Monolithic 0D windkessel 3D structure coupling with applications in cardiac mechanics

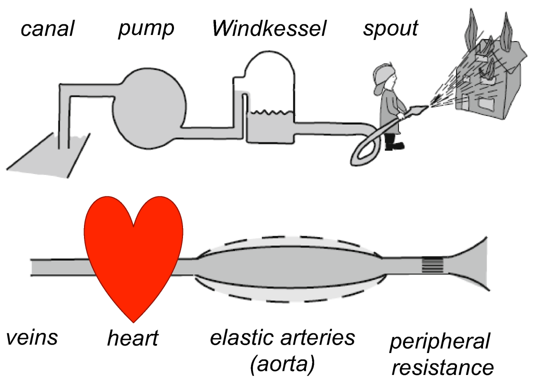
Marc Hirschvogel, Michael W. Gee

Mechanics & High Performance Computing Group, Technische Universität München, Germany

Windkessel models in cardiac mechanics

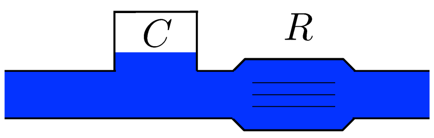
Windkessel = air chamber, used to transform an **oscillatory flow of a pump** into a rather continuous, **steady flow at the outlet** (spout)

The big elastic, **compliant arteries** in the human body act as windkessel for the unsteady output from the heart; windkessel effect governs the **pressure load onto the heart** during ejection

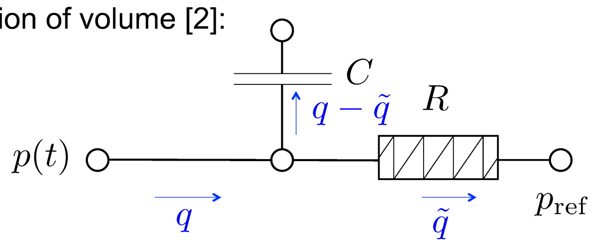


images from [1]

Most simple windkessel model relates pressure p to flow q by a **resistance** R and a **compliance** C (**2-element windkessel**), while the resistance is derived from Poiseuille flow through a rigid vessel and the compliance by considerations regarding conservation of volume [2]:



hydraulic circuit analogy



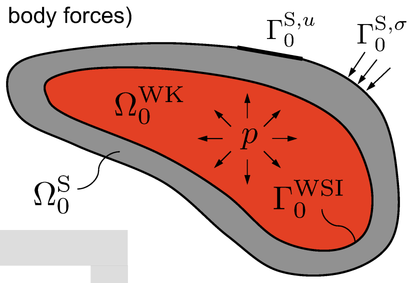
electric circuit analogy

$$C\frac{\mathrm{d}p}{\mathrm{d}t}+\frac{p-p_{\mathrm{ref}}}{R}=q$$

Monolithic windkessel-structure coupling

3D nonlinear elastodynamics (in absence of prescribed body forces)

$$\begin{aligned}\mathrm{Div}(\boldsymbol{FS})-\rho_0\ddot{\boldsymbol{u}}&=\boldsymbol{0}\quad \text{in } \Omega_0^{\mathrm{S}}\times[0,T] \qquad \boldsymbol{S}=\frac{\partial\Psi}{\partial\boldsymbol{E}}\\ \boldsymbol{u}&=\hat{\boldsymbol{u}}\quad \text{on } \Gamma_0^{\mathrm{S},u}\times[0,T]\\ \boldsymbol{T}&=\hat{\boldsymbol{T}}\quad \text{on } \Gamma_0^{\mathrm{S},\sigma}\times[0,T]\\ \boldsymbol{u}(\boldsymbol{X},0)=\hat{\boldsymbol{u}},\quad \dot{\boldsymbol{u}}(\boldsymbol{X},0)&=\hat{\dot{\boldsymbol{u}}}\quad \text{in } \Omega_0^{\mathrm{S}}\\ \boldsymbol{T}&=p\,J\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{N}\quad \text{on } \Gamma_0^{\mathrm{WSI}}\times[0,T]\end{aligned}$$



2-element windkessel (exemplary)

$$C\frac{\mathrm{d}p}{\mathrm{d}t}+\frac{p-p_{\mathrm{ref}}}{R}-q(\boldsymbol{u})=0\quad \text{in } \Omega_0^{\mathrm{WK}}\times[0,T]$$

with the flux as material time derivative of the current enclosed volume:

$$q(\boldsymbol{u})=-\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_{\Omega^{\mathrm{WK}}}\mathrm{d}v=-\frac{\mathrm{D}}{\mathrm{D}t}\frac{1}{3}\int\limits_{\Gamma^{\mathrm{WSI}}}\boldsymbol{x}\cdot\boldsymbol{n}\,\mathrm{d}a=-\frac{1}{3}\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_{\Gamma_0^{\mathrm{WSI}}}(\boldsymbol{u}+\boldsymbol{X})\cdot J\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{N}\,\mathrm{d}A$$

Final nonlinear system of equations to solve in each time step n after **finite element discretization in space** and **finite difference discretization in time** (e.g. Generalized-Alpha time integration for the nonlinear solid, One-Step-Theta time integration for the windkessel equation):

$$\mathbf{R}(\mathbf{d}_{n+1},p_{n+1})=\begin{bmatrix}\mathbf{r}^{\mathrm{S}}(\mathbf{d}_{n+1},p_{n+1})\\ \mathbf{r}^{\mathrm{WK}}(\mathbf{d}_{n+1},p_{n+1})\end{bmatrix}=$$

$$\begin{bmatrix}\mathbf{M}\,\mathbf{a}_{n+1-\alpha_{\mathrm{m}}}+\mathbf{C}\,\mathbf{v}_{n+1-\alpha_{\mathrm{f}}}+\mathbf{F}_{\mathrm{int};n+1-\alpha_{\mathrm{f}}}(\mathbf{d}_{n+1})-\mathbf{F}_{\mathrm{ext};n+1-\alpha_{\mathrm{f}}}(\mathbf{d}_{n+1},p_{n+1})\\ C\,\overset{\circ}{p}_{n+\theta}+\frac{p_{n+\theta}-p_{\mathrm{ref}}}{R}-q_{n+\theta}(\mathbf{d}_{n+1})\end{bmatrix}\overset{!}{=}\mathbf{0}$$

Linearized monolithic system to be solved in each Newton iteration i:

$$\begin{bmatrix}\frac{\partial\mathbf{r}^{\mathrm{S}}_{n+1}}{\partial\mathbf{d}_{n+1}} & \frac{\partial\mathbf{r}^{\mathrm{S}}_{n+1}}{\partial p_{n+1}}\\ \frac{\partial\mathbf{r}^{\mathrm{WK}}_{n+1}}{\partial\mathbf{d}_{n+1}} & \frac{\partial\mathbf{r}^{\mathrm{WK}}_{n+1}}{\partial p_{n+1}}\end{bmatrix}^{i}\begin{bmatrix}\Delta\mathbf{d}_{n+1}\\ \Delta p_{n+1}\end{bmatrix}^{i+1}=-\begin{bmatrix}\mathbf{r}^{\mathrm{S}}_{n+1}\\ \mathbf{r}^{\mathrm{WK}}_{n+1}\end{bmatrix}^{i}$$

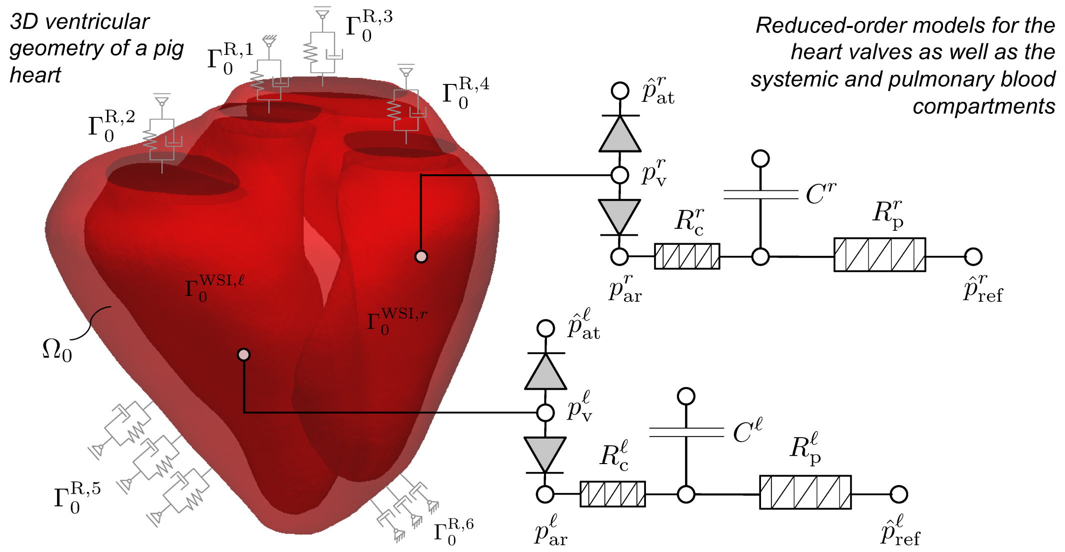
$$\left.\frac{\partial\mathbf{r}^{\mathrm{S}}_{n+1}}{\partial\mathbf{d}_{n+1}}\right|^{i}=\left[\frac{1-\alpha_{\mathrm{m}}}{\beta\Delta t^2}\mathbf{M}+\frac{(1-\alpha_{\mathrm{f}})\gamma}{\beta\Delta t}\mathbf{C}+(1-\alpha_{\mathrm{f}})\frac{\partial\mathbf{F}_{\mathrm{int}}(\mathbf{d}_{n+1})}{\partial\mathbf{d}_{n+1}}-(1-\alpha_{\mathrm{f}})\frac{\partial\mathbf{F}_{\mathrm{ext}}(\mathbf{d}_{n+1},p_{n+1})}{\partial\mathbf{d}_{n+1}}\right]\Bigg|^{i}$$

$$\left.\frac{\partial\mathbf{r}^{\mathrm{S}}_{n+1}}{\partial p_{n+1}}\right|^{i}=-(1-\alpha_{\mathrm{f}})\left.\frac{\partial\mathbf{F}_{\mathrm{ext}}(\mathbf{d}_{n+1},p_{n+1})}{\partial p_{n+1}}\right|^{i}$$

$$\left.\frac{\partial\mathbf{r}^{\mathrm{WK}}_{n+1}}{\partial p_{n+1}}\right|^{i}=\left[C\frac{\partial\overset{\circ}{p}_{n+\theta}}{\partial\overset{\circ}{p}_{n+1}}\frac{\partial\overset{\circ}{p}_{n+1}}{\partial p_{n+1}}+\frac{1}{R}\frac{\partial p_{n+\theta}}{\partial p_{n+1}}\right]\Bigg|^{i}=\left[C\frac{1}{\Delta t}+\frac{1}{R}\theta\right]\Bigg|^{i}$$

$$\left.\frac{\partial\mathbf{r}^{\mathrm{WK}}_{n+1}}{\partial\mathbf{d}_{n+1}}\right|^{i}=-\left.\frac{\partial q_{n+\theta}(\mathbf{d}_{n+1})}{\partial\mathbf{d}_{n+1}}\right|^{i}=\left.\frac{\partial V(\mathbf{d}_{n+1})}{\partial\mathbf{d}_{n+1}}\frac{1}{\Delta t}\right|^{i}$$

Active cardiac mechanics: Problem setting



Structure (heart) – Weak balance equation in terms of Principle of Virtual Work:

$$\int\limits_{\Omega_0}\rho_0\ddot{\boldsymbol{u}}\cdot\delta\boldsymbol{u}\,\mathrm{d}V+\int\limits_{\Omega_0}\boldsymbol{S}:\delta\boldsymbol{E}\,\mathrm{d}V+\sum_{j=1}^{6}\int\limits_{\Gamma_0^{\mathrm{R},j}}\mathbb{I}\,(k_j\boldsymbol{u}+c_j\boldsymbol{v})\cdot\delta\boldsymbol{u}\,\mathrm{d}A=$$

$$=\sum_{i=\ell,r}\int\limits_{\Gamma_0^{\mathrm{WSI},i}}p_{\mathrm{v}}^{i}J\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{N}\cdot\delta\boldsymbol{u}\,\mathrm{d}A$$

Robin boundary conditions: springs and dashpots serving as embedding tissue model for the outer heart surfaces

$\mathbb{I}=\mathbf{1}$ if $j=1$ or $j=6$, else $\mathbb{I}=\boldsymbol{N}\otimes\boldsymbol{N}$

Passive plus **active** stress [3]: $\boldsymbol{S}=\frac{\partial\Psi}{\partial\boldsymbol{E}}+\tau_{\mathrm{a}}\,\boldsymbol{f}_0\otimes\boldsymbol{f}_0,\quad \dot{\tau}_{\mathrm{a}}=-|u|\tau_{\mathrm{a}}+\sigma_0|u|_+$

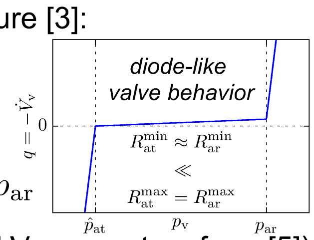
Anisotropic passive material law for myocardium (2 distinct directions) [4]:

$$\Psi^{\mathrm{myo}}=\frac{a}{2b}e^{b(\bar{I}_C-3)}+\sum_{i=\mathrm{f,s}}\frac{a_i}{2b_i}\left(e^{b_i(I\!V_i-1)^2}-1\right)+\frac{a_{\mathrm{fs}}}{2b_{\mathrm{fs}}}\left(e^{b_{\mathrm{fs}}V\!I\!I\!I_{\mathrm{fs}}^2}-1\right)+\Psi_{\mathrm{vol}}(\kappa,J)$$

$$I_C=\mathrm{tr}\boldsymbol{C},\quad I\!V_{\mathrm{f}}=\boldsymbol{f}_0\cdot\boldsymbol{C}\boldsymbol{f}_0,\quad I\!V_{\mathrm{s}}=\boldsymbol{s}_0\cdot\boldsymbol{C}\boldsymbol{s}_0,\quad V\!I\!I\!I_{\mathrm{fs}}=\boldsymbol{f}_0\cdot\boldsymbol{C}\boldsymbol{s}_0$$

Valve law relating arterial and atrial to ventricular pressure [3]:

$$q(\boldsymbol{u})=\begin{cases}\frac{1}{R_{\mathrm{at}}^{\mathrm{min}}}(p_{\mathrm{v}}-\hat{p}_{\mathrm{at}}), & p_{\mathrm{v}}\leq\hat{p}_{\mathrm{at}}\\ \frac{1}{R_{\mathrm{at}}^{\mathrm{max}}}(p_{\mathrm{v}}-\hat{p}_{\mathrm{at}}), & \hat{p}_{\mathrm{at}}\leq p_{\mathrm{v}}\leq p_{\mathrm{ar}}\\ \frac{1}{R_{\mathrm{ar}}^{\mathrm{min}}}(p_{\mathrm{v}}-p_{\mathrm{ar}})+\frac{1}{R_{\mathrm{at}}^{\mathrm{max}}}(p_{\mathrm{ar}}-\hat{p}_{\mathrm{at}}), & p_{\mathrm{v}}\geq p_{\mathrm{ar}}\end{cases}$$



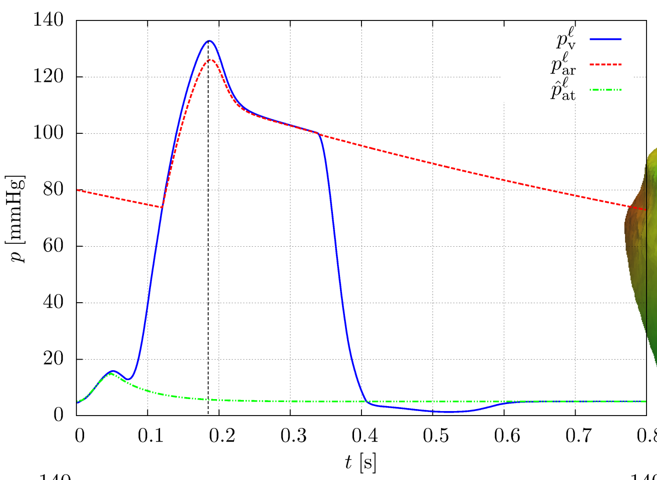
3-element windkessel model for the arterial pressure (LV parameters from [5]):

$$C\dot{p}_{\mathrm{ar}}+\frac{p_{\mathrm{ar}}-\hat{p}_{\mathrm{ref}}}{R_{\mathrm{p}}}=\begin{cases}0, & p_{\mathrm{v}}\leq p_{\mathrm{ar}}\\ \left(1+\frac{R_{\mathrm{c}}}{R_{\mathrm{p}}}\right)q(\boldsymbol{u})+R_{\mathrm{c}}C\dot{q}(\boldsymbol{u}), & p_{\mathrm{v}}>p_{\mathrm{ar}}\end{cases}$$

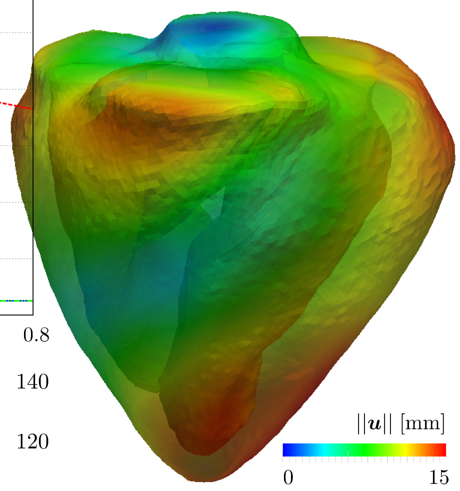
Prescribed atrial pressure $\hat{p}_{\mathrm{at}}(t)$ such that atrial contraction can be simulated

Prestressing [6] of ventricles to low end diastolic pressure $p_{\mathrm{v}}^{0}=\hat{p}_{\mathrm{at}}(t=t_0)$

Results

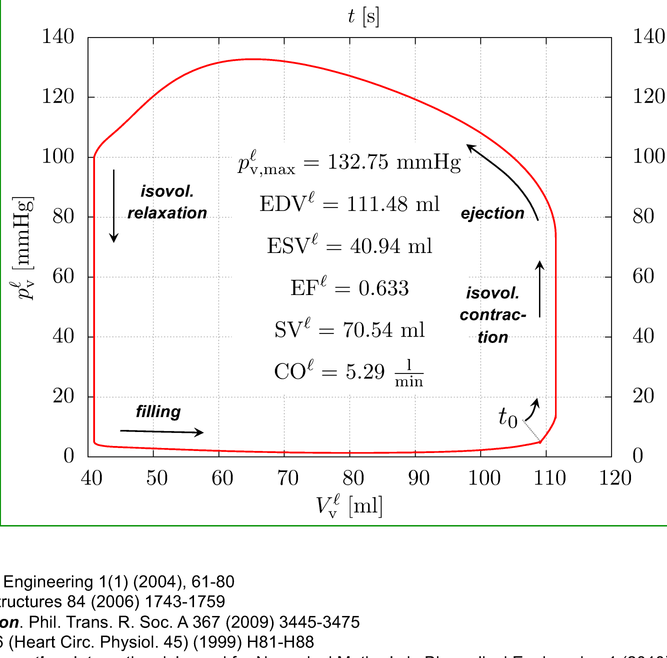


Left systemic pressures over cardiac cycle time



"patient-specific" pig heart, snapshot at time of maximum contraction ($t\approx0.2$ s), color is absolute value of displacements

finite element edge length 2 mm
178 567 TET10 elements
(804 489 degrees of freedom)



Left ventricular pressure over volume: Physiologically meaningful results that correspond well to experimental data [7]

References

[1] Westerhof, N.; Lankhaar, J.-W.; Westerhof, B. E. ***The arterial Windkessel***. Med. Biol. Eng. Comput. 47(2) (2009) 131-141

[2] Olufsen, M.S.; Nadim, A. ***On deriving lumped models for blood flow and pressure in the systemic arteries***. Mathematical Biosciences and Engineering 1(1) (2004), 61-80

[3] Sainte-Marie, J.; Chapelle, D.; Cimrman, R.; Sorine, M. ***Modeling and estimation of the cardiac electromechanical activity***. Computers & Structures 84 (2006) 1743-1759

[4] Holzapfel, G.A.; Ogden, R.W. ***Constitutive modelling of passive myocardium: a structurally based framework for material characterization***. Phil. Trans. R. Soc. A 367 (2009) 3445-3475

[5] Stergiopulos, N.; Westerhof, B.E.; Westerhof, N. ***Total arterial intertance as the fourth element of the windkessel model***. Am. J. Physiol. 276 (Heart Circ. Physiol. 45) (1999) H81-H88

[6] Gee, M.W.; Förster, Ch.; Wall, W.A. ***A computational strategy for prestressing patient-specific biomechanical problems under finite deformation***. International Journal for Numerical Methods in Biomedical Engineering 1 (2010) 52-72

[7] Schlosser, T.; Pagonidis, K.; Herborn, C.U.; Hunold, P.; Waltering, K.U.; Lauenstein, T.C.; Barkhausen, J. ***Assessment of left-ventricular parameters using 16-MDCT and new software for endocardial and epicardial border delineation***. AJR Am. J. Roentgenol. 184(3) (2005) 765-773