Lévy-driven Volterra equations in space and time

Carsten Chong[*]

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Abstract

We investigate nonlinear stochastic Volterra equations in space and time that are driven by Lévy bases. Under a Lipschitz condition on the nonlinear term, we give existence and uniqueness criteria in weighted function spaces that depend on integrability properties of the kernel and the characteristics of the Lévy basis. Particular attention is devoted to equations with stationary solutions, or more generally, to equations with infinite memory, that is, where the time domain of integration starts at minus infinity. Here, in contrast to the case where time is positive, the usual integrability conditions on the kernel are no longer sufficient for the existence and uniqueness of solutions, but we have to impose additional size conditions on the kernel and the Lévy characteristics. Furthermore, once the existence of a solution is guaranteed, we analyse its asymptotic stability, that is, whether its moments remain bounded when time goes to infinity. Stability is proved whenever kernel and characteristics are small enough, or the nonlinearity of the equation exhibits a fractional growth of order strictly smaller than one. The results are applied to the stochastic heat equation for illustration.

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[*]Center for Mathematical Sciences, Technische Universität München, Boltzmannstraße 3, 85748 Garching, Germany, e-mail: carsten.chong@tum.de, url: www.statistics.ma.tum.de

1 Introduction

In this paper we investigate stochastic tempo–spatial Volterra equations of the following form:

$$Y(t,x) = Y_0(t,x) + \int_I \int_{\mathbb{R}^d} G(t,x;s,y)\sigma(Y(s,y))\,\Lambda(\mathrm{d}s,\mathrm{d}y), \quad (t,x) \in I \times \mathbb{R}^d. \tag{1.1}$$

Here, Y_0 is a given stochastic process, I is a real time interval, G a deterministic kernel function and σ a deterministic function. Apart from Y_0, the stochasticity of (1.1) comes from its integrator Λ, which is an infinitely divisible independently scattered random measure, or a *Lévy basis* for short.

While the theory of deterministic Volterra equations is very well studied by now (see, for example, the monograph [17]), the literature on Volterra equations with stochastic integrators is considerably smaller. If no space is involved, [26] proves existence and uniqueness for general semimartingale integrators under differentiability assumptions on the kernel G. In the special case of Lévy-driven stochastic delay equations, the asymptotic behaviour of solutions and the existence of stationary solutions are discussed in [28]. As soon as the kernel becomes explosive, existence and uniqueness results have been found for Brownian integrators, see [12, 13, 32]. In the tempo–spatial case, singular kernels are typically encountered in the theory of stochastic PDEs, with two main approaches having become established in this context: on the one hand, there is the functional analytic approach that treats infinite-dimensional stochastic evolution equations as ordinary SDEs with irregular coefficients driven by Hilbert or Banach space-valued Lévy processes; see, for instance, [24] for an excellent account on this subject; or see the recent paper [19] for the treatment of Volterra-type equations within this framework. On the other hand, there is the random field approach that directly considers (1.1) as a scalar-valued equation driven by a multi-parameter Lévy noise. In the Gaussian case, the two approaches have been compared in [15], in the general Lévy case, this problem seems to be open.

Since our treatment of (1.1) will be within the random field approach, we review the existing literature in this field in more detail: based on the seminal work [31], which uses equations of type (1.1) in order to solve certain stochastic PDEs driven by Gaussian white noise, several attempts have been made to generalize Walsh's method to other noise types. One possibility is, for instance, to consider Gaussian noise that is white in time but coloured in space, which is proposed in [14]. Leaving the Gaussian world, [2, 3] study the stochastic heat equation driven by Lévy white noise. However, since both references still employ the L^2-theory of Walsh, they are confronted with the uncomfortable fact that the stochastic heat equation will have no solutions in dimensions greater than 1, cf. [31, pp. 328ff.]. This is due to the bad integrability properties of the heat kernel that plays the role of G in (1.1): it is square-integrable only for $d = 1$.

Therefore, the passage from the L^2- to an L^p-framework, $p \in (0, 2]$, is inevitable. The first paper that discusses Lévy-driven stochastic PDEs in an L^p-framework with $p \in [1, 2]$ is, to our best knowledge, [29]. Under the usual Lipschitz condition on σ, existence and uniqueness for (1.1) are proved when G is the heat kernel and Λ a homogeneous Lévy basis that is either a martingale measure or of locally finite variation. In [21, 22] a specific equation that goes beyond the results of [29] is studied: they take the non-Lipschitz coefficient $\sigma(x) = x^\beta$ with $\beta \neq 1$ and an α-stable spectrally positive Lévy basis for Λ, where $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$, respectively. Finally, [5] treats the Lipschitz case with α-stable Λ where $\alpha \neq 1$. In all articles mentioned so far, the time horizon is $I = \mathbb{R}_+$.

Let us also point out that processes of the form (1.1) are closely related to a class of random fields that are called *ambit processes* and have found applications in physics, finance, biology among other disciplines; see [6, 7, 8, 25] for more details. This class of processes takes the form

$$Y(t,x) = \mu + \int_{A(t,x)} G(t,x;s,y)\sigma(s,y)\,\Lambda(\mathrm{d}s,\mathrm{d}y), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \tag{1.2}$$

where $A(t,x)$, the so-called ambit sets, are certain subsets of $\mathbb{R}\times\mathbb{R}^d$, $\mu\in\mathbb{R}$ is a constant and σ is some given random field. As we can see, the major difference to (1.1) is that the random field σ in (1.2) is given by a function of Y in (1.1). Once a solution to (1.1) is found, it is a special type of ambit processes. For the connection between ambit processes and stochastic PDEs, we refer to [7].

The paper is organized as follows: after we have provided all necessary background information in Section 2, we start to discuss (1.1) in Section 3 for $I=\mathbb{R}_+$. In Theorem 3.1 we establish existence and uniqueness conditions for (1.1) in L^p-spaces for $p\in(0,2]$ under Lipschitz conditions on σ. They generalize the results mentioned in the literature review to kernels G that need not be of convolution type or related to stochastic PDEs, as well as to Lévy bases that are combinations of martingale and finite variation parts, and whose characteristics are potentially inhomogeneous in space and time. The most stringent condition in Theorem 3.1 is that, loosely speaking, Λ must have a moment structure that is at least as nice as its variation structure. This, for instance, a priori excludes any stable Lévy basis. An extension to such cases is provided in Theorem 3.5 if Λ only has finitely many large jumps on finite time intervals. Using localization methods as in [5], we are able to reduce the situation to the framework of Theorem 3.1 and prove existence and uniqueness of solutions this way. Beyond that, if σ has sublinear growth, we prove that they have finite L^p-moments for some $p\in(0,2]$.

In Section 4, we extend the results from Section 3 to the case of infinite memory, which, to our knowledge, has not been considered before in the literature. More precisely, we investigate existence and uniqueness for (1.1) when $I=\mathbb{R}$ (Theorem 4.4), which turns out to be much more involved than the case $I=[0,\infty)$. First, the method of Theorem 3.5 will no longer work, that is, Λ is required to have a good moment structure. Second, and more importantly, an explicit size condition on G, σ and Λ comes into play, which is already a characteristic feature of deterministic Volterra equations, see Example 4.1. Therefore, detailed L^p-estimates for the stochastic integral in (1.1) are required. Furthermore, under certain conditions on Y_0, one can improve the results by using weighted L^p-spaces. If G is a kernel of convolution form and Λ is homogeneous in space and time, the stationarity of the solution is discussed in Theorem 4.8. Section 4 is round off with some results concerning the L^p-continuity of the solution Y and its continuous dependence on Y_0; see Theorem 4.7.

In Section 5 we assume that we have already found a solution to (1.1) that is L^p-bounded up to time T for every $T\in\mathbb{R}_+$. We want to address the question when the solution remains L^p-bounded as $T\to\infty$. An affirmative answer is given under two types of conditions (Theorem 5.2): first, if G, σ and Λ are small enough, a feature that we have already encountered in Theorem 4.4 and that is also similar to the conditions in [28] in the context of stationary solutions to stochastic delay equations; and second, if the function σ is of sublinear growth. Both conditions are intrinsic for Volterra-type equations as a deterministic example shows, see Example 5.1.

In Sections 3 to 5, we illustrate all our results by means of the stochastic heat equation, see Examples 3.4, 3.8, 4.9 and 5.3.

Finally, Section 6 contains several lemmata needed for the proof of the main theorems, which is carried out in Section 7.

2 Preliminaries

We begin with a table of frequently used notations and abbreviations:

$\mathbb{R}_+$	the set $[0,\infty)$ of *positive* real numbers, while *strict positivity* excludes 0;
$\bar{\mathbb{R}}$	the extended real line $\mathbb{R}\cup\{\pm\infty\}$;
$\mathbb{N}$	the set $\{1,2,\ldots\}$ of natural numbers;
I	either $I=\mathbb{R}_+$ or $I=\mathbb{R}$;
I_T	$I\cap(-\infty,T]$ for some $T\in\mathbb{R}\cup\{\infty\}$;
p^*	$p\vee 1$ for $p\in[0,\infty)$;
$\lvert z\rvert_s^r$	$\lvert z\rvert^r\mathbb{1}_{\{\lvert z\rvert>1\}}+\lvert z\rvert^s\mathbb{1}_{\{\lvert z\rvert\leq 1\}}$ for $r,s,z\in\mathbb{R}$;
$\mathbb{B}$	a stochastic basis $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\in I},\mathbb{P})$ satisfying the usual hypotheses of right-continuity and completeness that is large enough to support all random elements of this paper;
$\tilde{\Omega}$	$\tilde{\Omega}:=\Omega\times I\times\mathbb{R}^d$ for some $d\in\mathbb{N}\cup\{0\}$ with the convention $\mathbb{R}^0:=\{1\}$;
$\tilde{\mathcal{P}}$	depending on the context, either the *tempo–spatial predictable* σ*-field* $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$ where $\mathcal{P}$ is the usual predictable σ-field and $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$, or the class of *predictable* (i.e. $\tilde{\mathcal{P}}$-measurable) mappings $\tilde{\Omega}\to\bar{\mathbb{R}}$;
$\tilde{\mathcal{P}}_\mathrm{b}$	the collection of all sets $A\in\tilde{\mathcal{P}}$ such that there exists $k\in\mathbb{N}$ with $A\subseteq\Omega\times(I\cap[-k,k])\times[-k,k]^d$;
$\mathcal{B}_\mathrm{b}$	the collection of all bounded Borel sets in $I\times\mathbb{R}^d$;
$[\![R,S]\!]$	$\{(\omega,t)\in\Omega\times I\colon R(\omega)\leq t\leq S(\omega)\}$ for two $\mathbb{F}$-stopping times R,S, analogously for the other stochastic intervals;
$\lvert\mu\rvert$	the total variation measure of a signed Borel measure μ;
$x+A$	$\{x+a\colon a\in A\}$ for $x\in\mathbb{R}^d$ and $A\subseteq\mathbb{R}^d$;
A^c	$\mathbb{R}^d\setminus A$ for $A\subseteq\mathbb{R}^d$;
$\mathrm{diam}(A)$	$\sup\{\lvert x-y\rvert\colon x,y\in A\}$ for $A\subseteq\mathbb{R}^d$;
$(x,y]$	$\{z\in\mathbb{R}^d\colon x_i<z_i\leq y_i \text{ for all } i=1,\ldots,d\}$ for $x,y\in\mathbb{R}^d$;
L^p	the usual spaces $L^p(\Omega,\mathcal{F},\mathbb{P})$ for $p\in[0,\infty)$ endowed with the topologies induced by $\lVert X\rVert_{L^p}:=\mathbb{E}[\lvert X\rvert^p]^{1/p^*}$ for $p\in(0,\infty)$ and $\lVert X\rVert_{L^0}:=\mathbb{E}[\lvert X\rvert\wedge 1]$ for $p=0$;

In model (1.1), Λ will always be a *Lévy basis* on $I\times\mathbb{R}^d$, that is, a mapping $\Lambda\colon\tilde{\mathcal{P}}_\mathrm{b}\to L^0$ with the following properties:

(1) $\Lambda(\emptyset)=0$ a.s.

(2) For every sequence $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets in $\tilde{\mathcal{P}}_\mathrm{b}$ with $\bigcup_{i=1}^\infty A_i\in\tilde{\mathcal{P}}_\mathrm{b}$ we have

$$\Lambda\left(\bigcup_{i=1}^\infty A_i\right)=\sum_{i=1}^\infty\Lambda(A_i)\quad\text{in } L^0.$$

(3) For all $A\in\tilde{\mathcal{P}}_\mathrm{b}$ with $A\subseteq\Omega\times I_t\times\mathbb{R}^d$ for some $t\in I$, the random variable $\Lambda(A)$ is $\mathcal{F}_t$-measurable.

(4) For all $A\in\tilde{\mathcal{P}}_\mathrm{b}$, $t\in I$ and $\Omega_0\in\mathcal{F}_t$, we have

$$\Lambda\big(A\cap(\Omega_0\times(t,\infty)\times\mathbb{R}^d)\big)=\mathbb{1}_{\Omega_0}\Lambda\big(A\cap(\Omega\times(t,\infty)\times\mathbb{R}^d)\big)\quad\text{a.s.}$$

(5) If $(B_i)_{i\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{B}_\mathrm{b}$, then $(\Lambda(\Omega\times B_i))_{i\in\mathbb{N}}$ is a sequence of independent random variables. Furthermore, if $B\in\mathcal{B}_\mathrm{b}$ satisfies $B\subseteq(t,\infty)\times\mathbb{R}^d$ for some $t\in I$, then $\Lambda(\Omega\times B)$ is independent of $\mathcal{F}_t$.

(6) For all $B\in\mathcal{B}_\mathrm{b}$, $\Lambda(\Omega\times B)$ has an infinitely divisible distribution.

(7) For all $t\in I$ and $k\in\mathbb{N}$ we have $\Lambda(\Omega\times\{t\}\times[-k,k]^d)=0$ a.s.

Lévy bases are originally called infinitely divisible independently scattered random measures in [27]; the short terminology has been introduced in [6]. Conditions (3) and (4) are added to ensure that Lévy bases are "adapted" to the underlying stochastic bases, see e.g. [11]. Just as Lévy processes are semimartingales in the purely temporal case, Lévy bases are random measures, that is, stochastic integrators in space–time. In other words, it is possible to develop an Itô stochastic integration theory for Lévy bases. Let us briefly recall this; all details can be found in [9, Chap. 3] and [10]. Starting with simple integrands $H\in\mathcal{S}$, that is, $H=\sum_{i=1}^r a_i\mathbb{1}_{A_i}$ with $r\in\mathbb{N}$, real numbers a_i and sets $A_i\in\tilde{\mathcal{P}}_\mathrm{b}$, we define the stochastic integral in the canonical way:

$$\int_I\int_{\mathbb{R}^d} H(t,x)\,\Lambda(\mathrm{d}t,\mathrm{d}x) := \sum_{i=1}^r a_i\Lambda(A_i).$$

Given a general predictable function $H\in\tilde{\mathcal{P}}$, we introduce the *Daniell mean*

$$\|H\|_\Lambda := \sup_{S\in\mathcal{S},|S|\leq|H|}\left\|\int_I\int_{\mathbb{R}^d} S(t,x)\,\Lambda(\mathrm{d}t,\mathrm{d}x)\right\|_{L^0},$$

and define the class of *integrable* functions $L^0(\Lambda)$ as the closure of $\mathcal{S}$ under the Daniell mean $\|\cdot\|_\Lambda$. This is to say that $H\in\tilde{\mathcal{P}}$ is integrable w.r.t. Λ if and only if there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{S}$ such that $\|H-S_n\|_\Lambda\to 0$ as $n\to\infty$. Then the *stochastic integral*

$$\int_I\int_{\mathbb{R}^d} H(t,x)\,\Lambda(\mathrm{d}t,\mathrm{d}x) := \lim_{n\to\infty}\int_I\int_{\mathbb{R}^d} S_n(t,x)\,\Lambda(\mathrm{d}t,\mathrm{d}x)$$

as a limit in probability exists and does not depend on the chosen sequence $(S_n)_{n\in\mathbb{N}}$. Moreover, defining

$$H\cdot\Lambda_t := \int_{I_t}\int_{\mathbb{R}^d} H(s,y)\,\Lambda(\mathrm{d}s,\mathrm{d}y),\quad t\in I,$$

the process $H\cdot\Lambda=(H\cdot\Lambda_t)_{t\in I}$ has a modification that is a semimartingale on I. In the case $I=\mathbb{R}$, we mean by this that $X_{-\infty}:=\lim_{t\downarrow-\infty}X_t$ exists as a limit in probability, and for all bijective increasing functions $\phi\colon\mathbb{R}_+\to[-\infty,\infty)$ the process $X^\phi:=(X_{\phi(t)})_{t\in\mathbb{R}_+}$ is a usual semimartingale with respect to $(\mathcal{F}_{\phi(t)})_{t\in\mathbb{R}_+}$. For later reference, we shall mention that its quadratic variation process is defined by $[X]_t:=[X^\phi]_{\phi^{-1}(t)}$ for $t\in\bar{\mathbb{R}}$. Finally, given a function $H\in\tilde{\mathcal{P}}$, one can define a new random measure $H.\Lambda$ by setting

$$\begin{gathered} K\in L^0(H.\Lambda) :\Leftrightarrow KH\in L^0(\Lambda),\\ \int_I\int_{\mathbb{R}^d} K(t,x)\,(H.\Lambda)(\mathrm{d}t,\mathrm{d}x) := \int_I\int_{\mathbb{R}^d} K(t,x)H(t,x)\,\Lambda(\mathrm{d}t,\mathrm{d}x). \end{gathered}\tag{2.1}$$

This indeed defines a random measure $H.\Lambda$ if there exists a sequence $(A_k)_{k\in\mathbb{N}}\subseteq\tilde{\mathcal{P}}$ with $A_k\uparrow\tilde{\Omega}$ such that $\mathbb{1}_{A_k}\in L^0(H.\Lambda)$ for all $k\in\mathbb{N}$.

Every Lévy basis Λ has a canonical decomposition of the following form, see e.g. [11, Thm. 3.2]:

$$\Lambda(\mathrm{d}t,\mathrm{d}x) = B(\mathrm{d}t,\mathrm{d}x) + \Lambda^{\mathrm{c}}(\mathrm{d}t,\mathrm{d}x) + \int_{\mathbb{R}} z\mathbb{1}_{\{|z|\leq 1\}}\,(\mu-\nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) + \int_{\mathbb{R}} z\mathbb{1}_{\{|z|>1\}}\,\mu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z), \tag{2.2}$$

where the ingredients are as follows:

(1) B is a deterministic σ-finite signed Borel measure on $I\times\mathbb{R}^d$.

(2) Λ^{c}, the continuous part of Λ in the usual sense ([10, Thm. 4.13]), is a Gaussian random measure with variance measure C, which means that it is itself a Lévy basis and $\Lambda^{\mathrm{c}}(\Omega\times B)$ has a normal distribution with mean 0 and variance $C(B)$ for every $B\in\mathcal{B}_{\mathrm{b}}$.

(3) μ is a Poisson measure on $I\times\mathbb{R}^d\times\mathbb{R}$ relative to $\mathbb{F}$ with intensity measure ν, see [18, Def. II.1.20].

Moreover, we have a representation

$$\begin{aligned} B(\mathrm{d}t,\mathrm{d}x) &= b(t,x)\,\lambda(\mathrm{d}t,\mathrm{d}x), \qquad\qquad C(\mathrm{d}t,\mathrm{d}x) = c(t,x)\,\lambda(\mathrm{d}t,\mathrm{d}x),\\ \nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) &= \pi(t,x,\mathrm{d}z)\,\lambda(\mathrm{d}t,\mathrm{d}x), \end{aligned} \tag{2.3}$$

with measurable functions $b\colon I\times\mathbb{R}^d\to\mathbb{R}$, $c\colon I\times\mathbb{R}^d\to\mathbb{R}_+$, a transition kernel π from $I\times\mathbb{R}^d$ to $\mathbb{R}$ such that $\pi(t,x,\cdot)$ is a Lévy measure for each (t,x), and a positive σ-finite measure λ on $I\times\mathbb{R}^d$ satisfying $\lambda(\{t\}\times\mathbb{R}^d)=0$ for all $t\in I$.

If π satisfies

$$\int_{|z|>1} |z|\,\pi(t,x,\mathrm{d}z) < \infty, \tag{2.4}$$

$$\text{or}\quad \int_{|z|\leq 1} |z|\,\pi(t,x,\mathrm{d}z) < \infty, \quad \text{respectively,} \tag{2.5}$$

for all $(t,x)\in I\times\mathbb{R}^d$, then it makes sense to introduce the *mean measure* (resp. *drift measure*)

$$B_1(\mathrm{d}t,\mathrm{d}x) := b_1(t,x)\,\lambda(\mathrm{d}t,\mathrm{d}x), \qquad b_1(t,x) := b(t,x) + \int_{\mathbb{R}} z\mathbb{1}_{\{|z|>1\}}\,\pi(t,x,\mathrm{d}z), \tag{2.6}$$

$$B_0(\mathrm{d}t,\mathrm{d}x) := b_0(t,x)\,\lambda(\mathrm{d}t,\mathrm{d}x), \qquad b_0(t,x) := b(t,x) - \int_{\mathbb{R}} z\mathbb{1}_{\{|z|\leq 1\}}\,\pi(t,x,\mathrm{d}z). \tag{2.7}$$

If in the first case we have $b_1(t,x)=0$ for all $(t,x)\in I\times\mathbb{R}^d$, then Λ is called a *martingale Lévy basis*, which will be denoted by $\Lambda\in\mathcal{M}$; if in the second case we have $b_0(t,x)=0$ for all $(t,x)\in I\times\mathbb{R}^d$, then Λ is called a *Lévy basis without drift*. Next, Λ is called *symmetric* if for all $(t,x)\in I\times\mathbb{R}^d$ we have $b(t,x)=0$ and the Lévy measure $\pi(t,x,\cdot)$ is symmetric. Furthermore, Λ is called a *homogeneous* Lévy basis if λ is the Lebesgue measure on $I\times\mathbb{R}^d$ and b, c and π do not depend on $(t,x)\in I\times\mathbb{R}^d$. In this case, a function $\phi\in\tilde{\mathcal{P}}$ is *jointly stationary with* Λ if for arbitrary $n\in\mathbb{N}$, $(h,\eta)\in\mathbb{R}\times\mathbb{R}^d$, points $(t_1,x_1),\dots,(t_n,x_n)\in I\times\mathbb{R}^d$ and pairwise disjoint sets $B_1,\dots,B_n\in\mathcal{B}_{\mathrm{b}}$, we have

$$(\phi(t_i,x_i),\Lambda(B_i)\colon i=1,\dots,n, t_i{+}h\in I) \stackrel{d}{=} (\phi(t_i{+}h,x_i{+}\eta),\Lambda(B_i{+}(h,\eta))\colon i=1,\dots,n, t_i{+}h\in I).$$

Let us come back to Equation (1.1). We first clarify what we mean by a solution Y to (1.1):

Definition 2.1 Equation (1.1) is said to have a *solution* if there exists a predictable process $Y \in \tilde{\mathcal{P}}$ such that for all $(t,x) \in I\times\mathbb{R}^d$ the stochastic integral on the right-hand side of (1.1) is well defined and equation (1.1) holds a.s. We identify two solutions Y_1 and Y_2 if for all $(t,x) \in I \times \mathbb{R}^d$ we have $Y_1(t,x) = Y_2(t,x)$ a.s. □

In order to construct solutions to (1.1), we introduce some spaces of stochastic processes. Let $w\colon I \times \mathbb{R}^d \to \mathbb{R}$ be a *weight function*, that is, a strictly positive measurable function. We denote by $L^{\infty,w}_I$ the Banach space of all measurable functions $f\colon I \times \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\|f\|_{L^{\infty,w}_I} := \sup_{(t,x)\in I\times\mathbb{R}^d} \frac{|f(t,x)|}{w(t,x)} < \infty. \tag{2.8}$$

Similarly, for $p \in (0,\infty)$, $B^{p,w}_I$ is the space of all $\phi \in \tilde{\mathcal{P}}$ with

$$\|\phi\|_{B^{p,w}_I} := \sup_{(t,x)\in I\times\mathbb{R}^d} \left(\frac{\mathbb{E}[|\phi(t,x)|^p]}{w(t,x)}\right)^{1/(p\vee 1)} < \infty. \tag{2.9}$$

If $f \in L^{\infty,w}_{I_T}$ or $\phi \in B^{p,w}_{I_T}$ for all $T \in I$, then we write $f \in L^{\infty,w}_{I,\mathrm{loc}}$ or $\phi \in B^{p,w}_{I,\mathrm{loc}}$, respectively. In the special case $w \equiv 1$, we use the notations L^∞_I, $L^\infty_{I,\mathrm{loc}}$, B^p_I and $B^p_{I,\mathrm{loc}}$.

Before we proceed to the main results of this paper, we recall how stochastic PDEs can be treated in the framework of (1.1). Let $I \subset \mathbb{R}$ be an interval, U an open subset of $\mathbb{R}^d$ with boundary ∂U and P a polynomial in $1+d$ variables. Given some deterministic coefficient σ and some Lévy basis Λ, they give rise to the following formal equation:

$$P(\partial_t, \partial_1, \ldots, \partial_d)Y(t,x) = \sigma(Y(t,x))\dot{\Lambda}(t,x), \quad (t,x) \in I \times U, \tag{2.10}$$

where $\dot{\Lambda} = \partial_t\partial_1 \ldots \partial_d\Lambda$ is the formal derivative of Λ, its noise. Usually, (2.10) is subjected to some boundary conditions on $\partial(I \times U)$. Of course, the derivative of Λ is not defined except in trivial cases, so a strong solution to (2.10) will not exist. Going back to [31] is the idea of constructing a so-called *mild solution* to (2.10). For this method to work, one has to assume that the operator P possesses a Green's function on $I \times U$. Then a mild solution to (2.10) is nothing but a solution in the sense of Definition 2.1 to (1.1), where G is the Green's function and Y_0 a term that only depends on the boundary conditions posed on $\partial(I \times U)$.

Remark 2.2 While the notion of a solution as in Definition 2.1 is very common in the theory of stochastic PDEs, it is different to the standard notion of solutions to (ordinary) SDEs: let $I = \mathbb{R}_+$ and $d = 0$, that is, space contains only one point, and consider $G(t,1;s,1) = g(s)\mathbb{1}_{\{s\leq t\}}$ with some smooth function g. Then Equation (1.1) is equivalent to the SDE

$$\mathrm{d}Y(t) = g(t)\sigma(Y(t-))\,\Lambda(\mathrm{d}t), \quad t \geq 0, \quad Y(0) = Y_0, \tag{2.11}$$

where Λ is a semimartingale with independent increments. Ordinary SDE theory tells us that Equation (2.11) has a càdlàg solution Y that is unique up to indistinguishability. In contrast, a solution in the sense of Definition 2.1 would be the predictable version $Y(\cdot-)$, and uniqueness is only understood up to modifications. The reason why we have chosen this slightly different notion of a solution is that we are particularly interested in the case where G in Equation (1.1) has singularities. In such cases, Equation (1.1) permits no càdlàg solutions. □

3 Existence and uniqueness results on $I = \mathbb{R}_+$

The goal of this section is to provide sufficient conditions under which there exists a (unique) solution to (1.1) on the interval $I = \mathbb{R}_+$. It is clear that everything in this section holds analogously if we replace $I = [0, \infty)$ by $I = [a, \infty)$ with some $a \in \mathbb{R}$. As mentioned in the Introduction, the forthcoming theorem generalizes the results of [29] to potentially inhomogeneous Lévy bases and kernels different from the heat kernel. It holds under the following list of assumptions:

Assumption A Let $p \in (0, 2]$ and the predictable characteristics of Λ be given by (2.3). We impose the following conditions:

(1) $Y_0 \in B^p_{[0,\infty),\mathrm{loc}}$.

(2) There exists $C_{\sigma,1} \in \mathbb{R}_+$ such that $|\sigma(x) - \sigma(y)| \leq C_{\sigma,1}|x - y|$ for all $x, y \in \mathbb{R}$.

(3) $G\colon (\mathbb{R}_+ \times \mathbb{R}^d)^2 \to \mathbb{R}$ is a measurable function such that $G(t, \cdot; s, \cdot) \equiv 0$ whenever $s > t$.

(4) If $p < 2$, then Λ has no Gaussian part: $c(t, x) = 0$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. If $p = 2$, then we assume for all $T \in \mathbb{R}_+$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |G(t, x; s, y)|^2 c(s, y)\, \lambda(\mathrm{d}s, \mathrm{d}y) < \infty. \tag{3.1}$$

(5) For all $T \in \mathbb{R}_+$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} |G(t, x; s, y)z|^p\, \nu(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) < \infty. \tag{3.2}$$

(6) Recall the definition of b_1 and b_0 from (2.6) and (2.7). If $p \geq 1$, assume that ν satisfies (2.4) and that for all $T \in \mathbb{R}$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |G(t, x; s, y)b_1(s, y)|\, \lambda(\mathrm{d}s, \mathrm{d}y) < \infty; \tag{3.3}$$

if $p < 1$, assume that ν satisfies (2.5) and that $b_0(t, x) = 0$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

(7) Define for $(t, x), (s, y) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$G^A(t, x; s, y) := |G(t, x; s, y)|^p \left(\int_{\mathbb{R}} |z|^p\, \pi(s, y, \mathrm{d}z) + c(s, y) \right) + |G(t, x; s, y)b_1(s, y)| \mathbb{1}_{\{p\geq1\}},$$

and assume that for every $T \in \mathbb{R}_+$ and $\epsilon > 0$ there exists $k \in \mathbb{N}$ together with a subdivision $\mathcal{T}\colon 0 = t_0 < t_1 < \ldots < t_{k+1} = T$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \sup_{i=0,\ldots,k} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} G^A(t, x; s, y)\, \lambda(\mathrm{d}s, \mathrm{d}y) < \epsilon. \tag{3.4}$$

□

Theorem 3.1. *Let Assumption A be valid. Then Equation* (1.1) *has a unique solution in* $B^p_{[0,\infty),\mathrm{loc}}$.

The conditions of Assumption A simplify a lot if G and Λ are *quasi-stationary*, that is,

$$|G(t,x;s,y)| \leq g(t-s,x-y), \quad \lambda(\mathrm{d}t,\mathrm{d}x) = \mathrm{d}(t,x), \quad b,c \in L^{\infty}_{[0,\infty),\mathrm{loc}}, \quad \pi(t,x,\mathrm{d}z) \leq \pi_0(\mathrm{d}z), \tag{3.5}$$

where $g\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a positive measurable function.

Corollary 3.2. *Suppose that (3.5) holds and that Assumption A(1), (2) and (3) are given. Furthermore, assume that we have for some $p \in (0,2]$*

$$b_0 \equiv 0 \text{ if } p<1, \quad c \equiv 0 \text{ if } p<2, \quad \int_{\mathbb{R}} |z|^p \, \pi_0(\mathrm{d}z) < \infty, \tag{3.6}$$

and for all $T \in \mathbb{R}_+$

$$\int_0^T \int_{\mathbb{R}^d} g^p(t,x)\,\mathrm{d}(t,x) < \infty, \quad \text{and} \quad \int_0^T \int_{\mathbb{R}^d} g(t,x)\,\mathrm{d}(t,x) < \infty \text{ if } p \geq 1 \text{ and } \Lambda \notin \mathcal{M}. \tag{3.7}$$

Then all conditions of Assumption A are satisfied and Theorem 3.1 holds.

Remark 3.3 (1) Assumption A and Theorem 3.1 are special cases of Assumption C and Theorem 4.4, respectively, which we will discuss in Section 4. In fact, Theorem 3.1 follows if we take $I = [0,\infty)$ and $w \equiv 1$ in Theorem 4.4.

(2) Conditions (4), (5) and (6) in Assumption A are conditions on the joint size of G and the three characteristics of Λ, respectively. Although they are valid for many interesting examples, especially condition (5) might be too restrictive: it is violated as soon as the moment structure of Λ is worse than its variation structure, which, for instance, occurs if Λ is an α-stable Lévy basis with $\alpha \in (0,2)$; see also the last condition in (3.6). Theorem 3.5 below provides, under some additional hypotheses, an extension of Theorem 3.1 that includes such cases.

(3) The following observation follows from Corollary 3.2: in the quasi-stationary case (3.5), condition (7) in Assumption A is already implied by conditions (4), (5) and (6). In other words, condition (7) is a smallness assumption on the non-stationary part of G and the characteristics of Λ.

(4) As we shall see in the more general Theorem 4.4 in Section 4, it actually suffices that the left-hand side of (3.4) can be made smaller than some fixed constant that does not depend on $\mathcal{T}$. Due to the previous remark, however, this fact is not that important in the case $I = [0,\infty)$ (in the case $I = \mathbb{R}$, it is!).

□

Next, we apply Theorem 3.1 and its corollary to the stochastic heat equation. In fact, this equation will serve as our toy example and will be carried through the whole paper and revisited after each main theorem: see the Examples 3.8, 4.9 and 5.3.

Example 3.4 We consider the stochastic heat equation on $\mathbb{R}_+ \times \mathbb{R}^d$, that is, (2.10) with P given by $P(t,x) = t - \sum_{i=1}^d x_i^2 + a$, $a \in \mathbb{R}$, and some Lipschitz coefficient σ. The Green's function is the heat kernel

$$G_a(t,x;s,y) = g_a(t-s,x-y) = \frac{\exp\left(-\frac{|x-y|^2}{4(t-s)} - a(t-s)\right)}{(4\pi(t-s))^{d/2}} \mathbb{1}_{\{s<t\}}. \tag{3.8}$$

We pose an initial condition at time $t = 0$, that is, we require $Y(0, x) = y_0(x)$, where $y_0 \colon \mathbb{R}^d \to \mathbb{R}$ is some bounded continuous and, for simplicity, deterministic function. Then the correct term for Y_0 in (1.1) is

$$Y_0(t, x) := \int_{\mathbb{R}^d} g_a(t, x - y) y_0(y) \, \mathrm{d}y, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{3.9}$$

The stochastic heat equation on $I = \mathbb{R}_+$ then reads as

$$Y(t, x) = Y_0(t, x) + \int_0^t \int_{\mathbb{R}^d} g_a(t - s, x - y) \sigma(Y(s, y)) \, \Lambda(\mathrm{d}s, \mathrm{d}y), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{3.10}$$

Let us determine sufficient conditions for existence and uniqueness of solutions to (3.10): assuming that the characteristics of Λ satisfy (3.5), we have to check the conditions of Corollary 3.2: (1) and (2) of Assumption A are clear. Since

$$\int_0^T \int_{\mathbb{R}^d} g_a^p(s, y) \, \mathrm{d}(s, y) < \infty \text{ for all } T \in \mathbb{R}_+ \iff p < 1 + 2/d, \tag{3.11}$$

we obtain existence and uniqueness for the stochastic heat equation (3.10) on $I = \mathbb{R}_+$ if (3.6) holds with some $0 < p < 1 + 2/d$. In particular, this excludes the choice $p = 2$ and therefore the possibility of taking a non-zero Gaussian part whenever $d \geq 2$. □

As pointed out in Remark 3.3(2), Theorem 3.1 excludes any Lévy basis that has the property that for every $p \in (0, 2]$

$$\lambda\left(\left\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \colon \int_{\mathbb{R}} |z|^p \, \pi(t, x, \mathrm{d}z) = \infty\right\}\right) > 0. \tag{3.12}$$

We now discuss a possibility to circumvent this.

Assumption B Consider the following hypotheses:

(1) Assumption A(2) and (3) are valid.

(2) There exists some $q \in (0, 2]$ such that for all $n \in \mathbb{N}$ conditions (4)–(7) of Assumption A are valid when p is replaced by q and ν is replaced by

$$\nu^n(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z) := \mathbb{1}_{\{|z| \leq n\}} \, \nu(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z).$$

Of course, b_1 is changed accordingly.

(3) For all $T \in \mathbb{R}_+$ we have $\nu([0, T] \times \mathbb{R}^d \times [-1, 1]^{\mathrm{c}}) < \infty$.

(4) $Y_0 \in \tilde{\mathcal{P}}$ and there are stopping times $(T_n)_{n \in \mathbb{N}}$ with $T_n \uparrow \infty$ a.s. and $Y_0 \mathbb{1}_{[\![0, T_n]\!]} \in B^q_{[0,\infty),\mathrm{loc}}$ for all $n \in \mathbb{N}$.

(5) There exist $\gamma \in (0, 1)$ and $C_{\sigma,2} \in \mathbb{R}_+$ such that $|\sigma(x)| \leq |\sigma(0)| + C_{\sigma,2}|x|^\gamma$ for all $x \in \mathbb{R}$.

(6) There exists $p \in (0, 2)$ satisfying $p < q$ and $q\gamma \leq p$ such that $Y_0 \in B^p_{[0,\infty),\mathrm{loc}}$.

(7) For all $T \in \mathbb{R}_+$

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}} |G(t, x; s, y) z|_q^p \, \nu(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) < \infty.$$

(8) If $p \geq 1$, (3.3) holds.

(9) If $p < 1$, there exist exponents $\alpha \in (-\infty, 2], \beta \in [0, \infty)$ with the following properties:

(9a) For all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $A \in [1,\infty)$ and $a \in (0,1]$ we have

$$\left| b(t,x) - \int_{\mathbb{R}} z \mathbb{1}_{\{|z| \in (a,1]\}}\, \pi(t,x,\mathrm{d}z) \right| \leq F_0(t,x) a^{1-\alpha}, \tag{3.13}$$

$$\left| b(t,x) + \int_{\mathbb{R}} z \mathbb{1}_{\{|z| \in (1,A]\}}\, \pi(t,x,\mathrm{d}z) \right| \leq F_1(t,x) A^{1-\beta} \tag{3.14}$$

for some positive measurable functions $F_0, F_1 \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$.

(9b) For all $T \in \mathbb{R}_+$ we have

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} (F_0(s,y) \vee F_1(s,y)) |G(t,x;s,y)|_\beta^\alpha \, \lambda(\mathrm{d}s, \mathrm{d}y) < \infty. \tag{3.15}$$

(9c) $(\alpha \vee \beta)\gamma \leq p$.

(10) The partition property (3.4) holds with G^B instead of G^A, where for $(t,x),(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$\begin{aligned} G^B(t,x;s,y) &:= |G(t,x;s,y)|^2 c(s,y) + \int_{\mathbb{R}} |G(t,x;s,y)z|_q^p \, \pi(s,y,\mathrm{d}z) \\ &\quad + \begin{cases} |G(t,x;s,y) b_1(s,y)|, & p \geq 1, \\ (F_0(s,y) \vee F_1(s,y)) |G(t,x;s,y)|_\beta^\alpha, & p < 1 \end{cases}, \end{aligned} \tag{3.16}$$

□

Theorem 3.5. *(1) Suppose that conditions (1)–(4) of Assumption B are true. Then there exists a unique solution to Equation* (1.1) *among those* $Y \in \tilde{\mathcal{P}}$ *for which there exist stopping times* $(T_n)_{n\in\mathbb{N}}$ *with* $T_n \uparrow \infty$ *a.s. such that* $Y \mathbb{1}_{[\![0,T_n]\!]} \in B^q_{[0,\infty),\mathrm{loc}}$ *for all* $n \in \mathbb{N}$.

(2) If in addition also conditions (5)–(10) of Assumption B are valid, then the solution Y *from part (1) belongs to* $B^p_{[0,\infty),\mathrm{loc}}$.

Remark 3.6 (1) Part (1) of this theorem relies on some stopping time techniques that have already been used in [5] to construct solutions to (1.1) driven by α-stable noise with $\alpha \neq 1$. Theorem 3.5 extends this result to more general Lévy bases and, more importantly, provides in part (2) conditions under which this solution belongs to the space $B^p_{[0,\infty),\mathrm{loc}}$.

(2) The smaller the growth index γ of σ is, the smaller can p be chosen and therefore, the weaker the conditions (6)–(9) of Assumption B are. For α-stable Lévy bases with $\alpha \in (0,2)$, any $\gamma \in (0,1)$ and $p < q$ will suffice.

(3) If $p < 1$, condition (9) of Assumption B looks quite technical but is actually only a very mild assumption. In the next Corollary 3.7 where we treat the quasi-stationary case, it is already implied by condition (6) below.

(4) Remark 3.3(3) holds analogously: see the next corollary.

(5) For the second condition of Assumption B, if $p \geq 1$, one has to check Assumption A(6) for different replacements of b_1 as n varies, which are usually non-zero even when $\Lambda \in \mathcal{M}$.

(6) The most stringent condition in Assumption B is (3): it requires the intensity of large jumps of Λ to decay quickly enough in space. For example, it is typically *not* enough to have $\pi(t,x,\mathrm{d}z) = \pi_0(\mathrm{d}z)$. See Corollary 3.7 and Example 3.8 for more details.

□

Again we reformulate Assumption B in the quasi-stationary case:

Corollary 3.7. *Assume that G and Λ satisfy* (3.5)*, but with the stronger condition*

$$\pi(t,x,\mathrm{d}z) \leq \pi_1(t,x)\,\pi_0(\mathrm{d}z), \quad \pi_1 \in L^\infty_{[0,\infty),\mathrm{loc}}, \quad \int_0^T \int_{\mathbb{R}^d} \pi_1(t,x)\,\mathrm{d}(t,x) < \infty \tag{3.17}$$

for all $T \in \mathbb{R}_+$. Then part (1) of Theorem 3.5 holds if:

(1) Assumption B(1) and (4) are valid.

(2) For some $q \in (0,2]$ conditions (3.6) *and* (3.7) *hold with p replaced by q and π_0 replaced by $\mathbb{1}_{\{|z|\leq 1\}}\,\pi_0(\mathrm{d}z)$.*

(3) If $q \geq 1$, either $\int_0^T \int_{\mathbb{R}^d} g(t,x)\,\mathrm{d}(t,x) < \infty$ for all $T \in \mathbb{R}_+$, or Λ is symmetric.

Part (2) of the same theorem holds if additionally:

(4) σ satisfies the growth condition of Assumption B(5) with $\gamma \in (0,1)$.

(5) There exists $p \in (0,2)$ with $p < q$ and $q\gamma \leq p$ such that $Y_0 \in B^p_{[0,\infty),\mathrm{loc}}$.

(6) $\int_{\mathbb{R}} |z|^p_q\,\pi_0(\mathrm{d}z) < \infty$ and $\int_0^T \int_{\mathbb{R}^d} |g(t,x)|^q_p\,\mathrm{d}(t,x) < \infty$ for all $T \in \mathbb{R}_+$.

For illustration purposes we go through the conditions of Theorem 3.5 and Corollary 3.7 for the stochastic heat equation.

Example 3.8 (Continuation of Example 3.4) Our aim is to extend the findings of Example 3.4 when Λ has bad moment properties in the sense of (3.12). For simplicity we assume that the characteristics of Λ are within the setting of Corollary 3.7, that is, they satisfy (3.5) and (3.17). As before, σ is a Lipschitz continuous function and the equation of interest is (3.10) with Y_0 given by (3.9). In view of (3.11), it is immediate to see that Corollary 3.7 yields the following conditions for part (1) of Theorem 3.5 to hold:

$$\int_{[-1,1]} |z|^q\,\pi_0(\mathrm{d}z) < \infty \text{ for some } 0 < q < 1 + 2/d, \quad c \equiv 0 \text{ if } d \geq 2, \quad b_0 \equiv 0 \text{ if } q < 1. \tag{3.18}$$

Furthermore, if σ has growth of order $\gamma \in (0,1)$ and

$$\int_{|z|>1} |z|^p\,\pi_0(\mathrm{d}z) < \infty \text{ for some } p < 1 + 2/d \text{ with } p < q \text{ and } q\gamma \leq p, \tag{3.19}$$

then the solution Y belongs to $B^p_{[0,\infty),\mathrm{loc}}$. Indeed, this claim follows from Corollary 3.7 and the fact that for all $p, q \in (0,\infty)$ we have

$$\int_0^T \int_{\mathbb{R}^d} |g_a(t,x)|^q_p\,\mathrm{d}(t,x) < \infty \tag{3.20}$$

for all $T \in \mathbb{R}_+$ if and only if $q \in (0, 1 + 2/d)$ (p does not matter). From (3.19) we also see the following: the smaller the growth order γ of σ is, the fewer moments π_0 is required to have.

At last, we give some further explanation for the integrability condition on π_1 given in (3.17). We assume that $\pi(t, x, \mathrm{d}z) = \pi_1(t, x)\pi_0(\mathrm{d}z)$ with a Lévy measure π_0 of unbounded support. Then it is obvious to see that we cannot take $\pi_1 \equiv 1$, that is, a homogeneous noise Λ, but have to choose π_1 with sufficient decay in space. For instance, if there exists some exponent $r \in \mathbb{R}$ such that for all $T \in \mathbb{R}_+$ we have $\pi_1(t, x) \leq C_T|x|^{-r}$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and some constant $C_T \in \mathbb{R}_+$, then we need for (3.17) that $r > d$, a condition that is stronger in higher dimensions. Finally, (3.17) is always met if π_1 is bounded and vanishes outside a compact in $\mathbb{R}^d$, which corresponds to a noise that only acts locally. In particular, this assumption is very natural if we consider the stochastic heat equation on bounded domains as, for instance, in [3, 5, 31]. □

Remark 3.9 Theorem 3.1 and 3.5 can actually be extended to even more general random measures than Lévy bases. Let us consider a random measure M on $\mathbb{R}_+ \times \mathbb{R}^d$ that is defined by

$$
\begin{aligned}
M(\mathrm{d}t, \mathrm{d}x) &= b(t, x)\,\mathrm{d}(t, x) + \rho(t, x)\,W(\mathrm{d}t, \mathrm{d}x) + \int_E \underline{\delta}(t, x, z)\,(\mathfrak{p} - \mathfrak{q})(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z) \\
&\quad + \int_E \overline{\delta}(t, x, z)\,\mathfrak{p}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z),
\end{aligned}
\tag{3.21}
$$

where $(E, \mathcal{E})$ is an arbitrary Polish space equipped with its Borel σ-field, $b, \rho \in \tilde{\mathcal{P}}$, $\delta = \underline{\delta} + \overline{\delta} = \delta \mathbb{1}_{\{|\delta| \leq 1\}} + \delta \mathbb{1}_{\{|\delta| > 1\}}$ is an $\tilde{\mathcal{P}} \otimes \mathcal{E}$-measurable function, W is a Gaussian random measure with the Lebesgue measure on $I \times \mathbb{R}^d$ as variance measure, $\mathfrak{p}$ is a homogeneous Poisson random measure on $I \times \mathbb{R}^d \times E$ relative to the filtration $\mathbb{F}$ with intensity measure $\mathfrak{q}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z) = \mathrm{d}t\,\mathrm{d}x\,\lambda(\mathrm{d}z)$ where λ is a σ-finite infinite atomless measure on $(E, \mathcal{E})$. Moreover, all ingredients are such that $M(\Omega \times (I \cap (-k, k]) \times (-k, k]^d)$ is well defined for all $k \in \mathbb{N}$. Such a measure M can be viewed as the space–time analogue of Itô semimartingales. We impose the following conditions on the coefficients (these are classical in the semimartingale setting, cf. [1, Chap. 6]): there exist positive constants $(\beta_N)_{N \in \mathbb{N}}$, a sequence of stopping times $(\tau_N)_{N \in \mathbb{N}}$ increasing to infinity a.s., and deterministic positive measurable functions $j_N(z)$ such that for all $(\omega, t, x) \in \tilde{\mathcal{P}}$ with $t \leq \tau_N(\omega)$ we have

(1) $|b(\omega, t, x)|, |c(\omega, t, x)| \leq \beta_N$,

(2) $|\delta(\omega, t, x, z)|^p \leq j_N(z)$ and $\int_E j_N(z)\,\lambda(\mathrm{d}z) < \infty$.

Then with obvious changes to Assumptions A and B, respectively, Theorems 3.1 and 3.5 also apply to Equation (1.1) when driven by the random measure M as given in (3.21). □

4 Existence and uniqueness results on $I = \mathbb{R}$

While Section 3 deals with Equation (1.1) on $I = [0, \infty)$, this section investigates the case $I = \mathbb{R}$. In particular, we obtain conditions for Equation (1.1) to possess a stationary solution. In order to demonstrate the difference between the two cases $I = [0, \infty)$ and $I = \mathbb{R}$, we analyse the following deterministic example.

Example 4.1 Let $\lambda \in \mathbb{R}$ and consider the following equation:

$$
v(t) = 1 + \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} v(s)\,\mathrm{d}s, \quad t \in \mathbb{R}. \tag{4.1}
$$

By standard computation one can show the following: if $\lambda \leq 0$, Equation (4.1) has no solution; if $\lambda > 0$ and $\lambda \neq 1$, then the solutions to (4.1) are

$$v(t) = c\mathrm{e}^{(1-\lambda)t} + \frac{\lambda}{\lambda - 1}, \quad c \in \mathbb{R};$$

if $\lambda = 1$, the solutions are

$$v(t) = t + c, \quad c \in \mathbb{R}.$$

We draw some important conclusions, also regarding possibilities and limitations for Equation (1.1) with $I = \mathbb{R}$:

(1) The reason why (4.1) possesses no solution for $\lambda \leq 0$ is simply the non-integrability of the kernel:

$$\int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}s = \int_{0}^{\infty} \mathrm{e}^{-\lambda s}\,\mathrm{d}s = \infty. \tag{4.2}$$

(2) If Equation (4.1) has a solution, it has uncountably many. If $\lambda \in (1, \infty)$, only one solution is in $L^{\infty}_{\mathbb{R},\mathrm{loc}}$, namely if $c = 0$. The reason for this is that the integral of the kernel given in (4.2) is smaller than 1. In this case the uniqueness of solutions in $L^{\infty}_{\mathbb{R},\mathrm{loc}}$ follows from Lemma 6.4(2). Thus, in the stochastic case of (1.1), we can expect existence and uniqueness of solutions in $B^{p}_{\mathbb{R},\mathrm{loc}}$ only if the quantities (3.1), (3.2) and (3.3) are *small enough* (not only finite) in a sense to be made precise.

(3) In contrast to the case $\lambda \in (1, \infty)$, we have for $\lambda \in (0, 1)$ that all solutions belong to $L^{\infty}_{\mathbb{R},\mathrm{loc}}$ and for $\lambda = 1$ that no solution belongs to $L^{\infty}_{\mathbb{R},\mathrm{loc}}$. Furthermore, in these cases, all solutions start with strictly negative values at $-\infty$. This is somewhat surprising given the fact that all ingredients of (4.1) (the exponential kernel, the constant driving force and the Lebesgue measure as integrator) are positive. This phenomenon is typical when the integral of the kernel in (4.2) becomes greater or equal to one: the kernel is too large to allow for a positive solution. Finally, none of the solutions can be found via a Picard iteration scheme (since the Picard iterates are always positive when the input factors are). Thus, if the kernel in (1.1) is too large in a certain sense, we will not be able to construct a solution in general.

(4) Under certain circumstances, however, one can make the kernel size smaller (which then implies the existence and uniqueness of solutions) by considering Volterra equations in weighted spaces. For instance, consider the following modification of Equation (4.1):

$$v(t) = \mathrm{e}^{\alpha t} + \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} v(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.3}$$

with $\alpha, \lambda \in \mathbb{R}$ satisfying $\lambda > 0$ and $\alpha + \lambda > 1$. The family of solutions in this case is

$$v(t) = \frac{\alpha + \lambda}{\alpha + \lambda - 1}\mathrm{e}^{\alpha t} + c\mathrm{e}^{(1-\lambda)t}, \quad t \in \mathbb{R}, \quad c \in \mathbb{R}. \tag{4.4}$$

First note that positive solutions do exist, namely, when $c \geq 0$. Furthermore, with $w(t) := \mathrm{e}^{\alpha t}$, we have

$$\int_{-\infty}^{t} w^{-1}(t)\mathrm{e}^{-\lambda(t-s)} w(s)\,\mathrm{d}s = \int_{-\infty}^{t} \mathrm{e}^{-(\alpha+\lambda)(t-s)}\,\mathrm{d}s = (\alpha + \lambda)^{-1} < 1.$$

That is, by Lemma 6.4(2), there exists a unique solution to (4.3) in $L^{\infty,w}_{\mathbb{R},\mathrm{loc}}$, which corresponds to the case $c = 0$ in (4.4). Roughly speaking, this device was possible because the force function $e^{\alpha t}$ is small enough at $-\infty$ (the constant function in (4.1) was obviously *not* small enough). This motivates us to work in the weighted spaces $B^{p,w}_{\mathbb{R},\mathrm{loc}}$ for Equation (1.1) on $I = \mathbb{R}$.

□

We are about to formulate a set of conditions that generalizes those of Assumption A and leads to the existence and uniqueness of solutions for Equation (1.1) on arbitrary intervals, in particular on $I = \mathbb{R}$. In order to do so, we need the following definition.

Definition 4.2 Let $p \in (0, \infty)$.

(1) For $p \in (0, 1)$ we set $C^{\mathrm{BDG}}_p := 1$.

(2) For $p \in [1, \infty)$ we denote by C^{BDG}_p the smallest positive number such that for all local martingales $(M_t)_{t\in\mathbb{R}_+}$ w.r.t. $\mathbb{F}$ we have

$$\sup_{t\geq 0} \|M_t\|_{L^p} \leq C^{\mathrm{BDG}}_p \|[M]^{1/2}_\infty\|_{L^p}. \tag{4.5}$$

□

Remark 4.3 We make some comments on Definition 4.2:

(1) The Burkholder-Davis-Gundy inequality ensures the finiteness of C^{BDG}_p for $p \in [1, \infty)$. Of course, inequality (4.5) becomes false in general for $p < 1$; the definition above for $p \in (0, 1)$ is merely for notational convenience. Moreover, the inequality for $p \in [1, \infty)$ is usually stated with the supremum inside the L^p-norm on the left-hand side of (4.5). However, this may enlarge the optimal constant C^{BDG}_p.

(2) The choice $I = \mathbb{R}_+$ is unimportant: a straightforward time change argument shows that C^{BDG}_p remains optimal for any other non-trivial interval $I \subseteq \mathbb{R}$.

(3) For $p \in [1, \infty)$, the actual value of C^{BDG}_p is not known in general. We are only interested in the case $p \in [1, 2]$, for which the following results are available: $C^{\mathrm{BDG}}_p \leq \sqrt{8p}$ for $p \in (1, 2)$, $C^{\mathrm{BDG}}_2 = 1$ (cf. [9, Eq. (4.2.3)]) and $C^{\mathrm{BDG}}_1 = 2$ (cf. [23, Thm. 8.7]).

□

Assumption C Let $0 < p \leq 2$, $I \subseteq \mathbb{R}$ be an interval and $w\colon I \times \mathbb{R}^d \to \mathbb{R}$ be a weight function. We impose the following conditions:

(1) $Y_0 \in B^{p,w}_{I,\mathrm{loc}}$.

(2) There exists $C_{\sigma,1} \in \mathbb{R}_+$ such that $|\sigma(x) - \sigma(y)| \leq C_{\sigma,1}|x - y|$ for all $x, y \in \mathbb{R}$.

(3) $G\colon (I \times \mathbb{R}^d)^2 \to \mathbb{R}$ is a measurable function such that $G(t, \cdot; s, \cdot) \equiv 0$ whenever $s > t$.

(4) If $p < 2$, then Λ has no Gaussian part: $c(t, x) = 0$ for all $(t, x) \in I \times \mathbb{R}^d$. If $p = 2$, then we assume for all $T \in I$

$$\sup_{(t,x)\in I_T\times\mathbb{R}^d} \int_I \int_{\mathbb{R}^d} w^{-1}(t, x)|G(t, x; s, y)|^2 c(s, y)(w(s, y) \vee \sigma(0))\, \lambda(\mathrm{d}s, \mathrm{d}y) < \infty. \tag{4.6}$$

(5) For all $T \in I$

$$\sup_{(t,x)\in I_T\times\mathbb{R}^d} \int_I \int_{\mathbb{R}^d} \int_{\mathbb{R}} w^{-1}(t,x)|G(t,x;s,y)z|^p (w(s,y)\vee\sigma(0))\,\nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) < \infty. \tag{4.7}$$

(6) Recall the definition of b_1 and b_0 from (2.6) and (2.7). If $p \geq 1$, assume that ν satisfies (2.4) and that for all $T \in I$

$$\sup_{(t,x)\in I_T\times\mathbb{R}^d} \int_I \int_{\mathbb{R}^d} w^{-1}(t,x)|G(t,x;s,y)b_1(s,y)|(w(s,y)\vee\sigma(0))\,\lambda(\mathrm{d}s,\mathrm{d}y) < \infty; \tag{4.8}$$

if $p < 1$, assume that ν satisfies (2.5) and that $b_0(t,x) = 0$ for all $(t,x) \in I \times \mathbb{R}^d$.

(7) If $p \geq 1$ and $\Lambda \notin \mathcal{M}$, assume that (6) also holds with w replaced by the constant function 1.

(8) Define for $(t,x),(s,y) \in I \times \mathbb{R}^d$

$$\begin{aligned}
\bar{G}^{C,1}(t,x;s,y) &:= (C_{\sigma,1}C_p^{\mathrm{BDG}})^p |G(t,x;s,y)|^p \left(\int_{\mathbb{R}} |z|^p\,\pi(s,y,\mathrm{d}z) + c(s,y)\right),\\
\bar{G}^{C,2}(t,x;s,y) &:= C_{\sigma,1}^p \left(\int_I \int_{\mathbb{R}^d} |G(t,x;s,y)b_1(s,y)|\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{p-1} |G(t,x;s,y)b_1(s,y)|\mathbb{1}_{\{p\geq 1\}},\\
G^{C,1}(t,x;s,y) &:= w^{-1}(t,x)\bar{G}^{C,1}(t,x;s,y)w(s,y),\\
G^{C,2}(t,x;s,y) &:= w^{-1}(t,x)\bar{G}^{C,2}(t,x;s,y)w(s,y),
\end{aligned} \tag{4.9}$$

and assume that for every $T \in I$ there exists $k \in \mathbb{N}$ together with a subdivision $\mathcal{T}\colon \inf I = t_0 < t_1 < \ldots < t_{k+1} = T$ such that

$$\sup_{(t,x)\in I_T\times\mathbb{R}^d} \sup_{i=0,\ldots,k} \sum_{l=1}^{2} \left(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} G^{C,l}(t,x;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/(p\vee 1)} < 1. \tag{4.10}$$

□

Theorem 4.4. *Under Assumption C there exists a unique solution to Equation* (1.1) *in* $B^{p,w}_{I,\mathrm{loc}}$.

In the quasi-stationary case, Assumption C simplifies a lot:

Corollary 4.5. *Let* $I = \mathbb{R}$, $w \equiv 1$ *and Assumption C(1), (2) and (3) be valid. We assume that* G *and* Λ *satisfy*

$$|G(t,x;s,y)| \leq g(t-s,x-y),\quad \lambda(\mathrm{d}t,\mathrm{d}x) = \mathrm{d}(t,x),\quad b,c \in L^\infty_{\mathbb{R}},\quad \pi(t,x,\mathrm{d}z) \leq \pi_0(\mathrm{d}z) \tag{4.11}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ *and some positive measurable* $g\colon \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}$. *Furthermore, we suppose that for some* $p \in (0,2]$ *we have*

$$b_0 \equiv 0 \text{ if } p < 1,\quad c \equiv 0 \text{ if } p < 2,\quad \zeta_p := \int_{\mathbb{R}} |z|^p\,\pi_0(\mathrm{d}z) < \infty, \tag{4.12}$$

and that the following size condition is fulfilled: if $p \in (0,1)$, *then*

$$C_{\sigma,1}^p \zeta_p \int_0^\infty \int_{\mathbb{R}^d} g^p(t,x)\,\mathrm{d}(t,x) < 1, \tag{4.13}$$

and if $p \in [1,2]$, *then*

$$C_{\sigma,1}\left[C_p^{\mathrm{BDG}} \left((\zeta_p + \|c\|_{L^\infty_{\mathbb{R}}}) \int_0^\infty \int_{\mathbb{R}^d} g^p(t,x)\,\mathrm{d}(t,x)\right)^{1/p} + \|b_1\|_{L^\infty_{\mathbb{R}}} \int_0^\infty \int_{\mathbb{R}^d} g(t,x)\,\mathrm{d}(t,x)\right] < 1. \tag{4.14}$$

Then all conditions of Assumption C are satisfied and Theorem 4.4 holds.

We write down some important observations:

Remark 4.6 (1) There is a fundamental difference between condition (7) of Assumption A and condition (8) of Assumption C. For instance, consider the quasi-stationary case in Corollary 3.2 and Corollary 4.5, where they reduce to (3.7) and either (4.13) or (4.14). While in the former case we only need certain integrability properties of the kernel, we explicitly have to care about the size of the integrals involved in the latter case, which is also the size condition we have mentioned in Example 4.1(2). Also notice that this is related to the fact that in the case $I = \mathbb{R}$, we typically cannot make the left-hand side of (4.10) as small as we want by refining the subdivision $\mathcal{T}$ since the first interval $(t_0, t_1] = (-\infty, t_1]$ always has infinite length. So whereas condition (7) of Assumption A is quite natural for $I = [0, \infty)$, the analogous condition for $I = \mathbb{R}$ would be very restrictive.

(2) By the nature of Equation (1.1), the size condition (8) of Assumption C is "symmetric" in G, σ and Λ.

(3) In Theorem 4.4 uniqueness does not hold in $\tilde{\mathcal{P}}$: see Equation (4.1) with $\lambda \in (1, \infty)$.

□

The next theorem reports some basic properties of the solution found in Theorem 4.4:

Theorem 4.7. *Let Assumption C be valid and Y be the unique solution to Equation* (1.1) *in* $B^{p,w}_{I,\mathrm{loc}}$.

(1) For $(t,x), (\tau,\xi), (s,y) \in I \times \mathbb{R}^d$ define

$$\tilde{G}(t,x;\tau,\xi;s,y) := \bigg(|G(t,x;s,y) - G(\tau,\xi;s,y)|^p \left(\int_{\mathbb{R}} |z|^p \,\pi(s,y,\mathrm{d}z) + c(s,y) \right) + |[G(t,x;s,y) - G(\tau,\xi;s,y)] b_1(s,y)| \mathbb{1}_{\{p\geq 1\}} \bigg) w(s,y). \tag{4.15}$$

If for all $(t,x) \in I \times \mathbb{R}^d$

$$\int_I \int_{\mathbb{R}^d} \tilde{G}(t,x;\tau,\xi;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) \to 0 \tag{4.16}$$

whenever $(\tau,\xi) \to (t,x)$, then Y is an L^p-continuous process, that is,

$$\mathbb{E}[|Y(t,x) - Y(\tau,\xi)|^p] \to 0, \quad \textit{whenever } (\tau,\xi) \to (t,x). \tag{4.17}$$

(2) Assume the case of Corollary 4.5 with $G(t,x;s,y) = g(t-s, x-y)$. Then (4.16) *and therefore the conclusion of (1) hold automatically.*

(3) Y depends continuously on Y_0. In other words, if Y and Y' are the solutions to (1.1) *with $Y_0, Y_0' \in B^{p,w}_{I,\mathrm{loc}}$ as force functions, respectively, then there exists a constant $C_{I,T,w} \in \mathbb{R}_+$ that may depend on I, T and w, but is independent of Y_0, Y_0' such that*

$$\|Y - Y'\|_{B^{p,w}_{I_T}} \leq C_{I,T,w} \|Y_0 - Y_0'\|_{B^{p,w}_{I_T}}. \tag{4.18}$$

One of our basic motivations for studying Equation (1.1) on $I = \mathbb{R}$ is to construct stationary solutions. We show that if G is of convolution form and Λ is homogeneous over space and time, then the stationarity of the solution in Theorem 4.4 follows naturally.

Theorem 4.8. *Assume that $G(t,x;s,y) = g(t-s,x-y)$ and that Λ is a homogeneous Lévy basis, satisfying the assumptions of Corollary 4.5. Furthermore, suppose that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ we have*

$$s \downarrow t, \quad y \downarrow x \text{ (i.e. } y_i \downarrow x_i \text{ for all } i = 1,\dots,d) \quad \Longrightarrow \quad g(s,y) \to g(t,x), \tag{4.19}$$

or that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ implication (4.19) holds with $\downarrow$ replaced by $\uparrow$. If Y_0 is L^p-continuous and jointly stationary with Λ, then also Y and Λ are jointly stationary.

Example 4.9 (Continuation of Examples 3.4 and 3.8) While the number a in (3.8) did not play any role in Examples 3.4 and 3.8, this changes when we consider the stochastic heat equation on $I = \mathbb{R}$. Let $p \in (0, 1 + 2/d)$ and set $p(d) := (1-p)d/2$. Then we have the following trichotomy: for $a > 0$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} g_a^p(t,x) \, \mathrm{d}(t,x) = (4\pi)^{p(d)} p^{-\frac{d}{2}} (ap)^{-1-p(d)} \Gamma(1 + p(d)); \tag{4.20}$$

for $a = 0$ we have for $T \in \mathbb{R}_+$

$$\int_0^T \int_{\mathbb{R}^d} g_0^p(t,x) \, \mathrm{d}(t,x) = \frac{(4\pi)^{p(d)} p^{-\frac{d}{2}}}{1 + p(d)} T^{1+p(d)}, \tag{4.21}$$

which is of polynomial growth when $T \to \infty$; finally, if $a < 0$, we have

$$\int_0^T \int_{\mathbb{R}^d} g_a^p(t,x) \, \mathrm{d}(t,x) = (4\pi)^{p(d)} p^{-\frac{d}{2}} \int_0^T \mathrm{e}^{-apt} t^{p(d)} \, \mathrm{d}t, \tag{4.22}$$

which grows faster than e^{-apT} as $T \to \infty$. Thus, in the latter two cases, for Theorem 4.4 to be applicable, the characteristics of Λ must decay fast enough at $-\infty$ to ensure the integrability conditions (4), (5) and (6) of Assumption C.

We will only focus on the case $a > 0$. Given sufficiently strong decay properties of Λ at $-\infty$, the subsequent arguments can easily be transferred to the other two cases. First, we assume that $w \equiv 1$ and that (1) and (2) of Assumption C hold. We further suppose the quasi-stationary case of (4.11), and that the following conditions hold:

$$p < 1 + \frac{2}{d}, \quad b_0 \equiv 0 \text{ if } p < 1, \quad c \equiv 0 \text{ if } p < 2, \quad \zeta_p := \int_{\mathbb{R}} |z|^p \, \pi_0(\mathrm{d}z) < \infty. \tag{4.23}$$

The only condition left is the size condition (4.13) for $p \in (0,1)$ and (4.14) for $p \in [1,2]$, respectively, before we can apply Corollary 4.5. By (4.20), they are equivalent to

$$\zeta_p C_{\sigma,1}^p (4\pi)^{p(d)} p^{-\frac{d}{2}} (ap)^{-1-p(d)} \Gamma(1 + p(d)) < 1 \tag{4.24}$$

in the case $p \in (0,1)$, and to

$$C_{\sigma,1} \left[C_p^{\mathrm{BDG}} \left((\zeta_p + \|c\|_{L^\infty_{\mathbb{R}}}) (4\pi)^{p(d)} p^{-\frac{d}{2}} (ap)^{-1-p(d)} \Gamma(1 + p(d)) \right)^{1/p} + \|b_1\|_{L^\infty_{\mathbb{R}}} a^{-1} \right] < 1 \tag{4.25}$$

in the case $p \in [1,2]$.

Finally, we would like to demonstrate how weighted spaces can be useful in Theorem 4.4. Let $a > 0$ and $p \in (0, 1 + 2/d)$ as before and define $w(t,x) := \mathrm{e}^{\eta t}$ with $\eta \in \mathbb{R}$ satisfying $ap + \eta > 0$. Assume that $Y_0 \in B^{p,w}_{\mathbb{R},\mathrm{loc}}$ and, if $\eta < 0$, that $\sigma(0) = 0$. Since

$$\sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^d} \int_{-\infty}^t \int_{\mathbb{R}^d} w^{-1}(t,x) g_a^p(t-s, x-y) w(s,y) \, \mathrm{d}(s,y) = \int_0^\infty \int_{\mathbb{R}^d} g_a^p(s,y) \mathrm{e}^{-\eta s} \, \mathrm{d}(s,y)$$
$$= (4\pi)^{p(d)} p^{-\frac{d}{2}} (ap + \eta)^{-1-p(d)} \Gamma(1 + p(d)),$$

we have that in the conditions (4.24) and (4.25), the term ap is now replaced by $ap+\eta$. We draw two conclusions: if Y_0 is sufficiently small at $-\infty$, meaning $Y_0 \in B^{p,w}_{\mathbb{R},\mathrm{loc}}$ for some $\eta > 0$, then the conditions (4.24) and (4.25) can be relaxed by using $ap+\eta$ instead of ap. Contrarily, if $\sigma(0) = 0$, $\eta < 0$, and the left-hand side of (4.24) or (4.25), respectively, remains smaller than 1 with $ap+\eta$ instead of ap, then one can even construct solutions with $Y_0 \in B^{p,w}_{\mathbb{R},\mathrm{loc}}$ that diverges at $-\infty$. □

5 Asymptotic stability

In Theorems 3.1, 3.5 and 4.4 we have established solutions to (1.1) that belong to the space $B^{p,w}_{I,\mathrm{loc}}$. In this section we will give criteria under which they even belong to the space $B^{p,w}_{I}$. Our primary focus is on the case where $\sup I = +\infty$, that is, we want to investigate whether solutions to (1.1) are asymptotically L^p-stable. Moreover, we shall replace the Lipschitz condition on σ, which was essential in Sections 3 and 4, by another growth condition, which, as we shall see, will determine the asymptotic behaviour of the solution. Of course, due to the possible non-Lipschitzianity of σ, we now have to *assume* the existence of a solution in $B^{p,w}_{I,\mathrm{loc}}$. In fact, this approach allows us to include solutions to (1.1) with non-Lipschitz σ which go beyond the results of the Sections 3 and 4 but are, for instance, studied in [21, 22].

Let us again start with a deterministic example that highlights the main features of the behaviour at infinity.

Example 5.1 Let $g \in L^1_{[0,\infty)}$, $f \in L^\infty_{[0,\infty)}$ and $v \in L^\infty_{[0,\infty),\mathrm{loc}}$ be positive functions satisfying

$$v(t) = f(t) + \int_0^t g(t-s)v^\gamma(s)\,\mathrm{d}s, \quad t \in \mathbb{R}_+, \tag{5.1}$$

with $\gamma \in (0,1]$. The question is under what conditions we have $v \in L^\infty_{[0,\infty)}$. It turns out that there is a fundamental difference between the cases $\gamma \in (0,1)$ and $\gamma = 1$. In the former case, we always have $v \in L^\infty_{[0,\infty)}$. In fact, if we denote the convolution on the right-hand side of (5.1) by $(g * v^\gamma)(t)$, iteration of (5.1) yields

$$v = f + g * v^\gamma = f + g * (f + g * v^\gamma)^\gamma = f + g * (f + g * (f + g * v^\gamma)^\gamma)^\gamma = \ldots$$

Using Young's inequality, we obtain

$$\begin{aligned}\|v\|_{L^\infty_{[0,T]}} &\leq \|f\|_{L^\infty_{[0,\infty)}} + \|g\|_{L^1_{[0,\infty)}} \|v\|^\gamma_{L^\infty_{[0,T]}} \leq \|f\|_{L^\infty_{[0,\infty)}} + \|g\|_{L^1_{[0,\infty)}} (\|f\|_{L^\infty_{[0,\infty)}} + \|g\|_{L^1_{[0,\infty)}} \|v\|^\gamma_{L^\infty_{[0,T]}})^\gamma \\ &\leq \|f\|_{L^\infty_{[0,\infty)}} + \|g\|_{L^1_{[0,\infty)}} (\|f\|_{L^\infty_{[0,\infty)}} + \|g\|_{L^1_{[0,\infty)}} (\|f\|_{L^\infty_{[0,\infty)}} + \|g\|_{L^1_{[0,\infty)}} \|v\|^\gamma_{L^\infty_{[0,T]}})^\gamma)^\gamma \leq \ldots,\end{aligned}$$

or, equivalently, for every $T \in [0,\infty)$ and $n \in \mathbb{N}$

$$\|v\|_{L^\infty_{[0,T]}} \leq a_n(T), \quad \text{where} \quad a_1(T) := \|v\|_{L^\infty_{[0,T]}}, \quad a_{n+1}(T) := \|f\|_{L^\infty_{[0,\infty)}} + \|g\|_{L^1_{[0,\infty)}} (a_n(T))^\gamma.$$

By induction it can be shown that $0 \leq a_n(T) \leq a \vee a_1(T)$, where a is the unique solution in $(0,\infty)$ of the equation

$$a - \|f\|_{L^\infty_{[0,\infty)}} - \|g\|_{L^1_{[0,\infty)}} a^\gamma = 0.$$

Note that a does not depend on T, so we conclude that $\limsup_{n\to\infty} a_n(T) \leq a$ and $\|v\|_{L^\infty_{[0,T]}} \leq a$ for all $T \in [0,\infty)$. Hence we have $v \in L^\infty_{[0,\infty)}$ with $\|v\|_{L^\infty_{[0,\infty)}} \leq a$.

The situation is totally different for $\gamma = 1$. Then (5.1) becomes

$$v(t) = f(t) + \int_0^t g(t-s)v(s)\,\mathrm{d}s, \quad t \in \mathbb{R}_+, \tag{5.2}$$

which is the well known *renewal equation*. If $f \in L^\infty_{[0,\infty)}$, one can show under some technical assumptions that the unique solution v to (5.2) exhibits the following behaviour: if $\|g\|_{L^1_{[0,\infty)}} < 1$, we have $v \in L^\infty_{[0,\infty)}$; if $\|g\|_{L^1_{[0,\infty)}} = 1$, the boundedness of v depends on whether $f \in L^1_{[0,\infty)}$ or not; if $\|g\|_{L^1_{[0,\infty)}} > 1$, then $v(t) \to \infty$ exponentially fast as $t \to \infty$. For precise statements with the required assumptions, we refer to [4, Chap. V], especially to the Theorems V.4.3 and V.7.1 and Proposition V.7.4.

In summary, whereas locally bounded solutions to (5.1) with $\gamma \in (0,1)$ are automatically globally bounded as soon as $f \in L^\infty_{[0,\infty)}$ and $g \in L^1_{[0,\infty)}$, the behaviour of the solution to (5.2) at infinity strongly depends on the *size* of $\|g\|_{L^1_{[0,\infty)}}$. For a formalization of this example see also Lemma 6.5 for $\gamma \in (0,1)$ and Lemma 6.4 for $\gamma = 1$. □

For Equation (1.1) the precise requirements are the following:

Assumption D Let $p \in (0,2]$, I be an interval and $w\colon I \times \mathbb{R}^d \to \mathbb{R}$ be a weight function satisfying $\sup_{(t,x)\in I\times\mathbb{R}^d} w^{-1}(t,x) < \infty$. We assume the following hypotheses:

(1) $Y_0 \in B^{p,w}_I$.

(2) $\sigma\colon \mathbb{R} \to \mathbb{R}$ satisfies $|\sigma(x)| \leq |\sigma(0)| + C_{\sigma,2}|x|^\gamma$ for all $x \in \mathbb{R}$ with some $\gamma \in (0,1]$.

(3) Either $c(t,x) = 0$ for all $(t,x) \in I \times \mathbb{R}^d$, or we have $2\gamma \leq p$ and

$$\sup_{(t,x)\in I\times\mathbb{R}^d} \int_I \int_{\mathbb{R}^d} w^{-1}(t,x)|G(t,x;s,y)|^2 w(s,y)c(s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) < \infty. \tag{5.3}$$

(4) There exists $q \in (0,2]$ with $p \leq q$ and $q\gamma \leq p$ such that

$$\sup_{(t,x)\in I\times\mathbb{R}^d} \int_I \int_{\mathbb{R}^d} \int_{\mathbb{R}} w^{-1}(t,x)|G(t,x;s,y)z|^p_q w(s,y)\,\nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z) < \infty. \tag{5.4}$$

(5) If $p \geq 1$, then ν satisfies (2.4) and

$$\sup_{(t,x)\in I\times\mathbb{R}^d} \int_I \int_{\mathbb{R}^d} w^{-1}(t,x)|G(t,x;s,y)b_1(s,y)|w(s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) < \infty, \tag{5.5}$$

and (5.5) also holds with $w \equiv 1$; if $p < 1$, then there exist $\alpha \in (-\infty, 2], \beta \in [0,\infty)$ satisfying (3.13), (3.14) (with $\mathbb{R}_+$ replaced by I) and $(\alpha \vee \beta)\gamma \leq p$ such that

$$\sup_{(t,x)\in I\times\mathbb{R}^d} \int_I \int_{\mathbb{R}^d} (F_0(s,y) \vee F_1(s,y))|G(t,x;s,y)|^\alpha_\beta\,\lambda(\mathrm{d}s,\mathrm{d}y) < \infty. \tag{5.6}$$

(6) At least one of the following three cases occurs:

(6a) We have $\gamma < 1$, $q\gamma < p$, $2\gamma < p$ if $c \not\equiv 0$ and $(\alpha \vee \beta)\gamma < p$ if $p < 1$.

(6b) We have $p \in [1,2]$, and if we define for $(t,x),(s,y) \in I \times \mathbb{R}^d$

$$\begin{aligned}
\bar{G}^{D,1}(t,x;s,y) &:= 2^{p-1}\left(\int_I \int_{\mathbb{R}^d} |G(t,x;s,y)b_1(s,y)|\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{p-1} |G(t,x;s,y)b_1(s,y)|,\\
\bar{G}^{D,2}(t,x;s,y) &:= 2(C_p^{\mathrm{BDG}})^2 |G(t,x;s,y)|^2 c(s,y),\\
\bar{G}^{D,3}(t,x;s,y) &:= 2^{p-1}(C_p^{\mathrm{BDG}})^p \int_{\mathbb{R}} |G(t,x;s,y)z|^p \mathbb{1}_{\{|G(t,x;s,y)z|>1\}}\,\pi(s,y,\mathrm{d}z),\\
\bar{G}^{D,4}(t,x;s,y) &:= 2^{q-1}(C_p^{\mathrm{BDG}})^q \int_{\mathbb{R}} |G(t,x;s,y)z|^q \mathbb{1}_{\{|G(t,x;s,y)z|\leq 1\}}\,\pi(s,y,\mathrm{d}z),\\
G^{D,l}(t,x;s,y) &:= w^{-1}(t,x)G^{D,l}(t,x;s,y)w(s,y), \quad l=1,2,3,4,
\end{aligned} \tag{5.7}$$

then there exists a partition of I into pairwise disjoint intervals $I_1,\dots,I_k$ such that

$$\sup_{(t,x)\in I\times\mathbb{R}^d} \sup_{j=1,\dots,k} \sum_{l=1}^{4} C_{\sigma,2} \left(\int_{I_j}\int_{\mathbb{R}^d} G^{D,l}(t,x;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/p} < 1. \tag{5.8}$$

(6c) We have $p \in (0,1)$, and if we define for $(t,x),(s,y) \in I \times \mathbb{R}^d$

$$\begin{aligned}
G^{D,1}(t,x;s,y) &:= 2^{(\alpha\vee\beta\vee 1)-1}(F_0(s,y)\vee F_1(s,y))|G(t,x;s,y)|_\beta^\alpha,\\
G^{D,2}(t,x;s,y) &:= 2^{p+1}|G(t,x;s,y)|^2 c(s,y),\\
G^{D,3}(t,x;s,y) &:= 2^p 2^{(q\vee 1)-1}\int_{\mathbb{R}} |G(t,x;s,y)z|_q^p\,\pi(s,y,\mathrm{d}z),
\end{aligned} \tag{5.9}$$

and

$$r_1 := \alpha\vee\beta, \quad r_2 := 2, \quad r_3 := 1, \tag{5.10}$$

then there exists a partition of I into pairwise disjoint intervals $I_1,\dots,I_k$ such that

$$\sup_{(t,x)\in I\times\mathbb{R}^d} \sup_{j=1,\dots,k} \sum_{l=1}^{3} C_{\sigma,2}^{r_l} \int_{I_j}\int_{\mathbb{R}^d} G^{D,l}(t,x;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) < 1. \tag{5.11}$$

□

Theorem 5.2. *Let Assumption D be valid. If Equation* (1.1) *has a solution* $Y \in B^{p,w}_{I,\mathrm{loc}}$*, it automatically also belongs to* $B^{p,w}_I$*.*

For quasi-stationary G and Λ, there is no significant simplification of Assumption D possible. Thus, we directly move to an example study.

Example 5.3 (Continuation of Examples 3.4, 3.8 and 4.9) Let $I=[0,\infty)$, $a=0$ and $w\equiv 1$. We assume that $Y \in B^p_{[0,\infty),\mathrm{loc}}$ solves

$$Y(t,x) = Y_0(t,x) + \int_0^t \int_{\mathbb{R}^d} g_0(t-s,x-y)\sigma(Y(s,y))\,\Lambda(\mathrm{d}s,\mathrm{d}y), \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d,$$

where Y_0 is given by (3.9) and σ satisfies condition (2) of Assumption D with $\gamma\in(0,1]$. We want to find conditions that guarantee $Y \in B^p_{[0,\infty)}$. Let us check the requirements of Assumption D. (1) and (2) are clear. For (3), (4) and (5), the key observation is the following: for $p,q\in(0,2]$

$$\int_0^\infty \int_{\mathbb{R}^d} |g_0(s,y)|_q^p\,\mathrm{d}(s,y) < \infty \iff p\in(0,1+2/d) \text{ and } q\in(1+2/d,2]. \tag{5.12}$$

As a consequence of the last condition, unless in trivial cases, the classical stochastic heat equation with $a = 0$ will be asymptotically unstable in dimensions 1 and 2. Only in dimensions $d \geq 3$ there is a chance for asymptotic stability. We pose the following conditions:

$$\begin{aligned} &\lambda(\mathrm{d}t, \mathrm{d}x) = \mathrm{d}(t, x), \quad \pi(t, x, \mathrm{d}z) \leq \pi_0(\mathrm{d}z), \quad p \in (0, 1 + 2/d), \quad q \in (1 + 2/d, 2], \\ &q\gamma \leq p, \quad c \equiv 0, \quad b_1 \equiv 0 \text{ if } p \geq 1, \quad \Lambda \text{ is symmetric if } p < 1, \quad \int_{\mathbb{R}} |z|_p^q \, \pi_0(\mathrm{d}z) < \infty. \end{aligned} \tag{5.13}$$

We notice that $\gamma = 1$ is not possible, and that $\int_{\mathbb{R}} |z|^p \, \pi_0(\mathrm{d}z) < \infty$ is no longer sufficient, but π_0 must have a moment structure that is strictly better than its variation structure. Moreover, c must be 0; if $p \geq 1$, only $\Lambda \in \mathcal{M}$ is possible; and if $p < 1$, Λ is required to have no drift and a symmetric Lévy measure. All this is because g_0 is not L^p-integrable on $\mathbb{R}_+ \times \mathbb{R}^d$ for any $p \in (0, 2]$. One readily sees that (5.13) implies conditions (3), (4) and (5). So if (6a) holds, we obtain $Y \in B^p_{[0,\infty)}$. In the case of (6b) or (6c), again a size condition has to be verified, which is analogous to the calculations in Example 4.9. We leave the details to the reader. Note that in this example we have $\gamma < 1$, and therefore (6b) or (6c) is only needed in rare situations. Finally, for $a > 0$ we refer the reader to the calculations in Example 4.9 again which can be re-used. In particular, one can find conditions for asymptotic stability in dimensions 1 and 2 this time. □

6 A series of lemmata

This section contains several lemmata that will play a crucial role in proving the main theorems in Section 7. First, we investigate the stochastic integral mapping in Equation (1.1): fix some $\phi_0 \in \tilde{\mathcal{P}}$ and define for a predictable process $\phi \in \tilde{\mathcal{P}}$ the process $J(\phi)$ by

$$J(\phi)(t, x) := \phi_0(t, x) + \int_I \int_{\mathbb{R}^d} G(t, x; s, y)\sigma(\phi(s, y)) \, \Lambda(\mathrm{d}s, \mathrm{d}y) \tag{6.1}$$

for all $(t, x) \in I \in \mathbb{R}^d$ for which the stochastic integral exists, and set $J(\phi)(t, x) := +\infty$ otherwise. The next lemma, which is of crucial importance for all main results in this paper, relates the moment structure of $J(\phi)$ to that of ϕ.

Lemma 6.1. *Let $w\colon I \times \mathbb{R}^d \to \mathbb{R}$ be a weight function.*

(1) Suppose that Assumption C holds with $p \in (0, 2]$ and recall the definition of $G^{C,1}$ and $G^{C,2}$ in (4.9). Then for all $\phi \in \tilde{\mathcal{P}}$ and $(t, x) \in I \times \mathbb{R}^d$, we have

$$\begin{aligned} \frac{\|J(\phi)(t, x)\|_{L^p}}{(w(t, x))^{1/(p\vee1)}} &\leq \frac{\|\phi_0(t, x)\|_{L^p}}{(w(t, x))^{1/(p\vee1)}} \\ &+ \sum_{l=1}^{2} \left(\int_I \int_{\mathbb{R}^d} \frac{G^{C,l}(t, x; s, y)}{C^p_{\sigma,1}} \left(\frac{|\sigma(0)|^{p\wedge1} + C^{p\wedge1}_{\sigma,1} \|\phi(s, y)\|_{L^p}}{(w(s, y))^{1/(p\vee1)}} \right)^{p\vee1} \lambda(\mathrm{d}s, \mathrm{d}y) \right)^{1/(p\vee1)}, \end{aligned} \tag{6.2}$$

where in the case $C_{\sigma,1} = 0$ we use the convention $0/0 := 1$.

(2) Furthermore, still under Assumption C, we have for any $\phi_1, \phi_2 \in \tilde{\mathcal{P}}$ for which the right-hand side of (6.2) is finite that

$$\begin{aligned} &\frac{\|J(\phi_1)(t, x) - J(\phi_2)(t, x)\|_{L^p}}{(w(t, x))^{1/(p\vee1)}} \\ &\leq \sum_{l=1}^{2} \left(\int_I \int_{\mathbb{R}^d} G^{C,l}(t, x; s, y) \left(\frac{\|\phi_1(s, y) - \phi_2(s, y)\|_{L^p}}{(w(s, y))^{1/(p\vee1)}} \right)^{p\vee1} \lambda(\mathrm{d}s, \mathrm{d}y) \right)^{1/(p\vee1)}. \end{aligned} \tag{6.3}$$

(3) Let Assumption B or Assumption D be valid with $p \in [1,2]$. In the first case let $I = [0,\infty)$ and $w \equiv 1$. Then the following holds for all $\phi \in \tilde{\mathcal{P}}$ and $(t,x) \in I \times \mathbb{R}^d$:

$$\begin{aligned}\frac{\|J(\phi)(t,x)\|_{L^p}}{(w(t,x))^{1/p}} &\leq \frac{\|\phi_0(t,x)\|_{L^p}}{(w(t,x))^{1/p}} + \frac{2[1+|\sigma(0)|+C_{\sigma,2}]}{(w(t,x))^{1/p}} \\ &\quad + (|\sigma(0)|+C_{\sigma,2}) \sum_{l=1}^{4} \left(\int_I \int_{\mathbb{R}^d} G^{D,l}(t,x;s,y)(w(s,y))^{-1}\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p} \\ &\quad + \sum_{l=1}^{4} C_{\sigma,2} \left(\int_I \int_{\mathbb{R}^d} \frac{G^{D,l}(t,x;s,y)}{(w(s,y))^{1-\rho}} \left(\frac{\|\phi(s,y)\|_{L^p}}{(w(s,y))^{1/p}} \right)^{p\rho} \lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}, \qquad (6.4)\end{aligned}$$

where $G^{D,l}$ is defined by (5.7), and ρ can be chosen as $\rho = (q \vee 2\mathbb{1}_{\{c\not\equiv 0\}})\gamma/p$ or $\rho = 1$.

(4) Let Assumption B or Assumption D be valid with $p \in (0,1)$. In the first case let $I = [0,\infty)$ and $w \equiv 1$. Then for all $\phi \in \tilde{\mathcal{P}}$ and $(t,x) \in I \times \mathbb{R}^d$

$$\begin{aligned}\frac{\|J(\phi)(t,x)\|_{L^p}}{w(t,x)} &\leq \frac{\|\phi_0(t,x)\|_{L^p}}{w(t,x)} + \frac{2^{p+1}+1}{w(t,x)} \\ &\quad + \sum_{l=1}^{3} (|\sigma(0)|_0^{r_l} + C_{\sigma,2}^{r_l}) \int_I \int_{\mathbb{R}^d} G^{D,l}(t,x;s,y)(w(s,y))^{-1}\,\lambda(\mathrm{d}s,\mathrm{d}y) \\ &\quad + \sum_{l=1}^{3} C_{\sigma,2}^{r_l} \int_I \int_{\mathbb{R}^d} \frac{G^{D,l}(t,x;s,y)}{(w(s,y))^{1-\rho}} \left(\frac{\|\phi(s,y)\|_{L^p}}{w(s,y)} \right)^{\rho} \lambda(\mathrm{d}s,\mathrm{d}y). \qquad (6.5)\end{aligned}$$

where $G^{D,l}$ and r_l are given by (5.9) and (5.10), and $\rho = (q \vee 2\mathbb{1}_{\{c\not\equiv 0\}} \vee \alpha \vee \beta)\gamma/p$ or $\rho = 1$.

Proof. It suffices to prove the lemma for $w \equiv 1$: the general case follows if we divide the equations (6.2), (6.3) and (6.4) by $w^{1/(p\vee 1)}$. Throughout the proof, $(t,x) \in I \times \mathbb{R}^d$ is fixed, and the abbreviations $\Phi(s,y) := G(t,x;s,y)[\sigma(\phi_1(s,y)) - \sigma(\phi_2(s,y))]$ and $\Psi(s,y) := G(t,x;s,y)\sigma(\phi(s,y))$ are used. Moreover, in the numerous integrals below, we will often drop the integration variables and use the shorthand notations $\iint_t := \int_{I_t}\int_{\mathbb{R}^d}$ and $\iiint_t := \int_{I_t}\int_{\mathbb{R}^d}\int_{\mathbb{R}}$.

a) We first prove (2) when $p \geq 1$. To this end, we decompose

$$\begin{aligned}\Lambda(\mathrm{d}t,\mathrm{d}x) &= \left[\Lambda^c(\mathrm{d}t,\mathrm{d}x) + \int_{\mathbb{R}} z\,(\mu-\nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)\right] + \left[B(\mathrm{d}t,\mathrm{d}x) + \int_{\mathbb{R}} z\mathbb{1}_{\{|z|>1\}}\,\nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)\right] \\ &=: M(\mathrm{d}t,\mathrm{d}x) + B_1(\mathrm{d}t,\mathrm{d}x), \qquad (6.6)\end{aligned}$$

and obtain that $\|J(\phi_1)(t,x) - J(\phi_2)(t,x)\|_{L^p}$ is bounded by

$$\|J^{(1)}(\phi_1)(t,x) - J^{(1)}(\phi_2)(t,x)\|_{L^p} + \|J^{(2)}(\phi_1)(t,x) - J^{(2)}(\phi_2)(t,x)\|_{L^p},$$

where $J^{(1)}$ and $J^{(2)}$ are defined as in (6.1) with Λ replaced by M and B_1, respectively. For the $J^{(2)}$-part, Hölder's inequality yields

$$\begin{aligned}\|J^{(2)}(\phi_1)(t,x) - J^{(2)}(\phi_2)(t,x)\|_{L^p} &\leq C_{\sigma,1} \left[\left(\iint_t |G|\,\mathrm{d}|B_1| \right)^{p-1} \iint_t |G| \mathbb{E}[|\phi_1 - \phi_2|^p]\,\mathrm{d}|B_1| \right]^{1/p} \\ &= \left(\iint_t G^{C,2}(t,x;s,y) \|\phi_1(s,y) - \phi_2(s,y)\|_{L^p}^p\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}. \qquad (6.7)\end{aligned}$$

For the $J^{(1)}$-part, we assume for the moment that the process

$$N_\tau := \iint_\tau G(t,x;s,y)[\sigma(\phi_1(s,y)) - \sigma(\phi_2(s,y))]\,M(\mathrm{d}s,\mathrm{d}y) = \Phi\cdot M_\tau, \quad \tau\in I, \tag{6.8}$$

which is well defined by assumption, is a local martingale. Then we have by Definition 4.2 and the assumption that $c\equiv 0$ for $p<2$

$$\begin{aligned}
&\|J^{(1)}(\phi_1)(t,x) - J^{(1)}(\phi_2)(t,x)\|_{L^p}\\
&\leq C_p^{\mathrm{BDG}}\|[N]_t^{1/2}\|_{L^p} = C_p^{\mathrm{BDG}}\left\|\left(\iiint_t |\Phi z|^2\,\mathrm{d}\mu + \iint_t |\Phi|^2\,\mathrm{d}C\right)^{1/2}\right\|_{L^p}\\
&\leq C_p^{\mathrm{BDG}}\mathbb{E}\left[\iiint_t |\Phi z|^p\,\mathrm{d}\mu + \iint_t |\Phi|^2\,\mathrm{d}C\right]^{1/p} = C_p^{\mathrm{BDG}}\mathbb{E}\left[\iiint_t |\Phi z|^p\,\mathrm{d}\nu + \iint_t |\Phi|^2\,\mathrm{d}C\right]^{1/p}\\
&\leq \left(\iint_t G^{C,1}(t,x;s,y)\|\phi_1(s,y)-\phi_2(s,y)\|_{L^p}^p\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/p}.
\end{aligned} \tag{6.9}$$

Equations (6.7) and (6.9) together imply (6.3) for $p\in[1,2]$. It remains to discuss whether N in (6.8) is a local martingale. Without loss of generality, we may assume that the right-hand side of (6.9) is finite; otherwise (6.3) becomes trivial. Let $\epsilon>0$ and $H\in\tilde{\mathcal{P}}$ be a bounded function satisfying $|H(\omega,s,y)|\leq\epsilon|\Phi(\omega,s,y)|$ pointwise for all $(\omega,s,y)\in\Omega\times I\times\mathbb{R}^d$. Then $H\cdot M$ is a martingale such that we have by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}
\sup_{\tau\in I}\|H\cdot M_\tau\|_{L^p} &\leq C_p^{\mathrm{BDG}}\left\|\left(\iiint_t |Hz|^2\,\mathrm{d}\mu + \iint_t |H|^2\,\mathrm{d}C\right)^{1/2}\right\|_{L^p}\\
&\leq \epsilon C_p^{\mathrm{BDG}}\left\|\left(\iiint_t |\Phi z|^2\,\mathrm{d}\mu + \iint_t |\Phi|^2\,\mathrm{d}C\right)^{1/2}\right\|_{L^p}.
\end{aligned} \tag{6.10}$$

The right-hand side of (6.10) is finite by (6.9). Moreover, as $\epsilon\downarrow 0$, it goes to 0 independently of H. Thus, [10, Prop. 4.9b] is applicable (the extension of this proposition to intervals I different from $I=\mathbb{R}_+$ is straightforward) and shows that N is indeed a local martingale.

b) We prove (2) when $p<1$. By hypothesis, Λ is Lévy basis without drift. Thus, using the basic estimate $|\sum_{i=1}^\infty a_i|^p \leq \sum_{i=1}^\infty |a_i|^p$, we obtain

$$\begin{aligned}
&\|J(\phi_1)(t,x) - J(\phi_2)(t,x)\|_{L^p} = \left\|\iiint_t \Phi z\,\mathrm{d}\mu\right\|_{L^p} \leq \mathbb{E}\left[\iiint_t |\Phi z|^p\,\mathrm{d}\mu\right]\\
&\leq C_{\sigma,1}^p\iiint_t |Gz|^p\mathbb{E}[|\phi_1-\phi_2|^p]\,\mathrm{d}\nu = \iint_t G^{C,1}(t,x;s,y)\|\phi_1(s,y)-\phi_2(s,y)\|_{L^p}\,\lambda(\mathrm{d}s,\mathrm{d}y),
\end{aligned}$$

which is (6.3).

c) Because the Lipschitz condition on σ implies $|\sigma(x)|\leq|\sigma(0)|+C_{\sigma,1}|x|$ for all $x\in\mathbb{R}$, (1) can be deduced in complete analogy to a) and b).

d) We prove (3). To this end, we again consider the decomposition $\Lambda = M + B_1$ as in (6.6). Using Definition 4.2, Jensen's inequality and the hypothesis that $q\gamma\leq p$ and $2\gamma\mathbb{1}_{\{c\not\equiv 0\}}\leq p$, we obtain

$$\begin{aligned}
&\|\Psi\cdot M_t\|_{L^p} \leq C_p^{\mathrm{BDG}}\left\|\left(\iiint_t |\Psi z|^2\,\mathrm{d}\mu + \iint_t |\Psi|^2\,\mathrm{d}C\right)^{1/2}\right\|_{L^p}\\
&\leq C_p^{\mathrm{BDG}}\left(\mathbb{E}\left[\iiint_t |\Psi z|^q\mathbb{1}_{\{|Gz|\leq 1\}}\,\mathrm{d}\nu\right]^{p/q} + \mathbb{E}\left[\iiint_t |\Psi z|^p\mathbb{1}_{\{|Gz|>1\}}\,\mathrm{d}\nu + \left(\iint_t |\Psi|^2\,\mathrm{d}C\right)^{p/2}\right]\right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\le C_p^{\mathrm{BDG}}\Bigg[\bigg(2^{q-1}\iiint_t |Gz|^q\mathbb{1}_{\{|Gz|\le 1\}}(|\sigma(0)|^q + C_{\sigma,2}^q\|\phi\|_{L^p}^{q\gamma})\,\mathrm{d}\nu\bigg)^{1/q}\\
&\qquad + \bigg(2^{p-1}\iiint_t |Gz|^p\mathbb{1}_{\{|Gz|>1\}}(|\sigma(0)|^p + C_{\sigma,2}^p\|\phi\|_{L^p}^{p\gamma})\,\mathrm{d}\nu\bigg)^{1/p}\\
&\qquad + \bigg(2\iint_t |G|^2(|\sigma(0)|^2 + C_{\sigma,2}^2\|\phi\|_{L^p}^{2\gamma})\,\mathrm{d}C\bigg)^{1/2}\Bigg]\\
&\le C_p^{\mathrm{BDG}}\Bigg[\bigg(2^{q-1}\iiint_t |Gz|^q\mathbb{1}_{\{|Gz|\le 1\}}(|\sigma(0)|^q + C_{\sigma,2}^q + C_{\sigma,2}^q\|\phi\|_{L^p}^{p\rho})\,\mathrm{d}\nu\bigg)^{1/q}\\
&\qquad + \bigg(2^{p-1}\iiint_t |Gz|^p\mathbb{1}_{\{|Gz|>1\}}(|\sigma(0)|^p + C_{\sigma,2}^p + C_{\sigma,2}^p\|\phi\|_{L^p}^{p\rho})\,\mathrm{d}\nu\bigg)^{1/p}\\
&\qquad + \bigg(2\iint_t |G|^2(|\sigma(0)|^2 + C_{\sigma,2}^2 + C_{\sigma,2}^2\|\phi\|_{L^p}^{p\rho})\,\mathrm{d}C\bigg)^{1/2}\Bigg]\\
&\le (|\sigma(0)| + C_{\sigma,2})\left[2 + \sum_{l=2}^{4}\bigg(\int_I\int_{\mathbb{R}^d} G^{D,l}(t,x;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y)\bigg)^{1/p}\right]\\
&\qquad + 2 + \sum_{l=2}^{4} C_{\sigma,2}\bigg(\int_I\int_{\mathbb{R}^d} G^{D,l}(t,x;s,y)\|\phi(s,y)\|_{L^p}^{p\rho}\,\lambda(\mathrm{d}s,\mathrm{d}y)\bigg)^{1/p}.
\end{aligned}
\tag{6.11}
$$

Again, one can justify that $\Psi\cdot\Lambda$ is indeed a well defined local martingale whenever the right-hand side of (6.4) is finite. For the B_1-integral, another application of Hölder's inequality demonstrates

$$
\begin{aligned}
\|(\Psi\cdot B_1)_t\|_{L^p} &\le \left[2^{p-1}\bigg(\iint_t |G|\,\mathrm{d}|B_1|\bigg)^{p-1}\iint_t |G|(|\sigma(0)|^p + C_{\sigma,2}^p\|\phi\|_{L^p}^{p\gamma})\,\mathrm{d}|B_1|\right]^{1/p}\\
&\le \left[2^{p-1}\bigg(\iint_t |G|\,\mathrm{d}|B_1|\bigg)^{p-1}\iint_t |G|(|\sigma(0)|^p + C_{\sigma,2}^p + C_{\sigma,2}^p\|\phi\|_{L^p}^{p\rho})\,\mathrm{d}|B_1|\right]^{1/p}.
\end{aligned}
\tag{6.12}
$$

Equation (6.4) now follows from (6.11) and (6.12).

e) We consider the last part (4). In this case we directly use the canonical decomposition of $\Psi\cdot\Lambda$:

$$
\Psi\cdot\Lambda_t = \Psi\cdot\Lambda_t^{\mathrm{c}} + \iiint_t \Psi z\mathbb{1}_{\{|\Psi z|\le 1\}}\,\mathrm{d}(\mu-\nu) + \iiint_t \Psi z\mathbb{1}_{\{|\Psi z|>1\}}\,\mathrm{d}\mu + B_t^{\Psi\cdot\Lambda} =: J^1 + J^2 + J^3 + J^4,
$$

where

$$
B^{\Psi\cdot\Lambda}(\mathrm{d}t,\mathrm{d}x) = \Psi(t,x)\left[b(t,x) + \int_{\mathbb{R}} z(\mathbb{1}_{\{|\Psi(t,x)z|\le 1\}} - \mathbb{1}_{\{|z|\le 1\}})\,\pi(t,x,\mathrm{d}z)\right]\lambda(\mathrm{d}t,\mathrm{d}x).
$$

We begin with J^1:

$$
\begin{aligned}
\|J^1\|_{L^p} &\le (C_1^{\mathrm{BDG}})^p\mathbb{E}\left[\bigg(\iint_t |\Psi|^2\,\mathrm{d}C\bigg)^{1/2}\right]^p \le 2^p\bigg(\iint_t \mathbb{E}[|\Psi|^2]\,\mathrm{d}C\bigg)^{p/2}\\
&\le 2^p\bigg(1 + 2\iint_t G^2(|\sigma(0)|^2 + C_{\sigma,2}^2\|\phi\|_{L^p}^{2\gamma/p})\,\mathrm{d}C\bigg)\\
&\le 2^p\bigg(1 + 2\iint_t G^2(|\sigma(0)|^2 + C_{\sigma,2}^2 + C_{\sigma,2}^2\|\phi\|_{L^p}^{\rho})\,\mathrm{d}C\bigg).
\end{aligned}
$$

For the jumps part, we obtain

$$\|J^2+J^3\|_{L^p} \le (C_1^{\mathrm{BDG}})^p \mathbb{E}\left[\left(\iiint_t |\Psi z|^2 \mathbb{1}_{\{|\Psi z|\le 1\}}\,\mathrm{d}\mu\right)^{1/2}\right]^p + \mathbb{E}\left[\iiint_t |\Psi z|^p \mathbb{1}_{\{|\Psi z|>1\}}\,\mathrm{d}\mu\right]$$

$$\le 2^p \left(\iiint_t \mathbb{E}[|\Psi z|^q \mathbb{1}_{\{|\Psi z|\le 1\}}]\,\mathrm{d}\nu\right)^{p/2} + \iiint_t \mathbb{E}[|\Psi z|^p \mathbb{1}_{\{|\Psi z|>1\}}]\,\mathrm{d}\nu$$

$$\le 2^p \left(1+\iiint_t \mathbb{E}[|\Psi z|_q^p]\,\mathrm{d}\nu\right) \le 2^p\left(1+2^{(q\vee 1)-1}\iiint_t |Gz|_q^p(|\sigma(0)|_0^q + C_{\sigma,2}^q \|\phi\|_{L^p}^{q\gamma/p})\,\mathrm{d}\nu\right)$$

$$\le 2^p\left(1+2^{(q\vee 1)-1}\iiint_t |Gz|_q^p(|\sigma(0)|_0^q + C_{\sigma,2}^q + C_{\sigma,2}^q\|\phi\|_{L^p}^{\rho})\,\mathrm{d}\nu\right).$$

Finally, since

$$|J^4| \le \iint_t |\Psi(s,y)| \left| b(s,y) + \int_{\mathbb{R}} \left[z\mathbb{1}_{\{|z|\in(1,|\Psi(s,y)|^{-1}]\}} - z\mathbb{1}_{\{|z|\in(|\Psi(s,y)|^{-1},1]\}}\right] \pi(s,y,\mathrm{d}z)\right| \lambda(\mathrm{d}s,\mathrm{d}y),$$

we deduce the following bound for J^4 from Assumption B(9) or Assumption D(4), respectively:

$$\|J^4\|_{L^p} \le \mathbb{E}\left[\iint_t |\Psi|(|\Psi|^{\beta-1}F_1\mathbb{1}_{\{|\Psi|\le 1\}} + |\Psi|^{\alpha-1}F_0\mathbb{1}_{\{|\Psi|>1\}})\,\mathrm{d}\lambda\right]^p \le \left(\iint_t (F_0\vee F_1)\mathbb{E}[|\Psi|_\beta^\alpha]\,\mathrm{d}\lambda\right)^p$$

$$\le 1+2^{(\alpha\vee\beta\vee 1)-1}\iint_t (F_0\vee F_1)|G|_\beta^\alpha\left(|\sigma(0)|_0^{\alpha\vee\beta} + C_{\sigma,2}^{\alpha\vee\beta}\|\phi\|_{L^p}^{(\alpha\vee\beta)\gamma/p}\right)\mathrm{d}\lambda$$

$$\le 1+2^{(\alpha\vee\beta\vee 1)-1}\iint_t (F_0\vee F_1)|G|_\beta^\alpha\left(|\sigma(0)|_0^{\alpha\vee\beta} + C_{\sigma,2}^{\alpha\vee\beta} + C_{\sigma,2}^{\alpha\vee\beta}\|\phi\|_{L^p}^{\rho}\right)\mathrm{d}\lambda.$$

In combination with the estimates for J^1, J^2 and J^3, this finishes the proof of (6.5). □

The next lemma allows us to take good versions of the stochastic integral process (6.1):

Lemma 6.2. *For every $\phi\in\tilde{\mathcal{P}}$ there exists a predictable modification of $J(\phi)$, that is, a $(-\infty,\infty]$-valued process $\bar{J}(\phi)\in\tilde{\mathcal{P}}$ such that for each $(t,x)\in I\times\mathbb{R}^d$ we have $J(\phi)(t,x)=\bar{J}(\phi)(t,x)$ a.s.*

Proof. The set A of all $(t,x)\in I\times\mathbb{R}^d$ for which $G(t,x;\cdot,\cdot)\sigma(\phi)$ is integrable w.r.t. Λ is deterministic by definition, and by [11, Thm. 4.1] and Fubini's theorem also measurable. It follows that there exists a measurable modification $J^{\mathrm{m}}(\phi)$ of $J(\phi)$: set $J^{\mathrm{m}}(\phi)=\infty$ on A^{c} and use [20, Thm. 1] on A. Next, define ${}^{\mathrm{p}}J(\phi)$ as the extended predictable projection of $J^{\mathrm{m}}(\phi)$ in the sense of [18, Thm. I.2.28]. By [30, Prop. 3] we may choose ${}^{\mathrm{p}}J(\phi)(t,x)$ measurably in x. And indeed, ${}^{\mathrm{p}}J(\phi)$ is still a modification of $J(\phi)$ since for each $(t,x)\in I\times\mathbb{R}^d$ we have a.s.

$${}^{\mathrm{p}}J(\phi)(t,x)=\mathbb{E}[J^{\mathrm{m}}(\phi)(t,x)\,|\,\mathcal{F}_{t-}] = \int_{I_t}\int_{\mathbb{R}^d} G(t,x;s,y)\sigma(\phi(s,y))\,\Lambda(\mathrm{d}s,\mathrm{d}y) = J(\phi)(t,x).$$

□

We proceed with a discretization result for stochastic integrals:

Lemma 6.3. *Let $I\subseteq\mathbb{R}$ be an interval and $w\equiv 1$, and assume that G, σ and Λ satisfy (2)–(6) of Assumption C. Fix some $(t,x)\in I\times\mathbb{R}^d$ and assume that $G(t,x;\cdot,\cdot)$ has the following properties: for all $(s,y)\in I_t,\times\mathbb{R}^d$ we have*

$$r\uparrow s,\quad z\uparrow y \ (\textit{i.e. } z_i\uparrow y_i \textit{ for all } i=1,\dots,d) \quad\Longrightarrow\quad G(t,x;r,z)\to G(t,x;s,y), \qquad (6.13)$$

and for some $\epsilon > 0$ *the function* $G^*_\epsilon(t,x;s,y) := \sup_{r\in I, s-\epsilon<r\le s, |y-z|<\epsilon} |G(t,x;r,z)|$ *satisfies*

$$\int_{I_t}\int_{\mathbb{R}^d} \left(|G^*_\epsilon(t,x;s,y)|^p \left(\int_{\mathbb{R}} |z|^p \,\pi(s,y,\mathrm{d}z) + c(s,y) \right) \right.$$
$$\left. + |G^*_\epsilon(t,x;s,y) b_1(s,y)| \mathbb{1}_{\{p\ge 1\}} \right) \lambda(\mathrm{d}s,\mathrm{d}y) < \infty. \tag{6.14}$$

Moreover, we specify discretization schemes for both time and space: first, we choose for each $N\in\mathbb{N}$ *a number* $k(N)\in\mathbb{N}\cup\{\infty\}$ *of time points* $(s^N_i)_{i=1}^{k(N)} \subseteq I_t$ *such that*

$$s^N_i < s^N_{i+1}, \quad \textit{and} \quad s^N_1 \downarrow \inf I, \quad s^N_{k(N)} \uparrow t, \quad \sup_{i=1,\ldots,k(N)-1} |s^N_{i+1} - s^N_i| \downarrow 0 \quad \textit{as } N\uparrow\infty;$$

and second, we fix for each $N\in\mathbb{N}$ *a number* $l(N)\in\mathbb{N}\cup\{\infty\}$ *of non-empty pairwise disjoint hyperrectangles* $(Q^N_j = (a^N_j, b^N_j])_{j=1}^{l(N)} \subseteq \mathbb{R}^d$ *satisfying*

$$\bigcup_{j=1}^{l(N)} Q^N_j \uparrow \mathbb{R}^d \quad \textit{and} \quad \sup_{j=1,\ldots,l(N)} \mathrm{diam}(Q^N_j) \downarrow 0 \quad \textit{as } N\uparrow\infty.$$

(1) If $\phi\in B^p_{I,\mathrm{loc}}$ *is an* L^p*-continuous process (cf.* (4.17)*), then the stochastic integral* $J(\phi)(t,x)$ *is well defined and*

$$\phi_0(t,x) + \sum_{i=1}^{k(N)-1}\sum_{j=1}^{l(N)} G(t;x;s^N_i,a^N_j)\sigma(\phi(s^N_i,a^N_j))\Lambda((s^N_i,s^N_{i+1}]\times Q^N_j) \to J(\phi)(t,x) \tag{6.15}$$

in L^p *as* $N\to\infty$.

(2) The statement of (1) remains true if we replace $\uparrow$ *in* (6.13) *by* $\downarrow$*, and at the same time replace* $G(t,x;s^N_i,a^N_j)$ *by* $G(t,x;s^N_{i+1},b^N_j)$ *in* (6.15).

Proof. Part (2) is proved in the same fashion as part (1). That the stochastic integral $J(\phi)(t,x)$ exists, is a consequence of Lemma 6.1(1), the assumptions posed on G and Λ, and the fact that $\phi\in B^p_{I,\mathrm{loc}}$. To prove (6.15), let us call its left-hand side $J^N(\phi)(t,x)$. It follows that

$$J^N(\phi)(t,x) = \phi_0(t,x) + \int_{I_t}\int_{\mathbb{R}^d} H^N(t,x;s,y)\,\Lambda(\mathrm{d}s,\mathrm{d}y), \quad \text{where}$$
$$H^N(t,x;s,y) = \sum_{i=1}^{k(N)-1}\sum_{j=1}^{l(N)} G(t;x;s^N_i,a^N_j)\sigma(\phi(s^N_i,a^N_j))\mathbb{1}_{(s^N_i,s^N_{i+1}]\times Q^N_j}(s,y).$$

We notice that $H^N(t,x;s,y)=0$ if (s,y) does not belong to the set $A^N := (s^N_1, s^N_{k(N)}]\times\bigcup_{j=1}^{l(N)} Q^N_j$, and that for each $(s,y)\in(\inf I,t)\times\mathbb{R}^d$ we have $\mathbb{1}_{(A^N)^c}(s,y)\to 0$ as $N\to\infty$. Now, we distinguish between two cases: first, if $p<1$, or $p\ge 1$ and $\Lambda\in\mathcal{M}$, then similar calculations as done for

Lemma 6.1(2) lead to (set $H(t,x;s,y) := G(t,x;s,y)\sigma(\phi(s,y))$)

$$\begin{aligned}
&\mathbb{E}[|J(\phi)(t,x) - J^N(\phi)(t,x)|^p] \\
&\leq (C_p^{\mathrm{BDG}})^p \int_{I_t}\int_{\mathbb{R}^d} \mathbb{E}[|H(t,x;s,y) - H^N(t,x;s,y)|^p] \left(\int_{\mathbb{R}} |z|^p\,\pi(s,y,\mathrm{d}z) + c(s,y)\right) \lambda(\mathrm{d}s,\mathrm{d}y) \\
&\leq \int_{(A^N)^c} \frac{G^{C,1}(t,x;s,y)}{C_{\sigma,1}^p} \mathbb{E}[|\sigma(\phi(s,y))|^p]\,\lambda(\mathrm{d}s,\mathrm{d}y) \\
&\quad + \iint_{A^N} \frac{G^{C,1}(t,x;s,y)}{C_{\sigma,1}^p} \sum_{i,j} \mathbb{E}[|\sigma(\phi(s,y)) - \sigma(\phi(s_i^N,a_j^N))|^p] \mathbb{1}_{(s_i^N,s_{i+1}^N]\times Q_j^N}(s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) \\
&\quad + (C_p^{\mathrm{BDG}})^p \iint_{A^N} \sum_{i,j} |G(t,x;s,y) - G(t,x;s_i^N,a_j^N)|^p \mathbb{E}[|\sigma(\phi(s_i^N,a_j^N))|^p] \mathbb{1}_{(s_i^N,s_{i+1}^N]\times Q_j^N}(s,y) \\
&\quad \cdot \left(\int_{\mathbb{R}} |z|^p\,\pi(s,y,\mathrm{d}z) + c(s,y)\right) \lambda(\mathrm{d}s,\mathrm{d}y) \\
&=: I_1^N + I_2^N + I_3^N. \qquad (6.16)
\end{aligned}$$

Since $\phi \in B^p_{I,\mathrm{loc}}$ and $G^{C,1}$ is integrable w.r.t. λ by hypothesis, $I_1^N \to 0$ as $N \to \infty$ by dominated convergence. Next, as a consequence of the L^p-continuity of ϕ and the refining properties of our discretization scheme, the sum within I_2^N goes to 0 pointwise for each $(s,y) \in I_t \times \mathbb{R}^d$. Moreover, this sum is majorized by $2\|\sigma(\phi)\|_{B^p_{I_t}}$ such that also $I_2^N \to 0$ as $N \to \infty$. Regarding I_3^N, we obtain as an upper bound

$$\begin{aligned}
I_3^N \leq (C_p^{\mathrm{BDG}})^p \|\sigma(\phi)\|_{B^p_{I_t}} \iint_{A^N} &\left| G(t,x;s,y) - \sum_{i,j} G(t,x;s_i^N,a_j^N) \mathbb{1}_{(s_i^N,s_{i+1}^N]\times Q_j^N}(s,y) \right|^p \\
&\cdot \left(\int_{\mathbb{R}} |z|^p\,\pi(s,y,\mathrm{d}z) + c(s,y)\mathbb{1}_{\{p=2\}}\right) \lambda(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

Because of (6.13), the integrand in the last line goes to 0 as $N \to \infty$, pointwise for $(s,y) \in I_t \times \mathbb{R}^d$. Moreover, it is dominated by $2G^*_\epsilon$, when ϵ is chosen according to (6.14) and N is large enough such that $\sup_{i=1,\dots,k(N)-1} |s_{i+1}^N - s_i^N|$ and $\sup_{j=1,\dots,l(N)} \mathrm{diam}(Q_j^N)$ are smaller than ϵ. By dominated convergence, we conclude $I_3^N \to 0$ as $N \to \infty$.

It remains to discuss the case $p \geq 1$ and $\Lambda \notin \mathcal{M}$. As in Lemma 6.1(2), we decompose $\Lambda = M + B_1$, where M is a martingale measure and B_1 the drift measure. For M we can apply the calculations above. For B_1 we obtain an analogous decomposition as in (6.16): $G^{C,1}$ is replaced by $G^{C,2}$, and instead of the Burkholder-Davis-Gundy constants, the factor

$$\left(\int_{I_t}\int_{\mathbb{R}^d} \sum_{i,j} |G(t,x;s,y) - G(t,x;s_i^N,a_j^N)| \mathbb{1}_{(s_i^N,s_{i+1}^N]\times Q_j^N}(s,y)\,|B_1|(\mathrm{d}s,\mathrm{d}y)\right)^{p-1}$$

appears. But this also goes to 0 as $N \to \infty$, as desired. □

The next lemma concerns the solvability of deterministic integral equations and provides a comparison result. Certainly, there is a huge literature on deterministic Volterra equations, but we did not find a reference completely satisfying our purposes. Thus, we decided to include the proof, which is also very instructive for the proofs of the main theorems below.

Lemma 6.4. *Let $I \subseteq \mathbb{R}$ be an interval and λ a positive measure on $(I \times \mathbb{R}^d, \mathcal{B}(I \times \mathbb{R}^d))$ and $p \in [1,\infty)$. Further suppose that for every $l \in \mathbb{N}$ we have a positive measurable function $G^{(l)}\colon (I \times$*

$\mathbb{R}^d)^2 \to \mathbb{R}$ with $G^{(l)}(t,\cdot;s,\cdot) \equiv 0$ for $s > t$. Moreover, assume that there exists $k \in \mathbb{N}$ and a partition of I into pairwise disjoint intervals $I_1,\dots,I_k$ such that

$$\rho := \sup_{(t,x)\in I\times\mathbb{R}^d} \sup_{j=1,\dots,k} \sum_{l=1}^{\infty} \left(\int_{I_j} \int_{\mathbb{R}^d} G^{(l)}(t,x;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p} < 1. \tag{6.17}$$

Then the following statements hold:

(1) Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of positive functions in L_I^∞ satisfying

$$v_{n+1}(t,x) \le \sum_{l=1}^{\infty} \left(\int_I \int_{\mathbb{R}^d} G^{(l)}(t,x;s,y)(v_n(s,y))^p\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}, \quad n \in \mathbb{N}. \tag{6.18}$$

Then $\sum_{n=1}^\infty \|v_n\|_{L_I^\infty}$ is finite. In particular, v_n converges in L_I^∞ to 0.

(2) For every positive $f \in L_I^\infty$ the equation

$$v(t,x) = f(t,x) + \sum_{l=1}^{\infty} \left(\int_I \int_{\mathbb{R}^d} G^{(l)}(t,x;s,y)(v(s,y))^p\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}, \quad (t,x) \in I\times\mathbb{R}^d, \tag{6.19}$$

has a unique solution $v \in L_I^\infty$. Furthermore, this solution v is positive.

(3) If $\bar v \in L_I^\infty$ is a positive function satisfying

$$\bar v(t,x) \le f(t,x) + \sum_{l=1}^{\infty} \left(\int_I \int_{\mathbb{R}^d} G(t,x;s,y)(\bar v(s,y))^p\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}, \quad (t,x) \in I\times\mathbb{R}^d, \tag{6.20}$$

then we have $\bar v(t,x) \le v(t,x)$ for all $(t,x) \in I \times \mathbb{R}^d$. In particular, if $f \equiv 0$, then $v \equiv \bar v \equiv 0$.

Proof. a) We start with (1). Let $I = I_1 \cup \ldots \cup I_k$ be as in the hypothesis and suppose that the intervals I_j are arranged in increasing order (i.e. $\sup I_j = \inf I_{j+1}$). Furthermore, define for $\phi \in L_I^\infty$ and $(t,x) \in I \times \mathbb{R}^d$

$$\|\phi\|_{G^{(l)},p}(t,x) := \left(\int_I \int_{\mathbb{R}^d} G^{(l)}(t,x;s,y)|\phi(s,y)|^p\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}$$

$$\|\phi\|_{G^{(l)},p,j}(t,x) := \left(\int_{I_j} \int_{\mathbb{R}^d} G^{(l)}(t,x;s,y)|\phi(s,y)|^p\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}, \quad l \in \mathbb{N}, \quad j = 1,\dots,k. \tag{6.21}$$

Obviously, we have $\|\phi\|_{G^{(l)},p}(t,x) \le \sum_{j=1}^k \|\phi\|_{G^{(l)},p,j}(t,x)$ for each $(t,x) \in I \times \mathbb{R}^d$ and $l \in \mathbb{N}$. Hence, it follows from (6.18) that

$$v_{n+1} \le \sum_{l=1}^{\infty} \|v_n\|_{G^{(l)},p} \le \sum_{j=1}^{k} \sum_{l=1}^{\infty} \|v_n\|_{G^{(l)},p,j}, \tag{6.22}$$

an equation that holds pointwise for all $(t,x) \in I \times \mathbb{R}^d$ and for all $n \in \mathbb{N}$. Iterating (6.22) n times, together with the subadditivity of the functional $\|\cdot\|_{G^{(l)},p,j}$, yields

$$\begin{aligned} v_{n+1} &\le \sum_{j_1=1}^{k} \sum_{l_1=1}^{\infty} \|v_n\|_{G^{(l_1)},p,j_1} \le \sum_{j_1,j_2=1}^{k} \sum_{l_1=1}^{\infty} \left\| \sum_{l_2=1}^{\infty} \|v_{n-1}\|_{G^{(l_2)},p,j_2} \right\|_{G^{(l_1)},p,j_1} \le \ldots \\ &\le \sum_{j_1,\dots,j_n=1}^{k} \sum_{l_1=1}^{\infty} \left\| \sum_{l_2=1}^{\infty} \left\| \ldots \sum_{l_n=1}^{\infty} \|v_1\|_{G^{(l_n)},p,j_n} \ldots \right\|_{G^{(l_2)},p,j_2} \right\|_{G^{(l_1)},p,j_1}. \end{aligned} \tag{6.23}$$

Observe that the Volterra property of G implies that on the right-hand side of (6.23), only those summands are non-zero for which $j_1 \geq \ldots \geq j_n$. Since there are exactly $\binom{n+k-1}{n}$ such sequences, and $\sup_{j=1,\ldots,k} \left\| \sum_{l=1}^\infty \|1\|_{G^{(l)},p,j} \right\|_{L^\infty_I} = \rho$, we deduce that

$$\sum_{n=1}^\infty \|v_n\|_{L^\infty_I} \leq \|v_1\|_{L^\infty_I} \sum_{n=0}^\infty \binom{n+k-1}{n} \rho^n < \infty \tag{6.24}$$

by the ratio test and the fact that $\rho < 1$.

b) Next we prove (2) and construct a solution to (6.19) by Picard iteration. Define $v^0(t,x) = f(t,x)$ and for $n \in \mathbb{N}$,

$$v^n(t,x) := f(t,x) + \sum_{l=1}^\infty \left(\int_I \int_{\mathbb{R}^d} G^{(l)}(t,x;s,y)(v^{n-1}(s,y))^p \, \lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}, \quad (t,x) \in I \times \mathbb{R}^d. \tag{6.25}$$

Since G satisfies (6.17), f belongs to L^∞_I, and both functions are positive, v^n is by induction again a positive function in L^∞_I. Now form the difference sequence $u^n := |v^{n+1} - v^n|$ for $n \in \mathbb{N}$, which satisfies property (6.18) by the reverse triangle inequality. By (1), $\sum_{n=1}^\infty \|u^n\|_{L^\infty_I} < \infty$, in other words, v as the limit in L^∞_I of v^n exists. Of course, v is positive. Moreover, taking the limit on both sides of (6.25), we conclude that v indeed satisfies (6.19). The uniqueness part follows by applying part (1) to the difference of two solutions in L^∞_I.

c) For $\phi \in L^\infty_I$ set $I_f(\phi) := f + \sum_{l=1}^\infty \|\phi\|_{G^{(l)},p}$, which again belongs to L^∞_I. By (6.25), we have $v^n = I_f^{(n)}(f)$, which is the n-fold iteration $I_f(I_f(\ldots I_f(f)\ldots))$. Moreover, by (6.20),

$$\bar{v} \leq I_f(\bar{v}) \leq I_f\big(I_f(\bar{v})\big) \leq \ldots \leq I_f^{(n)}(\bar{v}) \leq I_f^{(n-1)}(f) + I_0^{(n)}(\bar{v}) = v^{n-1} + I_0^{(n)}(\bar{v}).$$

As shown in a), v^{n-1} converges to v uniformly on $I \times \mathbb{R}^d$. In addition, $I_0^{(n)}(\bar{v})$ is less or equal to the right-hand side of (6.23) when v_1 is replaced by $\bar{v}$. Thus, the considerations in a) show that $I_0^{(n)}(\bar{v}) \leq \|\bar{v}\|_{L^\infty_I} \binom{n+k-1}{n} \rho^n \to 0$ as $n \to \infty$, which implies (3). □

The next lemma concerns the asymptotic behaviour of deterministic Volterra equations with a fractional nonlinearity:

Lemma 6.5. *Let I, p and $G^{(l)}$ be as in Lemma 6.4. Further suppose that $f \in L^\infty_I$ is a positive function and*

$$\theta := \sup_{(t,x)\in I\times\mathbb{R}^d} \sum_{l=1}^\infty \left(\int_I \int_{\mathbb{R}^d} G^{(l)}(t,x;s,y) \, \lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p} < \infty.$$

Moreover, we assume that $v \in L^\infty_{I,\mathrm{loc}}$ is positive and satisfies

$$v(t,x) \leq f(t,x) + \left(\sum_{l=1}^\infty \int_I \int_{\mathbb{R}^d} G^{(l)}(t,x;s,y)(v(s,y))^{p\gamma} \, \lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/p}, \quad (t,x) \in I \times \mathbb{R}^d, \tag{6.26}$$

for some $\gamma \in (0,1)$. Then $v \in L^\infty_I$ with $\|v\|_{L^\infty_I} \leq a$, where a is the unique strictly positive solution to the equation $a - \|f\|_{L^\infty_I} - \theta a^\gamma = 0$.

Proof. The proof is a straightforward generalization of the arguments given in Example 5.1. We include it for the sake of completeness. Fix $T \in I$ and recall the definition of $\|\cdot\|_{G^{(l)},p}$ and $I_f(\cdot)$ from the proof of Lemma 6.4. By (6.26), it follows that

$$\|v\|_{L^\infty_{I_T}} \leq \|I_f(v^\gamma)\|_{L^\infty_{I_T}} \leq I_{\|f\|_{L^\infty_I}}(\|v\|^\gamma_{L^\infty_{I_T}}).$$

By iteration of the last inequality, we deduce that $\|v\|_{L^\infty_{I_T}} \leq a_n(T)$ for all $n \in \mathbb{N}$ where $a_1(T) := \|v\|_{L^\infty_{I_T}}$ and $a_{n+1}(T) = I_{\|f\|_{L^\infty_I}}((a_n(T))^\gamma) = \|f\|_{L^\infty_I} + \theta(a_n(T))^\gamma$ for $n \in \mathbb{N}$. Straightforward analysis reveals that $\limsup_{n\to\infty} a_n(T) \leq a$, a number independent of T. Hence, $\|v\|_{L^\infty_I} \leq a$. □

7 Proof of the main theorems

Proof of Theorem 3.1. We show that Theorem 3.1 is a special case of Theorem 4.4, or more precisely, that Assumption A is contained in Assumption C: setting $I = \mathbb{R}_+$ and $w \equiv 1$ in Assumption C, it is not hard to see that the first six conditions break down to conditions (1)–(6) of Assumption A, and that condition (7) of Assumption C becomes superfluous. The only thing to show is that (3.4) implies (4.10). To this end, fix $T \in \mathbb{R}_+$, define

$$\epsilon := 2^{-(p\vee 1)} \left[(C_{\sigma,1} C_p^{\mathrm{BDG}})^p + C_{\sigma,1}^p \left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |G(t,x;s,y) b_1(s,y)| \,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{p-1} \right]^{-1},$$

and let $\mathcal{T}$ be a subdivision of $[0,T]$ such that (3.4) holds. Then we have for all $(t,x) \in [0,T]\times\mathbb{R}^d$ and $i = 0,\ldots,k$ that

$$\begin{aligned}
&\sum_{l=1}^{2} \left(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} G^{C,l}(t,x;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/(p\vee 1)} \\
&\leq 2 \left(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} G^{C,1}(t,x;s,y) + G^{C,2}(t,x;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/(p\vee 1)} \\
&\leq \epsilon^{-1/(p\vee 1)} \left(\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} G^{A}(t,x;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y) \right)^{1/(p\vee 1)} < 1,
\end{aligned}$$

which is (4.10). □

Proof of Corollary 3.2. We check the conditions of Assumption A. (1), (2) and (3) are also assumed in the corollary. Regarding (4), (5) and (6), it is easy to see that because of (3.5), conditions (3.1), (3.2) and (3.3) split into separated conditions for both G and Λ, which are fulfilled thanks to (3.6) and (3.7), respectively. Only (7) if left to be verified. Let $T \in \mathbb{R}_+$ be arbitrary and define $t_n^i := i/n^2$ for $n \in \mathbb{N}$ and $i = 0,\ldots,Tn^2$. Then, using the notation

$$g^A := \left(\int_{\mathbb{R}} |z|^p \,\pi_0(\mathrm{d}z) + \|c\|_{L^\infty_{[0,T]}} \right) g^p + \|b_1\|_{L^\infty_{[0,T]}} g \mathbb{1}_{\{p\geq 1\}}, \tag{7.1}$$

we have for all $(t,x) \in [0,T]\times\mathbb{R}^d$

$$\int_{t_n^i}^{t_n^{i+1}} \int_{\mathbb{R}^d} G^A(t,x;s,y)\,\mathrm{d}(s,y) \leq \int_{t_n^i}^{t_n^{i+1}} \int_{\mathbb{R}^d} g^A(t-s,x-y)\,\mathrm{d}(s,y) \leq \int_{(t-t_n^{i+1})\vee 0}^{(t-t_n^i)\vee 0} \int_{\mathbb{R}^d} g^A(s,y)\,\mathrm{d}(s,y).$$

The right-hand side becomes arbitrarily small as $n \to \infty$, uniformly in $(t,x) \in [0,T]\times\mathbb{R}^d$ and $i = 0,\ldots,Tn^2-1$. If not, there would exist some $\epsilon > 0$ as well as for each $n \in \mathbb{N}$ some $\tau_n \in [0,T]$ and $i(n) \in \{0,\ldots,Tn^2-1\}$ such that

$$\int_{(\tau_n - t_n^{i(n)+1})\vee 0}^{(\tau_n - t_n^{i(n)})\vee 0} \int_{\mathbb{R}^d} g^A(s,y)\,\mathrm{d}(s,y) \geq \epsilon.$$

This, however, would contradict the dominated convergence theorem and the Borel-Cantelli lemma since $|((\tau_n - t_n^{i(n)}) \vee 0) - ((\tau_n - t_n^{i(n)+1}) \vee 0)| \leq |t_n^{i(n)+1} - t_n^{i(n)}| = 1/n^2$. Thus, Corollary 3.2 is proved. □

Proof of Theorem 3.5. a) We first prove the existence of a solution to (1.1). To this end, define

$$T_n := \inf\{t > 0 \colon |\Lambda(\{t\} \times \mathbb{R}^d)| > n\}, \quad n \in \mathbb{N}.$$

Assumption B(3) implies that $(T_n)_{n\in\mathbb{N}}$ is a sequence of stopping times such that we have $T_n > 0$ a.s. for each $n \in \mathbb{N}$ and $T_n \uparrow +\infty$ a.s. as $n \to \infty$. Next, we introduce for each $n \in \mathbb{N}$ a truncation of Λ in the following sense:

$$\begin{aligned}\Lambda^n(\mathrm{d}t, \mathrm{d}x) := B(\mathrm{d}t, \mathrm{d}x) + \Lambda^{\mathrm{c}}(\mathrm{d}t, \mathrm{d}x) + \int_{\mathbb{R}} z \mathbb{1}_{\{|z|\leq 1\}} \,(\mu - \nu)(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z)\\ + \int_{\mathbb{R}} z \mathbb{1}_{\{1<|z|\leq n\}} \,\mu(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z).\end{aligned}$$

By Assumption B(4) we may assume without loss of generality that $Y_0 \in B^q_{[0,\infty),\mathrm{loc}}$. Consequently, thanks to Assumption B(1) and (2) and Theorem 3.1, Equation (1.1) with Λ^n as driving noise has a unique solution $Y^n \in B^q_{[0,\infty),\mathrm{loc}}$. We claim that $Y := Y^1 \mathbb{1}_{[\![0,T_1]\!]} + \sum_{n=2}^\infty Y^n \mathbb{1}_{]\!]T_{n-1},T_n]\!]}$ is a solution to the original equation (1.1) with Λ. The predictability of Y is clear. Now fix a (non-random) time $T \in \mathbb{R}_+$ and define

$$\Omega^n_T := \left\{\omega \in \Omega \colon \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\Lambda(\{(t,x)\})(\omega)| \in [0,n]\right\}, \quad n \in \mathbb{N}.$$

By Assumption B(3) the sequence $(\Omega^n_T)_{n\in\mathbb{N}}$ increases to Ω up to a $\mathbb{P}$-null set. Moreover, we have $\mathbb{1}_{[\![0,T_k]\!]}(t) Y^k(t,x) = \mathbb{1}_{[\![0,T_k]\!]}(t) Y^n(t,x)$ a.s. for all $n \in \mathbb{N}$ and $k = 1, \dots, n$ as a consequence of the uniqueness statement of Theorem 3.1 and the fact that $\mathbb{P}[T_k = t] = 0$. Now part (1) of Theorem 3.5 follows from the observation that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$ we have a.s.

$$\begin{aligned}&\mathbb{1}_{\Omega^n_T} \int_0^t \int_{\mathbb{R}^d} G(t,x;s,y)\sigma(Y(s,y))\,\Lambda(\mathrm{d}s,\mathrm{d}y)\\ &= \mathbb{1}_{\Omega^n_T} \int_0^t \int_{\mathbb{R}^d} G(t,x;s,y) \left(\sigma(Y^1(s,y))\mathbb{1}_{[\![0,T_1]\!](s)} + \sum_{k=2}^n \sigma(Y^k(s,y))\mathbb{1}_{]\!]T_{k-1},T_k]\!]}(s)\right) \Lambda^n(\mathrm{d}s,\mathrm{d}y)\\ &= \mathbb{1}_{\Omega^n_T} \int_0^t \int_{\mathbb{R}^d} G(t,x;s,y)\sigma(Y^n(s,y))\,\Lambda^n(\mathrm{d}s,\mathrm{d}y) = \mathbb{1}_{\Omega^n_T} Y^n(t,x) = \mathbb{1}_{\Omega^n_T} Y(t,x).\end{aligned}$$

To be utterly precise, for the transition from the second to the third line to be true, we must show that $J(\phi)$ and $J(\phi')$ as defined in (6.1) are modifications of each other as soon as ϕ and ϕ' are. But this follows from (6.3). Finally, the uniqueness statement follows from that of Theorem 3.1 by localization.

b) Next, we verify that the solution Y found in a) belongs to $B^p_{[0,\infty),\mathrm{loc}}$ if also (5)–(10) of Assumption B are valid. We only carry out the proof for $p \geq 1$. The case $p < 1$ can be proved in the same fashion. Let $T \in \mathbb{R}_+$ and observe from a) that Y equals Y^n on Ω^n_T. Define $v^n(t,x) := \|Y^n(t,x)\|_{L^p}$ for $(t,x) \in [0,T] \times \mathbb{R}^d$, which is always finite because $Y^n \in B^q_{[0,\infty),\mathrm{loc}}$. Moreover, if we define $G^{D,l}$ as in (5.7) with $w \equiv 1$, then we have for all $(t,x) \in [0,T] \times \mathbb{R}^d$ according to Lemma 6.1(3) with $\rho = 1$

$$\|Y(t,x)\mathbb{1}_{\Omega^n_T}\|_{L^p} \leq v^n(t,x) \leq f(t,x) + \sum_{l=1}^4 C_{\sigma,2} \left(\int_0^t \int_{\mathbb{R}^d} G^{D,l}(t,x;s,y)(v^n(s,y))^p \,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/p}, \tag{7.2}$$

where f is the sum of the first three summands on the right-hand side of (6.4). A priori, $G^{D,l}$ may depend on n since it involves the underlying underlying Lévy measure ν^n. However, it is obvious that inequality (6.4) remains true if we use the original Lévy measure ν to form $G^{D,l}$: the right-hand side of (7.2) will only be enlarged. In this case, (7.2) falls into the category of Lemma 6.4(3). Indeed, Assumption B(10) guarantees that $f \in L^\infty_{[0,T]}$, and that the key assumption (6.17) is met (note that the different constants appearing in $G^{D,l}$ compared to G^B are irrelevant because G^B satisfies the partition property (3.4) for all $\epsilon > 0$). Thus, we have $v^n(t,x) \leq v(t,x)$ where $v \in L^\infty_{[0,T]}$ is again independent of n and is the solution of the corresponding Volterra equation if we replace the second inequality sign in (7.2) by equality. Taking the limit $n \to \infty$, we conclude

$$\|Y(t,x)\|_{L^p} = \lim_{n\to\infty} \|Y(t,x)\mathbb{1}_{\Omega^n_T}\|_{L^p} \leq v(t,x),$$

that is, $Y \in B^p_{[0,\infty),\mathrm{loc}}$. □

Proof of Corollary 3.7. a) We begin with the first statement, for which we need to verify (2) and (3) of Assumption B. That (2) holds, follows from the proof of Corollary 3.2, where we have shown that (3.5), (3.6) and (3.7) imply the validity of Assumption A(4)–(7). Notice that in the quasi-stationary case, it suffices to check Assumption B(2) only for $n = 1$ because $\int_{1<|z|\leq n} |z|^q \,\pi_0(\mathrm{d}z)$ is always finite and condition (3) of Corollary 3.7 is in force. That (3) of Assumption B holds, is due to (3.17):

$$\nu([0,T]\times\mathbb{R}^d\times[-1,1]^c) \leq \int_0^T \int_{\mathbb{R}^d} \pi_1(t,x)\,\mathrm{d}(t,x)\pi_0(|z|>1) < \infty.$$

b) For the second part we must prove (5)–(10) of Assumption B. (5) and (6) hold by hypothesis. Furthermore, since $p < q$ implies $|ab|^p_q \leq |a|^p_q |b|^q_p$ for all $a,b \in \mathbb{R}$, we have by (3.5)

$$\begin{aligned}&\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}} |G(t,x;s,y)z|^p_q \,\nu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\\ &\quad\leq \|\pi_1\|_{L^\infty_{[0,T]}} \int_{\mathbb{R}} |z|^p_q \,\pi_0(\mathrm{d}z) \int_0^T \int_{\mathbb{R}^d} |g(t,x)|^q_p \,\mathrm{d}(t,x) < \infty,\end{aligned}$$

which implies (7) of Assumption B. Next, (8) is a direct consequence of condition (3) of the corollary. For (9) we choose $\alpha = q$ and $\beta = p$, which clearly satisfy (9c). For (9a) and (9b) first observe that

$$\left| b(t,x) + \int_{\mathbb{R}} z\mathbb{1}_{\{|z|\in(1,A]\}}\,\pi(t,x,\mathrm{d}z)\right| \leq \|b\|_{L^\infty_{[0,T]}} + \|\pi_1\|_{L^\infty_{[0,T]}} \int_{|z|>1} |z|^p \,\pi_0(\mathrm{d}z) A^{1-p} \leq F_1 A^{1-p}$$

holds for all $A \in [1,\infty)$ if $F_1 \in \mathbb{R}_+$ is chosen large enough. Second, if $q < 1$, we have $b_0 \equiv 0$ by (3.6), which means that

$$\begin{aligned}\left| b(t,x) - \int_{\mathbb{R}} z\mathbb{1}_{\{|z|\in(a,1]\}}\,\pi(t,x,\mathrm{d}z)\right| &= \left| \int_{\mathbb{R}} z\mathbb{1}_{\{|z|\in(0,a]\}}\,\pi(t,x,\mathrm{d}z)\right|\\ &\leq \|\pi_1\|_{L^\infty_{[0,T]}} \int_{|z|\leq 1} |z|^q \,\pi_0(\mathrm{d}z) a^{1-q}.\end{aligned}$$

Finally, if $q \geq 1$ we have

$$\left| b(t,x) - \int_{\mathbb{R}} z\mathbb{1}_{\{|z|\in(a,1]\}}\,\pi(t,x,\mathrm{d}z)\right| \leq \|b\|_{L^\infty_{[0,T]}} + \|\pi_1\|_{L^\infty_{[0,T]}} \int_{|z|\leq 1} |z|^q \,\pi_0(\mathrm{d}z) a^{1-q} \leq F_0 a^{1-q}$$

for all $a \in (0,1]$ and some constant $F_0 \in \mathbb{R}_+$. Finally, condition (10) holds by the same arguments used in the proof of Corollary 3.2. □

Proof of Theorem 4.4. We base the proof on a Picard iteration scheme, which parallels the construction of a solution to (6.19) in Lemma 6.4. We define processes $Y^n \in \tilde{\mathcal{P}}$ inductively as follows: starting with $Y^0(t,x) := Y_0(t,x)$, we assume that $Y^{n-1} \in B^{p,w}_{I,\mathrm{loc}}$ has already been constructed for some $n \in \mathbb{N}$. Define for each $(t,x) \in I \times \mathbb{R}^d$

$$Y^n(t,x) := Y_0(t,x) + \int_I \int_{\mathbb{R}^d} G(t,x;s,y)\sigma(Y^{n-1}(s,y))\,\Lambda(\mathrm{d}s,\mathrm{d}y), \tag{7.3}$$

hereby choosing a predictable version of Y^n, cf. Lemma 6.2. Let $T \in I$. Then we have by Lemma 6.1(1) for all $(t,x) \in I_T \times \mathbb{R}^d$

$$\begin{aligned}\frac{\|Y^n(t,x)\|_{L^p}}{(w(t,x))^{1/(p\vee1)}} &\leq \frac{\|Y_0(t,x)\|_{L^p}}{(w(t,x))^{1/(p\vee1)}}\\ &+\sum_{l=1}^2\left(\int_I\int_{\mathbb{R}^d}\frac{G^{C,l}(t,x;s,y)}{C^p_{\sigma,1}}\left(\frac{|\sigma(0)|^{p\wedge1}+C^{p\wedge1}_{\sigma,1}\|Y^{n-1}(s,y)\|_{L^p}}{(w(s,y))^{1/(p\vee1)}}\right)^{p\vee1}\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/(p\vee1)},\end{aligned}$$

which is finite by Assumption C. Thus, $Y^n \in B^{p,w}_{I,\mathrm{loc}}$ for all $n \in \mathbb{N}$. Next, Lemma 6.1(2) implies that $u^n := Y^n - Y^{n-1}$ satisfies

$$\frac{\|u^{n+1}(t,x)\|_{L^p}}{(w(t,x))^{1/(p\vee1)}} \leq \sum_{l=1}^2\left(\int_I\int_{\mathbb{R}^d}G^{C,l}(t,x;s,y)\left(\frac{\|u^n(s,y)\|_{L^p}}{(w(s,y))^{1/(p\vee1)}}\right)^{p\vee1}\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/(p\vee1)} \tag{7.4}$$

for all $(t,x) \in I \times \mathbb{R}^d$, which is a recursive relation as in Lemma 6.4(1). Note that the key hypothesis (6.17) is fulfilled because of Assumption C(8). We conclude that $\sum_{n=1}^\infty \|u^n\|_{B^{p,w}_{I_T}} < \infty$, in other words, Y^n converges in $B^{p,w}_{I_T}$ to some limit Y. Applying Lemma 6.1(2) to $\phi_1 := Y$ and $\phi_2 := Y^{n-1}$, the convergence $Y^{n-1} \to Y$ also implies that $J(Y^{n-1}) = Y^n \to J(Y)$ in $B^{p,w}_{I_T}$, that is, Y indeed satisfies (1.1). The uniqueness of the solution to (1.1) follows if we substitute u^n in (7.4) by the difference of two solutions. Since $T \in I$ is arbitrary, Theorem 4.4 follows. □

Proof of Corollary 4.5. We verify Assumption C for $I = \mathbb{R}$ and $w \equiv 1$. (1), (2) and (3) hold by hypothesis; (4), (5) and (6) are consequences of (4.11), (4.12), (4.13) and (4.14). Moreover, condition (7) of Assumption C is redundant such that it remains to verify (8). To this end, define

$$\begin{aligned}g^{C,1} &:= (C_{\sigma,1}C_p^{\mathrm{BDG}})^p(\zeta_p + \|c\|_{L^\infty_{\mathbb{R}}})g^p,\\ g^{C,2} &:= (C_{\sigma,1}\|b_1\|_{L^\infty_{\mathbb{R}}})^p\left(\int_0^\infty\int_{\mathbb{R}^d}g(t,x)\,\mathrm{d}(t,x)\right)^{p-1}g\mathbb{1}_{\{p\geq1\}}.\end{aligned}$$

Then, for any subdivision $\mathcal{T}\colon -\infty = t_0 < \ldots < t_{k+1} = T$, all $(t,x) \in (-\infty,T] \times \mathbb{R}^d$ and $i = 0,\ldots,k$, we have by (4.13) and (4.14)

$$\begin{aligned}\sum_{l=1}^2\left(\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}^d}G^{C,l}(t,x;s,y)\,\mathrm{d}(s,y)\right)^{1/(p\vee1)} &\leq \sum_{l=1}^2\left(\int_{-\infty}^t\int_{\mathbb{R}^d}g^{C,l}(t-s,x-y)\,\mathrm{d}(s,y)\right)^{1/(p\vee1)}\\ &= \sum_{l=1}^2\left(\int_0^\infty\int_{\mathbb{R}^d}g^{C,l}(t,x)\,\mathrm{d}(t,x)\right)^{1/(p\vee1)} < 1.\end{aligned}$$

□

Proof of Theorem 4.7. a) Fix $T \in I$ and choose $(t,x),(\tau,\xi) \in I_T \times \mathbb{R}^d$. Then similar calculations as in Lemma 6.1(2) lead to

$$\begin{aligned}\|Y(t,x)-Y(\tau,\xi)\|_{L^p} &\le \sum_{l=1}^2 \left(\int_I\int_{\mathbb{R}^d} \tilde G^{(l)}(t,x;\tau,\xi;s,y)\left(\frac{\|\sigma(Y(s,y))\|_{L^p}}{(w(s,y))^{1/(p\vee 1)}}\right)^{p\vee 1}\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/(p\vee 1)}\\ &\le \|\sigma(Y)\|_{B^{p,w}_{I_T}} \sum_{l=1}^2\left(\int_I\int_{\mathbb{R}^d}\tilde G^{(l)}(t,x;\tau,\xi;s,y)\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/(p\vee 1)},\end{aligned}$$

where

$$\begin{aligned}\tilde G^{(1)}(t,x;\tau,\xi;s,y) &:= (C_p^{\mathrm{BDG}})^p|G(t,x;s,y)-G(\tau,\xi;s,y)|^p\left(\int_{\mathbb{R}}|z|^p\,\pi(s,y,\mathrm{d}z)+c(s,y)\right)w(s,y),\\ \tilde G^{(2)}(t,x;\tau,\xi;s,y) &:= \left(\int_I\int_{\mathbb{R}^d}|[G(t,x;s,y)-G(\tau,\xi;s,y)]b_1(s,y)|\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{p-1}\\ &\qquad\cdot|[G(t,x;s,y)-G(\tau,\xi;s,y)]b_1(s,y)|w(s,y)\mathbb{1}_{\{p\ge 1\}}.\end{aligned}$$

The claim now follows from (4.16) because Assumption C(7) implies

$$\begin{aligned}&\sup_{(t,x),(\tau,\xi)\in I_T\times\mathbb{R}^d}\left(\int_I\int_{\mathbb{R}^d}|[G(t,x;s,y)-G(\tau,\xi;s,y)]b_1(s,y)|\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{p-1}\\ &\le 2\sup_{(t,x)\in I_T\times\mathbb{R}^d}\left(\int_I\int_{\mathbb{R}^d}|G(t,x;s,y)b_1(s,y)|\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{p-1}<\infty.\end{aligned}$$

b) In the situation of Corollary 4.5 with G in convolution form, we have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\tilde G(t,x;\tau,\xi;s,y)\,\mathrm{d}(s,y) &\le (\zeta_p+\|c\|_{L^\infty_{\mathbb{R}}})\int_{\mathbb{R}}\int_{\mathbb{R}^d}|g(t-s,x-y)-g(\tau-s,\xi-y)|^p\,\mathrm{d}(s,y)\\ &\quad+\|b_1\|_{L^\infty_{\mathbb{R}}}\mathbb{1}_{\{p\ge1\}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}|g(t-s,x-y)-g(\tau-s,\xi-y)|\,\mathrm{d}(s,y)\\ &=(\zeta_p+\|c\|_{L^\infty_{\mathbb{R}}})\int_{\mathbb{R}}\int_{\mathbb{R}^d}|g(s+h,y+\eta)-g(s,y)|^p\,\mathrm{d}(s,y)\\ &\quad+\|b_1\|_{L^\infty_{\mathbb{R}}}\mathbb{1}_{\{p\ge1\}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}|g(s+h,y+\eta)-g(s,y)|\,\mathrm{d}(s,y)\to 0\end{aligned}$$

because $(h,\eta)=(|t-\tau|,|x-\xi|)\to 0$, cf. [16, Lemma 0.12].

c) Let $T\in I$, $\bar p := p\vee 1$ and define $v(t,x) := w^{-1/\bar p}(t,x)\|Y(t,x)-Y'(t,x)\|_{L^p}$ as well as $v_0(t,x) := w^{-1/\bar p}(t,x)\|Y_0(t,x)-Y_0'(t,x)\|_{L^p}$. Furthermore, choose $k\in\mathbb{N}$ and a partition $I_T = I_1\cup\ldots\cup I_k$ such that (4.10) is satisfied. Next, recall from (6.21) the definition of $\|\phi\|_{G^{(l)},\bar p}(t,x)$ and $\|\phi\|_{G^{(l)},\bar p,j}(t,x)$ for $(t,x)\in I_T\times\mathbb{R}^d$, $l=1,2$ and $j=1,\ldots,k$. From Lemma 6.1(2) we deduce

$$v\le v_0+\sum_{l=1}^2\|v\|_{G^{(l)},\bar p}\le v_0+\sum_{j=1}^k\sum_{l=1}^2\|v\|_{G^{(l)},\bar p,j}. \tag{7.5}$$

By the same arguments as in the proof of Lemma 6.4(1), iterating (7.5) N times produces

$$\|v\|_{L^\infty_{I_T}}\le\|v_0\|_{L^\infty_{I_T}}\sum_{n=0}^{N-1}\binom{n+k-1}{n}\rho^n+\|v\|_{L^\infty_{I_T}}\binom{N+k-1}{N}\rho^N,$$

with $\rho < 1$ being the left-hand side of (4.10). Letting $N \to \infty$ leads to the assertion. □

Proof of Theorem 4.8. It suffices to prove the case where (4.19) holds. Since $Y \in B^p_{\mathbb{R},\mathrm{loc}}$ is constructed as the limit of the Picard iterates Y^n in (7.3), it suffices to prove that Y^n, Y_0 and Λ are jointly stationary for all $n \in \mathbb{N}$. By induction, we assume that Y^{n-1} is jointly stationary with Λ and Y_0 (that Y_0 is, holds by assumption). First, we assume that g is bounded and has compact support in $\mathbb{R}_+ \times \mathbb{R}^d$, which obviously implies that (6.14) holds for arbitrary $\epsilon > 0$. Moreover, Y^{n-1} is L^p-continuous because Y^0 is by hypothesis and thus also Y^{n-1} for general n by the same arguments as in the proof of Theorem 4.7(2). Next, we fix $(t,x),(h,\eta) \in \mathbb{R} \times \mathbb{R}^d$ and define for $N \in \mathbb{N}$ and $i = 0,\ldots,N^2$ the time points $s_i^N := t - N + i/N$. Moreover, we set $Q_N := (0,(1/N,\ldots,1/N)]$ and $\Gamma_N := \{(i_1/N,\ldots,i_d/N)\colon i_1,\ldots,i_d \in \{-N^2,\ldots,N^2\}\}$. Lemma 6.3 now gives

$$
\begin{aligned}
&Y^n(t+h,x+\eta) = Y_0(t+h,x+\eta) + \int_{-\infty}^{t+h}\int_{\mathbb{R}^d} g(t+h-s,x+\eta-y)\sigma(Y^{n-1}(s,y)\,\Lambda(\mathrm{d}s,\mathrm{d}y)\\
&= Y_0(t+h,x+\eta) + \int_{-\infty}^{t}\int_{\mathbb{R}^d} g(t-s,x-y)\sigma(Y^{n-1}(s+h,y+\eta)\,\Lambda(h+\mathrm{d}s,\eta+\mathrm{d}y)\\
&= Y_0(t+h,x+\eta) + L^p\text{--}\lim_{N\to\infty}\sum_{i=0}^{N^2-1}\sum_{y_j^N\in\Gamma_N} g(t-s_i^N,x-y_j^N)\sigma(Y^{n-1}(s_i^N+h,y_j^N+\eta))\\
&\qquad \cdot\Lambda\big((s_i^N+h,s_{i+1}^N+h)\times(y_j^N+\eta+Q_N)\big)\\
&\stackrel{d}{=} Y_0(t,x) + L^p\text{--}\lim_{N\to\infty}\sum_{i=0}^{N^2-1}\sum_{y_j^N\in\Gamma_N} g(t-s_i^N,x-y_j^N)\sigma(Y^{n-1}(s_i^N,y_j^N))\Lambda\big((s_i^N,s_{i+1}^N)\times(y_j^N+Q_N)\big)\\
&= Y^n(t,x).
\end{aligned}
$$

The calculation remains valid when we consider joint distributions with Y_0 and Λ, and when we extend it to n space–time points. So the theorem is proved for bounded functions g with compact support. For general functions g we notice that property (4.19) implies that we can write $g = \sum_{i=1}^\infty g_i$ where each g_i is bounded with compact support. The theorem follows since the calculation above is invariant under summation and taking limits. □

Proof of Theorem 5.2. Let $Y \in B^{p,w}_{I,\mathrm{loc}}$ be a solution to (1.1). Then we have $v \in L^\infty_{I,\mathrm{loc}}$ where v is defined by $v(t,x) := w^{-1/(p\vee 1)}(t,x)\|Y(t,x)\|_{L^p}$. The claim is that v also belongs to L^∞_I. We only consider the case $p \in [1,2]$, the case $p \in (0,1)$ can be treated analogously. First, we suppose that Assumption D(6a) holds. In this case, it follows from Lemma 6.1(3) that there exists some $\rho \in (0,1)$ with

$$
v(t,x) \le f(t,x) + \sum_{l=1}^4 C_{\sigma,2}\left(\int_I\int_{\mathbb{R}^d} G^{D,l}(t,x;s,y)(w(s,y))^{\rho-1}(v(s,y))^{p\rho}\,\lambda(\mathrm{d}s,\mathrm{d}y)\right)^{1/p}, \quad (7.6)
$$

where f denotes the sum of the first three terms on the right-hand side of (6.4). By hypothesis, the functions w^{-1}, $w^{-1/p}$ and $w^{\rho-1}$ are uniformly bounded on $I \times \mathbb{R}^d$, which means that f belongs to L^∞_I. Consequently, Lemma 6.5 together with (3), (4) and (5) of Assumption D shows that $v \in L^\infty_I$. Now suppose that Assumption D(6b) holds. Then, by replacing r in (7.6) by 1, the claim follows from Lemma 6.4(3) and assumption (5.8). □

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