STABLE RECOVERY FROM THE MAGNITUDE OF SYMMETRIZED FOURIER MEASUREMENTS

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ABSTRACT

In this note we show that stable recovery of complex-valued signals $\mathbf{x} \in \mathbb{C}^n$ up to a global sign can be achieved from the magnitudes of $4n - 1$ Fourier measurements when a certain *symmetrization and zero-padding* is performed before measurement ($4n - 3$ is possible in certain cases). For real signals, symmetrization itself is linear and therefore our result is in this case a statement on uniform phase retrieval. Since complex conjugation is involved, such measurement procedure is not complex–linear but recovery is still possible from magnitudes of linear measurements on, for example, $(\mathrm{Re}(\mathbf{x}), \mathrm{Im}(\mathbf{x}))$.

1. INTRODUCTION

Recovering a signal from intensity (magnitude) measurements is known as the *phase retrieval problem*. This problem has a long history beginning in the 70's by GERCHBERG and SAXTON [1] and later by FIENUP [2], who gave explicit reconstruction algorithms for the phase from magnitude Fourier measurements. Since the magnitude of a linear measurement cannot distinguish between numbers of unit modulus, stability and injectivity for such measurements can only hold up to a global phase resp. sign, i.e. up to a factor $e^{i\omega}$ resp. ± 1. One of the challenging tasks in phase retrieval is to determine the necessary and sufficient number of linear measurements for stability or injectivity. For example, CANDES et.al. [3] have shown stable reconstruction of any n−dimensional complex-valued signal from the magnitude of $\mathcal{O}(n \log n)$ linear Gaussian-random measurements. A more principal result from BALAN et al. in [4] shows that a generic frame exists with injectivity at $4n-2$ measurements. Moreover, they could give a fast reconstruction algorithm in [5]. Using projection methods, MONDRAGON and VORONINSKI could even show in [6] injectivity from $4n-3$ generic linear measurements. In a recent result [7], BANDEIRA et al. conjecture that $4n - 4$ linear measurements are necessary for injectivity. However, a practical construction and implementation of measurements at this limiting number seems to be rather hard, but it serves as an ultimate theoretical bound.

More recently, non-linear or interference–based approaches are considered to provide unique phase reconstruction. For example, WANG [8] presented a method where interference with a known signal $\mathbf{y} \in \mathbb{C}^n$ helps to recover a signal $\mathbf{x} \in \mathbb{C}^n$ up to a global sign from only $3n$ Fourier measurements $|\mathbf{F}(\mathbf{x} + \omega\mathbf{y})|^2$ where $\omega \in \mathbb{C}$ is a root of unity. For *real* k–sparse signals, ELDAR and MENDELSON [9] established stable recovery from $\mathcal{O}(k \log(n/k))$ subgaussian random measurements with high probability. A very recent result [10] from EHLER, FORNASIER and SIGL even extends this to the complex case and provides an explicit reconstruction algorithm. LU and VETTERLI also use sparsity for spectral factorization of real valued impulse responses [11]. Moreover, they also give a reconstruction algorithm. A recent result by WANG and XU [12] states injectivity for k−sparse complex-valued signals from $4k - 2$ generic measurements as long as $k < n$. Unfortunately, so far (to the authors knowledge) there doesn't exists a constructive or deterministic frame providing a recovery or even stable recovery.

In this contribution, we will show a concrete measurement procedure allowing stable recovery of any vector $\mathbf{x} \in \mathbb{C}^n$ with $x_0 \in \mathbb{R}$ up to a *global sign* from magnitudes of $4n - 3$ measurements. The measurements can be implemented as linear mappings on, for example, $(\mathrm{Re}(\mathbf{x}), \mathrm{Im}(\mathbf{x}))$ or $(\mathbf{x}, \bar{\mathbf{x}})$. We want to stress the fact, that our measurements are *not complex–linear*, since we perform a conjugate symmetrization on the signal to obtain equality between auto-correlation and auto-convolution, which allows magnitude measurements from $4n - 3$ linear Fourier measurements. However, this will have implications on certain (compressive) signal processing tasks since such type of measurements occur prior to I/Q–down conversion into a suitable complex baseband model. To prove stability for magnitude Fourier measurements on auto-convolutions, we will use our result in [13] for the (s, f)−sparse zero-padded circular convolution. In view of sparsity, zero padding can also be seen as a particular structured sparse signal subclass in $\mathbb{C}^{4n-3}$.

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2. CIRCULAR CONVOLUTIONS, CORRELATIONS AND THE RNMP

Let $(\mathbf{F})_{kl} := n^{-\frac{1}{2}} \exp(-i2\pi\frac{kl}{n})$ be the $k,l \in \{0,\dots,n-1\}$ elements of the $n\times n$ discrete Fourier transform (DFT) matrix. If dimension of a matrix is important it also will occur as a subscript, i.e. here $\mathbf{F} = \mathbf{F}_n$. As well-known, $\mathbf{F}$ is unitary and $\mathbf{\Gamma} := \mathbf{F}^2$ denotes time-reversal given by its action:

$$\mathbf{\Gamma}\cdot(x_0,\dots,x_{n-1})^T := (x_0,x_{n-1},\dots,x_1)^T$$

In particular $\mathbf{\Gamma}$ is an involution, i.e. $\mathbf{\Gamma}^2 = \mathbf{F}^4 = 1$. The circular convolution $\sum_{l=0}^{n-1} x_l y_{k\ominus l}$ ($\ominus$ and $\oplus$ mean $\pm$ modulo n) of two vectors $\mathbf{x},\mathbf{y} \in \mathbb{C}^n$ is a symmetric bilinear mapping given as:

$$\mathbf{x} \circledast \mathbf{y} = \sqrt{n}\mathbf{F}^*(\mathbf{F}\mathbf{x} \odot \mathbf{F}\mathbf{y}) = \mathbf{y} \circledast \mathbf{x} \tag{1}$$

and $\mathbf{x}\circledast\mathbf{x}$ is called (circular) auto-convolution. Similarly, the circular correlation $\sum_{l=0}^{n-1} x_l \bar{y}_{k\oplus l}$ is defined as $\mathbf{x} \circledast \mathbf{y} := \mathbf{x} \circledast \mathbf{\Gamma}\bar{y}$ and we have that Fourier transform of the *auto-correlation*:

$$\begin{aligned} \mathbf{F}(\mathbf{x}\circledast\mathbf{x}) &= \sqrt{n}\mathbf{F}\mathbf{x} \odot \mathbf{F}\mathbf{\Gamma}\overline{\mathbf{x}} \\ &= \sqrt{n}\mathbf{F}\mathbf{x} \odot \overline{\mathbf{F}\mathbf{x}} = \sqrt{n}|\mathbf{F}\mathbf{x}|^2 \end{aligned} \tag{2}$$

is given as the squared magnitudes of the Fourier transform of $\mathbf{x}$. Furthermore, $(\mathbf{S}^i)_{kl} = \delta_{k\oplus i,l}$ denotes the elements of ith power of the unit right shift operator $\mathbf{S}$.

In [13] and [14] we have established a stability statement for zero-padded sparse circular convolutions. Let $\text{supp}(\mathbf{x}) := \{i : x_i \neq 0\}$ be the support of a vector in the canonical basis and $\Sigma_k^n := \{\mathbf{x} \in \mathbb{C}^n : |\text{supp}(\mathbf{x})| \leq k\}$ be the k–sparse vectors. We have the following result on the *restricted norm multiplicativity property* (RNMP) for the circular convolution of sparse zero-padded signals (see [13] for the general definition):

Theorem 1 (RNMP for circular convolutions, [15, 14])**.** *Let $s,f,n \in \mathbb{N}$ with $s \leq f \leq n$. Then there exists a constant $\alpha_{n'} > 0$ with $n' = n'(s,f,n) := \min\{\tilde{n}(s,f),n\}$, such that for all $\mathbf{x} \in \Sigma_s^n, \mathbf{y} \in \Sigma_f^n$ it holds*

$$\alpha_{n'}\|\mathbf{x}\|\,\|\mathbf{y}\| \leq \|(\mathbf{x},\mathbf{0}) \circledast (\mathbf{y},\mathbf{0})\| \leq \sqrt{s}\,\|\mathbf{x}\|\,\|\mathbf{y}\|, \tag{3}$$

where $(\mathbf{x},\mathbf{0}),(\mathbf{y},\mathbf{0}) \in \mathbb{C}^{2n-1}$ denotes the vectors padded by $n-1$ zeros.

Note that for sufficiently small s and f the constant $\alpha_{n'}$ depends *solely* on the sparsity and not on the ambient dimension n [14][1]. Furthermore, without additional restrictions, zero padding is necessary to obtain a lower bound strictly greater than zero (see for example also [15] for an explicit example here). In fact, Theorem 1 is a statement on regular convolutions. However, it is natural to expect also a bound without zero padding in prime dimension. Moreover, from $\mathbf{x}\circledast\mathbf{y} = \mathbf{S}\mathbf{x}\circledast\mathbf{S}\mathbf{y}$ follows that (3) holds whenever the zeros are contained in a cyclic block of size $n-1$.

[1]Our first approach on an explicit formula for $\tilde{n}(s,f)$ in [15] has been corrected in [14]

3. RECOVERY FROM THE MAGNITUDE OF SYMMETRIZED FOURIER MEASUREMENTS

Our contribution is motivated by the framework given in [13] on bilinear maps. Let $B(\mathbf{x},\mathbf{y})$ be a symmetric bilinear map and denote its diagonal part by $A(\mathbf{x}) = B(\mathbf{x},\mathbf{x})$. Obviously there holds the binomial-type formula:

$$A(\mathbf{x}_1) - A(\mathbf{x}_2) = B(\mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1+\mathbf{x}_2) \tag{4}$$

establishing that such $\mathbf{x}_1$ and $\mathbf{x}_2$ can be (stable) distinguished modulo global sign on the basis of $A(\mathbf{x}_1)$ and $A(\mathbf{x}_2)$ whenever $B(\mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1+\mathbf{x}_2)$ is well-separated from zero. More precisely, such a condition is given by the RNMP (given in (3) for the special case $B(\mathbf{x},\mathbf{y}) = \mathbf{x}\circledast\mathbf{y}$ to be considered here). Since $B(\mathbf{x},\mathbf{y}) = \mathbf{x}\circledast\mathbf{y}$ is symmetric one can therefore combine (4), Theorem 1 and the compression results in [13] with a union bound over $\binom{n}{k}$ possible support cases of Σ_k^n. It then follows that each (zero-padded) k–sparse $\mathbf{x}$ for sufficiently large n can be stable recovered modulo global sign from $O(k\log n)$ compressive i.i.d. subgaussian (and suitable generalizations based on concentration properties) samples of its circular auto-convolution (which itself is at most k^2–sparse). However, more important is the estimation of $\mathbf{x}$ based on measurements on its auto-correlation $\mathbf{x}\circledast\mathbf{x}$. In particular, for Fourier measurements this corresponds to the observation of intensity, see (2). But, circular correlation $\mathbf{x}\circledast\mathbf{y}$ is only symmetric when $\mathbf{x} = \mathbf{\Gamma}\bar{\mathbf{x}}$ (if and only if and the same also for $\mathbf{y}$). In general, a symmetrization $\mathcal{S}:\mathbb{C}^n \to \mathbb{C}^{2n-1}$ is therefore necessary here:

$$\mathcal{S}(\mathbf{x}) := (\overbrace{x_0,x_1,\dots,x_{n-1}}^{=\mathbf{x}},\overbrace{\bar{x}_{n-1},\dots,\bar{x}_1}^{=:\mathbf{x}_-^\circ})^T. \tag{5}$$

Linear phase filters: The impulse response $\mathbf{h} := \mathbf{S}_{2n-1}^{n-1}\mathcal{S}(\mathbf{x}) = (\overline{x_{n-1}},\dots,\overline{x_1},x_0,x_1,\dots,x_{n-1})^T \in \mathbb{C}^{2n-1}$, defines an odd-length *linear-phase filter* $H(z) = \sum_{k=0}^{2n-2} h_k z^{-k}$ for $z \in \mathbb{C}$ if $x_0 \in \mathbb{R}$, since we have $\overline{h_0} = h_{2n-2} \neq 0$ and

$$\overline{h_k} = h_{2n-2-k} \quad \text{for} \quad k \in \{0,\dots,2n-2\}. \tag{6}$$

If $x_{n-1} \neq 0$, then the impulse response or filter is called *Hermitian* or *conjugate symmetric* of order $2n-2$, see e.g. [16, Cha.2]. Hence, by the shift-invariance we get[2] for $\mathbf{x} \in \mathbb{C}_0^n$

$$\begin{aligned} A(\mathbf{x}) &= \mathcal{S}(\mathbf{x}) \circledast \mathcal{S}(\mathbf{x}) \\ &= \mathbf{S}_{2n-1}^{n-1}\mathcal{S}(\mathbf{x}) \circledast \mathbf{S}_{2n-1}^{n-1}\mathcal{S}(\mathbf{x}) = \mathbf{h}\circledast\mathbf{h}, \end{aligned} \tag{7}$$

which is the circular auto-convolution of a linear-phase filter.

Let us stress the fact, that the symmetrization map is linear only for *real* vectors $\mathbf{x}$ since complex conjugation is involved. On the other hand, $\mathcal{S}$ can obviously be written as a linear map

[2] Note, that $\mathbf{S}_{2n-1}^{n-1}$ centers the impulse response such that it becomes a causal FIR filter.

on vectors like $(\mathrm{Re}(\mathbf{x}), \mathrm{Im}(\mathbf{x}))$ or $(\mathbf{x}, \bar{\mathbf{x}})$. Now, for $x_0 = \bar{x}_0$ the symmetry condition $\mathcal{S}(\mathbf{x}) = \mathbf{\Gamma}\overline{\mathcal{S}(\mathbf{x})}$ is fulfilled (note that here $\mathbf{\Gamma} = \mathbf{\Gamma}_{2n-1}$):

$$\mathcal{S}(\mathbf{x}) = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}_-^\circ \end{pmatrix} = \mathbf{\Gamma} \begin{pmatrix} \overline{\mathbf{x}} \\ \mathbf{x}_-^\circ \end{pmatrix} = \mathbf{\Gamma} \overline{\begin{pmatrix} \mathbf{x} \\ \mathbf{x}_-^\circ \end{pmatrix}} = \mathbf{\Gamma}\overline{\mathcal{S}(\mathbf{x})}. \tag{8}$$

Let us abbreviate therefore $\mathbb{C}_0^n := \{\mathbf{x} \in \mathbb{C}^n : x_0 \in \mathbb{R}\}$. Thus, for $\mathbf{x}, \mathbf{y} \in \mathbb{C}_0^n$, circular correlation of (conjugate) symmetrized vectors is symmetric and agrees with the circular convolution. To apply Theorem 1 we define the *zero-padded symmetrization* (first zero padding, then symmetrization) $\mathcal{S}_z : \mathbb{C}^n \to \mathbb{C}^{4n-3}$ by:

$$\mathcal{S}_z(\mathbf{x}) := \mathcal{S}\begin{pmatrix} \mathbf{x} \\ \mathbf{0}_{n-1} \end{pmatrix}, \tag{9}$$

Theorem 2. *Let $n \in \mathbb{N}$, then $\tilde{n} = 4n - 3$ absolute-square Fourier measurements of zero padded symmetrized vectors in $\mathbb{C}^{\tilde{n}}$, given by (9), are stable up to a global sign for $\mathbf{x} \in \mathbb{C}_0^n$, i.e. for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}_0^n$ it holds*

$$\left\| |\mathbf{F}\mathcal{S}_z(\mathbf{x}_1)|^2 - |\mathbf{F}\mathcal{S}_z(\mathbf{x}_2)|^2 \right\| \geq c \, \|\mathcal{S}_z(\mathbf{x}_1 - \mathbf{x}_2)\| \, \|\mathcal{S}_z(\mathbf{x}_1 + \mathbf{x}_2)\| \tag{10}$$

with $c = c(\tilde{n}) = \alpha_{\tilde{n}}/\sqrt{\tilde{n}} > 0$ and $\mathbf{F} = \mathbf{F}_{\tilde{n}}$.

Note that we have:

$$2\|\mathbf{x}\|^2 \geq \|\mathcal{S}_z(\mathbf{x})\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{x}_-^\circ\|^2 \geq \|\mathbf{x}\|^2$$

Thus, $\mathcal{S}_z(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$ and the stability in distinguishing $\mathbf{x}_1$ and $\mathbf{x}_2$ up to a global sign follows from the RHS of (10) and reads explicitly as:

$$\left\| |\mathbf{F}\mathcal{S}_z(\mathbf{x}_1)|^2 - |\mathbf{F}\mathcal{S}_z(\mathbf{x}_2)|^2 \right\| \geq c \, \|\mathbf{x}_1 - \mathbf{x}_2\| \, \|\mathbf{x}_1 + \mathbf{x}_2\| \, . \tag{11}$$

Proof. For symmetrized vectors $\mathcal{S}_z(\mathbf{x})$, auto-convolution agrees with auto-correlation and we get from (2):

$$\mathbf{F}(A(\mathbf{x})) = \mathbf{F}(\mathcal{S}_z(\mathbf{x}) \circledast \mathcal{S}_z(\mathbf{x})) = \sqrt{\tilde{n}} \, |\mathbf{F}\mathcal{S}_z(\mathbf{x})|^2 \, . \tag{12}$$

Putting things together we get for every $\mathbf{x} \in \mathbb{C}_0^n$:

$$\begin{aligned} \sqrt{\tilde{n}} \left\| |\mathbf{F}\mathcal{S}_z(\mathbf{x}_1)|^2 - |\mathbf{F}\mathcal{S}_z(\mathbf{x}_2)|^2 \right\| & \\ &= \|\mathbf{F}(A(\mathbf{x}_1) - A(\mathbf{x}_2))\| \\ \mathbf{F} \text{ is unitary} \to &= \|A(\mathbf{x}_1) - A(\mathbf{x}_2)\| \\ &\overset{(4)}{=} \|\mathcal{S}_z(\mathbf{x}_1 - \mathbf{x}_2) \circledast \mathcal{S}_z(\mathbf{x}_1 + \mathbf{x}_2)\| \\ \text{Thm 1} \to &\geq \alpha_{\tilde{n}} \, \|\mathcal{S}_z(\mathbf{x}_1 - \mathbf{x}_2)\| \cdot \|\mathcal{S}_z(\mathbf{x}_1 + \mathbf{x}_2)\| \end{aligned} \tag{13}$$

In the last step we use that Theorem 1 applies whenever the non-zero entries are contained in a cyclic block of length $2n - 1$. □

In the *real case* (10) is equivalent to a *stable linear embedding* in $\mathbb{R}^{4n-3}$ up to a global sign (see here also [9] where ELDAR and MENDELSON used the ℓ^1−norm on the left side) and therefore this is an *explicit phase retrieval statement* for *real* signals. Recently, stable recovery also in the complex case up to a global phase from the same number of subgaussian measurements has been achieved by EHLER et al. in [10]. Here the difference term is lifted to a symmetric matrix difference and the ℓ^2−norm to the Frobenius norm. Both results hold with exponential high probability whereby our result is deterministic. Nevertheless, Ehler et.al. could show by [10, Thm.3.1] the convergence of a greedy algorithm if the deterministic non-linear measurements fulfill a stable embedding and the signals obey a sufficient decay in magnitude. Since our Theorem 2 guarantees for the non-linear Fourier type measurements in (10) a stable embedding, there result ensure a recovery by a greedy algorithm. But, since $\mathcal{S}_z$ is *not complex-linear* Theorem 2 cannot directly be compared with the usual complex phase retrieval results. On the other hand, such an approach can now indeed distinguish the complex phase by the Fourier measurements and symmetrization provides injectivity for magnitude Fourier measurements up to a global sign. To get rid of the odd definition $\mathbb{C}_0^n$ one could symmetrize (and zero padding) $\mathbf{x} \in \mathbb{C}^n$ also by:

$$\mathcal{S}_z(\mathbf{x}) := (\mathbf{0}_n, x_0, \ldots, x_{n-1}, \bar{x}_{n-1}, \ldots, \bar{x}_0, \mathbf{0}_{n-1})^T \in \mathbb{C}^{4n-1} \tag{14}$$

again satisfying $\mathcal{S}_z(\mathbf{x}) = \mathbf{\Gamma}_{4n-1}\overline{\mathcal{S}_z(\mathbf{x})}$ at the price of two further dimensions. Hence, we also have:

Corrolary 1. *Let $n \in \mathbb{N}$, then $\tilde{n} = 4n - 1$ absolute-square Fourier measurements of zero padded and symmetrized vectors given by (14) are stable up to a global sign for $\mathbf{x} \in \mathbb{C}^n$, i.e. for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^n$ it holds*

$$\left\| |\mathbf{F}\mathcal{S}_z(\mathbf{x}_1)|^2 - |\mathbf{F}\mathcal{S}_z(\mathbf{x}_2)|^2 \right\| \geq 2c \, \|\mathbf{x}_1 - \mathbf{x}_2\| \, \|\mathbf{x}_1 + \mathbf{x}_2\| \tag{15}$$

with $c = c(\tilde{n}) = \alpha_{\tilde{n}}/\sqrt{\tilde{n}} > 0$ and $\mathbf{F} = \mathbf{F}_{\tilde{n}}$.

The proof of it is along the same steps as in Theorem 2. The direct extension to sparse signals as in [13] seems to be difficult since randomly chosen Fourier samples do not provide a sufficient measure of concentration property without further randomization.

4. CONCLUSION

In this note we have shown stable recovery (up to a global sign) of a signal $\mathbf{x}$ from magnitude measurements on the Fourier transform of its symmetrization $\mathcal{S}_z(\mathbf{x})$. For real signals this procedure is linear and establishes therefore a phase retrieval method up to global sign. However, also in the complex case this has practical relevance and system design implications when considering linear measurements on $(\mathrm{Re}(\mathbf{x}), \mathrm{Im}(\mathbf{x}))$ (or $(\mathbf{x}, \bar{\mathbf{x}})$). Our result is deterministic and uniform, i.e. it guarantees recovery up to a global sign for any vector $\mathbf{x} \in \mathbb{C}^n$. Finally, the constant in the stability result depends only on the sparsity of $\mathbf{x}$ indicating a possible further reduction of the number of observations in the Fourier domain also in this case.

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