2-\varepsilon} \|\delta q\|^2_{H^{3/2+\varepsilon}(I)}. \end{aligned}$$

Proof. As in the proof of Lemma 2.3.25 we have to introduce an intermediate solution. Let $F = F(q)$, $\delta F = F'(q)(\delta q)$ and let $\tilde{S}_h(q,\delta q) \in V_{h,0}$ be the solution to

$$a_h(F)(\tilde{S}_h(q,\delta q), v_h) = l^\delta(F, \delta F, S(q))(v_h) \qquad \forall v_h \in V_{h,0},$$

i.e. $\tilde{S}_h(q,\delta q)$ is the Ritz-projection of $S'(q)(\delta q)$. Now we split the error

$$\left\| S'(q)(\delta q) - S_h'(q)(\delta q) \right\|_{H_0^1(\Omega_{0,h})} \le \left\| S'(q)(\delta q) - \tilde{S}_h(q,\delta q) \right\|_{H_0^1(\Omega_{0,h})} + \left\| \tilde{S}_h(q,\delta q) - S_h'(q)(\delta q) \right\|_{H_0^1(\Omega_{0,h})}.$$

Using an estimate similar to (2.80) shows that we can apply Theorem A.2.23 and obtain

$$\left\| S'(q)(\delta q) - \tilde{S}_h(q,\delta q) \right\|_{H_0^1(\Omega_{0,h})} \le c_\varepsilon h^{1/2-\varepsilon} \|\delta q\|_{H^{3/2+\varepsilon}(I)}.$$

Now let $e_h = \left(\tilde{S}_h(q,\delta q) - S_h'(q,\delta q)\right)$, then $e_h \in V_{h,0}$ solves

$$\begin{aligned} a_h(F)(e_h, v_h) &= \left(\nabla\left(S_h(q) - S(q)\right), A'_{F,\delta F}\cdot\nabla v_h\right)_h \\ &\quad + \left(S_h(q) - S(q), v_h \operatorname{div}\left(\gamma_F\, \mathrm{D}T_F^{-1}\cdot\delta F\right)\right)_h \qquad \forall v_h \in V_{h,0}, \end{aligned}$$

and Lemma 2.3.26 yields

$$\begin{aligned} \|e_h\|_{H_0^1(\Omega_{0,h})} &\le c\,\|S(q) - S_h(q)\|_{H^1(\Omega_{0,h})}\left(\left\|A'_{F,\delta F}\right\|_{L^\infty(\Omega_0)} + \|\delta F\|_{H^1(\Omega_0)}\right) \\ &\le c_\varepsilon h^{1/2-\varepsilon}, \end{aligned}$$

which proves the first part of this lemma. The estimate for the second part can be proven using the same methods. □

Lemma 2.3.28. *For $q \in Q^{\mathrm{ad}}$ and $\delta q \in Q$ it holds that*

$$\begin{aligned} |j(q) - j_h(q)| &\le c_\varepsilon h^{1-\varepsilon}, \\ \left|j'(q)(\delta q) - j_h'(q)(\delta q)\right| &\le c_\varepsilon h^{1/2-\varepsilon}\,\|\delta q\|_{H^2(I)}, \\ \left|j''(q)(\delta q,\delta q) - j_h''(q)(\delta q,\delta q)\right| &\le c_\varepsilon h^{1/2-\varepsilon}\,\|\delta q\|_{H^2(I)}^2. \end{aligned}$$

Proof. First note that $|\Omega_0\backslash\Omega_{0,h}| \le ch^2$, and using Lemma 2.3.26 we get

$$\begin{aligned} |j(q) - j_h(q)| &= \frac{1}{2}\left|\int_{\Omega_0}\left(S(q) - u_d\circ T_F\right)^2\gamma_F\,\mathrm{d}x - \int_{\Omega_{0,h}}\left(S_h(q) - u_d\circ T_F\right)^2\gamma_F\,\mathrm{d}x\right| \\ &\le \frac{1}{2}\left|\int_{\Omega_0\backslash\Omega_{0,h}}\left(S(q) - u_d\circ T_F\right)^2\gamma_F\,\mathrm{d}x\right| \\ &\quad + \frac{1}{2}\left|\int_{\Omega_{0,h}}\left(\left(S(q) - u_d\circ T_F\right)^2 - \left(S_h(q) - u_d\circ T_F\right)^2\right)\gamma_F\,\mathrm{d}x\right| \\ &\le c\left(h^2 + \|S(q) - S_h(q)\|_{L^1(\Omega_{0,h})}\right) \\ &\le c_\varepsilon h^{1-\varepsilon}. \end{aligned}$$

The second and the third part of this lemma can be proven in the same way using Lemma 2.3.20 and Lemma 2.3.27. □

2.3.2.3. Estimates between the state-discretized case and the fully discretized case

Within this subsubsection we are going to estimate the error induced by the additional discretization of the transformation F. If not stated otherwise we will always assume that $k > 0$ is chosen sufficiently small.

Lemma 2.3.29. *For $q \in Q^{\mathrm{ad}}$, $F = F(q)$, $F_k = F_k(q)$ and $p \in [4,\infty]$ it holds that*

$$\|F - F_k\|_{W^{1,p}(\Omega_0)} \le ck^{\frac{1}{2}+\frac{2}{p}}\,\|q\|_{H^2(I)}.$$

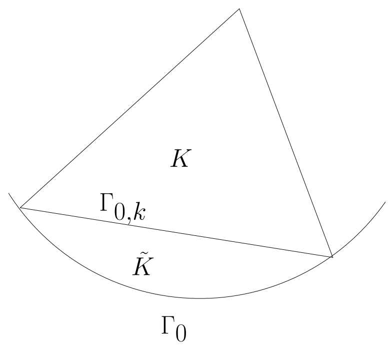

Figure 2.2.: The boundary triangle K and its extension $\tilde{K}$.

Proof. The case $p = 4$ is a direct consequence of Theorem A.2.1 and the continuous embedding $H^2(I) \hookrightarrow W^{7/4,4}(I)$.

Now let $i_k\colon C(\overline{\Omega_0}) \to V_k$ be the pointwise interpolation operator. With the triangulation π_k of $\Omega_{0,k}$ we may associate a triangulation $\tilde{\pi}_k$ of Ω_0, where all the boundary triangles K are replaced by their curved extension $\tilde{K}$, cf. Figure 2.2. Due to the smoothness of Ω_0 and $|\Omega_0 \backslash \Omega_{0,k}| \le ck^2$ it follows that the family $\{\tilde{\pi}_k\}_{k>0}$ also fulfills the usual regularity assumptions. Hence we may use an inverse inequality, it follows that

$$
\begin{aligned}
\|F - F_k\|_{W^{1,\infty}(\Omega_0)} &\le \|F - i_k F\|_{W^{1,\infty}(\Omega_0)} + \|i_k F - F_k\|_{W^{1,\infty}(\Omega_0)} \\
&\le ck^{1/2} \|F\|_{W^{3/2,\infty}(\Omega_0)} + ck^{-1/2} \|i_k F - F_k\|_{W^{1,4}(\Omega_0)} \\
&\le ck^{1/2} \|F\|_{W^{3/2,\infty}(\Omega_0)} + ck^{-1/2} \Big(\|F - i_k F\|_{W^{1,4}(\Omega_0)} + \|F - F_k\|_{W^{1,4}(\Omega_0)} \Big) \\
&\le ck^{1/2} \|F\|_{W^{3/2,\infty}(\Omega_0)} + ck^{-1/2} \Big(k \|F\|_{W^{2,4}(\Omega_0)} + k \|q\|_{H^2(I)} \Big) \\
&\le ck^{1/2} \|q\|_{H^2(I)} ,
\end{aligned}
$$

where we used the continuous embeddings $H^{5/2}(\Omega_0) \hookrightarrow W^{3/2,\infty}(\Omega_0)$, $H^{5/2}(\Omega_0) \hookrightarrow W^{2,4}(\Omega_0)$ and the case $p = 4$. What is left follows with interpolation. □

Lemma 2.3.30. *For $q \in Q^{\mathrm{ad}}$ and $\delta q \in Q$ it holds that*

$$
\begin{aligned}
\|S_{h,k}(q)\|_{H_0^1(\Omega_{0,h})} &\le c, \\
\left\|S'_{h,k}(q)(\delta q)\right\|_{H_0^1(\Omega_{0,h})} &\le c \|\delta q\|_{H^2(I)} , \\
\left\|S''_{h,k}(q)(\delta q, \delta q)\right\|_{H_0^1(\Omega_{0,h})} &\le c \|\delta q\|_{H^2(I)}^2 .
\end{aligned}
$$

Proof. Let $F = F(q)$ and $F_k = F_k(q)$. From Lemma 2.3.29 it follows that $A_{F_k} \to A_F$ in $L^\infty(\Omega_0)$ for $k \to 0$, hence the matrices $\left\{ A_{F_k(q)} \,\middle|\, q \in Q^{\mathrm{ad}} \right\}$ are uniformly elliptic for k sufficiently small. This lemma now follows from Definition 2.3.4. □

Lemma 2.3.31. *For $q \in Q^{\mathrm{ad}}$ it holds that*

$$
\|S_h(q) - S_{h,k}(q)\|_{H^1(\Omega_{0,h})} \le ck^{1/2}.
$$

Proof. Let $F = F(q)$, $F_k = F_k(q)$ and $e = (S_h(q) - S_{h,k}(q)) \in V_{h,0}$. We get

$$\begin{aligned}
\|e\|^2_{H^1(\Omega_{0,h})} &\leq a_h(F)(e,e) \\
&\leq |l_h(F)(e) - l_h(F_k)(e)| + |a_h(F_k)(S_{h,k}(q),e) - a_h(F)(S_{h,k}(q),e)| \\
&\leq c\left(\|F - F_k\|_{H^1(\Omega_0)} \|e\|_{L^2(\Omega_0)} + \|S_{h,k}(q)\|_{H^1(\Omega_{0,h})} \|F - F_k\|_{W^{1,\infty}(\Omega_0)} \|e\|_{H^1(\Omega_{0,h})}\right),
\end{aligned}$$

and using Lemma 2.3.29 and Lemma 2.3.30 we arrive at

$$\|e\|_{H^1(\Omega_{0,h})} \leq ck^{1/2}.$$

□

Lemma 2.3.32. *For $q \in Q^{\mathrm{ad}}$ and $\delta q \in Q$ it holds that*

$$\begin{aligned}
\left\|S'_h(q)(\delta q) - S'_{h,k}(q)(\delta q)\right\|_{H^1(\Omega_{0,h})} &\leq ck^{1/2} \|\delta q\|_{H^2(I)}, \\
\left\|S''_h(q)(\delta q, \delta q) - S''_{h,k}(q)(\delta q, \delta q)\right\|_{H^1(\Omega_{0,h})} &\leq ck^{1/2} \|\delta q\|^2_{H^2(I)}.
\end{aligned}$$

Proof. This lemma can be proven analogously to Lemma 2.3.31. □

Lemma 2.3.33. *For $q \in Q^{\mathrm{ad}}$ and $\delta q \in Q$ it holds that*

$$\begin{aligned}
\|S_{h,k}(q)\|_{H^{3/2-\varepsilon}(\Omega_{0,h})} &\leq c_\varepsilon, \\
\left\|S'_{h,k}(q)(\delta q)\right\|_{H^{3/2-\varepsilon}(\Omega_{0,h})} &\leq c_\varepsilon \|\delta q\|_{H^{3/2+\varepsilon}(I)}, \\
\left\|S''_{h,k}(q)(\delta q, \delta q)\right\|_{H^{3/2-\varepsilon}(\Omega_{0,h})} &\leq c_\varepsilon \|\delta q\|^2_{H^{3/2+\varepsilon}(I)}.
\end{aligned}$$

Proof. This lemma follows with Lemma 2.3.25, Lemma 2.3.31 and Lemma 2.3.32 as well as an inverse estimate, Theorem A.1.23. □

Lemma 2.3.34. *For $q \in Q^{\mathrm{ad}}$ and $\delta q \in Q$ it holds that*

$$\begin{aligned}
|j_h(q) - j_{h,k}(q)| &\leq ck^{1/2}, \\
\left|j'_h(q)(\delta q) - j'_{h,k}(q)(\delta q)\right| &\leq ck^{1/2} \|\delta q\|_{H^2(I)}, \\
\left|j''_h(q)(\delta q, \delta q) - j''_{h,k}(q)(\delta q, \delta q)\right| &\leq ck^{1/2} \|\delta q\|^2_{H^2(I)}.
\end{aligned}$$

Proof. This lemma follows with Definition 2.3.7, Definition 2.3.8 and the previous lemmata of this subsubsection. □

Lemma 2.3.35. *For $q, p \in Q^{\mathrm{ad}}$ and $\delta q \in Q$ it holds that*

$$\begin{aligned}
\|S_{h,k}(q) - S_{h,k}(p)\|_{H^1_0(\Omega_{0,h})} &\leq c\|p - q\|_{H^2(I)}, \\
\left\|S'_{h,k}(q)(\delta q) - S'_{h,k}(q)(\delta q)\right\|_{H^1_0(\Omega_{0,h})} &\leq c\|p - q\|_{H^2(I)} \|\delta q\|_{H^2(I)}.
\end{aligned}$$

Proof. Let $F_k = F_k(q)$, $E_k = F_k(p)$ and $e = (S_{h,k}(q) - S_{h,k}(p))$, then it holds that

$$\begin{aligned}
c\|e\|^2_{H^1_0(\Omega_{0,h})} &\leq a_h(F_k)(e,e) \\
&\leq |l_h(F_k)(e) - l_h(E_k)(e)| + |a_h(E_k)(S_{h,k}(p),e) - a_h(F_k)(S_{h,k}(p),e)| \\
&\leq c\left(\|F_k - E_k\|_{H^1(\Omega_0)} \|e\|_{L^2(\Omega_0)} + \|S_{h,k}(p)\|_{H^1(\Omega_{0,h})} \|F_k - E_k\|_{W^{1,\infty}(\Omega_0)} \|e\|_{H^1(\Omega_{0,h})}\right).
\end{aligned}$$

As $q \mapsto F_k(q)$ is linear, it follows with Lemma 2.3.29 that

$$\begin{aligned}\|F_k - E_k\|_{W^{1,\infty}(\Omega_0)} &\le c\,\|F(q-p)\|_{W^{1,\infty}(\Omega_0)}\\ &\le c\,\|F(q-p)\|_{H^{5/2}(\Omega_0)}\\ &\le c\,\|q-p\|_{H^2(I)}\,,\end{aligned}$$

and the first part of this lemma follows. The second part follows in a likewise manner. □

The following lemma can be proven by a direct calculation using Lemma 2.3.35.

Lemma 2.3.36. *For $q, p \in Q^{\mathrm{ad}}$ and $\delta q \in Q$ it holds that*

$$\left|j'_{h,k}(q)(\delta q) - j'_{h,k}(p)(\delta q)\right| \le c\,\|p-q\|_{H^2(I)}\,\|\delta q\|_{H^2(I)}\,.$$

2.3.3. A-priori error estimates for the optimal control

As already mentioned, due to the low regularity of the transformation matrix A_F for general $q \in Q^{\mathrm{ad}}$ it is not possible to generally show convergence rates of optimal order. However, as the optimal control has some improved regularity, it is possible to show optimal order of convergence in that case. In what follows let $\overline{q}$ be a fixed optimal solution to (2.27) with corresponding optimal state $S(\overline{q})$, optimal transformation $\overline{F} = F(\overline{q})$ and optimal discretized transformation $\overline{F}_k = F_k(\overline{q})$. We shortly summarize the known regularity.

Lemma 2.3.37. *It holds that $\overline{q} \in H^4(I)$, $S(\overline{q}) \in W^{2,\infty}(\Omega_0)$ and $\overline{F} \in H^{9/2}(\Omega_0)$.*

Proof. The regularity for $\overline{q}$ follows from Theorem 2.1.40, the regularity for $\overline{F}$ follows with Theorem A.1.28 and the regularity for $S(\overline{q})$ follows with [48], Theorem 6.3.2.1 and Remark 6.3.2.4. □

Lemma 2.3.38. *Let $p \in (1,\infty)$ and $s \in [0,1]$, then it holds that*

$$\left\|\overline{F} - \overline{F}_k\right\|_{W^{s,p}(\Omega_0)} \le c_p k^{2-s}.$$

Proof. With Lemma 2.3.37 it holds that $\overline{q} \in H^4(I) \hookrightarrow W^{2,\infty}(I)$, and this lemma follows with Theorem A.2.1. □

Lemma 2.3.39. *For $\delta q \in Q$ it holds that*

$$\left\|S'(\overline{q})(\delta q)\right\|_{W^{2,4}(\Omega_0)} \le c\,\|\delta q\|_{H^2(I)}\,.$$

Proof. Let $\delta F = F'(\overline{q})(\delta q)$, because of $S(\overline{q}) \in W^{2,\infty}(\Omega_0)$ it holds that

$$-\left(\nabla S(\overline{q}), A'_{\overline{F},\delta F}\cdot\nabla v\right) = \left(\operatorname{div}\left(A'_{\overline{F},\delta F}\cdot\nabla S(\overline{q})\right), v\right) \qquad \forall v \in C_0^\infty(\Omega_0).$$

As a result, the right hand side in Definition 2.3.2 for $S'(\overline{q})(\delta q)$ is a functional in $L^4(\Omega_0)$, and with Theorem A.1.38 it follows that

$$\begin{aligned}\left\|S'(\overline{q})(\delta q)\right\|_{W^{2,4}(\Omega_0)} &\le c\left(\left\|f\circ T_{\overline{F}}\operatorname{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right\|_{L^4(\Omega_0)} + \left\|\left(\nabla f\circ T_{\overline{F}}\right)^T\cdot\delta F\gamma_{\overline{F}}\right\|_{L^4(\Omega_0)}\right)\\ &\quad + c\left(\left\|\operatorname{div}\left(A'_{\overline{F},\delta F}\cdot\nabla S(\overline{q})\right)\right\|_{L^4(\Omega_0)} + \left\|S(\overline{q})\operatorname{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right\|_{L^4(\Omega_0)}\right)\\ &\le c\,\|\delta F\|_{W^{2,4}(\Omega_0)} \le c\,\|\delta F\|_{H^{5/2}(\Omega_0)}\\ &\le c\,\|\delta q\|_{H^2(I)}\,.\end{aligned}$$

□

2.3.3.1. Optimal estimates between the continuous case and the state-discretized case

Lemma 2.3.40. *For $\delta q \in Q$ and $s \in [0,1]$ it holds that*

$$\|S(\overline{q}) - S_h(\overline{q})\|_{H^s(\Omega_0)} \leq ch^{2-s}, \tag{2.81}$$

$$\left\|S'(\overline{q})(\delta q) - S_h'(\overline{q})(\delta q)\right\|_{H^s(\Omega_0)} \leq ch^{2-s} \|\delta q\|_{H^2(I)} . \tag{2.82}$$

Proof. Estimate (2.81) is a direct consequence of Theorem A.2.1. For the second part, (2.82), we follow an idea presented in [73]. Let $\delta F = F'(\overline{q})(\delta q)$ and let $\tilde{S}_h(\overline{q}, \delta q) \in V_{h,0}$ be the solution to

$$a_h(\overline{F})(\tilde{S}_h(\overline{q}, \delta q), v_h) = l_h^\delta(\overline{F}, \delta F, S(\overline{q}))(v_h) \qquad \forall v_h \in V_{h,0}.$$

As $\tilde{S}_h(\overline{q}, \delta q)$ is the Ritz-projection of $S'(\overline{q})(\delta q)$, Lemma 2.3.39 implies

$$\left\|S'(\overline{q})(\delta q) - \tilde{S}_h(\overline{q}, \delta q)\right\|_{H^s(\Omega_0)} \leq ch^{2-s} \|\delta q\|_{H^2(I)} ,$$

for $s \in [0,1]$. Now let $e = \left(\tilde{S}_h(\overline{q}, \delta q) - S_h'(\overline{q})(\delta q)\right) \in V_{h,0}$, then e solves

$$\begin{aligned} a_h(\overline{F})(e, v_h) = &\left(\nabla\left(S_h(\overline{q}) - S(\overline{q})\right), A'_{\overline{F},\delta F} \cdot \nabla v_h\right)_h \\ &+ \left(S_h(\overline{q}) - S(\overline{q}), v_h \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right)_h \qquad \forall v_h \in V_{h,0}, \end{aligned}$$

which yields

$$\begin{aligned} \|e\|_{H^1(\Omega_0)} &\leq \|S(\overline{q}) - S_h(\overline{q})\|_{H^1(\Omega_0)} \left(\left\|A'_{\overline{F},\delta F}\right\|_{L^\infty(\Omega_0)} + \left\|\operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right\|_{L^2(\Omega_0)}\right) \\ &\leq ch \|\delta q\|_{H^2(I)} . \end{aligned}$$

It remains to estimate the L^2-error. Let $z \in H_0^1(\Omega_0)$ and $z_h \in V_{h,0}$ be the solutions to

$$\begin{aligned} a(\overline{F})(v, z) &= (e, v)_h & \forall v \in H_0^1(\Omega_0), \\ a_h(\overline{F})(v_h, z_h) &= (e, v_h)_h & \forall v_h \in V_{h,0}. \end{aligned}$$

We have $z \in H^2(\Omega_0)$ and $\|z\|_{H^2(\Omega_0)} \leq c \|e\|_{L^2(\Omega_0)}$, it holds that

$$\begin{aligned} \|e\|^2_{L^2(\Omega_0)} &= a_h(\overline{F})(e, z_h) \\ &= \left(\nabla\left(S_h(\overline{q}) - S(\overline{q})\right), A'_{\overline{F},\delta F} \cdot \nabla z_h\right)_h + \left(S_h(\overline{q}) - S(\overline{q}), z_h \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right)_h \\ &= \left(\nabla\left(S_h(\overline{q}) - S(\overline{q})\right), A'_{\overline{F},\delta F} \cdot \nabla z\right)_h + \left(S_h(\overline{q}) - S(\overline{q}), z \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right)_h \\ &+ \left(\nabla\left(S_h(\overline{q}) - S(\overline{q})\right), A'_{\overline{F},\delta F} \cdot \nabla\left(z_h - z\right)\right)_h \\ &+ \left(S_h(\overline{q}) - S(\overline{q}), (z_h - z) \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right)_h . \end{aligned} \tag{2.83}$$

The estimate (2.81) and general finite element error estimates yield

$$\begin{aligned} \left|\left(S_h(\overline{q}) - S(\overline{q}), z \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right)_h\right| &\leq ch^2 \|\delta q\|_{H^2(I)} \|e\|_{L^2(\Omega_0)} , \\ \left|\left(\nabla\left(S_h(\overline{q}) - S(\overline{q})\right), A'_{\overline{F},\delta F} \cdot \nabla\left(z_h - z\right)\right)_h\right| &\leq ch^2 \|\delta q\|_{H^2(I)} \|e\|_{L^2(\Omega_0)} , \\ \left|\left(S_h(\overline{q}) - S(\overline{q}), (z_h - z) \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right)_h\right| &\leq ch^4 \|\delta q\|_{H^2(I)} \|e\|_{L^2(\Omega_0)} , \end{aligned}$$

and it remains to estimate the last part on the right hand side of (2.83). Using Theorem A.1.5 and Theorem A.2.1 it holds that

$$\begin{aligned}\left|\left(\nabla\left(S_h(\overline{q})-S(\overline{q})\right), A'_{\overline{F},\delta F}\cdot\nabla z\right)_h\right| &= \left|\left(S_h(\overline{q})-S(\overline{q}), \operatorname{div}\left(A'_{\overline{F},\delta F}\cdot\nabla z\right)\right)_h\right| \\ &\leq \|S_h(\overline{q})-S(\overline{q})\|_{L^4(\Omega_0)}\left\|A'_{\overline{F},\delta F}\cdot\nabla z\right\|_{W^{1,4/3}(\Omega_0)} \\ &\leq \|S_h(\overline{q})-S(\overline{q})\|_{L^4(\Omega_0)}\left\|A'_{\overline{F},\delta F}\right\|_{W^{1,4}(\Omega_0)}\|\nabla z\|_{H_0^1(\Omega_0)} \\ &\leq ch^2\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_0)},\end{aligned}$$

and the case $s \in (0,1)$ follows with interpolation. □

Lemma 2.3.41. *For $\delta q \in Q$ it holds that*

$$\left|j'(\overline{q})(\delta q) - j'_h(\overline{q})(\delta q)\right| \leq ch^2\|\delta q\|_{H^2(I)}.$$

Proof. The proof for this lemma is similar to the proof of Lemma 2.3.28, the improved order of convergence is due to Lemma 2.3.40. □

2.3.3.2. Optimal estimates between the state-discretized case and the fully discretized case

In what follows we are going to prove a better convergence rate with respect to k, which will be done using Taylor's theorem.

Lemma 2.3.42. *The mappings*

$$\begin{aligned}\gamma\colon W^{1,\infty}(\Omega_0) &\to L^\infty(\Omega_0), \\ \gamma(F) &= \gamma_F,\end{aligned} \qquad \text{and} \qquad \begin{aligned}A\colon W^{1,\infty}(\Omega_0) &\to L^\infty(\Omega_0), \\ A(F) &= A_F,\end{aligned}$$

are at least three times continuously Fréchet-differentiable.

Proof. This lemma follows from Theorem A.1.10, Theorem A.1.11 and Theorem A.1.13. □

Lemma 2.3.43. *Let $p \in (1,\infty)$, then it holds that*

$$\|\overline{F}-\overline{F}_k\|_{L^p(\Gamma_{0,k})} \leq c_p k^2.$$

Proof. Let $\widetilde{\overline{q}\,n}$ be defined as in (A.24) and let the operator Q_k be defined as in (A.34). It holds that

$$\|\overline{F}-\overline{F}_k\|_{L^p(\Gamma_{0,k})} \leq \left\|\overline{F}-\widetilde{\overline{q}\,n}\right\|_{L^p(\Gamma_{0,k})} + \left\|\widetilde{\overline{q}\,n}-\overline{F}_k\right\|_{L^p(\Gamma_{0,k})}.$$

As $\overline{F}\big|_{\Gamma_0} = \overline{q}\,n$ we may use Lemma A.2.3 to obtain

$$\left\|\overline{F}-\widetilde{\overline{q}\,n}\right\|_{L^p(\Gamma_{0,k})} \leq c_p k^2.$$

In addition it holds that $\overline{F}_k\big|_{\Gamma_{0,k}} = Q_k(\widetilde{\overline{q}\,n})$, hence

$$\left\|\widetilde{\overline{q}\,n}-\overline{F}_k\right\|_{L^p(\Gamma_{0,k})} = \left\|\widetilde{\overline{q}\,n}-Q_k(\widetilde{\overline{q}\,n})\right\|_{L^p(\Gamma_{0,k})},$$

and the result follows with standard interpolation results. □

Lemma 2.3.44. *Let $p \in (1,\infty)$, then it holds that*

$$\left\|\overline{F}-\overline{F}_k\right\|_{L^p(\Gamma_0)} \leq c_p k^2.$$

Proof. Let $x \in \Gamma_0$ and let $x_k \in \Gamma_{0,k}$ be the orthogonal projection of x onto $\Gamma_{0,k}$, cf. (A.23). Now it holds that

$$\left(\overline{F}-\overline{F}_k\right)(x) = \left(\overline{F}-\overline{F}_k\right)(x_k) + \int_0^1 \left(\overline{F}-\overline{F}_k\right)'(x_k + t\,(x-x_k))\,(x-x_k)\,\mathrm{d}t.$$

As $|x-x_k| \leq ck^2$ and $\left\|\overline{F}-\overline{F}_k\right\|_{W^{1,\infty}(\Omega_0)} \leq ck^{1/2}$ due to (A.28) and Lemma 2.3.29, this lemma follows with Lemma 2.3.43 and (A.25). □

Lemma 2.3.45. *Let $p \in (1,\infty)$, then it holds that*

$$\left\|\overline{F}-\overline{F}_k\right\|_{L^p(\Gamma_{0,h})} \leq c_p \left(h^2+k^2\right).$$

Proof. The proof for this lemma is similar to the proof of Lemma 2.3.44. Let $x_h \in \Gamma_{0,h}$ and let $x \in \Gamma_0$ such that x_h is the orthogonal projection of x onto $\Gamma_{0,h}$. It holds that

$$\left(\overline{F}-\overline{F}_k\right)(x_h) = \left(\overline{F}-\overline{F}_k\right)(x) + \int_0^1 \left(\overline{F}-\overline{F}_k\right)'(x + t\,(x_h-x))\,(x_h-x)\,\mathrm{d}t.$$

Again we use the fact that $|x-x_h| \leq ch^2$ and $\left\|\overline{F}-\overline{F}_k\right\|_{W^{1,\infty}(\Omega_0)} \leq ck^{1/2}$ to obtain

$$\left|\left(\overline{F}-\overline{F}_k\right)(x_h)\right| \leq \left|\left(\overline{F}-\overline{F}_k\right)(x)\right| + ck^{1/2}h^2,$$

and end up with

$$\left\|\overline{F}-\overline{F}_k\right\|_{L^p(\Gamma_{0,h})} \leq c\left(\left\|\overline{F}-\overline{F}_k\right\|_{L^p(\Gamma_0)} + k^{1/2}h^2\right).$$

The proof is finished using Lemma 2.3.44 and Young's inequality. □

Lemma 2.3.46. *Let $p \in (1,\infty)$ and $v \in W^{1,p}(\Omega_{0,h})$, then it holds that*

$$\left|\left(v, \gamma_{\overline{F}} - \gamma_{\overline{F}_k}\right)_h\right| \leq c_p\left(h^2+k^2\right)\|v\|_{W^{1,p}(\Omega_{0,h})}.$$

Proof. Let $\overline{F} = \left(\overline{F}_1, \overline{F}_2\right)^T$ and $\overline{F}_k = \left(\overline{F}_{k,1}, \overline{F}_{k,2}\right)^T$, it holds that

$$\begin{aligned}
\gamma_{\overline{F}} - \gamma_{\overline{F}_k} &= \partial_x(\overline{F}_1 - \overline{F}_{k,1}) + \partial_y(\overline{F}_2 - \overline{F}_{k,2}) \\
&+ \partial_x(\overline{F}_1 - \overline{F}_{k,1})\,\partial_y\overline{F}_2 + \partial_y\overline{F}_1\,\partial_x(\overline{F}_{k,2} - \overline{F}_2) \\
&- \partial_x(\overline{F}_1 - \overline{F}_{k,1})\,\partial_y(\overline{F}_2 - \overline{F}_{k,2}) - \partial_y(\overline{F}_{k,1} - \overline{F}_1)\,\partial_x(\overline{F}_2 - \overline{F}_{k,2}) \\
&+ \partial_x\overline{F}_1\,\partial_y(\overline{F}_2 - \overline{F}_{k,2}) + \partial_y(\overline{F}_{k,1} - \overline{F}_1)\,\partial_x\overline{F}_2.
\end{aligned} \tag{2.84}$$

In what follows let $q = p/(p-1) \in (1,\infty)$ be the conjugate index to p. Now we use Green's theorem and get

$$\begin{aligned}
\left|\left(v, \partial_x(\overline{F}_1 - \overline{F}_{k,1})\right)_h\right| &\leq \left|\left(\partial_x v, \overline{F}_1 - \overline{F}_{k,1}\right)_h\right| + \left|\langle v, (\overline{F}_1 - \overline{F}_{k,1})n_y\rangle_h\right| \\
&\leq \|v\|_{W^{1,p}(\Omega_{0,h})}\left\|\overline{F}_1 - \overline{F}_{k,1}\right\|_{L^q(\Omega_{0,h})} \\
&+ c\,\|v\|_{L^p(\Gamma_{0,h})}\left\|\overline{F}_1 - \overline{F}_{k,1}\right\|_{L^q(\Gamma_{0,h})} \\
&\leq c_p\left(h^2+k^2\right)\|v\|_{W^{1,p}(\Omega_{0,h})},
\end{aligned} \tag{2.85}$$

where we used Lemma 2.3.38, the trace theorem and Lemma 2.3.45. Using the same references we also get

$$\begin{aligned}
\left|\left(v, \partial_x(\overline{F}_1 - \overline{F}_{k,1})\,\partial_y \overline{F}_2\right)_h\right| &\le \left|\left(\partial_x v\,\partial_y \overline{F}_2 + v\partial^2_{xy}\overline{F}_2, \overline{F}_1 - \overline{F}_{k,1}\right)_h\right| \\
&\quad + \left|\langle v\partial_y \overline{F}_2, (\overline{F}_1 - \overline{F}_{k,1})n_y\rangle_h\right| \\
&\le c\,\|v\|_{W^{1,p}(\Omega_{0,h})}\,\|\overline{F}_2\|_{W^{2,\infty}(\Omega_{0,h})}\,\|\overline{F}_1 - \overline{F}_{k,1}\|_{L^q(\Omega_{0,h})} \\
&\quad + c\,\|v\|_{L^p(\Gamma_{0,h})}\,\|\overline{F}_2\|_{W^{1,\infty}(\Gamma_{0,h})}\,\|\overline{F}_1 - \overline{F}_{k,1}\|_{L^q(\Gamma_{0,h})} \\
&\le c_p\left(h^2 + k^2\right)\|v\|_{W^{1,p}(\Omega_{0,h})},
\end{aligned} \tag{2.86}$$

where we used $\overline{F} \in H^{9/2}(\Omega_h) \hookrightarrow W^{2,\infty}(\Omega_{0,h})$. Let $q' = (2p)/(p-1) \in (1,\infty)$, and using Hölder's inequality we get

$$\begin{aligned}
&\left|\left(v, \partial_x(\overline{F}_1 - \overline{F}_{k,1})\,\partial_y(\overline{F}_2 - \overline{F}_{k,2})\right)_h\right| \\
&\le \|v\|_{L^p(\Omega_{0,h})}\,\|\overline{F}_1 - \overline{F}_{k,1}\|_{W^{1,q'}(\Omega_{0,h})}\,\|\overline{F}_2 - \overline{F}_{k,2}\|_{W^{1,q'}(\Omega_{0,h})} \\
&\le c\,\|v\|_{W^{1,p}(\Omega_{0,h})}\,\|\overline{F} - \overline{F}_k\|^2_{W^{1,q'}(\Omega_0)} \\
&\le c_p k^2\,\|v\|_{W^{1,p}(\Omega_{0,h})},
\end{aligned} \tag{2.87}$$

where we again used Lemma 2.3.38. □

Lemma 2.3.47. *Let $v, w \in H^2(\Omega_{0,h})$, then it holds that*

$$\left|\left(\nabla v, \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot\nabla w\right)_h\right| \le c\left(h^2 + k^2\right)\|v\|_{H^2(\Omega_{0,h})}\,\|w\|_{H^2(\Omega_{0,h})}.$$

Proof. Let $\overline{\delta F} = \left(\overline{F}_k - \overline{F}\right)$. With Lemma 2.3.42 and Theorem A.1.12 it follows that

$$\left(\nabla v, \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot\nabla w\right)_h = \left(\nabla v, \left(A'_{\overline{F},\overline{\delta F}} + R_2(\overline{F}, \overline{\delta F})\right)\cdot\nabla w\right)_h,$$

where $R_2(\cdot,\cdot)$ is the remainder term from Taylor's theorem. Using the estimate (A.2) for $R_2(\cdot,\cdot)$, the representation of $A''_{\overline{F}+\tau\,\overline{\delta F},\overline{\delta F},\overline{\delta F}}$ as shown in (2.21) and Lemma 2.3.38 we get

$$\begin{aligned}
&\left|\left(\nabla v, R_2(\overline{F}, \overline{\delta F})\cdot\nabla w\right)_h\right| \\
&\le \|v\|_{W^{1,4}(\Omega_{0,h})}\,\|w\|_{W^{1,4}(\Omega_{0,h})}\,\|R_2(\overline{F}, \overline{\delta F})\|_{L^2(\Omega_{0,h})} \\
&\le c\,\|v\|_{H^2(\Omega_{0,h})}\,\|w\|_{H^2(\Omega_{0,h})}\left(\|\overline{F}\|^5_{W^{1,14}(\Omega_{0,h})} + \|\overline{F}_k\|^5_{W^{1,14}(\Omega_{0,h})}\right)\|\overline{F} - \overline{F}_k\|^2_{W^{1,14}(\Omega_{0,h})} \\
&\le ck^2\,\|v\|_{H^2(\Omega_{0,h})}\,\|w\|_{H^2(\Omega_{0,h})}.
\end{aligned}$$

From (2.20) we know that

$$A'_{\overline{F},\overline{\delta F}} = \operatorname{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T^{-1}_{\overline{F}}\cdot\overline{\delta F}\right)\mathrm{D}T^{-1}_{\overline{F}}\cdot\mathrm{D}T^{-T}_{\overline{F}} - \mathrm{D}T^{-1}_{\overline{F}}\cdot\mathrm{D}\overline{\delta F}\cdot A_{\overline{F}} - A_{\overline{F}}\cdot\mathrm{D}\overline{\delta F}^T\cdot\mathrm{D}T^{-T}_{\overline{F}}. \tag{2.88}$$

Partial integration yields

$$\begin{aligned}
\left|\left(\nabla v, \operatorname{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T^{-1}_{\overline{F}}\cdot\overline{\delta F}\right)\mathrm{D}T^{-1}_{\overline{F}}\cdot\mathrm{D}T^{-T}_{\overline{F}}\cdot\nabla w\right)_h\right| &\le \left|\left(\gamma_{\overline{F}}\,\mathrm{D}T^{-1}_{\overline{F}}\cdot\overline{\delta F}, \nabla\left(\nabla v^T\cdot\mathrm{D}T^{-1}_{\overline{F}}\cdot\mathrm{D}T^{-T}_{\overline{F}}\cdot\nabla w\right)\right)_h\right| \\
&\quad + \left|\langle\nabla v^T\cdot\mathrm{D}T^{-1}_{\overline{F}}\cdot\mathrm{D}T^{-T}_{\overline{F}}\cdot\nabla w, \gamma_{\overline{F}}\overline{\delta F}^T\cdot\mathrm{D}T^{-T}_{\overline{F}}\cdot n\rangle_h\right|.
\end{aligned}$$

Now we use $\mathrm{D}T_{\overline{F}}^{-1}, \gamma_{\overline{F}} \in H^{7/2}(\Omega_{0,h}) \hookrightarrow C^2(\overline{\Omega_{0,h}})$ as well as $W^{1,4}(\Omega_{0,h}) \hookrightarrow H^2(\Omega_{0,h})$ and get

$$\begin{aligned}\left|\left(\nabla v, \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \overline{\delta F}\right) \mathrm{D}T_{\overline{F}}^{-1} \cdot \mathrm{D}T_{\overline{F}}^{-T} \cdot \nabla w\right)_h\right| &\le c \left\|\overline{\delta F}\right\|_{L^4(\Omega_{0,h})} \|v\|_{H^2(\Omega_{0,h})} \|w\|_{H^2(\Omega_{0,h})} \\ &\quad + c \left\|\overline{\delta F}\right\|_{L^3(\Gamma_{0,h})} \|\nabla v\|_{L^3(\Gamma_{0,h})} \|\nabla w\|_{L^3(\Gamma_{0,h})} \\ &\le c\left(h^2 + k^2\right) \|v\|_{H^2(\Omega_{0,h})} \|w\|_{H^2(\Omega_{0,h})},\end{aligned}$$

where we used Lemma 2.3.38, Lemma 2.3.45 and $H^1(\Omega_{0,h})\big|_{\Gamma_{0,h}} \hookrightarrow L^3(\Gamma_{0,h})$. For the two remaining terms within (2.88) we use Green's theorem once more, and using a similar estimate as before we arrive at

$$\begin{aligned}&\left|\left(\nabla v, \left(\mathrm{D}T_{\overline{F}}^{-1} \cdot \mathrm{D}\overline{\delta F} \cdot A_{\overline{F}} + A_{\overline{F}} \cdot \mathrm{D}\overline{\delta F}^T \cdot \mathrm{D}T_{\overline{F}}^{-T}\right) \cdot \nabla w\right)_h\right| \\ &\le c \left|\left(\mathrm{D}\left(\nabla v^T \cdot \mathrm{D}T_{\overline{F}}^{-1} \cdot A_{\overline{F}} \cdot \nabla w\right), \overline{\delta F}\right)_h\right| \\ &+ c \left|\langle \nabla v^T \cdot \mathrm{D}T_{\overline{F}}^{-1} \cdot A_{\overline{F}} \cdot \nabla w, \overline{\delta F}^T \cdot n\rangle_h\right| \\ &\le c\left(h^2 + k^2\right) \|v\|_{H^2(\Omega_{0,h})} \|w\|_{H^2(\Omega_{0,h})}.\end{aligned}$$

□

Lemma 2.3.48. *Let $s \in [0,1]$, then it holds that*

$$\|S_h(\overline{q}) - S_{h,k}(\overline{q})\|_{H^s(\Omega_{0,h})} \le c\left(h^{2-s} + k^{2-s}\right).$$

Proof. Let $e = (S_h(\overline{q}) - S_{h,k}(\overline{q})) \in V_{h,0}$, using Lemma 2.3.33 and Lemma 2.3.38 it follows that

$$\begin{aligned}\|e\|_{H^1(\Omega_{0,h})}^2 &\le a_h(\overline{F})(e,e) \\ &\le \left|l_h(\overline{F})(e) - l_h(\overline{F}_k)(e)\right| + \left|a_h(\overline{F}_k)(S_{h,k}(\overline{q}), e) - a_h(\overline{F})(S_{h,k}(\overline{q}), e)\right| \\ &\le c\left(\left\|\overline{F} - \overline{F}_k\right\|_{H^1(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})} + \left\|\overline{F} - \overline{F}_k\right\|_{W^{1,6}(\Omega_{0,h})} \|S_{h,k}(\overline{q})\|_{W^{1,3}(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})}\right) \\ &\le ck \|e\|_{H^1(\Omega_{0,h})},\end{aligned}$$

which proves the estimate for $s = 1$.

Now we estimate the L^2-error. Let $z \in H_0^1(\Omega_{0,h})$ and $z_h \in V_{h,0}$ be the solutions to

$$a_h(\overline{F})(v,z) = (e,v)_h \qquad \forall v \in H_0^1(\Omega_{0,h}), \tag{2.89}$$

$$a_h(\overline{F})(v_h,z_h) = (e,v_h)_h \qquad \forall v_h \in V_{h,0}. \tag{2.90}$$

Due to the regularity of $A_{\overline{F}}$ it holds that $z \in H^2(\Omega_{0,h})$ and $\|z\|_{H^2(\Omega_{0,h})} \le c\|e\|_{L^2(\Omega_{0,h})}$, where the constant c does not depend on h or $\Omega_{0,h}$, for the H^2-estimate just depends on the diameter of the domain, cf. [51], Theorem 9.1.26, and [69]. Now it holds that

$$\begin{aligned}\|e\|_{L^2(\Omega_{0,h})}^2 &= a_h(\overline{F})(e, z_h) \\ &= l_h(\overline{F})(z_h) - l_h(\overline{F}_k)(z_h) + a_h(\overline{F}_k)(S_{h,k}(\overline{q}), z_h) - a_h(\overline{F})(S_{h,k}(\overline{q}), z_h) \\ &\le \left|l_h(\overline{F})(z - z_h) - l_h(\overline{F}_k)(z - z_h)\right| + \left|l_h(\overline{F})(z) - l_h(\overline{F}_k)(z)\right| \\ &+ \left|a_h(\overline{F})(S(\overline{q}) - S_{h,k}(\overline{q}), z_h) - a_h(\overline{F}_k)(S(\overline{q}) - S_{h,k}(\overline{q}), z_h)\right| \\ &+ \left|a_h(\overline{F})(S(\overline{q}), z - z_h) - a_h(\overline{F}_k)(S(\overline{q}), z - z_h)\right| \\ &+ \left|a_h(\overline{F})(S(\overline{q}), z) - a_h(\overline{F}_k)(S(\overline{q}), z)\right|,\end{aligned} \tag{2.91}$$

and we will estimate each of the five terms on the right hand side of (2.91) separately. The three terms including the differences $(S(\overline{q}) - S_{h,k}(\overline{q}))$ and $(z - z_h)$ can be estimated in a straightforward way. Using Lemma 2.3.38 and standard finite element error estimates if follows that

$$\begin{aligned}\left|l_h(\overline{F})(z - z_h) - l_h(\overline{F}_k)(z - z_h)\right| &\leq c \left\|\overline{F} - \overline{F}_k\right\|_{H^1(\Omega_{0,h})} \|z - z_h\|_{L^2(\Omega_{0,h})} \\ &\leq ch^2 k \|e\|_{L^2(\Omega_{0,h})},\end{aligned} \tag{2.92}$$

and

$$\begin{aligned}&\left|a_h(\overline{F})(S(\overline{q}) - S_{h,k}(\overline{q}), z_h) - a_h(\overline{F}_k)(S(\overline{q}) - S_{h,k}(\overline{q}), z_h)\right| \\ &\leq c \left\|\overline{F} - \overline{F}_k\right\|_{W^{1,6}(\Omega_{0,h})} \|S(\overline{q}) - S_{h,k}(\overline{q})\|_{H^1(\Omega_{0,h})} \|z_h\|_{W^{1,3}(\Omega_{0,h})} \\ &\leq c(h + k)k \|e\|_{L^2(\Omega_{0,h})}.\end{aligned} \tag{2.93}$$

In order to estimate the second factor within (2.93) we used Lemma 2.3.40 and the first part of this lemma, for the last factor we used the estimate

$$\|z_h\|_{W^{1,3}(\Omega_{0,h})} \leq c \|z_h\|_{H^{4/3}(\Omega_{0,h})} \leq c \|z\|_{H^{4/3}(\Omega_{0,h})} \leq c \|z\|_{H^2(\Omega_{0,h})}, \tag{2.94}$$

which is due to Theorem A.1.24. As the triangulations $\{\pi_h\}_{h>0}$ are assumed to be quasiuniform and as Ω_0 is sufficiently smooth, the constants within (2.94) do not depend on h. In the same way it holds that

$$\begin{aligned}&\left|a_h(\overline{F})(S(\overline{q}), z - z_h) - a_h(\overline{F}_k)(S(\overline{q}), z - z_h)\right| \\ &\leq c \left\|\overline{F} - \overline{F}_k\right\|_{W^{1,4}(\Omega_{0,h})} \|S(\overline{q})\|_{W^{1,4}(\Omega_{0,h})} \|z - z_h\|_{H^1(\Omega_{0,h})} \\ &\leq chk \|e\|_{L^2(\Omega_{0,h})}.\end{aligned} \tag{2.95}$$

The two remaining parts on the right hand side of (2.91) require more care. First, it holds that

$$\left|l_h(\overline{F})(z) - l_h(\overline{F}_k)(z)\right| \leq \left|\left(f \circ T_{\overline{F}} - f \circ T_{\overline{F}_k}, z\gamma_{\overline{F}_k}\right)_h\right| + \left|\left(f \circ T_{\overline{F}} z, \gamma_{\overline{F}} - \gamma_{\overline{F}_k}\right)_h\right|, \tag{2.96}$$

and for the first part it holds that

$$\begin{aligned}\left|\left(f \circ T_{\overline{F}} - f \circ T_{\overline{F}_k}, z\gamma_{\overline{F}_k}\right)_h\right| &\leq c \left(\left|T_{\overline{F}} - T_{\overline{F}_k}\right|, \left|z\gamma_{\overline{F}_k}\right|\right)_h \\ &\leq c \left\|\overline{F} - \overline{F}_k\right\|_{L^2(\Omega_{0,h})} \|z\|_{L^2(\Omega_{0,h})} \left\|\gamma_{\overline{F}_k}\right\|_{L^\infty(\Omega_{0,h})} \\ &\leq ck^2 \|e\|_{L^2(\Omega_{0,h})}.\end{aligned} \tag{2.97}$$

For the second part we use Lemma 2.3.46 and $\left(f \circ T_{\overline{F}} z\right) \in H_0^1(\Omega_{0,h})$ to get

$$\begin{aligned}\left|\left(f \circ T_{\overline{F}} z, \gamma_{\overline{F}} - \gamma_{\overline{F}_k}\right)_h\right| &\leq c\left(h^2 + k^2\right) \left\|f \circ T_{\overline{F}} z\right\|_{H^1(\Omega_{0,h})} \\ &\leq c\left(h^2 + k^2\right) \|e\|_{L^2(\Omega_{0,h})}.\end{aligned} \tag{2.98}$$

Now we use Lemma 2.3.46, Lemma 2.3.47 and get

$$\begin{aligned}\left|a_h(\overline{F})(S(\overline{q}), z) - a_h(\overline{F}_k)(S(\overline{q}), z)\right| &\leq \left|\left(\nabla S(\overline{q}), \left(A_{\overline{F}} - A_{\overline{F}_k}\right) \cdot \nabla z\right)_h\right| + \left|\left(S(\overline{q}) z, \gamma_{\overline{F}} - \gamma_{\overline{F}_k}\right)_h\right| \\ &\leq c\left(h^2 + k^2\right) \|S(\overline{q})\|_{H^2(\Omega_{0,h})} \|z\|_{H^2(\Omega_{0,h})} \\ &\leq c\left(h^2 + k^2\right) \|e\|_{L^2(\Omega_{0,h})}.\end{aligned}$$

Now Young's inequality finally proves the case $s = 0$, and what is left follows with interpolation. □

Lemma 2.3.49. *For $\delta q \in Q$ and $s \in [0,1]$ it holds that*

$$\left\| S_h'(\overline{q})(\delta q) - S_{h,k}'(\overline{q})(\delta q) \right\|_{H^s(\Omega_{0,h})} \leq c\left(h^{2-s} + k^{2-s}\right) \|\delta q\|_{H^2(I)}.$$

Proof. The main idea of the following proof is similar to the proof of Lemma 2.3.48 but more involved due to a more complicated right hand side in the equations for $S_h'(\overline{q})(\delta q)$ and $S_{h,k}'(\overline{q})(\delta q)$. Again, let $e = \left(S_h'(\overline{q})(\delta q) - S_{h,k}'(\overline{q})(\delta q)\right) \in V_{h,0}$, $\delta F = F'(\overline{q})(\delta q)$ and $\delta F_k = F_k'(\overline{q})(\delta q)$. It holds that

$$\begin{aligned} c\|e\|^2_{H^1(\Omega_{0,h})} &\leq a_h(\overline{F})(e,e) \\ &\leq \left| l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(e) - l_h^\delta(\overline{F}_k, \delta F_k, S_{h,k}(\overline{q}))(e) \right| \\ &\quad + \left| a_h(\overline{F})(S_{h,k}'(\overline{q})(\delta q), e) - a_h(\overline{F}_k)(S_{h,k}'(\overline{q})(\delta q), e) \right|. \end{aligned} \tag{2.99}$$

For the second part on the right hand side of (2.99) we use Lemma 2.3.38 and Lemma 2.3.33 to get

$$\begin{aligned} &\left| a_h(\overline{F})(S_{h,k}'(\overline{q})(\delta q), e) - a_h(\overline{F}_k)(S_{h,k}'(\overline{q})(\delta q), e) \right| \\ &\leq c\left\|\overline{F} - \overline{F}_k\right\|_{W^{1,6}(\Omega_{0,h})} \left\|S_{h,k}'(\overline{q})(\delta q)\right\|_{W^{1,3}(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})} \\ &\leq ck\|\delta q\|_{H^2(I)} \|e\|_{H^1(\Omega_{0,h})}. \end{aligned}$$

For the first part it holds that

$$\begin{aligned} &\left| l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(e) - l_h^\delta(\overline{F}_k, \delta F_k, S_{h,k}(\overline{q}))(e) \right| \\ &\leq \left| \left(f\circ T_{\overline{F}}, e\,\mathrm{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right)_h - \left(f\circ T_{\overline{F}_k}, e\,\mathrm{div}\left(\gamma_{\overline{F}_k}\,\mathrm{D}T_{\overline{F}_k}^{-1}\cdot\delta F_k\right)\right)_h \right| \qquad (2.100) \\ &\quad + \left| \left(\left(\nabla f\circ T_{\overline{F}}\right)^T\cdot\delta F, e\gamma_{\overline{F}}\right)_h - \left(\left(\nabla f\circ T_{\overline{F}_k}\right)^T\cdot\delta F_k, e\gamma_{\overline{F}_k}\right)_h \right| \qquad (2.101) \\ &\quad + \left| \left(\nabla S_h(\overline{q}), A'_{\overline{F},\delta F}\cdot\nabla e\right)_h - \left(\nabla S_{h,k}(\overline{q}), A'_{\overline{F}_k,\delta F_k}\cdot\nabla e\right)_h \right| \qquad (2.102) \\ &\quad + \left| \left(S_h(\overline{q}), e\,\mathrm{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right)_h - \left(S_{h,k}(\overline{q}), e\,\mathrm{div}\left(\gamma_{\overline{F}_k}\,\mathrm{D}T_{\overline{F}_k}^{-1}\cdot\delta F_k\right)\right)_h \right|. \qquad (2.103) \end{aligned}$$

The differences (2.100) and (2.101) can be estimated through telescoping, we get

$$\begin{aligned} \left| \left(f\circ T_{\overline{F}}, e\,\mathrm{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right)_h - \left(f\circ T_{\overline{F}_k}, e\,\mathrm{div}\left(\gamma_{\overline{F}_k}\,\mathrm{D}T_{\overline{F}_k}^{-1}\cdot\delta F_k\right)\right)_h \right| &\leq ck\|\delta q\|_{H^2(I)}\|e\|_{H^1(\Omega_{0,h})}, \\ \left| \left(\left(\nabla f\circ T_{\overline{F}}\right)^T\cdot\delta F, e\gamma_{\overline{F}}\right)_h - \left(\left(\nabla f\circ T_{\overline{F}_k}\right)^T\cdot\delta F_k, e\gamma_{\overline{F}_k}\right)_h \right| &\leq ck\|\delta q\|_{H^2(I)}\|e\|_{H^1(\Omega_{0,h})}. \end{aligned}$$

The third part, (2.102), is a bit more complicated. It holds that

$$\begin{aligned} &\left| \left(\nabla S_h(\overline{q}), A'_{\overline{F},\delta F}\cdot\nabla e\right)_h - \left(\nabla S_{h,k}(\overline{q}), A'_{\overline{F}_k,\delta F_k}\cdot\nabla e\right)_h \right| \\ &\leq \left| \left(\nabla\left(S_h(\overline{q}) - S_{h,k}(\overline{q})\right), A'_{\overline{F},\delta F}\cdot\nabla e\right)_h \right| + \left| \left(\nabla S_{h,k}(\overline{q}), A'_{\overline{F},\delta F-\delta F_k}\cdot\nabla e\right)_h \right| \qquad (2.104) \\ &\quad + \left| \left(\nabla S_{h,k}(\overline{q}), \left(A'_{\overline{F},\delta F_k} - A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla e\right)_h \right|, \end{aligned}$$

with

$$\begin{aligned}
\left|\left(\nabla\left(S_h(\overline{q}) - S_{h,k}(\overline{q})\right), A'_{\overline{F},\delta F} \cdot \nabla e\right)_h\right| &\leq \|S_h(\overline{q}) - S_{h,k}(\overline{q})\|_{H^1(\Omega_{0,h})} \left\|A'_{\overline{F},\delta F}\right\|_{L^\infty(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})} \\
&\leq ck \|\delta q\|_{H^2(I)} \|e\|_{H^1(\Omega_{0,h})}, \\
\left|\left(\nabla S_{h,k}(\overline{q}), A'_{\overline{F},\delta F - \delta F_k} \cdot \nabla e\right)_h\right| &\leq \|S_{h,k}(\overline{q})\|_{W^{1,3}(\Omega_{0,h})} \left\|A'_{\overline{F},\delta F - \delta F_k}\right\|_{L^6(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})} \\
&\leq c \|\delta F - \delta F_k\|_{W^{1,6}(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})} \\
&\leq ck \|\delta q\|_{H^2(I)} \|e\|_{H^1(\Omega_{0,h})}.
\end{aligned}$$

In order to estimate the last part of (2.104) we again use Taylor's theorem. Fix δF_k and define

$$\begin{aligned}
&G\colon W^{1,\infty}(\Omega_{0,h}) \to L^\infty(\Omega_{0,h}), \\
&G(F) = A'_{F,\delta F_k},
\end{aligned}$$

then G is at least twice continuously differentiable due to Lemma 2.3.42. Using Theorem A.1.12 it follows that

$$A'_{\overline{F}_k,\delta F_k} - A'_{\overline{F},\delta F_k} = A''_{\overline{F},\overline{F}_k - \overline{F},\delta F_k} + \tilde{R}_2(\overline{F}, \overline{F}_k - \overline{F}, \delta F_k), \tag{2.105}$$

and the estimate (A.2) for the remainder term $\tilde{R}_2$ reads as

$$\begin{aligned}
\left\|\tilde{R}_2(\overline{F}, \overline{F}_k - \overline{F}, \delta F_k)\right\|_{L^\infty(\Omega_0)} &\leq c \sup_{0<\tau<1} \left\|A'''_{\overline{F}+\tau(\overline{F}_k - \overline{F}),\overline{F}_k - \overline{F},\overline{F}_k - \overline{F},\delta F_k}\right\|_{L^\infty(\Omega_0)} \\
&\leq c \|\delta F_k\|_{W^{1,\infty}(\Omega_{0,h})} \left\|\overline{F} - \overline{F}_k\right\|^2_{W^{1,\infty}(\Omega_{0,h})}.
\end{aligned} \tag{2.106}$$

Equation (2.105) yields

$$\begin{aligned}
\left|\left(\nabla S_{h,k}(\overline{q}), \left(A'_{\overline{F},\delta F_k} - A'_{\overline{F}_k,\delta F_k}\right) \cdot \nabla e\right)_h\right| &\leq \left|\left(\nabla S_{h,k}(\overline{q}), A''_{\overline{F},\delta F_k,\overline{F}_k - \overline{F}} \cdot \nabla e\right)_h\right| \\
&\quad + \left|\left(\nabla S_{h,k}(\overline{q}), \tilde{R}_2(\overline{F}, \overline{F}_k - \overline{F}, \delta F_k) \cdot \nabla e\right)_h\right|,
\end{aligned}$$

with

$$\begin{aligned}
\left|\left(\nabla S_{h,k}(\overline{q}), A''_{\overline{F},\overline{F}_k - \overline{F},\delta F_k} \cdot \nabla e\right)_h\right| &\leq \|S_{h,k}(\overline{q})\|_{W^{1,3}(\Omega_{0,h})} \left\|A''_{\overline{F},\overline{F}_k - \overline{F},\delta F_k}\right\|_{L^6(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})} \\
&\leq c \|\delta F_k\|_{W^{1,12}(\Omega_{0,h})} \left\|\overline{F}_k - \overline{F}\right\|_{W^{1,12}(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})} \\
&\leq ck \|\delta q\|_{H^2(I)} \|e\|_{H^1(\Omega_{0,h})}.
\end{aligned}$$

Using the estimate (2.106), the fact that $\delta F_k = F_k(\delta q)$ and Lemma 2.3.29 we arrive at

$$\begin{aligned}
\left|\left(\nabla S_{h,k}(\overline{q}), \tilde{R}_2(\overline{F}, \overline{F}_k - \overline{F}, \delta F_k) \cdot \nabla e\right)_h\right| &\leq c \|S_{h,k}(\overline{q})\|_{H^1(\Omega_{0,h})} \left\|\tilde{R}_2(\overline{F}, \overline{F}_k - \overline{F}, \delta F_k)\right\|_{L^\infty(\Omega_0)} \|e\|_{H^1(\Omega_{0,h})} \\
&\leq c \|\delta F_k\|_{W^{1,\infty}(\Omega_{0,h})} \left\|\overline{F} - \overline{F}_k\right\|^2_{W^{1,\infty}(\Omega_{0,h})} \|e\|_{H^1(\Omega_{0,h})} \\
&\leq ck \|\delta q\|_{H^2(I)} \|e_h\|_{H^1(\Omega_{0,h})}.
\end{aligned}$$

The last term within this estimate, (2.103), can again be estimated through telescoping,

$$\left|\left(S_h(\overline{q}), e\operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right)_h - \left(S_{h,k}(\overline{q}), e\operatorname{div}\left(\gamma_{\overline{F}_k}\, \mathrm{D}T_{\overline{F}_k}^{-1}\cdot\delta F_k\right)\right)_h\right| \le ck\left\|\delta q\right\|_{H^2(I)}\left\|e\right\|_{H^1(\Omega_{0,h})},$$

which finishes the proof for $s = 1$.

We proceed with the L^2-error, which can be proven similarly to the proof of Lemma 2.3.48. Let again $z \in H_0^1(\Omega_{0,h})$ and $z_h \in V_{h,0}$ be the solutions to

$$\begin{aligned} a_h(\overline{F})(v, z) &= (e, v)_h && \forall v \in H_0^1(\Omega_{0,h}),\\ a_h(\overline{F})(v_h, z_h) &= (e, v_h)_h && \forall v_h \in V_{h,0}. \end{aligned}$$

Again it holds that $z \in H^2(\Omega_{0,h})$ with $\|z\|_{H^2(\Omega_{0,h})} \le c\,\|e\|_{L^2(\Omega_0)}$, where the constant c is independent of h. With Theorem A.1.27 it follows that

$$\|z - z_h\|_{H^s(\Omega_{0,h})} \le ch^{2-s}\left\|e\right\|_{L^2(\Omega_{0,h})},$$

for $s \in [0, 3/2)$. We get

$$\begin{aligned} \|e\|^2_{L^2(\Omega_{0,h})} &= a_h(\overline{F})(e, z_h) = a_h(\overline{F})(S_h'(\overline{q})(\delta q), z_h) - a_h(\overline{F})(S_{h,k}'(\overline{q})(\delta q), z_h)\\ &= l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(z_h) - l_h^\delta(\overline{F}_k, \delta F_k, S_{h,k}(\overline{q}))(z_h)\\ &\quad + a_h(\overline{F}_k)(S_{h,k}'(\overline{q})(\delta q), z_h) - a_h(\overline{F})(S_{h,k}'(\overline{q})(\delta q), z_h)\\ &\le \left|l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(z_h - z) - l_h^\delta(\overline{F}_k, \delta F_k, S_{h,k}(\overline{q}))(z_h - z)\right| && (2.107)\\ &\quad + \left|l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(z) - l_h^\delta(\overline{F}_k, \delta F_k, S_{h,k}(\overline{q}))(z)\right| && (2.108)\\ &\quad + \left|a_h(\overline{F}_k)(S_{h,k}'(\overline{q})(\delta q) - S'(\overline{q})(\delta q), z_h) - a_h(\overline{F})(S_{h,k}'(\overline{q})(\delta q) - S'(\overline{q})(\delta q), z_h)\right| && (2.109)\\ &\quad + \left|a_h(\overline{F}_k)(S'(\overline{q})(\delta q), z_h - z) - a_h(\overline{F})(S'(\overline{q})(\delta q), z_h - z)\right| && (2.110)\\ &\quad + \left|a_h(\overline{F}_k)(S'(\overline{q})(\delta q), z) - a_h(\overline{F})(S'(\overline{q})(\delta q), z)\right|. && (2.111) \end{aligned}$$

Again we can estimate each term separately. The first part, (2.107), can be estimated as in the first part of this proof which yields

$$\begin{aligned} \left|l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(z_h - z) - l_h^\delta(\overline{F}_k, \delta F_k, S_{h,k}(\overline{q}))(z_h - z)\right| &\le ck\left\|\delta q\right\|_{H^2(I)}\left\|z - z_h\right\|_{H^1(\Omega_{0,h})}\\ &\le chk\left\|\delta q\right\|_{H^2(I)}\left\|e\right\|_{L^2(\Omega_0)}. \end{aligned}$$

The following part, (2.108), needs to be split once more, we have

$$\begin{aligned} &\left|l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(z) - l_h^\delta(\overline{F}_k, \delta F_k, S_{h,k}(\overline{q}))(z)\right|\\ &\le \left|l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(z) - l_h^\delta(\overline{F}, \delta F, S_{h,k}(\overline{q}))(z)\right| && (2.112)\\ &\quad + \left|l_h^\delta(\overline{F}, \delta F, S_{h,k}(\overline{q}) - S(\overline{q}))(z) - l_h^\delta(\overline{F}_k, \delta F_k, S_{h,k}(\overline{q}) - S(\overline{q}))(z)\right| && (2.113)\\ &\quad + \left|l_h^\delta(\overline{F}, \delta F, S(\overline{q}))(z) - l_h^\delta(\overline{F}_k, \delta F_k, S(\overline{q}))(z)\right|. && (2.114) \end{aligned}$$

We start with (2.112) and get

$$\begin{aligned} \left|l_h^\delta(\overline{F}, \delta F, S_h(\overline{q}))(z) - l_h^\delta(\overline{F}, \delta F, S_{h,k}(\overline{q}))(z)\right| &\le \left|\left(\nabla\left(S_h(\overline{q}) - S_{h,k}(\overline{q})\right), A_{\overline{F},\delta F}'\cdot\nabla z\right)_h\right|\\ &\quad + \left|\left(S_h(\overline{q}) - S_{h,k}(\overline{q}), z\operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right)_h\right|, \end{aligned}$$

which results in

$$\begin{aligned}\left|\left(\nabla\left(S_h(\overline{q})-S_{h,k}(\overline{q})\right), A'_{\overline{F},\delta F}\cdot\nabla z\right)_h\right| &= \left|\left(S_h(\overline{q})-S_{h,k}(\overline{q}), \operatorname{div}\left(A'_{\overline{F},\delta F}\cdot\nabla z\right)\right)_h\right| \\ &\le c\left\|S_h(\overline{q})-S_{h,k}(\overline{q})\right\|_{L^2(\Omega_{0,h})}\left\|A'_{\overline{F},\delta F}\right\|_{L^\infty(\Omega_{0,h})}\left\|\nabla^2 z\right\|_{L^2(\Omega_{0,h})} \\ &+ c\left\|S_h(\overline{q})-S_{h,k}(\overline{q})\right\|_{L^2(\Omega_{0,h})}\left\|A'_{\overline{F},\delta F}\right\|_{W^{1,4}(\Omega_{0,h})}\left\|\nabla z\right\|_{L^4(\Omega_{0,h})}.\end{aligned}$$

Now we use $\|S_h(\overline{q})-S_{h,k}(\overline{q})\|_{L^2(\Omega_{0,h})}\le c\left(h^2+k^2\right)$ due to Lemma 2.3.40 and Lemma 2.3.48, the continuous embeddings $W^{1,4}(\Omega_{0,h})\hookrightarrow L^\infty(\Omega_{0,h})$, $H^2(\Omega_{0,h})\hookrightarrow W^{1,4}(\Omega_{0,h})$ and the estimate

$$\left\|A'_{\overline{F},\delta F}\right\|_{W^{1,4}(\Omega_{0,h})}\le c\left\|\delta F\right\|_{W^{2,4}(\Omega_{0,h})}\le c\left\|\delta F\right\|_{H^{5/2}(\Omega_0)}\le c\left\|\delta q\right\|_{H^2(I)},$$

to obtain

$$\begin{aligned}\left|\left(\nabla\left(S_h(\overline{q})-S_{h,k}(\overline{q})\right), A'_{\overline{F},\delta F}\cdot\nabla z\right)_h\right| &\le c\left(h^2+k^2\right)\left\|\delta q\right\|_{H^2(I)}\left\|z\right\|_{H^2(\Omega_{0,h})} \\ &\le c\left(h^2+k^2\right)\left\|\delta q\right\|_{H^2(I)}\left\|e\right\|_{L^2(\Omega_{0,h})}.\end{aligned}$$

In addition,

$$\begin{aligned}\left|\left(S_h(\overline{q})-S_{h,k}(\overline{q}), z\operatorname{div}\left(\gamma_{\overline{F}}\operatorname{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right)_h\right| &\le c\left\|S_h(\overline{q})-S_{h,k}(\overline{q})\right\|_{L^2(\Omega_{0,h})}\left\|z\right\|_{L^\infty(\Omega_{0,h})}\left\|\delta F\right\|_{H^1(\Omega_{0,h})} \\ &\le c\left(h^2+k^2\right)\left\|\delta q\right\|_{H^2(I)}\left\|e\right\|_{L^2(\Omega_{0,h})},\end{aligned}$$

hence

$$\left|l_h^\delta(\overline{F},\delta F,S_h(\overline{q}))(z)-l_h^\delta(\overline{F},\delta F,S_{h,k}(\overline{q}))(z)\right|\le c\left(h^2+k^2\right)\left\|\delta q\right\|_{H^2(I)}\left\|e\right\|_{L^2(\Omega_{0,h})}.$$

For the next two estimates we use Theorem A.1.12 and Lemma 2.3.42 and get

$$\begin{aligned}A'_{\overline{F},\delta F}-A'_{\overline{F}_k,\delta F_k} &= A'_{\overline{F},\delta F-\delta F_k}+A'_{\overline{F},\delta F_k}-A'_{\overline{F}_k,\delta F_k} \\ &= A'_{\overline{F},\delta F-\delta F_k}+A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k}+\tilde{R}_2(\overline{F},\overline{F}_k-\overline{F},\delta F_k),\end{aligned}\tag{2.115}$$

similar to (2.105). Now we estimate (2.113) via

$$\begin{aligned}&\left|l_h^\delta(\overline{F},\delta F,S_{h,k}(\overline{q})-S(\overline{q}))(z)-l_h^\delta(\overline{F}_k,\delta F_k,S_{h,k}(\overline{q})-S(\overline{q}))(z)\right| \\ &\le\left|\left(f\circ T_{\overline{F}}, z\operatorname{div}\left(\gamma_{\overline{F}}\operatorname{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right)_h-\left(f\circ T_{\overline{F}_k}, z\operatorname{div}\left(\gamma_{\overline{F}_k}\operatorname{D}T_{\overline{F}_k}^{-1}\cdot\delta F_k\right)\right)_h\right| \\ &+\left|\left(\left(\nabla f\circ T_{\overline{F}}\right)^T\cdot\delta F, z\gamma_{\overline{F}}\right)_h-\left(\left(\nabla f\circ T_{\overline{F}_k}\right)^T\cdot\delta F_k, z\gamma_{\overline{F}_k}\right)_h\right| \\ &+\left|\left(\nabla\left(S(\overline{q})-S_{h,k}(\overline{q})\right),\left(A'_{\overline{F},\delta F}-A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla z\right)_h\right| \\ &+\left|\left(\left(S(\overline{q})-S_{h,k}(\overline{q})\right)z, \operatorname{div}\left(\gamma_{\overline{F}}\operatorname{D}T_{\overline{F}}^{-1}\cdot\delta F-\gamma_{\overline{F}_k}\operatorname{D}T_{\overline{F}_k}^{-1}\cdot\delta F_k\right)\right)_h\right|.\end{aligned}$$

The first two expression can be estimated using standard telescoping arguments, for the third line we use (2.115) to get

$$\begin{aligned}
&\left|\left(\nabla\left(S(\overline{q})-S_{h,k}(\overline{q})\right),\left(A'_{\overline{F},\delta F}-A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla z\right)_h\right|\\
&\leq\left|\left(\nabla\left(S(\overline{q})-S_{h,k}(\overline{q})\right),A'_{\overline{F},\delta F-\delta F_k}\cdot\nabla z\right)_h\right|\\
&+\left|\left(\nabla\left(S(\overline{q})-S_{h,k}(\overline{q})\right),A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k}\cdot\nabla z\right)_h\right|\\
&+\left|\left(\nabla\left(S(\overline{q})-S_{h,k}(\overline{q})\right),\tilde{R}_2(\overline{F},\overline{F}_k-\overline{F},\delta F_k)\cdot\nabla z\right)_h\right|,
\end{aligned}$$

where it holds that

$$\begin{aligned}
&\left|\left(\nabla\left(S(\overline{q})-S_{h,k}(\overline{q})\right),A'_{\overline{F},\delta F-\delta F_k}\cdot\nabla z\right)_h\right|\\
&\leq\|S(\overline{q})-S_{h,k}(\overline{q})\|_{H^1(\Omega_{0,h})}\left\|A'_{\overline{F},\delta F-\delta F_k}\right\|_{L^4(\Omega_{0,h})}\|z\|_{W^{1,4}(\Omega_{0,h})}\\
&\leq c\,(h+k)\,\|\delta F-\delta F_k\|_{W^{1,4}(\Omega_{0,h})}\|z\|_{W^{1,4}(\Omega_{0,h})}\\
&\leq c\,(h+k)\,k\,\|\delta q\|_{W^{2-1/4,4}(I)}\|z\|_{H^2(\Omega_{0,h})}\\
&\leq c\,(h+k)\,k\,\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_{0,h})},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left(\nabla\left(S(\overline{q})-S_{h,k}(\overline{q})\right),A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k}\cdot\nabla z\right)_h\right|\\
&\leq\|S(\overline{q})-S_{h,k}(\overline{q})\|_{H^1(\Omega_{0,h})}\left\|A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k}\right\|_{L^4(\Omega_{0,h})}\|z\|_{W^{1,4}(\Omega_{0,h})}\\
&\leq c\,(h+k)\left\|\overline{F}-\overline{F}_k\right\|_{W^{1,4}(\Omega_{0,h})}\|\delta F_k\|_{W^{1,\infty}(\Omega_{0,h})}\|z\|_{H^2(\Omega_{0,h})}\\
&\leq c\,(h+k)\,k\,\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_{0,h})}.
\end{aligned}$$

For the last part we use the estimate (2.106) and get

$$\begin{aligned}
&\left|\left(\nabla\left(S(\overline{q})-S_{h,k}(\overline{q})\right),\tilde{R}_2(\overline{F},\overline{F}_k-\overline{F},\delta F_k)\cdot\nabla z\right)_h\right|\\
&\leq\|S(\overline{q})-S_{h,k}(\overline{q})\|_{H^1(\Omega_{0,h})}\left\|\tilde{R}_2(\overline{F},\overline{F}_k-\overline{F},\delta F_k)\right\|_{L^\infty(\Omega_{0,h})}\|z\|_{H^1(\Omega_{0,h})}\\
&\leq c\,(h+k)\left\|\overline{F}-\overline{F}_k\right\|^2_{W^{1,\infty}(\Omega_{0,h})}\|\delta F_k\|_{W^{1,\infty}(\Omega_{0,h})}\|z\|_{H^2(\Omega_{0,h})}\\
&\leq c\,(h+k)\,k\,\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_{0,h})}.
\end{aligned}$$

Finally we estimate (2.114),

$$\begin{aligned}
&\left|l_h^\delta(\overline{F},\delta F,S(\overline{q}))(z)-l_h^\delta(\overline{F}_k,\delta F_k,S(\overline{q}))(z)\right|\\
&\leq\left|\left(f\circ T_{\overline{F}},z\operatorname{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F\right)\right)_h-\left(f\circ T_{\overline{F}_k},z\operatorname{div}\left(\gamma_{\overline{F}_k}\,\mathrm{D}T_{\overline{F}_k}^{-1}\cdot\delta F_k\right)\right)_h\right| && (2.116)\\
&+\left|\left(\left(\nabla f\circ T_{\overline{F}}\right)^T\cdot\delta F,z\gamma_{\overline{F}}\right)_h-\left(\left(\nabla f\circ T_{\overline{F}_k}\right)^T\cdot\delta F_k,z\gamma_{\overline{F}_k}\right)_h\right| && (2.117)\\
&+\left|\left(\nabla S(\overline{q}),\left(A'_{\overline{F},\delta F}-A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla z\right)_h\right| && (2.118)\\
&+\left|\left(S(\overline{q})z,\operatorname{div}\left(\gamma_{\overline{F}}\,\mathrm{D}T_{\overline{F}}^{-1}\cdot\delta F-\gamma_{\overline{F}_k}\,\mathrm{D}T_{\overline{F}_k}^{-1}\cdot\delta F_k\right)\right)_h\right|. && (2.119)
\end{aligned}$$

The expressions (2.116) and (2.117) can again be estimated through telescoping, and with (2.115) one can estimate (2.118) via

$$\begin{aligned}\left|\left(\nabla S(\overline{q}),\left(A'_{\overline{F},\delta F}-A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla z\right)_h\right| &\leq \left|\left(\nabla S(\overline{q}),A'_{\overline{F},\delta F-\delta F_k}\cdot\nabla z\right)_h\right|+\left|\left(\nabla S(\overline{q}),A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k}\cdot\nabla z\right)_h\right|\\ &+\left|\left(\nabla S(\overline{q}),\tilde{R}_2(\overline{F},\overline{F}_k-\overline{F},\delta F_k)\cdot\nabla z\right)_h\right|.\end{aligned}$$

Using the same method as in the proof of Lemma 2.3.47 one can show that

$$\left|\left(\nabla S(\overline{q}),A'_{\overline{F},\delta F-\delta F_k}\cdot\nabla z\right)_h\right|\leq c\left(h^2+k^2\right)\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_{0,h})}.$$

It also holds that

$$\left|\left(\nabla S(\overline{q}),A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k}\cdot\nabla z\right)_h\right|\leq\left|\left(\nabla S(\overline{q}),A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k-\delta F}\cdot\nabla z\right)_h\right|+\left|\left(\nabla S(\overline{q}),A''_{\overline{F},\overline{F}_k-\overline{F},\delta F}\cdot\nabla z\right)_h\right|$$

with

$$\begin{aligned}\left|\left(\nabla S(\overline{q}),A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k-\delta F}\cdot\nabla z\right)_h\right| &\leq\|S(\overline{q})\|_{W^{1,4}(\Omega_{0,h})}\left\|A''_{\overline{F},\overline{F}_k-\overline{F},\delta F_k-\delta F}\right\|_{L^2(\Omega_{0,h})}\|z\|_{W^{1,4}(\Omega_{0,h})}\\ &\leq c\left\|\overline{F}-\overline{F}_k\right\|_{W^{1,4}(\Omega_{0,h})}\|\delta F_k-\delta F\|_{W^{1,4}(\Omega_{0,h})}\|z\|_{W^{1,4}(\Omega_{0,h})}\\ &\leq ck^2\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_{0,h})}.\end{aligned}$$

The estimate

$$\left|\left(\nabla S(\overline{q}),A''_{\overline{F},\overline{F}_k-\overline{F},\delta F}\cdot\nabla z\right)_h\right|\leq c\left(h^2+k^2\right)\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_{0,h})},$$

can be shown using partial integration similar to the proof of Lemma 2.3.47. Using the representation (2.106) we get

$$\begin{aligned}\left|\left(\nabla S(\overline{q}),\tilde{R}_2(\overline{F},\overline{F}_k-\overline{F},\delta F_k)\cdot\nabla z\right)_h\right| &\leq\|S(\overline{q})\|_{W^{1,4}(\Omega_{0,h})}\left\|\tilde{R}_2(\overline{F},\overline{F}_k-\overline{F},\delta F_k)\right\|_{L^2(\Omega_{0,h})}\|z\|_{W^{1,4}(\Omega_{0,h})}\\ &\leq c\left\|\tilde{R}_2(\overline{F},\overline{F}_k-\overline{F},\delta F_k)\right\|_{L^2(\Omega_{0,h})}\|z\|_{H^2(\Omega_{0,h})}\\ &\leq c\|\delta F_k\|_{W^{1,6}(\Omega_{0,h})}\left\|\overline{F}-\overline{F}_k\right\|^2_{W^{1,6}(\Omega_{0,h})}\|z\|_{H^2(\Omega_{0,h})}\\ &\leq ck^2\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_{0,h})}.\end{aligned}$$

The remaining part (2.119) can be estimated via

$$\left|\left(S(\overline{q})z,\operatorname{div}\left(\gamma_{\overline{F}}\operatorname{D}T^{-1}_{\overline{F}}\cdot\delta F-\gamma_{\overline{F}_k}\operatorname{D}T^{-1}_{\overline{F}_k}\cdot\delta F_k\right)\right)_h\right|=\left|\left(\nabla\left(S(\overline{q})z\right),\gamma_{\overline{F}}\operatorname{D}T^{-1}_{\overline{F}}\cdot\delta F-\gamma_{\overline{F}_k}\operatorname{D}T^{-1}_{\overline{F}_k}\cdot\delta F_k\right)_h\right|,$$

and using Lemma 2.3.38 one can proceed with telescoping as in the previous cases, as $(S(\overline{q})z)\in H^2(\Omega_{0,h})$ one can also use Lemma 2.3.46 and ends up with

$$\left|\left(S(\overline{q})z,\operatorname{div}\left(\gamma_{\overline{F}}\operatorname{D}T^{-1}_{\overline{F}}\cdot\delta F-\gamma_{\overline{F}_k}\operatorname{D}T^{-1}_{\overline{F}_k}\cdot\delta F_k\right)\right)_h\right|\leq c\left(h^2+k^2\right)\|\delta q\|_{H^2(I)}\|e\|_{L^2(\Omega_{0,h})}.$$

Now we return to the original estimation. For (2.109) it holds that

$$\begin{aligned}
&\left|a_h(\overline{F}_k)(S'_{h,k}(\overline{q})(\delta q) - S'(\overline{q})(\delta q), z_h) - a_h(\overline{F})(S'_{h,k}(\overline{q})(\delta q) - S'(\overline{q})(\delta q), z_h)\right| \\
&\leq \left|\left(\nabla\left(S'_{h,k}(\overline{q})(\delta q) - S'(\overline{q})(\delta q)\right), \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot \nabla z_h\right)_h\right| \\
&\leq \left\|S'(\overline{q})(\delta q) - S'_{h,k}(\overline{q})(\delta q)\right\|_{H^1(\Omega_{0,h})} \left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^6(\Omega_{0,h})} \|z_h\|_{W^{1,3}(\Omega_{0,h})} \\
&\leq c\,(h+k)\,\|\delta q\|_{H^2(I)} \left\|\overline{F} - \overline{F}_k\right\|_{W^{1,6}(\Omega_{0,h})} \|z\|_{H^2(\Omega_{0,h})} \\
&\leq c\,(h+k)\,k\,\|\delta q\|_{H^2(I)} \|e\|_{L^2(\Omega_{0,h})}\,.
\end{aligned}$$

Expression (2.110) can be estimated as follows,

$$\begin{aligned}
&\left|a_h(\overline{F}_k)(S'(\overline{q})(\delta q), z_h - z) - a_h(\overline{F})(S'(\overline{q})(\delta q), z_h - z)\right| \\
&\leq \left|\left(\nabla S'(\overline{q})(\delta q), \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot \nabla\left(z - z_h\right)\right)_h\right| \\
&\leq \left\|S'(\overline{q})(\delta q)\right\|_{W^{1,4}(\Omega_{0,h})} \left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^4(\Omega_{0,h})} \|z - z_h\|_{H^1(\Omega_{0,h})} \\
&\leq ch\,\|\delta q\|_{H^2(I)} \left\|\overline{F} - \overline{F}_k\right\|_{W^{1,4}(\Omega_{0,h})} \|z\|_{H^2(\Omega_{0,h})} \\
&\leq chk\,\|\delta q\|_{H^2(I)} \|e\|_{L^2(\Omega_{0,h})}\,.
\end{aligned}$$

The last part, (2.111), can again be estimated using Lemma 2.3.46 and Lemma 2.3.47. We can apply these lemmata because of $S'(\overline{q})(\delta q) \in H^2(\Omega_{0,h})$, $(z\,S'(\overline{q})(\delta q)) \in H^1(\Omega_{0,h})$, and get

$$\begin{aligned}
\left|a_h(\overline{F}_k)(S'(\overline{q})(\delta q), z) - a_h(\overline{F})(S'(\overline{q})(\delta q), z)\right| &\leq c\left(h^2 + k^2\right)\left\|S'(\overline{q})(\delta q)\right\|_{H^2(\Omega_{0,h})} \|z\|_{H^2(\Omega_{0,h})} \\
&\leq c\left(h^2 + k^2\right)\|\delta q\|_{H^2(I)} \|e\|_{L^2(\Omega_{0,h})}\,,
\end{aligned}$$

which finishes the L^2-estimate, and what is left follows with interpolation. □

Lemma 2.3.50. *For $\delta q \in Q$ it holds that*

$$\left|j'_h(\overline{q})(\delta q) - j'_{h,k}(\overline{q})(\delta q)\right| \leq c\left(h^2 + k^2\right)\|\delta q\|_{H^2(I)}\,.$$

Proof. This lemma follows with Definition 2.3.7, Definition 2.3.8, Lemma 2.3.48 and Lemma 2.3.49. □

2.3.4. Second order optimality conditions

Within this subsection we are going to prove some optimality conditions of second order and some quadratic-growth conditions which hold in the optimal continuous and discretized solutions. At first we have to make the following assumption regarding the coercivity of the second derivative of the reduced cost functional.

Assumption 2.3.51. We assume that

$$j''(\overline{q})(\delta q, \delta q) > 0 \qquad\qquad \forall \delta q \in Q\setminus\{0\}\,.$$

Remark 2.3.52. As j is twice continuously differentiable, we know that there holds a necessary optimality condition of second order,

$$j''(\overline{q})(\delta q, \delta q) \geq 0 \qquad \forall \delta q \in Q. \tag{2.120}$$

Hence, Assumption 2.3.51 is just slightly stronger than the second order optimality condition (2.120) and therefore reasonable to assume. In addition, numerical optimization algorithms are more likely to fail if Assumption 2.3.51 is not satisfied.

We also recall the first order optimality conditions, where $\overline{q}_\sigma$, $\overline{q}_{\sigma,h}$ and $\overline{q}_{\sigma,h,k}$ are arbitrary local optimal solutions to (2.59), (2.66) and (2.73), respectively.

$$\begin{aligned} j'(\overline{q})(\delta q) &= 0 \quad \forall \delta q \in Q, \\ j'(\overline{q}_\sigma)(\delta q_\sigma) &= 0 \quad \forall \delta q_\sigma \in Q_\sigma, \\ j_h'(\overline{q}_{\sigma,h})(\delta q_\sigma) &= 0 \quad \forall \delta q_\sigma \in Q_\sigma, \\ j_{h,k}'(\overline{q}_{\sigma,h,k})(\delta q_\sigma) &= 0 \quad \forall \delta q_\sigma \in Q_\sigma. \end{aligned} \tag{2.121}$$

In what follows we are going to prove that all optimal solutions fulfilling Assumption 2.3.51 also fulfill a seemingly stronger form of coercivity. The following lemmata and proofs have been inspired by [30].

Lemma 2.3.53. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $(\delta q_n)_{n\in\mathbb{N}} \subset Q$. If $\delta q_n \to \delta q$ in $H^{3/2+\varepsilon}(I)$, then it holds that*

$$S'(q)(\delta q_n) \to S'(q)(\delta q) \qquad \text{in } H_0^1(\Omega_0), \tag{2.122}$$

$$S''(q)(\delta q_n, \delta q_n) \to S''(q)(\delta q, \delta q) \qquad \text{in } H_0^1(\Omega_0). \tag{2.123}$$

Proof. Let $F = F(q)$, $\delta F = F'(q)(\delta q)$ and $\delta F_n = F'(q)(\delta q_n)$. With Lemma 2.1.2 it follows that $\delta F_n \to \delta F$ in $H^{2+\varepsilon}(\Omega_0)$, and with Theorem A.1.5 it follows that

$$\left.\begin{aligned} A'_{F,\delta F_n} &\to A'_{F,\delta F}, \\ \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F_n\big) &\to \operatorname{div}\big(\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\big), \end{aligned}\right. \qquad \text{in} \qquad H^{1+\varepsilon}(\Omega_0) \hookrightarrow C(\overline{\Omega_0}).$$

As a result, the right hand side in Definition 2.3.2 converges in $H^{-1}(\Omega_0)$, and (2.122) follows with standard H^1-stability results. The second part, (2.123), is proven analogously to the first part. In order to show that the right hand side in Definition 2.3.2 converges in $H^{-1}(\Omega_0)$ one has to use (2.122), Lemma 2.1.6 and the fact that the trace of a matrix, trace: $X^{2\times 2} \to X$ is continuous for every Banach space X. □

Lemma 2.3.54. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $(\delta q_n)_{n\in\mathbb{N}} \subset Q$ with $\delta q_n \to \delta q$ in $H^{3/2+\varepsilon}(I)$. Let $m \colon Q^{\mathrm{ad}} \times Q \to \mathbb{R}$ and $n \colon Q^{\mathrm{ad}} \times Q \to \mathbb{R}$ be defined via*

$$\begin{aligned} m(q)(\delta q) &= j'(q)(\delta q) - \alpha(q, \delta q)_{H^2(I)}, \\ n(q)(\delta q) &= j''(q)(\delta q, \delta q) - \alpha(\delta q, \delta q)_{H^2(I)}. \end{aligned}$$

Then it holds that

$$\left.\begin{aligned} m(q)(\delta q_n) &\to m(q)(\delta q), \\ n(q)(\delta q_n) &\to n(q)(\delta q), \end{aligned}\right. \qquad \text{for } n \to \infty.$$

Proof. This lemma is a direct consequence of Definition 2.3.6 and Lemma 2.3.53. ☐

Lemma 2.3.55. *Let $q \in Q^{\text{ad}}$, $\delta q \in Q$ and $(\delta q_n)_{n\in\mathbb{N}} \subset Q$. If $\delta q_n \rightharpoonup \delta q$ in $H^2(I)$ then*

$$\begin{aligned} j'(q)(\delta q_n) &\to j'(q)(\delta q), \\ j''(q)(\delta q, \delta q) &\leq \liminf_{n\to\infty} j''(q)(\delta q_n, \delta q_n). \end{aligned}$$

Proof. As $H^2(I)$ is compactly embedded into $H^{3/2+\varepsilon}(I)$ for $\varepsilon < 1/2$, we get $\delta q_n \to \delta q$ in $H^{3/2+\varepsilon}(I)$. As $(q, \delta q_n)_{H^2(I)} \to (q, \delta q)_{H^2(I)}$, the first part follows from the first part of Lemma 2.3.54. The squared H^2-norm is a continuous and convex functional on $H^2(I)$ and therefore weakly lower semi-continuous, hence $\|\delta q\|^2_{H^2(I)} \leq \liminf_{n\to\infty} \|\delta q_n\|^2_{H^2(I)}$, and the second part follows from the second part of Lemma 2.3.54. ☐

Lemma 2.3.56. *Let $q \in Q^{\text{ad}}$, $\delta q \in Q$, $(\delta q_n)_{n\in\mathbb{N}} \subset Q$ and $\delta q_n \rightharpoonup \delta q$ in $H^2(I)$. If*

$$\lim_{n\to\infty} j''(q)(\delta q_n, \delta q_n) = j''(q)(\delta q, \delta q),$$

then

$$\delta q_n \to \delta q \qquad \text{in } H^2(I).$$

Proof. Again we get $\delta q_n \to \delta q$ in $H^{3/2+\varepsilon}(I)$ for $\varepsilon < 1/2$. With the second part of Lemma 2.3.54 it follows that $\|\delta q_n\|_{H^2(I)} \to \|\delta q\|_{H^2(I)}$. The result follows from the fact that within Hilbert spaces, strong convergence is equivalent to weak convergence plus convergence of the norm. ☐

Theorem 2.3.57. *Let $\overline{q} \in Q^{\text{ad}}$ be a solution of (2.27) fulfilling Assumption 2.3.51. Then there exists $\beta > 0$ such that*

$$j''(\overline{q})(\delta q, \delta q) \geq \beta \|\delta q\|^2_{H^2(I)} \qquad \forall \delta q \in Q. \tag{2.124}$$

Proof. Assume that (2.124) does not hold. Then there exists a sequence $(\delta q_n)_{n\in\mathbb{N}} \subset Q$ with

$$\|\delta q_n\|_{H^2(I)} = 1$$

and

$$j''(\overline{q})(\delta q_n, \delta q_n) < \frac{1}{n}.$$

Possibly after extracting a subsequence we get the existence of an element $\overline{\delta q} \in Q$ with $\delta q_n \rightharpoonup \overline{\delta q}$ in $H^2(I)$. We get

$$0 \leq j''(\overline{q})(\overline{\delta q}, \overline{\delta q}) \leq \liminf_{n\to\infty} j''(\overline{q})(\delta q_n, \delta q_n) \leq \limsup_{n\to\infty} j''(\overline{q})(\delta q_n, \delta q_n) \leq \limsup_{n\to\infty} \frac{1}{n} = 0. \tag{2.125}$$

The first inequality is just the necessary optimality condition of second order (2.120), and the second inequality is due to Lemma 2.3.55. Equation (2.125) yields

$$j''(\overline{q})(\delta q_n, \delta q_n) \to j''(\overline{q})(\overline{\delta q}, \overline{\delta q}) = 0.$$

As a result, Assumption 2.3.51 implies $\overline{\delta q} = 0$, whereas Lemma 2.3.56 implies $\delta q_n \to \overline{\delta q}$ in $H^2(I)$, which contradicts the fact that $\|\delta q_n\|_{H^2(I)} = 1$. ☐

Lemma 2.3.58. *Let $q \in Q^{\mathrm{ad}}$. If there exists $\beta > 0$ such that*

$$j''(q)(\delta q, \delta q) \geq \beta \|\delta q\|^2_{H^2(I)} \qquad \forall \delta q \in Q,$$

then there exists $\delta > 0$ such that for all $p \in Q^{\mathrm{ad}}$ with $\|q - p\|_{H^2(I)} \leq \delta$ it holds that

$$j''(p)(\delta q, \delta q) \geq \frac{\beta}{2} \|\delta q\|^2_{H^2(I)} \qquad \forall \delta q \in Q.$$

Proof. With Lemma 2.3.24 we get the existence of $c_0 > 0$ such that

$$\begin{aligned}
j''(p)(\delta q, \delta q) &= j''(q)(\delta q, \delta q) + j''(p)(\delta q, \delta q) - j''(q)(\delta q, \delta q)\\
&\geq j''(q)(\delta q, \delta q) - \left| j''(p)(\delta q, \delta q) - j''(q)(\delta q, \delta q)\right|\\
&\geq \beta \|\delta q\|^2_{H^2(I)} - c_0 \|q - p\|_{H^2(I)} \|\delta q\|^2_{H^2(I)}\\
&= \Big(\beta - c_0 \|q - p\|_{H^2(I)}\Big) \|\delta q\|^2_{H^2(I)},
\end{aligned}$$

and the result follows for $\delta \leq \frac{\beta}{2c_0}$. □

Lemma 2.3.59. *Let $\overline{q} \in Q^{\mathrm{ad}}$ be a solution of (2.27). Then the following two statements are equivalent.*

- *There exists $\beta_1 > 0$ such that*

$$j''(\overline{q})(\delta q, \delta q) \geq \beta_1 \|\delta q\|^2_{H^2(I)} \qquad \forall \delta q \in Q. \tag{2.126}$$

- *There exist $\beta_2, \delta > 0$ such that*

$$j(p) \geq j(\overline{q}) + \beta_2 \|p - \overline{q}\|^2_{H^2(I)} \qquad \forall p \in Q^{\mathrm{ad}}\colon\ \|p - \overline{q}\|_{H^2(I)} \leq \delta. \tag{2.127}$$

Proof. If (2.126) holds, then due to Theorem A.1.12 we have for some $t \in [0, 1]$ that

$$\begin{aligned}
j(p) &= j(\overline{q}) + j'(\overline{q})(p - \overline{q}) + \frac{1}{2} j''(\overline{q} + t(p - \overline{q}))(p - \overline{q}, p - \overline{q})\\
&= j(\overline{q}) + \frac{1}{2} j''(\overline{q} + t(p - \overline{q}))(p - \overline{q}, p - \overline{q})\\
&\geq j(\overline{q}) + \frac{\beta_1}{4} \|p - \overline{q}\|^2_{H^2(I)},
\end{aligned}$$

whereas in the second step we used the first order optimality condition (2.121), in the third step we used Lemma 2.3.58.

If the second assertion, (2.127), holds, then $\overline{q}$ is a solution to

$$\min_{\substack{q \in Q^{\mathrm{ad}}\\ \|q - \overline{q}\|_{H^2(I)} \leq \delta}} \Big(j(q) - \beta_2 \|q - \overline{q}\|^2_{H^2(I)} \Big),$$

and the necessary optimality condition of second order yields

$$j''(\overline{q})(\delta q, \delta q) - 2\beta_2 \|\delta q\|^2_{H^2(I)} \geq 0 \qquad \forall \delta q \in Q.$$ □

Lemma 2.3.60. *There exists $\varepsilon > 0$ such that for all $q_\sigma \in Q_\sigma$ with $\|q_\sigma - \overline{q}_\sigma\|_{H^2(I)} \leq \varepsilon$, where $\overline{q}_\sigma$ is a local optimal control of* (2.59) *with $\|\overline{q}_\sigma - \overline{q}\|_{H^2(I)} \leq \varepsilon$, it holds that*

$$j(q_\sigma) \geq j(\overline{q}_\sigma) + \frac{\beta}{4} \|q_\sigma - \overline{q}_\sigma\|^2_{H^2(I)}.$$

Proof. As in the proof of Lemma 2.3.59 we use a taylor expansion around $\overline{q}_\sigma$. There exists $t \in [0, 1]$ such that for $\xi = t\overline{q}_\sigma + (1 - t)q_\sigma$ it holds that

$$\begin{aligned}
j(q_\sigma) &= j(\overline{q}_\sigma) + j'(\overline{q}_\sigma)(\overline{q}_\sigma - q_\sigma) + \frac{1}{2} j''(\xi)(\overline{q}_\sigma - q_\sigma, \overline{q}_\sigma - q_\sigma) \\
&= j(\overline{q}_\sigma) + \frac{1}{2} j''(\xi)(\overline{q}_\sigma - q_\sigma, \overline{q}_\sigma - q_\sigma) \\
&\geq j(\overline{q}_\sigma) + \frac{\beta}{4} \|\overline{q}_\sigma - q_\sigma\|^2_{H^2(I)}.
\end{aligned}$$

In the first step we used the first order optimality condition (2.121), in the second step we used Lemma 2.3.58 and the fact that $\|\overline{q} - \xi\|_{H^2(I)}$ can be made arbitrarily small by a suitable choice of $\varepsilon > 0$. □

We can also prove similar statements regarding the discrete cost functional j_h.

Lemma 2.3.61. *There exists $\varepsilon > 0$ such that for all $p \in Q^{\text{ad}}$ with $\|\overline{q} - p\|_{H^2(I)} \leq \varepsilon$ and all h sufficiently small it holds that*

$$j_h''(p)(\delta q, \delta q) \geq \frac{\beta}{4} \|\delta q\|^2_{H^2(I)} \qquad \forall \delta q \in Q.$$

Proof. Using Lemma 2.3.28 we get the existence of $c_0 > 0$ such that

$$\begin{aligned}
j_h''(p)(\delta q, \delta q) &\geq j''(p)(\delta q, \delta q) - \left| j''(p)(\delta q, \delta q) - j_h''(p)(\delta q, \delta q) \right| \\
&\geq \frac{\beta}{2} \|\delta q\|^2_{H^2(I)} - c_0 h^{1/4} \|\delta q\|^2_{H^2(I)} \\
&= \left(\frac{\beta}{2} - c_0 h^{1/4} \right) \|\delta q\|^2_{H^2(I)} \\
&\geq \frac{\beta}{4} \|\delta q\|^2_{H^2(I)},
\end{aligned}$$

which holds for

$$h \leq \left(\frac{\beta}{4c_0} \right)^4. \qquad \square$$

Lemma 2.3.62. *There exists $\varepsilon > 0$ such that for all $q_{\sigma,h} \in Q_\sigma$ with $\left\|q_{\sigma,h} - \overline{q}_{\sigma,h}\right\|_{H^2(I)} \leq \varepsilon$, where $\overline{q}_{\sigma,h}$ is a local optimal control of* (2.66) *with $\left\|\overline{q} - \overline{q}_{\sigma,h}\right\|_{H^2(I)} \leq \varepsilon$, it holds that*

$$j_h(q_{\sigma,h}) \geq j_h(\overline{q}_{\sigma,h}) + \frac{\beta}{8} \left\|q_{\sigma,h} - \overline{q}_{\sigma,h}\right\|^2_{H^2(I)}.$$

Proof. We use a taylor expansion around $\overline{q}_{\sigma,h}$, it holds with $\xi_h = t\overline{q}_{\sigma,h} + (1-t)q_{\sigma,h}$ for a $t \in [0,1]$ that

$$\begin{aligned} j_h(q_{\sigma,h}) &= j_h(\overline{q}_{\sigma,h}) + j_h'(\overline{q}_{\sigma,h})(\overline{q}_{\sigma,h} - q_{\sigma,h}) + \frac{1}{2} j_h''(\xi_h)(\overline{q}_{\sigma,h} - q_{\sigma,h}, \overline{q}_{\sigma,h} - q_{\sigma,h}) \\ &= j_h(\overline{q}_{\sigma,h}) + \frac{1}{2} j_h''(\xi_h)(\overline{q}_{\sigma,h} - q_{\sigma,h}, \overline{q}_{\sigma,h} - q_{\sigma,h}) \\ &\geq j_h(\overline{q}_{\sigma,h}) + \frac{\beta}{8} \left\| \overline{q}_{\sigma,h} - q_{\sigma,h} \right\|_{H^2(I)}^2 . \end{aligned}$$

In the first step we used the first order optimality condition (2.121), in the second step we used Lemma 2.3.61 and the fact that $\|\overline{q} - \xi_h\|_{H^2(I)}$ can be made arbitrarily small by a suitable choice of $\varepsilon > 0$. □

Finally we also state the versions for the fully discretized cost functional $j_{h,k}$.

Lemma 2.3.63. *There exists $\varepsilon > 0$ such that for all $p \in Q^{\text{ad}}$ with $\|\overline{q} - p\|_{H^2(I)} \leq \varepsilon$ and all h, k sufficiently small it holds that*

$$j_{h,k}''(p)(\delta q, \delta q) \geq \frac{\beta}{8} \|\delta q\|_{H^2(I)}^2 \qquad \forall \delta q \in Q.$$

Proof. Using Lemma 2.3.34 we get the existence of $c_0 > 0$ such that

$$\begin{aligned} j_{h,k}''(p)(\delta q, \delta q) &\geq j_h''(p)(\delta q, \delta q) - \left| j_h''(p)(\delta q, \delta q) - j_{h,k}''(p)(\delta q, \delta q) \right| \\ &\geq \frac{\beta}{4} \|\delta q\|_{H^2(I)}^2 - c_0 k^{1/2} \|\delta q\|_{H^2(I)}^2 \\ &= \left(\frac{\beta}{4} - c_0 k^{1/2} \right) \|\delta q\|_{H^2(I)}^2 . \end{aligned}$$

Hence, for $k \leq \left(\frac{\beta}{8c_0}\right)^2$ and h sufficiently small such that Lemma 2.3.61 is applicable it holds that

$$j_{h,k}''(p)(\delta q, \delta q) \geq \frac{\beta}{8} \|\delta q\|_{H^2(I)}^2 .$$ □

The following lemma can be proven in the same way as Lemma 2.3.60 and Lemma 2.3.62.

Lemma 2.3.64. *There exists $\varepsilon > 0$ such that for all $q_{\sigma,h,k} \in Q_\sigma$ with $\left\| q_{\sigma,h,k} - \overline{q}_{\sigma,h,k} \right\|_{H^2(I)} \leq \varepsilon$, where $\overline{q}_{\sigma,h,k}$ is a local optimal control of* (2.73) *with $\left\| \overline{q} - \overline{q}_{\sigma,h,k} \right\|_{H^2(I)} \leq \varepsilon$, it holds that*

$$j_{h,k}(q_{\sigma,h,k}) \geq j_{h,k}(\overline{q}_{\sigma,h,k}) + \frac{\beta}{16} \left\| q_{\sigma,h,k} - \overline{q}_{\sigma,h,k} \right\|_{H^2(I)}^2 .$$

2.3.5. On the existence of a converging subsequence

In order to prove Theorem 2.3.1 we need to show that for every fixed $\overline{q}$ there exist sequences of local optimal controls $(\overline{q}_\sigma)_{\sigma>0} \subset Q_\sigma^{\text{ad}}$, $\left(\overline{q}_{\sigma,h}\right)_{\sigma,h>0} \subset Q_\sigma^{\text{ad}}$ and $\left(\overline{q}_{\sigma,h,k}\right)_{\sigma,h,k>0} \subset Q_\sigma^{\text{ad}}$ to the discretized problems (2.59), (2.66) and (2.73), respectively, that converge to $\overline{q}$ in $H^2(I)$ for $\sigma, h, k \to 0$. What follows adapts a method presented in [28]. At first we fix $\overline{q}$ as the solution to the continuous optimal control problem and let

$$Q_{\sigma,\varepsilon}^{\text{ad}} = \left\{ q_\sigma \in Q_\sigma^{\text{ad}} \,\middle|\, \|\overline{q} - q_\sigma\|_{H^2(I)} \leq \varepsilon \right\}, \tag{2.128}$$

where $\varepsilon > 0$ is chosen sufficiently small such that Lemma 2.3.59, (2.127), holds for all $q_\sigma \in Q^{\text{ad}}_{\sigma,\varepsilon}$. Let $i_\sigma \colon Q \to Q_\sigma$ be the interpolation operator from Definition 2.2.1, with Lemma 2.2.2 it follows that $Q_\sigma \ni i_\sigma \overline{q} \to \overline{q}$ in $H^2(I)$, hence $Q^{\text{ad}}_{\sigma,\varepsilon} \neq \emptyset$ for $\sigma = \sigma(\varepsilon)$ sufficiently small, where we have to assume that $\overline{q} \in \operatorname{int}(Q^{\text{ad}})$. Let $\overline{q}_{\sigma,\varepsilon}$ be the optimal solution to

$$\min_{q_{\sigma,\varepsilon} \in Q^{\text{ad}}_{\sigma,\varepsilon}} j(q_{\sigma,\varepsilon}). \tag{2.129}$$

Lemma 2.3.65. *For $\sigma \to 0$ it holds that $j(\overline{q}_{\sigma,\varepsilon}) \to j(\overline{q})$.*

Proof. We have

$$j(\overline{q}_{\sigma,\varepsilon}) \geq j(\overline{q}),$$

due to the definition of $\overline{q}$, and using Lemma 2.3.24 we get

$$j(\overline{q}_{\sigma,\varepsilon}) - j(\overline{q}) \leq j(i_\sigma \overline{q}) - j(\overline{q}) \leq c \left\| i_\sigma \overline{q} - \overline{q} \right\|_{H^2(I)} \to 0 \qquad \text{for } \sigma \to 0. \qquad \square$$

Lemma 2.3.66. *For $\sigma \to 0$ it holds that $\left\| \overline{q}_{\sigma,\varepsilon} - \overline{q} \right\|_{H^2(I)} \to 0$.*

Proof. Using Lemma 2.3.59 it follows that

$$j(\overline{q}_{\sigma,\varepsilon}) - j(\overline{q}) \geq \frac{\beta}{2} \left\| \overline{q}_{\sigma,\varepsilon} - \overline{q} \right\|^2_{H^2(I)},$$

and the proof follows with Lemma 2.3.65. □

Lemma 2.3.67. *For σ sufficiently small, $\overline{q}_{\sigma,\varepsilon}$ is a local solution to the partially discretized problem* (2.59).

Proof. We have to show that all elements $q_\sigma \in Q^{\text{ad}}_\sigma$ which are sufficiently close to $\overline{q}_{\sigma,\varepsilon}$ are also elements of $Q^{\text{ad}}_{\sigma,\varepsilon}$. Choose σ such that $\left\| \overline{q}_{\sigma,\varepsilon} - \overline{q} \right\|_{H^2(I)} \leq \frac{\varepsilon}{2}$. Now, if $q_\sigma \in Q^{\text{ad}}_\sigma$ and $\left\| q_\sigma - \overline{q}_{\sigma,\varepsilon} \right\| \leq \frac{\varepsilon}{2}$ it holds that

$$\left\| \overline{q} - q_\sigma \right\|_{H^2(I)} \leq \left\| \overline{q} - \overline{q}_{\sigma,\varepsilon} \right\|_{H^2(I)} + \left\| \overline{q}_{\sigma,\varepsilon} - q_\sigma \right\|_{H^2(I)} \leq \varepsilon,$$

which shows that $q_\sigma \in Q^{\text{ad}}_{\sigma,\varepsilon}$. □

Now fix $\overline{q}_\sigma$ as a local solution to (2.59), close to $\overline{q}$, and define

$$Q^{\text{ad}}_{\sigma,h,\varepsilon} = \left\{ q_\sigma \in Q^{\text{ad}}_\sigma \,\middle|\, \left\| q_\sigma - \overline{q}_\sigma \right\|_{H^2(I)} \leq \varepsilon \right\}, \tag{2.130}$$

where σ and ε have to be chosen sufficiently small such that Lemma 2.3.60 holds in $\overline{q}_\sigma$ for all $q_\sigma \in Q^{\text{ad}}_{\sigma,h,\varepsilon}$. Now let $\overline{q}_{\sigma,h,\varepsilon}$ be the optimal solution to

$$\min_{q_{\sigma,h,\varepsilon} \in Q^{\text{ad}}_{\sigma,h,\varepsilon}} j_h(q_{\sigma,h,\varepsilon}). \tag{2.131}$$

Lemma 2.3.68. *For $h \to 0$ it holds that $j_h(\overline{q}_{\sigma,h,\varepsilon}) \to j(\overline{q}_\sigma)$.*

Proof. It holds that

$$j_h(\overline{q}_{\sigma,h,\varepsilon}) \leq j_h(\overline{q}_\sigma),$$

and Lemma 2.3.28 shows

$$j_h(\overline{q}_{\sigma,h,\varepsilon}) - j(\overline{q}_\sigma) \leq j_h(\overline{q}_\sigma) - j(\overline{q}_\sigma) \leq ch^{1/2}.$$

Because of

$$j(\overline{q}_\sigma) \leq j(\overline{q}_{\sigma,h,\varepsilon}),$$

it follows that

$$j_h(\overline{q}_{\sigma,h,\varepsilon}) - j(\overline{q}_\sigma) \geq j_h(\overline{q}_{\sigma,h,\varepsilon}) - j(q_{\sigma,h,\varepsilon}) \geq -ch^{1/2}.$$ □

Lemma 2.3.69. *For $h \to 0$ it holds that $j(\overline{q}_{\sigma,h,\varepsilon}) \to j(\overline{q}_\sigma)$.*

Proof. It holds that

$$\left|j(\overline{q}_{\sigma,h,\varepsilon}) - j(\overline{q}_\sigma)\right| \leq \left|j(\overline{q}_{\sigma,h,\varepsilon}) - j_h(\overline{q}_{\sigma,h,\varepsilon})\right| + \left|j_h(\overline{q}_{\sigma,h,\varepsilon}) - j(\overline{q}_\sigma)\right| \to 0,$$

where for the first term we used Lemma 2.3.28, for the second term we used Lemma 2.3.68. □

Lemma 2.3.70. *For $h \to 0$ it holds that $\left\|\overline{q}_{\sigma,h,\varepsilon} - \overline{q}_\sigma\right\|_{H^2(I)} \to 0$.*

Proof. Using Lemma 2.3.60 it holds that

$$j(\overline{q}_{\sigma,h,\varepsilon}) - j(\overline{q}_\sigma) \geq \frac{\beta}{4}\left\|\overline{q}_{\sigma,h,\varepsilon} - \overline{q}_\sigma\right\|^2_{H^2(I)},$$

and the proof follows with Lemma 2.3.69. □

Lemma 2.3.71. *For h sufficiently small, $\overline{q}_{\sigma,h,\varepsilon}$ is also a local solution to the partially discretized problem* (2.66).

Proof. We have to show that all elements $q_\sigma \in Q_\sigma^{\mathrm{ad}}$, which are sufficiently close to $\overline{q}_{\sigma,h,\varepsilon}$ are also elements of $Q^{\mathrm{ad}}_{\sigma,h,\varepsilon}$. Choose h such that $\left\|\overline{q}_{\sigma,h,\varepsilon} - \overline{q}_\sigma\right\|_{H^2(I)} \leq \frac{\varepsilon}{2}$. Now, if $q_\sigma \in Q_\sigma^{\mathrm{ad}}$ and $\left\|q_\sigma - \overline{q}_{\sigma,h,\varepsilon}\right\| \leq \frac{\varepsilon}{2}$, then it holds that

$$\left\|\overline{q}_\sigma - q_\sigma\right\|_{H^2(I)} \leq \left\|\overline{q}_\sigma - \overline{q}_{\sigma,h,\varepsilon}\right\|_{H^2(I)} + \left\|\overline{q}_{\sigma,h,\varepsilon} - q_\sigma\right\|_{H^2(I)} \leq \varepsilon,$$

which gives $q_\sigma \in Q^{\mathrm{ad}}_{\sigma,h,\varepsilon}$. □

At last, fix $\overline{q}_{\sigma,h}$ as a local optimal solution to (2.66) and define

$$Q^{\mathrm{ad}}_{\sigma,h,k,\varepsilon} = \left\{ q_\sigma \in Q^{\mathrm{ad}}_\sigma \,\middle|\, \left\|q_\sigma - \overline{q}_{\sigma,h}\right\|_{H^2(I)} \leq \varepsilon \right\}, \tag{2.132}$$

where σ, h and ε are chosen sufficiently small such that Lemma 2.3.62 holds in $\overline{q}_{\sigma,h}$ for all $q_\sigma \in Q^{\mathrm{ad}}_{\sigma,h,k,\varepsilon}$. Now let $\overline{q}_{\sigma,h,k,\varepsilon}$ be the optimal solution to

$$\min_{q_{\sigma,h,k,\varepsilon} \in Q^{\mathrm{ad}}_{\sigma,h,k,\varepsilon}} j_{h,k}(q_{\sigma,h,k,\varepsilon}). \tag{2.133}$$

Lemma 2.3.72. *For $k \to 0$ it holds that $j_{h,k}(\overline{q}_{\sigma,h,k,\varepsilon}) \to j_h(\overline{q}_{\sigma,h})$.*

Proof. It holds that

$$j_{h,k}(\overline{q}_{\sigma,h,k,\varepsilon}) \leq j_{h,k}(\overline{q}_{\sigma,h}),$$

and using Lemma 2.3.34 we conclude that

$$j_{h,k}(\overline{q}_{\sigma,h,k,\varepsilon}) - j_h(\overline{q}_{\sigma,h}) \leq j_{h,k}(\overline{q}_{\sigma,h}) - j_h(\overline{q}_{\sigma,h}) \leq ck^{1/2}.$$

Because of

$$j_h(\overline{q}_{\sigma,h}) \leq j_h(\overline{q}_{\sigma,h,k,\varepsilon})$$

it follows that

$$j_{h,k}(\overline{q}_{\sigma,h,k,\varepsilon}) - j_h(\overline{q}_{\sigma,h}) \geq j_{h,k}(\overline{q}_{\sigma,h,k,\varepsilon}) - j_h(\overline{q}_{\sigma,h,k,\varepsilon}) \geq -ck^{1/2}.$$ □

Lemma 2.3.73. *For $k \to 0$ it holds that $j_h(\overline{q}_{\sigma,h,k,\varepsilon}) \to j_h(\overline{q}_{\sigma,h})$.*

Proof. It holds that

$$\left|j_h(\overline{q}_{\sigma,h,k,\varepsilon}) - j_h(\overline{q}_{\sigma,h})\right| \leq \left|j_h(\overline{q}_{\sigma,h,k,\varepsilon}) - j_{h,k}(\overline{q}_{\sigma,h,k,\varepsilon})\right| + \left|j_{h,k}(\overline{q}_{\sigma,h,k,\varepsilon}) - j_h(\overline{q}_{\sigma,h})\right| \to 0,$$

where for the first term we used Lemma 2.3.34, for the second term we used Lemma 2.3.72. □

Lemma 2.3.74. *For $k \to 0$ it holds that $\left\|\overline{q}_{\sigma,h,k,\varepsilon} - \overline{q}_{\sigma,h}\right\|_{H^2(I)} \to 0$.*

Proof. Using Lemma 2.3.73 it follows that

$$j_h(\overline{q}_{\sigma,h,k,\varepsilon}) - j_h(\overline{q}_{\sigma,h}) \geq \frac{\beta}{8} \left\|\overline{q}_{\sigma,h,k,\varepsilon} - \overline{q}_{\sigma}\right\|^2_{H^2(I)},$$

and the proof follows with Lemma 2.3.62. □

Lemma 2.3.75. *For k sufficiently small, $\overline{q}_{\sigma,h,k,\varepsilon}$ is also a local solution to the fully discretized problem* (2.73).

Proof. We have to show that all elements $q_\sigma \in Q^{\mathrm{ad}}_\sigma$, which are sufficiently close to $\overline{q}_{\sigma,h,k,\varepsilon}$ are also elements of $Q^{\mathrm{ad}}_{\sigma,h,k,\varepsilon}$. Choose k such that $\left\|\overline{q}_{\sigma,h,k,\varepsilon} - \overline{q}_{\sigma}\right\|_{H^2(I)} \leq \frac{\varepsilon}{2}$. Then, for $q_\sigma \in Q^{\mathrm{ad}}_\sigma$ and $\left\|q_\sigma - \overline{q}_{\sigma,h,k,\varepsilon}\right\|_{H^2(I)} \leq \frac{\varepsilon}{2}$ it holds that

$$\left\|\overline{q}_{\sigma,h} - q_\sigma\right\|_{H^2(I)} \leq \left\|\overline{q}_{\sigma,h} - \overline{q}_{\sigma,h,k,\varepsilon}\right\|_{H^2(I)} + \left\|\overline{q}_{\sigma,h,k,\varepsilon} - q_\sigma\right\|_{H^2(I)} \leq \varepsilon,$$

which shows that $q_\sigma \in Q^{\mathrm{ad}}_{\sigma,h,k,\varepsilon}$. □

Lemma 2.3.76. *There exist sequences $(\overline{q}_\sigma)_{\sigma>0}$, $\left(\overline{q}_{\sigma,h}\right)_{\sigma,h>0}$ and $\left(\overline{q}_{\sigma,h,k}\right)_{\sigma,h,k>0}$ of local optimal solutions to* (2.59), (2.66) *and* (2.73), *respectively, with*

$$\lim_{\sigma\to 0} \|\overline{q}_\sigma - \overline{q}\|_{H^2(I)} = \lim_{\sigma,h\to 0} \left\|\overline{q}_{\sigma,h} - \overline{q}\right\|_{H^2(I)} = \lim_{\sigma,h,k\to 0} \left\|\overline{q}_{\sigma,h,k} - \overline{q}\right\|_{H^2(I)} = 0.$$

Proof. The existence of $(\overline{q}_\sigma)_{\sigma>0}$ follows with Lemma 2.3.66 and Lemma 2.3.67, the existence of $\left(\overline{q}_{\sigma,h}\right)_{\sigma,h>0}$ follows with Lemma 2.3.70, Lemma 2.3.71 and the first part of this lemma, and the existence of $\left(\overline{q}_{\sigma,h,k}\right)_{\sigma,h,k>0}$ follows with Lemma 2.3.74, Lemma 2.3.75 and the first two parts of this lemma. □

No we can finally finish the proof of Theorem 2.3.1.

Proof. Let $\overline{q}$ be an optimal control for (2.27) and let $\overline{q}_{\sigma,h,k}$ be an optimal control for (2.73) for σ, h and k sufficiently small, such that Lemma 2.3.63 holds for $\overline{q}_{\sigma,h,k}$. The existence of such a $\overline{q}_{\sigma,h,k}$ is guaranteed by Lemma 2.3.76. Now there exists $t \in [0,1]$ such that with $\xi = t\overline{q} + (1-t)\overline{q}_{\sigma,h,k}$ it holds that

$$\begin{aligned} c\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2 &\leq j''_{h,k}(\xi)(\overline{q}-\overline{q}_{\sigma,h,k},\overline{q}-\overline{q}_{\sigma,h,k}) \\ &= j'_{h,k}(\overline{q})(\overline{q}-\overline{q}_{\sigma,h,k}) - j'_{h,k}(\overline{q}_{\sigma,h,k})(\overline{q}-\overline{q}_{\sigma,h,k}) \\ &= j'_{h,k}(\overline{q})(\overline{q}-\overline{q}_{\sigma,h,k}) - j'_{h,k}(\overline{q}_{\sigma,h,k})(\overline{q}-i_\sigma\overline{q}), \end{aligned}$$

where we used the first order optimality condition in $\overline{q}_{\sigma,h,k}$. Using the first order optimality condition in $\overline{q}$ we get

$$\begin{aligned} c\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2 &\leq j'_{h,k}(\overline{q})(\overline{q}-\overline{q}_{\sigma,h,k}) - j'(\overline{q})(\overline{q}-\overline{q}_{\sigma,h,k}) \\ &\quad + j'_{h,k}(\overline{q})(\overline{q}-i_\sigma\overline{q}) - j'_{h,k}(\overline{q}_{\sigma,h,k})(\overline{q}-i_\sigma\overline{q}) \\ &\quad + j'(\overline{q})(\overline{q}-i_\sigma\overline{q}) - j'_{h,k}(\overline{q})(\overline{q}-i_\sigma\overline{q}), \end{aligned}$$

and using Lemma 2.3.41, Lemma 2.3.50 and Lemma 2.3.36 we arrive at

$$\begin{aligned} \left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2 &\leq c\left(h^2+k^2\right)\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)} \\ &\quad + c\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}\left\|\overline{q}-i_\sigma\overline{q}\right\|_{H^2(I)} \\ &\quad + c\left(h^2+k^2\right)\left\|\overline{q}-i_\sigma\overline{q}\right\|_{H^2(I)}. \end{aligned}$$

With Young's inequality we get the existence of a $c_1 > 0$ such that

$$\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2 \leq c_1\left(\left(h^2+k^2\right)^2 + \left\|\overline{q}-i_\sigma\overline{q}\right\|_{H^2(I)}^2\right) + \frac{1}{2}\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2.$$

For $a, b \geq 0$ it holds that $\sqrt{a^2+b^2} \leq a+b$, and using Lemma 2.2.2 we finally end up with

$$\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)} \leq c\left(\sigma^2+h^2+k^2\right).$$

□

3. Optimization of eigenvalues

The aim of this chapter is to apply the framework introduced in Chapter 2 to a shape optimization problem where the cost functional includes eigenvalues of a partial differential operator.

This chapter is organized as follows. In Section 3.1 we present the problem under consideration and apply the framework presented in Section 2.1 to this problem, i.e. we transform the problem onto a fixed reference domain, show existence of an optimal solution and prove some regularity and differentiability results. As even normalized eigenfunctions are only defined up to their sign, Section 3.2 deals with estimating the difference between two eigenfunctions corresponding to different controls. In Section 3.3 we discretize the original problem using finite elements similar to Section 2.2. Finally, in Section 3.4 we prove an a-priori error estimate for the error between the optimal control and a sequence of local optimal controls to the fully discretized problem.

3.1. The problem

Within this chapter we focus on the maximization of the distance between the first two eigenvalues of a partial differential operator corresponding to the transmission problem. This choice of the cost functional may be motivated as follows. Multiple eigenvalues are in general no longer Fréchet-differentiable with respect to domain perturbations, this irregularity is also responsible for some physical effects. In the context of musical instruments, for example, it is possible to hear some undesired interferences if some of the lower eigenvalues are too close to each other. For a more detailed investigation onto that topic we refer to [43].

The exact definition of the cost functional to be minimized will be given in (3.10), we start with some preliminaries. In what follows, the notation will be very similar to that in Chapter 2. As in Section 2.1, let $q \in Q = H^2_{\text{per}}(I)$ with $I = (0, 2\pi)$ be the control variable, and let

$$\Omega_q = \left\{ (x, y) \in \mathbb{R}^2 \middle| -2 < x, y < 2 \right\} \subset \mathbb{R}^2,$$

be the interior of a square with side length 4, centered at the origin and sides parallel to the axes. Let Ω_q be divided into an inner, star-shaped domain,

$$\Omega_{q,0} = \left\{ (x, y) \in \mathbb{R}^2 \middle| r < 1 + q(\varphi),\, r = \sqrt{x^2 + y^2},\, \varphi = \arg(x + iy) \right\},$$

and the outer domain,

$$\Omega_{q,1} = \Omega_q \backslash \overline{\Omega_{q,0}},$$

see Figure 3.1. In order to exclude a possible degeneracy of the domain $\Omega_{q,0}$ we fix $\overline{\varepsilon} > 0$ and define

$$\overline{Q}^{\text{ad}} = \left\{ q \in Q \middle| q(\varphi) \geq -1 + \overline{\varepsilon}\ \forall \varphi \in I \text{ and } \overline{\Omega_{q,0}} \subset \Omega_q \right\}. \tag{3.1}$$

As $H^2(I) \hookrightarrow C^{1,1/2}(\overline{I})$, (3.1) is well-defined. Now let $d > 0$ be a constant which shall remain fixed

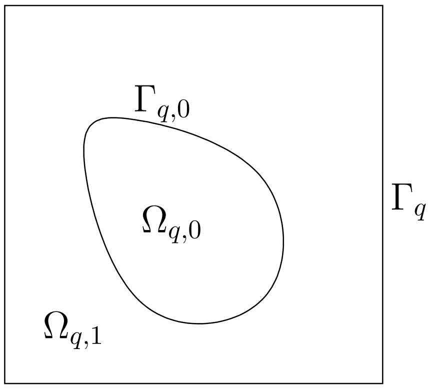

Figure 3.1.: The original domain Ω_q

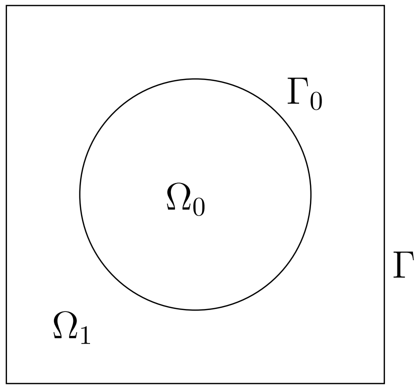

Figure 3.2.: The transformed domain Ω

throughout this chapter and let $\tilde{L}\colon H_0^1(\Omega_q) \to H^{-1}(\Omega_q)$ be the partial differential operator such that for $u \in H_0^1(\Omega_q)$ and $f \in H^{-1}(\Omega_q)$ the equation $\tilde{L}u = f$ shall be a formulation for u being the unique weak solution to

$$\begin{cases} -d\,\Delta u = f & \text{in } \Omega_{q,0}, \qquad -\Delta u = f \quad \text{in } \Omega_{q,1}, \\ [u]_q = 0 & \text{on } \Gamma_{q,0} = \partial\Omega_{q,0}, \qquad u = 0 \quad \text{on } \Gamma_q = \partial\Omega_q, \\ d\,\partial_n u_{q,-} = \partial_n u_{q,+} & \text{on } \Gamma_{q,0}, \end{cases} \tag{3.2}$$

where $[u]_q$ is defined as follows. For $x \in \Gamma_{q,0}$ let

$$u_{q,+}(x) = \lim_{\substack{y \to x \\ y \in \Omega_{q,1}}} u(y), \qquad u_{q,-}(x) = \lim_{\substack{y \to x \\ y \in \Omega_{q,0}}} u(y), \tag{3.3}$$

be the function values when approaching $\Gamma_{q,0}$ from either $\Omega_{q,1}$ or $\Omega_{q,0}$ in a nontangential way, cf. [40], and let

$$[u]_q = u_{q,+} - u_{q,-},$$

be the jump of u over $\Gamma_{q,0}$. It can easily be derived that the weak formulation of (3.2) reads as

$$(\nabla u, \nabla v)_{\Omega_{q,1}} + d\,(\nabla u, \nabla v)_{\Omega_{q,0}} = (f, v)_{\Omega_q} \qquad \forall v \in H_0^1(\Omega_q). \tag{3.4}$$

As $H_0^1(\Omega_q)$ is compactly embedded into $L^2(\Omega_q)$, it follows that $\tilde{L}^{-1}$ is a compact and self-adjoint operator on $L^2(\Omega_q)$. Hence, for each fixed $q \in \overline{Q}^{\mathrm{ad}}$, the spectral theorem for compact operators, Theorem A.1.2, yields the existence of a sequence $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{R}^+$ with $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ (counted with multiplicity) and

$$\lim_{i\to\infty} \lambda_i = \infty,$$

and a sequence of eigenfunctions $(u_i)_{i\in\mathbb{N}} \subset H_0^1(\Omega_q)$ with

$$\tilde{L}u_i = \lambda_i u_i \qquad \forall i \in \mathbb{N}, \tag{3.5}$$

and eigenfunctions to different eigenvalues are orthogonal with respect to the L^2-scalar product. In order to compute the i-th eigenvalue for general $i \in \mathbb{N}$ one may use the following lemma, a proof can be found in the survey article [15], Chapter 7.

Lemma 3.1.1. *Let V and H be two real Hilbert spaces with dense and continuous embedding $V \hookrightarrow H$. Let $a\colon V \times V \to \mathbb{R}$ and $b\colon H \times H \to \mathbb{R}$ be two symmetric and continuous bilinear forms. Let $a(\cdot,\cdot)$ be V-elliptic, i.e. there exists $\alpha > 0$ such that $a(v,v) \geq \alpha \|v\|_V^2$ for all $v \in V$, and let $b(\cdot,\cdot)$ define a scalar product on H. For $i \in \mathbb{N}$, let $V^{(i)}$ denote the set of all subspaces of V of dimension i. Then the i-th eigenvalue corresponding to the equation*

$$a(u_i, v) = \lambda_i\, b(u_i, v) \qquad \forall v \in V,$$

is given via

$$\lambda_i = \min_{E \in V^{(i)}} \max_{v \in E} \frac{a(v,v)}{b(v,v)}, \tag{3.6}$$

where the minimum with respect to the subspace is attained for E being the subspace spanned by the first i eigenfunctions, and the maximum with respect to the element of that subspace is attained for v being an eigenfunction to λ_i.

Now let

$$\mu_q = 1 + (d-1)\chi_{\Omega_{q,0}},$$

with $\chi_{\Omega_{q,0}}$ being the characteristic function of $\Omega_{q,0}$, and

$$a_q(u,v) = (\nabla u, \mu_q \nabla v)_{\Omega_q}, \tag{3.7}$$

$$b_q(u,v) = (u,v)_{\Omega_q}. \tag{3.8}$$

Then the weak formulation of (3.5), including a normalizing condition, reads as

$$\begin{cases} a_q(u_i, v) = \lambda_i\, b_q(u_i, v) \quad \forall v \in H_0^1(\Omega_q), \\ b_q(u_i, u_i) = 1. \end{cases} \tag{3.9}$$

For the rest of this chapter we consider the variational problem. The problem under consideration is now given via

$$\min_{q \in \overline{Q}^{\text{ad}}} j(q) = \lambda_1(q) - \lambda_2(q) + \frac{\alpha}{2} \|q\|_{H^2(I)}^2, \tag{3.10}$$

subject to (3.6) and (3.9), where $\alpha > 0$ is a given constant.

Remark 3.1.2. With $\lambda_i(q)$ for $i \in \mathbb{N}$ and $q \in Q^{\text{ad}}$ we will always denote the i-th eigenvalue for a given control q, which can be computed via (3.6).

In order to prove the existence of a solution to (3.10) we will at first show that j is uniformly bounded from below. This follows from the fact that $\lambda_2(q)$ is uniformly bounded from above for $q \in \overline{Q}^{\text{ad}}$, which is a direct consequence of the following lemma.

Lemma 3.1.3. *Let $i \in \mathbb{N}$, then there exists $c = c(i) > 0$ such that $\lambda_i(q) \in (0, c]$ for all $q \in \overline{Q}^{\text{ad}}$.*

Proof. As all the eigenvalues are known to be positive we just have to prove the upper bound. Let $\tilde{\lambda}_i$ denote the i-th eigenvalue for the Laplacian on Ω_q which does not depend on q. Using Lemma 3.1.1 it follows that

$$\begin{aligned} \lambda_i(q) &= \min_{E \in V^{(i)}} \max_{v \in E} \frac{(\nabla v, \mu_q \nabla v)}{(v,v)} \\ &\leq \max\{1, d\} \min_{E \in V^{(i)}} \max_{v \in E} \frac{(\nabla v, \nabla v)}{(v,v)} \\ &= \max\{1, d\}\, \tilde{\lambda}_i. \end{aligned}$$

□

Remark 3.1.4. The eigenvalues for the Laplacian on a rectangle can be computed exactly. For example, on the square $D = (-2,2)\times(-2,2)$ the set of eigenvalues is given by $\left\{ \frac{\pi^2}{16}\left(m^2+n^2\right) \middle| \, m,n \in \mathbb{N} \right\}$.

Lemma 3.1.3 ensures that

$$\lim_{\|q\|_{H^2(I)}\to\infty} j(q) = \infty.$$

It follows that there exists $\tilde{C} = \tilde{C}(\alpha)$ such that we can restrict the search for a minimum onto the set

$$Q^{\mathrm{ad}} = \left\{ q \in Q \middle| \, \|q\|_{H^2(I)} \le \tilde{C} \right\}. \tag{3.11}$$

As within Subsection 3.1.1 we have to assume that the constant $\tilde{C}$ is sufficiently small, it is reasonable to assume that $Q^{\mathrm{ad}} \subset \overline{Q}^{\mathrm{ad}}$, i.e. the elements of Q^{ad} are not degenerated in the sense of (3.1). Before we continue in proving the existence of a solution to (3.10) we will apply a transformation argument similar to Subsection 2.1.1.

3.1.1. Transformation of the problem

In order to solve (3.10) we will use a transformation T_F to transform the equation (3.9) onto a partitioned reference domain, see Figure 3.2. Let $\Omega = \Omega_q$, let Ω_0 be the open unit circle centered at the origin and let $\Omega_1 = \Omega\backslash\overline{\Omega_0}$. Let F be the weak solution to

$$\begin{cases} -\Delta F = 0 & \text{in } \Omega_j,\ j \in \{0,1\}, \\ F = 0 & \text{on } \Gamma = \partial\Omega, \\ F = q\,n & \text{on } \Gamma_0 = \partial\Omega_0, \end{cases} \tag{3.12}$$

where n shall always denote the outer unit normal with respect to Ω_0. Similar to Subsubsection 2.1.3.2, it is possible to reformulate (3.12) in variational form which then shall be denoted with

$$\mathcal{G}(q,G) = (F,G) \qquad\qquad \forall G \in L^2(\Omega), \tag{3.13}$$

where $\mathcal{G}\colon L^2(\Gamma_0)\times L^2(\Omega) \to \mathbb{R}$ is bilinear and continuous. Let $T_F = \mathrm{Id} + F$ be the transformation, it now holds that

$$\Omega_{q,j} = T_F(\Omega_j),$$

for $j \in \{0,1\}$, where we also have to assume that $\tilde{C}$ from (3.11) is chosen sufficiently small such that T_F is a bijection from Ω_q onto Ω for all $q \in Q^{\mathrm{ad}}$, cf. Assumption 2.1.11 and the lemmata cited therein.

Remark 3.1.5. With $F(q)$ we will always denote the solution to (3.12) for a given control $q \in Q^{\mathrm{ad}}$.

Lemma 3.1.6. *Let $q, p \in Q^{\mathrm{ad}}$ with corresponding transformations F and E, respectively. Then it holds that*

$$F_0 = F|_{\Omega_0} \in H^{5/2}(\Omega_0) \hookrightarrow W^{2,4}(\Omega_0) \hookrightarrow C^{1,1/2}(\overline{\Omega_0}), \tag{3.14}$$

$$F_1 = F|_{\Omega_1} \in W^{2,4}(\Omega_1) \hookrightarrow C^{1,1/2}(\overline{\Omega_1}), \tag{3.15}$$

$$F \in W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega}), \text{ and } \|F\|_{W^{1,\infty}(\Omega)} \le c_\varepsilon \|q\|_{H^{3/2+\varepsilon}(I)}, \tag{3.16}$$

$$\|F - E\|_{W^{1,\infty}(\Omega)} \le c_\varepsilon \|q - p\|_{H^{3/2+\varepsilon}(I)}. \tag{3.17}$$

Proof. The regularity results for F_0 and F_1, (3.14) and (3.15), follow with Theorem A.1.28, Theorem A.1.38 and Theorem A.1.4. As F_0 and F_1 are both continuous and coincide on the boundary Γ_0 due to (3.12) it follows that F is continuous on Ω, its regularity can be seen as follows. Let $x, y \in \Omega$. If either $x, y \in \Omega_0$ or $x, y \in \Omega_1$, then the regularity result within (3.16) follows from the regularity of F_0 and F_1. Now, without loss of generality, let $x \in \Omega_0$, $y \in \Omega_1$ and let $z \in \Gamma_0$ be the intersection of the line segment $\overline{xy}$ with Γ_0, $|x-y| = |x-z| + |z-y|$. In addition, let L_0 and L_1 be the Lipschitz constants of F_0 and F_1, respectively. Then it holds that

$$\begin{aligned} |F(x) - F(y)| &\leq |F(x) - F(z)| + |F(z) - F(y)| \\ &\leq L_0 \,|x-z| + L_1 \,|z-y| \\ &\leq \max\{L_0, L_1\} \,(|x-z| + |z-y|) \\ &= \max\{L_0, L_1\} \,|x-y| \,. \end{aligned}$$

From the regularity results cited above it follows that $q \in C^{1,\varepsilon}(I)$ is sufficient for F_0 and F_1 to be Lipschitz, and L_0 and L_1 continuously depend on $\|q\|_{C^{1,\varepsilon}(I)}$. Because of $H^{3/2+\varepsilon}(I) \hookrightarrow C^{1,\varepsilon}(\overline{I})$ we end up with

$$|F(x) - F(y)| \leq c_\varepsilon \, \|q\|_{H^{3/2+\varepsilon}(I)} \, |x-y| \,,$$

which proves (3.16). The last assertion, (3.17), follows with (3.16) and the fact that $q \mapsto F(q)$ is linear. □

For given $q \in Q^{\mathrm{ad}}$ with $F = F(q)$ and transformation T_F it is now possible to transform (3.9) onto the reference domain, which then reads as

$$\begin{cases} (\nabla u_i(q), \mu A_F \cdot \nabla v) = \lambda_i(q)\, (u_i(q), v\gamma_F) \quad \forall v \in H_0^1(\Omega), \\ \quad (u_i(q), u_i(q)\gamma_F) = 1, \end{cases} \tag{3.18}$$

with $\mu = 1 + (d-1)\chi_{\Omega_0}$, where χ_{Ω_0} is the characteristic function of Ω_0, $\gamma_F = \det(\mathrm{D}T_F)$ and $A_F = \mathrm{D}T_F^{-1} \cdot \mathrm{D}T_F^{-T}\gamma_F$. Regularity and differentiability results concerning these functions can be proven similar to Lemma 2.1.6. In what follows we will use the following abbreviations,

$$a(F)(u,v) = (\nabla u, \mu A_F \cdot \nabla v)\,, \tag{3.19}$$

$$b(F)(u,v) = (u, v\gamma_F)\,, \tag{3.20}$$

such that (3.18) can be rewritten as

$$\begin{cases} \quad a(F)(u_i(q), v) = \lambda_i(q)\, b(F)(u_i(q), v) \quad \forall v \in H_0^1(\Omega), \\ b(F)(u_i(q), u_i(q)) = 1. \end{cases}$$

Remark 3.1.7. Let $u_i(q)$ denote the i-th eigenfunction for given $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$, which can be computed via (3.18).

The transformed problem now reads as

$$\min_{q \in Q^{\mathrm{ad}}} j(q) = \lambda_1(q) - \lambda_2(q) + \frac{\alpha}{2} \|q\|_{H^2(I)}^2 \,, \tag{3.21}$$

subject to (3.13), (3.6) and (3.18).

3.1.2. On the existence of eigenfunctions

Although the existence of real eigenvalues and eigenfunctions for the original equation (3.2) is well-known, here we give rigorous proofs for their existence in the transformed setting (3.18). The approach presented within this subsection will also be needed in the context of error estimation later on.

Definition 3.1.8. For given $q \in Q^{\mathrm{ad}}$ and $F = F(q)$, let $L = L_q \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ be the differential operator related to the bilinear form (3.19),

$$Lu = -\operatorname{div}(\mu A_F \cdot \nabla u)\,.$$

Furthermore, let $L^{-1} \colon H^{-1}(\Omega) \to H_0^1(\Omega)$ be the inverse of L with respect to the scalar product induced by the bilinear form $b(F)(\cdot,\cdot)$, i.e. $u = L^{-1}f$ is defined as the unique solution to

$$(\nabla u, \mu A_F \cdot \nabla v) = (f, v\gamma_F)_{H^{-1},H_0^1} \qquad \forall v \in H_0^1(\Omega). \tag{3.22}$$

The following spaces will be used in order to simplify notation. The equivalence of the norms follows from Subsection 3.1.1.

Definition 3.1.9. For fixed $q \in Q^{\mathrm{ad}}$ and $F = F(q)$, let $L_b^2(\Omega) = L_b^2(\Omega, q)$ be the space $L^2(\Omega)$ equipped with the scalar product induced by (3.20),

$$(u, v)_{L_b^2(\Omega)} = (u, v\gamma_F)\,.$$

The $L_b^2(\Omega)$-norm is equivalent to the standard $L^2(\Omega)$-norm.

Definition 3.1.10. For fixed $q \in Q^{\mathrm{ad}}$ and $F = F(q)$, let $H_{0,a}^1(\Omega) = H_{0,a}^1(\Omega, q)$ be the space $H_0^1(\Omega)$ equipped with the scalar product induced by (3.19),

$$(u, v)_{H_{0,a}^1(\Omega)} = (\nabla u, \mu A_F \cdot \nabla v)\,.$$

The $H_{0,a}^1(\Omega)$-norm is equivalent to the standard $H_0^1(\Omega)$-norm.

Lemma 3.1.11. *Let $q \in Q^{\mathrm{ad}}$, then the operator L^{-1} from Definition 3.1.8 is compact from $L^2(\Omega)$ onto $H_0^1(\Omega)$.*

Proof. Let $\tilde{L}^{-1}$ be the solution operator for (3.2). From [95], Theorem 5 and Remark 5.1, it follows that $\tilde{L}^{-1}$ maps $L^2(\Omega)$ onto $H_0^{3/2-\varepsilon}(\Omega)$ for $\varepsilon > 0$. As $H_0^{3/2-\varepsilon}(\Omega)$ is compactly embedded into $H_0^1(\Omega)$ for $\varepsilon < 1/2$, it follows that $\tilde{L}^{-1}$ is compact from $L^2(\Omega)$ onto $H_0^1(\Omega)$. Because $L^{-1}(f) = \left(\tilde{L}^{-1}(f \circ T_F^{-1})\right) \circ T_F$ is the concatenation of linear and compact operators, the result follows. □

As L^{-1} is selfadjoint and compact over $H_0^1(\Omega)$ due to Lemma 3.1.11 and hence also over $H_{0,a}^1(\Omega)$, it follows from Theorem A.1.2 that there exists a sequence of eigenvalues $(\nu_i)_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$ with 0 as only limit point, and a sequence of eigenfunctions $(u_i)_{i\in\mathbb{N}} \subset H_{0,a}^1(\Omega)$ with

$$L^{-1}u_i = \nu_i u_i.$$

Taking the $H_{0,a}^1(\Omega)$-scalar product on both sides yields

$$\left(\nabla\left(L^{-1}u_i\right), \mu A_F \cdot \nabla v\right) = \left(\nabla\left(\nu_i u_i\right), \mu A_F \cdot \nabla v\right) \qquad \forall v \in H_{0,a}^1(\Omega).$$

Setting $\lambda_i = \nu_i^{-1}$ and using the definition of L^{-1} we arrive at

$$(\nabla u_i, \mu A_F \cdot \nabla v) = \lambda_i \,(u_i, v\gamma_F) \qquad \forall v \in H^1_{0,a}(\Omega).$$

From

$$\lambda_i \,(u_i, u_j\gamma_F) = (\nabla u_i, \mu A_F \cdot \nabla u_j) = \lambda_j \,(u_i, u_j\gamma_F)\,,$$

it also follows that the eigenfunctions are mutually orthogonal,

$$a(F)(u_i, u_j) = b(F)(u_i, u_j) = 0, \tag{3.23}$$

for $i \neq j$.

3.1.3. Existence of a solution

Within this subsection we are going to prove that the variational problem (3.10) has a solution. As the original problem is equivalent to the transformed problem (3.21), we will show the existence of a minimizer just for the transformed one. First we need a continuity result for the eigenvalues, the following theorem can be found in [56], Theorem 2.3.1.

Theorem 3.1.12. *Let T_1 and T_2 be two self-adjoint, compact and positive operators on a separable Hilbert space V. Let $i \in \mathbb{N}$, and let $\nu_i(T_1)$ and $\nu_i(T_2)$ be their i-th eigenvalues, respectively. Then it holds that*

$$\begin{aligned} |\nu_i(T_1) - \nu_i(T_2)| &\leq \sup_{v\in V} \frac{(v, (T_1 - T_2)v)_V}{\|v\|_V^2} \\ &\leq \sup_{v\in V} \frac{\|(T_1 - T_2)(v)\|_V}{\|v\|_V} = \|T_1 - T_2\|_V\,. \end{aligned}$$

Lemma 3.1.13. *Let $i \in \mathbb{N}$ and let $q, p \in Q^{\mathrm{ad}}$ with corresponding transformations F and E, respectively. Then it holds that*

$$\begin{aligned} |\lambda_i(q) - \lambda_i(p)| &\leq c \sup_{u\in H^1_0(\Omega)} \frac{|(\nabla u, \mu\,(A_F - A_E)\cdot \nabla u)| + \left|\left(u^2, \gamma_F - \gamma_E\right)\right|}{\|u\|^2_{H^1_0(\Omega)}} \\ &\leq c_\varepsilon \,\|q - p\|_{H^{3/2+\varepsilon}(I)}\,. \end{aligned}$$

Proof. This lemma follows from Theorem 3.1.12 and Lemma 3.4.6. □

Theorem 3.1.14. *Problem* (3.21) *has a solution.*

Proof. Let $(q_n)_{n\in\mathbb{N}} \subset Q^{\mathrm{ad}}$ be a minimizing sequence with

$$\lim_{n\to\infty} j(q_n) = \inf_{q\in Q^{\mathrm{ad}}} j(q) = \bar{j}.$$

As Q^{ad} is a bounded, closed and convex subset of the Hilbert space Q it is weakly sequentially compact. It follows that there exists $\bar{q} \in Q^{\mathrm{ad}}$ and a subsequence of $(q_n)_{n\in\mathbb{N}}$, denoted in the same way, with

$$\begin{aligned} q_n &\rightharpoonup \bar{q} && \text{in } H^2(I), \\ q_n &\to \bar{q} && \text{in } H^{2-\varepsilon}(I), \end{aligned}$$

where the strong convergence follows from the fact that $H^2(I)$ is compactly embedded into $H^{2-\varepsilon}(I)$. With Lemma 3.1.13 it follows that $\lambda_i(q_n) \to \lambda_i(\overline{q})$ for $n \to \infty$ and $i \in \{1,2\}$. As the squared norm is lower semicontinuous it follows that

$$\liminf_{n\to\infty} \|q_n\|^2_{H^2(I)} \geq \|\overline{q}\|^2_{H^2(I)},$$

hence

$$\liminf_{n\to\infty} j(q_n) \geq j(\overline{q}),$$

and from the definition of $\overline{j}$ it follows that

$$j(\overline{q}) = \overline{j}.$$

□

Remark 3.1.15. As $q \mapsto \lambda_i(q)$ is highly nonlinear, the optimal control $\overline{q}$ need not be unique.

3.1.4. Regularity of the eigenfunctions

The aim of this subsection is to investigate in the regularity of the eigenfunctions, i.e. the solutions (u_i, λ_i) to (3.18),

$$(\nabla u_i, \mu A_F \cdot \nabla v) = \lambda_i (u_i, v\gamma_F) \qquad \forall v \in H^1_0(\Omega),$$

where it is known that $A_F \in C^{0,1/2}(\overline{\Omega_j})$ for $j \in \{0,1\}$. Here we will prove some general regularity results for u_i, in a later section we will show that the optimal control $\overline{q}$ possesses some even higher regularity which will also improve the regularity of the associated optimal eigenfunctions. As we just focus on the regularity of the eigenfunctions, we omit the normalizing condition in (3.18) within this subsection.

Lemma 3.1.16. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$ and $u_i = u_i(q)$, then it holds that*

$$\|u_i\|_{H^1_0(\Omega)} \leq c_i \|u_i\|_{L^2(\Omega)}.$$

Proof. Let $F = F(q)$ and $\lambda_i = \lambda_i(q)$, as all the matrices μA_F are uniformly elliptic for $q \in Q^{\mathrm{ad}}$ it follows that

$$\begin{aligned} c\|u_i\|^2_{H^1_0(\Omega)} &\leq a(F)(u_i,u_i) = \lambda_i b(F)(u_i,u_i) \\ &\leq c\lambda_i \|u_i\|^2_{L^2(\Omega)}, \end{aligned}$$

and the proof follows with Lemma 3.1.3. □

Lemma 3.1.17. *There exists $p \in (2,\infty)$ such that for all $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$ it holds that $u_i(q) \in W^{1,p}(\Omega)$ and*

$$\|u_i\|_{W^{1,p}(\Omega)} \leq c_{i,p} \|u_i\|_{L^2(\Omega)}.$$

Proof. Let $F = F(q)$ and $\lambda_i = \lambda_i(q)$. Again we use the fact that for $q \in Q^{\mathrm{ad}}$ the ellipticity constants of the matrices μA_F can be bounded uniformly. The existence of such a $p > 2$ now follows from Theorem A.1.32. From the cited theorem it also follows that

$$\begin{aligned} \|u_i\|_{W^{1,p}(\Omega)} &\leq c_p \|\lambda_i u_i \gamma_F\|_{L^p(\Omega)} \leq c_{i,p} \|u_i\|_{L^p(\Omega)} \\ &\leq c_{i,p} \|u_i\|_{H^1_0(\Omega)} \leq c_{i,p} \|u_i\|_{L^2(\Omega)}, \end{aligned}$$

where we used the continuous embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p < \infty$ in dimension $n = 2$ and Lemma 3.1.16. □

The following lemma can be proven by a direct calculation.

Lemma 3.1.18. *Let $f \in C^{1,\alpha}(Y,Z)$ and $g \in C^{1,\alpha}(X,Y)$ for some $\alpha \in (0,1]$ and closed subsets X, Y and Z of some Banach spaces. Then it holds that $f \circ g \in C^{1,\alpha}(X,Z)$ and*

$$\|f \circ g\|_{C^{1,\alpha}(X,Z)} \leq c \|f\|_{C^{1,\alpha}(Y,Z)} \|g\|_{C^{1,\alpha}(X,Y)}.$$

Lemma 3.1.19. *Let $q \in Q^{\mathrm{ad}}$, $F = F(q)$, $j \in \{0,1\}$ and $K \subset\subset \Omega_j$. Then $F|_K$ is analytic.*

Proof. This lemma is a direct consequence of Weyl's lemma, cf. [108], Lemma 2, and the fact that F is weakly harmonic in Ω_j for $j \in \{0,1\}$. □

Lemma 3.1.20. *Let $i \in \mathbb{N}$, $q \in Q^{\mathrm{ad}}$, $u_i = u_i(q)$, $j \in \{0,1\}$ and let $K \subset\subset \Omega_j$ be sufficiently smooth. Then it holds that $u_i \in C^{1,1/2}(K)$ for $i \in \mathbb{N}$ and there exists $c_i = c_i(K)$ such that*

$$\|u_i\|_{C^{1,1/2}(K)} \leq c_i \|u_i\|_{L^2(\Omega)}.$$

Proof. Let $F = F(q)$ and $K' = T_F^{-1}(K)$. Due to Lemma 3.1.19, $F|_K$ is analytic, and as T_F is bijective it follows that K' is sufficiently smooth. On K' it holds that $u_{q,i} = u_i \circ T_F^{-1}$ solves $-\Delta u_{q,i} = \tilde{\lambda} u_{q,i}$, where $\tilde{\lambda}_i = \lambda_i$ or $\tilde{\lambda}_i = \frac{\lambda_i}{d}$, depending on whether j is either 0 or 1. Using the results presented in [56], Section 1.2.4 and the references cited therein it follows that $\|u_{i,q}\|_{W^{2,4}(K')} \leq c(i,K) \|u_{i,q}\|_{L^2(\Omega_q)}$. The regularity result and the estimate for $u_i|_K = u_{i,q} \circ T_F$ follow with the continuous embedding $W^{2,4}(K') \hookrightarrow C^{1,1/2}(\overline{K'})$, Lemma 3.1.18 and Lemma 3.1.6. □

Next we are going to prove a result dealing with the regularity of the eigenfunctions up to the boundary Γ_0. The following theorem can be found in [81], Corollary 1.3.

Theorem 3.1.21. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,\alpha}$-boundary Γ with $\alpha \in (0,1)$. Let $L \in \mathbb{N}$ and for $1 \leq m \leq L$ let Ω_m be a subdomain of Ω with $C^{1,\alpha}$-boundary and $\overline{\Omega} = \bigcup_{m=1}^{L} \overline{\Omega_m}$. For $1 \leq m \leq L$ let $A^{(m)} \in C^{0,\mu}(\overline{\Omega_m})$ with $\mu \in (0,1]$ be a symmetric and positive definite matrix, and let the matrix A be defined via $A|_{\Omega_m} = A^{(m)}$. Suppose that $0 < c_1 \leq A \leq c_2 < \infty$ on Ω in the sense of symmetric and positive definite matrices. In a likewise manner, let $h^{(m)} \in C^{0,\mu}(\Omega_m)$ and $h|_{\Omega_m} = h^{(m)}$. At last, let $f \in L^\infty(\Omega)$ and $g \in C^{1,\mu}(\Gamma)$. Then the restriction of the weak solution u to*

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u) = f + \operatorname{div}(h) & \text{in } \Omega, \\ u = g & \text{on } \Gamma, \end{cases} \tag{3.24}$$

onto Ω_m belongs to $C^{1,\alpha'}(\overline{\Omega_m})$ for $0 < \alpha' \leq \min\left\{\mu, \frac{\alpha}{(\alpha+1)n}\right\}$ and there holds the estimate

$$\max_{1 \leq m \leq L} \|u\|_{C^{1,\alpha'}(\overline{\Omega_m})} \leq c \left(\|f\|_{L^\infty(\Omega)} + \max_{1 \leq m \leq L} \left\| h^{(m)} \right\|_{C^{0,\alpha'}(\overline{\Omega_m})} + \|g\|_{C^{1,\alpha'}(\Gamma)} \right),$$

with the constant c being independent of f, $h^{(m)}$ and g.

Coming back to our situation, we obtain:

Corollary 3.1.22. *Let $i \in \mathbb{N}$, $q \in Q^{\mathrm{ad}}$, $u_i = u_i(q)$, $\varepsilon > 0$, and let $\Omega_\varepsilon = \{x \in \Omega |\operatorname{dist}(x,\Gamma) > \varepsilon\}$. Then it holds that*

$$u_i|_{\Omega_0} \in C^{1,1/6}(\overline{\Omega_0}), \qquad \|u_i\|_{C^{1,1/6}(\overline{\Omega_0})} \leq c_i \|u_i\|_{L^2(\Omega)},$$

and

$$u_i|_{\Omega_1 \cap \Omega_\varepsilon} \in C^{1,1/6}(\overline{\Omega_1 \cap \Omega_\varepsilon}), \qquad \|u_i\|_{C^{1,1/6}(\overline{\Omega_1 \cap \Omega_\varepsilon})} \leq c_{i,\varepsilon} \|u_i\|_{L^2(\Omega)}.$$

Proof. Let $K \subset\subset \Omega_1$ be sufficiently smooth such that $\partial\Omega_\varepsilon \subset K$. Lemma 3.1.20 now yields

$$\|u_i\|_{C^{1,1/6}(\partial\Omega_\varepsilon)} \leq c\,\|u_i\|_{C^{1,1/2}(K)} \leq c_i\,\|u_i\|_{L^2(\Omega)}\,.$$

This corollary now follows with Lemma 3.1.17 which ensures that $u_i \in L^\infty(\Omega)$ and Theorem 3.1.21. □

The following lemmata are proven in order to show $H^{3/2-\varepsilon}(\Omega) \cap W^{1,p}(\Omega)$-regularity of u_i.

Lemma 3.1.23. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$, $u_i = u_i(q)$, $p \in [2,\infty)$ and $\varepsilon > 0$. Then there exists*

$$u_{i,\Gamma} \in W_0^{1,p}(\Omega) \cap W^{1+1/p-\varepsilon,p}(\Omega),$$

such that $u_{i,\Gamma}|_{\Gamma_0} = u_i|_{\Gamma_0}$ and

$$\|u_{i,\Gamma}\|_{W^{1+1/p-\varepsilon,p}(\Omega)} \leq c_{i,\varepsilon,p}\,\|u_i\|_{L^2(\Omega)}\,.$$

Proof. With Corollary 3.1.22 it follows that $u_i|_{\Gamma_0} \in C^{1,1/6}(\Gamma_0) \hookrightarrow W^{7/6,p}(\Gamma_0)$ for all $p \leq \infty$. In addition, for $\varepsilon > 0$ sufficiently small let the annulus K be defined as

$$K = \{x \in \Omega_1 |\, \mathrm{dist}(x,\Gamma_0) \leq \varepsilon\} \subset \Omega_1.$$

Using Theorem A.1.3 it follows that for $p \in [6/5,\infty)$ there exists a function $u_{i,\Gamma} \in L^1(\Omega_0 \cup K)$ with the following properties.

$$\begin{aligned} u_{i,\Gamma}|_{\Omega_0} &\in W^{7/6+1/p,p}(\Omega_0) \hookrightarrow C^{1,1/6-1/p}(\overline{\Omega_0}),\\ u_{i,\Gamma}|_{\Gamma_0} &= u_i|_{\Gamma_0}\,,\\ \partial_n u_{i,\Gamma}|_{\Gamma_0} &= 0,\\ \|u_{i,\Gamma}\|_{W^{7/6+1/p,p}(\Omega_0)} &\leq c_p\,\|u_i\|_{W^{7/6,p}(\Gamma_0)}\,, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} u_{i,\Gamma}|_K &\in W^{7/6+1/p,p}(K) \hookrightarrow C^{1,1/6-1/p}(K),\\ u_{i,\Gamma}|_{\partial K\setminus\Gamma_0} &= \partial_n u_{i,\Gamma}|_{\partial K\setminus\Gamma_0} = 0,\\ \|u_{i,\Gamma}\|_{W^{7/6+1/p,p}(K)} &\leq c_p\,\|u_i\|_{W^{7/6,p}(\Gamma_0)}\,. \end{aligned} \tag{3.26}$$

As $u_{i,\Gamma}$ is continuous along Γ_0, it follows that

$$\begin{aligned} \|u_{i,\Gamma}\|^p_{W^{1,p}(K\cup\Omega_0)} &= \|u_{i,\Gamma}\|^p_{W^{1,p}(K)} + \|u_{i,\Gamma}\|^p_{W^{1,p}(\Omega_0)}\\ &\leq c_p\,\|u_i\|^p_{W^{7/6,p}(\Gamma_0)}\\ &\leq c_p\,\|u_i\|^p_{C^{1,1/6}(\overline{\Omega_0})}\\ &\leq c_{i,p}\,\|u_i\|^p_{L^2(\Omega)}\,, \end{aligned} \tag{3.27}$$

where we used Corollary 3.1.22. From the definition of fractional norms (B.1) it now follows that

$$
\begin{aligned}
|u_{i,\Gamma}|^p_{W^{1+1/p-\varepsilon,p}(K\cup\Omega_0)} &= \int_{K\cup\Omega_0}\int_{K\cup\Omega_0} \frac{|\nabla u_{i,\Gamma}(x)-\nabla u_{i,\Gamma}(y)|^p}{|x-y|^{2+p(1/p-\varepsilon)}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le c_p\left(\left(|u_{i,\Gamma}|^p_{W^{1+1/p-\varepsilon,p}(K)} + |u_{i,\Gamma}|^p_{W^{1+1/p-\varepsilon,p}(\Omega_0)}\right) + \int_K\int_{\Omega_0} \frac{|\nabla u_{i,\Gamma}(x)-\nabla u_{i,\Gamma}(y)|^p}{|x-y|^{2+p(1/p-\varepsilon)}}\,\mathrm{d}x\,\mathrm{d}y\right) \\
&\le c_{\varepsilon,p}\left\|u_{i,\Gamma}\right\|^p_{W^{7/6,p}(\Gamma_0)} + \max\left\{\left\|u_{i,\Gamma}\right\|^p_{C^1(K)}, \left\|u_{i,\Gamma}\right\|^p_{C^1(\overline{\Omega_0})}\right\}\int_K\int_{\Omega_0}\frac{1}{|x-y|^{2+p(1/p-\varepsilon)}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le c_{\varepsilon,p}\left(\left\|u_{i,\Gamma}\right\|^p_{C^{1,1/6}(\Gamma_0)} + \max\left\{\left\|u_{i,\Gamma}\right\|^p_{W^{7/6+1/7,7}(K)}, \left\|u_{i,\Gamma}\right\|^p_{W^{7/6+1/7,7}(\overline{\Omega_0})}\right\}\left\|1_{\Omega_0}\right\|^p_{W^{1/p-\varepsilon,p}(K\cup\Omega_0)}\right) \\
&\le c_{\varepsilon,p}\left(\left\|u_{i,\Gamma}\right\|^p_{C^{1,1/6}(\overline{\Omega_0})} + \left\|u_i\right\|^p_{W^{7/6,7}(\Gamma_0)}\left\|1_{\Omega_0}\right\|^p_{W^{1/p-\varepsilon,p}(K\cup\Omega_0)}\right) \\
&\le c_{\varepsilon,p}\left\|u_i\right\|^p_{C^{1,1/6}(\overline{\Omega_0})}\left(1+\left\|1_{\Omega_0}\right\|^p_{W^{1/p-\varepsilon,p}(K\cup\Omega_0)}\right) \\
&\le c_{i,\varepsilon,p}\left\|u_i\right\|^p_{L^2(\Omega)},
\end{aligned}
\tag{3.28}
$$

where we used Corollary 3.1.22 and the fact that the characteristic function of every bounded C^1-domain is an element of $W^{1/p-\varepsilon,p}(\mathbb{R}^2)$, cf. [101], Proposition 2.1, and [9]. If we extend $u_{i,\Gamma}$ by zero to the whole domain Ω, one can repeat the steps undertaken in (3.27) and (3.28) to show $W^{1+1/p-\varepsilon,p}(\Omega)$-regularity as well as the stability estimate and thus finish this proof. □

Lemma 3.1.24. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$, $u_i = u_i(q)$ and $p < \infty$. Then it holds that $u_i \in W^{1,p}(\Omega)$ and*

$$\left\|u_i\right\|_{W^{1,p}(\Omega)} \le c_{i,p}\left\|u_i\right\|_{L^2(\Omega)}.$$

Proof. Let $\tilde{u} = (u_i - u_{i,\Gamma})$ with $u_{i,\Gamma}$ defined as in Lemma 3.1.23. Then $\tilde{u}$ is the weak solution to

$$
\begin{cases}
-\operatorname{div}(\mu A_F \cdot \nabla\tilde{u}) = \lambda_i u_i \gamma_F + \operatorname{div}(\mu A_F\cdot\nabla u_{i,\Gamma}) & \text{in } \Omega_j, \\
\tilde{u} = 0 & \text{on } \partial\Omega_j,
\end{cases}
\tag{3.29}
$$

for $j \in \{0,1\}$. As μ is constant on Ω_j, one can apply Theorem A.1.31 and get $\tilde{u} \in W_0^{1,p}(\Omega_j)$, as well as

$$
\begin{aligned}
\left\|\tilde{u}\right\|_{W^{1,p}(\Omega_j)} &\le c_p\left(\left\|\lambda_i u_i\gamma_F\right\|_{L^p(\Omega_j)} + \left\|A_F\cdot\nabla u_{i,\Gamma}\right\|_{L^p(\Omega_j)}\right) \\
&\le c_{i,p}\left(\left\|u_i\right\|_{H^1(\Omega_j)} + \left\|A_F\right\|_{L^\infty(\Omega_j)}\left\|u_{i,\Gamma}\right\|_{W^{1,p}(\Omega_j)}\right) \\
&\le c_{i,p}\left\|u_i\right\|_{L^2(\Omega)},
\end{aligned}
$$

where we used Lemma 3.1.16, Lemma 3.1.6 and Lemma 3.1.23. As $\tilde{u} \in W_0^{1,p}(\Omega_j)$ it also follows that $\tilde{u} \in W_0^{1,p}(\Omega)$, and the result follows. □

Lemma 3.1.25. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$ and $u_i = u_i(q)$. Then it holds that $u_i \in H^{3/2-\varepsilon}(\Omega)$ and*

$$\left\|u_i\right\|_{H^{3/2-\varepsilon}(\Omega)} \le c_{i,\varepsilon}\left\|u_i\right\|_{L^2(\Omega)}.$$

Proof. As in the proof of Lemma 3.1.24 let $\tilde{u} = (u_i - u_{i,\Gamma})$, then $\tilde{u}$ is the weak solution to (3.29) for $j \in \{0,1\}$. As both subdomains of Ω are Lipschitz, Theorem A.1.30 proves $\tilde{u}|_{\Omega_j} \in H^{3/2-\varepsilon}(\Omega_j)$ and

$$\begin{aligned}\|\tilde{u}\|_{H^{3/2-\varepsilon}(\Omega_j)} &\leq c_\varepsilon \left(\|\lambda_i u_i \gamma_F\|_{L^2(\Omega_j)} + \|A_F \cdot \nabla u_{i,\Gamma}\|_{H^{1/2-\varepsilon}(\Omega_j)} \right) \\ &\leq c_{i,\varepsilon} \left(\|u_i\|_{L^2(\Omega)} + \|A_F\|_{H^{3/2}(\Omega_j)} \|u_{i,\Gamma}\|_{H^{3/2-\varepsilon}(\Omega_j)} \right) \\ &\leq c_{i,\varepsilon} \|u_i\|_{L^2(\Omega)} ,\end{aligned}$$

where we used Lemma 3.1.23 and Theorem A.1.5. It remains to prove $H^{3/2-\varepsilon}(\Omega)$-regularity. This can be done in exactly the same way as shown in (3.28) within the proof of Lemma 3.1.23. □

3.1.5. Differentiability of the eigenvalues

It is well-known that in the case of a sufficiently smooth domain, the eigenvalues with multiplicity one are Fréchet-differentiable with respect to smooth domain perturbations, whereas eigenvalues with a higher multiplicity are only Gâteaux-differentiable, cf. [54]. In the following subsection we are going to prove differentiability of the eigensystem with respect to domain perturbations. In order to do so we follow the approach presented in [34], where it is proven that eigenvalues are differentiable with respect to a specific boundary perturbation and also a representation for the derivative is given. Although our approach uses a transformation to a reference domain and our regularity assumptions differ, their proofs can be adapted to our case.

Assumption 3.1.26. We assume that for all $q \in Q^{\text{ad}}$, the eigenvalues $\lambda_1(q)$ and $\lambda_2(q)$ have multiplicity one.

Taking into account the cost functional (3.21) it is reasonable to assume that $\lambda_1(q) \neq \lambda_2(q)$ for all q sufficiently close to the optimal control $\overline{q}$. Another justification is the Krein-Rutman theorem (cf. [56], Theorem 1.2.5 and Theorem 1.2.6), which states that the first eigenvalue for a uniformly elliptic partial differential operator of second order is simple. However, we do have to admit that we did not find theoretical results supporting the claim that $\lambda_2(q) \neq \lambda_3(q)$ for all $q \in Q^{\text{ad}}$ with $\|\overline{q} - q\|_{H^2(I)}$ sufficiently small.

3.1.5.1. On the existence of the derivatives of λ_i and u_i

The proof of the existence of the derivatives of λ_i and u_i with respect to q relies on the implicit function theorem and Fredholm's alternative.

Theorem 3.1.27 (Fredholm's alternative)**.** *Let X be a Banach Space over $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let T be a compact operator on X with adjoint T', and let $\lambda \in \mathbb{K}$, $\lambda \neq 0$. Then exactly one of the following two possibilities holds true.*

- *The equation*

$$\lambda x - Tx = 0, \tag{3.30}$$

has $x = 0$ as its only solution and

$$\lambda x - Tx = y, \tag{3.31}$$

is uniquely solvable for every $y \in X$.

- *There exist* $n = \dim(\ker(\lambda \operatorname{Id} - T))$ *linear independent solutions to* (3.30)*, and the adjoint equation*

$$\lambda x' - T' x' = 0,$$

also has n linear independent solutions. Furthermore, there exists a solution to (3.31) *if and only if* $y \in (\ker(\lambda \operatorname{Id} - T'))^{\perp}$.

Proof. This theorem can be found in [3], Theorem 10.8. □

Lemma 3.1.28. *Let* $q \in Q^{\text{ad}}$, $F = F(q)$, $i \in \mathbb{N}$, *let* $(u_i = u_i(q), \lambda_i = \lambda_i(q))$ *be an eigenpair to the simple eigenvalue* λ_i *and let* $g \in H^{-1}(\Omega)$. *The equation*

$$(\nabla u, \mu A_F \cdot \nabla v) = \lambda_i \, (u, v\gamma_F) + (g, v)_{H^{-1}, H_0^1} \qquad \forall v \in H_0^1(\Omega) \tag{3.32}$$

has a solution $u \in H_0^1(\Omega)$ *if and only if* $(g, u_i)_{H^{-1}, H_0^1} = 0$.

Proof. Again, we use the operator L from Definition 3.1.8. Let $h = L^{-1}(g/\gamma_F)$, then equation (3.32) can be written as

$$(u, v)_{H^1_{0,a}(\Omega)} = \lambda_i \left(L^{-1} u, v \right)_{H^1_{0,a}(\Omega)} + (h, v)_{H^1_{0,a}(\Omega)} \qquad \forall v \in H^1_{0,a}(\Omega),$$

which can be written as

$$\nu_i u - L^{-1} u = \nu_i h \qquad \text{in } H^1_{0,a}(\Omega),$$

with $\nu_i = \lambda_i^{-1}$. With Theorem 3.1.27 it now follows that (3.32) has a solution if and only if

$$(h, u_i)_{H^1_{0,a}(\Omega)} = 0,$$

which reads as

$$0 = (\nabla h, \mu A_F \cdot \nabla u_i) = (g, u_i)_{H^{-1}, H_0^1}. \qquad \square$$

Theorem 3.1.29. *Let* $q \in Q^{\text{ad}}$, $\delta q \in Q$ *and* $i \in \mathbb{N}$ *such that* $\lambda_i(q)$ *is a simple eigenvalue. Then the mappings* $q \mapsto \lambda_i(q)$ *and* $q \mapsto u_i(q)$ *are at least two times continuously Fréchet-differentiable.*

Proof. Let $F = F(q)$ and let

$$B \colon H^2(I) \times H_0^1(\Omega) \times \mathbb{R} \to H^{-1}(\Omega) \times \mathbb{R},$$
$$B(q, u, \lambda) = \begin{pmatrix} -\operatorname{div}(\mu A_F \cdot \nabla u) - \lambda u \gamma_F \\ \int_\Omega u^2 \gamma_F \, \mathrm{d}x - 1 \end{pmatrix}.$$

The operator B is at least twice continuously differentiable, which can be shown similar to the proof of Lemma 2.1.24, and it follows that $B(q, u_i, \lambda_i) = 0$ if and only if u_i is a normalized eigenfunction with eigenvalue λ_i corresponding to the control q. Taking the derivative of B with respect to u and λ yields

$$\mathrm{D}_{u,\lambda} B(q, u_i, \lambda_i)(v, \vartheta) = \begin{pmatrix} -\operatorname{div}(\mu A_F \cdot \nabla v) - \lambda_i v \gamma_F - \vartheta u_i \gamma_F \\ 2 \int_\Omega u_i v \gamma_F \, \mathrm{d}x \end{pmatrix}.$$

Now we show that $\mathrm{D}_{u,\lambda}B(q,u_i,\lambda_i)$ is bijective, which can be done using Theorem 3.1.27 and Lemma 3.1.28 as follows. Let $(w,\tau) \in H^{-1}(\Omega) \times \mathbb{R}$ be arbitrary, we have to show that there exists $(v,\vartheta) \in H_0^1(\Omega) \times \mathbb{R}$ such that

$$\begin{cases} (\nabla v, \mu A_F \cdot \nabla\varphi) - \lambda_i\,(v,\varphi\gamma_F) = \vartheta\,(u_i,\varphi\gamma_F) + (w,\varphi)_{H^{-1},H_0^1} & \forall\varphi \in H_0^1(\Omega), \\ \qquad\qquad\qquad 2\,(u_i, v\gamma_F) = \tau. \end{cases} \tag{3.33}$$

If we set

$$\vartheta = -\,(w,u_i)_{H^{-1},H_0^1}\,,$$

then Lemma 3.1.28 yields the existence of $v_0 \in H_0^1(\Omega)$ such that the first equation within (3.33) is fulfilled for $v = v_0 + cu_i$ for all $c \in \mathbb{R}$. Setting

$$c = \frac{\tau}{2} - (u_i, v_0\gamma_F)\,,$$

makes v also fulfill the second equation within (3.33), and this theorem follows with the implicit function theorem, Theorem A.1.6. □

3.1.5.2. Representation of the derivatives λ_i' and u_i'

In this subsubsection we are going to find explicit representations for the derivatives of the eigenvalue and eigenfunction. Let $q \in Q^{\mathrm{ad}}$, $F = F(q)$, $\delta q \in Q$, $i \in \mathbb{N}$ and let $\lambda_i' = \lambda_i'(q)(\delta q)$, $\delta u_i = u_i'(q)(\delta q)$ and $\delta F = F'(q)(\delta q)$. Due to Theorem 3.1.29 we can differentiate (3.18) with respect to q, which yields

$$\begin{cases} (\nabla\delta u_i, \mu A_F \cdot \nabla v) = \lambda_i\,(\delta u_i, v\gamma_F) + \lambda_i'\,(u_i, v\gamma_F) \\ \qquad\qquad + \lambda_i\left(u_i, v\gamma_{F,\delta F}'\right) - \left(\nabla u_i, \mu A_{F,\delta F}' \cdot \nabla v\right) \quad \forall v \in H_0^1(\Omega), \\ 2\,(\delta u_i, u_i\gamma_F) + \left(u_i^2, \gamma_{F,\delta F}'\right) = 0. \end{cases} \tag{3.34}$$

Remark 3.1.30. From the first equation within (3.34) it follows that δu_i can formally be seen as a solution to

$$-\operatorname{div}(\mu A_F \cdot \nabla\delta u_i) = \lambda_i\delta u_i\gamma_F + \left(\lambda_i' u_i\gamma_F + \lambda_i u_i\gamma_{F,\delta F}' + \operatorname{div}\left(\mu A_{F,\delta F}' \cdot \nabla u_i\right)\right), \tag{3.35}$$

which is just a "perturbed" eigenvalue equation of the form

$$L\delta u_i = \lambda_i\delta u_i\gamma_F + g,$$

with $g = g(\lambda_i, u_i, q, \delta q) \in H^{-1}(\Omega)$. Solutions to (3.35) are not unique: if δu_i is a solution, then so is $\delta u_i + cu_i$ for all $c \in \mathbb{R}$. Instead, uniqueness is guaranteed through the second equation within (3.34).

Now using u_i as a test function in (3.18), we get

$$(\nabla u_i, \mu A_F \cdot \nabla u_i) = \lambda_i\left(u_i^2, \gamma_F\right),$$

and differentiation yields

$$2\,(\nabla u_i, \mu A_F \cdot \nabla\delta u_i) + \left(\nabla u_i, \mu A_{F,\delta F}' \cdot \nabla u_i\right) = \lambda_i'\left(u_i^2, \gamma_F\right) + 2\lambda_i\,(u_i\delta u_i, \gamma_F) + \lambda_i\left(u_i^2, \gamma_{F,\delta F}'\right). \tag{3.36}$$

As $\delta u_i \in H_0^1(\Omega)$ it holds that

$$\left(\nabla u_i, \mu A_F \cdot \nabla \delta u_i\right) = \lambda_i \left(u_i, \delta u_i \gamma_F\right). \tag{3.37}$$

Inserting (3.37) and the normalizing condition $\left(u_i^2, \gamma_F\right) = 1$ into (3.36) yields

$$\lambda_i'(q)(\delta q) = \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right) - \lambda_i \left(u_i^2, \gamma'_{F,\delta F}\right). \tag{3.38}$$

It can be seen that the computation of δu_i is not necessary in order to compute λ_i'. Expression (3.38) may be rewritten as a boundary integral, but in order to do so we need more regularity of the involved functions in order to justify partial integration. This will be shown in the next subsection.

3.1.6. Higher regularity of the optimal control

Our proof of the higher regularity of the optimal control exploits some first order optimality conditions, we therefore have to make the following assumption.

Assumption 3.1.31. We assume that the optimal control $\overline{q}$ under consideration is an element of the interior of Q^{ad}.

Due to Assumption 3.1.31, the first order optimality condition reads as

$$j'(\overline{q})(\delta q) = 0 \qquad \forall \delta q \in Q, \tag{3.39}$$

which is

$$\lambda_1'(\overline{q})(\delta q) - \lambda_2'(\overline{q})(\delta q) + \alpha \left(\overline{q}, \delta q\right)_{H^2(I)} = 0 \qquad \forall \delta q \in Q. \tag{3.40}$$

Lemma 3.1.32. *For every $q \in Q^{\mathrm{ad}}$ there exists $p_i = p_i(q) \in H^1(I)$ such that*

$$\lambda_i'(q)(\delta q) = \left(p_i, \delta q\right)_{H^1(I)} \qquad \forall \delta q \in Q.$$

Proof. Let $F = F(q)$, $\delta q \in Q$ and $\delta F = F'(q)(\delta q)$. With (3.38) it holds that

$$\begin{aligned} \lambda_i'(q)(\delta q) &= \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right) - \lambda_i \left(u_i^2, \gamma'_{F,\delta F}\right) \\ &= \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right)_{\Omega_0} + \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right)_{\Omega_1} \\ &\quad - \lambda_i \left(u_i^2, \gamma'_{F,\delta F}\right)_{\Omega_0} - \lambda_i \left(u_i^2, \gamma'_{F,\delta F}\right)_{\Omega_1}. \end{aligned} \tag{3.41}$$

Using Lemma 3.1.24 and the normalizing condition for u_i we can estimate the right hand side within (3.41) via

$$\begin{aligned} \left(u_i^2, \gamma'_{F,\delta F}\right)_{\Omega_j} &\leq \|u_i\|^2_{L^4(\Omega_j)} \left\|\gamma'_{F,\delta F}\right\|_{L^2(\Omega_j)} \leq c \|u_i\|^2_{H_0^1(\Omega_j)} \left\|\gamma_F \, \mathrm{D}T_F^{-1} \cdot \delta F\right\|_{H^1(\Omega_j)} \\ &\leq c \|u_i\|^2_{H_0^1(\Omega_j)} \|\delta F\|_{H^1(\Omega_j)} \leq c_i \|\delta F\|_{H^1(\Omega)} \\ &\leq c_i \|\delta q\|_{H^1(I)}, \end{aligned}$$

and in a similar way it holds that

$$\begin{aligned} \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right)_{\Omega_j} &\leq c \|u_i\|^2_{W^{1,4}(\Omega)} \left\|A'_{F,\delta F}\right\|_{L^2(\Omega_j)} \\ &\leq c \|\delta q\|_{H^1(I)}, \end{aligned}$$

for $j \in \{0, 1\}$. As $\delta q \mapsto \lambda_i'(q)(\delta q)$ is linear, the existence of such a p_i follows with the Riesz representation theorem. □

Lemma 3.1.33. *The optimal control $\overline{q} \in Q^{\mathrm{ad}}$ has the higher regularity $\overline{q} \in H^3(I)$.*

Proof. This lemma follows from (3.40), Lemma 3.1.32 and Lemma 2.1.36. □

Lemma 3.1.34. *For $\overline{F} = F(\overline{q})$ and $j \in \{0,1\}$ it holds that $\overline{F}\big|_{\Omega_j} \in W^{2,\infty}(\Omega_j)$.*

Proof. As $\overline{q} \in H^3(I)$ due to Lemma 3.1.33, Theorem A.1.28 yields $\overline{F}\big|_{\Omega_0} \in H^{7/2}(\Omega_0) \hookrightarrow W^{2,\infty}(\Omega_0)$. The regularity of $\overline{F}$ on Ω_1 follows with Theorem A.1.29. □

In order to derive higher regularity of $\overline{u}_i = u_i(\overline{q})$ we will need some regularity results concerning spaces with bounded mean oscillation.

Definition 3.1.35 (Campanato-John-Nirenberg space)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter d and let $\psi\colon [0,d] \to \mathbb{R}$ be a nonnegativ continuous function satisfying $r \leq c\psi(r)$ for some positive constant c. A function $f \in L^2(\Omega)$ is said to be an element of $\mathrm{BMO}_\Psi(\Omega)$, the space of bounded mean oscillation, if

$$|f|_{\mathrm{BMO}_\psi(\Omega)} = \sup_{\substack{x_0 \in \Omega \\ 0<\rho\leq d}} \frac{1}{\psi(\rho)} \left(\int_{\Omega(x_0,\rho)} \left| f(x) - (f)_{\Omega(x_0,\rho)} \right|^2 \,\mathrm{d}x \right)^{1/2} < \infty,$$

where $\Omega(x_0,\rho) = \Omega \cap Q_\rho(x_0)$ with $Q_\rho(x_0)$ being a cube with center x_0, sides parallel to the axis and side length equal to 2ρ. Furthermore,

$$(f)_D = \frac{1}{|D|} \int_D f \,\mathrm{d}x,$$

shall denote the mean value of f on D.

In what follows we will focus on the case where $\psi(\rho) = \rho^\alpha$ with $\alpha > 0$ sufficiently small. As mentioned in [109], the resulting spaces are called Campanato spaces. Furthermore, in that case it even holds that

$$\mathrm{BMO}_\psi(\Omega) = C^{0,\alpha}(\overline{\Omega}),$$

cf. [102], Example 1.

Definition 3.1.36 (Domains of class C^{k,BMO_ψ})**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that $\partial\Omega \in C^{k,\mathrm{BMO}_\psi}$ for $k \in \mathbb{N}$ if for any $x_0 \in \partial\Omega$ there exists a $C^{k-1,1}$-transformation $\mathcal{T}$ and a neighborhood $\mathcal{N}_{x_0}$ of x_0 such that

$$\mathcal{T}\colon \mathcal{N}_{x_0} \cap \Omega \to B_1^+(0),$$

where $B_1^+(0)$ is the unit ball with positive last coordinate, is one to one and onto with

$$\mathcal{T}\left(\mathcal{N}_{x_0} \cap \partial\Omega\right) = \overline{B_1^+(0)} \cap \{x_n = 0\}.$$

Moreover, the norms of $\mathcal{T}$, $\mathcal{T}^{-1}$ and their derivatives $\mathrm{D}^\nu\mathcal{T}$, $\mathrm{D}^\nu\left(\mathcal{T}^{-1}\right)$ are uniformly bounded in L^∞ and BMO_ψ for $|\nu| \leq k$.

From [37], Remark 3.2, it follows that domains which are locally the epigraph of a $C^{k,\alpha}$ function for $k \geq 1$ are of class $C^{k,\alpha}$. Furthermore, from the same source, Definition 3.1, it follows that if Ω is a domain of class $C^{k,\alpha}$, then it is also in C^{k,BMO_ψ} for $\psi(\rho) = \rho^\alpha$.

Theorem 3.1.37. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing $L \in \mathbb{N}$ disjoint subdomains $\Omega_m \subset\subset \Omega$ for $1 \le m \le L$, and let $\Omega_{L+1} = \Omega \backslash \bigcup_{m=1}^{L} \overline{\Omega_m}$. We consider weak solutions $u \in H^1(\Omega)$ to the equation*

$$-\operatorname{div}(A \cdot \nabla u) = -\operatorname{div}(f)\,, \tag{3.42}$$

where the matrix A is uniformly elliptic. Suppose that $\partial\Omega_m \in C^{k+1,\mathrm{BMO}_\psi}$ with $k \ge 1$,

$$A|_{\Omega_m}, f|_{\Omega_m} \in C^{k-1,1}(\overline{\Omega_m}) \qquad \text{and} \qquad \mathrm{D}^k A\Big|_{\Omega_m}, \mathrm{D}^k f\Big|_{\Omega_m} \in \mathrm{BMO}_\psi(\Omega_m).$$

Then for any $\Omega' \subset\subset \Omega$ it holds for the solution u to (3.42) that

$$u|_{\Omega_m} \in C^k(\overline{\Omega_m} \cap \Omega') \qquad \text{and} \qquad \mathrm{D}^{k+1} u \in \mathrm{BMO}_\psi(\Omega' \cap \Omega_m).$$

Proof. This theorem can be found in [109], Theorem 2.3, where it is assumed that the function ψ fulfills some additional assumptions. In [63], Remark 2.2, it is shown that these assumptions hold true for $\psi(\rho) = \rho^\alpha$ with $\alpha > 0$ sufficiently small. □

Lemma 3.1.38. *For $i \in \mathbb{N}$, $\overline{u}_i = u_i(\overline{q})$ and $j \in \{0,1\}$ it holds that $\overline{u}_i|_{\Omega_j} \in W^{2,\infty}(\Omega_j)$.*

Proof. With Theorem 3.1.37 it follows that $\overline{u}_i \in C^{2,\alpha}(\overline{\Omega_0}) \hookrightarrow W^{2,\infty}(\Omega_0)$. Now let $\overline{F} = F(\overline{q})$ and $\overline{u}_{i,q} = \overline{u}_i \circ T_{\overline{F}}^{-1}$ be an eigenfunction on the untransformed domain. As

$$\overline{u}_{q,i}|_{\Omega_{q,1}} \in H^{3/2-\varepsilon}(\Omega_{q,1}) \hookrightarrow L^\infty(\Omega_{q,1})$$

as shown in the proof of Lemma 3.1.11, it follows with Theorem A.1.38 that

$$\overline{u}_{q,i}|_{\Omega_{q,1}} \in W^{2,p}(\Omega_{q,1}) \hookrightarrow C^{1,\alpha}(\Omega_{q,1})$$

for all $p < \infty$ and $\alpha = 1 - 2/p > 0$. With Theorem A.1.29 it now follows that

$$\overline{u}_{q,1}|_{\Omega_{q,1}} \in W^{2,\infty}(\Omega_{q,1}).$$

Due to the regularity of $T_{\overline{F}}$ on Ω_1, cf. Lemma 3.1.34, it follows that

$$\overline{u}_i|_{\Omega_1} = \overline{u}_{q,1} \circ T_{\overline{F}}^{-1}\Big|_{\Omega_1} \in W^{2,\infty}(\Omega_1).$$

□

3.1.6.1. A representation of λ' as a boundary integral

Due to the higher regularity of the optimal eigenfunctions $\overline{u}_i$, equation (3.18) also holds in strong form (at least on each of the subdomains Ω_0 and Ω_1), therefore it is possible to rewrite expression (3.38) from above, the goal is to compute $\overline{\lambda}_i'(\overline{q})(\delta q)$ as a boundary integral over Γ_0. Let $\delta F = F'(\overline{q})(\delta q)$, using the same approach as in Subsubsection 2.1.3.4 one can show that

$$\begin{aligned}
\left(\nabla\overline{u}_i, \mu A'_{\overline{F},\delta F} \cdot \nabla\overline{u}_i\right) &= 2\left(\operatorname{div}\big(\mu A_{\overline{F}} \cdot \nabla\overline{u}_i\big), \nabla\overline{u}_i^T \cdot \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_0} \\
&\quad + 2\left(\operatorname{div}\big(\mu A_{\overline{F}} \cdot \nabla\overline{u}_i\big), \nabla\overline{u}_i^T \cdot \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_1} \\
&\quad - d\int_{\Gamma_0} \left|\mathrm{D}T_{\overline{F}_-}^{-T} \cdot \nabla\overline{u}_{i,-}\right|^2 \gamma_{\overline{F}_-} \delta F^T \cdot \mathrm{D}T_{\overline{F}_-}^{-T} \cdot n \,\mathrm{d}s \\
&\quad + \int_{\Gamma_0} \left|\mathrm{D}T_{\overline{F}_+}^{-T} \cdot \nabla\overline{u}_{i,+}\right|^2 \gamma_{\overline{F}_+} \delta F^T \cdot \mathrm{D}T_{\overline{F}_+}^{-T} \cdot n \,\mathrm{d}s,
\end{aligned} \tag{3.43}$$

where $\overline{u}_{i,-}$ and $\overline{u}_{i,+}$ shall denote $\overline{u}_i$ approaching Γ_0 from the inside of Ω_0 and Ω_1 respectively, cf. (3.3), and the same for $\overline{F}_-$ and $\overline{F}_+$. It holds that

$$\begin{aligned} 2\left(\operatorname{div}(\mu A_{\overline{F}} \cdot \nabla \overline{u}_i), \nabla \overline{u}_i^T \cdot \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_j} &= -2\overline{\lambda}_i \left(\overline{u}_i \gamma_{\overline{F}}, \nabla \overline{u}_i^T \cdot \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_j} \\ &= -\overline{\lambda}_i \left(\nabla\left(\overline{u}_i^2\right), \gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_j}, \end{aligned} \tag{3.44}$$

for $j \in \{0, 1\}$, and

$$-\overline{\lambda}_i \left(\overline{u}_i^2, \gamma'_{\overline{F},\delta F}\right) = -\overline{\lambda}_i \left(\overline{u}_i^2, \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right)_{\Omega_0} - \overline{\lambda}_i \left(\overline{u}_i^2, \operatorname{div}\left(\gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)\right)_{\Omega_1}. \tag{3.45}$$

Summing up (3.44) and (3.45) yields

$$\begin{aligned} &\sum_{j=0}^{1} \left(2\left(\operatorname{div}(\mu A_{\overline{F}} \cdot \nabla \overline{u}_i), \nabla \overline{u}_i^T \cdot \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right)_{\Omega_j} - \overline{\lambda}_i \left(\overline{u}_i^2, \gamma'_{\overline{F},\delta F}\right)_{\Omega_j} \right) \\ &= -\overline{\lambda}_i \int_{\Omega_0} \operatorname{div}\left(\overline{u}_i^2 \gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right) \mathrm{d}x - \overline{\lambda}_i \int_{\Omega_1} \operatorname{div}\left(\overline{u}_i^2 \gamma_{\overline{F}}\, \mathrm{D}T_{\overline{F}}^{-1} \cdot \delta F\right) \mathrm{d}x \\ &= -\overline{\lambda}_i \int_{\Gamma_0} \overline{u}_i^2 \gamma_{\overline{F}_-} \delta F^T \cdot \mathrm{D}T_{\overline{F}_-}^{-T} \cdot n \,\mathrm{d}s + \overline{\lambda}_i \int_{\Gamma_0} \overline{u}_i^2 \gamma_{\overline{F}_+} \delta F^T \cdot \mathrm{D}T_{\overline{F}_+}^{-T} \cdot n \,\mathrm{d}s \\ &- \overline{\lambda}_i \int_{\Gamma} \overline{u}_i^2 \gamma_{\overline{F}} \delta F^T \cdot \mathrm{D}T_{\overline{F}}^{-T} \cdot n \,\mathrm{d}s, \end{aligned} \tag{3.46}$$

where the last term vanishes due to $\overline{u}_i \in H_0^1(\Omega)$. Inserting (3.46) back into (3.43) finally yields

$$\begin{aligned} \overline{\lambda}_i'(\overline{q})(\delta q) = &- \int_{\Gamma_0} \left(d \left| \mathrm{D}T_{\overline{F}_-}^{-T} \cdot \nabla \overline{u}_{i,-} \right|^2 + \overline{\lambda}_i \overline{u}_i^2 \right) \gamma_{\overline{F}_-} \delta F^T \cdot \mathrm{D}T_{\overline{F}_-}^{-T} \cdot n \,\mathrm{d}s \\ &+ \int_{\Gamma_0} \left(\left| \mathrm{D}T_{\overline{F}_+^{-T}} \cdot \nabla \overline{u}_{i,+} \right|^2 + \overline{\lambda}_i \overline{u}_i^2 \right) \gamma_{\overline{F}_+} \delta F^T \cdot \mathrm{D}T_{\overline{F}_+}^{-T} \cdot n \,\mathrm{d}s. \end{aligned} \tag{3.47}$$

Remark 3.1.39. As $\delta F|_{\Gamma_0} = \delta q\, n$, it is not necessary to compute δF in order to compute $\overline{\lambda}_i'(\overline{q})(\delta q)$ via (3.47).

Lemma 3.1.40. *The optimal control $\overline{q} \in Q^{\mathrm{ad}}$ has the higher regularity $\overline{q} \in H^4(I)$.*

Proof. As $\delta F|_{\Gamma_0} = \delta q\, n$ and using the higher regularity of $\overline{u}_i$ and $\overline{F}$ as shown in Lemma 3.1.38 and Lemma 3.1.34, it follows similar to the proof of Lemma 3.1.32 that there exists $p_i = p_i(\overline{q}) \in L^2(I)$ with

$$\overline{\lambda}_i'(\overline{q})(\delta q) = (p_i, \delta q)_{L^2(I)} \qquad \forall \delta q \in Q,$$

and this lemma follows similar to Lemma 3.1.33. □

3.1.7. The second derivative

Within this subsection we are going to compute the second derivative of $u_i = u_i(q)$ and $\lambda_i = \lambda_i(q)$ with respect to perturbations in q, which exist due to Theorem 3.1.29. These explicit representations will be needed to prove error estimates within Section 3.4.

Taking the second derivative of the first equation of (3.18) with respect to q yields the equation for $\delta\tau u_i = u_i''(q)(\delta q, \tau q)$,

$$\begin{aligned}(\nabla\delta\tau u_i, \mu A_F \cdot \nabla v) = {}& \lambda_i \left(\delta\tau u_i, v\gamma_F\right) + \lambda''_{i,\delta q,\tau q} \left(u_i, v\gamma_F\right) + \lambda'_{i,\delta q} \left(\tau u_i, v\gamma_F\right) + \lambda'_{i,\delta q} \left(u_i, v\gamma'_{F,\tau F}\right) \\ & + \lambda'_{i,\tau q} \left(\delta u_i, v\gamma_F\right) + \lambda'_{i,\tau q} \left(u_i, v\gamma'_{F,\delta F}\right) \\ & + \lambda_i \left(\delta u_i, v\gamma'_{F,\tau F}\right) + \lambda_i \left(\tau u_i, v\gamma'_{F,\delta F}\right) + \lambda_i \left(u_i, v\gamma''_{F,\delta F,\tau F}\right) \\ & - \left(\nabla\delta u_i, \mu A'_{F,\tau F} \cdot \nabla v\right) - \left(\nabla\tau u_i, \mu A'_{F,\delta F} \cdot \nabla v\right) \\ & - \left(\nabla u_i, \mu A''_{F,\delta F,\tau F} \cdot \nabla v\right) \qquad \forall v \in H_0^1(\Omega);\end{aligned} \tag{3.48}$$

the abbreviations used are $\lambda_i = \lambda_i(q)$, $\lambda'_{i,\delta q} = \lambda_i'(q)(\delta q)$, $\lambda'_{i,\tau q} = \lambda_i'(q)(\tau q)$, $\lambda''_{i,\delta q,\tau q} = \lambda_i''(q)(\delta q, \tau q)$, $\delta u_i = u_i'(q)(\delta q)$ and $\tau u_i = u_i'(q)(\tau q)$. Note that (3.48) can again be regarded as a "perturbed" eigenfunction equation.

Using u_i itself as a test function within (3.18) and then taking the second derivative with respect to q yields

$$\begin{aligned}& 2\left(\nabla\tau u_i, \mu A_F \cdot \nabla\delta u_i\right) + 2\left(\nabla u_i, \mu A'_{F,\tau F} \cdot \nabla\delta u_i\right) + 2\left(\nabla u_i, \mu A_F \cdot \nabla\delta\tau u_i\right) \\ & + 2\left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla\tau u_i\right) + \left(\nabla u_i, \mu A''_{F,\delta F,\tau F} \cdot \nabla u_i\right) \\ & = \lambda''_{i,\delta q,\tau q}\left(u_i^2, \gamma_F\right) + 2\lambda'_{i,\delta q}\left(u_i, \tau u_i\gamma_F\right) + \lambda'_{i,\delta q}\left(u_i^2, \gamma'_{F,\tau F}\right) + 2\lambda'_{i,\tau q}\left(u_i, \delta u_i\gamma_F\right) \\ & + 2\lambda_i\left(\delta u_i, \tau u_i\gamma_F\right) + 2\lambda_i\left(u_i, \delta\tau u_i\gamma_F\right) + 2\lambda_i\left(u_i, \delta u_i\gamma'_{F,\tau F}\right) + \lambda'_{i,\tau q}\left(u_i^2, \gamma'_{F,\delta F}\right) \\ & + 2\lambda_i\left(u_i, \tau u_i\gamma'_{F,\delta F}\right) + \lambda_i\left(u_i^2, \gamma''_{F,\delta F,\tau F}\right).\end{aligned} \tag{3.49}$$

Using τu_i as a test function in (3.34), and vice versa for δu_i in the equation for τu_i, yields

$$\begin{aligned}\left(\nabla\tau u_i, \mu A_F \cdot \nabla\delta u_i\right) + \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla\tau u_i\right) = {}& \lambda'_{i,\delta q}\left(u_i, \tau u_i\gamma_F\right) \\ & + \lambda_i\left(\tau u_i, \delta u_i\gamma_F\right) + \lambda_i\left(u_i, \tau u_i\gamma'_{F,\delta F}\right),\end{aligned} \tag{3.50}$$

$$\begin{aligned}\left(\nabla\delta u_i, \mu A_F \cdot \nabla\tau u_i\right) + \left(\nabla u_i, \mu A'_{F,\tau F} \cdot \nabla\delta u_i\right) = {}& \lambda'_{i,\tau q}\left(u_i, \delta u_i\gamma_F\right) \\ & + \lambda_i\left(\delta u_i, \tau u_i\gamma_F\right) + \lambda_i\left(u_i, \delta u_i\gamma'_{F,\tau F}\right).\end{aligned} \tag{3.51}$$

The second derivative of the normalizing condition within (3.18) with respect to q reads as

$$2\left(\delta u_i, \tau u_i\gamma_F\right) + 2\left(u_i, \delta\tau u_i\gamma_F\right) + 2\left(u_i, \delta u_i\gamma'_{F,\tau F}\right) + 2\left(u_i, \tau u_i\gamma'_{F,\delta F}\right) + \left(u_i^2, \gamma''_{F,\delta F,\tau F}\right) = 0. \tag{3.52}$$

Now subtracting (3.50) and (3.51) twice from (3.49) finally yields

$$\begin{aligned}\lambda''_{i,\delta q,\tau q} = {}& \left(\nabla u_i, \mu A''_{F,\delta F,\tau F} \cdot \nabla u_i\right) - 2\left(\nabla\tau u_i, \mu A_F \cdot \nabla\delta u_i\right) \\ & - \lambda'_{i,\delta q}\left(u_i^2, \gamma'_{F,\delta F}\right) - \lambda'_{i,\tau q}\left(u_i^2, \gamma'_{F,\tau F}\right) + 2\lambda_i\left(\delta u_i, \tau u_i\gamma_F\right) - \lambda_i\left(u_i^2, \gamma''_{F,\delta F,\tau F}\right).\end{aligned} \tag{3.53}$$

3.2. Stability estimates for eigenvalues and eigenfunctions

In order to estimate the error between eigenfunctions and their discretized counterparts, the application of the "standard" techniques as done in Chapter 2 is not possible, this is due to the fact that eigenfunctions appear on the left-, as well as on the right hand side of the corresponding equation, cf. (3.18). Hence we have to deal with different concepts which will be presented in this section.

The results of Subsection 3.2.1 will be needed to estimate terms like $\|u_i(q) - u_i(p)\|$ for $q, p \in Q^{\text{ad}}$, whereas the results of Subsection 3.2.2 will be needed to estimate terms like $\|u_i'(q)(\delta q) - u_i'(p)(\delta q)\|$ for $q, p \in Q^{\text{ad}}$ and $\delta q \in Q$.

3.2.1. Gap between operators

If u is an eigenfunction of a linear partial differential operator L, then for arbitrary $c \in \mathbb{R}\setminus\{0\}$, cu is also an eigenfunction. Due to this fact it is not clear how to estimate the difference $\|u_{1,i} - u_{2,i}\|$ between the i-th eigenfunctions $u_{1,i}$ and $u_{2,i}$ corresponding to different differential operators L_1 and L_2, for even normalized eigenfunctions are only unique up to their sign. In order to deal with this difficulty we will have a closer look at the concept of the so-called gap between operators. What follows is mainly based on [10] and [22].

Definition 3.2.1. Let M and N be linear subspaces of a normed space Z. The gap *from* M to N is defined via

$$\delta(M,N) = \sup_{\substack{u\in M\\ \|u\|_Z=1}} \operatorname{dist}(u,N),$$

where for $u \in Z$ we have

$$\operatorname{dist}(u,N) = \inf_{v\in N} \|u - v\|_Z\,.$$

Furthermore, the gap *between* M and N is defined via

$$\hat{\delta}(M,N) = \max\left\{\delta(M,N), \delta(N,M)\right\}.$$

Lemma 3.2.2. *Let M and N be linear subspaces of a Hilbert space Z, and let P and Q be the orthogonal projections onto the closures of M and N, respectively. Then it holds that*

$$\begin{aligned}\delta(M,N) &= \|(1-Q)P\|_Z\,,\\ \hat{\delta}(M,N) &= \|P-Q\|_Z\,.\end{aligned}$$

Proof. This lemma can be found in [22], Theorem 2.2. □

Definition 3.2.3. Let $T\colon D(T) \subset X \to Y$ be a linear operator whose domain $D(T)$ is a subset of the Hilbert space X and maps onto the Hilbert space Y. The graph G of the operator T is defined as

$$G(T) = \{(u,Tu)|\ u \in D(T)\}\,.$$

Definition 3.2.4. Let X and Y be Hilbert spaces and let

$$\begin{aligned}S&\colon D(S) \subset X \to Y,\\ T&\colon D(T) \subset X \to Y,\end{aligned}$$

be linear operators mapping subsets of X onto Y. The gap *from* S to T is defined by

$$\delta(S,T) = \delta(G(S), G(T)),$$

whereas the gap *between* S and T is defined by

$$\hat{\delta}(S,T) = \hat{\delta}(G(S), G(T)).$$

More explicitly,

$$\delta(S,T) = \sup_{\substack{u\in D(S)\\ \|u\|_X^2+\|Su\|_Y^2=1}} \inf_{v\in D(T)} \left(\|u-v\|_X^2 + \|Su - Tv\|_Y^2\right)^{1/2}. \tag{3.54}$$

Lemma 3.2.5. *Let X be a Hilbert space and let S and T be selfadjoint on X. Then it holds that*

$$\delta(S,T) = \delta(T,S) = \hat{\delta}(S,T).$$

Proof. This lemma can be found in [22], Corollary 2.6. □

Theorem 3.2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let T be a selfadjoint operator over $L^2(\Omega)$ with compact resolvent bounded from below and let $i \in \mathbb{N}$ such that the i-th eigenvalue λ_i of T is simple. Then there exist $c_0, \delta_0 > 0$ such that for each selfadjoint operator S over $L^2(\Omega)$ whose compact resolvent is bounded from below, for which $\delta(S,T) < \delta_0$ and normalized eigenfunction $\tilde{u}_i$ corresponding to the i-th eigenvalue $\tilde{\lambda}_i$ of S, there exists a normalized eigenfunction u_i of T, corresponding to λ_i, such that*

$$\|u_i - \tilde{u}_i\|_{L^2(\Omega)} \leq c_0 \delta(S,T).$$

Proof. This theorem can be found in [22], Theorem 2.14. □

3.2.2. Stability estimates for affine eigenvectors

In what follows let X be a Hilbert space over $\mathbb{R}$ with scalar product $(\cdot,\cdot)_X$, norm $\|u\|_X = \sqrt{(u,u)_X}$, and let L be a compact linear operator over X. The ordered eigenvalues of L shall be denoted with $(\nu_i)_{i\in\mathbb{N}}$, where $\lim_{i\to\infty} \nu_i = 0$. The eigenspace corresponding to ν_i will be denoted with $N_i(L)$, its orthogonal complement $N_i(L)^\perp$ has to be understood with respect to the X-scalar product. From Theorem 3.1.27 it follows that for $g \in X$ and $i \in \mathbb{N}$ there exists a solution $u \in X$ to

$$Lu = \nu_i u + g, \tag{3.55}$$

if and only if $g \in N_i(L)^\perp$. This solution, if it exists, is not unique. If u solves (3.55), so does $u + cu_i$ for all $u_i \in N_i(L)$ and $c \in \mathbb{R}$. In what follows we are going to prove that there exists $c_i > 0$ such that for all $g \in N_i(L)^\perp$ there exists a solution to (3.55) with $\|u\|_X \leq c_i \|g\|_X$.

Lemma 3.2.7. *The subspace $N_i(L)^\perp$ is closed in X.*

Proof. With Theorem A.1.2 it follows that $N_i(L) \subset X$ is of finite dimension and closed. From [3], Lemma 7.17, it follows that $X = N_i(L) \oplus N_i(L)^\perp$. As $N_i(L)$ is closed, there exists a continuous orthogonal projection P onto $N_i(L)$ with $N_i(L)^\perp = \mathcal{N}(P)$, and this lemma follows with the closed complement theorem, cf. [3], Theorem 7.15. □

Lemma 3.2.8. *Let $g \in N_i(L)^\perp$ and let u_g be a solution to* (3.55). *Then u_g minimizes the X-norm among all solutions of* (3.55) *if and only if $u_g \in N_i(L)^\perp$.*

Proof. Let u_g be a solution to (3.55). Then u_g has minimal X-norm if and only if for all $u_i \in N_i(L)$, the solution to

$$\arg\min_{t\in\mathbb{R}} \|u_g + tu_i\|_X^2 , \tag{3.56}$$

is $t = 0$. The proof now follows by taking the first and second derivative of the squared norm within (3.56) with respect to t. □

Lemma 3.2.9. *Let $g \in N_i(L)^\perp$. Then there exists exactly one solution u_g to* (3.55) *that minimizes the X-norm among all solutions to* (3.55).

Proof. Let $u_{g,1}$ and $u_{g,2}$ be two solutions to (3.55) with minimal X-norm, and let $u_g = u_{g,1} - u_{g,2}$. With Lemma 3.2.8 it follows that $u_g \in N_i(L)^\perp$. As L is linear we get $Lu_g = \nu_i u_g$, hence $u_g \in N_i(L)$. It follows that $u_g \in N_i(L) \cap N_i(L)^\perp = \{0\}$, and the result follows. □

Corollary 3.2.10. *Let $g \in N_i(L)^\perp$, let u_g be an arbitrary solution to (3.55) and let $\{u_\nu^1, \dots, u_\nu^N\}$ be an orthogonal basis for $N_i(L)$. Then the solution $\overline{u}_g$ of (3.55) with minimal X-norm is given via*

$$\overline{u}_g = u_g - \sum_{i=1}^{N} \frac{\left(u_g, u_\nu^i\right)_X}{\|u_\nu^i\|_X^2} u_\nu^i.$$

Proof. By definition of $\overline{u}_g$ it follows that $\left(\overline{u}_g, u_\nu^i\right)_X = 0$ for all $i \in \{1, \dots, N\}$ and the result follows with Lemma 3.2.8. □

Definition 3.2.11. For $i \in \mathbb{N}$ let $T = T_i \colon N_i(L)^\perp \subset X \to X$, $Tg = u_g$ such that u_g is a solution to (3.55) corresponding to g with minimal X-norm, i.e. for all solutions $\tilde{u}_g$ to (3.55) with $u_g \neq \tilde{u}_g$ it holds that $\|u_g\|_X < \|\tilde{u}_g\|_X$.

Remark 3.2.12. The fact that the operator T from Definition 3.2.11 is well-defined follows with Theorem 3.1.27 and Lemma 3.2.9.

Lemma 3.2.13. *The operator T from Definition 3.2.11 is linear.*

Proof. Let $g, h \in N_i(L)^\perp$, let u_g and u_h be arbitrary solutions to the corresponding perturbed eigenvalue equations (3.55), let $\{u_\nu^1, \dots, u_\nu^N\}$ be an orthogonal basis for $N_i(L)$ and let $\alpha \in \mathbb{R}$. As

$$L(\alpha u_g) = \alpha L u_g = \alpha(\nu_i u_g + g) = \nu_i\,(\alpha u_g) + \alpha g,$$

it follows with Corollary 3.2.10 that

$$T(\alpha g) = \alpha u_g - \sum_{i=1}^{N} \frac{\left(\alpha u_g, u_\nu^i\right)_X}{\|u_\nu^i\|_X^2} u_\nu^i = \alpha \left(u_g - \sum_{i=1}^{N} \frac{\left(u_g, u_\nu^i\right)_X}{\|u_\nu^i\|_X^2} u_\nu^i \right) = \alpha T(g).$$

Furthermore,

$$L(u_g + u_h) = Lu_g + Lu_h = (\nu_i u_g + g) + (\nu_i u_h + h) = \nu_i\,(u_g + u_h) + (g + h),$$

and again we use Corollary 3.2.10 to get

$$\begin{aligned} T(g+h) &= (u_g + u_h) - \sum_{i=1}^{N} \frac{\left((u_g + u_h)\,, u_\nu^i\right)_X}{\|u_\nu^i\|_X^2} u_\nu^i \\ &= \left(u_g - \sum_{i=1}^{N} \frac{\left(u_g, u_\nu^i\right)_X}{\|u_\nu^i\|_X^2} u_\nu^i \right) + \left(u_h - \sum_{i=1}^{N} \frac{\left(u_h, u_\nu^i\right)_X}{\|u_\nu^i\|_X^2} u_\nu^i \right) \\ &= T(g) + T(h). \end{aligned}$$

□

Lemma 3.2.14. *Let T be as in Definition 3.2.11 and let $G(T) = \{\,(g, Tg) \mid g \in N_i(L)^\perp\} \subset (X \times X)$ be the graph of T. Then $G(T)$ is closed.*

Proof. Let $(g_n)_{n\in\mathbb{N}} \subset N_i(L)^\perp$, $u_n = T(g_n)$ with $g_n \to g$ and $u_n \to u$ in X for some elements $g, u \in X$. We have to show that $g \in N_i(L)^\perp$ and $u = Tg$. From Lemma 3.2.8 it follows that $(u_n)_{n\in\mathbb{N}} \subset N_i(L)^\perp$. As $N_i(L)^\perp$ is closed due to Lemma 3.2.7 it follows that $u, g \in N_i(L)^\perp$. As L is compact it follows that

$$Lu \leftarrow Lu_n = \nu_i u_n + g_n \to \nu_i u + g,$$

hence $u = Tg$. □

Lemma 3.2.15. *The operator T from Definition 3.2.11 is bounded.*

Proof. As T is a linear operator with closed graph due to Lemma 3.2.13 and Lemma 3.2.14, this lemma follows with the closed graph theorem, cf. [3], Theorem 5.9. □

Corollary 3.2.16. *There exists $c_i > 0$, independent of $g \in N_i(L)^\perp$, such that*

$$\|Tg\|_X \leq c_i \|g\|_X ,$$

for all $g \in N_i(L)^\perp$.

3.3. Discretization

Within this section we are using finite elements in order to discretize problem (3.21) with respect to the control, the state and the transformation. Most of what follows is similar to Section 2.2, where the original modelproblem has been discretized.

3.3.1. Discretization of the control

We split the interval $I = (0, 2\pi)$ into $N \in \mathbb{N}$ subintervals I_j for $j \in \{0, \ldots, N-1\}$ with maximal length σ, and introduce the space of (admissible) discretized controls as

$$\begin{aligned} Q_\sigma &= \left\{ q_\sigma \in Q |\ q_\sigma|_{I_j} \in \mathcal{P}^3(I_j)\, \forall j \in \{0, \ldots, N-1\} \right\}, \\ Q_\sigma^{\text{ad}} &= Q_\sigma \cap Q^{\text{ad}}, \end{aligned}$$

where $\mathcal{P}^3(I)$ shall denote the set of all polynomials of degree at most 3 over the interval I. The first partially discretized problem now reads as

$$\min_{q_\sigma \in Q_\sigma^{\text{ad}}} j(q_\sigma) = \lambda_1(q_\sigma) - \lambda_2(q_\sigma) + \frac{\alpha}{2} \|q_\sigma\|_{H^2(I)}^2 . \tag{3.57}$$

subject to

$$\begin{cases} \mathcal{G}(q_\sigma, G) = (F, G) & \forall G \in L^2(\Omega), \\ a(F)(u_i, v) = \lambda_i\, b(F)(u_i, v) & \forall v \in H_0^1(\Omega), \\ b(F)(u_i, u_i) = 1, \end{cases}$$

where $i \in \{1, 2\}$ and λ_i is the i-th eigenvalue given via (3.6).

3.3.2. Discretization of the state

For $h > 0$ let $\Omega_{0,h} \subset \Omega_0$ be a polygonal approximation of Ω_0 where we assume that all the vertices of $\Gamma_{0,h} = \partial\Omega_{0,h}$ lie on Γ_0. In addition, let $\Omega_{1,h} = \Omega\backslash\overline{\Omega_{0,h}} \supset \Omega_1$ be a polygonal approximation of Ω_1. Let $\{\pi_h\}_{h>0}$ be a family of admissible triangulations of Ω using triangles or quadrilaterals with maximal diameter h, fulfilling the usual regularity assumptions in the sense of Definition 2.2.4 and Remark 2.2.7. In addition we assume that each member of this family can be represented as the union of a triangulation of $\Omega_{0,h}$ with a triangulation of $\Omega_{1,h}$. We define the usual (bi)linear finite elements,

$$\begin{aligned} V_h &= \left\{ v_h \in H^1(\Omega) \middle|\ v_h|_{K_h} \in \mathcal{R}^1(K_h)\ \forall K_h \in \pi_h \right\}, \\ V_{h,0} &= V_h \cap H_0^1(\Omega), \end{aligned} \tag{3.58}$$

where $\mathcal{R}^1(K_h)$ is defined as in (2.62). Now let

$$\begin{aligned} \mu_h &= 1 + (d-1)\chi_{\Omega_{0,h}}, \\ a_h(F)(u,v) &= (\nabla u, \mu_h A_F \cdot \nabla v), \end{aligned} \tag{3.59}$$

with $\chi_{\Omega_{0,h}}$ being the characteristic function of $\Omega_{0,h}$. The following definition is similar to Definition 3.1.10.

Definition 3.3.1. For given $q \in Q^{\text{ad}}$, $F = F(q)$ and h sufficiently small, let $H^1_{0,a_h}(\Omega)$ be the space $H_0^1(\Omega)$ equipped with the scalar product

$$(u,v)_{H^1_{0,a_h}(\Omega)} = (\nabla u, \mu_h A_F \cdot \nabla v).$$

The $H^1_{0,a_h}(\Omega)$-norm is equivalent to the $H^1_0(\Omega)$-norm due to Lemma 3.4.37.

Definition 3.3.2. For given $q \in Q^{\text{ad}}$ with $F = F(q)$ let $\Pi_h \colon H_0^1(\Omega) \to V_{h,0}$ be defined as the projection with respect to $a_h(F)(\cdot,\cdot)$, i.e. for $u \in H_0^1(\Omega)$ it holds that

$$a_h(F)(u - \Pi_h u, v_h) = 0 \qquad \forall v_h \in V_{h,0}.$$

Definition 3.3.3. For the optimal control $\overline{q} \in Q^{\text{ad}}$ with $\overline{F} = F(\overline{q})$ let $\Pi_h^o \colon H_0^1(\Omega) \to V_{h,0}$ be defined as the projection with respect to $a(\overline{F})(\cdot,\cdot)$, i.e. for $u \in H_0^1(\Omega)$ it holds that

$$a(\overline{F})(u - \Pi_h^o u, v_h) = 0 \qquad \forall v_h \in V_{h,0}.$$

In addition, let $\lambda_{i,h}(q)$ be the i-th eigenvalue with respect to the bilinear forms $a_h(F)(\cdot,\cdot)$ and $b(F)(\cdot,\cdot)$ which can be computed via (3.6). The second partially discretized problem, where we additionally discretize the state, now reads as

$$\min_{q_\sigma \in Q_\sigma^{\text{ad}}} j_h(q_\sigma) = \lambda_{1,h}(q_\sigma) - \lambda_{2,h}(q_\sigma) + \frac{\alpha}{2}\|q_\sigma\|^2_{H^2(I)}, \tag{3.60}$$

subject to

$$\begin{cases} \mathcal{G}(q_\sigma, G) = (F, G) & \forall G \in L^2(\Omega), \\ a_h(F)(u_{i,h}, v_h) = \lambda_{i,h}\, b(F)(u_{i,h}, v_h) & \forall v_h \in V_{h,0}, \\ b(F)(u_{i,h}, u_{i,h}) = 1, \end{cases}$$

with $i \in \{1, 2\}$.

3.3.3. Discretization of the transformation

As in Subsection 3.3.2, let $\Omega_{0,k} \subset \Omega_0$ be a polygonal approximation to Ω_0, let $\Gamma_{0,k} = \partial\Omega_{0,k}$, let $\Omega_{1,k} = \Omega\backslash\overline{\Omega_{0,k}} \supset \Omega_1$ be a polygonal approximation to Ω_1 and let $\{\pi_k\}_{k>0}$ be a family of admissible triangulations of Ω using triangles or quadrilaterals with maximal diameter k, and fulfilling the usual regularity assumptions in the sense of Definition 2.2.4 and Remark 2.2.7. Again we assume that every triangulation π_k can be considered as the union of a triangulation of $\Omega_{0,k}$ with a triangulation of $\Omega_{1,k}$. Similar to (3.58) let

$$V_k = \left\{ v_k \in H^1(\Omega) \middle| \; v_k|_{K_k} \in \mathcal{R}^1(K_k) \; \forall K_k \in \pi_k \right\}. \tag{3.61}$$

In order to discretize the transformation we will use a discrete approximation $\mathcal{G}_k$ to the operator $\mathcal{G}$, defined similarly as in Subsection 2.2.3, cf. [13], Section 5. If $F_k = F_k(q)$ denotes the discrete transformation corresponding to the control q, then the fully discretized i-th eigenvalue $\lambda_{i,h,k}$ is given via (3.6) using the forms $a_h(F_k)(\cdot,\cdot)$ and $b(F_k)(\cdot,\cdot)$. The following two definitions are similar to Definition 3.1.9 and Definition 3.1.10

Definition 3.3.4. For fixed $q \in Q^{\text{ad}}$, $F_k = F_k(q)$ and k sufficiently small let $L^2_{b,k}(\Omega)$ be the space $L^2(\Omega)$ equipped with the scalar product

$$(u, v)_{L^2_{b,k}(\Omega)} = (u, v\gamma_{F_k}).$$

The $L^2_{b,k}(\Omega)$-norm is equivalent to the $L^2(\Omega)$-norm due to Assumption 3.4.58.

Definition 3.3.5. For fixed $q \in Q^{\text{ad}}$, $F_k = F_k(q)$ and h, k sufficiently small let $H^1_{0,a_{h,k}}(\Omega)$ be the space $H^1_0(\Omega)$ equipped with the scalar product

$$(u, v)_{H^1_{0,a_{h,k}}(\Omega)} = (\nabla u, \mu_h A_{F_k} \cdot \nabla v).$$

The $H^1_{0,a_{h,k}}(\Omega)$-norm is equivalent to the $H^1_0(\Omega)$-norm due to Lemma 3.4.37 and Assumption 3.4.58.

Definition 3.3.6. For given $q \in Q^{\text{ad}}$, $F = F(q)$ and $F_k = F_k(q)$ let $\Pi_k \colon H^1_0(\Omega) \to V_{h,0}$ be defined as

$$a_h(F_k)(\Pi_k u, v_h) = a(F)(u, v_h) \qquad\qquad \forall v_h \in V_{h,0}.$$

Finally, the fully discretized problem, where also the transformation is being discretized, reads as

$$\min_{q_\sigma \in Q^{\text{ad}}_\sigma} j_{h,k}(q_\sigma) = \lambda_{1,h,k}(q_\sigma) - \lambda_{2,h,k}(q_\sigma) + \frac{\alpha}{2} \|q_\sigma\|^2_{H^2(I)}, \tag{3.62}$$

subject to

$$\begin{cases} \qquad \mathcal{G}_k(q_\sigma, G_k) = (F_k, G_k) & \forall G_k \in V_k, \\ a_h(F_k)(u_{i,h}, v_h) = \lambda_{i,h,k}\, b(F_k)(u_{i,h}, v_h) & \forall v_h \in V_{h,0}, \\ b(F_k)(u_{i,h}, u_{i,h}) = 1, \end{cases}$$

where again $i \in \{1, 2\}$.

Theorem 3.3.7. *For $\sigma, h, k > 0$ the problems* (3.57), (3.60) *and* (3.62) *possess optimal solutions $\overline{q}_\sigma$, $\overline{q}_{\sigma,h}$ and $\overline{q}_{\sigma,h,k}$, respectively.*

Proof. This theorem can be proven similar to Theorem 2.2.3, Theorem 2.2.9 and Theorem 2.2.13. □

3.4. A-priori error estimates

The goal of this section is to prove error estimates for the H^2-error between the optimal control $\overline{q}$ for (3.21) and a sequence of optimal controls $\left(\overline{q}_{\sigma,h,k}\right)_{\sigma,h,k>0}$ for the fully discretized finite-element approximation (3.62). The main result of this section is the following theorem, the proof can be found on page 143.

Theorem 3.4.1. *Let $\overline{q} \in Q^{\mathrm{ad}}$ be a local optimal control to* (3.21)*. Then there exists a sequence $\left(\overline{q}_{\sigma,h,k}\right)_{\sigma,h,k>0}$ of local optimal controls to the fully discretized problem* (3.62) *such that for $\sigma, h, k > 0$ sufficiently small it holds that*

$$\left\|\overline{q} - \overline{q}_{\sigma,h,k}\right\|_{H^2(I)} \leq c\left(\sigma^2 + |\ln h|^{3/4} h^{3/2} + |\ln k|^{3/4} k^{3/2}\right).$$

Although Theorem 3.4.1 is of similar structure as the main result of Section 2.3, Theorem 2.3.1, there are some major differences in the methods used for its proof which are due to the specific structure of the eigenvalue equation. Again, if not explicitly stated otherwise, we wil always assume that σ, h and k are chosen sufficiently small. First we will restate the definitions of the (discretized) eigenpairs and their derivatives and we will also deal with the fact that, as already mentioned at the beginning of Subsection 3.2.1, even normalized eigenfunctions are only defined up to their sign.

Within the following three definitions, let $q \in Q^{\mathrm{ad}}$ and let $\delta q, \tau q \in Q$ be arbitrary.

Definition 3.4.2 (The continuous eigenpair)**.**

- Let $V^{(i)}$ be the set of all subspaces of $H_0^1(\Omega)$ with dimension i and let $F = F(q)$. It holds that
$$\lambda_i(q) = \min_{E\in V^{(i)}} \max_{u\in E} \frac{a(F)(u,u)}{b(F)(u,u)}. \tag{3.63}$$

- $u_i = S_i(q)$ is a solution of
$$\begin{cases} (\nabla u_i, \mu A_F \cdot \nabla v) = \lambda_i(q)\,(u_i, v\gamma_F) \quad \forall v \in H_0^1(\Omega), \\ \left(u_i^2, \gamma_F\right) = 1. \end{cases} \tag{3.64}$$

  For $q = 0$ we fix one of the two solutions of (3.64) as $S_i(0)$. For $q \in Q^{\mathrm{ad}} \backslash \{0\}$ with $\|q\|_{H^2(I)}$ sufficiently small let $S_i(q)$ be a solution to (3.64) such that Lemma 3.4.19 is applicable.

- It holds that
$$\lambda_i'(q)(\delta q) = \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla u_i\right) - \lambda_i(q)\left(u_i^2, \gamma'_{F,\delta F}\right), \tag{3.65}$$
  where $\delta F = F'(q)(\delta q)$.

- $\delta u_i = S_i'(q)(\delta q)$ is the solution of
$$\begin{cases} (\nabla \delta u_i, \mu A_F \cdot \nabla v) = \lambda_i(q)\,(\delta u_i, v\gamma_F) + \lambda_i'(q)(\delta q)\,(u_i, v\gamma_F) \\ \qquad\qquad + \lambda_i(q)\left(u_i, v\gamma'_{F,\delta F}\right) - \left(\nabla u_i, \mu A'_{F,\delta F} \cdot \nabla v\right) \quad \forall v \in H_0^1(\Omega), \\ 2\,(\delta u_i, u_i\gamma_F) + \left(u_i^2, \gamma'_{F,\delta F}\right) = 0. \end{cases} \tag{3.66}$$

- It holds that

$$\begin{aligned}\lambda_i''(q)(\delta q,\tau q) = {} & \left(\nabla u_i, \mu A''_{F,\delta F,\tau F}\cdot\nabla u_i\right) - 2\left(\nabla\tau u_i, \mu A_F\cdot\nabla\delta u_i\right)\\ & - \lambda_i'(q)(\delta q)\left(u_i^2,\gamma'_{F,\tau F}\right) - \lambda_i'(q)(\tau q)\left(u_i^2,\gamma'_{F,\delta F}\right)\\ & + 2\lambda_i(q)\left(\delta u_i,\tau u_i\gamma_F\right) - \lambda_i(q)\left(u_i^2,\gamma''_{F,\delta F,\tau F}\right),\end{aligned} \tag{3.67}$$

  with $\tau F = F'(q)(\tau q)$ and $\tau u_i = S_i'(q)(\tau q)$.

Definition 3.4.3 (The partially discretized eigenpair)**.**

- Let $V_{h,0}^{(i)}$ be the set of all subspaces of $V_{h,0}$ with dimension i and let $F = F(q)$. It holds that

$$\lambda_{i,h}(q) = \min_{E_h\in V_{h,0}^{(i)}}\max_{u_h\in E_h}\frac{a_h(F)(u_h,u_h)}{b(F)(u_h,u_h)}. \tag{3.68}$$

- $u_{i,h} = S_{i,h}(q)$ is a solution of

$$\begin{cases}\left(\nabla u_{i,h}, \mu_h A_F\cdot\nabla v_h\right) = \lambda_{i,h}(q)\left(u_{i,h}, v_h\gamma_F\right) \quad \forall v_h\in V_{h,0},\\ \left(u_{i,h}^2,\gamma_F\right) = 1,\\ b(F)(\Pi_h S_i(q), S_{i,h}(q)) \geq 0,\end{cases} \tag{3.69}$$

  where Π_h is defined as in Definition 3.3.2.

- It holds that

$$\lambda_{i,h}'(q)(\delta q) = \left(\nabla u_{i,h}, \mu_h A'_{F,\delta F}\cdot\nabla u_{i,h}\right) - \lambda_{i,h}(q)\left(u_{i,h}^2,\gamma'_{F,\delta F}\right), \tag{3.70}$$

  where $\delta F = F'(q)(\delta q)$.

- $\delta u_{i,h} = S_{i,h}'(q)(\delta q)$ is the solution to

$$\begin{cases}\begin{aligned}\left(\nabla\delta u_{i,h}, \mu_h A_F\cdot\nabla v_h\right) = {} & \lambda_{i,h}(q)\left(\delta u_{i,h}, v_h\gamma_F\right) + \lambda_{i,h}'(q)(\delta q)\left(u_{i,h}, v_h\gamma_F\right)\\ & + \lambda_{i,h}(q)\left(u_{i,h}, v_h\gamma'_{F,\delta F}\right)\\ & - \left(\nabla u_{i,h}, \mu_h A'_{F,\delta F}\cdot\nabla v_h\right) \qquad \forall v_h\in V_{h,0},\end{aligned}\\ 2\left(\delta u_{i,h}, u_{i,h}\gamma_F\right) + \left(u_{i,h}^2,\gamma'_{F,\delta F}\right) = 0.\end{cases} \tag{3.71}$$

- It holds that

$$\begin{aligned}\lambda_{i,h}''(q)(\delta q,\tau q) = {} & \left(\nabla u_{i,h}, \mu_h A''_{F,\delta F,\tau F}\cdot\nabla u_{i,h}\right) - 2\left(\nabla\tau u_{i,h}, \mu_h A_F\cdot\nabla\delta u_{i,h}\right)\\ & - \lambda_{i,h}'(q)(\delta q)\left(u_{i,h}^2,\gamma'_{F,\tau F}\right) - \lambda_{i,h}'(q)(\tau q)\left(u_{i,h}^2,\gamma'_{F,\delta F}\right)\\ & + 2\lambda_{i,h}(q)\left(\delta u_{i,h},\tau u_{i,h}\gamma_F\right) - \lambda_{i,h}(q)\left(u_{i,h}^2,\gamma''_{F,\delta F,\tau F}\right),\end{aligned} \tag{3.72}$$

  with $\tau F = F'(q)(\tau q)$ and $\tau u_{i,h} = S_{i,h}'(q)(\tau q)$.

Definition 3.4.4 (The fully discretized eigenpair).

- Let $V_{h,0}^{(i)}$ be the set of all subspaces of $V_{h,0}$ with dimension i and let $F_k = F_k(q)$. It holds that

$$\lambda_{i,h,k}(q) = \min_{E_h \in V_{h,0}^{(i)}} \max_{u_h \in E_h} \frac{a_h(F_k)(u_h, u_h)}{b(F_k)(u_h, u_h)}. \tag{3.73}$$

- $u_{i,h,k} = S_{i,h,k}(q)$ is a solution to

$$\begin{cases} (\nabla u_{i,h,k}, \mu_h A_{F_k} \cdot \nabla v_h) = \lambda_{i,h,k} \left(u_{i,h,k}, v_h \gamma_{F_k}\right) \quad \forall v_h \in V_{h,0}, \\ \left(u_{i,h,k}^2, \gamma_{F_k}\right) = 1, \\ b(F_k)(S_{i,h}(q), S_{i,h,k}(q)) \geq 0. \end{cases} \tag{3.74}$$

- It holds that

$$\lambda_{i,h,k}'(q)(\delta q) = \left(\nabla u_{i,h,k}, \mu_h A_{F_k,\delta F_k}' \cdot \nabla u_{i,h,k}\right) - \lambda_{i,h,k}(q) \left(u_{i,h,k}^2, \gamma_{F_k,\delta F_k}'\right), \tag{3.75}$$

  where $\delta F_k = F_k'(q)(\delta q)$.

- $\delta u_{i,h,k} = S_{i,h,k}'(q)(\delta q)$ is the solution to

$$\begin{cases} (\nabla \delta u_{i,h,k}, \mu_h A_{F_k} \cdot \nabla v_h) = \lambda_{i,h,k} \left(\delta u_{i,h,k}, v_h \gamma_{F_k}\right) + \lambda_{i,h,k}' \left(u_{i,h,k}, v_h \gamma_{F_k}\right) \\ \qquad + \lambda_{i,h,k} \left(u_{i,h,k}, v_h \gamma_{F_k,\delta F_k}'\right) \\ \qquad - \left(\nabla u_{i,h,k}, \mu_h A_{F_k,\delta F_k}' \cdot \nabla v_h\right) \qquad \forall v_h \in V_{h,0}, \\ 2\left(\delta u_{i,h,k}, u_{i,h} \gamma_{F_k}\right) + \left(u_{i,h,k}^2, \gamma_{F_k,\delta F_k}'\right) = 0. \end{cases} \tag{3.76}$$

- It holds that

$$\begin{aligned} \lambda_{i,h,k}''(q)(\delta q, \tau q) = {} & \left(\nabla u_{i,h,k}, \mu_h A_{F_k,\delta F_k,\tau F_k}'' \cdot \nabla u_{i,h,k}\right) - 2\left(\nabla \tau u_{i,h,k}, \mu_h A_{F_k} \cdot \nabla \delta u_{i,h,k}\right) \\ & - \lambda_{i,h,k}'(q)(\delta q) \left(u_{i,h,k}^2, \gamma_{F_k,\tau F_k}'\right) - \lambda_{i,h,k}'(q)(\tau q) \left(u_{i,h,k}^2, \gamma_{F_k,\delta F_k}'\right) \\ & + 2\lambda_{i,h,k} \left(\delta u_{i,h,k}, \tau u_{i,h,k} \gamma_{F_k}\right) - \lambda_{i,h,k} \left(u_{i,h,k}^2, \gamma_{F_k,\delta F_k,\tau F_k}''\right), \end{aligned} \tag{3.77}$$

  with $\tau F_k = F_k'(q)(\tau q)$ and $\tau u_{i,h,k} = S_{i,h,k}'(q)(\tau q)$.

For the ease of notation we introduce the following functionals which appear in the equations for the derivative of the (discrete) eigenfunctions.

Definition 3.4.5. For $q \in Q^{\text{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$ let

$$g_i = g_i(q, \delta q),\ g_{i,h} = g_{i,h}(q, \delta q),\ g_{i,h,k} = g_{i,h,k}(q, \delta q) \in H^{-1}(\Omega),$$

be defined via

$$\begin{aligned} (g_i, v)_{H^{-1},H_0^1} &= \lambda_i'(q)(\delta q)\left(u_i, v\gamma_F\right) + \lambda_i(q)\left(u_i, v\gamma_{F,\delta F}'\right) - \left(\nabla u_i, \mu A_{F,\delta F}' \cdot \nabla v\right), \\ (g_{i,h}, v)_{H^{-1},H_0^1} &= \lambda_{i,h}'(q)(\delta q)\left(u_{i,h}, v\gamma_F\right) + \lambda_{i,h}(q)\left(u_{i,h}, v\gamma_{F,\delta F}'\right) - \left(\nabla u_{i,h}, \mu_h A_{F,\delta F}' \cdot \nabla v\right), \\ (g_{i,h,k}, v)_{H^{-1},H_0^1} &= \lambda_{i,h,k}'(q)(\delta q)\left(u_{i,h,k}, v\gamma_{F_k}\right) + \lambda_{i,h,k}(q)\left(u_{i,h,k}, v\gamma_{F_k,\delta F_k}'\right) - \left(\nabla u_{i,h,k}, \mu_h A_{F_k,\delta F_k}' \cdot \nabla v\right), \end{aligned}$$

where $F = F(q)$, $F_k = F_k(q)$, $\delta F = F'(q)(\delta q)$ and $\delta F_k = F_k'(q)(\delta q)$.

As within Subsection 2.3.1 we need some general regularity and stability results for A_F and γ_F as well as their derivatives. The following lemma can be proven similar to Lemma 2.3.16, Lemma 2.3.17, Lemma 2.3.18 and Lemma 2.3.19, cf. Lemma 3.1.6.

Lemma 3.4.6. *Let $q, p \in Q^{\mathrm{ad}}$ with transformations F and E, respectively. Let $\delta q, \tau q \in Q$, $\delta F = F'(q)(\delta q)$, $\tau F = F'(q)(\tau q)$ and let $j \in \{0, 1\}$. Then it holds that*

$$\|A_F\|_{L^\infty(\Omega)} \le c, \qquad \|\gamma_F\|_{L^\infty(\Omega)} \le c,$$
$$\left\|A'_{F,\delta F}\right\|_{L^\infty(\Omega_j)} \le c\,\|\delta q\|_{H^2(I)}\,, \qquad \left\|\gamma'_{F,\delta F}\right\|_{L^\infty(\Omega_j)} \le c\,\|\delta q\|_{H^2(I)}\,,$$
$$\left\|A''_{F,\delta F,\tau F}\right\|_{L^\infty(\Omega_j)} \le c\,\|\delta q\|_{H^2(I)}\,\|\tau q\|_{H^2(I)}\,, \qquad \left\|\gamma''_{F,\delta F,\tau F}\right\|_{L^\infty(\Omega_j)} \le c\,\|\delta q\|_{H^2(I)}\,\|\tau q\|_{H^2(I)}\,,$$

and

$$\begin{aligned}
\|A_F - A_E\|_{L^\infty(\Omega)} &\le c\,\|q-p\|_{H^2(I)}\,,\\
\left\|A'_{F,\delta F} - A'_{E,\delta F}\right\|_{L^\infty(\Omega_j)} &\le c\,\|q-p\|_{H^2(I)}\,\|\delta q\|_{H^2(I)}\,,\\
\left\|A''_{F,\delta F,\tau F} - A''_{E,\delta F,\tau F}\right\|_{L^\infty(\Omega_j)} &\le c\,\|q-p\|_{H^2(I)}\,\|\delta q\|_{H^2(I)}\,\|\tau q\|_{H^2(I)}\,,\\
\|\gamma_F - \gamma_E\|_{L^\infty(\Omega)} &\le c\,\|q-p\|_{H^2(I)}\,,\\
\left\|\gamma'_{F,\delta F} - \gamma'_{E,\delta F}\right\|_{L^\infty(\Omega_j)} &\le c\,\|q-p\|_{H^2(I)}\,\|\delta q\|_{H^2(I)}\,,\\
\gamma''_{F,\delta F,\tau F}\big|_{\Omega_j} &= \gamma''_{E,\delta F,\tau F}\big|_{\Omega_j}\,.
\end{aligned}$$

The following theorem can be found in [3], Theorem 7.7.

Theorem 3.4.7. *Let X be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ with $\dim(X) \in \mathbb{N} \cup \{\infty\}$ and let $(e_i)_{1\le i\le \dim(X)}$ be an orthonormal basis for X. For all $x \in X$ it then holds that*

$$x = \sum_{i=1}^{\dim(X)} (x, e_i)_X\, e_i, \qquad \text{and} \qquad \|x\|_X^2 = \sum_{i=1}^{\dim(X)} |(x, e_i)_X|^2\,.$$

Corollary 3.4.8. *Let $q \in Q^{\mathrm{ad}}$, $F = F(q)$, $v \in L^2(\Omega)$ and $w \in H_0^1(\Omega)$. Then it holds that*

$$v = \sum_{i=1}^{\infty} \left(b(F)(v, S_i(q))\, S_i(q)\right) \qquad \text{and} \qquad \|v\|^2_{L^2_b(\Omega)} = \sum_{i=1}^{\infty} b(F)(v, S_i(q))^2,$$
$$w = \sum_{i=1}^{\infty} \left(\frac{1}{\lambda_i(q)} a(F)(w, S_i(q))\, S_i(q)\right) \qquad \text{and} \qquad \|w\|^2_{H^1_{0,a}(\Omega)} = \sum_{i=1}^{\infty} \left(\frac{1}{\lambda_i(q)} a(F)(w, S_i(q))^2\right),$$

and a similar statement holds for $v_h \in V_{h,0}$ and the orthonormal bases $(S_{i,h})_i$ and $(S_{i,h,k})_i$.

Lemma 3.4.9. *Let $g \in H^{-1}(\Omega)$ and $q \in Q^{\mathrm{ad}}$. Then, up to norm equivalence, it holds that*

$$\|g\|^2_{H^{-1}(\Omega)} = \sum_{i=1}^{\infty} \left(\frac{1}{\lambda_i(q)}\, (g, S_i(q))^2_{H^{-1},H_0^1}\right).$$

Proof. Let again $F = F(q)$ and $h = L^{-1}(g/\gamma_F)$ with L^{-1} as in Definition 3.1.8. Then it holds that

$$\begin{aligned}
\|g\|^2_{H^{-1}(\Omega)} &= \sup_{v\in H_0^1(\Omega)} \frac{(g, v)^2_{H^{-1},H_0^1}}{\|v\|^2_{H_0^1(\Omega)}} = \sup_{v\in H_0^1(\Omega)} \frac{a(F)(h, v)^2}{\|v\|^2_{H_0^1(\Omega)}}\\
&= \|h\|^2_{H^1_{0,a}(\Omega)} = \sum_{i=1}^{\infty} \left(\frac{1}{\lambda_i(q)} a(F)(h, S_i(q))^2\right) = \sum_{i=1}^{\infty} \left(\frac{1}{\lambda_i(q)}\, (g, S_i(q))^2_{H^{-1},H_0^1}\right).
\end{aligned}$$ □

Corollary 3.4.10. *Let $q \in Q^{\mathrm{ad}}$, $g \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$. Then it holds that*

$$\|u\|_{H_0^1(\Omega)}^2 \geq \sum_{i=1}^{\dim(V_{h,0})} \left(\frac{1}{\lambda_{i,h}(q)} a_h(F)(u, S_{i,h}(q))^2 \right), \tag{3.78}$$

$$\|g\|_{H^{-1}(\Omega)}^2 \geq \sum_{i=1}^{\dim(V_{h,0})} \left(\frac{1}{\lambda_{i,h}(q)} (g, S_{i,h}(q))_{H^{-1},H_0^1}^2 \right). \tag{3.79}$$

If $u \in V_{h,0}$, then equality holds within (3.78). *The statements* (3.78) *and* (3.79) *remain true if $\lambda_{i,h}(q)$ and $S_{i,h}(q)$ are replaced with $\lambda_{i,h,k}(q)$ and $S_{i,h,k}(q)$.*

The following lemma will be needed in order to use duality arguments later on.

Lemma 3.4.11. *Let $q \in Q^{\mathrm{ad}}$, $F = F(q)$, $f \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution to*

$$a(F)(u, v) = b(F)(f, v) \qquad \forall v \in H_0^1(\Omega).$$

Then it holds that $u \in H^{3/2-\varepsilon}(\Omega)$ and

$$\|u\|_{H^{3/2-\varepsilon}(\Omega)} \leq c_\varepsilon \|f\|_{L^2(\Omega)}.$$

Proof. Let T_F^{-1} be the inverse transformation for T_F, $T_F^{-1} \circ T_F = T_F \circ T_F^{-1} = \mathrm{Id}$, and let $\tilde{u} = u \circ T_F^{-1}$. Then $\tilde{u}$ is the solution to

$$a_q(\tilde{u}, \tilde{v}) = b_q(f \circ T_F^{-1}, \tilde{v}) \qquad \forall \tilde{v} \in H_0^1(\Omega_q),$$

cf. (3.7) and (3.8). With [95], Theorem 5 and Remark 5.1 it follows that $\tilde{u} \in H^{3/2-\varepsilon}(\Omega_q)$ and

$$\|\tilde{u}\|_{H^{3/2-\varepsilon}(\Omega_q)} \leq c_\varepsilon \left\| f \circ T_F^{-1} \right\|_{L^2(\Omega_q)}.$$

As $q \in Q^{\mathrm{ad}}$, it follows that $\|F(q)\|_{W^{1,\infty}(\Omega)}$ is sufficiently small, hence there exist $c_0, c_1 > 0$ such that

$$c_0 |\tilde{x} - \tilde{y}| \leq |T_F(\tilde{x}) - T_F(\tilde{y})| \leq c_1 |\tilde{x} - \tilde{y}| \qquad \forall \tilde{x}, \tilde{y} \in \Omega.$$

As T_F is also a bijection, let $x, y \in \Omega$ be arbitrary and let $\tilde{x} = T_F^{-1}(x)$, $\tilde{y} = T_F^{-1}(y)$. We get

$$c_1^{-1} |x - y| \leq \left|T_F^{-1}(x) - T_F^{-1}(y)\right| \leq c_0^{-1} |x - y| \qquad \forall x, y \in \Omega. \tag{3.80}$$

With (3.80) it follows that $\det\big(\mathrm{D}(T_F^{-1})\big)$ is uniformly bounded from above. Now it holds that

$$|u|_{H^{3/2-\varepsilon}(\Omega)}^2 = |\tilde{u} \circ T_F|_{H^{3/2-\varepsilon}(\Omega)}^2 = \int_\Omega \int_\Omega \frac{\left|\nabla\left(\tilde{u} \circ T_F\right)(x) - \nabla\left(\tilde{u} \circ T_F\right)(y)\right|^2}{|x - y|^{2+2(1/2-\varepsilon)}} \,\mathrm{d}x \,\mathrm{d}y$$

$$= \int_\Omega \int_\Omega \frac{|\nabla\tilde{u}(x) - \nabla\tilde{u}(y)|^2}{\left|T_F^{-1}(x) - T_F^{-1}(y)\right|^{2+2(1/2-\varepsilon)}} \det\big(\mathrm{D}(T_F^{-1}(x))\big) \det\big(\mathrm{D}(T_F^{-1}(y))\big) \,\mathrm{d}x \,\mathrm{d}y$$

$$= \int_\Omega \int_\Omega \frac{|\nabla\tilde{u}(x) - \nabla\tilde{u}(y)|^2}{|x - y|^{2+2(1/2-\varepsilon)}} \frac{|x - y|^{2+2(1/2-\varepsilon)}}{\left|T_F^{-1}(x) - T_F^{-1}(y)\right|^{2+2(1/2-\varepsilon)}} \det\big(\mathrm{D}(T_F^{-1}(x))\big) \det\big(\mathrm{D}(T_F^{-1}(y))\big) \,\mathrm{d}x \,\mathrm{d}y,$$

and again using (3.80) we arrive at

$$|u|^2_{H^{3/2-\varepsilon}(\Omega)} \le c \int_\Omega \int_\Omega \frac{|\nabla \tilde{u}(x) - \nabla \tilde{u}(y)|^2}{|x-y|^{2+2(1/2-\varepsilon)}} \,\mathrm{d}x \,\mathrm{d}y = c\, |\tilde{u}|^2_{H^{3/2-\varepsilon}(\Omega)} \,.$$

It also holds that

$$\begin{aligned} \left\| f \circ T_F^{-1} \right\|^2_{L^2(\Omega_q)} &= \int_{\Omega_q} \left(f \circ T_F^{-1} \right)^2 \,\mathrm{d}x = \int_\Omega f^2 \gamma_F \,\mathrm{d}x \\ &\le c \int_\Omega f^2 \,\mathrm{d}x = c\, \|f\|^2_{L^2(\Omega)} \,, \end{aligned}$$

and the result follows. □

3.4.1. A-priori error estimates for a general control

As within Section 2.3 we start with error estimates for a general control $q \in Q^{\mathrm{ad}}$, within Subsection 3.4.2 we will prove error estimates for the optimal control $\overline{q}$ which possesses a higher regularity and thus allows for higher convergence rates.

3.4.1.1. Estimates within the purely continuous case

Within this subsubsection we are going to investigate the error induced by the discretization of the control.

Lemma 3.4.12. *Let $q \in Q^{\mathrm{ad}}$, $p \in [1,\infty)$ and $i \in \mathbb{N}$. Then it holds that $S_i(q) \in W^{1,p}(\Omega) \cap H^{3/2-\varepsilon}(\Omega)$ and*

$$\|S_i(q)\|_{W^{1,p}(\Omega) \cap H^{3/2-\varepsilon}(\Omega)} \le c_{i,\varepsilon,p}.$$

Proof. This lemma is a direct consequence of Lemma 3.1.24, Lemma 3.1.25 and the normalizing condition within (3.64). □

Lemma 3.4.13. *For $q, p \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$ it holds that*

$$|\lambda_i(q) - \lambda_i(p)| \le c\, \|q - p\|_{H^2(I)} \,.$$

Proof. Let F and E be the transformations for q and p, respectively. Using Lemma 3.1.13 and Lemma 3.4.6 it follows that

$$\begin{aligned} |\lambda_i(q) - \lambda_i(p)| &\le c \Big(\|A_F - A_E\|_{L^\infty(\Omega)} + \|\gamma_F - \gamma_E\|_{L^2(\Omega)} \Big) \\ &\le c\, \|q - p\|_{H^2(I)} \,. \end{aligned}$$

□

Definition 3.4.14. For $q \in Q^{\mathrm{ad}}$ and $F = F(q)$ let $L_q \colon D(L_q) \subset L^2(\Omega) \to L^2(\Omega)$ be the self-adjoint operator corresponding to $a(F)(\cdot,\cdot)$, i.e.

$$L_q(u) = -\operatorname{div}(\mu A_F \cdot \nabla u) \,,$$

with domain

$$D(L_q) = \left\{ u \in H^1_0(\Omega) \,\middle|\, L_q(u) \in L^2(\Omega) \right\}.$$

In what follows we are going to estimate the gap $\delta(L_q, L_p)$, cf. Definition 3.2.4, for $q, p \in Q^{\mathrm{ad}}$.

Lemma 3.4.15. *Let $q, p \in Q^{\mathrm{ad}}$ with corresponding transformations F and E, respectively, and let $u \in D(L_q)$. Then there exists $v = v(u) \in D(L_p)$ and $c_1, c_2 > 0$, independent of p, q, u and v, such that the following conditions are simultaneously satisfied.*

$$-\operatorname{div}(\mu A_E \cdot \nabla v) = -\operatorname{div}(\mu A_F \cdot \nabla u) \qquad \text{in } L^2(\Omega), \tag{3.81}$$

$$\|v\|_{H_0^1(\Omega)} \leq c_1 \|u\|_{H_0^1(\Omega)}, \tag{3.82}$$

$$\|u - v\|_{H_0^1(\Omega)} \leq c_2 \|u\|_{H_0^1(\Omega)} \|q - p\|_{H^2(I)}. \tag{3.83}$$

Proof. Let $v \in H_0^1(\Omega)$ be the solution to

$$(\nabla v, \mu A_E \cdot \nabla \varphi) = (\nabla u, \mu A_F \cdot \nabla \varphi) \qquad \forall \varphi \in H_0^1(\Omega). \tag{3.84}$$

Due to the properties of the bilinear form related to $L_{\tilde{q}}$ for arbitrary $\tilde{q} \in Q^{\mathrm{ad}}$ it follows that such a $v \in H_0^1(\Omega)$ actually exists and $\|v\|_{H_0^1(\Omega)} \leq c \|u\|_{H_0^1(\Omega)}$, which shows (3.82). Furthermore, as $u \in D(L_q)$, the right hand side of (3.84) can be extended to a linear functional over $L^2(\Omega)$. Hence, also the left-hand side can be defined for test functions in $L^2(\Omega)$, which proves (3.81). At last it holds that

$$\begin{aligned} c \|\nabla (u - v)\|_{L^2(\Omega)}^2 &\leq (\nabla (u - v), \mu A_F \cdot \nabla (u - v)) \\ &= (\nabla u, \mu A_F \cdot \nabla (u - v)) - (\nabla v, \mu A_F \cdot \nabla (u - v)) \\ &= (\nabla v, \mu (A_E - A_F) \cdot \nabla (u - v)) \\ &\leq c \|v\|_{H_0^1(\Omega)} \|A_F - A_E\|_{L^\infty(\Omega)} \|\nabla (u - v)\|_{L^2(\Omega)}, \end{aligned}$$

and hence

$$\|\nabla (u - v)\|_{L^2(\Omega)} \leq c \|u\|_{H_0^1(\Omega)} \|q - p\|_{H^2(I)}. \qquad \square$$

Lemma 3.4.16. *Let $q \in Q^{\mathrm{ad}}$ and $F = F(q)$. There exists $c > 0$, independent of q, such that*

$$\sup_{\substack{u \in D(L_q) \\ \|u\|_{L^2(\Omega)}^2 + \|\operatorname{div}(\mu A_F \cdot \nabla u)\|_{L^2(\Omega)}^2 = 1}} \|u\|_{H_0^1(\Omega)} \leq c.$$

Proof. For $u \in D(L_q)$ it holds that

$$\begin{aligned} c \|u\|_{H_0^1(\Omega)}^2 &\leq (\nabla u, \mu A_F \cdot \nabla u) = -(\operatorname{div}(\mu A_F \cdot \nabla u), u) \\ &\leq \|\operatorname{div}(\mu A_F \cdot \nabla u)\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}, \end{aligned}$$

and using Young's inequality we end up with

$$\|u\|_{H_0^1(\Omega)} \leq c \sqrt{\|u\|_{L^2(\Omega)}^2 + \|\operatorname{div}(\mu A_F \cdot \nabla u)\|_{L^2(\Omega)}^2}. \qquad \square$$

Lemma 3.4.17. *There exists $c > 0$ such that for all $q, p \in Q^{\mathrm{ad}}$ it holds that*

$$\delta(L_q, L_p) \leq c \|q - p\|_{H^2(I)}.$$

Proof. Let again F and E be the transformations related to q and p, respectively. Using Definition 3.2.4 it follows that

$$\delta(L_q, L_p)$$
$$= \sup_{\substack{u\in D(L_q)\\ \|u\|^2_{L^2(\Omega)}+\|\mathrm{div}(\mu A_F\cdot\nabla u)\|^2_{L^2(\Omega)}=1}} \inf_{v\in D(L_p)} \left(\|u-v\|^2_{L^2(\Omega)} + \|\mathrm{div}(\mu A_F\cdot\nabla u) - \mathrm{div}(\mu A_E\cdot\nabla v)\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Let $v = v(u)$ be defined as in Lemma 3.4.15, we get

$$\delta(L_q, L_p) \le c\,\|q-p\|_{H^2(I)} \sup_{\substack{u\in D(L_q)\\ \|u\|^2_{L^2(\Omega)}+\|\mathrm{div}(\mu A_F\cdot\nabla u)\|^2_{L^2(\Omega)}=1}} \|u\|_{H^1_0(\Omega)},$$

and finish the proof with Lemma 3.4.16. □

Corollary 3.4.18. *For $q,p\in Q^{\mathrm{ad}}$ it holds that*

$$\hat{\delta}(L_q, L_p) \le c\,\|q-p\|_{H^2(I)}.$$

Lemma 3.4.19. *Let $q,p\in Q^{\mathrm{ad}}$ and $i\in\mathbb{N}$, then it holds that*

$$\|S_i(q)-S_i(p)\|_{L^2(\Omega)} \le c_i\,\|q-p\|_{H^2(I)}.$$

Proof. Let $v_i(q)$ and $v_i(p)$ be normalized i-th eigenfunctions in the sense of $L^2(\Omega)$ for L_q and L_p, i.e.

$$\begin{cases} L_q v_i(q) = \lambda_i(q)\,v_i(q),\\ \|v_i(q)\|_{L^2(\Omega)} = 1, \end{cases} \qquad \text{and} \qquad \begin{cases} L_p v_i(p) = \lambda_i(p)\,v_i(p),\\ \|v_i(p)\|_{L^2(\Omega)} = 1, \end{cases}$$

satisfying Theorem 3.2.6, i.e.

$$\|v_i(q)-v_i(p)\|_{L^2(\Omega)} \le c_i\delta(L_q,L_p) \le c_i\,\|q-p\|_{H^2(I)}, \tag{3.85}$$

where the second inequality is due to Lemma 3.4.17. Let F and E be the transformations for q and p, respectively, and let

$$\beta_{i,q} = \left(v_i(q)^2, \gamma_F\right)^{-1/2} \qquad \text{and} \qquad \beta_{i,p} = \left(v_i(p)^2, \gamma_E\right)^{-1/2}.$$

Then it holds that

$$S_i(q) = \beta_{i,q} v_i(q) \qquad \text{and} \qquad S_i(p) = \beta_{i,p} v_i(p),$$

and we have

$$\|S_i(q)-S_i(p)\|_{L^2(\Omega)} \le \beta_{i,q}\,\|v_i(q)-v_i(p)\|_{L^2(\Omega)} + |\beta_{i,q}-\beta_{i,p}|\,\|v_i(p)\|_{L^2(\Omega)}. \tag{3.86}$$

As γ_F is uniformly bounded from below and above for $q \in Q^{\mathrm{ad}}$, so is $\beta_{i,q}$. This allows to estimate the first part in (3.86) through (3.85), and it remains to estimate $|\beta_{i,q} - \beta_{i,p}|$. Using the uniform boundedness of $\beta_{i,q}$ and $\beta_{i,p}$ once again we get

$$\begin{aligned}
|\beta_{i,q} - \beta_{i,p}| &\leq c\frac{\beta_{i,q} + \beta_{i,p}}{\beta_{i,q}^2\,\beta_{i,p}^2}\,|\beta_{i,q} - \beta_{i,p}| = c\left|\beta_{i,p}^{-2} - \beta_{i,q}^{-2}\right| \\
&= c\left|\left(v_i(q)^2, \gamma_F\right) - \left(v_i(p)^2, \gamma_E\right)\right| \\
&\leq c\,\|v_i(q)\|_{L^2(\Omega)}^2\,\|\gamma_F - \gamma_E\|_{L^\infty(\Omega)} \\
&\quad + c\,\|v_i(q) + v_i(p)\|_{L^2(\Omega)}\,\|v_i(q) - v_i(p)\|_{L^2(\Omega)}\,\|\gamma_E\|_{L^\infty(\Omega)}\,,
\end{aligned}$$

and the second part can also be estimated using Lemma 3.4.6 and (3.85). □

Lemma 3.4.20. *For $q, p \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$ it holds that*

$$\|S_i(q) - S_i(p)\|_{H_0^1(\Omega)} \leq c_i\,\|q - p\|_{H^2(I)}\,.$$

Proof. Let again F and E be the transformation for q and p, we get

$$\begin{aligned}
c\,\|S_i(q) - S_i(p)\|_{H_0^1(\Omega)}^2 &\leq \left(\nabla\left(S_i(q) - S_i(p)\right), \mu A_F \cdot \nabla\left(S_i(q) - S_i(p)\right)\right) \\
&= \left(\nabla S_i(q), \mu A_F \cdot \nabla\left(S_i(q) - S_i(p)\right)\right) - \left(\nabla S_i(p), \mu A_F \cdot \nabla\left(S_i(q) - S_i(p)\right)\right) \\
&= \lambda_i(q)\left(S_i(q) - S_i(p), S_i(q)\gamma_F\right) - \lambda_i(p)\left(S_i(q) - S_i(p), S_i(p)\gamma_E\right) \\
&\quad + \left(\nabla S_i(p), \mu\left(A_E - A_F\right) \cdot \nabla\left(S_i(q) - S_i(p)\right)\right) \\
&\leq \|S_i(q) - S_i(p)\|_{L^2(\Omega)}\,\|\lambda_i(q)S_i(q)\gamma_F - \lambda_i(p)S_i(p)\gamma_E\|_{L^2(\Omega)} \\
&\quad + c\,\|S_i(p)\|_{H_0^1(\Omega)}\,\|A_F - A_E\|_{L^\infty(\Omega)}\,\|S_i(q) - S_i(p)\|_{H_0^1(\Omega)}\,,
\end{aligned}$$

and the result follows with Lemma 3.4.19, Lemma 3.4.13 and Lemma 3.4.6. □

Lemma 3.4.21. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Then it holds that*

$$\left|\lambda_i'(q)(\delta q)\right| \leq c_i\,\|\delta q\|_{H^2(I)}\,.$$

Proof. This lemma follows from (3.65) and Lemma 3.4.6. □

Lemma 3.4.22. *For $q, p \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$ it holds that*

$$\left|\lambda_i'(q)(\delta q) - \lambda_i'(p)(\delta q)\right| \leq c_i\,\|q - p\|_{H^2(I)}\,\|\delta q\|_{H^2(I)}\,.$$

Proof. Let F and E be the transformations corresponding to q and p, respectively. In addition, let $\delta F = F'(q)(\delta q) = F'(p)(\delta q)$, then it holds that

$$\begin{aligned}
\left|\lambda_i'(q)(\delta q) - \lambda_i'(p)(\delta q)\right| &= \left|\left(\nabla S_i(q), \mu A_{F,\delta F}' \cdot \nabla S_i(q)\right) - \left(\nabla S_i(p), \mu A_{E,\delta F}' \cdot \nabla S_i(p)\right)\right| \\
&\quad + \left|\lambda_i(q)\left(S_i(q)^2, \gamma_{F,\delta F}'\right) - \lambda_i(p)\left(S_i(p)^2, \gamma_{E,\delta F}'\right)\right|,
\end{aligned}$$

and the result follows with Lemma 3.4.13, Lemma 3.4.20 and Lemma 3.4.6. □

Lemma 3.4.23. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$, then it holds that*

$$\|g_i(q, \delta q)\|_{H^{-1}(\Omega)} \leq c_i\,\|\delta q\|_{H^2(I)}\,.$$

Proof. Let $F = F(q)$ and $\delta F = F'(q)(\delta q)$, then Definition 3.4.5, Lemma 3.4.6 and Lemma 3.4.21 yield

$$\begin{aligned}\|g_i(q,\delta q)\|_{H^{-1}(\Omega)} &\leq \left\|\lambda_i'(q)S_i(q)\gamma_F\right\|_{L^2(\Omega)} + \left\|\lambda_i(q)S_i(q)\gamma_{F,\delta F}'\right\|_{L^2(\Omega)} + \left\|\operatorname{div}\left(\mu A_{F,\delta F}' \cdot \nabla S_i(q)\right)\right\|_{H^{-1}(\Omega)} \\ &\leq c_i \|\delta q\|_{H^2(I)} .\end{aligned}$$

□

Lemma 3.4.24. *Let $q, p \in Q^{\text{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$, then it holds that*

$$\|g_i(q,\delta q) - g_i(p,\delta q)\|_{H^{-1}(\Omega)} \leq c_i \|q-p\|_{H^2(I)} \|\delta q\|_{H^2(I)} .$$

Proof. This lemma can be proven in a similar way to Lemma 3.4.23, we additionally have to use Lemma 3.4.22 and Lemma 3.4.20. □

The following lemmata will be needed to estimate the error between the derivative of the eigenfunctions with respect to the control variable.

Lemma 3.4.25. *Let $q \in Q^{\text{ad}}$, $F = F(q)$, $i \in \mathbb{N}$ and $g \in H^{-1}(\Omega)$ with $(g, S_i(q))_{H^{-1},H_0^1} = 0$. Then there exists a solution $\tilde{S}_i \in H_0^1(\Omega)$ to*

$$\left(\nabla \tilde{S}_i, \mu A_F \cdot \nabla v\right) = \lambda_i \left(\tilde{S}_i, v\gamma_F\right) + (g,v)_{H^{-1},H_0^1} \qquad \forall v \in H_0^1(\Omega), \tag{3.87}$$

with

$$\left\|\tilde{S}_i\right\|_{H_0^1(\Omega)} \leq c_i \|g\|_{H^{-1}(\Omega)} , \tag{3.88}$$

where c_i is independent of g. In addition it holds that

$$\left(\tilde{S}_i, S_i(q)\gamma_F\right) = 0. \tag{3.89}$$

Proof. Let $H_{0,a}^1(\Omega)$ be as in Definition 3.1.10 and let L^{-1} be the compact operator from Definition 3.1.8. Setting $h = L^{-1}(g/\gamma_F) \in H_{0,a}^1(\Omega)$ it follows that $(h, S_i(q))_{H_{0,a}^1(\Omega)} = 0$. It follows from Corollary 3.2.16 that there exists $\tilde{S}_i \in H_{0,a}^1(\Omega)$ as the solution to (3.87) with

$$\left\|\tilde{S}_i\right\|_{H_{0,a}^1(\Omega)} \leq c_i \|h\|_{H_{0,a}^1(\Omega)} .$$

As the norms of $H_{0,a}^1(\Omega)$ and $H_0^1(\Omega)$ are equivalent, (3.88) follows using standard stability estimates for L^{-1}. The orthogonality condition (3.89) follows from Subsection 3.2.2. □

Lemma 3.4.26. *Let $q \in Q^{\text{ad}}$, $\delta q \in Q$, $i \in \mathbb{N}$ and $p \in [1,\infty)$. Then there exists a solution $S_{i,0}'(q)(\delta q) \in W_0^{1,p}(\Omega) \cap H^{3/2-\varepsilon}(\Omega)$ to the first equation within* (3.66) *with*

$$\left\|S_{i,0}'(q)(\delta q)\right\|_{W^{1,p}(\Omega)\cap H^{3/2-\varepsilon}(\Omega)} \leq c_{i,\varepsilon,p} \|\delta q\|_{H^2(I)} .$$

Proof. Let $g_i \in H^{-1}(\Omega)$ be as in Definition 3.4.5. With (3.38) it follows that $(g_i, S_i(q))_{H^{-1},H_0^1} = 0$. Lemma 3.4.25 now yields the existence of such a $S_{i,0}'(q)(\delta q) \in H_0^1(\Omega)$ with

$$\left\|S_{i,0}'(q)(\delta q)\right\|_{H_0^1(\Omega)} \leq c_i \|g_i\|_{H^{-1}(\Omega)} \leq c_i \|\delta q\|_{H^2(I)} ,$$

where the last inequality is due to Lemma 3.4.23. In order to prove higher regularity of $S_{i,0}'(q)(\delta q)$ we refer to Subsection 3.1.6, where an analog result is proven for $S_i(q)$, cf. Lemma 3.4.12. This lemma can be proven following the same steps. □

Lemma 3.4.27. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$, $i \in \mathbb{N}$ and $p \in [1, \infty)$. Then it holds that*

$$\left\| S_i'(q)(\delta q) \right\|_{W^{1,p}(\Omega) \cap H^{3/2-\varepsilon}(\Omega)} \le c_{i,\varepsilon,p} \left\| \delta q \right\|_{H^2(I)} .$$

Proof. Let $F = F(q)$, $\delta F = F'(q)(\delta q)$ and let $S_{i,0}'(q)(\delta q)$ be as in Lemma 3.4.26. In order to fulfill the normalizing condition within (3.66) we have to find $t \in \mathbb{R}$ such that $S_i'(q)(\delta q) = S_{i,0}'(q)(\delta q) + t\, S_i(q)$ solves

$$2 \left(S_i'(q)(\delta q), S_i(q) \gamma_F \right) + \left(S_i(q)^2, \gamma_{F,\delta F}' \right) = 0,$$

and because of $\left(S_{i,0}'(q)(\delta q), S_i(q) \gamma_F \right) = 0$ due to Lemma 3.4.25 and Lemma 3.4.26 we have to set

$$t = -\frac{1}{2} \left(S_i(q)^2, \gamma_{F,\delta F}' \right),$$

with

$$|t| \le c \left\| \delta q \right\|_{H^2(I)} .$$

As a result,

$$\begin{aligned} \left\| S_i'(q)(\delta q) \right\|_{W^{1,p}(\Omega) \cap H^{3/2-\varepsilon}(\Omega)} &= \left\| S_{i,0}'(q)(\delta q) + t\, S_i(q) \right\|_{W^{1,p}(\Omega) \cap H^{3/2-\varepsilon}(\Omega)} \\ &\le \left\| S_{i,0}'(q)(\delta q) \right\|_{W^{1,p}(\Omega) \cap H^{3/2-\varepsilon}(\Omega)} + |t| \left\| S_i(q) \right\|_{W^{1,p}(\Omega) \cap H^{3/2-\varepsilon}(\Omega)} \\ &\le c_{i,\varepsilon,p} \left\| \delta q \right\|_{H^2(I)} , \end{aligned}$$

where we used Lemma 3.4.12 and Lemma 3.4.26. □

Lemma 3.4.28. *Let $q, p \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \{1, 2\}$. Let $S_{i,0}'(q)(\delta q), S_{i,0}'(p)(\delta q) \in H_0^1(\Omega)$ be defined as in Lemma 3.4.26, then it holds that*

$$\left\| S_{i,0}'(q)(\delta q) - S_{i,0}'(p)(\delta q) \right\|_{L^2(\Omega)} \le c \left\| q - p \right\|_{H^2(I)} \left\| \delta q \right\|_{H^2(I)} .$$

Proof. First note that due to Assumption 3.1.26 the expression $|\lambda_i(q) - \lambda_j(q)|$ is uniformly bounded from below for $i, j \in \{1, 2\}$ and $i \neq j$. Let the transformations for q and p be denoted with F and E, respectively. Using Definition 3.1.9 and Corollary 3.4.8 it follows that

$$\begin{aligned} \left\| S_{i,0}'(q)(\delta q) - S_{i,0}'(p)(\delta q) \right\|_{L_b^2(\Omega)}^2 &= \sum_{j \in \mathbb{N}} \left(S_{i,0}'(q)(\delta q) - S_{i,0}'(p)(\delta q), S_j(q) \right)_{L_b^2(\Omega)}^2 \\ &= \sum_{j \in \mathbb{N} \setminus \{i\}} \left(S_{i,0}'(q)(\delta q) - S_{i,0}'(p)(\delta q), S_j(q) \right)_{L_b^2(\Omega)}^2 \qquad (3.90) \\ &\quad + \left(S_{i,0}'(p)(\delta q), S_i(q) \right)_{L_b^2(\Omega)}^2 , \qquad (3.91) \end{aligned}$$

where we again used the fact that $\left(S_{i,0}'(q)(\delta q), S_i(q) \right)_{L_b^2(\Omega)} = 0$. We start by estimating (3.91), we get

$$\begin{aligned} \left(S_{i,0}'(p)(\delta q), S_i(q) \right)_{L_b^2(\Omega)}^2 &= \left(S_{i,0}'(p)(\delta q), S_i(q) - S_i(p) \right)_{L_b^2(\Omega)}^2 \\ &\le \left\| S_{i,0}'(p)(\delta q) \right\|_{L_b^2(\Omega)}^2 \left\| S_i(q) - S_i(p) \right\|_{L_b^2(\Omega)}^2 \qquad (3.92) \\ &\le c \left\| q - p \right\|_{H^2(I)}^2 \left\| \delta q \right\|_{H^2(I)}^2 , \end{aligned}$$

where we used Lemma 3.4.19 and Lemma 3.4.26. Now we estimate (3.90), the definition of $S_j(q)$ yields

$$a(F)(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S_j(q)) = \lambda_j(q)\, b(F)(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S_j(q)), \tag{3.93}$$

whereas using the definition of $S'_{i,0}(q)(\delta q)$ and $S'_{i,0}(p)(\delta q)$ yields

$$\begin{aligned}
&a(F)(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S_j(q)) \\
&= a(F)(S'_{i,0}(q)(\delta q), S_j(q)) - a(E)(S'_{i,0}(p)(\delta q), S_j(q)) + \left(\nabla S'_{i,0}(p)(\delta q), \mu\,(A_E - A_F)\cdot \nabla S_j(q)\right) \\
&= \lambda_i(q)\, b(F)(S'_{i,0}(q)(\delta q), S_j(q)) + (g_i(q,\delta q), S_j(q))_{H^{-1},H_0^1} - \lambda_i(p)\, b(E)(S'_{i,0}(p)(\delta q), S_j(q)) \\
&- (g_i(p,\delta q), S_j(q))_{H^{-1},H_0^1} + \left(\nabla S'_{i,0}(p)(\delta q), \mu\,(A_E - A_F)\cdot \nabla S_j(q)\right) \\
&= \lambda_i(q)\, b(F)(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S_j(q)) \\
&+ (\lambda_i(q) - \lambda_i(p))\; b(F)(S'_{i,0}(p)(\delta q), S_j(q)) + \lambda_i(p)\; \left(S'_{i,0}(p)(\delta q), S_j(q)\,(\gamma_F - \gamma_E)\right) \\
&+ (g_i(q,\delta q) - g_i(p,\delta q), S_j(q))_{H^{-1},H_0^1} + \left(\nabla S'_{i,0}(p)(\delta q), \mu\,(A_E - A_F)\cdot \nabla S_j(q)\right).
\end{aligned} \tag{3.94}$$

Combining (3.93) and (3.94) yields

$$\begin{aligned}
&\left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S_j(q)\right)_{L_b^2(\Omega)} \\
&= \frac{\lambda_i(q) - \lambda_i(p)}{\lambda_j(q) - \lambda_i(q)} \left(S'_{i,0}(p)(\delta q), S_j(q)\right)_{L_b^2(\Omega)} + \frac{\lambda_i(p)}{\lambda_j(q) - \lambda_i(q)} \left(S'_{i,0}(p)(\delta q), S_j(q)\,(\gamma_F - \gamma_E)\right) \\
&+ \frac{1}{\lambda_j(q) - \lambda_i(q)} \left(g_i(q,\delta q) - g_i(p,\delta q), S_j(q)\right)_{H^{-1},H_0^1} \\
&+ \frac{1}{\lambda_j(q) - \lambda_i(q)} \left(\nabla S'_{i,0}(p)(\delta q), \mu\,(A_E - A_F)\cdot \nabla S_j(q)\right).
\end{aligned}$$

Now let $v \in H_0^1(\Omega)$ be the unique solution to

$$(\nabla v, \mu A_F \cdot \nabla \varphi) = \left(\nabla S'_{i,0}(p)(\delta q), \mu\,(A_E - A_F)\cdot \nabla \varphi\right) \qquad \forall \varphi \in H_0^1(\Omega),$$

hence

$$\|v\|_{H_0^1(\Omega)} \le c \left\|S'_{i,0}(p)(\delta q)\right\|_{H_0^1(\Omega)} \|A_E - A_F\|_{L^\infty(\Omega)}. \tag{3.95}$$

As $|\lambda_i(q) - \lambda_j(q)|$ is uniformly bounded from below for $j \neq i$ we get

$$\begin{aligned}
&\left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S_j(q)\right)^2_{L_b^2(\Omega)} \\
&\le c_i \left(|\lambda_i(q) - \lambda_i(p)|^2 \left(S'_{i,0}(p)(\delta q), S_j(q)\right)^2_{L_b^2(\Omega)} + \left(S'_{i,0}(p)(\delta q)\frac{\gamma_F - \gamma_E}{\gamma_F}, S_j(q)\right)^2_{L_b^2(\Omega)} \right) \\
&+ c_i \left(\frac{1}{\lambda_j(q)} \left(g_i(q,\delta q) - g_i(p,\delta q), S_j(q)\right)^2_{H^{-1},H_0^1} + \frac{1}{\lambda_j(q)} \left(\nabla v, \mu A_F \cdot \nabla S_j(q)\right)^2 \right).
\end{aligned}$$

Summing up these terms and using Corollary 3.4.8 and Lemma 3.4.9 yields

$$\begin{aligned}
&\sum_{j \in \mathbb{N}\setminus\{i\}} \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S_j(q)\right)^2_{L_b^2(\Omega)} \\
&\le |\lambda_i(q) - \lambda_i(p)|^2 \left\|S'_{i,0}(p)(\delta q)\right\|^2_{H_0^1(\Omega)} + \left\|S'_{i,0}(p)(\delta q)\right\|^2_{L^2(\Omega)} \|\gamma_F - \gamma_E\|^2_{L^\infty(\Omega)} \\
&+ \|g_i(q,\delta q) - g_i(p,\delta q)\|^2_{H^{-1}(\Omega)} + \|v\|^2_{H_0^1(\Omega)},
\end{aligned}$$

which, together with the estimates from the previous lemmata, (3.95) and (3.92), proves the result. □

Lemma 3.4.29. *Let $q,p \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \{1,2\}$. Then it holds that*

$$\left\| S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right\|_{H_0^1(\Omega)} \le c \left\| q-p \right\|_{H^2(I)} \left\| \delta q \right\|_{H^2(I)}.$$

Proof. Let F and E denote the corresponding transformations to q and p, respectively. It then holds that

$$\begin{aligned}
&c \left\| S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right\|^2_{H_0^1(\Omega)} \\
&\le \left(\nabla \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right), \mu A_F \cdot \nabla \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right) \right) \\
&\le \left(\nabla \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right), \mu A_F \cdot \nabla S'_{i,0}(q)(\delta q) \right) - \left(\nabla \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right), \mu A_E \cdot \nabla S'_{i,0}(p)(\delta q) \right) \\
&- \left(\nabla \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right), \mu \left(A_F - A_E \right) \cdot \nabla S'_{i,0}(p)(\delta q) \right) \\
&= \lambda_i(q) \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S'_{i,0}(q)(\delta q)\gamma_F \right) + \left(g_i(q,\delta q), S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right)_{H^{-1},H_0^1} \\
&- \lambda_i(p) \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q), S'_{i,0}(p)(\delta q)\gamma_E \right) - \left(g_i(p,\delta q), S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right)_{H^{-1},H_0^1} \\
&- \left(\nabla \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right), \mu \left(A_F - A_E \right) \cdot \nabla S'_{i,0}(p)(\delta q) \right) \\
&= \left(\lambda_i(q)\, S'_{i,0}(q)(\delta q)\gamma_F - \lambda_i(p)\, S'_{i,0}(p)(\delta q)\gamma_E, S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right) \\
&+ \left(g_i(q,\delta q) - g_i(p,\delta q), S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right)_{H^{-1},H_0^1} \\
&- \left(\nabla \left(S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right), \mu \left(A_F - A_E \right) \cdot \nabla S'_{i,0}(p)(\delta q) \right) \\
&\le c \left(|\lambda_i(q) - \lambda_i(p)| + \left\| S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right\|_{L^2(\Omega)} + \left\| \gamma_F - \gamma_E \right\|_{L^2(\Omega)} \right) \left\| S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right\|_{L^2(\Omega)} \\
&+ \left\| g_i(q,\delta q) - g_i(p,\delta q) \right\|_{H^{-1}(\Omega)} \left\| S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right\|_{H_0^1(\Omega)} \\
&+ c \left\| S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right\|_{H_0^1(\Omega)} \left\| A_F - A_E \right\|_{L^\infty(\Omega)} \left\| S'_{i,0}(p)(\delta q) \right\|_{H_0^1(\Omega)},
\end{aligned}$$

and the result follows with Lemma 3.4.28, Lemma 3.4.13, Lemma 3.4.24, Lemma 3.4.26, Lemma 3.4.6 and Young's inequality. □

Lemma 3.4.30. *Let $q,p \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \{1,2\}$. Then it holds that*

$$\left\| S'_i(q)(\delta q) - S'_i(p)(\delta q) \right\|_{H_0^1(\Omega)} \le c \left\| q-p \right\|_{H^2(I)} \left\| \delta q \right\|_{H^2(I)}.$$

Proof. Let F and E be the transformations corresponding to q and p, respectively, with derivative $\delta F = F'(q)(\delta q) = F'(p)(\delta q)$. As in the proof of Lemma 3.4.27, let

$$t_q = -\frac{1}{2} \left(S_i(q)^2, \gamma'_{F,\delta F} \right) \qquad \text{and} \qquad t_p = -\frac{1}{2} \left(S_i(p)^2, \gamma'_{E,\delta F} \right),$$

such that

$$S'_i(q)(\delta q) = S'_{i,0}(q)(\delta q) + t_q\, S_i(q) \qquad \text{and} \qquad S'_i(p)(\delta q) = S'_{i,0}(p)(\delta q) + t_p\, S_i(p).$$

We use Lemma 3.4.20, Lemma 3.4.29 and Lemma 3.4.6 and get

$$\begin{aligned}
\left\| S'_i(q)(\delta q) - S'_i(p)(\delta q) \right\|_{H_0^1(\Omega)} &\le \left\| S'_{i,0}(q)(\delta q) - S'_{i,0}(p)(\delta q) \right\|_{H_0^1(\Omega)} \\
&+ |t_q - t_p| \left\| S_i(q) \right\|_{H_0^1(\Omega)} + |t_p| \left\| S_i(q) - S_i(p) \right\|_{H_0^1(\Omega)} \\
&\le c \left\| q-p \right\|_{H^2(I)} \left\| \delta q \right\|_{H^2(I)}.
\end{aligned}$$

□

Lemma 3.4.31. *For $q, p \in Q^{\text{ad}}$, $\delta q \in Q$ and $i \in \{1, 2\}$ it holds that*

$$\left|\lambda_i''(q)(\delta q, \delta q) - \lambda_i''(p)(\delta q, \delta q)\right| \le c \left\|q - p\right\|_{H^2(I)} \left\|\delta q\right\|_{H^2(I)}^2.$$

Proof. This lemma follows with representation (3.67) and the previous lemmata. □

Lemma 3.4.32. *For $q, p \in Q^{\text{ad}}$ and $\delta q \in Q$ it holds that*

$$\left|j'(q)(\delta q) - j'(p)(\delta q)\right| \le c \left\|q - p\right\|_{H^2(I)} \left\|\delta q\right\|_{H^2(I)},$$
$$\left|j''(q)(\delta q, \delta q) - j''(p)(\delta q, \delta q)\right| \le c \left\|q - p\right\|_{H^2(I)} \left\|\delta q\right\|_{H^2(I)}^2.$$

Proof. This lemma is a direct conclusion of Lemma 3.4.22 and Lemma 3.4.31. □

3.4.1.2. Estimates between the continuous case and the state-discretized case

Within this subsubsection we are going to estimate the error induced by the discretization of the state. We start with general results concerning the finite element approximation of eigenvalues and eigenfunctions, some of them are based on ideas presented in the survey article [15]. If not stated otherwise we will always assume that the discretization parameter h is chosen sufficiently small.

One possible definition of the convergence of a discretized eigenvalue problem to its continuous counterpart is as follows.

Definition 3.4.33. Let $i \in \mathbb{N}$ and let N_i and $N_{i,h}$ be the spaces spanned by the eigenfunctions and discrete eigenfunctions for the i-th eigenvalue, respectively. Let $m(i)$ denote the dimension of the space spanned by the first distinct i eigenspaces. Then we say that the discrete eigenvalue problem converges to the continuous one if, for any $\varepsilon > 0$ and $i > 0$, there exists $h_0 > 0$ such that for all $h < h_0$ we have

$$\max_{1 \le j \le m(i)} |\lambda_i - \lambda_{i,h}| \le \varepsilon \qquad \text{and} \qquad \hat{\delta}\left(\bigoplus_{j=1}^{m(i)} N_j, \bigoplus_{j=1}^{m(i)} N_{j,h}\right) \le \varepsilon.$$

Remark 3.4.34. It can be shown that Definition 3.4.33 includes convergence of eigenvalues and eigenfunctions with correct multiplicity and absence of spurious solutions. Using the notation introduced in Lemma 3.1.1 it further holds that, if the solution operator for the underlying equation is compact from H to V and $\Pi_h \colon V \to V_{h,0}$ as the elliptic projection associated with $a_h(F)(\cdot, \cdot)$ converges strongly to the identity operator from V to H, then convergence in the sense of Definition 3.4.33 holds, cf. [15], Proposition 7.4 and Proposition 7.6.

Lemma 3.4.35. *Let $q \in Q^{\text{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\lambda_{i,h}(q) \le c_i.$$

Proof. Using (3.68) this lemma can be proven similar to Lemma 3.1.3, where one has to use the fact that the i-th eigenvalue for the discrete Laplacian is bounded independently of h, which follows with Remark 3.4.34. □

Lemma 3.4.36. *Let $q \in Q^{\text{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\left\|S_{i,h}(q)\right\|_{H_0^1(\Omega)} \le c_i.$$

Proof. This lemma follows with Definition 3.4.3 and Lemma 3.4.35. □

Lemma 3.4.37. *Let $p \in [1, \infty]$. Then it holds that*

$$\|\mu - \mu_h\|_{L^p(\Omega)} \leq ch^{2/p}.$$

Proof. This lemma follows from the fact that $\mu, \mu_h \in L^\infty(\Omega)$,

$$|\{x \in \Omega|\ \mu(x) \neq \mu_h(x)\}| = |\Omega_0 \backslash \Omega_{0,h}| \leq ch^2,$$

and Hölder's inequality. □

Corollary 3.4.38. *Let $q \in Q^{\text{ad}}$, $F = F(q)$, $p_1, p_2, p_3 \in [1, \infty]$ with $1/p_1 + 1/p_2 + 1/p_3 = 1$, $v \in W^{1,p_1}(\Omega)$ and $w \in W^{1,p_2}(\Omega)$. Then it holds that*

$$|a(F)(v, w) - a_h(F)(v, w)| \leq ch^{2/p_3} \|v\|_{W_0^{1,p_1}(\Omega)} \|w\|_{W_0^{1,p_2}(\Omega)}.$$

Proof. This corollary follows with Theorem A.1.7 and Lemma 3.4.37. □

Lemma 3.4.39. *Let $q \in Q^{\text{ad}}$, $F = F(q)$, $i \in \mathbb{N}$ and $u_1, u_2 \in \operatorname{span}\{S_j(q)\}_{j=1}^i$. Then it holds that*

$$|a_h(F)(u_1, u_2) - a(F)(u_1, u_2)| \leq c_i h \|u_1\|_{W_0^{1,4}(\Omega)} \|u_2\|_{W_0^{1,4}(\Omega)}, \tag{3.96}$$

$$\left|\frac{a_h(F)(u_1, u_1)}{a(F)(u_1, u_1)}\right| = 1 + \mathcal{O}_i(h), \tag{3.97}$$

$$\left|\frac{a(F)(u_1, u_1)}{a_h(F)(u_1, u_1)}\right| = 1 + \mathcal{O}_i(h). \tag{3.98}$$

Proof. The first part, (3.96), follows with Corollary 3.4.38. The estimates (3.97) and (3.98) are immediate consequences of (3.96), Lemma 3.4.12 and the uniform ellipticity of A_F. □

Lemma 3.4.40. *Let $q \in Q^{\text{ad}}$ and $i \in \mathbb{N}$. Then it holds that*

$$\lambda_i(q) \leq (1 + c_i h)\, \lambda_{i,h}(q).$$

Proof. Let $F = F(q)$, using Definition 3.4.2 and Definition 3.4.3 we get

$$\begin{aligned}
\lambda_i(q) &= \min_{E \in V^{(i)}} \max_{u \in E} \frac{a(F)(u, u)}{b(F)(u, u)} \\
&= \min_{E \in V^{(i)}} \max_{u \in E} \left(\frac{a(F)(u, u)}{a_h(F)(u, u)} \frac{a_h(F)(u, u)}{b(F)(u, u)}\right) \\
&\leq (1 + c_i h) \min_{E_h \in V_{h,0}^{(i)}} \max_{u_h \in E_h} \frac{a_h(F)(u_h, u_h)}{b(F)(u_h, u_h)} \\
&= (1 + c_i h)\, \lambda_{i,h}(q),
\end{aligned}$$

where in the second step we used Lemma 3.4.39 and the fact that $V_{h,0}^{(i)} \subset V^{(i)}$. □

Lemma 3.4.41. *Let $q \in Q^{\text{ad}}$ and $i \in \mathbb{N}$. If the first i eigenvalues $(\lambda_j(q))_{1 \leq j \leq i}$ are simple, then it holds that*

$$\lambda_{i,h}(q) \leq (1 + c_{i,\varepsilon} h^{1-\varepsilon})\, \lambda_i(q).$$

Proof. Let $F = F(q)$, let $V^{(i)} = \bigoplus_{j=1}^{i} N_j$ be the space spanned by the first i eigenfunctions and let $N_h = \Pi_h V^{(i)}$ with Π_h as in Definition 3.3.2. Due to the coercivity of the bilinear form $a_h(F)(\cdot,\cdot)$ it follows that for h sufficiently small it holds that $\dim(N_h) = i$. Using N_h as testspace within (3.68) yields

$$\begin{aligned}
\lambda_{i,h}(q) &\leq \max_{u_h \in N_h} \frac{a_h(F)(u_h,u_h)}{b(F)(u_h,u_h)} = \max_{u \in V^{(i)}} \frac{a_h(F)(\Pi_h u, \Pi_h u)}{b(F)(\Pi_h u, \Pi_h u)} \\
&\leq \max_{u \in V^{(i)}} \frac{a_h(F)(u,u)}{b(F)(\Pi_h u, \Pi_h u)} = \max_{u \in V^{(i)}} \left(\frac{a_h(F)(u,u)}{a(F)(u,u)} \frac{a(F)(u,u)}{b(F)(u,u)} \frac{b(F)(u,u)}{b(F)(\Pi_h u, \Pi_h u)} \right) \\
&\leq (1 + c_i h)\, \lambda_i(q) \max_{u \in V^{(i)}} \frac{b(F)(u,u)}{b(F)(\Pi_h u, \Pi_h u)},
\end{aligned} \tag{3.99}$$

where we also used Lemma 3.4.39. It remains to estimate the last term on the right hand side of (3.99). Within Lemma 3.1.25 it has already been shown that $V^{(i)} \subset H^{3/2-\varepsilon}(\Omega)$, and a duality argument (cf. Lemma 3.4.11) proves that

$$\begin{aligned}
\|u - \Pi_h u\|_{L^2(\Omega)} &\leq c_\varepsilon h^{1-\varepsilon} \|u\|_{H^{3/2-\varepsilon}(\Omega)} \\
&\leq c_{i,\varepsilon} h^{1-\varepsilon} \|u\|_{L^2(\Omega)},
\end{aligned}$$

which yields

$$\|\Pi_h u\|_{L^2(\Omega)} \geq \|u\|_{L^2(\Omega)} \left(1 - c_{i,\varepsilon} h^{1-\varepsilon}\right). \tag{3.100}$$

Inserting (3.100) into (3.99) yields

$$\begin{aligned}
\lambda_{i,h}(q) &\leq (1 + c_i h) \left(\frac{1}{1 - c_{i,\varepsilon} h^{1-\varepsilon}} \right)^2 \lambda_i(q) \\
&\leq \left(1 + c_{i,\varepsilon} h^{1-\varepsilon}\right) \lambda_i(q),
\end{aligned}$$

where the second inequality is due to the fact that $(1-x)^{-2} = 1 + 2x + \mathcal{O}(x^2)$ for $|x| \ll 1$. □

Corollary 3.4.42. *Let $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$. Then it holds that*

$$|\lambda_i(q) - \lambda_{i,h}(q)| \leq c_{i,\varepsilon} h^{1-\varepsilon}.$$

Proof. This corollary follows with Lemma 3.4.40 and Lemma 3.4.41. □

Lemma 3.4.43. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Then it holds that*

$$\left|\lambda_{i,h}'(q)(\delta q)\right| \leq c_i \|\delta q\|_{H^2(I)}.$$

Proof. This lemma follows with (3.70), Lemma 3.4.35 and Lemma 3.4.6. □

Next we are going to estimate the error between an eigenfunction and its discrete counterpart. The following proof is based on ideas presented in [15] and [93].

Lemma 3.4.44. *Let $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$. Then it holds that*

$$\|S_i(q) - S_{i,h}(q)\|_{L^2(\Omega)} \leq c_{i,\varepsilon} h^{1-\varepsilon}.$$

Proof. Let $F = F(q)$ and

$$\rho_{i,h} = \max_{j \in \mathbb{N}\setminus\{i\}} \frac{\lambda_i(q)}{|\lambda_i(q) - \lambda_{j,h}(q)|}, \tag{3.101}$$

which is uniformly bounded from above for h sufficiently small since $\lambda_i(q)$ is a simple eigenvalue due to Assumption 3.1.26 and the fact that $\lambda_{j,h}(q) \to \lambda_j(q)$ for $h \to 0$ due to Corollary 3.4.42. In addition, let

$$\tilde{S}_{i,h}(q) = b(F)(\Pi_h S_i(q), S_{i,h}(q))\, S_{i,h}(q), \tag{3.102}$$

be the $L^2_b(\Omega)$-projection of $\Pi_h S_i(q)$ on the space spanned by $S_{i,h}(q)$, with $L^2_b(\Omega)$ as in Definition 3.1.9 and Π_h as in Definition 3.3.2. Now we have

$$\begin{aligned} \|S_i(q) - S_{i,h}(q)\|_{L^2_b(\Omega)} \leq \|S_i(q) - \Pi_h S_i(q)\|_{L^2_b(\Omega)} + \left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)} \\ + \left\|\tilde{S}_{i,h}(q) - S_{i,h}(q)\right\|_{L^2_b(\Omega)}, \end{aligned} \tag{3.103}$$

we start with examining the second term on the right hand side of (3.103). Using Corollary 3.4.8 it follows that

$$\Pi_h S_i(q) - \tilde{S}_{i,h}(q) = \sum_{j \in \mathbb{N}\setminus\{i\}} \Big((\Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)}\; S_{j,h}(q) \Big),$$

and

$$\left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|^2_{L^2_b(\Omega)} = \sum_{j \in \mathbb{N}\setminus\{i\}} (\Pi_h S_i(q), S_{j,h}(q))^2_{L^2_b(\Omega)}. \tag{3.104}$$

For the summands in (3.104) it holds that

$$\begin{aligned} (\Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)} &= \frac{1}{\lambda_{j,h}(q)}\, a_h(F)(\Pi_h S_i(q), S_{j,h}(q)) = \frac{1}{\lambda_{j,h}(q)}\, a_h(F)(S_i(q), S_{j,h}(q)) \\ &= \frac{\lambda_i(q)}{\lambda_{j,h}(q)}\, (S_i(q), S_{j,h}(q))_{L^2_b(\Omega)} \\ &+ \frac{1}{\lambda_{j,h}(q)}\, (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)), \end{aligned} \tag{3.105}$$

or equivalently

$$\lambda_{j,h}(q)\; (\Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)} = \lambda_i(q)\; (S_i(q), S_{j,h}(q))_{L^2_b(\Omega)} + (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)).$$

Subtracting $\lambda_i(q)\; (\Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)}$ on both sides gives

$$\begin{aligned} (\lambda_{j,h}(q) - \lambda_i(q))\; (\Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)} &= \lambda_i(q)\; (S_i(q) - \Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)} \\ &+ (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)), \end{aligned}$$

and hence

$$\begin{aligned}(\Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)} &= \frac{\lambda_i(q)}{\lambda_{j,h}(q)-\lambda_i(q)} \left(S_i(q) - \Pi_h S_i(q), S_{j,h}(q)\right)_{L^2_b(\Omega)} \\ &+ \frac{1}{\lambda_{j,h}(q)-\lambda_i(q)} \left(\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)\right).\end{aligned} \tag{3.106}$$

Now let $v_h \in V_{h,0}$ be the solution to

$$(\nabla v_h, \mu_h A_F \cdot \nabla \varphi_h) = (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla \varphi_h) \qquad \forall \varphi_h \in V_{h,0},$$

for which we have the stability estimate

$$\|v_h\|_{H^1_0(\Omega)} \le c\, \|\nabla S_i(q)\,(\mu_h - \mu)\|_{L^2(\Omega)} \le c_{i,\varepsilon} h^{1-\varepsilon},$$

where we used Lemma 3.4.12 and Lemma 3.4.37. Hence,

$$\begin{aligned}\frac{1}{\lambda_{j,h}(q)-\lambda_i(q)} \left(\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)\right) &= \frac{1}{\lambda_{j,h}(q)-\lambda_i(q)} \left(\nabla v_h, \mu_h A_F \cdot \nabla S_{j,h}(q)\right) \\ &= \frac{\lambda_{j,h}(q)}{\lambda_{j,h}(q)-\lambda_i(q)} \left(v_h, S_{j,h}(q)\right)_{L^2_b(\Omega)}.\end{aligned} \tag{3.107}$$

Inserting (3.107) into (3.106) yields

$$\begin{aligned}\left|(\Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)}\right| &\le \rho_{i,h} \left|(S_i(q) - \Pi_h S_i(q), S_{j,h}(q))_{L^2_b(\Omega)}\right| \\ &+ c_i \left|(v_h, S_{j,h})_{L^2_b(\Omega)}\right|,\end{aligned} \tag{3.108}$$

and using the estimate (3.108) within (3.104) shows

$$\begin{aligned}\left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|^2_{L^2_b(\Omega)} &\le c_i \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\rho^2_{i,h} \left(S_i(q) - \Pi_h S_i(q), S_{j,h}(q)\right)^2_{L^2_b(\Omega)}\right) \\ &+ c_i \sum_{j\in\mathbb{N}\setminus\{i\}} (v_h, S_{j,h})^2_{L^2_b(\Omega)} \\ &\le c_{i,\varepsilon} \left(\rho^2_{i,h} \|S_i(q) - \Pi_h S_i(q)\|^2_{L^2_b(\Omega)} + \|v_h\|^2_{L^2(\Omega)}\right) \\ &\le c_{i,\varepsilon} \left(\|S_i(q) - \Pi_h S_i(q)\|^2_{L^2_b(\Omega)} + h^{2-\varepsilon}\right).\end{aligned} \tag{3.109}$$

In order to estimate the third term within (3.103) we will show that

$$\left\|\tilde{S}_{i,h}(q) - S_{i,h}(q)\right\|_{L^2_b(\Omega)} \le \left\|\tilde{S}_{i,h}(q) - S_i(q)\right\|_{L^2_b(\Omega)}, \tag{3.110}$$

for (3.110) would imply that

$$\left\|\tilde{S}_{i,h}(q) - S_{i,h}(q)\right\|_{L^2_b(\Omega)} \le \|S_i(q) - \Pi_h S_i(q)\|_{L^2_b(\Omega)} + \left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)}. \tag{3.111}$$

It holds that

$$S_{i,h}(q) - \tilde{S}_{i,h}(q) = S_{i,h}(q)\left(1 - (\Pi_h S_i(q), S_{i,h}(q))_{L^2_b(\Omega)}\right), \tag{3.112}$$

and

$$\|S_i(q)\|_{L^2_b(\Omega)} - \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)} \le \left\|\tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)} \le \|S_i(q)\|_{L^2_b(\Omega)} + \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)}.$$

The normalizing conditions for $S_i(q)$ and $S_{i,h}(q)$ yield

$$1 - \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)} \le \left|(\Pi_h S_i(q), S_{i,h}(q))_{L^2_b(\Omega)}\right| \le 1 + \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)},$$

or

$$\left|\left|(\Pi_h S_i(q), S_{i,h}(q))_{L^2_b(\Omega)}\right| - 1\right| \le \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)}. \tag{3.113}$$

Due to (3.69) it holds that

$$(\Pi_h S_i(q), S_{i,h}(q))_{L^2_b(\Omega)} \ge 0,$$

and hence

$$\left|(\Pi_h S_i(q), S_{i,h}(q))_{L^2_b(\Omega)}\right| = (\Pi_h S_i(q), S_{i,h}(q))_{L^2_b(\Omega)}. \tag{3.114}$$

The estimates (3.112), (3.113) and (3.114) prove (3.110), and inserting the estimates (3.109) and (3.111) into (3.103) yields

$$\|S_i(q) - S_{i,h}(q)\|_{L^2_b(\Omega)} \le 2\left(1 + \rho_{i,h}\right)\|S_i(q) - \Pi_h S_i(q)\|_{L^2_b(\Omega)}. \tag{3.115}$$

In order to estimate the right hand side of (3.115), let $z \in H^1_0(\Omega)$ be the solution to

$$a(F)(v, z) = (S_i(q) - \Pi_h S_i(q), v)_{L^2_b(\Omega)} \qquad \forall v \in H^1_0(\Omega).$$

With Lemma 3.4.11 it follows that $z \in H^{3/2-\varepsilon}(\Omega)$ and

$$\|z\|_{H^{3/2-\varepsilon}(\Omega)} \le c_\varepsilon \|S_i(q) - \Pi_h S_i(q)\|_{L^2_b(\Omega)}.$$

Let $i_h z$ be the nodal interpolation of z, it holds that

$$\begin{aligned}
\|S_i(q) - \Pi_h S_i(q)\|^2_{L^2_b(\Omega)} &= a(F)(S_i(q) - \Pi_h S_i(q), z)\\
&= a_h(F)(S_i(q) - \Pi_h S_i(q), z) + \left(\nabla\left(S_i(q) - \Pi_h S_i(q)\right), \left(\mu - \mu_h\right) A_F \cdot \nabla z\right)\\
&= a_h(F)(S_i(q) - \Pi_h S_i(q), z - i_h z) + \left(\nabla\left(S_i(q) - \Pi_h S_i(q)\right), \left(\mu - \mu_h\right) A_F \cdot \nabla z\right)\\
&\le c_\varepsilon \|S_i(q) - \Pi_h S_i(q)\|_{H^1_0(\Omega)} h^{1/2-\varepsilon} \|z\|_{H^{3/2-\varepsilon}(\Omega)}\\
&\quad + c_\varepsilon \|S_i(q) - \Pi_h S_i(q)\|_{H^1_0(\Omega)} \|\mu - \mu_h\|_{L^{\frac{4}{1-2\varepsilon}}(\Omega)} \|z\|_{W^{1,\frac{4}{1+2\varepsilon}}(\Omega)}\\
&\le c_{i,\varepsilon} h^{1-\varepsilon} \|S_i(q) - \Pi_h S_i(q)\|_{L^2_b(\Omega)},
\end{aligned}$$

where we used Céa's lemma, Lemma 3.4.12, the continuous embedding $H^{3/2-\varepsilon}(\Omega) \hookrightarrow W^{1,\frac{4}{1+2\varepsilon}}(\Omega)$ and Lemma 3.4.37. □

Within the next lemma we are going to estimate the same error as in the previous lemma, but with respect to a stronger norm. As the proof is similar to the proof of Lemma 3.4.44 we will stick to the same notation. Another approach to prove this kind of estimates is presented in Subsection 3.4.2 within the proof of Lemma 3.4.101, where the same difference has to be estimated in case of the optimal control $\overline{q}$.

Lemma 3.4.45. *Let $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\|S_i(q) - S_{i,h}(q)\|_{H_0^1(\Omega)} \le c_{i,\varepsilon} h^{1/2-\varepsilon}.$$

Proof. Again let $F = F(q)$ and define

$$\tilde{S}_{i,h}(q) = b(F)(\Pi_h S_i(q), S_{i,h}(q))\, S_{i,h}(q) = \frac{1}{\lambda_{i,h}(q)} a_h(F)(\Pi_h S_i(q), S_{i,h}(q))\, S_{i,h}(q), \tag{3.116}$$

be the $L_b^2(\Omega)$-projection of $\Pi_h S_i(q)$ on the space spanned by $S_{i,h}(q)$, where Π_h is again as in Definition 3.3.2. Using the notation from Definition 3.3.1 we have

$$\begin{aligned} \|S_i(q) - S_{i,h}(q)\|_{H^1_{0,a_h}(\Omega)} \le \|S_i(q) - \Pi_h S_i(q)\|_{H^1_{0,a_h}(\Omega)} + \left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)} \\ + \left\|\tilde{S}_{i,h}(q) - S_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)}, \end{aligned} \tag{3.117}$$

and again we start with examining the second term. It holds that

$$\Pi_h S_i(q) - \tilde{S}_{i,h}(q) = \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\frac{1}{\lambda_{j,h}(q)} a_h(F)(\Pi_h S_i(q), S_{j,h}(q))\, S_{j,h}(q) \right),$$

and

$$\begin{aligned} \left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|^2_{H^1_{0,a_h}(\Omega)} &= \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\frac{1}{\lambda_{j,h}(q)} a_h(F)(\Pi_h S_i(q), S_{j,h}(q))^2 \right) \\ &= \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\lambda_{j,h}(q)\, (\Pi_h S_i(q), S_{j,h}(q))^2_{L_b^2(\Omega)} \right). \end{aligned} \tag{3.118}$$

Within the proof of Lemma 3.4.44 it has been shown that

$$\begin{aligned} (\lambda_{j,h}(q) - \lambda_i(q))\, (\Pi_h S_i(q), S_{j,h}(q))_{L_b^2(\Omega)} = \lambda_i(q)\, (S_i(q) - \Pi_h S_i(q), S_{j,h}(q))_{L_b^2(\Omega)} \\ + (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)), \end{aligned}$$

and hence

$$\begin{aligned} \lambda_{j,h}(q)\, (\Pi_h S_i(q), S_{j,h}(q))^2_{L_b^2(\Omega)} \le c \frac{\lambda_{j,h}(q)\, \lambda_i(q)^2}{|\lambda_{j,h}(q) - \lambda_i(q)|^2} (S_i(q) - \Pi_h S_i(q), S_{j,h}(q))^2_{L_b^2(\Omega)} \\ + c \frac{\lambda_{j,h}(q)}{|\lambda_{j,h}(q) - \lambda_i(q)|^2} (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q))^2. \end{aligned} \tag{3.119}$$

Let again $v_h \in V_{h,0}$ be the solution to

$$(\nabla v_h, \mu_h A_F \cdot \nabla \varphi_h) = (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{3.120}$$

for which we have the stability estimate

$$\|v_h\|_{H_0^1(\Omega)} \leq c \|\nabla S_i(q)\,(\mu_h - \mu)\|_{L^2(\Omega)} \leq c_i h^{1/2},$$

where we used Lemma 3.4.12 and Lemma 3.4.37. In addition, let $w_h \in V_{h,0}$ be the solution to

$$b(F)(w_h, \varphi_h) = b(F)(S_i(q) - \Pi_h S_i(q), \varphi_h) \qquad \forall \varphi_h \in V_{h,0}. \tag{3.121}$$

We get

$$\begin{aligned}\|w_h\|^2_{L^2_b(\Omega)} &= b(F)(S_i(q) - \Pi_h S_i(q), w_h)\\ &\leq \|S_i(q) - \Pi_h S_i(q)\|_{L^2_b(\Omega)} \|w_h\|_{L^2_b(\Omega)},\end{aligned}$$

and hence

$$\|w_h\|_{L^2_b(\Omega)} \leq c_{i,\varepsilon} h^{1-\varepsilon}.$$

Inserting (3.119) into (3.118) and using the definitions (3.120) and (3.121) shows that

$$\begin{aligned}\left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|^2_{H^1_{0,a_h}(\Omega)} &\leq c_i \sum_{j\in\mathbb{N}\setminus\{i\}} \left((w_h, S_{j,h}(q))^2_{L^2_b(\Omega)} + \frac{1}{\lambda_{j,h}(q)}\left(\nabla v_h, \mu_h A_F \cdot \nabla S_{j,h}(q)\right)^2\right)\\ &= c_i \left(\|w_h\|^2_{L^2(\Omega)} + \|v_h\|^2_{H_0^1(\Omega)}\right)\\ &\leq c_i h,\end{aligned}$$

and hence

$$\left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|_{H_0^1(\Omega)} \leq c_i h^{1/2}. \tag{3.122}$$

In order to estimate the third term within (3.117), we will show that

$$\left\|\tilde{S}_{i,h}(q) - S_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)} \leq \left\|\tilde{S}_{i,h}(q) - S_i(q)\right\|_{H^1_{0,a_h}(\Omega)} + c_\varepsilon h^{1/2-\varepsilon}, \tag{3.123}$$

for (3.123) would imply that

$$\begin{aligned}\left\|\tilde{S}_{i,h}(q) - S_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)} &\leq \|S_i(q) - \Pi_h S_i(q)\|_{H^1_{0,a_h}(\Omega)}\\ &\quad + \left\|\Pi_h S_i(q) - \tilde{S}_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)} + c_\varepsilon h^{1/2-\varepsilon}.\end{aligned} \tag{3.124}$$

It holds that

$$\begin{aligned}S_{i,h}(q) - \tilde{S}_{i,h}(q) &= S_{i,h}(q)\left(1 - (\Pi_h S_i(q), S_{i,h}(q))_{L^2_b(\Omega)}\right)\\ &= S_{i,h}(q)\left(1 - \frac{1}{\lambda_{i,h}(q)} a_h(F)(\Pi_h S_i(q), S_{i,h}(q))\right).\end{aligned} \tag{3.125}$$

Let $\|\cdot\|_{H^1_{0,a_h}(\Omega)}$ be the norm from Definition 3.3.1, the triangle inequality proves that

$$\begin{aligned}\|S_i(q)\|_{H^1_{0,a_h}(\Omega)} - \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)} &\leq \left\|\tilde{S}_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)} \\ &\leq \|S_i(q)\|_{H^1_{0,a_h}(\Omega)} + \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)}.\end{aligned}$$

By definition it holds that

$$\|S_{i,h}(q)\|_{H^1_{0,a_h}(\Omega)} = \sqrt{\lambda_{i,h}(q)}, \tag{3.126}$$

and

$$\begin{aligned}\|S_i(q)\|^2_{H^1_{0,a_h}(\Omega)} &= a_h(F)(S_i(q), S_i(q)) \\ &= a(F)(S_i(q), S_i(q)) + (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_i(q)) \\ &= \lambda_i(q) + (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_i(q)).\end{aligned}$$

With Lemma 3.4.39 and Lemma 3.4.12 it follows that

$$0 \leq (\nabla S_i(q), (\mu_h - \mu)\, A_F \cdot \nabla S_i(q)) \leq c_i h,$$

hence

$$\left| \|S_i(q)\|_{H^1_{0,a_h}(\Omega)} - \sqrt{\lambda_i(q)} \right| \leq c_i h^{1/2},$$

which yields

$$\begin{aligned}\sqrt{\lambda_i(q)} - \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{L^2_b(\Omega)} - c_i h^{1/2} &\leq \frac{1}{\sqrt{\lambda_{i,h}(q)}} \left|a_h(F)(\Pi_h S_i(q), S_{i,h}(q))\right| \\ &\leq \sqrt{\lambda_i(q)} + \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)} + c_i h^{1/2},\end{aligned}$$

or

$$\begin{aligned}\sqrt{\lambda_{i,h}(q)} \left| 1 - \frac{1}{\lambda_{i,h}(q)} \left|a_h(F)(\Pi_h S_i(q), S_{i,h}(q))\right| \right| &\leq \left\|S_i(q) - \tilde{S}_{i,h}(q)\right\|_{H^1_{0,a_h}(\Omega)} \\ &\quad + c_i \left(h^{1/2} + \sqrt{|\lambda_i(q) - \lambda_{i,h}(q)|} \right).\end{aligned}$$

With (3.69) it follows that

$$a_h(F)(\Pi_h S_i(q), S_{i,h}(q)) = \lambda_{i,h}(q)\, (\Pi_h S_i(q), S_{i,h}(q))_{L^2_b(\Omega)} \geq 0. \tag{3.127}$$

The estimate (3.123) now follows with (3.125), (3.126), (3.127) and Corollary 3.4.42. The remaining estimate for $\|S_i(q) - \Pi_h S_i(q)\|_{H^1_0(\Omega)}$ can again be shown using Céa's lemma and Lemma 3.4.12. □

Lemma 3.4.46. *Let $q \in Q^{\mathrm{ad}}$, $i \in \mathbb{N}$ and $p \in (1, 4)$. Then it holds that*

$$\|S_{i,h}(q)\|_{W^{1,p}(\Omega)} \leq c_{i,p}.$$

Proof. For $p \leq 2$ this statement follows with standard embedding theorems, Lemma 3.4.45 and Lemma 3.4.12. For $p > 2$ let $i_h S_i(q)$ be the nodal interpolation of $S_i(q)$. Using an inverse estimate it follows that

$$\begin{aligned}
\|S_{i,h}(q)\|_{W^{1,p}(\Omega)} &\leq \|i_h S_i(q)\|_{W^{1,p}(\Omega)} + \|i_h S_i(q) - S_{i,h}(q)\|_{W^{1,p}(\Omega)} \\
&\leq c\,\|S_i(q)\|_{W^{1,p}(\Omega)} + ch^{2(1/p-1/2)}\,\|i_h S_i(q) - S_{i,h}(q)\|_{H^1(\Omega)} \\
&\leq c\,\|S_i(q)\|_{W^{1,p}(\Omega)} + ch^{2/p-1}\left(\|S_i(q) - i_h S_i(q)\|_{H^1(\Omega)} + \|S_i(q) - S_{i,h}(q)\|_{H^1(\Omega)}\right) \\
&\leq c\,\|S_i(q)\|_{W^{1,p}(\Omega)} + c_{i,\varepsilon} h^{2/p-1} h^{1/2-\varepsilon}\,\|S_i(q)\|_{H^{3/2-\varepsilon}(\Omega)} \\
&\leq c_{i,p},
\end{aligned}$$

where in the last step we have to choose $\varepsilon = 2/p - 1/2 > 0$ for $p < 4$. □

Lemma 3.4.47. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Then it holds that*

$$\left|\lambda_i'(q)(\delta q) - \lambda_{i,h}'(q)(\delta q)\right| \leq c_{i,\varepsilon} h^{1/2-\varepsilon}\,\|\delta q\|_{H^2(I)}.$$

Proof. Let $F = F(q)$ and $\delta F = F'(q)(\delta q)$, it holds that

$$\begin{aligned}
\left|\lambda_i'(q)(\delta q) - \lambda_{i,h}'(q)(\delta q)\right| &\leq \left|\left(\nabla\left(S_i(q) - S_{i,h}(q)\right), \mu A'_{F,\delta F}\cdot\nabla\left(S_i(q) + S_{i,h}(q)\right)\right)\right| \\
&\quad + \left|\left(\lambda_i\, S_i(q)^2 - \lambda_{i,h}(q)\, S_{i,h}(q)^2, \gamma'_{F,\delta F}\right)\right| \\
&\quad + \left|\left(\nabla S_{i,h}(q), (\mu - \mu_h)\, A'_{F,\delta F}\cdot\nabla S_{i,h}(q)\right)\right|,
\end{aligned}$$

and the result follows with Lemma 3.4.45, Corollary 3.4.42, Lemma 3.4.6 and Lemma 3.4.37. □

Lemma 3.4.48. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$, $i \in \mathbb{N}$ and $g_{i,h}(q,\delta q) \in H^{-1}(\Omega)$ as in Definition 3.4.5. Then it holds that*

$$\|g_{i,h}(q,\delta q)\|_{H^{-1}(\Omega)} \leq c_i\,\|\delta q\|_{H^2(I)}.$$

Proof. Let $F = F(q)$ and $\delta F = F'(q)(\delta q)$, then it holds that

$$\begin{aligned}
\|g_{i,h}(q,\delta q)\|_{H^{-1}(\Omega)} &\leq \left\|\lambda_{i,h}'(q)(\delta q)\, S_{i,h}(q)\gamma_F + \lambda_{i,h}(q)\, S_{i,h}(q)\gamma'_{F,\delta F}\right\|_{L^2(\Omega)} \\
&\quad + \left\|\operatorname{div}\left(\mu_h A'_{F,\delta F}\cdot\nabla S_{i,h}(q)\right)\right\|_{H^{-1}(\Omega)} \\
&\leq c_i\,\|\delta q\|_{H^2(I)},
\end{aligned}$$

where we used Lemma 3.4.43 and Lemma 3.4.6. □

Lemma 3.4.49. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$, $i \in \mathbb{N}$ and $g_i(q,\delta q), g_{i,h}(q,\delta q) \in H^{-1}(\Omega)$ as in Definition 3.4.5. Then it holds that*

$$\|g_i(q,\delta q) - g_{i,h}(q,\delta q)\|_{H^{-1}(\Omega)} \leq c_{i,\varepsilon} h^{1/2-\varepsilon}\,\|\delta q\|_{H^2(I)}.$$

Proof. With $F = F(q)$ and $\delta F = F'(q)(\delta q)$ it holds that

$$\begin{aligned}
\|g_i(q,\delta q) - g_{i,h}(q,\delta q)\|_{H^{-1}(\Omega)} &\leq \left\|\lambda_i'(q)(\delta q)\, S_i(q)\gamma_F - \lambda_{i,h}'(q)(\delta q)\, S_{i,h}(q)\gamma_F\right\|_{L^2(\Omega)} \\
&\quad + \left\|\lambda_i(q)\, S_i(q)\gamma'_{F,\delta F} - \lambda_{i,h}(q)\, S_{i,h}(q)\gamma'_{F,\delta F}\right\|_{L^2(\Omega)} \\
&\quad + \left\|\operatorname{div}\left(\mu A'_{F,\delta F}\cdot\nabla S_i(q) - \mu_h A'_{F,\delta F}\cdot\nabla S_{i,h}(q)\right)\right\|_{H^{-1}(\Omega)},
\end{aligned}$$

and the result follows with Corollary 3.4.42 as well as Lemma 3.4.47, Lemma 3.4.45, Lemma 3.4.6 and Lemma 3.4.37. □

The subsequent lemmata will be needed to estimate the error between the continuous and the discrete derivative of the eigenfunction with respect to q.

Lemma 3.4.50. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Let $S'_{i,0}(q)(\delta q)$ and $S'_{i,h,0}(q)(\delta q)$ be the (discrete) solutions to the affine eigenvalue equations related to* (3.66) *and* (3.71) *without the normalizing conditions but with minimal $H^1_{0,a}(\Omega)$-, and $H^1_{0,a_h}(\Omega)$-norm, respectively, cf. Subsection 3.2.2. Then it holds that*

$$\left\|S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|_{L^2(\Omega)} \leq c_{i,\varepsilon} h^{1/2-\varepsilon} \left\|\delta q\right\|_{H^2(I)}.$$

Proof. Let $F = F(q)$. For arbitrary $j \in \mathbb{N}\setminus\{i\}$ it holds that

$$\begin{aligned} a(F)(S'_{i,0}(q)(\delta q), S_j(q)) &= b(F)(S'_{i,0}(q)(\delta q), S_j(q)) = 0, \\ a_h(F)(S'_{i,h,0}(q)(\delta q), S_{j,h}(q)) &= b(F)(S'_{i,h,0}(q)(\delta q), S_{j,h}(q)) = 0. \end{aligned} \tag{3.128}$$

Let Π_h as in Definition 3.3.2 and let

$$\begin{aligned} \tilde{\Pi} S'_{i,0}(q)(\delta q) &= \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\left(\Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\right)_{L^2_b(\Omega)} S_{j,h}(q)\right) \\ &= \Pi_h S'_{i,0}(q)(\delta q) - \left(\Pi_h S'_{i,0}(q)(\delta q), S_{i,h}(q)\right)_{L^2_b(\Omega)} S_{i,h}(q). \end{aligned} \tag{3.129}$$

Then we split the error,

$$\begin{aligned} \left\|S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} &\leq \left\|S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} \\ &\quad + \left\|\Pi_h S'_{i,0}(q)(\delta q) - \tilde{\Pi} S'_{i,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} \\ &\quad + \left\|\tilde{\Pi} S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|_{L^2_b(\Omega)}. \end{aligned} \tag{3.130}$$

Using a duality argument and Lemma 3.4.26 it follows that

$$\left\|S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} \leq c_{i,\varepsilon} h^{1-\varepsilon} \left\|\delta q\right\|_{H^2(I)}. \tag{3.131}$$

From (3.129) we get

$$\Pi_h S'_{i,0}(q)(\delta q) - \tilde{\Pi} S'_{i,0}(q)(\delta q) = \left(\Pi_h S'_{i,0}(q)(\delta q), S_{i,h}(q)\right)_{L^2_b(\Omega)} S_{i,h}(q),$$

hence

$$\begin{aligned} &\left\|\Pi_h S'_{i,0}(q)(\delta q) - \tilde{\Pi} S'_{i,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} \\ &= \left|\left(\Pi_h S'_{i,0}(q)(\delta q), S_{i,h}(q)\right)_{L^2_b(\Omega)}\right| \\ &= \left|\left(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,0}(q)(\delta q), S_{i,h}(q)\right)_{L^2_b(\Omega)} + \left(S'_{i,0}(q)(\delta q), S_{i,h}(q) - S_i(q)\right)_{L^2_b(\Omega)}\right| \\ &\leq \left\|\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} + \left\|S'_{i,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} \left\|S_i(q) - S_{i,h}(q)\right\|_{L^2_b(\Omega)} \\ &\leq c_{i,\varepsilon} h^{1-\varepsilon} \left\|\delta q\right\|_{H^2(I)}, \end{aligned}$$

where we used (3.131), Lemma 3.4.26 and Lemma 3.4.44. The last part within (3.130) remains. As

$$S'_{i,h,0}(q)(\delta q) = \sum_{j\in\mathbb{N}\setminus\{i\}} \Big(\big(S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} \, S_{j,h}(q) \Big),$$

it holds that

$$\begin{aligned} \tilde{\Pi} S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q) &= \sum_{j\in\mathbb{N}\setminus\{i\}} \Big(\big(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} \, S_{j,h}(q) \Big), \\ \Big\| \tilde{\Pi} S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q) \Big\|^2_{L^2_b(\Omega)} &= \sum_{j\in\mathbb{N}\setminus\{i\}} \big(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\big)^2_{L^2_b(\Omega)}, \end{aligned} \tag{3.132}$$

and we have to estimate each summand within (3.132). It holds that

$$\begin{aligned} &\big(\Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} = \frac{1}{\lambda_{j,h}(q)} a_h(F)(\Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)) \\ &= \frac{1}{\lambda_{j,h}(q)} a_h(F)(S'_{i,0}(q)(\delta q), S_{j,h}(q)) \\ &= \frac{\lambda_i(q)}{\lambda_{j,h}(q)} \big(S'_{i,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} + \frac{1}{\lambda_{j,h}(q)} \left(g_i(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0} \\ &+ \frac{1}{\lambda_{j,h}(q)} \left(a_h(F)(S'_{i,0}(q)(\delta q), S_{j,h}(q)) - a(F)(S'_{i,0}(q)(\delta q), S_{j,h}(q))\right) \\ &= \frac{\lambda_i(q)}{\lambda_{j,h}(q)} \big(\Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} + \frac{\lambda_i(q)}{\lambda_{j,h}(q)} \big(S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} \\ &+ \frac{1}{\lambda_{j,h}(q)} \left(g_i(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0} + \frac{1}{\lambda_{j,h}(q)} \big(\nabla S'_{i,0}(q)(\delta q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)\big), \end{aligned}$$

where we used Lemma 3.4.39 and Lemma 3.4.27. Hence

$$\begin{aligned} \big(\Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} &= \frac{\lambda_i(q)}{\lambda_{j,h}(q) - \lambda_i(q)} \big(S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} \\ &+ \frac{1}{\lambda_{j,h}(q) - \lambda_i(q)} \left(g_i(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0} \\ &+ \frac{1}{\lambda_{j,h}(q) - \lambda_i(q)} \big(\nabla S'_{i,0}(q)(\delta q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)\big). \end{aligned} \tag{3.133}$$

In addition,

$$\begin{aligned} \big(S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} &= \frac{1}{\lambda_{j,h}(q)} a_h(F)(S'_{i,h,0}(q)(\delta q), S_{j,h}(q)) \\ &= \frac{\lambda_{i,h}(q)}{\lambda_{j,h}(q)} \big(S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} + \frac{1}{\lambda_{j,h}(q)} \left(g_{i,h}(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0}, \end{aligned}$$

hence

$$\big(S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)} = \frac{1}{\lambda_{j,h}(q) - \lambda_{i,h}(q)} \left(g_{i,h}(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0}. \tag{3.134}$$

The equations (3.133) and (3.134) yield

$$
\begin{aligned}
&\left(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\right)_{L^2_b(\Omega)} \\
&= \frac{1}{\lambda_{j,h}(q) - \lambda_i(q)} \left(g_i(q,\delta q) - g_{i,h}(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0} \\
&+ \frac{\lambda_i(q) - \lambda_{i,h}(q)}{\left(\lambda_{j,h}(q) - \lambda_i(q)\right)\left(\lambda_{j,h}(q) - \lambda_{i,h}(q)\right)} \left(g_{i,h}(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0} \\
&+ \frac{\lambda_i(q)}{\lambda_{j,h}(q) - \lambda_i(q)} \left(S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\right)_{L^2_b(\Omega)} \\
&+ \frac{1}{\lambda_{j,h}(q) - \lambda_i(q)} \left(\nabla S'_{i,0}(q)(\delta q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)\right).
\end{aligned}
\tag{3.135}
$$

As $|\lambda_{j,h}(q) - \lambda_i(q)|$ and $|\lambda_{j,h}(q) - \lambda_{i,h}(q)|$ are uniformly bounded from below for $h \to 0$ and $j \neq i$, it holds that

$$
\begin{aligned}
&\left(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\right)^2_{L^2_b(\Omega)} \\
&\leq c_i \left(\frac{1}{\lambda_{j,h}(q)} \left(g_i(q,\delta q) - g_{i,h}(q,\delta q), S_{j,h}(q)\right)^2_{H^{-1},H^1_0} + |\lambda_i(q) - \lambda_{i,h}(q)|^2 \frac{1}{\lambda_{j,h}(q)} \left(g_{i,h}(q,\delta q), S_{j,h}(q)\right)^2_{H^{-1},H^1_0}\right) \\
&+ c_i \left(\left(S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\right)^2_{L^2_b(\Omega)} + \frac{1}{\lambda_{j,h}(q)} \left(\nabla S'_{i,0}(q)(\delta q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)\right)^2\right).
\end{aligned}
$$

Let $v \in H^1_0(\Omega)$ be the solution to

$$
(\nabla v, \mu_h A_F \nabla \varphi) = \left(\nabla S'_{i,0}(q)(\delta q), (\mu_h - \mu)\, A_F \cdot \nabla \varphi\right) \qquad \forall \varphi \in H^1_0(\Omega). \tag{3.136}
$$

Lemma 3.4.27 and Lemma 3.4.37 yield

$$
\|v\|_{H^1_0(\Omega)} \leq c \left\|(\mu - \mu_h)\, \nabla S'_{i,0}(q)(\delta q)\right\|_{L^2(\Omega)} \leq c h^{1/2} \|\delta q\|_{H^2(I)}. \tag{3.137}
$$

With (3.132) and Corollary 3.4.10 we conclude that

$$
\begin{aligned}
&\left\|\tilde{\Pi} S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|^2_{L^2_b(\Omega)} \\
&\leq c_i \left(\|g_i(q,\delta q) - g_{i,h}(q,\delta q)\|^2_{H^{-1}} + |\lambda_i(q) - \lambda_{i,h}(q)|^2 \|g_{i,h}(q,\delta q)\|^2_{H^{-1}}\right) \\
&+ c_i \left(\left\|S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q)\right\|^2_{L^2_b(\Omega)} + c h \|\delta q\|^2_{H^2(I)}\right),
\end{aligned}
$$

and finish this proof with Lemma 3.4.49, Lemma 3.4.42, Lemma 3.4.48 and (3.131). □

Lemma 3.4.51. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Let $S'_{i,0}(q)(\delta q)$ and $S'_{i,h,0}(q)(\delta q)$ as in Lemma 3.4.50. Then it holds that*

$$
\left\|S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|_{H^1_0(\Omega)} \leq c_{i,\varepsilon} h^{1/2-\varepsilon} \|\delta q\|_{H^2(I)}.
$$

Proof. The proof of this lemma is similar to the proof of Lemma 3.4.50. Let $F = F(q)$, let Π_h be defined as in Definition 3.3.2 and let

$$\begin{aligned}
\tilde{\Pi} S'_{i,0}(q)(\delta q) &= \sum_{j\in\mathbb{N}\setminus\{i\}} \Big(\big(\Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\big)_{L^2_b(\Omega)}\ S_{j,h}(q)\Big) \\
&= \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\frac{1}{\lambda_{j,h}(q)} a_h(F)(\Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q))\, S_{j,h}(q)\right) \\
&= \Pi_h S'_{i,0}(q)(\delta q) - \big(\Pi_h S'_{i,0}(q)(\delta q), S_{i,h}(q)\big)_{L^2_b(\Omega)}\ S_{i,h}(q) \\
&= \Pi_h S'_{i,0}(q)(\delta q) - \frac{1}{\lambda_{i,h}(q)} a_h(F)(\Pi_h S'_{i,0}(q)(\delta q), S_{i,h}(q))\, S_{i,h}(q).
\end{aligned} \tag{3.138}$$

Then we split the error,

$$\begin{aligned}
\left\| S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|_{H^1_{0,a_h}(\Omega)} \le{}& \left\| S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q)\right\|_{H^1_{0,a_h}(\Omega)} \\
&+ \left\| \Pi_h S'_{i,0}(q)(\delta q) - \tilde{\Pi} S'_{i,0}(q)(\delta q)\right\|_{H^1_{0,a_h}(\Omega)} \\
&+ \left\| \tilde{\Pi} S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|_{H^1_{0,a_h}(\Omega)}.
\end{aligned} \tag{3.139}$$

Using Lemma 3.4.26 it follows that

$$\left\| S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q)\right\|_{H^1_{0,a_h}(\Omega)} \le c_{i,\varepsilon} h^{1/2-\varepsilon} \left\|\delta q\right\|_{H^2(I)}. \tag{3.140}$$

From (3.138) we get

$$\Pi_h S'_{i,0}(q)(\delta q) - \tilde{\Pi} S'_{i,0}(q)(\delta q) = \big(\Pi_h S'_{i,0}(q)(\delta q), S_{i,h}(q)\big)_{L^2_b(\Omega)}\ S_{i,h}(q),$$

and

$$\begin{aligned}
&\left\| \Pi_h S'_{i,0}(q)(\delta q) - \tilde{\Pi} S'_{i,0}(q)(\delta q)\right\|_{H^1_{0,a_h}(\Omega)} \\
&\le c\sqrt{\lambda_{i,h}(q)}\, \left| \big(\Pi_h S'_{i,0}(q)(\delta q), S_{i,h}(q)\big)_{L^2_b(\Omega)}\right| \\
&\le c_i \left| \big(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,0}(q)(\delta q), S_{i,h}(q)\big)_{L^2_b(\Omega)} + \big(S'_{i,0}(q)(\delta q), S_{i,h}(q) - S_i(q)\big)_{L^2_b(\Omega)}\right| \\
&\le c_i \Big(\left\| \Pi_h S'_{i,0}(q)(\delta q) - S'_{i,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} + \left\| S'_{i,0}(q)(\delta q)\right\|_{L^2_b(\Omega)} \left\| S_i(q) - S_{i,h}(q)\right\|_{L^2_b(\Omega)}\Big) \\
&\le c_{i,\varepsilon} h^{1-\varepsilon} \left\|\delta q\right\|_{H^2(I)},
\end{aligned}$$

where we used (3.140) with a duality argument, Lemma 3.4.26 and Lemma 3.4.44. It remains to estimate the last part within (3.139). Because of

$$S'_{i,h,0}(q)(\delta q) = \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\frac{1}{\lambda_{j,h}(q)} a_h(F)(S'_{i,h,0}(q)(\delta q), S_{j,h}(q))\, S_{j,h}(q)\right),$$

it holds that

$$\tilde{\Pi} S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q) = \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\frac{1}{\lambda_{j,h}(q)} a_h(F)(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q))\, S_{j,h}(q)\right),$$

thus

$$\begin{aligned}
&\left\|\tilde{\Pi} S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|^2_{H^1_{0,a_h}(\Omega)} \\
&= \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\frac{1}{\lambda_{j,h}(q)} a_h(F)(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q))^2\right) \\
&= \sum_{j\in\mathbb{N}\setminus\{i\}} \left(\lambda_{j,h}(q)\left(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\right)^2_{L^2_b(\Omega)}\right).
\end{aligned} \tag{3.141}$$

The summands within (3.141) have already been estimated within the proof of Lemma 3.4.50, (3.135). It holds that

$$\begin{aligned}
&\sqrt{\lambda_{j,h}(q)}\left(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\right)_{L^2_b(\Omega)} \\
&= \frac{\sqrt{\lambda_{j,h}(q)}}{\lambda_{j,h}(q) - \lambda_i(q)} \left(g_i(q,\delta q) - g_{i,h}(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0} \\
&+ \frac{\sqrt{\lambda_{j,h}(q)}\,(\lambda_i(q) - \lambda_{i,h}(q))}{(\lambda_{j,h}(q) - \lambda_i(q))\,(\lambda_{j,h}(q) - \lambda_{i,h}(q))} \left(g_{i,h}(q,\delta q), S_{j,h}(q)\right)_{H^{-1},H^1_0} \\
&+ \frac{\lambda_i(q)\sqrt{\lambda_{j,h}(q)}}{\lambda_{j,h}(q) - \lambda_i(q)} \left(S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\right)_{L^2_b(\Omega)} \\
&+ \frac{\sqrt{\lambda_{j,h}(q)}}{\lambda_{j,h}(q) - \lambda_i(q)} \left(\nabla S'_{i,0}(q)(\delta q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)\right).
\end{aligned}$$

Again we use the fact that $|\lambda_{j,h}(q) - \lambda_i(q)|$ and $|\lambda_{j,h}(q) - \lambda_{i,h}(q)|$ are uniformly bounded from below for $h \to 0$ and $j \neq i$ and get

$$\begin{aligned}
&\lambda_{j,h}(q)\left(\Pi_h S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q), S_{j,h}(q)\right)^2_{L^2_b(\Omega)} \\
&\leq c_i \left(\frac{1}{\lambda_{j,h}(q)} \left(g_i(q,\delta q) - g_{i,h}(q,\delta q), S_{j,h}(q)\right)^2_{H^{-1},H^1_0} + |\lambda_i(q) - \lambda_{i,h}(q)|^2 \frac{1}{\lambda_{j,h}(q)} \left(g_{i,h}(q,\delta q), S_{j,h}(q)\right)^2_{H^{-1},H^1_0}\right) \\
&+ c_i \left(\left(S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q), S_{j,h}(q)\right)^2_{L^2_b(\Omega)} + \frac{1}{\lambda_{j,h}(q)} \left(\nabla S'_{i,0}(q)(\delta q), (\mu_h - \mu)\, A_F \cdot \nabla S_{j,h}(q)\right)^2\right).
\end{aligned}$$

Defining $v \in H^1_0(\Omega)$ as in (3.136), inserting this relation into (3.141), using Corollary 3.4.10 and the estimate (3.137) yields

$$\begin{aligned}
&\left\|\tilde{\Pi} S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|^2_{H^1_{0,a_h}(\Omega)} \\
&\leq c_i \left(\|g_i(q,\delta q) - g_{i,h}(q,\delta q)\|^2_{H^{-1}(\Omega)} + |\lambda_i(q) - \lambda_{i,h}(q)|^2 \|g_{i,h}(q,\delta q)\|^2_{H^{-1}(\Omega)}\right) \\
&+ c_i \left(\left\|S'_{i,0}(q)(\delta q) - \Pi_h S'_{i,0}(q)(\delta q)\right\|^2_{L^2_b(\Omega)} + h\,\|\delta q\|^2_{H^2(I)}\right),
\end{aligned}$$

and we finish the proof with Lemma 3.4.49, Lemma 3.4.42, Lemma 3.4.48 and (3.140). □

Lemma 3.4.52. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Then it holds that*

$$\left\|S'_i(q)(\delta q) - S'_{i,h}(q)(\delta q)\right\|_{H^1_0(\Omega)} \leq c_{i,\varepsilon} h^{1/2-\varepsilon} \|\delta q\|_{H^2(I)}.$$

Proof. Let $F = F(q)$, $\delta F = F'(q)(\delta q)$, and let $S'_{i,0}(q)(\delta q)$ and $S'_{i,h,0}(q)(\delta q)$ be defined as in Lemma 3.4.50. It follows that there exist $t_1, t_2 \in \mathbb{R}$ such that

$$S'_i(q)(\delta q) = S'_{i,0}(q)(\delta q) + t_1\, S_i(q) \qquad \text{and} \qquad S'_{i,h}(q)(\delta q) = S'_{i,h,0}(q)(\delta q) + t_2\, S_{i,h}(q).$$

Using the normalizing conditions for $S_i(q)$ and $S_{i,h}(q)$ it follows that

$$t_1 = -\frac{1}{2}\left(S_i(q)^2, \gamma'_{F,\delta F}\right) \qquad \text{and} \qquad t_2 = -\frac{1}{2}\left(S_{i,h}(q)^2, \gamma'_{F,\delta F}\right),$$

we get

$$\begin{aligned}\left\|S'_i(q)(\delta q) - S'_{i,h}(q)(\delta q)\right\|_{H^1_0(\Omega)} \le{}& \left\|S'_{i,0}(q)(\delta q) - S'_{i,h,0}(q)(\delta q)\right\|_{H^1_0(\Omega)} + |t_1 - t_2|\, \|S_i(q)\|_{H^1_0(\Omega)} \\ &+ |t_2|\, \|S_i(q) - S_{i,h}(q)\|_{H^1_0(\Omega)},\end{aligned}$$

and the result follows with Lemma 3.4.51, Lemma 3.4.45 and Lemma 3.4.6. □

Lemma 3.4.53. *Let $q \in Q^{\text{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Then it holds that*

$$\left|\lambda''_i(q)(\delta q, \delta q) - \lambda''_{i,h}(q)(\delta q, \delta q)\right| \le c_{i,\varepsilon} h^{1/2-\varepsilon}\, \|\delta q\|^2_{H^2(I)}.$$

Proof. Let $F = F(q)$ and $\delta F = F'(q)(\delta q)$. With (3.53) and (3.72) it follows that

$$\begin{aligned}&\left|\lambda''_i(q)(\delta q, \delta q) - \lambda''_{i,h}(q)(\delta q, \delta q)\right| \\ &\le \left|\left(\nabla\left(S_i(q) - S_{i,h}(q)\right), \mu_h A''_{F,\delta F,\delta F} \cdot \nabla\left(S_i(q) + S_{i,h}(q)\right)\right)\right| \\ &+ 2\left|\left(\nabla\left(S'_i(q)(\delta q) - S'_{i,h}(q)(\delta q)\right), \mu_h A_F \cdot \nabla\left(S'_i(q)(\delta q) + S'_{i,h}(q)(\delta q)\right)\right)\right| \\ &+ 2\left|\left(\lambda_i(q)\, S'_i(q)(\delta q)^2 - \lambda_{i,h}(q)\, S'_{i,h}(q)(\delta q)^2, \gamma_F\right)\right| + \left|\left(\lambda_i(q)\, S_i(q)^2 - \lambda_{i,h}(q)\, S_{i,h}(q), \gamma''_{F,\delta F,\delta F}\right)\right| \\ &+ \left|\left(\nabla S_i(q), (\mu - \mu_h)\, A''_{F,\delta F,\delta F} \cdot \nabla S_i(q)\right)\right| + 2\left|\left(\nabla S'_i(q)(\delta q), (\mu - \mu_h)\, A_F \cdot \nabla S'_i(q)(\delta q)\right)\right|,\end{aligned}$$

and the result follows with Corollary 3.4.42, Lemma 3.4.47, Lemma 3.4.45, Lemma 3.4.52, Lemma 3.4.6 and general stability estimates. □

Lemma 3.4.54. *For $q \in Q^{\text{ad}}$ and $\delta q \in Q$ it holds that*

$$\begin{aligned}\left|j'(q)(\delta q) - j'_h(q)(\delta q)\right| &\le c_{i,\varepsilon} h^{1/2-\varepsilon}\, \|\delta q\|_{H^2(I)}, \\ \left|j''(q)(\delta q, \delta q) - j''_h(q)(\delta q, \delta q)\right| &\le c_{i,\varepsilon} h^{1/2-\varepsilon}\, \|\delta q\|^2_{H^2(I)}.\end{aligned}$$

Proof. This lemma follows with Lemma 3.4.47 and Lemma 3.4.53. □

3.4.1.3. Estimates between the state-discretized case and the fully discretized case

Within this subsubsection we are going to estimate the error induced by the additional discretization of the transformation F. If not stated otherwise we will always assume that the discretization parameter k is chosen sufficiently small.

The proof of the following lemma is based on ideas presented in [32].

Lemma 3.4.55. *Let $p \in [2,4]$ and let $u \in W^{1,p}(\Omega)$ with $u|_{\Omega_j} \in W^{2,p}(\Omega_j)$ for $j \in \{0,1\}$. Furthermore, let $i_k \colon C(\overline{\Omega}) \to V_k$ be the pointwise interpolation operator and let $s \in [0,1]$. If $p = 2$ then it holds that*

$$\|u - i_k u\|_{H^1(\Omega)} \leq c k^{2-s} |\ln k|^{1/2} \left(\|u\|_{H^1(\Omega)} + \|u\|_{H^2(\Omega_0)} + \|u\|_{H^2(\Omega_1)} \right),$$

whereas for $p > 2$ it holds that

$$\|u - i_k u\|_{W^{1,p}(\Omega)} \leq c_p k^{2-s} \left(\|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{2,p}(\Omega_0)} + \|u\|_{W^{2,p}(\Omega_1)} \right).$$

Proof. The proof for $p = 2$ can be found in [32], Lemma 2.1. The cited proof can also be generalized to the case $p > 2$. In that case it holds that $W^{1,p}(\Omega_j) \hookrightarrow L^\infty(\Omega_j)$, thus the log-term disappears. □

Lemma 3.4.56. *Let $u \in W^{1,\infty}(\Omega)$ with $u|_{\Omega_j} \in W^{3/2,\infty}(\Omega_j)$ for $j \in \{0,1\}$. Then it holds that*

$$\|u - i_k u\|_{W^{1,\infty}(\Omega)} \leq c k^{1/2} \left(\|u\|_{W^{1,\infty}(\Omega)} + \|u\|_{W^{3/2,\infty}(\Omega_0)} + \|u\|_{W^{3/2,\infty}(\Omega_1)} \right).$$

Proof. This lemma can be proven using the same ideas as presented in the proof of Lemma 3.4.55. □

Within the following lemma we are going to estimate the error between the continuous transformation and its discrete counterpart in $W^{1,p}$ for some p, the error in L^p is estimated within Lemma 3.4.84.

Lemma 3.4.57. *Let $q \in Q^{\text{ad}}$, $F = F(q)$, $F_k = F_k(q)$ and $p \in [4, \infty]$. Then it holds that*

$$\|F - F_k\|_{H^1(\Omega)} \leq ck \|q\|_{H^2(I)}, \tag{3.142}$$

and

$$\|F - F_k\|_{W^{1,p}(\Omega)} \leq c_p k^{2/p} \|q\|_{H^2(I)}. \tag{3.143}$$

Proof. Let $\tilde{F} \in W^{2,4}(\Omega)$ be a $W^{2,4}$-stable extension of $F|_{\Omega_1}$ onto Ω, the extension of such a $\tilde{F}$ follows with Lemma A.2.2. Using Lemma A.2.22 and Theorem A.2.1 it follows that

$$\left\|\tilde{F} - F_k\right\|_{H^1(\Omega_{1,k})} \leq ck \|q\|_{H^2(I)} \qquad \text{and} \qquad \|F - F_k\|_{H^1(\Omega_{0,k})} \leq ck \|q\|_{H^2(I)},$$

it remains to estimate $\left\|F - \tilde{F}\right\|_{H^1(\Omega_0 \setminus \Omega_{0,k})}$. Hölder's inequality and the fact that $|\Omega_0 \setminus \Omega_{0,k}| \leq ck^2$ prove

$$\|F\|_{H^1(\Omega_0 \setminus \Omega_{0,k})} \leq ck \|F\|_{W^{1,\infty}(\Omega_0)} \leq ck \|q\|_{H^2(I)},$$

and

$$\left\|\tilde{F}\right\|_{H^1(\Omega_0 \setminus \Omega_{0,k})} \leq ck \left\|\tilde{F}\right\|_{W^{1,\infty}(\Omega_0)} \leq ck \left\|\tilde{F}\right\|_{W^{2,4}(\Omega_0)} \leq ck \|q\|_{H^2(I)},$$

which proves (3.142). Using the same notation and references as before it holds that

$$\left\| \tilde{F} - F_k \right\|_{W^{1,4}(\Omega_{1,k})} \leq c \left| \ln k \right|^{1/2} k \left\| q \right\|_{H^2(I)} \qquad \text{and} \qquad \left\| F - F_k \right\|_{W^{1,4}(\Omega_{0,k})} \leq ck \left\| q \right\|_{H^2(I)},$$

and again it remains to estimate $\left\| F - \tilde{F} \right\|_{W^{1,4}(\Omega_0 \setminus \Omega_{0,k})}$. We have

$$\left\| F \right\|_{W^{1,4}(\Omega_0 \setminus \Omega_{0,k})} \leq ck^{1/2} \left\| F \right\|_{W^{1,\infty}(\Omega_0)} \leq ck^{1/2} \left\| q \right\|_{H^2(I)},$$

and

$$\left\| \tilde{F} \right\|_{W^{1,4}(\Omega_0 \setminus \Omega_{0,k})} \leq ck^{1/2} \left\| \tilde{F} \right\|_{W^{1,\infty}(\Omega_0)} \leq ck^{1/2} \left\| q \right\|_{H^2(I)}.$$

As $\left| \ln k \right|^{1/2} k \leq k^{1/2}$ for k sufficiently small, the proof for $p = 4$ in (3.143) is finished. The proof for $p = \infty$ is similar to the second part of the proof of Lemma 2.3.29, it holds that

$$\left\| F - F_k \right\|_{W^{1,\infty}(\Omega)} \leq \left\| F - i_k F \right\|_{W^{1,\infty}(\Omega)} + \left\| i_k F - F_k \right\|_{W^{1,\infty}(\Omega)}. \tag{3.144}$$

For the first part on the right hand side of (3.144) we use Lemma 3.4.56, for the second part we use an inverse estimate and get

$$\begin{aligned} \left\| F - F_k \right\|_{W^{1,\infty}(\Omega)} &\leq ck^{1/2} \left(\left\| F \right\|_{W^{1,\infty}(\Omega)} + \left\| F \right\|_{W^{3/2,\infty}(\Omega_0)} + \left\| F \right\|_{W^{3/2,\infty}(\Omega_1)} \right) \\ &\quad + ck^{-1/2} \left\| i_k F - F_k \right\|_{W^{1,4}(\Omega_0)} \\ &\leq ck^{1/2} \left\| q \right\|_{H^2(I)} + ck^{-1/2} \left(\left\| F - i_k F \right\|_{W^{1,4}(\Omega)} + \left\| F - F_k \right\|_{W^{1,4}(\Omega)} \right), \end{aligned}$$

and now we use Lemma 3.4.55 and the first part of this Lemma to finally obtain

$$\begin{aligned} \left\| F - F_k \right\|_{W^{1,\infty}(\Omega)} &\leq ck^{1/2} \left\| q \right\|_{H^2(I)} + ck^{-1/2} k \left(\left\| F \right\|_{W^{1,4}(\Omega)} + \left\| F \right\|_{W^{2,4}(\Omega_0)} + \left\| F \right\|_{W^{2,4}(\Omega_1)} \right) \\ &\quad + ck^{-1/2} k^{1/2} \left\| q \right\|_{H^2(I)} \\ &\leq c \left\| q \right\|_{H^2(I)}, \end{aligned}$$

and what is left follows with interpolation. □

Assumption 3.4.58. We assume that there exist $c_1, c_2, c_3 > 0$ such that for all $q \in Q^{\text{ad}}$ it holds that $\gamma_{F_k(q)} \geq c_3$ on Ω, and the eigenvalues of the matrix $A_{F_k(q)}$ are elements of the interval $[c_1, c_2]$.

Remark 3.4.59. With Lemma 3.4.57 it follows that Assumption 3.4.58 holds if the constant $\tilde{C}$ from (3.11) is sufficiently small.

Lemma 3.4.60. *Let $q \in Q^{\text{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\lambda_{i,h,k}(q) \leq c_i.$$

Proof. Let $F_k = F_k(q)$. With Assumption 3.4.58 it follows that there exist $c_1, c_2 > 0$, independent of q, such that for all $u \in H_0^1(\Omega)$ it holds that

$$\begin{aligned} a_h(F_k)(u,u) &\leq c_1 \left(\nabla u, \nabla u \right), \\ b(F_k)(u,u) &\geq c_2 \left(u, u \right). \end{aligned}$$

With (3.73) it now follows that

$$\begin{aligned}\lambda_{i,h,k}(q) &= \min_{E_h \in V_{h,0}^{(i)}} \max_{u_h \in E_h} \frac{a_h(F_k)(u_h, u_h)}{b(F_k)(u_h, u_h)} \\ &\leq \frac{c_1}{c_2} \min_{E_h \in V_{h,0}^{(i)}} \max_{u_h \in E_h} \frac{(\nabla u_h, \nabla u_h)}{(u_h, u_h)} \\ &\leq c_i,\end{aligned}$$

where we used the fact that the eigenvalues for the discrete Laplacian on Ω are bounded independently of h, cf. Lemma 3.4.35. □

Lemma 3.4.61. *Let $q \in Q^{\text{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\|S_{i,h,k}(q)\|_{H_0^1(\Omega)} \leq c_i.$$

Proof. This lemma follows with (3.4.4) and Lemma 3.4.60. □

Lemma 3.4.62. *Let $q \in Q^{\text{ad}}$, $F = F(q)$, $F_k = F_k(q)$, $p_1, p_2, p_3 \in [1, \infty]$ and $p_3 \geq 4$ such that $1/p_1 + 1/p_2 + 1/p_3 = 1$. If $v \in W^{1,p_1}(\Omega)$ and $w \in W^{1,p_2}(\Omega)$, then it holds that*

$$|a_h(F)(v, w) - a_h(F_k)(v, w)| \leq c_{p_3} k^{2/p_3} \|v\|_{W_0^{1,p_1}(\Omega)} \|w\|_{W_0^{1,p_3}(\Omega)}.$$

If $v \in L^{p_1}(\Omega)$ and $w \in L^{p_2}(\Omega)$, then it holds that

$$|b(F)(v, w) - b(F_k)(v, w)| \leq c_{p_3} k^{2/p_3} \|v\|_{L^{p_1}(\Omega)} \|w\|_{L^{p_2}(\Omega)}.$$

Proof. This lemma follows with Theorem A.1.7 and Lemma 3.4.57. □

Lemma 3.4.63. *Let $q \in Q^{\text{ad}}$, $F = F(q)$, $F_k = F_k(q)$ and $i \in \mathbb{N}$. Let*

$$\tilde{u} \in \left(\operatorname{span}\{S_j(q)\}_{j=1}^{i} \cup \operatorname{span}\{S_{j,h}(q)\}_{j=1}^{i}\right),$$

then it holds that

$$\begin{aligned}\frac{a_h(F_k)(\tilde{u}, \tilde{u})}{a_h(F)(\tilde{u}, \tilde{u})} &= 1 + \mathcal{O}_{i,\varepsilon}(k^{1-\varepsilon}), \\ \frac{b(F)(\tilde{u}, \tilde{u})}{b(F_k)(\tilde{u}, \tilde{u})} &= 1 + \mathcal{O}_i(k).\end{aligned}$$

Proof. The proof to this lemma is similar to the proof of Lemma 3.4.39 and follows with Lemma 3.4.57 and Lemma 3.4.46. □

Lemma 3.4.64. *Let $q \in Q^{\text{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\lambda_{i,h,k}(q) \leq \left(1 + c_{i,\varepsilon} k^{1-\varepsilon}\right) \lambda_{i,h}(q).$$

Proof. Let $F = F(q)$, $F_k = F_k(q)$, and let $E_h = \operatorname{span}\{S_{j,h}(q)\}_{j=1}^i$ be the space spanned by the first i discrete eigenspaces with respect to $a_h(F)(\cdot,\cdot)$. With (3.73) it holds that

$$\begin{aligned}
\lambda_{i,h,k}(q) &\le \max_{u_h\in E_h} \frac{a_h(F_k)(u_h,u_h)}{b(F_k)(u_h,u_h)} \\
&= \max_{u_h\in E_h} \left(\frac{a_h(F)(u_h,u_h)}{b(F)(u_h,u_h)} \frac{a_h(F_k)(u_h,u_h)}{a_h(F)(u_h,u_h)} \frac{b(F)(u_h,u_h)}{b(F_k)(u_h,u_h)} \right) \\
&\le (1 + c_{i,\varepsilon} k^{1-\varepsilon})(1 + c_i k) \max_{u_h\in E_h} \frac{a_h(F)(u_h,u_h)}{b(F)(u_h,u_h)} \\
&\le (1 + c_{i,\varepsilon} k^{1-\varepsilon})\, \lambda_{i,h}(q),
\end{aligned}$$

where we used (3.68) and Lemma 3.4.63. □

Lemma 3.4.65. *Let $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\lambda_{i,h}(q) \le \left(1 + c_{i,\varepsilon}(h + k^{1-\varepsilon})\right) \lambda_{i,h,k}(q).$$

Proof. Let $F = F(q)$ and $F_k = F_k(q)$, with (3.63) it follows that

$$\begin{aligned}
\lambda_i(q) &= \min_{E\in V^{(i)}} \max_{u\in E} \frac{a(F)(u,u)}{b(F)(u,u)} \\
&= \min_{E\in V^{(i)}} \max_{u\in E} \left(\frac{a(F)(u,u)}{a_h(F)(u,u)} \frac{a_h(F)(u,u)}{a_h(F_k)(u,u)} \frac{b(F_k)(u,u)}{b(F)(u,u)} \frac{a_h(F_k)(u,u)}{b(F_k)(u,u)} \right) \\
&\le (1 + c_i h)(1 + c_{i,\varepsilon} k^{1-\varepsilon})(1 + c_i k) \min_{E\in V^{(i)}} \max_{u\in E} \frac{a_h(F_k)(u,u)}{b(F_k)(u,u)} \\
&\le (1 + c_i h)(1 + c_{i,\varepsilon} k^{1-\varepsilon})(1 + c_i k) \min_{E_h\in V_{h,0}^{(i)}} \max_{u_h\in E_h} \frac{a_h(F_k)(u_h,u_h)}{b(F_k)(u_h,u_h)} \\
&= (1 + c_i h)(1 + c_{i,\varepsilon} k^{1-\varepsilon})(1 + c_i k)\lambda_{i,h,k}(q) \\
&\le \left(1 + c_{i,\varepsilon}(h + k^{1-\varepsilon})\right) \lambda_{i,h,k}(q),
\end{aligned}$$

and the result follows with Lemma 3.4.41 and Lemma 3.4.63. □

Corollary 3.4.66. *Let $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$|\lambda_{i,h}(q) - \lambda_{i,h,k}(q)| \le c_{i,\varepsilon} \left(h + k^{1-\varepsilon}\right).$$

Proof. This corollary is a direct consequence of Lemma 3.4.64 and Lemma 3.4.65. □

Lemma 3.4.67. *Let $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\|S_{i,h}(q) - S_{i,h,k}(q)\|_{L^2(\Omega)} \le c_{i,\varepsilon} k^{1/2-\varepsilon}.$$

Proof. The following proof is based on the same ideas as the proof of Lemma 3.4.44. Let $F = F(q)$, $F_k = F_k(q)$ and

$$\tilde{S}_{i,h,k}(q) = b(F_k)(S_{i,h}(q), S_{i,h,k}(q))\, S_{i,h,k}(q). \tag{3.145}$$

Using Definition 3.3.4 we get

$$\|S_{i,h}(q) - S_{i,h,k}(q)\|_{L^2_{b,k}(\Omega)} \leq \left\|S_{i,h}(q) - \tilde{S}_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)} + \left\|\tilde{S}_{i,h,k}(q) - S_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)}. \tag{3.146}$$

We start with the first term on the right hand side within (3.146), we have

$$S_{i,h}(q) - \tilde{S}_{i,h,k}(q) = \sum_{j\in\mathbb{N}\setminus\{i\}} \Big((S_{i,h}(q), S_{j,h,k}(q))_{L^2_{b,k}(\Omega)} \, S_{j,h,k}(q) \Big),$$

and

$$\left\|S_{i,h}(q) - \tilde{S}_{i,h,k}(q)\right\|^2_{L^2_{b,k}(\Omega)} = \sum_{j\in\mathbb{N}\setminus\{i\}} (S_{i,h}(q), S_{j,h,k}(q))^2_{L^2_{b,k}(\Omega)}. \tag{3.147}$$

For the summands within (3.147) one can show that

$$\begin{aligned} b(F_k)(S_{i,h}(q), S_{j,h,k}(q)) = {} & \frac{\lambda_{i,h}(q)}{\lambda_{j,h,k}(q) - \lambda_{i,h}(q)} \left(b(F)(S_{i,h}(q), S_{j,h,k}(q)) - b(F_k)(S_{i,h}, S_{j,h,k}(q))\right) \\ & + \frac{1}{\lambda_{j,h,k}(q) - \lambda_{i,h}(q)} \left(a_h(F_k)(S_{i,h}(q), S_{j,h,k}(q)) - a_h(F)(S_{i,h}(q), S_{j,h,k}(q))\right). \end{aligned}$$

Now let $\varphi_1 \in L^2(\Omega)$ and $\varphi_2 \in H^1_0(\Omega)$ be the unique solutions to

$$\begin{aligned} b(F_k)(\varphi_1, v_1) &= b(F)(S_{i,h}(q), v_1) - b(F_k)(S_{i,h}(q), v_1) && \forall v_1 \in L^2(\Omega), \\ a_h(F_k)(\varphi_2, v_2) &= a_h(F_k)(S_{i,h}(q), v_2) - a_h(F)(S_{i,h}(q), v_2) && \forall v_2 \in H^1_0(\Omega). \end{aligned}$$

It holds that

$$\begin{aligned} \|\varphi_1\|_{L^2(\Omega)} &\leq \|S_{i,h}(q)\,(\gamma_F - \gamma_{F_k})\|_{L^2(\Omega)} \leq c_i k, \\ \|\varphi_2\|_{H^1_0(\Omega)} &\leq \|(A_F - A_{F_k}) \cdot \nabla S_{i,h}(q)\|_{L^2(\Omega)} \leq c_{i,\varepsilon} k^{1/2-\varepsilon}, \end{aligned}$$

and

$$\begin{aligned} b(F_k)(S_{i,h}(q), S_{j,h,k}(q)) = {} & \frac{\lambda_{i,h}(q)}{\lambda_{j,h,k}(q) - \lambda_{i,h}(q)} b(F_k)(\varphi_1, S_{j,h,k}(q)) \\ & + \frac{1}{\lambda_{j,h,k}(q) - \lambda_{i,h}(q)} a_h(F_k)(\varphi_2, S_{j,h,k}(q)), \end{aligned}$$

hence

$$\begin{aligned} \left\|S_{i,h}(q) - \tilde{S}_{i,h,k}(q)\right\|^2_{L^2_{b,k}(\Omega)} &\leq c\left(\|\varphi_1\|^2_{L^2(\Omega)} + \|\varphi_2\|^2_{H^1_0(\Omega)}\right) \\ &\leq c_{i,\varepsilon} k^{1-\varepsilon}. \end{aligned} \tag{3.148}$$

In order to estimate the second term within (3.146) we will show that

$$\left\|\tilde{S}_{i,h,k}(q) - S_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)} \leq \left\|\tilde{S}_{i,h,k}(q) - S_{i,h}(q)\right\|_{L^2_{b,k}(\Omega)}, \tag{3.149}$$

for (3.149) and (3.148) would finish this proof. We have

$$S_{i,h,k}(q) - \tilde{S}_{i,h,k}(q) = (1 - b(F_k)(S_{i,h}(q), S_{i,h,k}(q)))\, S_{i,h,k}(q),$$

and

$$\begin{aligned} \|S_{i,h}(q)\|_{L^2_{b,k}(\Omega)} - \left\|S_{i,h}(q) - \tilde{S}_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)} &\leq \left\|\tilde{S}_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)} \\ &\leq \|S_{i,h}(q)\|_{L^2_{b,k}(\Omega)} + \left\|S_{i,h}(q) - \tilde{S}_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)}, \end{aligned}$$

i.e.

$$1 - \left\|S_{i,h}(q) - \tilde{S}_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)} \leq |b(F_k)(S_{i,h}(q), S_{i,h,k}(q))| \leq 1 + \left\|S_{i,h}(q) - \tilde{S}_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)},$$

which reads as

$$||b(F_k)(S_{i,h}(q), S_{i,h,k}(q))| - 1| \leq \left\|S_{i,h}(q) - \tilde{S}_{i,h,k}(q)\right\|_{L^2_{b,k}(\Omega)}.$$

With (3.74) it follows that

$$|b(F_k)(S_{i,h}(q), S_{i,h,k}(q))| = b(F_k)(S_{i,h}(q), S_{i,h,k}(q)),$$

which finishes the proof. □

The following five lemmata can be proven in an analog way to Lemma 3.4.67 and the corresponding lemmata in Subsubsection 3.4.1.2.

Lemma 3.4.68. *Let $q \in Q^{\mathrm{ad}}$ and $i \in \mathbb{N}$, then it holds that*

$$\|S_{i,h}(q) - S_{i,h,k}(q)\|_{H^1_0(\Omega)} \leq c_{i,\varepsilon}\left(h + k^{1/2-\varepsilon}\right).$$

Lemma 3.4.69. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Then it holds that*

$$\left|\lambda'_{i,h}(q)(\delta q) - \lambda'_{i,h,k}(q)(\delta q)\right| \leq c_{i,\varepsilon}\left(h + k^{1/2-\varepsilon}\right)\|\delta q\|_{H^2(I)}.$$

Lemma 3.4.70. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Then it holds that*

$$\left\|S'_{i,h}(q)(\delta q) - S'_{i,h,k}(q)(\delta q)\right\|_{H^1_0(\Omega)} \leq c_{i,\varepsilon}\left(h + k^{1/2-\varepsilon}\right)\|\delta q\|_{H^2(I)}.$$

Lemma 3.4.71. *Let $q \in Q^{\mathrm{ad}}$, $\delta q \in Q$ and $i \in \mathbb{N}$. Then it holds that*

$$\left|\lambda''_{i,h}(q)(\delta q, \delta q) - \lambda''_{i,h,k}(q)(\delta q, \delta q)\right| \leq c_{i,\varepsilon}\left(h + k^{1/2-\varepsilon}\right)\|\delta q\|^2_{H^2(I)}.$$

Lemma 3.4.72. *Let $q \in Q^{\mathrm{ad}}$ and $\delta q \in Q$. Then it holds that*

$$\begin{aligned} \left|j'_h(q)(\delta q) - j'_{h,k}(q)(\delta q)\right| &\leq c_\varepsilon\left(h + k^{1/2-\varepsilon}\right)\|\delta q\|_{H^2(I)}, \\ \left|j''_h(q)(\delta q, \delta q) - j''_{h,k}(q)(\delta q, \delta q)\right| &\leq c_\varepsilon\left(h + k^{1/2-\varepsilon}\right)\|\delta q\|^2_{H^2(I)}. \end{aligned}$$

In order to prove Theorem 3.4.1 we also need a stability estimate for the derivative of the fully discretized cost functional. The following lemmata are needed to prove Lemma 3.4.78.

Lemma 3.4.73. *Let $q, p \in Q^{\text{ad}}$ and $i \in \mathbb{N}$. Then it holds that*

$$|\lambda_{i,h,k}(q) - \lambda_{i,h,k}(p)| \le c_i \|q - p\|_{H^2(I)}.$$

Proof. This lemma can be proven in the same way as Lemma 3.4.13. □

Lemma 3.4.74. *Let $q, p \in Q^{\text{ad}}$ and $i \in \mathbb{N}$. For $\|q-p\|_{H^2(I)}$, h and k sufficiently small it holds that*

$$\|S_{i,h,k}(q) - S_{i,h,k}(p)\|_{L^2_b(\Omega)} < \|S_{i,h,k}(q) + S_{i,h,k}(p)\|_{L^2_b(\Omega)}.$$

Proof. Using the triangle inequality and Lemma 3.4.19 we get

$$\begin{aligned} \|S_i(q) + S_i(p)\|_{L^2_b(\Omega)} &\ge 2\|S_i(q)\|_{L^2_b(\Omega)} - \|S_i(q) - S_i(p)\|_{L^2_b(\Omega)} \\ &\ge 2 - c_i \|q-p\|_{H^2(I)} \\ &\ge 1, \end{aligned} \tag{3.150}$$

for $\|q-p\|_{H^2(I)} \le c_i^{-1}$. Using (3.150), Lemma 3.4.44 and Lemma 3.4.67 we get

$$\begin{aligned} \|S_{i,h,k}(q) + S_{i,h,k}(p)\|_{L^2_b(\Omega)} &\ge \|S_i(q) + S_i(p)\|_{L^2_b(\Omega)} - \|S_i(q) - S_{i,h,k}(q)\|_{L^2_b(\Omega)} - \|S_i(p) - S_{i,h,k}(p)\|_{L^2_b(\Omega)} \\ &\ge 1 - c_i \left(h^{1/2} + k^{1/4}\right) \\ &\ge 1/2, \end{aligned}$$

for h, k sufficiently small. It also holds that

$$\begin{aligned} \|S_{i,h,k}(q) - S_{i,h,k}(p)\|_{L^2_b(\Omega)} &\le \|S_i(q) - S_i(p)\|_{L^2_b(\Omega)} + \|S_i(q) - S_{i,h,k}(q)\|_{L^2_b(\Omega)} + \|S_i(p) - S_{i,h,k}(p)\|_{L^2_b(\Omega)} \\ &\le c_i \left(\|q-p\|_{H^2(I)} + h^{1/2} + k^{1/4}\right), \end{aligned}$$

which can be made arbitrarily small for $\|q-p\|_{H^2(I)}$, h and k sufficiently small, and the result follows. □

Lemma 3.4.75. *Let $q, p \in Q^{\text{ad}}$ and $i \in \mathbb{N}$. For $\|q-p\|_{H^2(I)}$, h and k sufficiently small it holds that*

$$\|S_{i,h,k}(q) - S_{i,h,k}(p)\|_{L^2(\Omega)} \le c_i \|q-p\|_{H^2(I)}.$$

Proof. Although the sign of $S_{i,h,k}$ is defined via (3.74) and cannot be chosen arbitrarily, it follows with Lemma 3.4.74 that

$$\|S_{i,h,k}(q) - S_{i,h,k}(p)\|_{L^2_b(\Omega)} = \min\left\{\|S_{i,h,k}(q) - S_{i,h,k}(p)\|_{L^2_b(\Omega)}, \|S_{i,h,k}(q) + S_{i,h,k}(p)\|_{L^2_b(\Omega)}\right\},$$

and this lemma can be proven in the same way as Lemma 3.4.19. □

Lemma 3.4.76. *Let $q, p \in Q^{\text{ad}}$ and $i \in \mathbb{N}$. For $\|q-p\|_{H^2(I)}$, h and k sufficiently small it holds that*

$$\|S_{i,h,k}(q) - S_{i,h,k}(p)\|_{H^1(\Omega)} \le c_i \|q-p\|_{H^2(I)}.$$

Proof. Using Lemma 3.4.75, this lemma can be proven analogously to Lemma 3.4.20. □

Lemma 3.4.77. *Let $q, p \in Q^{\text{ad}}$ with $\|q-p\|_{H^2(I)}$, h and k sufficiently small, let $\delta q \in Q$ and $i \in \mathbb{N}$, then it holds that*

$$\left|\lambda'_{i,h,k}(q)(\delta q) - \lambda'_{i,h,k}(p)(\delta q)\right| \le c_i \|q-p\|_{H^2(I)} \|\delta q\|_{H^2(I)}.$$

Proof. Using Lemma 3.4.73, Lemma 3.4.75, Lemma 3.4.76 and Lemma 3.4.57, this lemma can be proven in the same way as Lemma 3.4.22. □

Lemma 3.4.78. *Let $q, p \in Q^{\text{ad}}$ with $\|q-p\|_{H^2(I)}$, h and k sufficiently small and let $\delta q \in Q$. Then it holds that*

$$\left|j'_{h,k}(q)(\delta q) - j'_{h,k}(p)(\delta q)\right| \le c \|q-p\|_{H^2(I)} \|\delta q\|_{H^2(I)}.$$

Proof. This lemma is a direct consequence of Lemma 3.4.77. □

3.4.2. A-priori error estimates for the optimal control

Within this subsection we are going to estimate the error between the optimal solution and its fully discretized counterpart. We start be recalling some regularity results for the optimal solution.

Lemma 3.4.79. *Let $\overline{q} \in Q^{\text{ad}}$ be an optimal control to (3.21) with corresponding transformation $\overline{F} = F(\overline{q})$ and eigenfunctions $(\overline{u}_i = u_i(\overline{q}))_{i\in\mathbb{N}}$. Then it holds that $\overline{q} \in H^4(I)$, $\overline{F}|_{\Omega_j} \in W^{2,\infty}(\Omega_j)$ for $j \in \{0,1\}$ and $\overline{u}_i|_{\Omega_j} \in W^{2,\infty}(\Omega_j)$ for $i \in \mathbb{N}$ and $j \in \{0,1\}$.*

Proof. This lemma is a direct consequence of Lemma 3.1.40, Lemma 3.1.34 and Lemma 3.1.38. □

Throughout this subsection let $\overline{q}$ be a fixed optimal control, and let $\overline{F} = F(\overline{q})$ and $\overline{F}_k = F_k(\overline{q})$ be the (discrete) optimal transformation.

In order to prove higher order of convergence with respect to the discretization of the state we will use a duality argument and thus need piecewise H^2-regularity of general solutions of the transmission problem with right hand side in $L^2(\Omega)$.

Definition 3.4.80. Let

$$H^2_{\text{pw}}(\Omega) = \left\{ u \in H^1_0(\Omega) \middle|\, u|_{\Omega_j} \in H^2(\Omega_j) \text{ for } j \in \{0,1\} \right\},$$

be the space of H^1_0-functions which are H^2-regular on each subdomain, equipped with the norm

$$\|u\|^2_{H^2_{\text{pw}}(\Omega)} = \|u\|^2_{H^1_0(\Omega)} + \|u\|^2_{H^2(\Omega_0)} + \|u\|^2_{H^2(\Omega_1)}.$$

Lemma 3.4.81. *Let $v \in H^2_{\text{pw}}(\Omega)$ and let $i_h \colon C(\overline{\Omega}) \to V_h$ be the nodal interpolation. Then it holds that*

$$\|v - i_h v\|_{L^2(\Omega)} + h \|\nabla(v - i_h v)\|_{L^2(\Omega)} \le c\, |\ln h|^{1/2} h^2 \|v\|_{H^2_{\text{pw}}(\Omega)}.$$

Proof. This lemma can be found in [32], Lemma 2.1. □

Theorem 3.4.82. *Let $\Omega' \subset \mathbb{R}^2$ be an open, bounded, convex and polygonal domain and let $\Omega_0' \subset \Omega'$ be open with C^2 boundary $\Gamma_0' \subset \Omega'$, and let $\Omega_1' = \Omega' \backslash \overline{\Omega_0'}$. Let $\alpha_0, \alpha_1 > 0$ and let $f \in L^2(\Omega')$. Then there exists a unique $\varphi \in H_0^1(\Omega')$ as the solution to*

$$(\nabla\varphi, \alpha_0 \nabla v)_{\Omega_0'} + (\nabla\varphi, \alpha_1 \nabla v)_{\Omega_1'} = (f, v)_{\Omega'} \qquad \forall v \in H_0^1(\Omega'),$$

and it holds the additional regularity $\varphi \in H^2_{\mathrm{pw}}(\Omega')$. Furthermore, there exists $c > 0$, independent of f and φ, such that

$$\|\varphi\|_{H^2_{\mathrm{pw}}(\Omega')} \leq c \|f\|_{L^2(\Omega')}.$$

Proof. This theorem can be found in [32], Theorem 2.1. Some more general results can also be found in [62], [70] and [23]. □

Lemma 3.4.83. *Let $f \in L^2(\Omega)$ and let $\varphi \in H_0^1(\Omega)$ be the unique solution to*

$$a(\overline{F})(\varphi, v) = \left(f, v\gamma_{\overline{F}}\right) \qquad \forall v \in H_0^1(\Omega).$$

Then it holds that $\varphi \in H^2_{\mathrm{pw}}(\Omega)$ and there exists $c > 0$, independent of f and φ, such that

$$\|\varphi\|_{H^2_{\mathrm{pw}}(\Omega)} \leq c \|f\|_{L^2(\Omega)}.$$

Proof. This lemma follows with Theorem 3.4.82 and a transformation argument. □

The following estimates will be needed in order to prove higher order of convergence with respect to the discretization of the transformation.

Lemma 3.4.84. *Let $q \in Q \cap W^{2,p}(I)$ for some $p \in [2,4]$, let $F = F(q)$ and $F_k = F_k(q)$. Then it holds that*

$$\|F - F_k\|_{L^p(\Omega)} \leq c_p |\ln k|^{1/2} k^2 \|q\|_{W^{2,p}(I)}.$$

Proof. As in the proof of Lemma 3.4.57, let $\tilde{F} \in W^{2,4}(\Omega)$ be a $W^{2,4}$-stable extension of $F|_{\Omega_1}$ onto Ω. With Lemma A.2.22 and Theorem A.2.1 we get

$$\left\|\tilde{F} - F_k\right\|_{L^p(\Omega_{1,k})} \leq c_p |\ln k|^{1/2} k^2 \|q\|_{W^{2,p}(I)} \quad \text{and} \quad \|F - F_k\|_{L^p(\Omega_{0,k})} \leq c_p k^2 \|q\|_{W^{2,p}(I)},$$

and it remains to estimate $\left\|F - \tilde{F}\right\|_{L^p(\Omega_0 \backslash \Omega_{0,k})}$. Let $x \in \Omega_0 \backslash \Omega_{0,k}$ be arbitrary, and let $\tilde{x} \in \Gamma_0$ such that the line from x to $\tilde{x}$ is orthogonal to $\Gamma_{0,k}$. Then it holds that $d(x, \tilde{x}) \leq ck^2$ and

$$\left(F - \tilde{F}\right)(x) = \left(F - \tilde{F}\right)(\tilde{x}) + \int_0^1 \left(F - \tilde{F}\right)'(x + t(\tilde{x} - x))(\tilde{x} - x)\,\mathrm{d}t.$$

Now we use the fact that $\left.\left(F - \tilde{F}\right)\right|_{\Gamma_0} = 0$, hence $\left(F - \tilde{F}\right)(\tilde{x}) = 0$, and get

$$\left|\left(F - \tilde{F}\right)(x)\right| \leq ck^2 \left(\|F\|_{W^{1,\infty}(\Omega_0)} + \left\|\tilde{F}\right\|_{W^{1,\infty}(\Omega_0)}\right).$$

Using Lemma 3.1.6 and the continuous embedding $W^{2,4}(\Omega_0) \hookrightarrow W^{1,\infty}(\Omega_0)$ we end up with

$$\left\|F - \tilde{F}\right\|_{L^\infty(\Omega_0 \backslash \Omega_{0,k})} \leq ck^2 \|q\|_{H^2(I)} \leq c_p k^2 \|q\|_{W^{2,p}(I)}.$$ □

Lemma 3.4.85. *Let $q \in W^{2,p}(I)$ with $p \in (1, \infty)$, $F = F(q)$ and $F_k = F_k(q)$. Then it holds that*

$$\begin{aligned} \|F - F_k\|_{L^p(\Gamma_{0,k})} &\leq c_p k^2 \|q\|_{W^{2,p}(I)}, \\ \|F - F_k\|_{L^p(\Gamma_0)} &\leq c_p k^2 \|q\|_{W^{2,p}(I)}, \\ \|F - F_k\|_{L^p(\Gamma_{0,h})} &\leq c_p \left(h^2 + k^2\right) \|q\|_{W^{2,p}(I)}. \end{aligned}$$

Proof. This lemma can be proven similar to Lemma 2.3.43, Lemma 2.3.44 and Lemma 2.3.45. Note that the estimate $\|F - F_k\|_{W^{1,\infty}(\Omega)} \leq c$ due to Lemma 3.4.57 is sufficient in order to prove these estimates. □

Lemma 3.4.86. *Let $v \in H^1(\Omega)$, then it holds that*

$$\left|\left(v, \gamma_{\overline{F}} - \gamma_{\overline{F}_k}\right)\right| \leq c \left(|\ln h|^{1/2} h^2 + |\ln k|^{1/2} k^2\right) \|v\|_{H^1(\Omega_0)}.$$

Proof. This lemma can be proven similar to Lemma 2.3.46, the terms corresponding to (2.84), (2.85) and (2.86) can be estimated using Lemma 3.4.84 and Lemma 3.4.85. The term corresponding to (2.87) can be estimated using Lemma 3.4.57 and Lemma A.1.15, including the usual choice of $p = |\ln k|$. □

Lemma 3.4.87. *Let $v \in H^1(\Omega)$, $\delta q \in Q$, $\delta F = F'(\overline{q})(\delta q)$ and $\delta F_k = F_k'(\overline{q})(\delta q)$. Then it holds that*

$$\left|\left(v, \gamma'_{\overline{F},\delta F} - \gamma'_{\overline{F}_k,\delta F_k}\right)\right| \leq c \left(|\ln h|^{1/2} h^2 + |\ln k|^{1/2} k^2\right) \|\delta q\|_{H^2(I)} \|v\|_{H^1(\Omega_0)}.$$

Proof. Let $\overline{F} = \left(\overline{F}_1, \overline{F}_2\right)^T$, and analogously for $\overline{F}_k$, δF and δF_k. On each subdomain Ω_j for $j \in \{0, 1\}$ it holds that

$$\begin{aligned} &\gamma'_{\overline{F},\delta F} - \gamma'_{\overline{F}_k,\delta F_k} \\ &= \partial_x \left(\delta F_1 - \delta F_{k,1}\right) + \partial_y \left(\delta F_2 - \delta F_{k,2}\right) \\ &+ \partial_x \overline{F}_1\, \partial_y \left(\delta F_2 - \delta F_{k,2}\right) + \partial_x \left(\overline{F}_1 - \overline{F}_{k,1}\right) \partial_y \left(\delta F_{k,2} - \delta F_2\right) + \partial_x \left(\overline{F}_1 - \overline{F}_{k,1}\right) \partial_y \delta F_2 \\ &+ \partial_y \overline{F}_2\, \partial_x \left(\delta F_1 - \delta F_{k,1}\right) + \partial_y \left(\overline{F}_2 - \overline{F}_{k,2}\right) \partial_x \left(\delta F_{k,1} - \delta F_1\right) + \partial_y \left(\overline{F}_2 - \overline{F}_{k,2}\right) \partial_x \delta F_1 \\ &- \partial_x \overline{F}_2\, \partial_y \left(\delta F_1 - \delta F_{k,1}\right) - \partial_x \left(\overline{F}_2 - \overline{F}_{k,2}\right) \partial_y \left(\delta F_{k,1} - \delta F_1\right) - \partial_x \left(\overline{F}_2 - \overline{F}_{k,2}\right) \partial_y \delta F_1 \\ &- \partial_y \overline{F}_1\, \partial_x \left(\delta F_2 - \delta F_{k,2}\right) - \partial_y \left(\overline{F}_1 - \overline{F}_{k,1}\right) \partial_x \left(\delta F_{k,2} - \delta F_2\right) - \partial_y \left(\overline{F}_1 - \overline{F}_{k,1}\right) \partial_x \delta F_2, \end{aligned}$$

and this lemma can be proven similar to Lemma 3.4.86. □

Lemma 3.4.88. *Let $u, v \in H^2_{\mathrm{pw}}(\Omega)$, then it holds that*

$$\left|\left(\nabla u, \left(\mu A_{\overline{F}} - \mu_h A_{\overline{F}_k}\right) \cdot \nabla v\right)\right| \leq c \left(|\ln h|\, h^2 + |\ln k|\, k^2\right) \|u\|_{H^2_{\mathrm{pw}}(\Omega)} \|v\|_{H^2_{\mathrm{pw}}(\Omega)}.$$

Proof. It holds that

$$\begin{aligned} \left|\left(\nabla u, \left(\mu A_{\overline{F}} - \mu_h A_{\overline{F}_k}\right) \cdot \nabla v\right)\right| &\leq \left|\left(\nabla u, \mu \left(A_{\overline{F}} - A_{\overline{F}_k}\right) \cdot \nabla v\right)\right| && (3.151) \\ &+ \left|\left(\nabla u, \left(\mu - \mu_h\right) A_{\overline{F}_k} \cdot \nabla v\right)\right|. && (3.152) \end{aligned}$$

For the first part, (3.151), we get

$$\left|\left(\nabla u, \mu\left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot\nabla v\right)\right| \le d\left|\left(\nabla u, \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot\nabla v\right)_{\Omega_0\setminus\Omega_{0,h}}\right| \tag{3.153}$$

$$+ d\left|\left(\nabla u, \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot\nabla v\right)_{\Omega_{0,h}}\right| \tag{3.154}$$

$$+ \left|\left(\nabla u, \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot\nabla v\right)_{\Omega_1}\right|. \tag{3.155}$$

For (3.153) we use Lemma A.1.15 and get

$$\begin{aligned}\left|\left(\nabla u, \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot\nabla v\right)_{\Omega_0\setminus\Omega_{0,h}}\right| &\le \|u\|_{W^{1,p}(\Omega_0)}\|v\|_{W^{1,p}(\Omega_0)}\left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^\infty(\Omega_0)}\|1\|_{L^{p/(p-2)}(\Omega_0\setminus\Omega_{0,h})}\\ &\le cph^{2-4/p}\|u\|_{H^2_{\text{pw}}(\Omega)}\|v\|_{H^2_{\text{pw}}(\Omega)}\\ &\le c\left|\ln h\right| h^2\|u\|_{H^2_{\text{pw}}(\Omega)}\|v\|_{H^2_{\text{pw}}(\Omega)},\end{aligned}$$

where we choose $p = |\ln h|$. As (3.154) as well as (3.155) can be estimated in the same way, we will focus on the first one. Similar to Lemma 2.3.47 we set $\overline{\delta F} = \left(\overline{F}_k - \overline{F}\right)$ and use Taylor's theorem to get

$$\left(\nabla u, \left(A_{\overline{F}} - A_{\overline{F}_k}\right)\cdot\nabla v\right)_{\Omega_{0,h}} = \left(\nabla u, \left(A'_{\overline{F},\overline{\delta F}} + R_2(\overline{F},\overline{\delta F})\right)\cdot\nabla v\right)_{\Omega_{0,h}}.$$

With Lemma A.1.15 we get

$$\begin{aligned}\left|\left(\nabla u, R_2(\overline{F},\overline{\delta F})\cdot\nabla v\right)_{\Omega_{0,h}}\right| &\le \|u\|_{W^{1,p}(\Omega_0)}\|v\|_{W^{1,p}(\Omega_0)}\left\|R_2(\overline{F},\overline{\delta F})\right\|_{L^{p/(p-2)}(\Omega_{0,h})}\\ &\le cp\|u\|_{H^2(\Omega_{0,h})}\|v\|_{H^2(\Omega_{0,h})}\left\|\overline{F} - \overline{F}_k\right\|^2_{L^{2p/(p-2)}(\Omega_{0,h})}.\end{aligned} \tag{3.156}$$

Setting $p = |\ln k|$ and using Lemma 3.4.57 yields

$$\left|\left(\nabla u, R_2(\overline{F},\overline{\delta F})\cdot\nabla v\right)_{\Omega_{0,h}}\right| \le c\left|\ln k\right| k^2\|u\|_{H^2(\Omega)}\|v\|_{H^2(\Omega)}.$$

The estimation of the term $\left|\left(\nabla u, A'_{\overline{F},\overline{\delta F}}\cdot\nabla v\right)_{\Omega_0}\right|$ can be done as in the proof of Lemma 2.3.47, where we also have to use Lemma 3.4.85 and Lemma A.1.15. In order to estimate (3.152) we use Lemma 3.4.37 as well as Lemma A.1.15 and get

$$\begin{aligned}\left|\left(\nabla u, (\mu - \mu_h)A_{\overline{F}_k}\cdot\nabla v\right)\right| &\le c\|u\|_{W^{1,p}(\Omega)}\|\mu - \mu_h\|_{L^{\frac{p}{p-2}}(\Omega)}\|v\|_{W^{1,p}(\Omega)}\\ &\le cph^{2-4/p}\|u\|_{H^2_{\text{pw}}(\Omega)}\|v\|_{H^2_{\text{pw}}(\Omega)},\end{aligned}$$

and setting $p = |\ln h|$ finishes the proof. □

Lemma 3.4.89. *Let $u, v \in H^2_{\text{pw}}(\Omega)$, $\delta q \in Q$, $\delta F = F'(\overline{q})(\delta q)$ and $\delta F_k = F'_k(\overline{q})(\delta q)$. Then it holds that*

$$\left|\left(\nabla u, \left(\mu A'_{\overline{F},\delta F} - \mu_h A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla v\right)\right| \le c\left(\left|\ln h\right| h^2 + \left|\ln k\right| k^2\right)\|u\|_{H^2_{\text{pw}}(\Omega)}\|v\|_{H^2_{\text{pw}}(\Omega)}\|\delta q\|_{H^2(I)}.$$

Proof. This lemma can be proven similarly to Lemma 3.4.88. □

Lemma 3.4.90. *For $i \in \mathbb{N}$ and $\overline{u}_1, \overline{u}_2 \in \operatorname{span}\{S_j(\overline{q})\}_{j=1}^i$ it holds that*

$$\begin{aligned} \left|a(\overline{F})(\overline{u}_1,\overline{u}_2) - a_h(\overline{F})(\overline{u}_1,\overline{u}_2)\right| &\leq c_i h^2, \\ \left|a_h(\overline{F})(\overline{u}_1,\overline{u}_2) - a_h(\overline{F}_k)(\overline{u}_1,\overline{u}_2)\right| &\leq c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right). \end{aligned}$$

Proof. The first estimate is a direct consequence of Lemma 3.4.79 and Corollary 3.4.38, the second estimate follows with the first one and Lemma 3.4.88. □

Lemma 3.4.91. *Let $i \in \mathbb{N}$ and $\overline{u}_1, \overline{u}_2 \in \operatorname{span}\{S_j(\overline{q})\}_{j=1}^i$. Then it holds that*

$$\left|b(\overline{F})(\overline{u}_1,\overline{u}_2) - b(\overline{F}_k)(\overline{u}_1,\overline{u}_2)\right| \leq c_i \left(|\ln h|^{1/2} h^2 + |\ln k|^{1/2} k^2\right).$$

Proof. This lemma follows with Lemma 3.4.86. □

Next we will prove two approximation results including the two operators Π_h^o and Π_k from Definition 3.3.3 and Definition 3.3.6. A statement similar to the following lemma can also be found in [32], Theorem 2.2.

Lemma 3.4.92. *Let $v \in H^2_{\text{pw}}(\Omega)$ and let Π_h^o be as in Definition 3.3.3. For $s \in [0,1]$ it holds that*

$$\|v - \Pi_h^o v\|_{H^s(\Omega)} \leq c\, |\ln h|^{\frac{2-s}{2}}\, h^{2-s}\, \|v\|_{H^2_{\text{pw}}(\Omega)}.$$

Proof. Using Céa's lemma and Lemma 3.4.81 we get

$$\|v - \Pi_h^o v\|_{H^1_0(\Omega)} \leq c\, |\ln h|^{1/2}\, h\, \|v\|_{H^2_{\text{pw}}(\Omega)}.$$

Now let $z \in H^1_0(\Omega)$ be the weak solution to

$$a(\overline{F})(v,z) = (v - \Pi_h^o v, w) \qquad \forall w \in H^1_0(\Omega).$$

We get

$$\begin{aligned} \|v - \Pi_h^o v\|^2_{L^2(\Omega)} &= a(\overline{F})(v - \Pi_h^o v, z) \\ &= a(\overline{F})(v - \Pi_h^o v, z - i_h z) \\ &\leq c\, \|v - \Pi_h^o v\|_{H^1_0(\Omega)}\, \|z - i_h z\|_{H^1_0(\Omega)} \\ &\leq c\, |\ln h|^{1/2}\, h\, \|v\|_{H^2_{\text{pw}}(\Omega)}\, |\ln h|^{1/2}\, h\, \|z\|_{H^2_{\text{pw}}(\Omega)} \\ &\leq c\, |\ln h|\, h^2\, \|v - \Pi_h^o v\|_{L^2(\Omega)}\, \|v\|_{H^2_{\text{pw}}(\Omega)}, \end{aligned}$$

where we used Lemma 3.4.83 and Lemma 3.4.81. What is left follows with interpolation. □

Lemma 3.4.93. *Let $i \in \mathbb{N}$ and $s \in [0,1]$, then it holds that*

$$\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^s(\Omega)} \leq c_i \left(|\ln h|^{\frac{2-s}{2}}\, h^{2-s} + |\ln k|^{\frac{2-s}{2}}\, k^{2-s}\right).$$

Proof. Using the coercivity of $a_h(\overline{F}_k)(\cdot,\cdot)$ we get

$$\begin{aligned}
& c\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|^2_{H^1_0(\Omega)} \le a_h(\overline{F}_k)(S_i(\overline{q}) - \Pi_k S_i(\overline{q}), S_i(\overline{q}) - \Pi_k S_i(\overline{q})) \\
&= a_h(\overline{F}_k)(S_i(\overline{q}), S_i(\overline{q})) + a_h(\overline{F}_k)(\Pi_k S_i(\overline{q}), \Pi_k S_i(\overline{q})) - 2a_h(\overline{F}_k)(S_i(\overline{q}), \Pi_k S_i(\overline{q})) \\
&= a(\overline{F})(S_i(\overline{q}), \Pi_k S_i(\overline{q}) - S_i(\overline{q})) - a_h(\overline{F}_k)(S_i(\overline{q}), \Pi_k S_i(\overline{q}) - S_i(\overline{q})) && (3.157)\\
&+ a(\overline{F})(S_i(\overline{q}) - \Pi_h^o S_i(\overline{q}), S_i(\overline{q}) - \Pi_k S_i(\overline{q})) && (3.158)\\
&+ a(\overline{F})(S_i(\overline{q}) - \Pi_h^o S_i(\overline{q}), \Pi_k S_i(\overline{q}) - S_i(\overline{q})) - a_h(\overline{F}_k)(S_i(\overline{q}) - \Pi_h^o S_i(\overline{q}), \Pi_k S_i(\overline{q}) - S_i(\overline{q})) && (3.159)\\
&+ a(\overline{F})(S_i(\overline{q}) - \Pi_h^o S_i(\overline{q}), S_i(\overline{q})) - a_h(\overline{F}_k)(S_i(\overline{q}) - \Pi_h^o S_i(\overline{q}), S_i(\overline{q})), && (3.160)
\end{aligned}$$

and it remains to estimate each of these four terms. For (3.157) we get

$$\begin{aligned}
&\left|\left(\nabla S_i(\overline{q}), \left(\mu A_{\overline{F}} - \mu_h A_{\overline{F}_k}\right) \cdot \nabla\left(\Pi_k S_i(\overline{q}) - S_i(\overline{q})\right)\right)\right| \\
&\le c\,\|S_i(\overline{q})\|_{W^{1,\infty}(\Omega)} \left(\left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^2(\Omega)} + \|\mu - \mu_h\|_{L^2(\Omega)}\right) \|\Pi_k S_i(\overline{q}) - S_i(\overline{q})\|_{H^1_0(\Omega)} \\
&\le c_i\,(h+k)\,\|\Pi_k S_i(\overline{q}) - S_i(\overline{q})\|_{H^1_0(\Omega)},
\end{aligned}$$

where we used Lemma 3.4.37 and Lemma 3.4.57. For (3.158) we use Lemma 3.4.92 and obtain

$$\begin{aligned}
\left|a(\overline{F})(S_i(\overline{q}) - \Pi_h^o S_i(\overline{q}), S_i(\overline{q}) - \Pi_k S_i(\overline{q}))\right| &\le c\,\|S_i(\overline{q}) - \Pi_h^o S_i(\overline{q})\|_{H^1_0(\Omega)} \|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^1_0(\Omega)} \\
&\le c_i\,|\ln h|^{1/2}\, h\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^1_0(\Omega)}.
\end{aligned}$$

For (3.159) we get

$$\begin{aligned}
&\left|\left(\nabla\left(S_i(\overline{q}) - \Pi_h^o S_i(\overline{q})\right), \left(\mu A_{\overline{F}} - \mu_h A_{\overline{F}_k}\right) \cdot \nabla\left(S_i(\overline{q}) - \Pi_k S_i(\overline{q})\right)\right)\right| \\
&\le c\,\|S_i(\overline{q}) - \Pi_h^o S_i(\overline{q})\|_{H^1_0(\Omega)} \left(\left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^\infty(\Omega)} + \|\mu - \mu_h\|_{L^\infty(\Omega)}\right) \|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^1_0(\Omega)} \\
&\le c_i\,|\ln h|^{1/2}\, h\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^1_0(\Omega)}.
\end{aligned}$$

At last we estimate (3.160) via

$$\begin{aligned}
&\left|\left(\nabla\left(S_i(\overline{q}) - \Pi_h^o S_i(\overline{q})\right), \left(\mu A_{\overline{F}} - \mu_h A_{\overline{F}_k}\right) \cdot \nabla S_i(\overline{q})\right)\right| \\
&\le c\,\|S_i(\overline{q}) - \Pi_h^o S_i(\overline{q})\|_{H^1_0(\Omega)} \left(\left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^2(\Omega)} + \|\mu - \mu_h\|_{L^2(\Omega)}\right) \|S_i(\overline{q})\|_{W^{1,\infty}(\Omega)} \\
&\le c_i\,|\ln h|^{1/2}\, h\,(h+k),
\end{aligned}$$

and the case $s = 1$ follows with Young's inequality.

Now let $z \in H^1_0(\Omega)$ be the solution to

$$a(\overline{F})(v,z) = b(\overline{F})(S_i(\overline{q}) - \Pi_k S_i(\overline{q}), v) \qquad \forall v \in H^1_0(\Omega).$$

Due to Lemma 3.4.83 it holds that $z \in H^2_{\mathrm{pw}}(\Omega)$ and $\|z\|_{H^2_{\mathrm{pw}}(\Omega)} \leq c\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{L^2(\Omega)}$. We get

$$
\begin{aligned}
& c\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|^2_{L^2(\Omega)} \leq a(\overline{F})(S_i(\overline{q}) - \Pi_k S_i(\overline{q}), z) \\
& = a(\overline{F})(S_i(\overline{q}), z) - a(\overline{F})(\Pi_k S_i(\overline{q}), z) \\
& = a(\overline{F})(S_i(\overline{q}), \Pi^o_h z) - a(\overline{F})(\Pi_k S_i(\overline{q}), z) + a(\overline{F})(S_i(\overline{q}), z - \Pi^o_h z) \\
& = a_h(\overline{F}_k)(\Pi_k S_i(\overline{q}) - S_i(\overline{q}), \Pi^o_h z - z) - a(\overline{F})(\Pi_k S_i(\overline{q}) - S_i(\overline{q}), \Pi^o_h z - z) && (3.161)\\
& + a_h(\overline{F}_k)(\Pi_k S_i(\overline{q}) - S_i(\overline{q}), z) - a(\overline{F})(\Pi_k S_i(\overline{q}) - S_i(\overline{q}), z) && (3.162)\\
& + a(\overline{F})(S_i(\overline{q}) - \Pi_k S_i(\overline{q}), z - \Pi^o_h z) && (3.163)\\
& + a_h(\overline{F}_k)(S_i(\overline{q}), \Pi^o_h z - z) - a(\overline{F})(S_i(\overline{q}), \Pi^o_h z - z) && (3.164)\\
& + a(\overline{F}_k)(S_i(\overline{q}), z) - a(\overline{F})(S_i(\overline{q}), z). && (3.165)
\end{aligned}
$$

Again we have to estimate each of these five terms separately. For (3.161) we use the first part of this lemma and Lemma 3.4.92 and get

$$
\begin{aligned}
& \left|\left(\nabla\left(S_i(\overline{q}) - \Pi_k S_i(\overline{q})\right), \left(\mu A_{\overline{F}} - \mu_h A_{\overline{F}_k}\right) \cdot \nabla\left(z - \Pi^o_h z\right)\right)\right| \\
& \leq c\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^1_0(\Omega)} \left(\left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^\infty(\Omega)} + \|\mu - \mu_h\|_{L^\infty(\Omega)}\right) \|z - \Pi^o_h z\|_{H^1_0(\Omega)} \\
& \leq c_i\,|\ln h|^{1/2}\,h\left(|\ln h|^{1/2}\,h + |\ln k|^{1/2}\,k\right)\|z\|_{H^2_{\mathrm{pw}}(\Omega)} \\
& \leq c_i\left(|\ln h|\,h^2 + |\ln k|\,k^2\right)\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{L^2(\Omega)}\,.
\end{aligned}
$$

In order to estimate (3.162), let $p_0 = |\ln k|$, $p_1 = |\ln h|$ and choose q_j such that $p_j^{-1} + q_j^{-1} = 1/2$ for $j \in \{0, 1\}$. We obtain

$$
\begin{aligned}
& \left|\left(\nabla\left(S_i(\overline{q}) - \Pi_k S_i(\overline{q})\right), \left(\mu A_{\overline{F}} - \mu_h A_{\overline{F}_k}\right) \cdot \nabla z\right)\right| \\
& \leq c\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^1_0(\Omega)} \left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^{q_0}(\Omega)} \|z\|_{W^{1,p_0}(\Omega)} \\
& + c\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^1_0(\Omega)} \|\mu - \mu_h\|_{L^{q_1}(\Omega)} \|z\|_{W^{1,p_1}(\Omega)} \\
& \leq c_i\left(|\ln h|^{1/2}\,h + |\ln k|^{1/2}\,k\right)\left(|\ln h|^{1/2}\,h^{1-\frac{2}{|\ln h|}} + |\ln k|\,k^{1-\frac{2}{|\ln k|}}\right)\|z\|_{H^2_{\mathrm{pw}}(\Omega)} \\
& \leq c_i\left(|\ln h|\,h^2 + |\ln k|\,k^2\right)\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{L^2(\Omega)}\,.
\end{aligned}
$$

Expression (3.163) can be estimated via

$$
\begin{aligned}
\left|\left(\nabla\left(S_i(\overline{q}) - \Pi_k S_i(\overline{q})\right), \mu A_{\overline{F}} \cdot \nabla\left(z - \Pi^o_h z\right)\right)\right| & \leq c\,\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{H^1_0(\Omega)} \|z - \Pi^o_h z\|_{H^1_0(\Omega)} \\
& \leq c_i\left(|\ln h|^{1/2}\,h + |\ln k|^{1/2}\,k\right)|\ln h|^{1/2}\,h\,\|z\|_{H^2_{\mathrm{pw}}(\Omega)} \\
& \leq c_i\left(|\ln h|\,h^2 + |\ln k|\,k^2\right)\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{L^2(\Omega)}\,.
\end{aligned}
$$

For (3.164) we get

$$
\begin{aligned}
& \left|\left(\nabla S_i(\overline{q}), \left(\mu A_{\overline{F}} - \mu_h A_{\overline{F}_k}\right) \cdot \nabla\left(z - \Pi^o_h z\right)\right)\right| \\
& \leq c\,\|S_i(\overline{q})\|_{W^{1,\infty}(\Omega)} \left(\left\|A_{\overline{F}} - A_{\overline{F}_k}\right\|_{L^2(\Omega)} + \|\mu - \mu_h\|_{L^2(\Omega)}\right)\|z - \Pi^o_h z\|_{H^1_0(\Omega)} \\
& \leq c_i\,(h + k)\,|\ln h|^{1/2}\,h\,\|z\|_{H^2_{\mathrm{pw}}(\Omega)} \\
& \leq c_i\left(|\ln h|\,h^2 + |\ln k|\,k^2\right)\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{L^2(\Omega)}\,.
\end{aligned}
$$

Finally, for (3.165) we use Lemma 3.4.88 and get

$$\begin{aligned}\left|\left(\nabla S_i(\overline{q}),\left(\mu A_{\overline{F}}-\mu_h A_{\overline{F}_k}\right)\cdot\nabla z\right)\right| &\le c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right)\|z\|_{H^2_{\text{pw}}(\Omega)}\\ &\le c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right)\|S_i(\overline{q})-\Pi_k S_i(\overline{q})\|_{L^2(\Omega)}.\end{aligned}$$

Collecting these five estimates finishes the case $s=0$, and what is left follows with interpolation. □

Lemma 3.4.94. *For $i\in\mathbb{N}$ it holds that*

$$\lambda_i(\overline{q})\le\left(1+c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right)\right)\lambda_{i,h,k}(\overline{q}).$$

Proof. This lemma can be proven similar to Lemma 3.4.65, the higher rate of convergence follows with Lemma 3.4.90 and Lemma 3.4.91. □

Lemma 3.4.95. *For $i\in\mathbb{N}$ it holds that*

$$\lambda_{i,h,k}(\overline{q})\le\left(1+c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right)\right)\lambda_i(\overline{q}).$$

Proof. This proof is based on the same idea as the proof of Lemma 3.4.41. Let again $V^{(i)}=\bigoplus_{j=1}^i N_j$ be the space spanned by the first i eigenfunctions and let $N_k=\Pi_k V^{(i)}$ with Π_k as in Definition 3.3.6. For h and k sufficiently small it follows that $\dim(N_k)=i$, and (3.73) yields

$$\begin{aligned}\lambda_{i,h,k}(\overline{q})&\le\max_{u_h\in N_k}\frac{a_h(\overline{F}_k)(u_h,u_h)}{b(\overline{F}_k)(u_h,u_h)}=\max_{u\in V^{(i)}}\frac{a_h(\overline{F}_k)(\Pi_k u,\Pi_k u)}{b(\overline{F}_k)(\Pi_k u,\Pi_k u)}\\ &=\max_{u\in V^{(i)}}\frac{a(\overline{F})(u,\Pi_k u)}{b(\overline{F}_k)(\Pi_k u,\Pi_k u)}\\ &=\max_{u\in V^{(i)}}\left(\frac{a(\overline{F})(u,u)}{b(\overline{F})(u,u)}\frac{b(\overline{F})(u,u)}{b(\overline{F}_k)(\Pi_k u,\Pi_k u)}\frac{a(\overline{F})(u,\Pi_k u)}{a(\overline{F})(u,u)}\right),\end{aligned}\tag{3.166}$$

where in the second line we used the definition of Π_k. With Lemma 3.4.93 it follows that

$$\frac{b(\overline{F})(u,u)}{b(\overline{F}_k)(\Pi_k u,\Pi_k u)}\le 1+c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right).\tag{3.167}$$

Partial integration yields

$$a(\overline{F})(u,u-\Pi_k u)=-d\left(\operatorname{div}\left(A_{\overline{F}}\cdot\nabla u\right),u-\Pi_k u\right)_{\Omega_0}-\left(\operatorname{div}\left(A_{\overline{F}}\cdot\nabla u\right),u-\Pi_k u\right)_{\Omega_1},$$

where the boundary terms vanish due to the transformed version of (3.2). With Lemma 3.4.93 we get

$$\begin{aligned}\left|a(\overline{F})(u,u-\Pi_k u)\right|&\le c\,\|u-\Pi_k u\|_{L^2(\Omega)}\\ &\le c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right),\end{aligned}$$

and hence

$$\frac{a(\overline{F})(u,\Pi_k u)}{a(\overline{F})(u,u)}\le 1+c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right).\tag{3.168}$$

Inserting (3.167) and (3.168) into (3.166) finally yields

$$\begin{aligned}\lambda_{i,h,k}(\overline{q}) &\leq \left(1 + c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right)\right)^2 \max_{u\in V^{(i)}} \frac{a(\overline{F})(u,u)}{b(\overline{F})(u,u)} \\ &\leq \left(1 + c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right)\right) \lambda_i(\overline{q}). \qquad \square\end{aligned}$$

Corollary 3.4.96. *For $i \in \mathbb{N}$ it holds that*

$$|\lambda_i(\overline{q}) - \lambda_{i,h,k}(\overline{q})| \leq c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right).$$

Proof. This corollary is a direct consequence of Lemma 3.4.94 and Lemma 3.4.95. □

Lemma 3.4.97. *For $i \in \mathbb{N}$ it holds that*

$$|\lambda_i(\overline{q}) - \lambda_{i,h}(\overline{q})| \leq c_i\, |\ln h|\, h^2.$$

Proof. This lemma can be proven using the same ideas and methods as presented in the proofs of Lemma 3.4.94, Lemma 3.4.95 and Corollary 3.4.96. □

Our next goal is to estimate the error $\|S_i(\overline{q}) - S_{i,h}(\overline{q})\|_{L^2(\Omega)}$. The proof for this estimation is similar to the proof for the non-optimal case within Lemma 3.4.44. The following two lemmata are needed in order to show higher order of convergence in this optimal case.

Lemma 3.4.98. *Let Π_h be defined as in Definition 3.3.2, let $i \in \mathbb{N}$ and $s \in [0,1]$. Then it holds that*

$$\|S_i(\overline{q}) - \Pi_h S_i(\overline{q})\|_{H^s(\Omega)} \leq c_i\, |\ln h|^{\frac{2-s}{2}}\, h^{2-s}.$$

Proof. The case $s = 1$ is due to Céa's lemma, let $i_h\colon C(\overline{\Omega}) \to V_h$ be the nodal interpolation operator, then it holds that

$$\begin{aligned}c\,\|S_i(\overline{q}) - \Pi_h S_i(\overline{q})\|^2_{H^1_0(\Omega)} &\leq a_h(\overline{F})(S_i(\overline{q}) - \Pi_h S_i(\overline{q}), S_i(\overline{q}) - \Pi_h S_i(\overline{q})) \\ &= a_h(\overline{F})(S_i(\overline{q}) - \Pi_h S_i(\overline{q}), S_i(\overline{q}) - i_h S_i(\overline{q})) \\ &\leq c\,\|S_i(\overline{q}) - \Pi_h S_i(\overline{q})\|_{H^1_0(\Omega)}\, \|S_i(\overline{q}) - i_h S_i(\overline{q})\|_{H^1_0(\Omega)},\end{aligned}$$

and this part follows with Lemma 3.4.81. The case $s = 0$ uses a duality argument but needs some additional care due to the definition of Π_h. Let $z \in H^1_0(\Omega)$ be the solution to

$$a(\overline{F})(v,z) = (S_i(\overline{q}) - \Pi_h S_i(\overline{q}), v) \qquad \forall v \in H^1_0(\Omega).$$

With Lemma 3.4.83 it follows that $z \in H^2_{\mathrm{pw}}(\Omega)$ and $\|z\|_{H^2_{\mathrm{pw}}(\Omega)} \leq c\,\|S_i(\overline{q}) - \Pi_h S_i(\overline{q})\|_{L^2(\Omega)}$. We get

$$\begin{aligned}\|S_i(\overline{q}) - \Pi_h S_i(\overline{q})\|^2_{L^2(\Omega)} &= a(\overline{F})(S_i(\overline{q}) - \Pi_h S_i(\overline{q}), z) \\ &= a_h(\overline{F})(S_i(\overline{q}) - \Pi_h S_i(\overline{q}), z) \\ &\quad + \left(\nabla\left(S_i(\overline{q}) - \Pi_h S_i(\overline{q})\right), (\mu - \mu_h)\, A_{\overline{F}} \cdot \nabla z\right) \\ &= a_h(\overline{F})(S_i(\overline{q}) - \Pi_h S_i(\overline{q}), z - i_h z) \qquad (3.169) \\ &\quad + \left(\nabla\left(S_i(\overline{q}) - \Pi_h S_i(\overline{q})\right), (\mu - \mu_h)\, A_{\overline{F}} \cdot \nabla z\right). \qquad (3.170)\end{aligned}$$

We use Lemma 3.4.81 and the first part of this lemma to estimate (3.169),

$$\begin{aligned} a_h(\overline{F})(S_i(\overline{q}) - \Pi_h S_i(\overline{q}), z - i_h z) &\leq c \left\| S_i(\overline{q}) - \Pi_h S_i(\overline{q}) \right\|_{H_0^1(\Omega)} \left\| z - i_h z \right\|_{H_0^1(\Omega)} \\ &\leq c_i \left| \ln h \right|^{1/2} h \left| \ln h \right|^{1/2} h \left\| z \right\|_{H^2_{\mathrm{pw}}(\Omega)} \\ &\leq c_i \left| \ln h \right| h^2 \left\| S_i(\overline{q}) - \Pi_h S_i(\overline{q}) \right\|_{L^2(\Omega)} . \end{aligned}$$

In order to estimate (3.170) we use Lemma 3.4.37 and Lemma A.1.15,

$$\begin{aligned} &\left(\nabla \left(S_i(\overline{q}) - \Pi_h S_i(\overline{q})\right), (\mu - \mu_h) A_{\overline{F}} \cdot \nabla z\right) \\ &\leq c \left\| S_i(\overline{q}) - \Pi_h S_i(\overline{q}) \right\|_{H_0^1(\Omega)} \left\| \mu - \mu_h \right\|_{L^{\frac{2p}{p-2}}(\Omega)} \left\| A_{\overline{F}} \right\|_{L^\infty(\Omega_0)} \left\| z \right\|_{W^{1,p}(\Omega_0)} \\ &\leq c_i \left| \ln h \right|^{1/2} h h^{1-2/p} p^{1/2} \left\| z \right\|_{H^2(\Omega_0)} , \end{aligned}$$

and by setting $p = |\ln h|$ we arrive at

$$\begin{aligned} \left(\nabla \left(S_i(\overline{q}) - \Pi_h S_i(\overline{q})\right), (\mu - \mu_h) A_{\overline{F}} \cdot \nabla z\right) &\leq c_i \left| \ln h \right| h^2 \left\| z \right\|_{H^2_{\mathrm{pw}}(\Omega)} \\ &\leq c_i \left| \ln h \right| h^2 \left\| S_i(\overline{q}) - \Pi_h S_i(\overline{q}) \right\|_{L^2(\Omega)} , \end{aligned}$$

which finishes the case $s = 0$, and what is left follows with interpolation. □

Lemma 3.4.99. *Let $i \in \mathbb{N}$ and let $v_h \in V_{h,0}$ be the solution to*

$$\left(\nabla v_h, \mu_h A_{\overline{F}} \cdot \nabla \varphi_h\right) = \left(\nabla S_i(\overline{q}), (\mu_h - \mu) A_{\overline{F}} \cdot \nabla \varphi_h\right) \qquad \forall \varphi_h \in V_{h,0}. \tag{3.171}$$

Then it holds that

$$\left\| v_h \right\|_{L^2(\Omega)} + \left| \ln h \right|^{1/2} h \left\| v_h \right\|_{H_0^1(\Omega)} \leq c_i \left| \ln h \right|^{1/2} h^2.$$

Proof. Using (3.171) and Lemma 3.4.37 it follows that

$$\begin{aligned} \left\| v_h \right\|^2_{H_0^1(\Omega)} &\leq c \left(\nabla S_i(\overline{q}), (\mu_h - \mu) A_{\overline{F}} \cdot \nabla v_h\right) \\ &\leq c \left\| S_i(\overline{q}) \right\|_{W^{1,\infty}(\Omega_0)} \left\| \mu - \mu_h \right\|_{L^2(\Omega)} \left\| A_{\overline{F}} \right\|_{L^\infty(\Omega_0)} \left\| v_h \right\|_{H_0^1(\Omega)} \\ &\leq c_i h \left\| v_h \right\|_{H_0^1(\Omega)} , \end{aligned}$$

hence

$$\left\| v_h \right\|_{H_0^1(\Omega)} \leq c_i h. \tag{3.172}$$

Now let $z \in H_0^1(\Omega)$ be the solution to

$$a(\overline{F})(\varphi, z) = (v_h, \varphi) \qquad \forall \varphi \in H_0^1(\Omega).$$

With Lemma 3.4.83 it follows that $z \in H^2_{\mathrm{pw}}(\Omega)$ and $\|z\|_{H^2_{\mathrm{pw}}(\Omega)} \leq c \|v_h\|_{L^2(\Omega)}$, we get

$$\begin{aligned} \left\| v_h \right\|^2_{L^2(\Omega)} &= a(\overline{F})(v_h, z) \\ &= a_h(\overline{F})(v_h, z) + \left(\nabla v_h, (\mu - \mu_h) A_{\overline{F}} \cdot \nabla z\right) . \end{aligned} \tag{3.173}$$

For the first term on the right hand side of (3.173) we get

$$a_h(\overline{F})(v_h, z) = a_h(\overline{F})(v_h, z - i_h z) + a_h(\overline{F})(v_h, i_h z). \tag{3.174}$$

We use (3.172) and Lemma 3.4.81 and get

$$\begin{aligned} a_h(\overline{F})(v_h, z - i_h z) &\le c \|v_h\|_{H_0^1(\Omega)} \|z - i_h z\|_{H_0^1(\Omega)} \\ &\le c_i |\ln h|^{1/2} h^2 \|z\|_{H^2_{\mathrm{pw}}(\Omega)} \\ &\le c_i |\ln h|^{1/2} h^2 \|v_h\|_{L^2(\Omega)} . \end{aligned}$$

For the remaining term within (3.174) we get

$$\begin{aligned} &a_h(\overline{F})(v_h, i_h z) = \big(\nabla S_i(\overline{q}), (\mu_h - \mu)\, A_{\overline{F}} \cdot \nabla i_h z\big) \\ &= \big(\nabla S_i(\overline{q}), (\mu - \mu_h)\, A_{\overline{F}} \cdot \nabla (z - i_h z)\big) + \big(\nabla S_i(\overline{q}), (\mu_h - \mu)\, A_{\overline{F}} \cdot \nabla z\big) \\ &\le \|S_i(\overline{q})\|_{W^{1,\infty}(\Omega_0)} \left\|A_{\overline{F}}\right\|_{L^\infty(\Omega_0)} \left(\|\mu - \mu_h\|_{L^2(\Omega)} \|z - i_h z\|_{H_0^1(\Omega)} + \|\mu - \mu_h\|_{L^{\frac{p}{p-1}}(\Omega)} \|z\|_{W^{1,p}(\Omega_0)} \right) \\ &\le c \left(|\ln h|^{1/2} h^2 + h^{2-2/p} p^{1/2} \right) \|z\|_{H^2_{\mathrm{pw}}(\Omega)} \\ &\le c |\ln h|^{1/2} h^2 \|v_h\|_{L^2(\Omega)} . \end{aligned}$$

Finally, the second term on the right hand side of (3.173) can be estimated via

$$\begin{aligned} \left|\big(\nabla v_h, (\mu - \mu_h)\, A_{\overline{F}} \cdot \nabla z\big)\right| &\le \|v_h\|_{H_0^1(\Omega)} \|\mu - \mu_h\|_{L^{\frac{2p}{p-2}}(\Omega)} \left\|A_{\overline{F}}\right\|_{L^\infty(\Omega)} \|z\|_{W^{1,p}(\Omega_0)} \\ &\le c \|v_h\|_{H_0^1(\Omega)} h^{1-2/p} p^{1/2} \|z\|_{H^2(\Omega_0)} \\ &\le c |\ln h|^{1/2} h \|v_h\|_{H_0^1(\Omega)} \|v_h\|_{L^2(\Omega)} \\ &\le c |\ln h|^{1/2} h^2 \|v_h\|_{L^2(\Omega)} , \end{aligned}$$

and inserting these estimates into (3.173) finishes the proof. □

Lemma 3.4.100. *For $i \in \mathbb{N}$ it holds that*

$$\|S_i(\overline{q}) - S_{i,h}(\overline{q})\|_{L^2(\Omega)} \le c_i |\ln h|\, h^2.$$

Proof. This lemma can be shown in the same way as Lemma 3.4.44, the higher order of convergence follows with Lemma 3.4.98 and Lemma 3.4.99. □

Lemma 3.4.101. *For $i \in \mathbb{N}$ it holds that*

$$\|S_i(\overline{q}) - S_{i,h}(\overline{q})\|_{H^1(\Omega)} \le c_i |\ln h|^{1/2} h.$$

Proof. It holds that

$$\begin{aligned} &c \|S_i(\overline{q}) - S_{i,h}(\overline{q})\|^2_{H_0^1(\Omega)} \le a_h(\overline{F})(S_i(\overline{q}) - S_{i,h}(\overline{q}), S_i(\overline{q}) - S_{i,h}(\overline{q})) \\ &= a_h(\overline{F})(S_i(\overline{q}), S_i(\overline{q})) - 2a_h(\overline{F})(S_i(\overline{q}), S_{i,h}(\overline{q})) + a_h(\overline{F})(S_{i,h}(\overline{q}), S_{i,h}(\overline{q})) \\ &= a(\overline{F})(S_i(\overline{q}), S_i(\overline{q})) + \big(\nabla S_i(\overline{q}), (\mu_h - \mu)\, A_{\overline{F}} \cdot \nabla S_i(\overline{q})\big) - 2a_h(\overline{F})(\Pi_h S_i(\overline{q}), S_{i,h}(\overline{q})) \\ &+ a_h(\overline{F})(S_{i,h}(\overline{q}), S_{i,h}(\overline{q})) \\ &\le \lambda_i(\overline{q}) + c_i h^2 - 2\lambda_{i,h}(\overline{q})\, b(\overline{F})(\Pi_h S_i(\overline{q}), S_{i,h}(\overline{q})) + \lambda_{i,h}(\overline{q}) \\ &= \lambda_i(\overline{q}) - \lambda_{i,h}(\overline{q}) - 2\lambda_{i,h}(\overline{q})\, b(\overline{F})(\Pi_h S_i(\overline{q}) - S_{i,h}(\overline{q}), S_{i,h}(\overline{q})) + c_i h^2 \\ &\le |\lambda_i(\overline{q}) - \lambda_{i,h}(\overline{q})| + 2c_i \left(\|\Pi_h S_i(\overline{q}) - S_i(\overline{q})\|_{L^2(\Omega)} + \|S_i(\overline{q}) - S_{i,h}(\overline{q})\|_{L^2(\Omega)} \right) + c_i h^2, \end{aligned}$$

and we finish the proof using Lemma 3.4.97, Lemma 3.4.98 and Lemma 3.4.100. □

Lemma 3.4.102. *For $i \in \mathbb{N}$ it holds that*

$$\|S_i(\overline{q}) - S_{i,h,k}(\overline{q})\|_{L^2(\Omega)} \leq c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right).$$

Proof. This proof is similar to the proof of Lemma 3.4.44. Let Π_k be defined as in Definition 3.3.6 and let

$$\tilde{S}_{i,h,k}(\overline{q}) = b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{i,h,k}(\overline{q}))\, S_{i,h,k}(\overline{q}),$$

we have

$$\begin{aligned} \|S_i(\overline{q}) - S_{i,h,k}(\overline{q})\|_{L^2_{b,k}(\Omega)} \leq \|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{L^2_{b,k}(\Omega)} + \left\|\Pi_k S_i(\overline{q}) - \tilde{S}_{i,h,j}(\overline{q})\right\|_{L^2_{b,k}(\Omega)} \\ + \left\|\tilde{S}_{i,h,k}(\overline{q}) - S_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)}. \end{aligned} \tag{3.175}$$

For the first term on the right hand side of (3.175) we use Lemma 3.4.93 and get

$$\|S_i(\overline{q}) - \Pi_k S_i(\overline{q})\|_{L^2_{b,k}(\Omega)} \leq c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right). \tag{3.176}$$

We now concentrate on the second term on the right hand side of (3.175). It holds that

$$\Pi_k S_i(\overline{q}) - \tilde{S}_{i,h,j}(\overline{q}) = \sum_{j \in \mathbb{N}\setminus\{i\}} \left(b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{j,h,k}(\overline{q}))\, S_{j,h,k}(\overline{q})\right),$$

hence

$$\left\|\Pi_k S_i(\overline{q}) - \tilde{S}_{i,h,j}(\overline{q})\right\|^2_{L^2_{b,k}(\Omega)} = \sum_{j \in \mathbb{N}\setminus\{i\}} \left|b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{j,h,k}(\overline{q}))\right|^2. \tag{3.177}$$

For the summands within (3.177) it holds that

$$\begin{aligned} b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{j,h,k}(\overline{q})) &= \frac{1}{\lambda_{j,h,k}(\overline{q})} a_h(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{j,h,k}(\overline{q})) = \frac{1}{\lambda_{j,h,k}(\overline{q})} a(\overline{F})(S_i(\overline{q}), S_{j,h,k}(\overline{q})) \\ &= \frac{\lambda_i(\overline{q})}{\lambda_{j,h,k}(\overline{q})} b(\overline{F})(S_i(\overline{q}), S_{j,h,k}(\overline{q})), \end{aligned}$$

hence

$$\lambda_{j,h,k}(\overline{q})\, b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{j,h,k}(\overline{q})) = \lambda_i(\overline{q})\, b(\overline{F})(S_i(\overline{q}), S_{j,h,k}(\overline{q})),$$

which yields

$$b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{j,h,k}(\overline{q}) = \frac{\lambda_i(\overline{q})}{\lambda_{j,h,k}(\overline{q}) - \lambda_i(\overline{q})} \left(b(\overline{F})(S_i(\overline{q}), S_{j,h,k}(\overline{q})) - b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{j,h,k}(\overline{q}))\right).$$

Using Lemma 3.4.86 and Lemma 3.4.93 one can show that

$$\left\|\Pi_k S_i(\overline{q}) - \tilde{S}_{i,h,j}(\overline{q})\right\|_{L^2_{b,k}(\Omega)} \leq c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right), \tag{3.178}$$

and it remains to estimate the last term on the right hand side of (3.175). In order to do so we will show that

$$\left\|\tilde{S}_{i,h,k}(\overline{q}) - S_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)} \leq \left\|S_i(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)} + c_i\left(\left|\ln h\right| h^2 + \left|\ln k\right| k^2\right), \tag{3.179}$$

for (3.179), the triangle inequality, (3.176) and (3.178) would prove the final claim. It holds that

$$S_{i,h,k}(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q}) = S_{i,h,k}(\overline{q})\left(1 - b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{i,h,k}(\overline{q}))\right),$$

and

$$\begin{aligned}\left\|S_i(\overline{q})\right\|_{L^2_{b,k}(\Omega)} - \left\|S_i(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)} &\leq \left\|\tilde{S}_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)} \\ &\leq \left\|S_i(\overline{q})\right\|_{L^2_{b,k}(\Omega)} + \left\|S_i(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)}.\end{aligned}$$

Lemma 3.4.86 and (3.64) prove that

$$\left|\left\|S_i(\overline{q})\right\|_{L^2_{b,k}(\Omega)} - 1\right| \leq c_i\left(\left|\ln h\right| h^2 + \left|\ln k\right| k^2\right),$$

and using the normalizing condition for $S_{i,h,k}(\overline{q})$ it follows that

$$\begin{aligned}&1 - c_i\left(\left|\ln h\right| h^2 + \left|\ln k\right| k^2\right) - \left\|S_i(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)} \\ &\leq \left|b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{i,h,k}(\overline{q}))\right| \\ &\leq 1 + c_i\left(\left|\ln h\right| h^2 + \left|\ln k\right| k^2\right) + \left\|S_i(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)},\end{aligned}$$

hence

$$\left|\left|b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{i,h,k}(\overline{q}))\right| - 1\right| - c_i\left(\left|\ln h\right| h^2 + \left|\ln k\right| k^2\right) \leq \left\|S_i(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q})\right\|_{L^2_{b,k}(\Omega)}.$$

It holds that

$$\begin{aligned}b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{i,h,k}(\overline{q})) = b(\overline{F}_k)(S_{i,h}(\overline{q}), S_{i,h,k}(\overline{q})) + b(\overline{F}_k)(S_i(\overline{q}) - S_{i,h}(\overline{q}), S_{i,h,k}(\overline{q})) \\ + b(\overline{F}_k)(\Pi_k S_i(\overline{q}) - S_i(\overline{q}), S_{i,h,k}(\overline{q})).\end{aligned} \tag{3.180}$$

With (3.74) it follows that

$$b(\overline{F}_k)(S_{i,h}(\overline{q}), S_{i,h,k}(\overline{q})) \geq 0, \tag{3.181}$$

and with the help of Lemma 3.4.100 one can show that

$$\left|b(\overline{F}_k)(S_i(\overline{q}) - S_{i,h}(\overline{q}), S_{i,h,k}(\overline{q}))\right| \leq c_i\left(\left|\ln h\right| h^2 + \left|\ln k\right| k^2\right). \tag{3.182}$$

In addition, Lemma 3.4.93 proves

$$\left|b(\overline{F}_k)(\Pi_k S_i(\overline{q}) - S_i(\overline{q}), S_{i,h,k}(\overline{q}))\right| \leq c_i\left(\left|\ln h\right| h^2 + \left|\ln k\right| k^2\right). \tag{3.183}$$

Inserting (3.181), (3.182) and (3.183) into (3.180) proves

$$b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{i,h,k}(\overline{q})) \geq -c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right),$$

where c_i is a positive real number. It follows that

$$\begin{aligned}\left\| S_{i,h,k}(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q}) \right\|_{L^2_{b,k}(\Omega)} &= \left|1 - b(\overline{F}_k)(\Pi_k S_i(\overline{q}), S_{i,h,k}(\overline{q}))\right| \\ &\leq \left\| S_i(\overline{q}) - \tilde{S}_{i,h,k}(\overline{q}) \right\|_{L^2_{b,k}(\Omega)} + c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right). \qquad \square\end{aligned}$$

Lemma 3.4.103. *For $i \in \mathbb{N}$ it holds that*

$$\|S_{i,h}(\overline{q}) - S_{i,h,k}(\overline{q})\|_{H^1_0(\Omega)} \leq c_i \left(|\ln h|^{1/2}\, h + |\ln k|^{1/2}\, k\right).$$

Proof. We use the same ideas as presented in the proof of Lemma 3.4.101 and get

$$\begin{aligned}
&c\,\|S_{i,h}(\overline{q}) - S_{i,h,k}(\overline{q})\|^2_{H^1(\Omega)} \leq a_h(\overline{F}_k)(S_{i,h}(\overline{q}) - S_{i,h,k}(\overline{q}), S_{i,h}(\overline{q}) - S_{i,h,k}(\overline{q})) \\
&= a_h(\overline{F}_k)(S_{i,h}(\overline{q}), S_{i,h}(\overline{q})) - 2a_h(\overline{F}_k)(S_{i,h}(\overline{q}), S_{i,h,k}(\overline{q})) + a_h(\overline{F}_k)(S_{i,h,k}(\overline{q}), S_{i,h,k}(\overline{q})) \\
&= a_h(\overline{F})(S_{i,h}(\overline{q}), S_{i,h}(\overline{q})) + \left(\nabla S_{i,h}(\overline{q}), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla S_{i,h}(\overline{q})\right) \\
&\quad - 2\lambda_{i,h,k}(\overline{q})\, b(\overline{F}_k)(S_{i,h}(\overline{q}), S_{i,h,k}(\overline{q})) + \lambda_{i,h,k}(\overline{q}) \\
&= \lambda_{i,h}(\overline{q}) - \lambda_{i,h,k}(\overline{q}) + 2\lambda_{i,h,k}(\overline{q})\, b(\overline{F}_k)(S_{i,h,k}(\overline{q}) - S_{i,h}(\overline{q}), S_{i,h,k}(\overline{q})) \\
&\quad + \left(\nabla S_{i,h}(\overline{q}), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla S_{i,h}(\overline{q})\right) \\
&\leq |\lambda_{i,h}(\overline{q}) - \lambda_{i,h,k}(\overline{q})| + c_i \left(\|S_i(\overline{q}) - S_{i,h,k}(\overline{q})\|_{L^2(\Omega)} + \|S_i(\overline{q}) - S_{i,h,k}(\overline{q})\|_{L^2(\Omega)}\right) \\
&\quad + \left(\nabla S_{i,h}(\overline{q}), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla S_{i,h}(\overline{q})\right).
\end{aligned} \tag{3.184}$$

The first term on the right hand side of (3.184) can be estimated using Lemma 3.4.97 and Corollary 3.4.96. The second part can be estimated using Lemma 3.4.100 and Lemma 3.4.102, we get

$$\|S_{i,h}(\overline{q}) - S_{i,h,k}(\overline{q})\|^2_{H^1_0(\Omega)} \leq c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right) + c\left(\nabla S_{i,h}(\overline{q}), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla S_{i,h}(\overline{q})\right).$$

The remaining term on the right hand side of (3.184) can be estimated via

$$\begin{aligned}
&\left(\nabla S_{i,h}(\overline{q}), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla S_{i,h}(\overline{q})\right) \\
&= \left(\nabla \left(S_{i,h}(\overline{q}) - S_i(\overline{q})\right), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla \left(S_{i,h}(\overline{q}) - S_i(\overline{q})\right)\right) && (3.185) \\
&+ 2\left(\nabla S_i(\overline{q}), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla \left(S_{i,h}(\overline{q}) - S_i(\overline{q})\right)\right) && (3.186) \\
&+ \left(\nabla S_i(\overline{q}), \left(\mu_h A_{\overline{F}_k} - \mu A_{\overline{F}}\right) \cdot \nabla S_i(\overline{q})\right) && (3.187) \\
&+ \left(\nabla S_i(\overline{q}), (\mu - \mu_h)\, A_{\overline{F}} \cdot \nabla S_i(\overline{q})\right). && (3.188)
\end{aligned}$$

For (3.185) we get

$$\begin{aligned}
&\left|\left(\nabla \left(S_{i,h}(\overline{q}) - S_i(\overline{q})\right), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla \left(S_{i,h}(\overline{q}) - S_i(\overline{q})\right)\right)\right| \\
&\leq c\,\|S_{i,h}(\overline{q}) - S_i(\overline{q})\|^2_{H^1_0(\Omega)} \left\|A_{\overline{F}_k} - A_{\overline{F}}\right\|_{L^\infty(\Omega)} \\
&\leq c_i \left(|\ln h|\, h^2 + |\ln k|\, k^2\right).
\end{aligned}$$

For the second part, (3.186), we get

$$\begin{aligned}
&\left|\left(\nabla S_i(\overline{q}), \mu_h \left(A_{\overline{F}_k} - A_{\overline{F}}\right) \cdot \nabla \left(S_{i,h}(\overline{q}) - S_i(\overline{q})\right)\right)\right| \\
&\leq c \left\|S_i(\overline{q})\right\|_{W^{1,\infty}(\Omega)} \left\|A_{\overline{F}_k} - A_{\overline{F}}\right\|_{L^2(\Omega)} \left\|S_{i,h}(\overline{q}) - S_i(\overline{q})\right\|_{H_0^1(\Omega)} \\
&\leq c_i k \left(|\ln h|^{1/2} h + |\ln k|^{1/2} k\right) \\
&\leq c_i \left(|\ln h| \, h^2 + |\ln k| \, k^2\right).
\end{aligned}$$

The third term, (3.187), can be estimated using Lemma 3.4.88, and the last part, (3.188), can be estimated using Lemma 3.4.37. □

Lemma 3.4.104. *For $\delta q \in Q$ and $i \in \mathbb{N}$ it holds that*

$$\left|\lambda_i'(\overline{q})(\delta q) - \lambda_{i,h,k}'(\overline{q})(\delta q)\right| \leq c_i \left(|\ln h|^{3/4} h^{3/2} + |\ln k|^{3/4} k^{3/2}\right) \|\delta q\|_{H^2(I)}.$$

Proof. Let $\delta F = F'(\overline{q})(\delta q)$ and $\delta F_k = F_k'(q)(\delta q)$. With (3.65) and (3.75) it follows that

$$\begin{aligned}
&\left|\lambda_i'(\overline{q})(\delta q) - \lambda_{i,h,k}'(\overline{q})(\delta q)\right| \\
&\leq \left|\left(\nabla S_i(\overline{q}), \mu A_{\overline{F},\delta F}' \cdot \nabla S_i(\overline{q})\right) - \left(\nabla S_{i,h,k}(\overline{q}), \mu_h A_{\overline{F}_k,\delta F_k}' \cdot \nabla S_{i,h,k}(\overline{q})\right)\right| && (3.189) \\
&+ \left|\lambda_i(\overline{q}) \left(S_i(\overline{q})^2, \gamma_{\overline{F},\delta F}'\right) - \lambda_{i,h,k}(\overline{q}) \left(S_{i,h,k}(\overline{q})^2, \gamma_{\overline{F}_k,\delta F_k}'\right)\right|. && (3.190)
\end{aligned}$$

We start with estimating (3.189) and get

$$\begin{aligned}
&\left(\nabla S_i(\overline{q}), \mu A_{\overline{F},\delta F}' \cdot \nabla S_i(\overline{q})\right) - \left(\nabla S_{i,h,k}(\overline{q}), \mu_h A_{\overline{F}_k,\delta F_k}' \cdot \nabla S_{i,h,k}(\overline{q})\right) \\
&= 2\left(\nabla \left(S_i(\overline{q}) - S_{i,h,k}(\overline{q})\right), \mu A_{\overline{F},\delta F}' \cdot \nabla S_i(\overline{q})\right) && (3.191) \\
&+ \left(\nabla \left(S_{i,h,k}(\overline{q}) - S_i(\overline{q})\right), (\mu - \mu_h) \left(A_{\overline{F},\delta F}' + A_{\overline{F}_k,\delta F_k}'\right) \cdot \nabla S_i(\overline{q})\right) && (3.192) \\
&+ \left(\nabla \left(S_{i,h,k}(\overline{q}) - S_i(\overline{q})\right), (\mu + \mu_h) \left(A_{\overline{F},\delta F}' - A_{\overline{F}_k,\delta F_k}'\right) \cdot \nabla S_i(\overline{q})\right) && (3.193) \\
&+ \left(\nabla S_i(\overline{q}), \left(\mu A_{\overline{F},\delta F}' - \mu_h A_{\overline{F}_k,\delta F_k}'\right) \cdot \nabla S_i(\overline{q})\right) && (3.194) \\
&+ \left(\nabla \left(S_{i,h,k}(\overline{q}) - S_i(\overline{q})\right), \mu_h A_{\overline{F}_k,\delta F_k}' \cdot \nabla \left(S_i(\overline{q}) - S_{i,h,k}(\overline{q})\right)\right). && (3.195)
\end{aligned}$$

The first part, (3.191), can be estimated in the following way.

$$\begin{aligned}
\left|\left(\nabla \left(S_i(\overline{q}) - S_{i,h,k}(\overline{q})\right), \mu A_{\overline{F},\delta F}' \cdot \nabla S_i(\overline{q})\right)\right| &\leq d \left|\left(\nabla \left(S_i(\overline{q}) - S_{i,h,k}(\overline{q})\right), A_{\overline{F},\delta F}' \cdot \nabla S_i(\overline{q})\right)_{\Omega_0}\right| \\
&+ \left|\left(\nabla \left(S_i(\overline{q}) - S_{i,h,k}(\overline{q})\right), A_{\overline{F},\delta F}' \cdot \nabla S_i(\overline{q})\right)_{\Omega_1}\right|. \qquad (3.196)
\end{aligned}$$

The terms on the right hand side of (3.196) can be estimated using partial integration, for $j \in \{0, 1\}$ it holds that

$$\begin{aligned}
\left|\left(\nabla \left(S_i(\overline{q}) - S_{i,h,k}(\overline{q})\right), A_{\overline{F},\delta F}' \cdot \nabla S_i(\overline{q})\right)_{\Omega_j}\right| &\leq \left|\left(\operatorname{div}\left(A_{\overline{F},\delta F}' \cdot \nabla S_i(\overline{q})\right), S_i(\overline{q}) - S_{i,h,k}(\overline{q})\right)_{\Omega_j}\right| && (3.197) \\
&+ \left|\langle S_i(\overline{q}) - S_{i,h,k}(\overline{q}), \nabla S_i(\overline{q})^T \cdot A_{\overline{F},\delta F}' \cdot n\rangle_{\Gamma_0}\right|. && (3.198)
\end{aligned}$$

For (3.197) we have

$$\left|\left(\operatorname{div}\left(A'_{\overline{F},\delta F}\cdot\nabla S_i(\overline{q})\right), S_i(\overline{q})-S_{i,h,k}(\overline{q})\right)_{\Omega_j}\right| \le c_i\left\|A'_{\overline{F},\delta F}\right\|_{H^1(\Omega_j)}\left\|S_i(\overline{q})-S_{i,h,k}(\overline{q})\right\|_{L^2(\Omega_j)}$$
$$\le \left(|\ln h|\,h^2+|\ln k|\,k^2\right)\|\delta q\|_{H^2(I)},$$

whereas (3.198) can be estimated via

$$\begin{aligned}
&\left|\langle S_i(\overline{q})-S_{i,h,k}(\overline{q}),\nabla S_i(\overline{q})^T\cdot A'_{\overline{F},\delta F}\cdot n\rangle_{\Gamma_0}\right|\\
&\le \|S_i(\overline{q})-S_{i,h,k}(\overline{q})\|_{L^2(\Gamma_0)}\|S_i(\overline{q})\|_{W^{1,\infty}(\Gamma_0)}\left\|A'_{\overline{F},\delta F}\right\|_{L^2(\Gamma_0)}\\
&\le c_i\|S_i(\overline{q})-S_{i,h,k}(\overline{q})\|_{L^2(\Omega)}^{1/2}\|S_i(\overline{q})-S_{i,h,k}(\overline{q})\|_{H^1(\Omega)}^{1/2}\left\|A'_{\overline{F},\delta F}\right\|_{H^1(\Omega)}\\
&\le c_i\left(|\ln h|^{3/4}h^{3/2}+|\ln k|^{3/4}k^{3/2}\right)\|\delta q\|_{H^2(I)},
\end{aligned}$$

where we used (A.38).

The term (3.192) may be estimated as follows.

$$\begin{aligned}
&\left|\left(\nabla\left(S_{i,h,k}(\overline{q})-S_i(\overline{q})\right),(\mu-\mu_h)\left(A'_{\overline{F},\delta F}+A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla S_i(\overline{q})\right)\right|\\
&\le \|S_i(\overline{q})-S_{i,h,k}(\overline{q})\|_{H_0^1(\Omega)}\|\mu-\mu_h\|_{L^2(\Omega)}\left(\left\|A'_{\overline{F},\delta F}\right\|_{L^\infty(\Omega)}+\left\|A'_{\overline{F}_k,\delta F_k}\right\|_{L^\infty(\Omega)}\right)\|S_i(\overline{q})\|_{W^{1,\infty}(\Omega)}\\
&\le c_i h\left(|\ln h|^{1/2}h+|\ln k|^{1/2}k\right)\|\delta q\|_{H^2(I)}.
\end{aligned}$$

The term (3.193) can be estimated via

$$\begin{aligned}
&\left|\left(\nabla\left(S_{i,h,k}(\overline{q})-S_i(\overline{q})\right),(\mu+\mu_h)\left(A'_{\overline{F},\delta F}-A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla S_i(\overline{q})\right)\right|\\
&\le c\|S_i(\overline{q})-S_{i,h,k}(\overline{q})\|_{H_0^1(\Omega)}\left\|A'_{\overline{F},\delta F}-A'_{\overline{F}_k,\delta F_k}\right\|_{L^2(\Omega)}\|S_i(\overline{q})\|_{W^{1,\infty}(\Omega)}\\
&\le c_i\left(|\ln h|^{1/2}h+|\ln k|^{1/2}k\right)k\|\delta q\|_{H^2(I)}.
\end{aligned}$$

For the fourth term (3.194) we use Lemma 3.4.89 and get

$$\left|\left(\nabla S_i(\overline{q}),\left(\mu A'_{\overline{F},\delta F}-\mu_h A'_{\overline{F}_k,\delta F_k}\right)\cdot\nabla S_i(\overline{q})\right)\right| \le c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right)\|\delta q\|_{H^2(I)},$$

and for (3.195) we get

$$\begin{aligned}
&\left|\left(\nabla\left(S_{i,h,k}(\overline{q})-S_i(\overline{q})\right),\mu_h A'_{\overline{F}_k,\delta F_k}\cdot\nabla\left(S_i(\overline{q})-S_{i,h,k}(\overline{q})\right)\right)\right|\\
&\le c\|S_i(\overline{q})-S_{i,h,k}(\overline{q})\|_{H_0^1(\Omega)}^2\left\|A'_{\overline{F}_k,\delta F_k}\right\|_{L^\infty(\Omega)}\\
&\le c_i\left(|\ln h|\,h^2+|\ln k|\,k^2\right)\|\delta q\|_{H^2(I)}.
\end{aligned}$$

It remains to estimate the second part within the original estimation, (3.190). We get

$$\lambda_i(\overline{q})\left(S_i(\overline{q})^2,\gamma'_{\overline{F},\delta F}\right)-\lambda_{i,h,k}(\overline{q})\left(S_{i,h,k}(\overline{q})^2,\gamma'_{\overline{F}_k,\delta F_k}\right)$$
$$=\left(\lambda_i(\overline{q})-\lambda_{i,h,k}(\overline{q})\right)\left(S_i(\overline{q})^2,\gamma'_{\overline{F},\delta F}\right) \tag{3.199}$$
$$+\lambda_{i,h,k}(\overline{q})\left(S_i(\overline{q})^2-S_{i,h,k}(\overline{q})^2,\gamma'_{\overline{F}_k,\delta F_k}\right) \tag{3.200}$$
$$+\lambda_{i,h,k}(\overline{q})\left(S_i(\overline{q})^2,\gamma'_{\overline{F},\delta F}-\gamma'_{\overline{F}_k,\delta F_k}\right). \tag{3.201}$$

For the first term, (3.199), we use Corollary 3.4.96 and get

$$|\lambda_i(\overline{q}) - \lambda_{i,h,k}(\overline{q})| \left(S_i(\overline{q})^2, \gamma'_{\overline{F},\delta F} \right) \leq c_i \left(|\ln h| \, h^2 + |\ln k| \, k^2 \right) \|\delta q\|_{H^2(I)} .$$

For (3.200) we use Lemma 3.4.102 and obtain

$$\begin{aligned} \lambda_{i,h,k}(\overline{q}) \left(S_i(\overline{q})^2 - S_{i,h,k}(\overline{q})^2, \gamma'_{\overline{F}_k,\delta F_k} \right) &= \lambda_{i,h,k}(\overline{q}) \left(S_i(\overline{q}) - S_{i,h,k}(\overline{q}), \left(S_i(\overline{q}) + S_{i,h,k}(\overline{q})\right) \gamma'_{\overline{F}_k,\delta F_k} \right) \\ &\leq c_i \left(|\ln h| \, h^2 + |\ln k| \, k^2 \right) \|\delta q\|_{H^2(I)} , \end{aligned}$$

and (3.201) can be estimated using Lemma 3.4.87. □

Lemma 3.4.105. *For $\delta q \in Q$ and $i \in \mathbb{N}$ it holds that*

$$\left| j'(\overline{q})(\delta q) - j'_{h,k}(\overline{q})(\delta q) \right| \leq c \left(|\ln h|^{3/4} h^{3/2} + |\ln k|^{3/4} k^{3/2} \right) \|\delta q\|_{H^2(I)} .$$

Proof. This lemma is a direct consequence of Lemma 3.4.104. □

3.4.3. Second order optimality conditions and converging subsequences

Within this subsection we are going to state some optimality conditions of second order for both the continuous as well as the (partially) discretized cost functional. As these results can be proven in a similar way to the corresponding results in Section 2.3, we just present the results and will omit the proofs.

Again we have to start with the following assumption.

Assumption 3.4.106. We assume that

$$j''(\overline{q})(\delta q, \delta q) > 0 \qquad\qquad \forall \delta q \in Q \setminus \{0\} .$$

As in the proof of Theorem 2.3.57 within Subsection 2.3.4 it is possible to show that Assumption 3.4.106 implies the strict convexity of $j''(\overline{q})(\cdot,\cdot)$, and using the stability estimates for the error between the continuous cost functional and its discretized counterparts, Lemma 3.4.32, Lemma 3.4.54 and Lemma 3.4.72, the following theorem follows.

Theorem 3.4.107. *There exist $\beta > 0$ and $\varepsilon > 0$ such that for h, k sufficiently small and all $q \in Q^{\text{ad}}$ with $\|q - \overline{q}\|_{H^2(I)} < \varepsilon$ it holds that*

$$\begin{aligned} j''(q)(\delta q, \delta q) &\geq \beta \|\delta q\|^2_{H^2(I)} && \forall \delta q \in Q, \\ j''_h(q)(\delta q, \delta q) &\geq \beta \|\delta q\|^2_{H^2(I)} && \forall \delta q \in Q, \\ j''_{h,k}(q)(\delta q, \delta q) &\geq \beta \|\delta q\|^2_{H^2(I)} && \forall \delta q \in Q. \end{aligned}$$

As in Subsection 2.3.5 one can introduce auxiliary problems like (2.128), and using the results obtained in Subsubsection 3.4.1.1, Subsubsection 3.4.1.2 and Subsubsection 3.4.1.3 the following theorem can be proven.

Theorem 3.4.108. *There exist sequences $(\overline{q}_\sigma)_{\sigma>0}$, $\left(\overline{q}_{\sigma,h}\right)_{\sigma,h>0}$, $\left(\overline{q}_{\sigma,h,k}\right)_{\sigma,h,k>0}$ of local optimal solutions to* (3.57), (3.60) *and* (3.62)*, respectively, such that*

$$\lim_{\sigma\to 0} \|\overline{q} - \overline{q}_\sigma\|_{H^2(I)} = \lim_{\sigma,h\to 0} \left\|\overline{q} - \overline{q}_{\sigma,h}\right\|_{H^2(I)} = \lim_{\sigma,h,k\to 0} \left\|\overline{q} - \overline{q}_{\sigma,h,k}\right\|_{H^2(I)} = 0.$$

We are now able to finish the proof of the main theorem of this section, Theorem 3.4.1; what follows is similar to the proof of Theorem 2.3.1 on page 63.

Proof. Let $\overline{q}$ be an optimal control for (3.21) and let $\overline{q}_{\sigma,h,k}$ be an optimal control for (3.62) for σ, h and k sufficiently small, such that Theorem 3.4.107 holds for $\overline{q}_{\sigma,h,k}$ and Lemma 3.4.78 holds for $\overline{q}$ and $\overline{q}_{\sigma,h,k}$. The existence of such a $\overline{q}_{\sigma,h,k}$ is guaranteed by Theorem 3.4.108. Now there exists $t \in [0,1]$ such that with $\xi = t\overline{q} + (1-t)\overline{q}_{\sigma,h,k}$ it holds that

$$\begin{aligned} c\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2 &\le j''_{h,k}(\xi)(\overline{q}-\overline{q}_{\sigma,h,k},\overline{q}-\overline{q}_{\sigma,h,k}) \\ &= j'_{h,k}(\overline{q})(\overline{q}-\overline{q}_{\sigma,h,k}) - j'_{h,k}(\overline{q}_{\sigma,h,k})(\overline{q}-\overline{q}_{\sigma,h,k}) \\ &= j'_{h,k}(\overline{q})(\overline{q}-\overline{q}_{\sigma,h,k}) - j'_{h,k}(\overline{q}_{\sigma,h,k})(\overline{q}-i_\sigma\overline{q}), \end{aligned}$$

where the second equality is due to the first order optimality condition in $\overline{q}_{\sigma,h,k}$ which reads as $j'_{h,k}(\overline{q}_{\sigma,h,k})(\delta q_\sigma) = 0$ for all $\delta q_\sigma \in Q_\sigma$. Next we use the first order optimality condition in $\overline{q}$, i.e. $j'(\overline{q})(\delta q) = 0$ for all $\delta q \in Q$, and get

$$\begin{aligned} c\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2 \le\ & j'_{h,k}(\overline{q})(\overline{q}-\overline{q}_{\sigma,h,k}) - j'(\overline{q})(\overline{q}-\overline{q}_{\sigma,h,k}) \\ &+ j'_{h,k}(\overline{q})(\overline{q}-i_\sigma\overline{q}) - j'_{h,k}(\overline{q}_{\sigma,h,k})(\overline{q}-i_\sigma\overline{q}) \\ &+ j'(\overline{q})(\overline{q}-i_\sigma\overline{q}) - j'_{h,k}(\overline{q})(\overline{q}-i_\sigma\overline{q}). \end{aligned}$$

Using Lemma 3.4.78 and Lemma 3.4.105 we arrive at

$$\begin{aligned} \left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2 \le\ & c\left(|\ln h|^{3/4} h^{3/2} + |\ln k|^{3/4} k^{3/2}\right)\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)} \\ &+ c\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)} \left\|\overline{q}-i_\sigma\overline{q}\right\|_{H^2(I)} \\ &+ c\left(|\ln h|^{3/4} h^{3/2} + |\ln k|^{3/4} k^{3/2}\right)\left\|\overline{q}-i_\sigma\overline{q}\right\|_{H^2(I)}. \end{aligned}$$

With Young's inequality we get the existence of a $c_1 > 0$ such that

$$\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2 \le c_1\left(\left(|\ln h|^{3/4} h^{3/2} + |\ln k|^{3/4} k^{3/2}\right)^2 + \left\|\overline{q}-i_\sigma\overline{q}\right\|_{H^2(I)}^2\right) + \frac{1}{2}\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)}^2.$$

For $a, b \ge 0$ it holds that $\sqrt{a^2+b^2} \le a+b$, and using Lemma 2.2.2 we finally end up with

$$\left\|\overline{q}-\overline{q}_{\sigma,h,k}\right\|_{H^2(I)} \le c\left(\sigma^2 + |\ln h|^{3/4} h^{3/2} + |\ln k|^{3/4} k^{3/2}\right). \qquad \square$$

4. Conclusion and perspectives

In this thesis we proved a-priori error estimates for finite element discretizations of two shape optimization problems with different cost functionals.

In the first part, Chapter 2, we investigated a shape optimization problem with tracking-type cost functional. We parametrized the class of star-shaped domains using periodic H^2-functions, which enabled us to use the standard methods of control theory. We used a Tikhonov-type term for regularization instead of a bound on an appropriate norm of q. Using the transformation approach we reformulated the original problem on a reference domain, where we stated the exact regularity assumptions needed for the transformation to exist. Using the optimality condition of first order, we proved higher regularity of the optimal control, which in turn enabled us to compute the derivative of the reduced cost functional as a boundary integral. The obtained representation is similar to the representations obtained by different methods like classical shape calculus or the level-set method. In order to obtain error estimates, we proved estimates for general controls as well as for the optimal control with higher regularity. Estimating the error with respect to the discretization of the control q and the state u can be done using standard arguments, for quadratic convergence in the optimal case we used various regularity results and duality arguments. In order to estimate the error with respect to the discretization of the transformation F we used Taylor's theorem and a result on the error of a finite element approximation on a non-polygonal domain. The existence of a sequence of local optimal controls to the fully discretized problem converging to the optimal control of the continuous problem is due to the error estimates for the non-optimal case, whereas the quadratic convergence within the final estimate is due to the error estimates for the optimal case.

The aim of the second part, Chapter 3, was to maximize the distance between the first two eigenvalues of an elliptic partial differential operator corresponding to the transmission problem, with respect to domain perturbations. Using the same methods as presented in Chapter 2 we transformed the problem on a reference domain, proved existence and higher regularity of the optimal control and showed how to compute the derivative of the reduced cost functional as a boundary integral. As solutions to the transmission problem have a jump of the normal derivative in the interior of their domain of definition, a special emphasis had to be put on regularity results for these kind of problems. We used the closed graph theorem to prove a stability result for the derivative of eigenfunction with respect to perturbations of the domain. Due to the special structure of eigenvalue equations, the methods usually used in finite element error estimation cannot be applied here. Instead, one uses Parseval's identity and the Bessel inequality to expand the error as weighted sum of certain eigenfunctions. We extended this method to also estimate the error between derivatives of eigenfunctions with respect to perturbations of the domain. Again having proved error estimates for the general as well as for the optimal case, we could proceed as within Chapter 2 in proving the existence of a converging sequence of optimal controls to the fully discretized problem and convergence rates.

In the Appendix, Chapter A, we presented some well-known functional analytic theorems and regularity results and showed how to generalize the Bramble-Hilbert lemma and inverse estimates for finite element ansatz functions onto fractional Sobolev spaces. In the second part of the Appendix we generalized a result regarding the error induced by a finite element discretization of a partial differential equation on a non-polygonal domain. For the proof we used regularity results in $W^{1,p}$, duality arguments and stability results for the Ritz projection on both convex and nonconvex domains. We showed that for convex domains the order of convergence is the same as one would expect on polygonal domains, whereas for nonconvex domains an additional logarithmic term enters the estimate.

There are several possible directions for future research on these topics:

- The main results of Chapter 2 and Chapter 3 rely on the higher regularity of the optimal controls, which can be shown using the assumption that no control constraints are active in the optimal control. Even in case of active control constraints one can often show higher regularity, cf. [71], thus it may be possible to derive error estimates of optimal order for that case as well.
- The elliptic partial differential operator within Chapter 2 may be exchanged with a parabolic one, and the shape of the domain may be time-dependent.
- It would be interesting to apply the framework for a-posteriori error estimation presented in [11] onto these shape optimization problems in order to get an adaptive refinement strategy for the discretization of the control.
- As has been shown within this thesis, given sufficient regularity of the control, the first derivative of the reduced cost functional may be represented as a boundary integral. Using the algorithm presented in [106] it should be possible to find a representation for the second derivative of the reduced cost functional as a boundary integral as well.
- Regarding eigenvalue problems, most publications deal with homogeneous Dirichlet conditions. Thus it would be interesting to consider a partial differential operator with Neumann or Robin boundary conditions within Chapter 3.

A. Appendix

This chapter is a collection of results which will be needed throughout in Chapter 2 and Chapter 3 but are not directly related to shape optimization. It is organized as follows.

Section A.1 contains some well-known functional analytic theorems, a generalization of the Bramble-Hilbert lemma for Sobolev spaces of fractional order, nonstandard inverse estimates for finite element ansatz functions as well as regularity results for elliptic partial differential equations, including the stability of the Ritz projection.

Section A.2 is devoted to a generalization of a result from [17] concerning finite element error estimates on non-polygonal domains to the $L^p/W^{1,p}$-case for some $p > 2$.

A.1. Some general theorems and regularity results

Theorem A.1.1 (Hahn-Banach theorem). *Let X be a normed space and let $Y \subset X$ be a subspace, equipped with the norm of X.*

- *For each $y' \in Y'$ there exists a $x' \in X'$ such that*

$$x' = y' \text{ in } Y \qquad \text{and} \qquad \|x'\|_{X'} = \|y'\|_{Y'}.$$

- *For each $x_0 \in X$ with $x_0 \neq 0$ there exists $x_0' \in X'$ with*

$$\|x_0'\|_{X'} = 1 \qquad \text{and} \qquad \left(x_0', x_0\right)_{X',X} = \|x_0\|_X.$$

Proof. These consequences of the Hahn-Banach theorem can be found in [3], Chapter 4. □

Theorem A.1.2 (Spectral theorem). *Let X be a Hilbert space over $\mathbb{R}$ and let $L \neq 0$ be a compact operator over X. Then it holds that:*

- *For the spectrum $\sigma(L)$ of L it holds that $\sigma(L)\backslash\{0\}$ consists of countably many eigenvalues with 0 as only possible accumulation point.*

- *For $\mu \in \sigma(L)\backslash\{0\}$ we have*

$$1 \leq n_\mu = \max\left\{n \in \mathbb{N} | \mathcal{N}\left((\mu\,\mathrm{I} - L)^{n-1}\right) \neq \mathcal{N}\left((\mu\,\mathrm{I} - L)^n\right)\right\} < \infty.$$

- *For $\mu \in \sigma(L)\backslash\{0\}$ it holds that*

$$X = \mathcal{N}\left((\mu\,\mathrm{I} - L)^{n_\mu}\right) \oplus \mathcal{R}\left((\mu\,\mathrm{I} - L)^{n_\mu}\right),$$

and both subspaces are closed.

- *If L is additionally self-adjoint, then there exists an orthonormal system $(e_i)_{i\in\mathbb{N}}$ and a sequence $(\mu_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ with*

$$Le_i = \mu_i e_i,$$

and it holds that

$$\lim_{i\to\infty} \mu_i = 0.$$

Proof. This theorem can be found in [3], Theorem 9.9, Theorem 10.12 and Remark 10.13. □

Theorem A.1.3 (Trace theorem)**.** *Let $\Omega \subset \mathbb{R}^2$ be bounded and open with a $C^{k,1}$ boundary Γ for a $k \in \mathbb{N}_0$. Let $s > 0$, $p \in (1,\infty)$ and let $(s - 1/p) \notin \mathbb{Z}$, $s \leq k+1$, $s - 1/p = l + \sigma$ with $\sigma \in (0,1)$ and $l \in \mathbb{N}_0$. Then the mapping*

$$u \mapsto \left\{ u|_\Gamma \,, \left.\frac{\partial u}{\partial n}\right|_\Gamma , \cdots , \left.\frac{\partial^l u}{\partial n^l}\right|_\Gamma \right\},$$

which is defined for $u \in C^{k,1}(\overline{\Omega})$, has a unique continuous extension as an operator from

$$W^{s,p}(\Omega) \text{ onto } \prod_{j=0}^{l} W^{s-j-1/p,p}(\Gamma),$$

and a similar statement holds in case that Γ is a curvilinear polygon of class $C^{k,1}$. This trace operator has a continuous right inverse which does not depend on p.

Proof. This version of the trace theorem can be found in [48], Theorem 1.5.1.2 and Theorem 1.5.2.1. □

Theorem A.1.4 (Embedding theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded and open Lipschitz domain. Let $m_1, m_2, k \geq 0$ be integers, let $p_1, p_2 \in [1,\infty)$ and let $\alpha \in [0,1]$. Then it holds that:*

- *If $m_1 - \frac{n}{p_1} \geq m_2 - \frac{n}{p_2}$ and $m_1 \geq m_2$, then it holds that $W^{m_1,p_1}(\Omega) \hookrightarrow W^{m_2,p_2}(\Omega)$.*
- *If $m_1 - \frac{n}{p_1} > m_2 - \frac{n}{p_2}$ and $m_1 > m_2$, then it holds that $W^{m_1,p_1}(\Omega) \hookrightarrow\hookrightarrow W^{m_2,p_2}(\Omega)$.*
- *If $m_1 - \frac{n}{p_1} \geq k + \alpha$ and $\alpha \in (0,1)$, then it holds that $W^{m_1,p_1}(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$.*
- *If $m_1 - \frac{n}{p_1} > k + \alpha$ then it holds that $W^{m_1,p_1}(\Omega) \hookrightarrow\hookrightarrow C^{k,\alpha}(\overline{\Omega})$.*

The first two statements remain true if both spaces $W^{m_i,p_i}(\Omega)$ are replaced by $W_0^{m_i,p_i}(\Omega)$ for $i \in \{1,2\}$. In addition, we use the notation $C^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega})$.

Proof. This theorem can be found in [3], Theorem 8.9 and Theorem 8.13. □

Theorem A.1.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open with Lipschitz boundary Γ, let $s_1, s_2 \geq s \geq 0$ and $p_1, p_2, p \in (1,\infty)$ such that either*

$$s_1 + s_2 - s \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0 \qquad \text{and} \qquad s_j - s > n\left(\frac{1}{p_j} - \frac{1}{p}\right) \qquad \text{for} \qquad j \in \{1,2\},$$

or

$$s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0 \qquad \text{and} \qquad s_j - s \geq n\left(\frac{1}{p_j} - \frac{1}{p}\right) \qquad \text{for} \qquad j \in \{1,2\}\,.$$

Then the mapping $(u, v) \mapsto uv$ *is continuous from* $W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega)$ *into* $W^{s,p}(\Omega)$.

Proof. This theorem can be found in [48], Theorem 1.4.4.2 and the comment afterward. □

Theorem A.1.6 (Implicit function theorem)**.** *Let* $B \in C^k(X^{\mathrm{ad}} \times Y^{\mathrm{ad}}, Z)$ *for* $k \in \mathbb{N}$*, where* Z *is a Banach space and* X^{ad}*,* Y^{ad} *are open subsets of the Banach spaces* X *and* Y*, respectively. Suppose that* $B(x^*, y^*) = 0$ *and let* $B'_y(x^*, y^*)$ *be continuously invertible. Then there exist neighborhoods* Θ *of* x^* *in* X*,* Φ *of* y^* *in* Y *and a map* $g \in C^k(\Theta, Y)$ *such that*

- $B(x, g(x)) = 0$ *for all* $x \in \Theta$*,*
- $B(x, y) = 0$*,* $(x, y) \in \Theta \times \Phi$ *implies* $y = g(x)$*,*
- $g'(x) = -\left(B_y(x, g(x))\right)^{-1} \circ B_x(x, g(x))$ *for* $x \in \Theta$*.*

Proof. This theorem can be found in [5], Theorem 2.3. □

Theorem A.1.7 (Generalized Hölder inequality)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded and open Lipschitz domain. Let* $k \in \mathbb{N}$*,* $p_i, q \in [1, \infty]$ *for* $1 \leq i \leq k$ *with* $\sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{q}$*. If* $f_i \in L^{p_i}(\Omega)$ *for* $1 \leq i \leq k$ *then it holds that*

$$\prod_{i=1}^{k} f_i \in L^q(\Omega)$$

and

$$\left\| \prod_{i=1}^{k} f_i \right\|_{L^q(\Omega)} \leq \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(\Omega)}\,.$$

Proof. A proof for this theorem can be found in [3], Lemma 1.16. □

Theorem A.1.8 (Generalized Young's inequality)**.** *Let* $p, q \in (1, \infty)$ *with* $1/p + 1/q = 1$ *and let* $\varepsilon > 0$*. Then there exists* $M_\varepsilon > 0$ *such that for all* $a, b \in \mathbb{R}$ *it holds that*

$$|ab| \leq \varepsilon\, |a|^p + M_\varepsilon\, |b|^q\,.$$

Proof. This result can be found in [3], (1-11). □

Theorem A.1.9 (Riesz-Thorin interpolation theorem)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a domain, let* $p_0, p_1 \in [1, \infty]$ *and let* $q_0, q_1 \in [1, \infty]$ *with* $p_0 \neq p_1$*,* $q_0 \neq q_1$*. If*

$$T\colon L^{p_0}(\Omega) \to L^{q_0}(\Omega),$$

and

$$T\colon L^{p_1}(\Omega) \to L^{q_1}(\Omega),$$

is a bounded linear operator with norm M_0 and M_1, respectively, then it holds that

$$T\colon L^p(\Omega) \to L^q(\Omega),$$

is also a bounded linear operator with norm M, $M \leq M_0^{1-\theta} M_1^{\theta}$, where $\theta \in (0,1)$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Proof. This theorem can be found in [14], Theorem 1.1.1. □

The following generalized differentiation rules can both be found in [110], Proposition 4.10 and Proposition 4.11.

Theorem A.1.10 (Chain rule)**.** *Let X, Y and Z be Banach spaces, let $x \in X$ with $f\colon U(x) \to Y$, where $U(x)$ is a neighborhood of x. Let $y = f(x)$, and $g\colon U(y) \to Z$ for a neighborhood $U(y)$ of y. Let $h\colon U(x) \to Z$, $h = g \circ f$ be the composite map. If $f'(x)$ and $g'(f(x))$ exist as Fréchet-derivatives, then h is Fréchet-differentiable at x and it holds that*

$$h'(x) = g'(f(x)) \circ f'(x). \tag{A.1}$$

If $f'(x)$ exists only as Gâteaux-derivative at x, then h is also Gâteaux-differentiable at x, and (A.1) *holds.*

Theorem A.1.11 (Product rule)**.** *Let X, X_1, X_2 and Y be Banach spaces and suppose that the mapping $B\colon X_1 \times X_2 \to Y$ is bilinear and bounded. Suppose further that the maps*

$$f_i\colon U_i(x) \subseteq X \to X_i, \qquad i \in \{1,2\},$$

are Fréchet-differentiable at x, where $U_i(x)$, $i \in \{1,2\}$, are neighborhoods of x. Then the mapping $h\colon X \to Y$, $h(x) = B(f_1(x), f_2(x))$ is Fréchet-differentiable at x and it holds that

$$h'(x)(\delta x) = B\left(f_1'(x)(\delta x), f_2(x)\right) + B\left(f_1(x), f_2'(x)(\delta x)\right) \qquad \forall \delta x \in X.$$

These results remains true if Fréchet-differentiability is replaced with Gâteaux-differentiability.

Theorem A.1.12 (Generalized Taylor's theorem)**.** *Let the mapping $f\colon U(x) \subset X \to Y$ be defined on an open and convex neighborhood $U(x)$ of x, and let X and Y be Banach spaces. Let $n \in \mathbb{N}$ and let $f', f'', \cdots, f^{(n)}$ exist as Fréchet-derivatives on $U(x)$, then it holds that*

$$f(x+h) = f(x) + \sum_{k=1}^{n-1} \left(\frac{1}{k!} f^{(k)}(x) h^k \right) + R_n(x,h),$$

where $R_n(x,h)$ is a remainder term with

$$\|R_n(x,h)\| \leq \frac{1}{n!} \sup_{0<\tau<1} \left\| f^{(n)}(x+\tau h) h^n \right\|, \tag{A.2}$$

where $\|\cdot\|$ is an arbitrary norm on Y. If $f^{(n)}$ is also continuous on $U(x)$, then

$$R_n(x,h) = \int_0^1 \frac{(1-\tau)^{n-1}}{(n-1)!} f^{(n)}(x+\tau h) h^n \,\mathrm{d}\tau.$$

Proof. This version of Taylor's theorem can be found in [110], Theorem 4.A. □

Theorem A.1.13 (Local inverse mapping theorem)**.** *Let X and Y be Banach spaces over $\mathbb{R}$, let $x_0 \in X$ and let $f\colon U(x_0) \to Y$, with $U(x_0) \subseteq X$ being a neighborhood of x_0, be a C^k-mapping where $k \in \mathbb{N} \cup \{\infty\}$. If $f'(x_0)\colon X \to Y$ is bijective, then f is a local C^k-diffeomorphism at x_0 and*

$$\left(f^{-1}\right)'(y) = f'(x)^{-1} \qquad \text{with } y = f(x),$$

for all x in a neighborhood of x_0.

Proof. This theorem can be found in [110], Theorem 4.F and Corollary 4.37. □

Theorem A.1.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, let $s \in \mathbb{R}$ and $p \in (1, \infty)$. Then the differentiation operator $\partial_{x_i}\colon W^{s,p}(\Omega) \to W^{s-1,p}(\Omega)$ is a linear and continuous functional unless $s = 1/p$.*

Proof. This theorem can be found in [48], Theorem 1.4.4.6. □

Lemma A.1.15. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists $c > 0$ such that for all $p > 2$ and $u \in L^p(\Omega) \cap H^1(\Omega)$ it holds that*

$$\|u\|_{L^p(\Omega)} \leq cp^{1/2} \|u\|_{H^1(\Omega)} .$$

Proof. This inequality can be found in [32], equation (2.16), and is based on results proven in [94]. □

A.1.1. The Bramble-Hilbert lemma for Sobolev spaces of fractional order

In the context of estimating interpolation errors, many results are based on the Bramble-Hilbert lemma. Within this subsection we are going to generalize this lemma to Sobolev spaces of fractional order. What follows is a generalization of the proof presented in [105], another version can also be found in [16], Chapter II, Theorem 6.3.

For the following three lemmata, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, let $p \in (1, \infty]$ and let $s = k + \sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in (0, 1)$.

Lemma A.1.16. *Let $u \in W^{s,p}(\Omega)$. If $\mathrm{D}^\alpha u$ is constant for each multiindex $\alpha \in \mathbb{R}^n$ with $|\alpha| = k$, then $u \in \mathcal{P}^k(\Omega)$.*

Proof. It is clear that $u \in W^{m,p}(\Omega)$ for all $m \in \mathbb{N}$, and hence $u \in C^\infty(\Omega)$. The proof follows with induction on k. □

Lemma A.1.17. *For every $u \in W^{s,p}(\Omega)$ there exists a unique $u_h \in \mathcal{P}^k(\Omega)$ such that for all $\alpha \in \mathbb{R}^n$ with $0 \leq |\alpha| \leq k$ it holds that*

$$\int_\Omega \mathrm{D}^\alpha (u - u_h) \,\mathrm{d}x = 0. \tag{A.3}$$

Proof. First we show uniqueness. Let $u_{h,1}, u_{h,2} \in \mathcal{P}^k(\Omega)$ be two polynomials satisfying (A.3), and let $\tilde{u}_h = u_{h,1} - u_{h,2}$. Then it holds that

$$\int_\Omega \mathrm{D}^\alpha \tilde{u}_h \,\mathrm{d}x = 0 \qquad \text{for } 0 \leq |\alpha| \leq k.$$

By induction it follows that $\tilde{u}_h = 0$, hence $u_{h,1} = u_{h,2}$. As the statement (A.3) is equivalent to a system of $\dim\left(\mathcal{P}^k(\Omega)\right)$ linear equations with the same number of unknowns, uniqueness implies existence. □

Lemma A.1.18. *There exists $c > 0$ such that for all $u \in W^{s,p}(\Omega)$ with*

$$\int_\Omega \mathrm{D}^\alpha u \,\mathrm{d}x = 0 \qquad \text{where } 0 \leq |\alpha| \leq k, \tag{A.4}$$

it holds that

$$\|u\|_{W^{s,p}(\Omega)} \leq c\,|u|_{W^{s,p}(\Omega)}\,. \tag{A.5}$$

Proof. Suppose that (A.5) does not hold, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{s,p}(\Omega)$ for which (A.4) holds and

$$\|u_n\|_{W^{s,p}(\Omega)} \geq n\,|u_n|_{W^{s,p}(\Omega)}\,.$$

Without loss of generality we may assume $\|u_n\|_{W^{s,p}(\Omega)} = 1$. As this sequence is bounded in $W^{s,p}(\Omega)$, we can extract a subsequence (still denoted $(u_n)_{n\in\mathbb{N}}$) such that $u_n \rightharpoonup u$ in $W^{s,p}(\Omega)$ for a $u \in W^{s,p}(\Omega)$, where for $p = \infty$ we have to use weak*-convergence. As $W^{s,p}(\Omega)$ is compactly embedded into $W^{k,p}(\Omega)$ we get $u_n \to u$ in $W^{k,p}(\Omega)$, and $(u_n)_{n\in\mathbb{N}}$ is Cauchy in $W^{k,p}(\Omega)$. As $|u_n|_{W^{s,p}(\Omega)} \to 0$, the sequence is also Cauchy in $W^{s,p}(\Omega)$, hence $u_n \to u$ in $W^{s,p}(\Omega)$. This implies

$$\|u\|_{W^{s,p}(\Omega)} = \lim_{n\to\infty} \|u_n\|_{W^{s,p}(\Omega)} = 1. \tag{A.6}$$

On the other hand, as $|u|_{W^{s,p}(\Omega)} = 0$ it follows from the definition of the $W^{s,p}$-seminorm that

$$\int_\Omega \int_\Omega \frac{|(\mathrm{D}^\alpha u)(x) - (\mathrm{D}^\alpha u)(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y = 0,$$

for all multiindices α with $|\alpha| = k$, hence $\mathrm{D}^\alpha u$ is constant almost everywhere and Lemma A.1.16 implies $u \in \mathcal{P}^k(\Omega)$. Furthermore,

$$\int_\Omega \mathrm{D}^\alpha u \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega \mathrm{D}^\alpha u_n \,\mathrm{d}x = 0,$$

for all α with $0 \leq |\alpha| \leq k$, which implies $u = 0$ as in the proof of Lemma A.1.17. This is a contradiction to (A.6). □

Theorem A.1.19 (Generalized Bramble-Hilbert lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, let $p \in (1,\infty]$ and let $s = k + \sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in (0,1]$. Let $F\colon W^{s,p}(\Omega) \to \mathbb{R}$ be a functional with*

$$\begin{aligned} |F(u)| &\leq c_0\,\|u\|_{W^{s,p}(\Omega)} && \forall u \in W^{s,p}(\Omega),\\ |F(u+v)| &\leq c_1\,(|F(u)| + |F(v)|) && \forall u,v \in W^{s,p}(\Omega),\\ F(p) &= 0 && \forall p \in \mathcal{P}^k(\Omega). \end{aligned}$$

Then there exists $c_2 > 0$ such that

$$|F(u)| \leq c_2\,|u|_{W^{s,p}(\Omega)} \qquad \forall u \in W^{s,p}(\Omega).$$

Proof. The case $\sigma = 1$ corresponds to the usual formulation and can be found in [105], Theorem 5.1. For $\sigma < 1$ let $u_h \in \mathcal{P}^k(\Omega)$ be the interpolation polynomial defined in Lemma A.1.17. It holds that

$$\begin{aligned} |F(u)| &= |F(u - u_h + u_h)| \\ &\le c\,(|F(u - u_h)| + |F(u_h)|) \\ &= c\,|F(u - u_h)| \\ &\le c\,\|u - u_h\|_{W^{s,p}(\Omega)} \\ &\le c\,|u - u_h|_{W^{s,p}(\Omega)} \\ &= c\,|u|_{W^{s,p}(\Omega)}\,. \end{aligned}$$

□

A.1.2. Generalized nonstandard finite element estimates

Inverse estimates, where a strong norm is estimated via a weaker norm, are often used in the context of error estimates. Given certain assumptions on the triangulation of the polygonal domain Ω_h, a typical estimate is

$$\|v_h\|_{H^1(\Omega_h)} \le ch^{-1}\,\|v_h\|_{L^2(\Omega_h)} \qquad \forall v_h \in V_h,$$

where V_h is the space of (bi)linear finite elements on Ω_h; a more general version can be found in [16], Chapter II, Theorem 6.8. We have $V_h \subset W^{1,\infty}(\Omega_h)$ and it even holds that $V_h \subset W^{1+1/p-\varepsilon,p}(\Omega_h)$ for all $p \in (1,\infty)$ and $\varepsilon > 0$. However, inverse estimates like

$$\|v_h\|_{H^{3/2-\varepsilon}(\Omega_h)} \le c_\varepsilon h^{-1/2+\varepsilon}\,\|v_h\|_{H^1(\Omega_h)} \qquad \forall v_h \in V_h,$$

are not considered in general, which is partly due to the fact that fractional norms contain nonlocal terms. Within the following subsection we are going to generalize the nonstandard estimates obtained in [12] and [20] to general $p \in [2,\infty)$.

Within this subsection, let $\Omega_h \subset \mathbb{R}^n$ with $n \in \{1,2\}$ be a polygonal domain and let $\{\pi_h\}_{h>0}$ be a family of triangulations of Ω_h satisfying the usual regularity assumptions in the sense of Remark 2.2.7. Let V_h be the space of bilinear finite elements over Ω_h with respect to the triangulation π_h, and let $V_{h,0} \subset V_h$ be the subspace of elements with zero boundary conditions, cf. (2.60) and (2.61). With K we will denote an element of π_h. In addition, let $p \in [2,\infty)$ be fixed from now on.

The statements of the following lemma can be found in [48], (1.3.2.12) and Theorem 1.4.4.3.

Lemma A.1.20. *Let*

$$\rho(x,\partial K) = \inf_{y\in\partial K} |x - y|\,,$$

be the distance from x to the boundary of K. For arbitrary $x \in K$ and $\sigma \in (0,1)$ it holds that

$$\int_{\Omega_h\setminus K} \frac{1}{|x-y|^{n+\sigma p}}\,\mathrm{d}y \le c_{\sigma,p}\frac{1}{\rho(x,\partial K)^{\sigma p}}.$$

In addition, let $\hat{K}$ be the reference cell and let $\sigma \in (0,1/p)$, then it holds that

$$\int_{\hat{K}} \frac{u^p}{\rho(x,\partial\hat{K})^{\sigma p}}\,\mathrm{d}x \le c_{\hat{K},\sigma,p}\,\|u\|_{W^{\sigma,p}(\hat{K})} \qquad \forall u \in W^{\sigma,p}(\hat{K}).$$

Lemma A.1.21. *Let $k \in \mathbb{N}_0$, $\sigma \in (0,1)$ and $u \in W^{k+\sigma,p}(\Omega_h)$. Then it holds that*

$$|u|^p_{W^{k+\sigma,p}(\Omega_h)} \le c_{\sigma,p} \sum_{|\alpha|=k} \sum_{K\in\pi_h} \left(|\mathrm{D}^\alpha u|^p_{W^{\sigma,p}(K)} + \int_K \frac{|\mathrm{D}^\alpha u|^p}{\rho(x,\partial K)^{\sigma p}} \,\mathrm{d}x \right).$$

Proof. It holds that

$$\begin{aligned} |u|^p_{W^{k+\sigma,p}(\Omega_h)} &= \sum_{|\alpha|=k} \int_{\Omega_h}\int_{\Omega_h} \frac{|\mathrm{D}^\alpha u(x) - \mathrm{D}^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{|\alpha|=k} \left(\sum_{K\in\pi_h} \int_K\int_K \frac{|\mathrm{D}^\alpha u(x) - \mathrm{D}^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y + \sum_{\substack{K,K'\in\pi_h\\ K\ne K'}} \int_K\int_{K'} \frac{|\mathrm{D}^\alpha u(x) - \mathrm{D}^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y \right). \end{aligned} \tag{A.7}$$

The first sum within (A.7) equals $\sum_{|\alpha|=k}\sum_{K\in\pi_h} |\mathrm{D}^\alpha u|^p_{W^{\sigma,p}(K)}$, whereas the second sum can be estimated as follows

$$\begin{aligned} &\sum_{\substack{K,K'\in\pi_h\\ K\ne K'}} \int_K\int_{K'} \frac{|\mathrm{D}^\alpha u(x) - \mathrm{D}^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y \\ &\le c_p \left(\sum_{\substack{K,K'\in\pi_h\\ K\ne K'}} \int_K\int_{K'} \frac{|\mathrm{D}^\alpha u(x)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y + \sum_{\substack{K,K'\in\pi_h\\ K\ne K'}} \int_K\int_{K'} \frac{|\mathrm{D}^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y \right), \end{aligned} \tag{A.8}$$

Using Lemma A.1.20, we get

$$\begin{aligned} \sum_{\substack{K,K'\in\pi_h\\ K\ne K'}} \int_K\int_{K'} \frac{|\mathrm{D}^\alpha u(x)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y &= \sum_{K\in\pi_h} \int_K |\mathrm{D}^\alpha u|^p \left(\int_{\Omega_h\setminus K} \frac{1}{|x-y|^{n+\sigma p}} \,\mathrm{d}y \right) \mathrm{d}x \\ &\le c_{\sigma,p} \sum_{K\in\pi_h} \int_K \frac{|\mathrm{D}^\alpha u|^p}{\rho(x,\partial K)^{\sigma p}} \,\mathrm{d}x, \end{aligned}$$

which finishes the proof since the right hand side in (A.8) is symmetric with respect to x and y. ☐

In what follows let $i_h \colon C(\overline{\Omega_h}) \to V_h$ be the nodal interpolation operator, and for $x \in K$ and $\varphi \in W^{s,p}(K)$ let $\hat{x} \in \hat{K}$ and $\hat{\varphi} \in W^{s,p}(\hat{K})$ be the transformed point and function, respectively, on the reference triangle.

Theorem A.1.22. *Let $\sigma \in (0, 1/p)$ and $\tau \in [\sigma, 1]$. Then it holds that*

$$|u - i_h u|_{W^{1+\sigma,p}(\Omega_h)} \le c_{\sigma,p} h^{\tau-\sigma} |u|_{W^{1+\tau,p}(\Omega_h)} \qquad \forall u \in W^{1+\tau,p}(\Omega_h).$$

Proof. From Lemma A.1.21 we get

$$|u - i_h u|^p_{W^{1+\sigma,p}(\Omega_h)} \le \sum_{K\in\pi_h} \left(|u - i_h u|^p_{W^{1+\sigma,p}(K)} + \int_K \frac{|\nabla(u - i_h u)|^p}{\rho(x,\partial K)^{\sigma p}} \,\mathrm{d}x \right).$$

Using a transformation argument we get

$$
\begin{aligned}
&|u - i_h u|^p_{W^{1+\sigma,p}(K)} + \int_K \frac{|\nabla(u - i_h u)|^p}{\rho(x, \partial K)^{\sigma p}} \,\mathrm{d}x \\
&\leq \left\|B_K^{-1}\right\|^{n+\sigma p} \|B_K\|^p \left|\det B_K\right|^2 \left|\hat{u} - \widehat{i_h u}\right|^p_{W^{1+\sigma,p}(\hat{K})} + \left\|B_K^{-1}\right\|^{\sigma p} \|B_K\|^p \left|\det B_K\right| \int_{\hat{K}} \frac{\left|\nabla(\hat{u} - \widehat{i_h u})\right|^p}{\rho(\hat{x}, \partial\hat{K})^{\sigma p}} \,\mathrm{d}\hat{x} \\
&\leq c_{\sigma,p} \left\|B_K^{-1}\right\|^{\sigma p} \|B_K\|^p \left|\det B_K\right| \left(\left|\hat{u} - \widehat{i_h u}\right|^p_{W^{1,p}(\hat{K})} + \left|\hat{u} - \widehat{i_h u}\right|^p_{W^{1+\sigma,p}(\hat{K})} \right),
\end{aligned}
$$

where in the last step we used Lemma A.1.20. Due to the equivalence of norms on finite dimensional spaces we get

$$
\begin{aligned}
\left|\hat{u} - \widehat{i_h u}\right|^p_{W^{1,p}(\hat{K})} + \left|\hat{u} - \widehat{i_h u}\right|^p_{W^{1+\sigma,p}(\hat{K})} &\leq c_p \left(|\hat{u}|^p_{W^{1,p}(\hat{K})} + |\hat{u}|^p_{W^{1+\sigma,p}(\hat{K})} + \left\|\widehat{i_h u}\right\|^p_{W^{1,p}(\hat{K})} \right) \\
&\leq c_p \left(|\hat{u}|^p_{W^{1,p}(\hat{K})} + |\hat{u}|^p_{W^{1+\sigma,p}(\hat{K})} + \|\hat{u}\|^p_{W^{1,p}(\hat{K})} \right) \\
&\leq c_{\sigma,p} \|\hat{u}\|^p_{W^{1+\sigma,p}(\hat{K})} .
\end{aligned}
$$

These estimates shows that we can apply the Bramble-Hilbert lemma, Theorem A.1.19, and end up with

$$
\begin{aligned}
&|u - i_h u|^p_{W^{1+\sigma,p}(K)} + \int_K \frac{|\nabla(u - i_h u)|^p}{\rho(x, \partial K)^{\sigma p}} \,\mathrm{d}x \\
&\leq c_{\sigma,p} \left\|B_K^{-1}\right\|^{\sigma p} \|B_K\|^p \left|\det B_K\right| |\hat{u}|^p_{W^{1+\sigma,p}(\hat{K})} \\
&\leq c_{\sigma,p} \left\|B_K^{-1}\right\|^{\sigma p} \|B_K\|^p \left|\det B_K\right| |\hat{u}|^p_{W^{1+\tau,p}(\hat{K})} \\
&\leq c_{\sigma,p} \left\|B_K^{-1}\right\|^{\sigma p} \|B_K\|^p \left|\det B_K\right| \|B_K\|^{n+\tau p} \left\|B_K^{-1}\right\|^p \left|\det B_K^{-1}\right|^2 |u|^p_{W^{1+\tau,p}(K)} \\
&\leq c_{\sigma,p} h^{p(\tau-\sigma)} |u|^p_{W^{1+\tau,p}(K)} .
\end{aligned}
$$

□

Theorem A.1.23. *Let $\sigma \in (0, 1/p)$ and $\tau \in [0, \sigma]$, then it holds that*

$$
|u_h|_{W^{1+\sigma,p}(\Omega_h)} \leq c_{\sigma,p} h^{\tau-\sigma} |u_h|_{W^{1+\tau,p}(\Omega_h)} \qquad \forall u_h \in V_h.
$$

Proof. With Lemma A.1.21 it holds that

$$
|u_h|^p_{W^{1+\sigma,p}(\Omega_h)} \leq c_{\sigma,p} \sum_{K \in \pi_h} \left(|u_h|^p_{W^{1+\sigma,p}(K)} + \int_K \frac{|\nabla u_h|^p}{\rho(x, \partial K)^{\sigma p}} \,\mathrm{d}x \right). \tag{A.9}
$$

Now we use the equivalence of norms on finite dimensional subspaces, Lemma A.1.20 and the quasi-uniformity of π_h to get

$$
\begin{aligned}
\sum_{K \in \pi_h} \left(|u_h|^p_{W^{1+\sigma,p}(K)} + \int_K \frac{|\nabla u_h|^p}{\rho(x, \partial K)^{\sigma p}} \,\mathrm{d}x \right) &\leq \left\|B_K^{-1}\right\|^{n+\sigma p} \left|\det B_K\right|^2 |\hat{u_h}|^p_{W^{1+\sigma,p}(\hat{K})} \\
&\quad + \left\|B_K^{-1}\right\|^{\sigma p} \left|\det B_K\right| \int_{\hat{K}} \frac{|\nabla \hat{u_h}|^p}{\rho(\hat{x}, \partial\hat{K})^{\sigma p}} \,\mathrm{d}\hat{x} \\
&\leq c_{\sigma,p} \left\|B_K^{-1}\right\|^{\sigma p} \left|\det B_K\right| \|\hat{u_h}\|^p_{W^{1,p}(\hat{K})} \\
&\leq c_{\sigma,p} \left\|B_K^{-1}\right\|^{\sigma p} \left|\det B_K\right| \left|\det B_K^{-1}\right| \|u_h\|^p_{W^{1,p}(K)} \\
&\leq c_{\sigma,p} h^{-\sigma p} \|u_h\|^p_{W^{1,p}(K)} .
\end{aligned} \tag{A.10}
$$

Combining the estimates (A.9) and (A.10) we get

$$|u_h|_{W^{1+\sigma,p}(\Omega_h)} \leq c_{\sigma,p} h^{-\sigma} \|u_h\|_{W^{1,p}(\Omega_h)}, \tag{A.11}$$

which proves this theorem for $\tau = 0$. The proof for $\tau \in (0, \sigma]$ is more complicated. For $K \in \pi_h$, let ω_K denote all the elements of π_h which share at least on vertex with K, i.e.

$$\omega_K = \left\{ K' \in \pi_h \middle| \overline{K} \cap \overline{K'} \neq \emptyset \right\},$$

and let S_K be defined via

$$S_K = \operatorname{int}\left(\bigcup_{K' \in \omega_K} \overline{K'} \right).$$

We have

$$\begin{aligned} |u_h|^p_{W^{1+\sigma,p}(\Omega_h)} &= \sum_{K,K' \in \pi_h} \int_K \int_{K'} \frac{|\nabla u_h(x) - \nabla u_h(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y \\ &= \sum_{\substack{K,K' \in \pi_h \\ K' \in \omega_K}} \int_K \int_{K'} \frac{|\nabla u_h(x) - \nabla u_h(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y \\ &\quad + \sum_{\substack{K,K' \in \pi_h \\ K' \notin \omega_K}} \int_K \int_{K'} \frac{|\nabla u_h(x) - \nabla u_h(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y. \end{aligned} \tag{A.12}$$

The second part within (A.12) can easily be estimated,

$$\begin{aligned} \sum_{\substack{K,K' \in \pi_h \\ K' \notin \omega_K}} \int_K \int_{K'} \frac{|\nabla u_h(x) - \nabla u_h(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y &\leq c_{\sigma,p} h^{p(\tau-\sigma)} \sum_{\substack{K,K' \in \pi_h \\ K' \notin \omega_K}} \int_K \int_{K'} \frac{|\nabla u_h(x) - \nabla u_h(y)|^p}{|x-y|^{n+\tau p}} \,\mathrm{d}x \,\mathrm{d}y \\ &\leq c_{\sigma,p} h^{p(\tau-\sigma)} |u_h|^p_{W^{1+\tau,p}(\Omega_h)}. \end{aligned}$$

With the definition of S_K it follows that

$$\sum_{\substack{K,K' \in \pi_h \\ K' \in \omega_K}} \int_K \int_{K'} \frac{|\nabla u_h(x) - \nabla u_h(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y \leq \sum_{K \in \pi_h} |u_h|^p_{W^{1+\sigma,p}(S_K)}. \tag{A.13}$$

Now let $\hat{S_K}$ be a reference domain similar to S_K, T the affine transformation that maps $\hat{S_K}$ to S_K and $\hat{u_h} = u_h \circ T$ be the pull-back of $\hat{S_K}$. We obtain, by using (A.11) and Theorem A.1.19,

$$\begin{aligned} |\hat{u_h}|_{W^{1+\sigma,p}(\hat{S_K})} &= \inf_{p \in \mathcal{P}^1(\hat{S_K})} |\hat{u_h} - p|_{W^{1+\sigma,p}(\hat{S_K})} \\ &\leq c_{\sigma,p} \inf_{p \in \mathcal{P}^1(\hat{S_K})} \|\hat{u_h} - p\|_{W^{1,p}(\hat{S_K})} \\ &\leq c_{\sigma,p} |\hat{u_h}|_{W^{1+\tau,p}(\hat{S_K})}. \end{aligned} \tag{A.14}$$

Combining (A.14) with a scaling argument yields

$$|u_h|_{W^{1+\sigma,p}(S_K)} \leq c_{\sigma,p} h^{\tau-\sigma} |u_h|_{W^{1+\tau,p}(S_K)},$$

which, together with (A.13), implies

$$\begin{aligned}\sum_{\substack{K,K'\in\pi_h\\K'\in\omega_K}} \int_K \int_{K'} \frac{|\nabla u_h(x) - \nabla u_h(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y &\leq c_{\sigma,p} h^{p(\tau-\sigma)} \sum_{K\in\pi_h} |u_h|^p_{W^{1+\tau,p}(S_K)} \\ &\leq c_{\sigma,p} h^{p(\tau-\sigma)} |u_h|^p_{W^{1+\tau,p}(S_K)}.\end{aligned} \tag{A.15}$$

The case $\tau \in (0,\sigma]$ now follows from (A.12) and (A.15). □

Theorem A.1.24. *Let $u \in H_0^{1+\sigma}(\Omega_h)$ for a $\sigma \in (0,1/2)$, let $a\colon H^1(\Omega_h) \times H^1(\Omega_h) \to \mathbb{R}$ be a continuous, H_0^1-coercive bilinear form, and let the Ritz-projection $u_h \in V_{h,0}$ be defined via $a(u-u_h, v_h) = 0$ for all $v_h \in V_{h,0}$. Then it holds that*

$$\|u_h\|_{H^{1+\sigma}(\Omega_h)} \leq c_\sigma \|u\|_{H^{1+\sigma}(\Omega_h)}.$$

Proof. As $H^{1+\varepsilon}(\Omega_h) \hookrightarrow C(\overline{\Omega_h})$ for $n \leq 2$, the pointwise interpolation $i_h u$ is well-defined. It holds that

$$\begin{aligned}\|u_h\|_{H^{1+\sigma}(\Omega_h)} &\leq \|u\|_{H^{1+\sigma}(\Omega_h)} + \|u-u_h\|_{H^{1+\sigma}(\Omega_h)} \\ &\leq \|u\|_{H^{1+\sigma}(\Omega_h)} + \|u - i_h u\|_{H^{1+\sigma}(\Omega_h)} + \|i_h u - u_h\|_{H^{1+\sigma}(\Omega_h)}.\end{aligned}$$

Now we use Theorem A.1.22 for $(u - i_h u)$ and Theorem A.1.23 for $(i_h u - u_h)$ to obtain

$$\begin{aligned}\|u_h\|_{H^{1+\sigma}(\Omega_h)} &\leq c_\sigma \|u\|_{H^{1+\sigma}(\Omega_h)} + c_\sigma h^{-\sigma} \|i_h u - u_h\|_{H^1(\Omega_h)} \\ &\leq c_\sigma \|u\|_{H^{1+\sigma}(\Omega_h)} + c_\sigma h^{-\sigma} \left(\|u - i_h u\|_{H^1(\Omega_h)} + \|u-u_h\|_{H^1(\Omega_h)} \right) \\ &\leq c_\sigma \|u\|_{H^{1+\sigma}(\Omega_h)} + c_\sigma h^{-\sigma} \|u - i_h u\|_{H^1(\Omega_h)} \\ &\leq c_\sigma \|u\|_{H^{1+\sigma}(\Omega_h)} + c_\sigma h^{-\sigma} h^{\sigma} \|u\|_{H^{1+\sigma}(\Omega_h)} \\ &\leq c_\sigma \|u\|_{H^{1+\sigma}(\Omega_h)},\end{aligned}$$

where we also used Céa's Lemma, $(u-u_h) \in H_0^1(\Omega_h)$ and standard interpolation estimates. □

Corollary A.1.25. *Let Ω be convex with polygonal approximation $\Omega_h \subset \Omega$. Let $u \in H_0^{1+\sigma}(\Omega)$ with $\sigma \in (0,1/2)$, and let $u_h \in V_{h,0}$ be defined as in Theorem A.1.24. Then it holds that*

$$\|u_h\|_{H^{1+\sigma}(\Omega_h)} \leq c_\sigma \|u\|_{H^{1+\sigma}(\Omega)}.$$

Proof. This corollary can be proven using the same method as in the proof of Theorem A.1.24, note that $u_h \in H_0^1(\Omega_h)$ can, via extension by zero, be regarded as a function in $H_0^1(\Omega)$. □

Theorem A.1.26. *Let Ω_h be convex and let $u \in W_0^{1+\sigma,p}(\Omega_h)$ for $\sigma \in (0,1/p)$. Let $A \in \mathbb{R}^{2\times 2}$ be sufficiently regular in the sense of Theorem A.1.41, and let $u_h \in V_{h,0}$ be the Ritz-projection defined via $(\nabla(u-u_h), A\cdot\nabla v_h)_h = 0$ for all $v_h \in V_{h,0}$. Then the Ritz-projection is stable in $W_0^{1+\sigma,p}(\Omega_h)$.*

Proof. Using the same arguments as in the proof for Theorem A.1.24 we arrive at

$$\|u_h\|_{W^{1+\sigma,p}(\Omega_h)} \leq c_\sigma \|u\|_{W^{1+\sigma,p}(\Omega_h)} + c_\sigma h^{-\sigma} \left(\|u - i_h u\|_{W^{1,p}(\Omega_h)} + \|u - u_h\|_{W^{1,p}(\Omega_h)} \right),$$

and the result follows with standard interpolation estimates and [92],

$$\begin{aligned} \|u_h\|_{W^{1+\sigma,p}(\Omega_h)} &\leq c_\sigma \|u\|_{W^{1+\sigma,p}(\Omega_h)} + c_\sigma h^{-\sigma} h^\sigma \|u\|_{W^{1+\sigma,p}(\Omega_h)} \\ &\leq c_\sigma \|u\|_{W^{1+\sigma,p}(\Omega_h)} . \end{aligned}$$

□

Theorem A.1.27. *Let Ω_h be convex and let $u \in W_0^{1,p}(\Omega_h) \cap W^{1+\sigma,p}(\Omega_h)$ with $\sigma \in (0,1]$. Let the matrix $A \in \mathbb{R}^{2\times 2}$ be sufficiently regular in the sense of Theorem A.1.41, and let $u_h \in V_{h,0}$ be the Ritz-projection, i.e. $(\nabla (u - u_h), A \cdot \nabla v_h)_h = 0$ for all $v_h \in V_{h,0}$. Then it holds that*

$$\|u - u_h\|_{W^{1+\tau,p}(\Omega_h)} \leq c_{\sigma,\tau} h^{\sigma-\tau} \|u\|_{W^{1+\sigma,p}(\Omega_h)} ,$$

for all $\tau \in (0, 1/p)$ with $\tau \leq \sigma$.

Proof. Let $v = u - i_h u$, with Theorem A.1.26 it follows that

$$\|u_h - i_h u\|_{W^{1+\tau,p}(\Omega_h)} \leq c_\tau \|u - i_h u\|_{W^{1+\tau,p}(\Omega_h)} ,$$

and the result follows with standard interpolation estimates. □

A.1.3. Regularity results for elliptic partial differential equations

Within this subsection we are going to present some general regularity results concerning the solutions of partial differential equations of second order.

Theorem A.1.28. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and open C^∞-domain. Let $s \geq 0$, $s \neq 1/2$, $f \in H^{s-1}(\Omega)$ and $g \in H^{s+1/2}(\Gamma)$. Then the weak solution u of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad\; u = g & \text{on } \Gamma, \end{cases}$$

belongs to $H^{s+1}(\Omega)$ and there holds the estimate $\|u\|_{H^{s+1}(\Omega)} \leq c_s \left(\|f\|_{H^{s-1}(\Omega)} + \|g\|_{H^{s+1/2}(\Gamma)} \right)$.

Proof. This theorem can be found in [51], Theorem 9.1.20. □

Theorem A.1.29. *Let $\Omega \subset \mathbb{R}^2$ be a rectangle, and let $u \in H_0^1(\Omega)$ be the weak solution to*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad\; u = 0 & \text{on } \Gamma. \end{cases}$$

If $f \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $f(x_i) = 0$ in all corner points x_i of Ω, then it holds that $u \in C^{3,\alpha}(\overline{\Omega})$.

Proof. This theorem follows from [31], Remark 1, and the references cited therein, [44] and [107]. □

Theorem A.1.30. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Let $1/2 \geq t > s > 0$, and let the coefficients of the matrix $A \in \mathbb{R}^{2\times 2}$ belong to $C^{0,t}(\overline{\Omega})$. If $f \in H^{-1+s}(\Omega)$, then there exists a unique $u \in H_0^{1+s}(\Omega)$ as the weak solution to*

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

and it holds that $\|u\|_{H^{1+s}(\Omega)} \leq c_s \|f\|_{H^{-1+s}(\Omega)}$.

Proof. This theorem follows with [51], Theorem 9.1.25, and [87]. □

Theorem A.1.31. *Let $\Omega \subset \mathbb{R}^2$ be bounded and either convex or C^1. Let $A \subset R^{2\times 2}$ be uniformly elliptic on $\overline{\Omega}$ with continuous coefficients. Let $p \in (1, \infty)$ and $f \in W^{-1,p}(\Omega)$. Then there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ of*

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

and there holds the estimate $\|u\|_{W_0^{1,p}(\Omega)} \leq c_p \|f\|_{W^{-1,p}(\Omega)}$.

Proof. This theorem can be found in [4], Theorem 1. □

Theorem A.1.32. *Let $\Omega \subset \mathbb{R}^2$ be bounded and either Lipschitz or with sufficiently smooth boundary Γ in the sense of Remark A.1.33. Let $A \subset \mathbb{R}^{2\times 2}$ be a uniformly elliptic matrix, and let $f \in W^{-1,p}(\Omega)$ for a $p \in (Q, P)$, where $P > 2$ depends on the ellipticity constant of A and $1/P + 1/Q = 1$. Then there exists a unique solution $u \in W_0^{1,p}(\Omega)$ of*

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

and it holds that

$$\|\nabla u\|_{L^p(\Omega)} \leq c_p \|f\|_{W^{-1,p}(\Omega)} .$$

For the constant P it holds that

$$\lim_{A \to \mathrm{I}} P(A) = \infty.$$

Proof. This theorem can be found in [86], Theorem 1. □

Remark A.1.33. The boundary Γ of the domain $\Omega \subset \mathbb{R}^2$ is sufficiently smooth in the sense of Theorem A.1.32 if the equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

has a unique solution $u \in W_0^{1,p}(\Omega)$ for all $f \in W^{-1,p}(\Omega)$ for that certain $p < \infty$, and it also holds that

$$\|u\|_{W^{1,p}(\Omega)} \leq c_p \|f\|_{W^{-1,p}(\Omega)} ,$$

where the constant c_p may depend on p but not on f. In [4] it is shown that for convex or C^1 domains this holds true for all $p \in (1, \infty)$; in [67], Theorem 0.5, it is shown that for every Lipschitz domain there exists $\varepsilon > 0$ such that this holds true for all $p \in \left(\frac{4+\varepsilon}{3+\varepsilon}, 4 + \varepsilon\right)$.

Theorem A.1.34. *Under the same assumptions on Ω, A, f and p as in Theorem A.1.31 and Theorem A.1.32, let $g \in W^{1-1/p,p}(\Gamma)$. Then there exists a unique $u \in W^{1,p}(\Omega)$ as the weak solution to*

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u) = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma, \end{cases} \tag{A.16}$$

and it also holds that $\|u\|_{W^{1,p}(\Omega)} \leq c_p \left(\|f\|_{W^{-1,p}(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)} \right)$.

Proof. This theorem follows with Theorem A.1.3, Theorem A.1.31 and Theorem A.1.32. □

Definition A.1.35. A domain $\Omega \subset \mathbb{R}^2$ is said to be a curvilinear, right-angled $C^{1,1}$-domain if its boundary is either $C^{1,1}$, or if there exist finitely many points $(x_i)_{1 \leq i \leq N}$ on the boundary Γ such that Γ is piecewise $C^{1,1}$, and the angle between the tangents in x_i is a right angle for all $i \in \{1, \ldots, N\}$.

Theorem A.1.36. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, curvilinear, right-angled $C^{1,1}$-domain, let the matrix $A \subset \mathbb{R}^{2\times 2}$ be uniformly elliptic with Lipschitz continuous coefficients, and let*

$$D\left(A, L^p(\Omega)\right) = \{u \in L^p(\Omega) | -\operatorname{div}(A \cdot \nabla u) \in L^p(\Omega)\}$$

be the domain of the maximal extension of the operator $u \mapsto -\operatorname{div}(A \cdot \nabla u)$ in $L^p(\Omega)$. Then,

$$u \mapsto \{-\operatorname{div}(A \cdot \nabla u)\,, u|_\Gamma\} \tag{A.17}$$

is an isomorphism from $D(A, L^p(\Omega))$ onto $L^p(\Omega) \times W^{-1/p,p}(\Gamma)$.

Proof. This theorem is proven in [48], Theorem 2.5.2.1, in the case of a $C^{1,1}$-domain. As that proof just relies on the fact that (A.17) is an isomorphism from $W^{2,q}(\Omega)$ onto $L^q(\Omega) \times W^{2-1/q,q}(\Gamma)$ for the conjugate index $q = p/(p-1)$, it can be extended to domains with right-angled vertices, cf. [48], Section 5.2. □

Corollary A.1.37. *Under the same assumptions on Ω and A as in Theorem A.1.36, let $f \in L^p(\Omega)$ and $g \in W^{-1/p,p}(\Gamma)$. Then there exists a unique $u \in L^p(\Omega)$ which weakly solves* (A.16) *and it holds that*

$$\|u\|_{L^p(\Omega)} \leq c_p \left(\|f\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} \right).$$

Theorem A.1.38. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, curvilinear, right-angled $C^{1,1}$-domain, let A be a uniformly elliptic matrix with Lipschitz continuous coefficients. Let $f \in L^p(\Omega)$ and $g \in W^{2-1/p,p}(\Gamma)$, then there exists a unique $u \in W^{2,p}(\Omega)$ which solves* (A.16) *and there holds the estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq c_p \left(\|f\|_{L^p(\Omega)} + \|g\|_{W^{2-1/p,p}(\Gamma)} \right).$$

Proof. This theorem can be found in [48], Theorem 2.4.2.5 and Section 5.2. □

Remark A.1.39. The estimate in Theorem A.1.38 is not explicitly mentioned in the source cited, but can be shown using either [3], Theorem 5.8 or the closed graph theorem, cf. [3], Theorem 5.9. Let $X = L^p(\Omega) \times W^{2-1/p,p}(\Gamma)$, $Y = W^{2,p}(\Omega)$ and $L\colon X \to Y$ be the linear solution operator to (A.16). Let $(f_n, g_n)_{n\in\mathbb{N}} \subset X$, $u_n = L(f_n, g_n) \in Y$ and $(f_n, g_n, u_n) \to (f, g, u)$ in $X \times Y$ for $n \to \infty$. It remains to show that $u = L(f, g)$. First it holds that

$$f \leftarrow f_n = -\operatorname{div}(A \cdot \nabla u_n) \to -\operatorname{div}(A \cdot \nabla u) \qquad \text{in } L^p(\Omega) \text{ for } n \to \infty,$$

hence $-\operatorname{div}(A \cdot \nabla u) = f$. Let $\gamma\colon W^{2,p}(\Omega) \to W^{2-1/p,p}(\Gamma)$ be the trace operator. It holds that

$$\begin{aligned}\|\gamma u - g\|_{W^{2-1/p,p}(\Gamma)} &\le \|\gamma u - \gamma u_n\|_{W^{2-1/p,p}(\Gamma)} + \|\gamma u_n - g\|_{W^{2-1/p,p}(\Gamma)} \\ &\le c\,\|u - u_n\|_{W^{2,p}(\Omega)} + \|g_n - g\|_{W^{2-1/p,p}(\Gamma)} \to 0 \qquad \text{for } n \to \infty,\end{aligned}$$

whereas we used the continuity of γ, Theorem A.1.3, $\gamma u_n = g_n$ by definition of u_n and g_n and the fact that $u_n \to u$, $g_n \to g$ in $W^{2,p}(\Omega)$ and $W^{2-1/p,p}(\Gamma)$, respectively. As a result, $u = L(f,g)$, which proves the continuity of L.

The following lemma can be proven using an interpolation argument, Corollary A.1.37 and Theorem A.1.38.

Lemma A.1.40. *Under the assumptions on Ω and the matrix A from Theorem A.1.38, let $f \in L^p(\Omega)$ and $g \in W^{s-1/p,p}(\Gamma)$ for a $s \in [0,2]$. Then there exists a unique weak solution $u \in W^{s,p}(\Omega)$ to* (A.16)*, and it holds that*

$$\|u\|_{W^{s,p}(\Omega)} \le c_p \left(\|f\|_{L^p(\Omega)} + \|g\|_{W^{s-1/p,p}(\Gamma)} \right).$$

A.1.4. On the stability of the Ritz-projection in general polygonal domains

It follows from Céa's lemma that the Ritz-projection of the solution to a uniformly elliptic linear partial differential equation of second order is stable in H^1 in both convex and non-convex domains. Within this subsection we are going to investigate on the stability of the Ritz-projection in $W^{1,p}$ for $p \neq 2$.

Theorem A.1.41. *Let $\Omega_h \subset \mathbb{R}^2$ be polygonal and convex and let the matrix $A \subset \mathbb{R}^{2\times 2}$ be symmetric, uniformly elliptic and Lipschitz such that there exists $\varepsilon > 0$ such that the mapping $u \mapsto -\operatorname{div}(A \cdot \nabla u)$ is a homeomorphism from $W_0^{1,q}(\Omega_h) \cap W^{2,q}(\Omega_h)$ onto $L^q(\Omega_h)$ for all $q \in (1, 2+\varepsilon]$. Let $u \in W_0^{1,p}(\Omega_h)$ for a $p \in [2,\infty]$, and let $u_h \in V_{h,0}$ be its Ritz-projection. Then it holds that*

$$\|u_h\|_{W_0^{1,p}(\Omega_h)} \le c\,\|u\|_{W_0^{1,p}(\Omega_h)},$$

where c is independent of u and p.

Proof. This theorem can be found in [92]. □

Remark A.1.42. The assumptions of Theorem A.1.41 are certainly true if A is the identity matrix, but they are also true if the transformation induced by the matrix A do not map the convex angles of Ω_h into non-convex ones, cf. [48], Section 5.2, which is certainly fulfilled if A is sufficiently close to the identity matrix.

In order to extend the result of Theorem A.1.41 onto non-convex domains, we need an assumption on the domain.

Assumption A.1.43. Let $\Omega_h \subset \mathbb{R}^2$ be a polygonal and non-convex domain with maximum interior angle $\alpha \in (0, 2\pi)$. Let $\{\pi_h\}_{h>0}$ be a family of triangulations of Ω_h fulfilling the usual regularity assumptions in the sense of Remark 2.2.7. We say that Ω_h fulfills Assumption A.1.43 if there exists a convex polygonal domain $\tilde{\Omega} \supset \Omega_h$, such that each triangulation π_h can be extended to a triangulation $\tilde{\pi}_h$ of $\tilde{\Omega}$, and the family $\{\tilde{\pi}_h\}_{h>0}$ is also quasi-uniform with the same constant c_0, cf. Definition 2.2.6.

Theorem A.1.44. *Let $\Omega_h \subset \mathbb{R}^2$ fulfill Assumption A.1.43. Let $u \in C(\overline{\Omega_h}) \cap W^{1,1}(\Omega_h)$, and let u_h be its Ritz-projection with respect to the Laplacian which interpolates u on the boundary Γ_h. Then it holds that*

$$\|u - u_h\|_{L^\infty(\Omega_h)} \leq c\,|\ln h| \inf_{v_h \in V_h} \|u - v_h\|_{L^\infty(\Omega_h)}\,.$$

If $u \in W^{1,\infty}(\Omega_h)$, then it holds that

$$\|u - u_h\|_{W^{1,\infty}(\Omega_h)} \leq c\,|\ln h| \inf_{v_h \in V_h} \|u - v_h\|_{W^{1,\infty}(\Omega_h)}\,.$$

Proof. This theorem can be found in [96], Theorem 2. □

Remark A.1.45. The statement of Theorem A.1.44 remains unchanged if the Ritz-projection is taken with respect to the differential operator induced by a uniformly elliptic matrix $A \in \mathbb{R}^{2\times 2}$ with Lipschitz coefficients. This holds due to the fact that all the result needed to prove Theorem A.1.44 rely on regularity results for the corresponding differential operator, cf. [88], [97] and [98], which are known to also holds in that case, cf. [48], Section 5.2.

Corollary A.1.46. *Let $\Omega_h \subset \mathbb{R}^2$ fulfill Assumption A.1.43. Let $p \in [2, \infty]$, and let the matrix $A \in \mathbb{R}^{2\times 2}$ be uniformly elliptic with Lipschitz coefficients. Let $u \in W^{1,p}(\Omega_h)$ with Ritz-projection u_h which interpolates u on the boundary,*

$$\left(\nabla\left(u - u_h\right), A \cdot \nabla v_h\right)_h = 0 \qquad \forall v_h \in V_{h,0}.$$

Then, for h sufficiently small, it holds that

$$\|u_h\|_{W^{1,p}(\Omega_h)} \leq c\,|\ln h|^{\frac{p-2}{p}} \|u\|_{W^{1,p}(\Omega_h)}\,.$$

Proof. The case $p = 2$ is well-known, the case $p = \infty$ follows with Theorem A.1.44 and Remark A.1.45, and what is left follows with interpolation, Theorem A.1.9. □

Within this last theorem we are going to collect the previous results and also extend them to the dual exponent.

Theorem A.1.47. *Let $\Omega_h \subset \mathbb{R}^2$ be a bounded and polygonal Lipschitz domain, let $p \in [4/3, 4]$ and let the matrix A be uniformly elliptic with Lipschitz continuous coefficients and sufficiently close to the identity matrix. If $f \in W^{-1,p}(\Omega_h)$, then there exists a unique $u \in W_0^{1,p}(\Omega_h)$ as the weak solution to*

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u) = f & \text{on } \Omega_h, \\ u = 0 & \text{in } \Gamma_h, \end{cases}$$

and it holds that

$$\|u\|_{W^{1,p}(\Omega_h)} \leq c_p \|f\|_{W^{-1,p}(\Omega_h)}\,. \tag{A.18}$$

Furthermore, for the Ritz-projection u_h of u it holds that

$$\|u_h\|_{W^{1,p}(\Omega_h)} \leq c_p\,|\ln h|^{\left|\frac{p-2}{p}\right|} \|u\|_{W^{1,p}(\Omega_h)}\,. \tag{A.19}$$

Proof. The existence and stability part for u, (A.18), follows with Theorem A.1.32 and Remark A.1.33. Now we concentrate on the stability estimate for the Ritz-projection (A.19). The case $p \geq 2$ is a direct consequence of Corollary A.1.46, we now focus on $p \in [4/3, 2)$. Let $q = p/(p-1) \in (2, 4]$ be the dual exponent. Due to Theorem A.1.1, there exists $s \in W^{-1,q}(\Omega_h)$ with $\|s\|_{W^{-1,q}(\Omega_h)} = 1$ and

$$(s, u - u_h)_{W^{-1,q}, W_0^{1,p}} = \|u - u_h\|_{W_0^{1,p}(\Omega_h)} .$$

Now let $w \in W_0^{1,q}(\Omega_h)$ be the solution to

$$(\nabla w, A \cdot \nabla v) = (s, v)_{W^{-1,q}, W_0^{1,p}} \qquad \forall v \in W_0^{1,p}(\Omega_h),$$

with Ritz-projection w_h. The existence of such a w is ensured by Remark A.1.33, it also holds that

$$\|w\|_{W^{1,q}(\Omega_h)} \leq c_p.$$

Using the first part of this theorem, we get

$$\begin{aligned}
\|u - u_h\|_{W_0^{1,p}(\Omega_h)} &= (s, u - u_h)_{W^{-1,q}, W_0^{1,p}} = (\nabla w, A \cdot \nabla (u - u_h))_h \\
&= (\nabla (w - w_h), A \cdot \nabla u)_h \leq c \|w - w_h\|_{W_0^{1,q}(\Omega_h)} \|u\|_{W_0^{1,p}(\Omega_h)} \\
&\leq c_p |\ln h|^{\frac{q-2}{q}} \|w\|_{W_0^{1,q}(\Omega_h)} \|u\|_{W_0^{1,p}(\Omega_h)} \\
&\leq c_p |\ln h|^{\frac{2-p}{p}} \|u\|_{W_0^{1,p}(\Omega_h)} .
\end{aligned}$$

□

A.2. A-priori error estimates for nonhomogeneous Dirichlet problems in curved domains

Most a-priori error estimates estimate the error between the continuous solution u of a partial differential equation and its discrete counterpart u_h in the L^2-, or the H^1-norm, where the underlying domain is in general polygonal and convex. In [17] error estimates where the domain no longer needs to be polygonal nor convex are proven. The discrete equation is formulated on a polygonal approximation of that curved domain and then extended onto the whole original domain. Within this section we are going to generalize these results in order to estimate the error in L^p and $W^{1,p}$ with $p > 2$.

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz, curvilinear $C^{1,1}$-domain which need not be convex and let Ω_h be a polygonal approximation of Ω, a precise definition will be given in Subsection A.2.1. We wish to estimate the error between the solution u and u_h to the following partial differential equation and its finite-element approximation:

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma, \end{cases} \qquad\qquad \begin{cases} L_h u_h = f & \text{in } \Omega_h, \\ u_h = g_h & \text{on } \Gamma_h, \end{cases} \tag{A.20}$$

where L is a uniformly elliptic partial differential operator of second order with discrete approximation L_h, f and g are given functions and g_h is an approximation to g on Γ_h. The needed regularity of the involved operators and functions are given below, the definition of g_h is given in Definition A.2.12. With the operator L we associate a symmetric matrix A in the sense that

$$Lu = -\operatorname{div}(A \cdot \nabla u), \tag{A.21}$$

where A is assumed to be uniformly elliptic on $\Omega\cup\Omega_h$ for all $h > 0$ sufficiently small. Equation (A.21) implies that (A.20) is just the strong form of

$$\begin{cases} (\nabla u, A\cdot\nabla v) = (f,v) \quad \forall v \in H_0^1(\Omega), \\ \qquad\qquad u|_\Gamma = g, \end{cases} \qquad \begin{cases} (\nabla u_h, A\cdot\nabla v_h)_h = (f,v_h)_h \quad \forall v_h \in V_{h,0}, \\ \qquad\qquad u_h|_{\Gamma_h} = g_h, \end{cases} \tag{A.22}$$

where $V_{h,0}$ is the usual space of (bi)linear finite elements with zero boundary conditions as specified in Subsection A.2.2. In what follows, we will focus on the following two cases:

- The matrix A has Lipschitz coefficients, $f \in L^p(\Omega)$ and $g \in W^{s,p}(\Gamma)$ for some $s > 0$ and $p \in (1,\infty)$.
- The matrix A has coefficients in $C^{0,1/2}(\overline{\Omega\cup\Omega_h})$, $f \in H^{-1/2+\varepsilon}(\Omega)$ and $g = 0$.

In addition, some regularity assumptions have to be imposed onto the boundary Γ which will be specified later on. Our main result, obtained for the first case, is the following theorem.

Theorem A.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, curvilinear, right-angled $C^{1,1}$-domain in the sense of Definition A.1.35, let A be a symmetric and uniformly elliptic matrix with Lipschitz coefficients, let*

$$p \in \begin{cases} [2,\infty) & \text{if } \Omega \text{ is convex,} \\ [2,4] & \text{else,} \end{cases}$$

and let $s \in [1-1/p, 2-1/p]$. Let u and u_h be defined as the solutions to (A.22) *with $f \in L^p(\Omega)$, $g \in W^{s,p}(\Gamma)$ and g_h defined as in* (A.2.12). *Then it holds that*

$$\begin{aligned} \|u-u_h\|_{L^p(\Omega)} &\le c_p c_\Omega(h)\left(h^2\|f\|_{L^p(\Omega)} + h^{s+1/p}\|g\|_{W^{s,p}(\Gamma)}\right), \\ \|u-u_h\|_{W^{1,p}(\Omega)} &\le c_p c_\Omega(h)\left(h\|f\|_{L^p(\Omega)} + h^{s-1+1/p}\|g\|_{W^{s,p}(\Gamma)}\right), \end{aligned}$$

where $c_\Omega(h)$ from Definition A.2.15 contributes a logarithmic factor if Ω is not convex.

Due to the approximation of curved boundaries through piecewise polygonal ones, it is sometimes necessary to extend functions onto a bigger domain.

Lemma A.2.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded with Lipschitz boundary, let $s > 0$ and $p \in (1,\infty)$. Then there exists a linear and continuous extension operator $E_s\colon W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^n)$ such that $E_s\phi|_\Omega = \phi$ for all $\phi \in W^{s,p}(\Omega)$. Furthermore E_s can be chosen independently of s.*

Proof. This lemma can be found in [48], Theorem 1.4.3.1. The independence of s is shown in [8] and [99]. □

From now on, if not stated otherwise, we will always assume that $p \in (1,\infty)$.

A.2.1. On the approximation of Ω with Ω_h

Within this subsection we give a precise definition of the polygonal domain Ω_h and show how to transform the boundary data g onto Γ_h. Let $\left(x^{(i)}\right)_{1\le i\le N}$ be a set of $N \in \mathbb{N}$ points on Γ and let $x^{(N+1)} = x^{(1)}$. Now let Ω_h be the polygonal domain with vertices $\left(x^{(i)}\right)_{1\le i\le N}$, and let $\Gamma_h^{(j)}$ be the edge from $x^{(j)}$ to $x^{(j+1)}$. We assume a quasiuniform distribution of the points $\left(x^{(i)}\right)_{1\le i\le N}$, i.e. there exists $c_0 > 0$ such that

$$\liminf_{N\to\infty}\left(\frac{\min_{1\le j\le N}\left|\Gamma_h^{(j)}\right|}{\max_{1\le j\le N}\left|\Gamma_h^{(j)}\right|}\right) \ge c_0.$$

In addition, let $\Gamma^{(j)}$ denote the part of Γ between $x^{(j)}$ and $x^{(j+1)}$, cf. Figure A.1. Let $h = \max_{1\le j\le N}\left|\Gamma_h^{(j)}\right|$ be the length of the longest edge of Ω_h. Let $n_h^{(j)}$ be the unit normal vector on $\Gamma_h^{(j)}$ pointing outwards

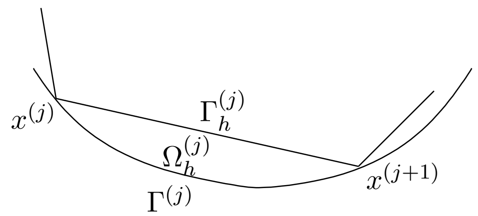

Figure A.1.: Approximation of Ω via Ω_h

and let $x_h(t)$ be a parametrization of $\Gamma_h^{(j)}$ by arc length. Furthermore let $\delta(x_h(t))$ be the distance between $x_h(t)$ and Γ along $n_h^{(j)}$ and let

$$X_h(t) = x_h(t) + \delta(x_h(t))\, n_h^{(j)}. \tag{A.23}$$

We assume that h is small enough such that $X_h(t)$ is well defined. Now, for an arbitrary function g defined on Γ, let

$$\begin{aligned} &\tilde{g}\colon \Gamma_h \to \mathbb{R}, \\ &\tilde{g}(x_h(t)) = g(X_h(t)) \quad \text{for } x_h(t) \in \Gamma_h^{(j)}, \end{aligned} \tag{A.24}$$

be the orthogonal projection onto Γ. This mapping also has an inverse, and as Γ is Lipschitz if follows that there exist constants $c_1, c_2 > 0$, independent of h and $p \in [1,\infty]$, such that

$$c_1 \|g\|_{L^p(\Gamma)} \le \|\tilde{g}\|_{L^p(\Gamma_h)} \le c_2 \|g\|_{L^p(\Gamma)}\,. \tag{A.25}$$

Now let $\Omega_h^{(j)}$ be the region bounded by $\Gamma^{(j)}$ and $\Gamma_h^{(j)}$, cf. Figure A.1. We rotate the coordinate system such that $\Gamma_h^{(i)}$ has its left endpoint at the origin, and it further holds that

$$\Gamma_h^{(j)} = \left\{ (x,y) \in \mathbb{R}^2 \middle|\, y = 0,\ x \in [0, c_1 h] \right\}, \tag{A.26}$$

$$\Gamma^{(j)} = \left\{ (x,y) \in \mathbb{R}^2 \middle|\, y = \delta(x) \ge 0,\ x \in [0, c_1 h] \right\}, \tag{A.27}$$

as well as

$$|\delta(x)| \le c_2 h^2, \qquad \left|\delta'(x)\right| \le c_3 h. \tag{A.28}$$

Let $\varphi \in W^{1,p}(\Omega \cup \Omega_h)$ be arbitrary and let

$$f_1 = \begin{pmatrix} 0 \\ \varphi^p \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 \\ y\varphi^p \end{pmatrix}.$$

We get

$$\operatorname{div}(f_1) = p\varphi^{p-1}\partial_y\varphi, \qquad \operatorname{div}(f_2) = \varphi^p + yp\varphi^{p-1}\partial_y\varphi,$$

and conclude with the divergence theorem that

$$\begin{aligned}\int_{\Omega_h^{(j)}} p\varphi^{p-1}\partial_y\varphi \,\mathrm{d}x &= \int_{\Gamma^{(j)}} \varphi^p(1+\delta_x'^2)^{-1/2}\,\mathrm{d}x - \int_{\Gamma_h^{(j)}} \varphi^p \,\mathrm{d}x,\\ \int_{\Omega_h^{(j)}} \varphi^p + yp\varphi^{p-1}\partial_y\varphi \,\mathrm{d}x &= \int_{\Gamma^{(j)}} y\varphi^p(1+\delta_x'^2)^{-1/2}\,\mathrm{d}x.\end{aligned} \tag{A.29}$$

Using Hölder's inequality, (A.28) and (A.29) leads to

$$\|\varphi\|^p_{L^p(\Omega_h^{(j)})} \le ch^2 \|\varphi\|^p_{L^p(\Gamma^{(j)})} + ph^2 \|\varphi\|^{p-1}_{L^p(\Omega_h^{(j)})} \|\partial_y\varphi\|_{L^p(\Omega_h^{(j)})},$$

and

$$\|\varphi\|^p_{L^p(\Gamma^{(j)})} \le p\|\varphi\|^{p-1}_{L^p(\Omega_h^{(j)})} \|\partial_y\varphi\|_{L^p(\Omega_h^{(j)})} + c\|\varphi\|^p_{L^p(\Gamma_h^{(j)})}.$$

Applying the generalized Young's inequality we end up with

$$\begin{aligned}\|\varphi\|^p_{L^p(\Omega_h^{(j)})} &\le c_p\left(h^2 \|\varphi\|^p_{L^p(\Gamma^{(j)})} + h^{2p}\|\partial_y\varphi\|^p_{L^p(\Omega_h^{(j)})}\right),\\ \|\varphi\|^p_{L^p(\Omega_h^{(j)})} &\le c_p\left(h^2 \|\varphi\|^p_{L^p(\Gamma_h^{(j)})} + h^{2p}\|\partial_y\varphi\|^p_{L^p(\Omega_h^{(j)})}\right).\end{aligned} \tag{A.30}$$

In case of $\varphi|_{\Gamma_h^{(j)}} = 0$ we also have

$$\|\varphi\|^p_{L^p(\Gamma^{(j)})} \le c_p h^{2p-2}\|\partial_y\varphi\|^p_{L^p(\Omega_h^{(j)})}. \tag{A.31}$$

Lemma A.2.3. *Let $s \in [1,2]$, $w \in W^{s,p}(\Omega)$, possibly extended with E_s. Let $g = w|_\Gamma$ denote the trace of w on Γ. Then it holds that*

$$\|w - \tilde{g}\|_{L^p(\Gamma_h)} \le c_p h^{2-\frac{4-2s}{p}} \|w\|_{W^{s,p}(\Omega)}.$$

Proof. Throughout this proof we will assume that the coordinate system is rotated as before, cf. (A.26) and (A.27). Now we set $\varphi(x,y) = w(x,y) - w(x,0)$ and use (A.25) and (A.31) to obtain

$$\|w-\tilde{g}\|^p_{L^p(\Gamma_h^{(j)})} = \int_{\Gamma_h^{(j)}} |\tilde{\varphi}|^p \,\mathrm{d}x \le c_p\|\varphi\|^p_{L^p(\Gamma^{(j)})} \le c_p h^{2p-2}\|\partial_y w\|^p_{L^p(\Omega_h^{(j)})}. \tag{A.32}$$

Applying (A.30) to $\partial_y w$ we also get

$$\|w-\tilde{g}\|^p_{L^p(\Gamma_h^{(j)})} \le c_p\left(h^{2p}\|\partial_y w\|^p_{L^p(\Gamma^{(j)})} + h^{4p-2}\|w\|^p_{W^{2,p}(\Omega_h^{(j)})}\right), \tag{A.33}$$

now we sum up (A.33) for all j and use Theorem A.1.3 in the sense that

$$\|w\|_{W^{1,p}(\Gamma)} \le c_p \|w\|_{W^{2,p}(\Omega)},$$

to prove the estimate for $s = 2$. Summing up (A.32) over all j yields the estimate for $s = 1$, and what is left follows by interpolation. □

Lemma A.2.4. *Let $w_1 \in W^{1,p}(\Omega)$ and $w_2 \in W^{1,p}(\Omega_h)$, both possibly extended by E_s. Then it holds that*

$$\begin{aligned} \|w_1\|^p_{L^p(\Omega_h \triangle \Omega)} &\le c_p \left(h^2 \|w_1\|^p_{L^p(\Gamma)} + h^{2p} \|w_1\|^p_{W^{1,p}(\Omega)} \right), \\ \|w_2\|^p_{L^p(\Omega_h \setminus \Omega)} &\le c_p \left(h^2 \|w_2\|^p_{L^p(\Gamma_h)} + h^{2p} \|w_2\|^p_{W^{1,p}(\Omega_h)} \right). \end{aligned}$$

Proof. This lemma follows from (A.30) by summing up for all j. □

A.2.2. Finite elements and interpolation results

Within this subsection we will introduce finite elements on the domain Ω_h as well as on the boundary Γ_h. We start with the discretization of the domain, let $\{\pi_h\}_{h>0}$ be a family of admissible triangulations of Ω_h in the sense of Definition 2.2.4 fulfilling the usual regularity assumptions in the sense of Remark 2.2.7. Let V_h and $V_{h,0}$ be the spaces of (bi)linear finite elements, where the latter is the one with homogeneous boundary values, cf. Definition 2.60 and Definition 2.61. The finite elements on the boundary are defined as follows.

Definition A.2.5. Let $S_h(\Gamma_h)$ consist of all continuous functions θ on Γ_h which are linear on each of the intervals $\Gamma_h^{(j)}$. The space $S_h(\Gamma)$ is defined as the space of transformed functions of $S_h(\Gamma_h)$ onto Γ, i.e.

$$S_h(\Gamma) = \left\{ \theta |\, \tilde{\theta} \in S_h(\Gamma_h) \right\}.$$

Furthermore, let $S_h^1(\Gamma)$ be the space of all functions $\theta^1 \in W^{2,\infty}(\Gamma)$ which are cubic polynomials by arclength on each of the intervals $\Gamma^{(j)}$.

We also introduce orthogonal projections with respect to the scalar product in L^2 over Γ_h and Γ, respectively, which will be denoted via

$$\begin{aligned} Q_h &\colon L^2(\Gamma_h) \to S_h(\Gamma_h), \\ \hat{Q}_h &\colon L^2(\Gamma) \to S_h(\Gamma), \end{aligned} \tag{A.34}$$

and

$$Q_h^1 \colon L^2(\Gamma) \to S_h^1(\Gamma). \tag{A.35}$$

The following lemma can be shown using the Bramble-Hilbert lemma, Theorem A.1.19.

Lemma A.2.6. *Let $w \in W^{s,p}(\Omega_h)$ with $s \in [1,2]$ and $p \in [2,\infty)$. Then there exists $w_h \in V_h$ such that*

$$\|w - w_h\|_{L^p(\Omega_h)} + h \|w - w_h\|_{W^{1,p}(\Omega_h)} \le c_p h^s \|w\|_{W^{s,p}(\Omega_h)}. \tag{A.36}$$

The estimate (A.36) even holds for $s \in [0,2]$ if the $W^{1,p}$-term is omitted.

Lemma A.2.7. *Let $w \in W^{s,p}(\Omega_h)$ for some $s \in [1,2]$ and let $\phi_h \in V_h$ be arbitrary, then it holds that*

$$\begin{aligned} &\inf_{\chi_h \in V_{h,0}} \left(\|w - \phi_h - \chi_h\|_{L^p(\Omega_h)} + h \|w - \phi_h - \chi_h\|_{W^{1,p}(\Omega_h)} \right) \\ &\le c_p \left(h^s \|w\|_{W^{s,p}(\Omega_h)} + h^{1/p} \|w - \phi_h\|_{L^p(\Gamma_h)} \right). \end{aligned}$$

Proof. Let $w_h \in V_h$ satisfy the approximation property from Lemma A.2.6, and choose $\chi_h \in V_{h,0}$ such that $\chi_h = w_h - \phi_h$ on all interior nodes of Ω_h. A direct calculation yields

$$\begin{aligned}\|w_h - \phi_h - \chi_h\|_{L^p(\Omega_h)} + h\,\|w_h - \phi_h - \chi_h\|_{W^{1,p}(\Omega_h)} &\le c_p h^{2/p} \left(\sum_{j=1}^{N} \left| w_h(x^{(j)}) - \phi_h(x^{(j)}) \right|^p \right)^{1/p} \\ &\le c_p h^{1/p}\,\|w_h - \phi_h\|_{L^p(\Gamma_h)}\,,\end{aligned}$$

where the last inequality follows from the equivalence of norms on the finite dimensional space $S_h(\Gamma_h)$ and a scaling factor due to the dependence on h. Now we use the triangle inequality and obtain

$$\begin{aligned}&\inf_{\chi_h \in V_{h,0}} \Big(\|w - \phi_h - \chi_h\|_{L^p(\Omega_h)} + h\,\|w - \phi_h - \chi_h\|_{W^{1,p}(\Omega_h)} \Big) \\ &\le \|w - w_h\|_{L^p(\Omega_h)} + h\,\|w - w_h\|_{W^{1,p}(\Omega_h)} + c_p h^{1/p}\,\|w_h - \phi_h\|_{L^p(\Gamma_h)}\,.\end{aligned} \tag{A.37}$$

Moreover, for $v \in W^{1,p}(\Omega_h)$ it holds that

$$\|v\|^p_{L^p(\Gamma_h)} \le c_p\,\|v\|^{p-1}_{L^p(\Omega_h)}\,\|v\|_{W^{1,p}(\Omega_h)}\,. \tag{A.38}$$

A proof for this estimate in the special case $p = 2$ can be found in [35], Dupont, Lemma 2.4. The proof can be adapted to the general case $p \in (1, \infty)$ in a straightforward way. Setting $v = w - w_h$ and using the generalized Young's inequality, (A.38) implies

$$h^{1/p}\,\|w - w_h\|_{L^p(\Gamma_h)} \le c_p \Big(\|w - w_h\|_{L^p(\Omega_h)} + h\,\|w - w_h\|_{W^{1,p}(\Omega_h)} \Big)\,. \tag{A.39}$$

Inserting (A.39) into (A.37) and using the triangle inequality yields

$$\begin{aligned}&\inf_{\chi_h \in V_{h,0}} \Big(\|w - \phi_h - \chi_h\|_{L^p(\Omega_h)} + h\,\|w - \phi_h - \chi_h\|_{W^{1,p}(\Omega_h)} \Big) \\ &\le c_p \Big(\|w - w_h\|_{L^p(\Omega_h)} + h\,\|w - w_h\|_{W^{1,p}(\Omega_h)} + h^{1/p}\,\|w - \phi_h\|_{L^p(\Gamma_h)} \Big)\,,\end{aligned}$$

and we finish the proof with the definition of w_h and Lemma A.2.6. □

Lemma A.2.8. *Let $w \in W^{1,p}_0(\Omega) \cap W^{s,p}(\Omega)$ for some $s \in [1, 2]$. If w is extended by E_s then it holds that*

$$\inf_{\chi_h \in V_{h,0}} \Big(\|w - \chi_h\|_{L^p(\Omega_h)} + h\,\|w - \chi_h\|_{W^{1,p}(\Omega_h)} \Big) \le c_p h^s\,\|w\|_{W^{s,p}(\Omega)}$$

Proof. This lemma follows with Lemma A.2.7 with $\phi_h = 0$ and the estimate (A.31). □

A.2.2.1. Interpolation results on the boundary

Using the Bramble-Hilbert lemma once again, the following approximation properties can be shown to hold for $g \in W^{s,p}(\Gamma)$ with $s \in [1, 2]$.

$$\begin{aligned}\inf_{\varphi_h \in S_h(\Gamma)} \Big(\|g - \varphi_h\|_{L^p(\Gamma)} + h\,\|g - \varphi_h\|_{W^{1,p}(\Gamma)} \Big) &\le c_p h^s\,\|g\|_{W^{s,p}(\Gamma)}\,, \\ \inf_{\varphi_h \in S_h(\Gamma)} \Big(\|g - \varphi_h\|_{L^p(\Gamma)} + h^{1-1/p}\,\|g - \varphi_h\|_{W^{1-1/p,p}(\Gamma)} \Big) &\le c_p h^s\,\|g\|_{W^{s,p}(\Gamma)}\,,\end{aligned} \tag{A.40}$$

where the first estimate even holds for $s \in [0, 2]$ if the $W^{1,p}$-term is omitted.

The following Lemmata are easy generalizations of results in [18] and [33].

Lemma A.2.9. *Let $g \in W^{s,p}(\Gamma)$ and $\varphi_h \in S_h(\Gamma)$, then it holds that*

$$\begin{aligned}
\left\|(\mathrm{Id} - \hat{Q}_h)g\right\|_{L^p(\Gamma)} &\le c_p h^s \|g\|_{W^{s,p}(\Gamma)} && \text{for } s \in [0,2]\,,\\
\left\|(\mathrm{Id} - \hat{Q}_h)g\right\|_{W^{1-1/p,p}(\Gamma)} &\le c_p h^{s-1+1/p} \|g\|_{W^{s,p}(\Gamma)} && \text{for } s \in [1-1/p,2]\,,\\
\left\|(\mathrm{Id} - \hat{Q}_h)g\right\|_{W^{-1/p,p}(\Gamma)} &\le c_p h^{s+1/p} \|g\|_{W^{s,p}(\Gamma)} && \text{for } s \in [0,2]\,,\\
\|\varphi_h\|_{W^{s,p}(\Gamma)} &\le c_p h^{-s} \|\varphi_h\|_{L^p(\Gamma)} && \text{for } s \in [0,1]\,.
\end{aligned}$$

Lemma A.2.10. *Let $g \in W^{s,p}(\Gamma)$ with $s \in [0,2]$ and $\varphi_h \in S_h^1(\Gamma)$, then it holds that*

$$\begin{aligned}
\|Q_h^1 g\|_{W^{s,p}(\Gamma)} &\le c_p \|g\|_{W^{s,p}(\Gamma)}\,,\\
\|(\mathrm{Id} - Q_h^1)g\|_{W^{-1/p,p}(\Gamma)} &\le c_p h^{s+1/p} \|g\|_{W^{s,p}(\Gamma)}\,,\\
\|(\mathrm{Id} - Q_h^1)g\|_{L^p(\Gamma)} &\le c_p h^{s} \|g\|_{W^{s,p}(\Gamma)}\,,\\
\|\varphi_h\|_{W^{s,p}(\Gamma)} &\le c_p h^{-s} \|\varphi\|_{L^p(\Gamma)}\,.
\end{aligned}$$

Lemma A.2.11. *Let Ω be a curvilinear $C^{1,1}$-domain and let $g \in L^p(\Gamma)$, then it holds that*

$$\left\|Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right\|_{L^p(\Gamma_h)} \le c_p h^2 \|g\|_{L^p(\Gamma)}\,.$$

Proof. Let t be the arclength parameter on Γ_h and let $J(t)$ be the Jacobian of the piecewise $C^{1,1}$-parametrization $t \mapsto X_h(t)$. Integration by substitution yields for every $\varphi \in L^1(\Gamma)$,

$$\int_\Gamma \varphi \,\mathrm{d}s = \int_{\Gamma_h} \tilde{\varphi} J \,\mathrm{d}t,$$

and, due to the regularity of Γ,

$$\max_{t\in[0,l_h]} |1 - J(t)| \le ch^2,$$

where l_h is the length of Γ_h. It is clear that if $\varphi \in S_h(\Gamma)$, then $\tilde{\varphi} \in S_h(\Gamma_h)$. Vice versa, if $\phi \in S_h(\Gamma_h)$, then there exists $\chi \in S_h(\Gamma)$ such that $\phi = \tilde{\chi}$. Let

$$\tilde{\chi} = \left(Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right)^{p-1},$$

and note that $\tilde{\chi} \in L^\infty(\Gamma_h)$. It holds that

$$\left\|Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right\|^p_{L^p(\Gamma_h)} = \int_{\Gamma_h} \left(Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right)\tilde{\chi}\,\mathrm{d}t = \int_{\Gamma_h} \left(Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right) Q_h\tilde{\chi}\,\mathrm{d}t,$$

where we used the fact that $\left(Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right) \in S_h(\Gamma_h)$ and (A.34). If $\chi_h \in S_h(\Gamma)$ is chosen such that $\tilde{\chi_h} = Q_h\tilde{\chi}$ then we get

$$\begin{aligned}\int_{\Gamma_h} \left(Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right) Q_h\tilde{\chi}\,\mathrm{d}t &= \int_{\Gamma_h} \tilde{g} Q_h\tilde{\chi}\,\mathrm{d}t - \int_{\Gamma_h} \widetilde{\hat{Q}_h g} Q_h\tilde{\chi}\,\mathrm{d}t \\ &= \int_{\Gamma} g\chi_h\,\mathrm{d}s - \int_{\Gamma} \hat{Q}_h g\chi_h\,\mathrm{d}s + \int_{\Gamma_h} \left(\tilde{g} - \widetilde{\hat{Q}_h g}\right) Q_h\tilde{\chi}\,(1 - J(t))\,\mathrm{d}t \\ &= \int_{\Gamma_h} \left(\tilde{g} - \widetilde{\hat{Q}_h g}\right) Q_h\tilde{\chi}\,(1 - J(t))\,\mathrm{d}t.\end{aligned}$$

Now we use the generalized Hölder's inequality to obtain

$$\left\|Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right\|^p_{L^p(\Gamma_h)} \leq \left(\max_{t\in[0,l_h]} |1 - J(t)|\right) \left\|\tilde{g} - \widetilde{\hat{Q}_h g}\right\|_{L^p(\Gamma_h)} \|Q_h\tilde{\chi}\|_{L^q(\Gamma_h)},$$

with the conjugate index $q = p/(p-1)$. Using (A.25) and the L^p-stability of the L^2-projection, cf. [33], we also get

$$\begin{aligned}\left\|\tilde{g} - \widetilde{\hat{Q}_h g}\right\|_{L^p(\Gamma_h)} &\leq c_p \left\|g - \hat{Q}_h g\right\|_{L^p(\Gamma)} \leq c_p \|g\|_{L^p(\Gamma)}, \\ \|Q_h\tilde{\chi}\|_{L^q(\Gamma_h)} \leq c_p \|\tilde{\chi}\|_{L^q(\Gamma_h)} &= \left\|Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right\|^{p-1}_{L^p(\Gamma_h)},\end{aligned}$$

and arrive at

$$\left\|Q_h\tilde{g} - \widetilde{\hat{Q}_h g}\right\|_{L^p(\Gamma_h)} \leq c_p h^2 \|g\|_{L^p(\Gamma)}. \qquad \square$$

Definition A.2.12. For $g \in L^2(\Gamma)$ we define the approximation of boundary data via

$$g_h = Q_h\tilde{g}.$$

A.2.3. A-priori error estimates

From now on, if not stated otherwise, let $\Omega \subset \mathbb{R}^2$ be a bounded, curvilinear, right-angled $C^{1,1}$-domain in the sense of Definition A.1.35. In addition, let the matrix A be symmetric and uniformly elliptic with Lipschitz coefficients.

Let $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and $a_h\colon H^1(\Omega_h) \times H^1(\Omega_h) \to \mathbb{R}$ be two bilinear forms defined via

$$\begin{aligned}a(u,v) &= \int_\Omega \nabla u^T \cdot A \cdot \nabla v\,\mathrm{d}x, \\ a_h(u,v) &= \int_{\Omega_h} \nabla u^T \cdot A \cdot \nabla v\,\mathrm{d}x.\end{aligned}$$

Now, as already mentioned in (A.22), let $u_h \in V_h$ be the solution of

$$\begin{cases} a_h(u_h, v_h) = (f, v_h)_h & \forall v_h \in V_{h,0}, \\ u_h = g_h & \text{on } \Gamma_h, \end{cases} \tag{A.41}$$

where f is extended by 0 onto $\Omega_h \backslash \Omega$. In addition, if $\Omega_h^{(j)} \not\subset \Omega_h$, then extend u_h as follows. Let $\tau_h^{(j)}$ be the triangle having $\Gamma_h^{(j)}$ as one of its sides, and define u_h on $\Omega_h^{(j)}$ to be the linear extension from $\tau_h^{(j)}$.

In order to prove Theorem A.2.1, we will consider the two cases $f = 0$ and $g = 0$ separately, we start with the first case.

Definition A.2.13. Let $u_0 \in W_0^{1,p}(\Omega)$ and $u_{h,0} \in V_{h,0}$ be the (weak) solutions to

$$\begin{cases} Lu_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma, \end{cases} \tag{A.42}$$

and

$$a_h(u_{h,0}, v_h) = (f, v_h)_h \qquad \forall v_h \in V_{h,0}. \tag{A.43}$$

The following lemma will be needed to estimate the error between u_0 and $u_{h,0}$.

Lemma A.2.14. *Let u_0 be extended by E_s, then it holds that*

$$a_h(u_0 - u_{h,0}, v_h) = -\left(\operatorname{div}(A \cdot \nabla u_0), v_h\right)_{\Omega_h \backslash \Omega} \qquad \forall v_h \in V_{h,0}.$$

Proof. Let $v_h \in V_{h,0}$ be arbitrary. Due to Lemma A.1.40 it holds that $u_0 \in W^{2,p}(\Omega)$, i.e. (A.42) holds in strong form, hence

$$(\nabla u_0, A \cdot \nabla v_h)_{\Omega \cap \Omega_h} = (f, v_h)_{\Omega \cap \Omega_h} + \langle v_h, \nabla u_0^T \cdot A \cdot n \rangle_{\partial(\Omega \cap \Omega_h)},$$

and

$$(\nabla u_0, A \cdot \nabla v_h)_{\Omega_h} = (f, v_h)_{\Omega \cap \Omega_h} + \langle v_h, \nabla u_0^T \cdot A \cdot n \rangle_{\partial(\Omega \cap \Omega_h)} + (\nabla u_0, A \cdot \nabla v_h)_{\Omega_h \backslash \Omega}.$$

In addition,

$$(\nabla u_{h,0}, A \cdot \nabla v_h)_{\Omega_h} = (f, v_h)_{\Omega_h}.$$

As f is extended by zero outside of Ω, it follows that $(f, v_h)_{\Omega_h} = (f, v_h)_{\Omega \cap \Omega_h}$, hence

$$\begin{aligned} a_h(u_0 - u_{h,0}, v_h) &= \left(\nabla \left(u_0 - u_{h,0}\right), A \cdot \nabla v_h\right)_{\Omega_h} \\ &= \langle v_h, \nabla u_0^T \cdot A \cdot n \rangle_{\partial(\Omega \cap \Omega_h)} + (\nabla u_0, A \cdot \nabla v_h)_{\Omega_h \backslash \Omega} \\ &= \langle v_h, \nabla u_0^T \cdot A \cdot n \rangle_{\partial(\Omega \cap \Omega_h)} - \left(\operatorname{div}(A \cdot \nabla u_0), v_h\right)_{\Omega_h \backslash \Omega} + \langle v_h \nabla u_0^T \cdot A \cdot n \rangle_{\partial(\Omega \backslash \Omega_h)} \\ &= \langle v_h, \nabla u_0^T \cdot A \cdot n \rangle_{\Gamma_h} - \left(\operatorname{div}(A \cdot \nabla u_0), v_h\right)_{\Omega_h \backslash \Omega} \\ &= -\left(\operatorname{div}(A \cdot \nabla u_0), v_h\right)_{\Omega_h \backslash \Omega}, \end{aligned}$$

where we used $v_h|_{\Gamma_h} = 0$. □

As in the original paper [17], we do not restrict ourselves to convex domains. However, in the case $p > 2$ there appear some logarithmic terms in the non-convex case.

Definition A.2.15. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz-domain. With $c_\Omega(h)$ we will denote constants such that

$$c_\Omega(h) = \begin{cases} c_\Omega, & \text{for convex domains } \Omega, \\ c_\Omega \left|\ln h\right|^{\tilde{p}}, & \text{for non-convex domains } \Omega, \end{cases}$$

where c_Ω is a constant depending on the domain Ω but not on h, and $\tilde{p} \geq 0$ is finite and uniformly bounded independent of h and Ω, cf. Theorem A.1.47.

Definition A.2.16. Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. With $p_{-1}(\Omega)$ we will denote the supremum over all $p \in [2, \infty]$ such that the Laplace problem with right hand side $f \in W^{-1,p}(\Omega)$ has a unique solution $u \in W_0^{1,p}(\Omega)$ which continuously depends on the data f for all $p \in (q_{-1}(\Omega), p_{-1}(\Omega))$, where

$$\frac{1}{p_{-1}(\Omega)} + \frac{1}{q_{-1}(\Omega)} = 1.$$

The following bounds follow from Theorem A.1.31 and Remark A.1.33.

$$p_{-1}(\Omega) \begin{cases} > 4 & \text{for arbitrary Lipschitz domains,} \\ = \infty & \text{for convex or } C^1\text{-domains.} \end{cases} \tag{A.44}$$

Lemma A.2.17. *Let $\Omega \subset \mathbb{R}^2$, $p \in [2, p_{-1}(\Omega_h))$ and $s \in [0, 1]$. Then it holds that*

$$\left\|u_0 - u_{h,0}\right\|_{W^{s,p}(\Omega_h)} \leq c_p c_{\Omega_h}(h) h^{2-s} \left\|f\right\|_{L^p(\Omega)}.$$

Proof. With Lemma A.1.40 it follows that $u_0 \in W^{2,p}(\Omega)$ and $\left\|u_0\right\|_{W^{2,p}(\Omega)} \leq c_p \left\|f\right\|_{L^p(\Omega)}$. For arbitrary $\chi_h \in V_{h,0}$ it holds that

$$\begin{aligned} c\left|u_0 - u_{h,0}\right|^p_{W^{1,p}(\Omega_h)} &\leq \Big(\nabla(u_0 - u_{h,0}), \left|\nabla(u_0 - u_{h,0})\right|^{p-2} A \cdot \nabla(u_0 - u_{h,0})\Big)_h \\ &= \Big(\nabla(u_0 - u_{h,0}), \left|\nabla(u_0 - u_{h,0})\right|^{p-2} A \cdot \nabla(u_0 - u_{h,0} - \chi_h)\Big)_h \\ &+ \Big(\nabla(u_0 - u_{h,0}), \left|\nabla(u_0 - u_{h,0})\right|^{p-2} A \cdot \nabla\chi_h\Big)_h. \end{aligned} \tag{A.45}$$

Now we estimate both terms on the right hand side of (A.45) separately. For the first part it holds that

$$\begin{aligned} &\left|\Big(\nabla(u_0 - u_{h,0}), \left|\nabla(u_0 - u_{h,0})\right|^{p-2} A \cdot \nabla(u_0 - u_{h,0} - \chi_h)\Big)_h\right| \\ &\leq c\left|u_0 - u_{h,0}\right|^{p-1}_{W^{1,p}(\Omega_h)} \left|u_0 - u_{h,0} - \chi_h\right|_{W^{1,p}(\Omega_0)}. \end{aligned} \tag{A.46}$$

Now we concentrate on the second part. Let $\varphi_h \in V_{h,0}$ be such that

$$\Big(\nabla v_h, \left|\nabla(u_0 - u_{h,0})\right|^{p-2} A \cdot \nabla\chi_h - A \cdot \nabla\varphi_h\Big)_h = 0 \qquad \forall v_h \in V_{h,0}.$$

The existence of such a φ_h as well as the stability estimate

$$\begin{aligned} \left\|\varphi_h\right\|_{W^{1,q}(\Omega_h)} &\leq c_p c_{\Omega_h}(h) \left\|\left|\nabla(u_0 - u_{h,0})\right|^{p-2} A \cdot \nabla\chi_h\right\|_{L^q(\Omega_h)} \\ &\leq c_p c_{\Omega_h}(h) \left\|u_0 - u_{h,0}\right\|^{p-2}_{W^{1,p}(\Omega_0)} \left\|\chi_h\right\|_{W^{1,p}(\Omega_h)}, \end{aligned} \tag{A.47}$$

follows from Theorem A.1.47. Using Lemma A.2.14 we get

$$\begin{aligned}
&\left(\nabla\left(u_0-u_{h,0}\right),\left|\nabla\left(u_0-u_{h,0}\right)\right|^{p-2}A\cdot\nabla\chi_h\right)_h\\
&=\left(\nabla\left(u_0-u_{h,0}\right),\left|\nabla\left(u_0-u_{h,0}\right)\right|^{p-2}A\cdot\nabla\chi_h-A\cdot\nabla\varphi_h\right)_h-\left(\operatorname{div}(A\cdot\nabla u_0),\varphi_h\right)_{\Omega_h\setminus\Omega}\\
&=\left(\nabla\left(u_0-u_{h,0}-\chi_h\right),\left|\nabla\left(u_0-u_{h,0}\right)\right|^{p-2}A\cdot\nabla\chi_h-A\cdot\nabla\varphi_h\right)_h-\left(\operatorname{div}(A\cdot\nabla u_0),\varphi_h\right)_{\Omega_h\setminus\Omega}.
\end{aligned}$$

As a result it holds that

$$\begin{aligned}
&\left|\left(\nabla\left(u_0-u_{h,0}\right),\left|\nabla\left(u_0-u_{h,0}\right)\right|^{p-2}A\cdot\nabla\chi_h\right)_h\right|\\
&\leq c\left|u_0-u_{h,0}-\chi_h\right|_{W^{1,p}(\Omega_h)}\left(\left|u_0-u_{h,0}\right|^{p-2}_{W^{1,p}(\Omega_h)}\left|\chi_h\right|_{W^{1,p}(\Omega_h)}+\left|\varphi_h\right|_{W^{1,q}(\Omega_h)}\right)\\
&\quad+c\left\|u_0\right\|_{W^{2,p}(\mathbb{R}^2)}\left\|\varphi_h\right\|_{L^q(\Omega_h\setminus\Omega)}.
\end{aligned}\tag{A.48}$$

Using Lemma A.2.2 and Lemma A.2.4 it follows that

$$\begin{aligned}
\left\|u_0\right\|_{W^{2,p}(\mathbb{R}^2)}\left\|\varphi_h\right\|_{L^q(\Omega_h\setminus\Omega)}&\leq c_ph^2\left\|u_0\right\|_{W^{2,p}(\Omega)}\left\|\varphi_h\right\|_{W^{1,q}(\Omega_h)}\\
&\leq c_ph^2\left\|f\right\|_{L^p(\Omega)}\left\|\varphi_h\right\|_{W^{1,q}(\Omega_h)}.
\end{aligned}\tag{A.49}$$

Inserting (A.46), (A.48) and (A.49) into (A.45) yields

$$\begin{aligned}
&\left|u_0-u_{h,0}\right|^{p}_{W^{1,p}(\Omega_h)}\\
&\leq c_p\left|u_0-u_{h,0}-\chi_h\right|_{W^{1,p}(\Omega_h)}\left(\left|u_0-u_{h,0}\right|^{p-1}_{W^{1,p}(\Omega_h)}+\left|u_0-u_{h,0}\right|^{p-2}_{W^{1,p}(\Omega_h)}\left|\chi_h\right|_{W^{1,p}(\Omega_h)}+\left|\varphi_h\right|_{W^{1,q}(\Omega_h)}\right)\\
&\quad+c_ph^2\left\|f\right\|_{L^p(\Omega)}\left\|\varphi_h\right\|_{W^{1,q}(\Omega_h)}.
\end{aligned}$$

Now we use the estimate (A.47) and get

$$\begin{aligned}
\left|u_0-u_{h,0}\right|^{2}_{W^{1,p}(\Omega_h)}\leq{}& c_pc_{\Omega_h}(h)\left|u_0-u_{h,0}-\chi_h\right|_{W^{1,p}(\Omega_h)}\left(\left|u_0-u_{h,0}\right|_{W^{1,p}(\Omega_h)}+\left|\chi_h\right|_{W^{1,p}(\Omega_h)}\right)\\
&+c_pc_{\Omega_h}(h)h^2\left\|f\right\|_{L^p(\Omega)}\left\|\chi_h\right\|_{W^{1,p}(\Omega_h)}\qquad\forall\chi_h\in V_{h,0}.
\end{aligned}\tag{A.50}$$

Now we fix χ_h as the infimum from Lemma A.2.7, and using Lemma A.2.3 we get

$$\begin{aligned}
\left|u_0-u_{h,0}-\chi_h\right|_{W^{1,p}(\Omega_h)}&\leq c_p\left(h\left\|u_0\right\|_{W^{2,p}(\Omega_h)}+h^{1/p-1}\left\|u_0\right\|_{L^p(\Gamma_h)}\right)\\
&\leq c_p\left(h\left\|u_0\right\|_{W^{2,p}(\Omega_h)}+h^{1/p-1}h^2\left\|u_0\right\|_{W^{2,p}(\Omega_h)}\right)\\
&\leq c_ph\left\|u_0\right\|_{W^{2,p}(\Omega_h)}.
\end{aligned}$$

Again we use the properties of E_s and get

$$\left|u_0-u_{h,0}-\chi_h\right|_{W^{1,p}(\Omega_h)}\leq c_ph\left\|f\right\|_{L^p(\Omega)}.\tag{A.51}$$

From the definition of χ_h it follows that

$$\begin{aligned}
\left\|\chi_h\right\|_{W^{1,p}(\Omega_h)}&\leq\left\|u_0-u_{h,0}-\chi_h\right\|_{W^{1,p}(\Omega_h)}+\left\|u_0-u_{h,0}\right\|_{W^{1,p}(\Omega_h)}\\
&\leq\left\|u_0-u_{h,0}-0\right\|_{W^{1,p}(\Omega_h)}+\left\|u_0-u_{h,0}\right\|_{W^{1,p}(\Omega_h)}\\
&\leq c\left\|u_0-u_{h,0}\right\|_{W^{1,p}(\Omega_h)}.
\end{aligned}\tag{A.52}$$

Inserting (A.51) and (A.52) into (A.50) finally yields

$$|u_0 - u_{h,0}|_{W^{1,p}(\Omega_h)} \leq c_p c_{\Omega_h}(h) h \|f\|_{L^p(\Omega_h)},$$

and it remains to prove the L^p-case.

Here we follow the proof presented in [17] and use a duality argument. Let $\varphi \in C_0^\infty(\Omega_h)$ be arbitrary and let $w \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ solve

$$\begin{cases} Lw = \varphi & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma, \end{cases}$$

where q is the conjugate index to p. Whenever necessary, regard w as being extended by 0 outside of Ω. Now, for $\chi_h \in V_{h,0}$ arbitrary it holds that

$$\begin{aligned} (u_0 - u_{h,0}, \varphi)_h &= (u_0 - u_{h,0}, Aw)_h + (u_0 - u_{h,0}, \varphi - Aw)_h \\ &= a_h(u_0 - u_{h,0}, w - \chi_h) + a_h(u_0 - u_{h,0}, \chi_h) \\ &- \langle u_0, \nabla w^T \cdot A \cdot n \rangle_h + (u_0 - u_{h,0}, \varphi - Aw)_h. \end{aligned} \tag{A.53}$$

We get

$$\begin{aligned} & a_h(u_0 - u_{h,0}, w - \chi_h) + a_h(u_0 - u_{h,0}, \chi_h) \\ & \leq c \|u_0 - u_{h,0}\|_{W^{1,p}(\Omega_h)} \|w - \chi_h\|_{W^{1,q}(\Omega_h)} + (Au_0 - f, \chi_h)_h \\ & \leq c \|u_0 - u_{h,0}\|_{W^{1,p}(\Omega_h)} \|w - \chi_h\|_{W^{1,q}(\Omega_h)} + c \|f\|_{L^p(\Omega)} \|\chi_h\|_{L^q(\Omega_h \setminus \Omega)} \\ & \leq c_p \left(\left(\|u_0 - u_{h,0}\|_{W^{1,p}(\Omega_h)} + h^2 \|f\|_{L^p(\Omega)} \right) \|w - \chi_h\|_{W^{1,q}(\Omega_h)} + h^2 \|f\|_{L^q(\Omega)} \|w\|_{W^{1,q}(\Omega)} \right), \end{aligned}$$

where we used Lemma A.2.4 and the triangle inequality. By taking χ_h as the infimum from Lemma A.2.8 we get

$$\begin{aligned} & a_h(u_0 - u_{h,0}, w - \chi_h) + a_h(u_0 - u_{h,0}, \chi_h) \\ & \leq c_p \left(h \|u_0 - u_{h,0}\|_{W^{1,p}(\Omega_h)} + h^2 \|f\|_{L^p(\Omega)} \right) \|\varphi\|_{L^q(\Omega)}. \end{aligned} \tag{A.54}$$

We also get

$$\begin{aligned} \left| \langle u_0, \nabla w^T \cdot A \cdot n \rangle_h \right| &\leq c \|u_0\|_{L^p(\Gamma_h)} \|w\|_{W^{1,q}(\Gamma_h)} \\ &\leq c_p h^2 \|u_0\|_{W^{2,p}(\Omega)} \|w\|_{W^{2,q}(\Omega)} \\ &\leq c_p h^2 \|f\|_{L^p(\Omega)} \|\varphi\|_{L^q(\Omega)}, \end{aligned} \tag{A.55}$$

where we used Theorem A.1.3 and Lemma A.2.3. Finally we get

$$\begin{aligned} (u_0 - u_{h,0}, \varphi - Aw)_h &\leq c \|u_0 - u_{h,0}\|_{L^p(\Omega_h \setminus \Omega)} \left(\|w\|_{W^{2,q}(\mathbb{R}^2)} + \|\varphi\|_{L^q(\Omega_h)} \right) \\ &\leq c_p \left(h^{2/p} \|u_0\|_{L^p(\Gamma_h)} + h^2 \|u_0 - u_{h,0}\|_{W^{1,p}(\Omega_h)} \right) \|\varphi\|_{L^q(\Omega_h)} \\ &\leq c_p \left(h^{2+2/p} \|f\|_{L^p(\Omega)} + h^2 \|u_0 - u_{h,0}\|_{W^{1,p}(\Omega_h)} \right) \|\varphi\|_{L^q(\Omega_h)}. \end{aligned} \tag{A.56}$$

Inserting the estimates (A.54), (A.55) and (A.56) into (A.53), we end up with

$$\frac{(u_0 - u_{h,0}, \varphi)_h}{\|\varphi\|_{L^q(\Omega_h)}} \leq c_p \left(h \|u_0 - u_{h,0}\|_{W^{1,p}(\Omega_h)} + h^2 \|f\|_{L^p(\Omega)} \right),$$

and using the first part of this lemma we conclude that

$$\|u_0 - u_{h,0}\|_{L^p(\Omega_h)} = \sup_{\varphi \in C_0^\infty(\Omega_h)} \frac{(u_0 - u_{h,0}, \varphi)_h}{\|\varphi\|_{L^q(\Omega_h)}} \leq c_p c_{\Omega_h}(h) h^2 \|f\|_{L^p(\Omega)},$$

and the rest follows by interpolation. □

Now let $v_h \in V_h$ and $\hat{u}^h, u^h \in W^{1,p}(\Omega)$ satisfy

$$\begin{cases} L\hat{u}^h = 0 & \text{in } \Omega, \\ \hat{u}^h = \hat{Q}_h g & \text{on } \Gamma, \end{cases} \tag{A.57}$$

$$\begin{cases} Lu^h = 0 & \text{in } \Omega, \\ u^h = Q_h^1 g & \text{on } \Gamma, \end{cases} \tag{A.58}$$

$$\begin{cases} a_h(v_h, w_h) = 0 & \forall w_h \in V_{h,0}, \\ v_h = \widetilde{\hat{Q}_h g} & \text{on } \Gamma_h. \end{cases} \tag{A.59}$$

Note that, because of $Q_h^1 g \in W^{2-1/p,p}(\Gamma)$, it holds that $u^h \in W^{2,p}(\Omega)$.

Lemma A.2.18. *Let $\Omega \subset \mathbb{R}^2$, $p \in [2, p_{-1}(\Omega_h))$ and $s \in [1-1/p, 2-1/p]$. Then it holds that*

$$\left\|\hat{u}^h - v_h\right\|_{L^p(\Omega_h)} \leq c_p c_{\Omega_h}(h) h^{s+1/p} \|g\|_{W^{s,p}(\Gamma)},$$
$$\left\|\hat{u}^h - v_h\right\|_{W^{1,p}(\Omega_h)} \leq c_p c_{\Omega_h}(h) h^{s-1+1/p} \|g\|_{W^{s,p}(\Gamma)}.$$

Proof. First we note that, within the estimate for $\left\|\hat{u}^h - v_h\right\|_{W^{1,p}(\Omega_h)}$, we can replace $\hat{u}^h$ by u^h, since

$$\left\|\hat{u}^h - u^h\right\|_{W^{1,p}(\Omega_h)} \leq \left\|E_s(\hat{u}^h - u^h)\right\|_{W^{1,p}(\mathbb{R}^2)} \leq c \left\|\hat{u}^h - u^h\right\|_{W^{1,p}(\Omega)} \leq c h^{s-1+1/p} \|g\|_{W^{s,p}(\Gamma)}, \tag{A.60}$$

for $s \in [1-1/p, 2-1/p]$, where we used Lemma A.1.34 and the properties of Q_h and Q_h^1. For arbitrary $\chi_h \in V_{h,0}$ it holds that

$$\begin{aligned} c\left|u^h - v_h\right|^p_{W^{1,p}(\Omega_h)} &\leq \left(\nabla(u^h - v_h), \left|\nabla(u^h - v_h)\right|^{p-2} A \cdot \nabla(u^h - v_h)\right)_h \\ &= \left(\nabla(u^h - v_h), \left|\nabla(u^h - v_h)\right|^{p-2} A \cdot \nabla(u^h - v_h - \chi_h)\right)_h \\ &+ \left(\nabla(u^h - v_h), \left|\nabla(u^h - v_h)\right|^{p-2} A \cdot \nabla\chi_h\right)_h, \end{aligned} \tag{A.61}$$

and we are now going to estimate both terms on the right hand side of (A.61) separately. For the first term it holds due to Hölder,

$$\begin{aligned}&\left(\nabla(u^h - v_h), \left|\nabla(u^h - v_h)\right|^{p-2} A \cdot \nabla(u^h - v_h - \chi_h)\right)_h \\ &\leq c \left|u^h - v_h\right|^{p-1}_{W^{1,p}(\Omega_h)} \left|u^h - v_h - \chi_h\right|_{W^{1,p}(\Omega_h)} .\end{aligned} \tag{A.62}$$

No we set χ_h to be the minimizing function from Lemma A.2.7 and get

$$\begin{aligned}&\left|u^h - v_h - \chi_h\right|_{W^{1,p}(\Omega_h)} \\ &\leq c_p \left(h \left\|u^h\right\|_{W^{2,p}(\Omega)} + h^{1/p-1} \left\|u^h - v_h\right\|_{L^p(\Omega_h)}\right) \\ &\leq c_p \left(h \left\|Q_h^1 g\right\|_{W^{2-1/p,p}(\Gamma)} + h^{1/p-1} \left(\left\|u^h - \widetilde{Q_h^1 g}\right\|_{L^p(\Gamma_h)} + \left\|\widetilde{Q_h^1 g} - \widetilde{\hat{Q}_h g}\right\|_{L^p(\Gamma_h)}\right)\right) \\ &\leq c_p \left(h \left\|Q_h^1 g\right\|_{W^{2-1/p,p}(\Gamma)} + h^{1/p-1} \left(h^{2-1/p} \left\|u^h\right\|_{W^{2,p}(\Omega)} + \left\|Q_h^1 g - \hat{Q}_h g\right\|_{L^p(\Gamma)}\right)\right) \\ &\leq c_p h^{k+1-1/p} \left\|g\right\|_{W^{k,p}(\Gamma)} ,\end{aligned} \tag{A.63}$$

where in the last step we used of the properties of Q_h^1 and $\hat{Q}_h$, cf. Lemma A.2.9 and Lemma A.2.10. It remains to estimate the second part within (A.61). As can be seen from its proof, Lemma A.2.14 also holds for the difference $\left(u^h - v_h\right)$, hence we have for arbitrary $\varphi_h \in V_{h,0}$,

$$\begin{aligned}&\left(\nabla(u^h - v_h), \left|\nabla(u^h - v_h)\right|^{p-2} A \cdot \nabla\chi_h\right)_h \\ &= \left(\nabla(u^h - v_h), \left|\nabla(u^h - v_h)\right|^{p-2} A \cdot \nabla\chi_h - A \cdot \nabla\varphi_h\right)_h - \left(\operatorname{div}\left(A \cdot \nabla u^h\right), \varphi_h\right)_{\Omega_h \setminus \Omega} \\ &= \left(\nabla(u^h - v_h - \chi_h), \left|\nabla(u^h - v_h)\right|^{p-2} A \cdot \nabla\chi_h - A \cdot \nabla\varphi_h\right)_h - \left(\operatorname{div}\left(A \cdot \nabla u^h\right), \varphi_h\right)_{\Omega_h \setminus \Omega} ,\end{aligned}$$

where the last equality holds if

$$\left(\nabla v_h, A \cdot \nabla\varphi_h - \left|\nabla(u^h - v_h)\right|^{p-2} A \cdot \nabla\chi_h\right)_h = 0 \qquad \forall v_h \in V_{h,0},$$

and it remains to show that such a $\varphi_h \in V_{h,0}$ actually exists. As $\left(\left|\nabla(u^h - v_h)\right|^{p-2} \nabla\chi_h\right) \in L^q(\Omega_h)$ with the conjugate index $q = p/(p-1)$, the existence of φ_h follows with Theorem A.1.47. Using Lemma A.2.2, Lemma A.2.4 and the property of the operator Q_h^1 it follows that

$$\begin{aligned}\left|\left(\operatorname{div}\left(A \cdot \nabla u^h\right), \varphi_h\right)_{\Omega_h \setminus \Omega}\right| &\leq c \left\|u^h\right\|_{W^{2,p}(\mathbb{R}^2)} \left\|\varphi_h\right\|_{L^q(\Omega_h \setminus \Omega)} \\ &\leq c_p h^2 \left\|u^h\right\|_{W^{2,p}(\Omega)} \left\|\varphi_h\right\|_{W^{1,q}(\Omega_h)} \\ &\leq c_p h^2 \left\|Q_h^1 g\right\|_{W^{2-1/p,p}(\Gamma)} \left\|\varphi_h\right\|_{W^{1,q}(\Omega_h)} \\ &\leq c_p h^{k+1/p} \left\|g\right\|_{W^{k,p}(\Gamma)} \left\|\varphi_h\right\|_{W^{1,q}(\Omega_h)} .\end{aligned}$$

Again we use the extended Hölder inequality and conclude

$$\begin{aligned}&\left(\nabla(u^h - v_h - \chi_h), \left|\nabla(u^h - v_h)\right|^{p-2} A\cdot\nabla\chi_h - A\cdot\nabla\varphi_h\right)_h\\&\leq c\left|u^h - v_h - \chi_h\right|_{W^{1,p}(\Omega_h)}\left|u^h - v_h\right|^{p-2}_{W^{1,p}(\Omega_h)}|\chi_h|_{W^{1,p}(\Omega)} + c\left|u^h - v_h - \chi_h\right|_{W^{1,p}(\Omega_h)}|\varphi_h|_{W^{1,q}(\Omega_h)}\\&+ c_p h^{k+1/p}\left\|g\right\|_{W^{2-1/p,p}(\Gamma)}\left\|\varphi_h\right\|_{W^{1,q}(\Omega_h)}.\end{aligned}$$

Due to the definition of χ_h as the infimum in Lemma A.2.7 and $0\in V_{h,0}$ it follows that

$$\begin{aligned}|\chi_h|_{W^{1,p}(\Omega_h)} &\leq \left|u^h - v_h - \chi_h\right|_{W^{1,p}(\Omega_h)} + \left|u^h - v_h\right|_{W^{1,p}(\Omega_h)}\\&\leq \left|u^h - v_h - 0\right|_{W^{1,p}(\Omega_h)} + \left|u^h - v_h\right|_{W^{1,p}(\Omega_h)}\\&\leq c\left|u^h - v_h\right|_{W^{1,p}(\Omega_h)}.\end{aligned}$$

Due to the definition of φ_h and the corresponding stability estimate we get

$$\begin{aligned}|\varphi_h|_{W^{1,q}(\Omega_h)} &\leq c_p c_{\Omega_h}(h)\left\|\left|\nabla(u^h - v_h)\right|^{p-2}\nabla\chi_h\right\|_{L^q(\Omega_h)}\\&\leq c_p c_{\Omega_h}(h)\left|u^h - v_h\right|^{p-1}_{W^{1,p}(\Omega_h)},\end{aligned}$$

which finally leads to

$$\begin{aligned}&\left(\nabla(u^h - v_h), \left|\nabla(u^h - v_h)\right|^{p-2} A\cdot\nabla\chi_h\right)_h\\&\leq c_p c_{\Omega_h}(h)\left|u^h - v_h\right|^{p-1}_{W^{1,p}(\Omega_h)}\left|u^h - v_h - \chi_h\right|_{W^{1,p}(\Omega_h)} + c_p h^{k+1/p}\left|u^h - v_h\right|^{p-1}_{W^{1,p}(\Omega_h)}\left\|g\right\|_{W^{k,p}(\Gamma)},\end{aligned}$$

and we finish the proof for the $W^{1,p}$-case with (A.63).

Now we deal with the L^p-case. For fixed $\varphi\in C_0^\infty(\Omega_h)$ let $w\in W_0^{1,q}(\Omega)\cap W^{2,q}(\Omega)$ solve

$$\begin{cases} Lw = \varphi & \text{in } \Omega,\\ w = 0 & \text{on } \Gamma,\end{cases}$$

where again q is the conjugate index to p. Then, for $\chi_h\in V_{h,0}$ it holds that

$$\begin{aligned}\left(\hat{u}^h - v_h, \varphi\right)_h &= \left(\hat{u}^h - v_h, Lw\right)_h + \left(\hat{u}^h - v_h, \varphi - Lw\right)_h\\&= a_h(\hat{u}^h - v_h, w - \chi_h) + a_h(\hat{u}^h - v_h, \chi_h)\\&- \langle\hat{u}^h - v_h, \nabla w^T\cdot A\cdot n\rangle_h + \left(\hat{u}^h - v_h, \varphi - Lw\right)_h.\end{aligned}\tag{A.64}$$

We get

$$\begin{aligned}&a_h(\hat{u}^h - v_h, w - \chi_h) + a_h(\hat{u}^h - v_h, \chi_h)\\&\leq c\left\|\hat{u}^h - v_h\right\|_{W^{1,p}(\Omega_h)}\left\|w - \chi_h\right\|_{W^{1,q}(\Omega_h)} + \left(Lu^h, \chi_h\right)_h + a_h(\hat{u}^h - u^h, \chi_h)\\&\leq c\left\|\hat{u}^h - v_h\right\|_{W^{1,p}(\Omega_h)}\left\|w - \chi_h\right\|_{W^{1,q}(\Omega_h)} + c\left\|u^h\right\|_{W^{2,p}(\Omega)}\left\|\chi_h\right\|_{L^q(\Omega_h\setminus\Omega)} + a_h(\hat{u}^h - u^h, \chi_h)\\&\leq c_p\left(\left\|\hat{u}_h - v_h\right\|_{W^{1,p}(\Omega_h)} + h^2\left\|Q_h^1 g\right\|_{W^{2-1/p,p}(\Gamma)}\right)\left\|w - \chi_h\right\|_{W^{1,q}(\Omega_h)}\\&+ c_p h^2\left\|Q_h^1 g\right\|_{W^{2-1/p,p}(\Gamma)}\left\|w\right\|_{W^{1,q}(\Omega)} + a_h(\hat{u}^h - u^h, \chi_h).\end{aligned}\tag{A.65}$$

Because of $w \in W_0^{1,q}(\Omega)$, we can extend it by 0 outside of Ω, we shall denote the extension by $\overline{w}$. Similar, extend χ_h by zero outside of Ω_h. We get

$$a_h(\hat{u}^h - u^h, \chi_h) = a_h(\hat{u}^h - u^h, \chi_h - \overline{w}) + a_h(\hat{u}^h - u^h, \overline{w}),$$

and since $\hat{u}^h - u^h$ is A-harmonic, $a(\hat{u}^h - u^h, \overline{w}) = 0$, it holds that

$$a_h(\hat{u}^h - u^h, \overline{w}) \leq c \left| E(\hat{u}^h - u^h) \right|_{W^{1,p}(\mathbb{R}^2)} \|\overline{w}\|_{W^{1,q}(\Omega_h \setminus \Omega)} . \tag{A.66}$$

From the definition of w and $\overline{w}$ we get

$$\begin{aligned} a_h(\hat{u}^h - u^h, \chi_h - \overline{w}) &\leq c \left| E(\hat{u}^h - u^h) \right|_{W^{1,p}(\mathbb{R}^2)} \|\overline{w} - \chi_h\|_{W^{1,q}(\Omega \cup \Omega_h)} \\ &\leq c \left| E(\hat{u}^h - u^h) \right|_{W^{1,p}(\mathbb{R}^2)} \left(\|w - \chi_h\|_{W^{1,q}(\Omega_h)} + \|w\|_{W^{1,q}(\Omega \triangle \Omega_h)} \right) . \end{aligned} \tag{A.67}$$

Inserting (A.66) and (A.67) into (A.65), we conclude

$$a_h(\hat{u}^h - u^h, \chi_h) \leq c \left\| \hat{u}^h - u^h \right\|_{W^{1,p}(\Omega)} \left(\|w - \chi_h\|_{W^{1,q}(\Omega_h)} + \|w\|_{W^{1,q}(\Omega \triangle \Omega_h)} \right) .$$

Using the first part of this lemma and Lemma A.2.4 we end up with

$$a_h(\hat{u}^h - u^h, \chi_h) \leq c \left\| \hat{Q}_h g - Q_h^1 g \right\|_{W^{1-1/p,p}(\Gamma)} \left(\|w - \chi_h\|_{W^{1,q}(\Omega_h)} + h \|w\|_{W^{2,q}(\Omega)} \right) . \tag{A.68}$$

The estimates (A.65) and (A.68) yield

$$\begin{aligned} &a_h(\hat{u}^h - v_h, w - \chi_h) + a_h(\hat{u}^h - v_h, \chi_h) \\ &\leq c_p \left(\left\| \hat{u}^h - v_h \right\|_{W^{1,p}(\Omega_h)} + h^2 \|Q_h^1 g\|_{W^{2-1/p,p}(\Gamma)} + \left\| \hat{Q}_h g - Q_h^1 g \right\|_{W^{1-1/p,p}(\Gamma)} \right) \|w - \chi_h\|_{W^{1,q}(\Omega_h)} \\ &+ c_p \left(h^2 \|Q_h^1 g\|_{W^{2-1/p,p}(\Gamma)} + h \left\| \hat{Q} g - Q_h^1 g \right\|_{W^{1-1/p,p}(\Gamma)} \right) \|w\|_{W^{2,q}(\Omega)} , \end{aligned}$$

and by taking χ_h as the infimum from Lemma A.2.7 we get

$$\begin{aligned} &a_h(\hat{u}^h - v_h, w - \chi_h) + a_h(\hat{u}^h - v_h, \chi_h) \\ &\leq c_p \left(h \left\| \hat{u}^h - v_h \right\|_{W^{1,p}(\Omega_h)} + h^2 \|Q_h^1 g\|_{W^{2-1/p,p}(\Gamma)} + h \left\| \hat{Q}_h g - Q_h^1 g \right\|_{W^{1-1/p,p}(\Gamma)} \right) \|\varphi\|_{L^q(\Omega)} \end{aligned} \tag{A.69}$$

For the next part within (A.64), we estimate

$$\begin{aligned} &\left| \langle \hat{u}^h - v_h, \nabla w^T \cdot A \cdot n \rangle_h \right| \\ &\leq c \left\| \hat{u}^h - v_h \right\|_{L^p(\Gamma_h)} \|\nabla w\|_{L^q(\Gamma_h)} \\ &\leq c_p \left\| \hat{u}^h - \widetilde{\gamma \hat{u}^h} \right\|_{L^p(\Gamma_h)} \|w\|_{W^{2,q}(\Omega)} \\ &\leq c_p \left(\left\| (\hat{u}^h - u^h) - \gamma(\widetilde{\hat{u}^h - u^h}) \right\|_{L^p(\Gamma_h)} + \left\| u^h - \widetilde{\gamma u^h} \right\|_{L^p(\Gamma_h)} \right) \|\varphi\|_{L^q(\Omega)} \\ &\leq c_p \left(h \left\| \hat{u}^h - u^h \right\|_{W^{1,p}(\Omega)} + h^2 \left\| u^h \right\|_{W^{2,p}(\Omega)} \right) \|\varphi\|_{L^q(\Omega)} \\ &\leq c_p \left(h \left\| \hat{Q}_h g - Q_h^1 g \right\|_{W^{1-1/p,p}(\Gamma)} + h^2 \|Q_h^1 g\|_{W^{2-1/p,p}(\Gamma)} \right) \|\varphi\|_{L^q(\Omega)} . \end{aligned} \tag{A.70}$$

As $Lw = \varphi$ in Ω, for the last term within (A.64) it holds that

$$\left(\hat{u}^h - v_h, \varphi - Lw\right)_h \leq c \left\|\hat{u}^h - v_h\right\|_{L^p(\Omega_h \setminus \Omega)} \|\varphi\|_{L^q(\Omega)},$$

and again it follows that

$$\left(\hat{u}^h - v_h, \varphi - Lw\right)_h \leq c_p \left(h \left\|\hat{Q}_h g - Q_h^1 g\right\|_{W^{1-1/p,p}(\Omega)} + h^2 \left\|Q_h^1 g\right\|_{W^{2-1/p,p}(\Gamma)}\right) \|\varphi\|_{L^q(\Omega)} \tag{A.71}$$

Combining the estimates (A.69), (A.70) and (A.71), inserting them into (A.64) and using Lemma A.2.9 and Lemma A.2.10 we get

$$\begin{aligned}
\left\|\hat{u}^h - v_h\right\|_{L^p(\Omega_h)} &= \sup_{\varphi \in C_0^\infty(\Omega_h)} \frac{\left(\hat{u}^h - v_h, \varphi\right)_h}{\|\varphi\|_{L^q(\Omega_h)}} \\
&\leq c_p \left(h \left\|\hat{u}^h - v_h\right\|_{W^{1,p}(\Omega_h)} + h \left\|\hat{Q}_h g - Q_h^1 g\right\|_{W^{1-1/p,p}(\Gamma)} + h^2 \left\|Q_h^1 g\right\|_{W^{2-1/p,p}(\Gamma)}\right) \\
&\leq c_p c_{\Omega_h}(h) h^{s+1/p} \|g\|_{W^{s,p}(\Gamma)},
\end{aligned}$$

for $s \in [1 - 1/p, 2 - 1/p]$, which finally completes the proof. □

Now let $u_H \in W^{1,p}(\Omega)$ be the weak solution to

$$\begin{cases} Lu_H = 0 & \text{in } \Omega, \\ u_H = g & \text{on } \Gamma. \end{cases} \tag{A.72}$$

Lemma A.2.19. *Let $p \in [2, p_{-1}(\Omega_h))$ and $s \in [1 - 1/p, 2 - 1/p]$, then it holds that*

$$\begin{aligned}
\|u_H - v_h\|_{L^p(\Omega_h)} &\leq c_p c_{\Omega_h}(h) h^{s+1/p} \|g\|_{W^{s,p}(\Gamma)}, \\
\|u_H - v_h\|_{W^{1,p}(\Omega_h)} &\leq c_p c_{\Omega_h}(h) h^{s-1+1/p} \|g\|_{W^{s,p}(\Gamma)},
\end{aligned}$$

where u_H may be extended to Ω_h by E_s if necessary.

Proof. From Lemma A.2.18 it is clear that we only have to estimates $\left\|u_H - \hat{u}^h\right\|_{W^{i,p}(\Omega)}$ for $i \in \{0, 1\}$. We have

$$\begin{aligned}
\left\|u_H - \hat{u}^h\right\|_{L^p(\Omega_h)} &\leq \left\|E(u_H - \hat{u}^h)\right\|_{L^p(\mathbb{R}^2)} \leq c_p \left\|u_H - \hat{u}^h\right\|_{L^p(\Omega)} \leq c_p \left\|g - \hat{Q}_h g\right\|_{W^{-1/p,p}(\Gamma)} \\
&\leq c_p h^{s+1/p} \|g\|_{W^{s,p}(\Gamma)},
\end{aligned}$$

for $s \in [1 - 1/p, 2 - 1/p]$, where we used Corollary A.1.37. The analog inequality holds true for $i = 1$, which, together with Lemma A.2.18, finishes the proof. □

To continue we need the following generalization of Poincaré's inequality.

Lemma A.2.20. *Let $p \in (1, \infty)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $u \in W^{1,p}(\Omega)$ with trace $u|_\Gamma = g \in W^{1-1/p,p}(\Gamma)$. There exists $c_p > 0$, independent of u and g, such that*

$$\|u\|_{W^{1,p}(\Omega)} \leq c_p \left(|u|_{W^{1,p}(\Omega)} + \|g\|_{L^p(\Gamma)}\right).$$

Proof. The following proof generalizes a proof presented in [35], Dupont, Lemma 2.7, to the case $p \neq 2$.

$$\begin{aligned}\|u\|^p_{L^p(\Omega)} &= \int_\Omega u^p\, \partial_{x_i}(x_i)\,\mathrm{d}x \\ &= -\int_\Omega \partial_{x_i}(u^p)\, x_i\,\mathrm{d}x + \int_\Gamma g^p x_i n_i\,\mathrm{d}s \\ &= -\int_\Omega p u^{p-1}\, \partial_{x_i}(u)\, x_i\,\mathrm{d}x + \int_\Gamma g^p x_i n_i\,\mathrm{d}s,\end{aligned}$$

where n_i shall denote the i-th component of the outer unit normal. Using Hölder's inequality, we get

$$\|u\|^p_{L^p(\Omega)} \le c_p \left(\|u\|^{p-1}_{L^p(\Omega)} |u|_{W^{1,p}(\Omega)} + \|g\|^p_{L^p(\Gamma)} \right),$$

and the generalized Young's inequality yields

$$\|u\|^p_{L^p(\Omega)} \le \frac{1}{2} \|u\|^p_{L^p(\Omega)} + c_p \left(|u|^p_{W^{1,p}(\Omega)} + \|g\|^p_{L^p(\Gamma)} \right). \qquad \square$$

It is well-known that harmonic functions minimize the H^1-seminorm among all functions with the same boundary conditions. Here we are going to generalize that result.

Lemma A.2.21. *Let $\Omega_h \subset \mathbb{R}^2$ be polygonal, let $p \in [2, p_{-1}(\Omega_h))$, let $A \in R^{2\times 2}$ be uniformly elliptic and Lipschitz and let $u_h \in V_h$ be a discrete A-harmonic function, i.e.*

$$(\nabla u_h, A \cdot \nabla v_h)_h = 0 \qquad \forall v_h \in V_{h,0}.$$

Then it holds that

$$|u_h|_{W^{1,p}(\Omega_h)} \le c_p c_{\Omega_h}(h) \inf_{\varphi_h \in V_{h,0}} |u_h - \varphi_h|_{W^{1,p}(\Omega_h)}.$$

Proof. Let $\chi_h \in V_{h,0}$ be the solution to

$$(\nabla \chi_h, A \cdot \nabla v_h)_h = \left(|\nabla u_h|^{p-2} \nabla u_h, A \cdot \nabla v_h \right)_h \qquad \forall v_h \in V_{h,0}.$$

The existence of such a χ_h is ensured by Theorem A.1.47, which also shows the following stability estimate,

$$|\chi_h|_{W^{1,q}(\Omega_h)} \le c_p\, c_{\Omega_h}(h) \left\| |\nabla u_h|^{p-2} \nabla u_h \right\|_{L^q(\Omega_h)} = c_p\, c_{\Omega_h}(h)\, |u_h|^{p-1}_{W^{1,p}(\Omega_h)}.$$

Let $\varphi_h \in V_{h,0}$ be arbitrary, it now holds that

$$\begin{aligned}|u_h|^p_{W^{1,p}(\Omega_h)} &\le c \left(\nabla u_h, |\nabla u_h|^{p-2} A \cdot \nabla u_h \right)_h \\ &= c \left(\nabla u_h, |\nabla u_h|^{p-2} A \cdot \nabla u_h - A \cdot \nabla \chi_h \right)_h \\ &= c \left(\nabla (u_h - \varphi_h), |\nabla u_h|^{p-2} A \cdot \nabla u_h - A \cdot \nabla \chi_h \right)_h,\end{aligned}$$

where in the first step we used the discrete A-harmonicity of u_h, in the second step we used the definition of χ_h. Now we use Hölder's inequality and get

$$\begin{aligned}\Big(\nabla(u_h-\varphi_h),|\nabla u_h|^{p-2}\,A\cdot\nabla u_h-A\cdot\nabla\chi_h\Big)_h &\le c\,|u_h-\varphi_h|_{W^{1,p}(\Omega_h)}\Big(|u_h|^{p-1}_{W^{1,p}(\Omega_h)}+|\chi_h|_{W^{1,q}(\Omega_h)}\Big)\\ &\le c_pc_{\Omega_h}(h)\,|u_h-\varphi_h|_{W^{1,p}(\Omega_h)}\,|u_h|^{p-1}_{W^{1,p}(\Omega_h)}\,,\end{aligned}$$

and hence

$$|u_h|_{W^{1,p}(\Omega_h)}\le c_pc_{\Omega_h}(h)\,|u_h-\varphi_h|_{W^{1,p}(\Omega_h)}\,.$$ □

Lemma A.2.22. *Let $p\in[2,p_{-1}(\Omega_h))$ and $s\in[1-1/p,2-1/p]$. Then it holds that*

$$\begin{aligned}\|u-u_h\|_{L^p(\Omega_h)}&\le c_pc_{\Omega_h}(h)\left(h^2\,\|f\|_{L^p(\Omega)}+h^{s+1/p}\,\|g\|_{W^{s,p}(\Gamma)}\right),\\ \|u-u_h\|_{W^{1,p}(\Omega_h)}&\le c_pc_{\Omega_h}(h)\left(h\,\|f\|_{L^p(\Omega)}+h^{s-1+1/p}\,\|g\|_{W^{s,p}(\Gamma)}\right).\end{aligned}$$

Proof. We set $u_{h,H}=u_h-u_{h,0}$, because of Lemma A.2.17 it remains to estimate $\|u_H-u_{h,H}\|_{W^{i,p}(\Omega_h)}$. For $i\in\{0,1\}$ we get

$$\|u_H-u_{h,H}\|_{W^{i,p}(\Omega_h)}\le\|u_H-v_h\|_{W^{i,p}(\Omega_h)}+\|v_h-u_{h,H}\|_{W^{i,p}(\Omega_h)}\,, \tag{A.73}$$

and with Lemma A.2.20 it follows that

$$\|v_h-u_{h,H}\|_{W^{1,p}(\Omega_h)}\le c_p\left(|v_h-u_{h,H}|_{W^{1,p}(\Omega_h)}+\|v_h-u_{h,H}\|_{L^p(\Gamma_h)}\right). \tag{A.74}$$

Since $v_h-u_{h,H}$ is discrete A-harmonic, it follows from Lemma A.2.21 that

$$|v_h-u_{h,H}|_{W^{1,p}(\Omega_h)}\le c_pc_{\Omega_h}(h)\,|v_h-u_{h,H}-\chi_h|_{W^{1,p}(\Omega_h)}\,,$$

for arbitrary $\chi_h\in V_{h,0}$. Thus, using Lemma A.2.7 with $w=0$ and $\phi_h=u_{h,H}-v_h$ we arrive at

$$\begin{aligned}|v_h-u_{h,H}|_{W^{1,p}(\Omega_h)}&\le c_pc_{\Omega_h}(h)\inf_{\chi_h\in V_{h,0}}|v_h-u_{h,H}-\chi_h|_{W^{1,p}(\Omega_h)}\\ &\le c_pc_{\Omega_h}(h)h^{1/p-1}\,\|v_h-u_{h,H}\|_{L^p(\Gamma_h)}\,.\end{aligned} \tag{A.75}$$

Combining (A.74) and (A.75) yields

$$\|v_h-u_{h,H}\|_{W^{1,p}(\Omega_h)}\le c_pc_{\Omega_h}(h)h^{1/p-1}\,\|v_h-u_{h,H}\|_{L^p(\Gamma_h)}\,.$$

By definition it holds that $(u_{h,H}-v_h)|_{\Gamma_h}=Q_hg-\widetilde{\hat Q_hg}$. In addition, $Q_h\widetilde{\hat Q_hg}=\widetilde{\hat Q_hg}=\widetilde{\hat Q_h\hat Q_hg}$. We set $G=g-\hat Q_hg$ and use Lemma A.2.11 to obtain

$$\begin{aligned}\|u_{h,H}-v_h\|_{L^p(\Gamma_h)}&=\left\|Q_hG-\widetilde{\hat Q_hG}\right\|_{L^p(\Gamma_h)}\\ &\le c_ph^2\,\|G\|_{L^p(\Gamma_h)}=c_ph^2\left\|(\mathrm{Id}-\hat Q_h)g\right\|_{L^p(\Gamma_h)}\,,\end{aligned}$$

and using Lemma A.2.9 we arrive at

$$\|v_h-u_{h,H}\|_{W^{1,p}(\Omega_h)}\le c_pc_{\Omega_h}(h)h^{s+1+1/p}\,\|g\|_{W^{s,p}(\Gamma)}\,, \tag{A.76}$$

for $s\in[0,2-1/p]$. Together with the splitting (A.73) at the beginning of this proof and Lemma A.2.17, this proof is finished. □

Now we are in the position to finally prove Theorem A.2.1.

Proof. Within the following proof, let $s \in [1-1/p, 2-1/p]$ be arbitrary. We start by splitting the error,

$$\|u-u_h\|_{W^{i,p}(\Omega)} \le \|u-u_h\|_{W^{i,p}(\Omega_h)} + \|u-u_h\|_{W^{i,p}(\Omega\setminus\Omega_h)}\,.$$

Because of Lemma A.2.22 it is sufficient to estimate the latter part. Setting $w^h = \left(u_0 + u^h\right) \in W^{2,p}(\Omega)$, we get

$$\|u-u_h\|_{W^{i,p}(\Omega\setminus\Omega_h)} \le \left\|u-w^h\right\|_{W^{i,p}(\Omega)} + \left\|w^h-u_h\right\|_{W^{i,p}(\Omega\setminus\Omega_h)}\,.$$

Now we use Lemma A.1.40 for $\left(u-w^h\right)$ and the properties of Q_h^1 to get

$$\left\|u-w^h\right\|_{W^{1,p}(\Omega)} \le c_p h^{s-1+1/p}\|g\|_{W^{s,p}(\Gamma)}\,, \tag{A.77}$$

$$\left\|u-w^h\right\|_{L^p(\Omega)} \le c_p h^{s+1/p}\|g\|_{W^{s,p}(\Gamma)}\,, \tag{A.78}$$

and it remains to estimate $\left\|w^h-u_h\right\|_{W^{i,p}(\Omega\setminus\Omega_h)}$. As u_h is linear on $\Omega_h^{(j)}$, its second derivatives vanish. It follows with (A.30) that

$$\left|w^h-u_h\right|^p_{W^{1,p}(\Omega_h^{(j)})} \le c_p\left(h^2\left|w^h-u_h\right|^p_{W^{1,p}(\Gamma_h^{(j)})} + h^{2p}\left\|w^h\right\|^p_{W^{2,p}(\Omega_h^{(j)})}\right). \tag{A.79}$$

If $\tau_h^{(j)}$ denotes the triangle which has $\Gamma_h^{(j)}$ as one of its vertices, then one can show, using (A.38) and Young's inequality, that

$$h\|v\|^p_{L^p(\Gamma_h^{(j)})} \le c_p\left(\|v\|^p_{L^p(\tau_h^{(j)})} + h^p|v|^p_{W^{1,p}(\tau_h^{(j)})}\right) \qquad \forall v \in W^{1,p}(\tau_h^{(j)}). \tag{A.80}$$

Setting $v = \nabla\left(w^h-u_h\right)$ and inserting the estimate (A.80) into (A.79) yields

$$\left|w^h-u_h\right|^p_{W^{1,p}(\Omega_h^{(j)})} \le c_p\left(h\left|w^h-u_h\right|^p_{W^{1,p}(\tau_h^{(j)})} + h^{p+1}\left\|w^h\right\|^p_{W^{2,p}(\Omega_h^{(j)}\cup\tau_h^{(j)})}\right). \tag{A.81}$$

Summing (A.81) over all j leads to

$$\left|w^h-u_h\right|_{W^{1,p}(\Omega\setminus\Omega_h)} \le c_p\left(h^{1/p}\left|w^h-u_h\right|_{W^{1,p}(\Omega_h)} + h^{1+1/p}\left\|w^h\right\|_{W^{2,p}(\Omega)}\right), \tag{A.82}$$

The first term on the right hand side of (A.82) can be estimated using the triangle inequality, Lemma A.2.22 and (A.77); the second term can be estimated using Theorem A.1.38 and the properties of Q_h^1, Lemma A.2.10. We arrive at

$$\left\|w^h-u_h\right\|_{W^{1,p}(\Omega\setminus\Omega_h)} \le c_p c_{\Omega_h}(h) h^{1/p}\left(h\|f\|_{L^p(\Omega)} + h^{s-1+1/p}\|g\|_{W^{s,p}(\Gamma)}\right),$$

and combining this estimate with (A.77) yields

$$\|u-u_h\|_{W^{1,p}(\Omega\setminus\Omega_h)} \le c_p c_{\Omega_h}(h)\left(h\|f\|_{L^p(\Omega)} + h^{s-1+1/p}\|g\|_{W^{s,p}(\Gamma)}\right).$$

It remains to prove bounds for $\left\|w^h-u_h\right\|_{L^p(\Omega\setminus\Omega_h)}$. Applying (A.30) and (A.80) directly to $\left(w^h-u_h\right)$ one can show, similar to (A.82), that

$$\left\|w^h-u_h\right\|_{L^p(\Omega\setminus\Omega_h)} \le c_p\left(h^{1/p}\left\|w^h-u_h\right\|_{L^p(\Omega_h)} + h^{1+1/p}\left\|w^h-u_h\right\|_{W^{1,p}(\Omega\cup\Omega_h)}\right). \tag{A.83}$$

We combine (A.82) and (A.83) to get

$$\begin{aligned}&\left\|w^h-u_h\right\|_{L^p(\Omega\setminus\Omega_h)}\\ &\le c_p\left(h^{1/p}\left\|w^h-u_h\right\|_{L^p(\Omega_h)} + h^{1+1/p}\left\|w^h-u_h\right\|_{W^{1,p}(\Omega_h)} + h^{2+2/p}\left\|w^h\right\|_{W^{2,p}(\Omega)}\right).\end{aligned} \tag{A.84}$$

Using the triangle inequality, Lemma A.2.22 and (A.78) proves

$$h^{1/p}\left\|w^h-u_h\right\|_{L^p(\Omega_h)} \le c_p c_{\Omega_h}(h) h^{1/p}\left(h^2\|f\|_{L^p(\Omega)} + h^{s+1/p}\|g\|_{W^{s,p}(\Gamma)}\right). \tag{A.85}$$

Using Theorem A.1.38 and the properties of Q_h^1 we arrive at

$$h^{2+2/p}\left\|w^h\right\|_{W^{2,p}(\Omega)} \le c_p h^{2/p}\left(h^2\|f\|_{L^p(\Omega)} + h^{s+1/p}\|g\|_{W^{s,p}(\Gamma)}\right). \tag{A.86}$$

It holds that $w^h-u_h = (u_0-u_{h,0}) + (u^h-v_h) + (v_h-u_{h,H})$. These terms can be estimated using Lemma A.2.17, Lemma A.2.18 and (A.60), and (A.76), respectively. We get

$$h^{1+1/p}\left\|\nabla(w^h-u_h)\right\|_{L^p(\Omega_h)} \le c_p c_{\Omega_h}(h) h^{1/p}\left(h^2\|f\|_{L^p(\Omega)} + h^{s+1/p}\|g\|_{W^{s,p}(\Gamma)}\right). \tag{A.87}$$

Using the estimates (A.85), (A.86) and (A.87) in (A.84), we get

$$\left\|w^h-u_h\right\|_{L^p(\Omega\setminus\Omega_h)} \le c_p c_{\Omega_h}(h) h^{1/p}\left(h^2\|f\|_{L^p(\Omega)} + h^{s+1/p}\|g\|_{W^{s,p}(\Gamma)}\right),$$

which finally results in

$$\|u-u_h\|_{L^p(\Omega\setminus\Omega_h)} \le c_p c_{\Omega_h}(h)\left(h^2\|f\|_{L^p(\Omega)} + h^{s+1/p}\|g\|_{W^{s,p}(\Gamma)}\right).$$

We finish the proof by noting that Ω_h is convex for all h sufficiently small if and only if Ω is convex. The bounds on p follow with (A.44). □

A.2.4. Finite element approximation on curved domains with irregular differential operators

As stated at the beginning of this section we are now going to prove some results concerning the case when the matrix A is not Lipschitz and thus does not admit H^2-regularity. Once again, let $\Omega \subset \mathbb{R}^2$ be a bounded and convex domain, let $f \in H^{-1/2-\varepsilon}(\Omega)$ and consider the following problem

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{A.88}$$

where the matrix A is uniformly positive definite with coefficients in $C^{0,1/2}(\overline{\Omega})$. With Theorem A.1.30 it follows that $u \in H^{3/2-\varepsilon}(\Omega)$ and

$$\|u\|_{H^{3/2-\varepsilon}(\Omega)} \leq c_\varepsilon \|f\|_{H^{-1/2-\varepsilon}(\Omega)} .$$

Using Céa's lemma it follows that

$$\|u - u_h\|_{H^1(\Omega)} \leq c_\varepsilon h^{1/2-\varepsilon}, \tag{A.89}$$

for the finite-element approximation u_h of u if Ω itself is polygonal. If this is not the case, one should not expect better approximation rates for one additionally has to approximate the curved boundaries. Furthermore, as shown in [91], Theorem 3.8, it holds that

$$\|u - u_h\|_{H^{-1}(\Omega)} \geq c \|u - u_h\|^2_{H^1_0(\Omega)} ,$$

and as estimate (A.89) is of optimal order, it follows that

$$\|u - u_h\|_{L^2(\Omega)} \leq c_\varepsilon h^{1-\varepsilon},$$

is the best order of convergence one can hope for in the general case.

Theorem A.2.23. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, let $u \in H^1_0(\Omega) \cap H^{3/2-\varepsilon}(\Omega)$ be the weak solution to* (A.88) *for given $f \in H^{-1/2-\varepsilon}(\Omega)$ and let $u_h \in V_{h,0}$ solve*

$$a_h(u_h, v_h) = (f, v_h)_h \qquad \forall v_h \in V_{h,0}.$$

If $s \in [0,1]$, then it holds that

$$\|u - u_h\|_{H^s(\Omega_h)} \leq c_\varepsilon h^{1-\varepsilon-s/2} \|f\|_{H^{-1/2-\varepsilon}(\Omega)} .$$

Proof. As Ω is convex it holds that $\Omega_h \subset \Omega$, hence every function in $H^1_0(\Omega_h)$ can, via extension by 0, be regarded as a function in $H^1_0(\Omega)$. Let $\chi_h \in V_{h,0}$ be arbitrary, it holds that

$$c \|\nabla (u - u_h)\|^2_{L^2(\Omega_h)} \leq a_h(u - u_h, u - u_h) = a_h(u - u_h, u - \chi_h),$$

hence

$$\|\nabla (u - u_h)\|_{L^2(\Omega_h)} \leq c \|\nabla (u - \chi_h)\|_{L^2(\Omega_h)} ,$$

and using Lemma A.2.8 we get

$$\|\nabla (u - u_h)\|_{L^2(\Omega_h)} \leq c_\varepsilon h^{1/2-\varepsilon} \|f\|_{H^{-1/2-\varepsilon}(\Omega)} .$$

Next the L^2-error has to be estimated. Let $\varphi \in C_0^\infty(\Omega_h)$ be arbitrary and let $w \in H_0^1(\Omega) \cap H^{3/2-\varepsilon}(\Omega)$ be the solution to

$$a(w, v) = (\varphi, v) \qquad \forall v \in H_0^1(\Omega),$$

again it holds that $\|w\|_{H^{3/2-\varepsilon}(\Omega)} \leq c_\varepsilon \|\varphi\|_{H^{-1/2-\varepsilon}(\Omega_h)}$. Now we have

$$(u - u_h, \varphi)_h = a(w, u - u_h) = a(w - w_h, u - u_h),$$

where $w_h \in V_{h,0}$ is the Ritz-projection of w. We end up with

$$\begin{aligned} (u - u_h, \varphi)_h &= a_h(w - w_h, u - u_h) + (\nabla (w - w_h), A \cdot \nabla (u - u_h))_{\Omega \setminus \Omega_h} \\ &= a_h(w - w_h, u - u_h) + (\nabla w, A \cdot \nabla u)_{\Omega \setminus \Omega_h} . \end{aligned} \tag{A.90}$$

The first term on the right hand side of (A.90) can be estimated as in first part of this proof, it holds that

$$\begin{aligned} a_h(w - w_h, u - u_h) &\leq c \|\nabla (w - w_h)\|_{L^2(\Omega_h)} \|\nabla (u - u_h)\|_{L^2(\Omega_h)} \\ &\leq c_\varepsilon h^{1-2\varepsilon} \|w\|_{H^{3/2-\varepsilon}(\Omega)} \|f\|_{H^{-1/2-\varepsilon}(\Omega)} \\ &\leq c_\varepsilon h^{1-2\varepsilon} \|\varphi\|_{L^2(\Omega_h)} \|f\|_{H^{-1/2-\varepsilon}(\Omega)} . \end{aligned} \tag{A.91}$$

The second term can be estimated using Hölder's generalized inequality. We get

$$(\nabla w, A \cdot \nabla u)_{\Omega \setminus \Omega_h} \leq \|1\|_{L^{\frac{2}{1-2\varepsilon}}(\Omega \setminus \Omega_h)} \|\nabla w\|_{L^{\frac{4}{1+2\varepsilon}}(\Omega)} \|A\|_{L^\infty(\Omega)} \|\nabla u\|_{L^{\frac{4}{1+2\varepsilon}}(\Omega)} . \tag{A.92}$$

Now we use the fact that $|\Omega \setminus \Omega_h| \leq ch^2$ and the continuous embedding $H^{3/2-\varepsilon}(\Omega) \hookrightarrow W^{1,4-\varepsilon}(\Omega)$, and end up with

$$\begin{aligned} (\nabla w, A \cdot \nabla u)_{\Omega \setminus \Omega_h} &\leq c_\varepsilon h^{1-2\varepsilon} \|w\|_{H^{3/2-\varepsilon}(\Omega)} \|u\|_{H^{3/2-\varepsilon}(\Omega)} \\ &\leq c_\varepsilon h^{1-2\varepsilon} \|\varphi\|_{L^2(\Omega_h)} \|f\|_{H^{-1/2-\varepsilon}(\Omega)} . \end{aligned} \tag{A.93}$$

Inserting (A.91) and (A.93) into (A.90) yields

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega_h)} &= \sup_{\varphi \in C_0^\infty(\Omega_h)} \frac{(u - u_h, \varphi)_h}{\|\varphi\|_{L^2(\Omega_h)}} \\ &\leq c_\varepsilon h^{1-2\varepsilon} \|f\|_{H^{-1/2-\varepsilon}(\Omega)} , \end{aligned}$$

and what is left follows with interpolation. □

B. Nomenclature

General notation and abbreviations

- Within this thesis we will follow the constant convention, i.e. c will always denote a constant with possibly different values upon different appearances. These constants shall, if not stated otherwise, be independent of other appearing functions. The explicit dependance upon a specific function or value X will be denoted with c_X.

- With ε we will always denote a positive real number which can be made arbitrarily small.

- For any Banach space X let X' denote its dual. The duality pairing will be denoted with $(\cdot,\cdot)_{X,X'}$.

- For any Hilbert space W let $(\cdot,\cdot)_W$ denote its scalar product.

- For X and Y being two normed spaces, let $\overline{X}^{\|\cdot\|_Y}$ denote the completion of the space X with respect to the norm of Y.

- The continuous embedding of the Banach space X into the Banach space Y will be denoted with $X \hookrightarrow Y$, if this embedding is also compact we will write $X \hookrightarrow\hookrightarrow Y$.

- Let $\mathbb{N}$ be the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

- For every Banach space X, let Id denote the identity function.

- For given $n \in \mathbb{N}$, let $\mathrm{I} \in \mathbb{R}^{n\times n}$ denote the identity matrix.

- For $T\colon X \to Y$ being a linear operator, let $\mathcal{N}(T) \subset X$ denote the nullspace of T and let $\mathcal{R}(T) \subset Y$ denote the range of T.

- For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ being a multiindex, let $|\alpha| = \sum_{i=1}^n \alpha_i$ and

$$\mathrm{D}^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} u.$$

- For an arbitrary subset $\mathcal{A} \subset \mathbb{R}^n$, let $|\mathcal{A}|$ denote its n-dimensional Lebesgue measure.

- For $\Omega \subset \mathbb{R}^n$ being a domain, its boundary $\partial\Omega$ will be denoted with Γ.

- For two sets A and B let $A \triangle B = (A \backslash B) \cup (B \backslash A)$ be the symmetric difference.

Hölder spaces

- For $k \in \mathbb{N}_0$, let $C^k(\Omega)$ be the set of all k-times continuously differentiable functions with norm

$$\|u\|_{C^k(\Omega)} = \max_{|\alpha|\leq k} \sup_{x\in\Omega} |\mathrm{D}^\alpha u(x)| .$$

- For $k \in \mathbb{N}_0$ and $\sigma \in (0,1]$, let $C^{k,\sigma}(\Omega)$ be the set of all k-times continuously differentiable functions whose derivatives of order k are Hölder-continuous with exponent σ. The norm is defined via

$$\|u\|_{C^{k,\sigma}(\Omega)} = \max\left\{\|u\|_{C^k(\Omega)}, |u|_{C^{k,\sigma}(\Omega)}\right\},$$

  with

$$|u|_{C^{k,\sigma}(\Omega)} = \max_{|\alpha|=k} \sup_{x,y\in\Omega} \frac{|\mathrm{D}^\alpha u(x) - \mathrm{D}^\alpha u(y)|}{|x-y|^\sigma}.$$

- Let

$$C^\infty(\Omega) = \bigcap_{k\in\mathbb{N}} C^k(\Omega),$$

  be the set of functions which are arbitrarily differentiable, and let

$$C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) |\, \operatorname{supp} u \subset\subset \Omega\},$$

  be the set of all arbitrarily differentiable functions with compact support in Ω.

Sobolev spaces

- For $\Omega \in \mathbb{R}^n$ and $p \in [1,\infty]$ let $L^p(\Omega)$ be the vector space of all (equivalence classes of) measurable functions u with $\|u\|_{L^p(\Omega)} < \infty$, where

$$\|u\|_{L^p(\Omega)}^p = \int_\Omega |u|^p \,\mathrm{d}x,$$

  for $p < \infty$, whereas for $p = \infty$ the norm is defined via

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| = \inf_{\substack{\mathcal{N}\subset\Omega \\ |\mathcal{N}|=0}} \sup_{x\in\Omega\setminus\mathcal{N}} |u(x)| .$$

- For $k \in \mathbb{N}$ let $W^{k,p}(\Omega)$ denote the vector space of all functions $u \in L^p(\Omega)$ such that the weak derivatives of u up to order k exist in $L^p(\Omega)$. For $p \in [1,\infty)$, the norm is defined via

$$\|u\|_{W^{k,p}(\Omega)}^p = \sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}^p ,$$

and a seminorm can be defined via

$$|u|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha|=k} \|\mathrm{D}^\alpha u\|^p_{L^p(\Omega)} \,.$$

For $p = \infty$ we have

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|\leq k} \|\mathrm{D}^\alpha u\|_{L^\infty(\Omega)} \,,$$
$$|u|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|= k} \|\mathrm{D}^\alpha u\|_{L^\infty(\Omega)} \,.$$

- Let $W^{k,p}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}}$. Within this space the seminorm $|\cdot|_{W^{k,p}(\Omega)}$ is equivalent to the full norm $\|\cdot\|_{W^{k,p}(\Omega)}$, hence let

$$\|u\|_{W^{k,p}_0(\Omega)} = |u|_{W^{k,p}(\Omega)} \,.$$

- For $s \in \mathbb{R}$, $s > 0$ and $s \notin \mathbb{N}$, the space $W^{s,p}(\Omega)$ can be defined via interpolation. Let $s = k + \sigma$ with $k = \lfloor s \rfloor$ and $\sigma \in (0,1)$, then again for $p < \infty$ it holds that

$$\|u\|^p_{W^{s,p}(\Omega)} = \|u\|^p_{W^{k,p}(\Omega)} + |u|^p_{W^{s,p}(\Omega)} \,,$$

where

$$|u|^p_{W^{s,p}(\Omega)} = \sum_{|\alpha|=k} \left(\int_\Omega \int_\Omega \frac{|\mathrm{D}^\alpha u(x) - \mathrm{D}^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y \right) . \tag{B.1}$$

For Ω sufficiently smooth it holds that $W^{s,\infty}(\Omega) = C^{k,\sigma}(\overline{\Omega})$.

- With a slight abuse of notation we may write $u \in W^{s,p}(\Omega)$ if $u \in (W^{s,p}(\Omega))^n$.
- For $s \in \mathbb{R}$ with $s < 0$ and $p \in (1,\infty]$, the space $W^{s,p}(\Omega)$ is defined as the dual space of $W^{-s,q}_0(\Omega)$, where $q \in [1,\infty)$ such that $1/p + 1/q = 1$.
- Let $H^s(\Omega) = W^{s,2}(\Omega)$, which is known to be a Hilbert space. For $k \in \mathbb{N}_0$ it holds that

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \left(\int_\Omega \mathrm{D}^\alpha u \, \mathrm{D}^\alpha v \,\mathrm{d}x \right),$$

whereas the scalar product on the boundary will be denoted with $\langle u,v\rangle_{H^k(\Gamma)}$. Let

$$(u,v)_\Omega = (u,v)_{L^2(\Omega)} \,.$$

If the domain Ω is fixed and there is no risk of confusion, let

$$(u,v) = (u,v)_{L^2(\Omega)} \,,$$
$$\langle u,v\rangle = \langle u,v\rangle_{L^2(\Gamma)} .$$

For Ω_h being a polygonal domain (possibly approximating Ω) with boundary Γ_h, let

$$\begin{aligned} (u,v)_h &= (u,v)_{L^2(\Omega_h)} \,, \\ \langle u,v\rangle_h &= \langle u,v\rangle_{L^2(\Gamma_h)} . \end{aligned} \tag{B.2}$$

If two polygonal domains are used simultaneously, we will denote the second one with Ω_k and boundary Γ_k, the notation (B.2) shall hold accordingly.

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