

Capacity Bounds for a Class of Diamond Networks

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Abstract—A class of diamond networks is studied where the broadcast component is modelled by two independent bit-pipes. New upper and lower bounds are derived on the capacity which improve previous bounds. The upper bound is in the form of a max-min problem, where the maximization is over a coding distribution and the minimization is over an auxiliary channel. The proof technique generalizes bounding techniques of Ozarow for the Gaussian multiple description problem (1981) and Kang and Liu for the Gaussian diamond network (2011). The bounds are evaluated for a Gaussian multiple access channel (MAC) and the binary adder MAC, and the capacity is found for interesting ranges of the bit-pipe capacities.

I. INTRODUCTION

The diamond network was introduced in [1] as a simple multi-hop network with broadcast and multiple access channel (MAC) components. The two-relay diamond network models the scenario where a source communicates with a sink through two relay nodes that do not have information of their own to communicate. The underlying challenge may be described as follows. In order to fully utilize the MAC to the receiver, we would ideally like full cooperation between the relay nodes. On the other hand, in order to communicate the maximum amount of information and better use the diversity that is offered by the relays, we would like to send independent information to the relay nodes over the broadcast channel. The problem of finding the capacity of this network is unresolved. Lower and upper bounds on the capacity are given in [1]. We remark that the problem is solved over linear deterministic relay networks, and the capacity of Gaussian relay networks has been approximated within a constant number of bits [2].

In this paper, we study a class of diamond networks where the broadcast channel is modelled by two independent bit-pipes. This problem was initially studied in [3] where lower and upper bounds were derived on the capacity. The best known upper bound for the problem is the cut-set bound, which does not fully capture the tradeoff between cooperation and diversity. Recently, [4] studied the network with a Gaussian MAC and proved a new upper bound which constrains the mutual information between the MAC inputs to improve the cut-set bound. The bounding technique in [4] is motivated by [5] that treats the Gaussian multiple description problem. Unfortunately, neither result seems to apply to discrete memoryless channels.

This paper is organized as follows. We state the problem setup in Section II. In Section III, we prove a new upper bound on the achievable rates by generalizing the bounding

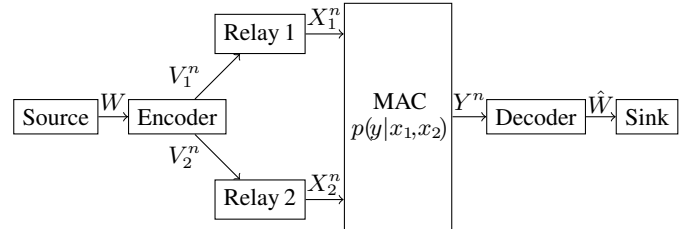


Fig. 1: Problem setup

technique of [4]. Our upper bound applies to the general class of discrete memoryless MACs, and strictly improves the cut-set bound. In Section IV, we improve the achievable rates of [3] by communicating a common piece of information from the source to both relays and using superposition coding and Marton's coding. Finally, we study our bounds for networks with a Gaussian MAC (Section V) and a binary adder MAC (Section VI). For both examples, we find conditions on the bit-pipe capacities such that the upper and lower bounds meet.

II. PRELIMINARIES

A. Notation

We use standard notation for random variables, e.g., X , probabilities, e.g., $p_X(x)$ or $p(x)$, entropies, e.g., $H(X)$ and $H(X|Y)$, and mutual information, e.g., $I(X; Y)$. We denote the sequence X_1, \dots, X_n by X^n . Sets are denoted by script letters and matrices are denoted by bold capital letters.

B. Model

Consider the diamond network in Fig. 1. A source communicates a message of rate R to a sink. The source is connected to two relays via noiseless bit-pipes of capacities C_1 and C_2 , and the relays communicate with the receiver over a MAC.

The source encodes a message W with nR bits into sequence V_1^n , which is available at encoder 1, and sequence V_2^n , which is available at encoder 2. V_1^n and V_2^n are such that $H(V_1^n) \leq nC_1$ and $H(V_2^n) \leq nC_2$. Each relay i , $i = 1, 2$, maps its received sequence V_i^n into a sequence X_i^n which is sent over the MAC. The MAC is characterized by its input alphabets \mathcal{X}_1 and \mathcal{X}_2 , output alphabet \mathcal{Y} , and transition probabilities $p(y|x_1, x_2)$, for each $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$. From the received sequence, the sink decodes an estimate \hat{W} of W . We are interested in characterizing the highest rate R that permits arbitrarily small positive error probability $\Pr(\hat{W} \neq W)$.

III. AN UPPER BOUND

The idea behind our upper bound is motivated by [4], [5]. The proposed bound is applicable not only to Gaussian channels, but also to general discrete memoryless channels, and it strictly improves the cut-set bound as we show via two examples.

In the network of Fig. 1, the cut-set bound gives

$$R \leq C_1 + C_2 \quad (1)$$

$$R \leq C_1 + I(X_2; Y|X_1) \quad (2)$$

$$R \leq C_2 + I(X_1; Y|X_2) \quad (3)$$

$$R \leq I(X_1 X_2; Y). \quad (4)$$

However, the bound in (1) disregards the potential correlation between the inputs of the MAC. A tighter bound on R is given in the following multi-letter form:

$$\begin{aligned} nR &\leq H(V_1^n, V_2^n) \\ &= H(V_1^n) + H(V_2^n) - I(V_1^n, V_2^n) \\ &\leq nC_1 + nC_2 - I(X_1^n; X_2^n). \end{aligned}$$

It is observed in [4] that $I(X_1^n; X_2^n)$ can be written in the following form for any random sequence U^n .

$$\begin{aligned} I(X_1^n; X_2^n) &= I(X_1^n X_2^n; U^n) - I(X_1^n; U^n | X_2^n) \\ &\quad - I(X_2^n; U^n | X_1^n) + I(X_1^n; X_2^n | U^n) \end{aligned}$$

Therefore, we have

$$\begin{aligned} nR &\leq nC_1 + nC_2 - I(X_1^n X_2^n; U^n) \\ &\quad + I(X_1^n; U^n | X_2^n) + I(X_2^n; U^n | X_1^n) \end{aligned} \quad (5)$$

where we have used $I(X_1^n; X_2^n | U^n) \geq 0$.

It may not be clear how to single-letterize the above bound. We proceed as follows. First, note that

$$nR \leq I(X_1^n X_2^n; Y^n) \leq I(X_1^n X_2^n; Y^n U^n). \quad (6)$$

Combining inequalities (5) and (6), we have

$$\begin{aligned} 2nR &\leq nC_1 + nC_2 + I(X_1^n X_2^n; Y^n | U^n) \\ &\quad + I(X_1^n; U^n | X_2^n) + I(X_2^n; U^n | X_1^n). \end{aligned} \quad (7)$$

Define U_i from X_{1i}, X_{2i}, Y_i through the channels $p_{U_i|X_{1i}X_{2i}Y_i}(u_i|x_{1i}, x_{2i}, y_i)$, $i = 1, 2, \dots, n$. With this choice of U_i we have the following chain of inequalities:

$$\begin{aligned} 2nR &\leq nC_1 + nC_2 + I(X_1^n X_2^n; Y^n | U^n) \\ &\quad + I(X_1^n; U^n | X_2^n) + I(X_2^n; U^n | X_1^n) \\ &\leq nC_1 + nC_2 + \sum_{i=1}^n I(X_1^n X_2^n; Y_i | U^n Y^{i-1}) \\ &\quad + \sum_{i=1}^n I(X_1^n; U_i | X_2^n U^{i-1}) + \sum_{i=1}^n I(X_2^n; U_i | X_1^n U^{i-1}) \\ &\stackrel{(a)}{\leq} nC_1 + nC_2 + \sum_{i=1}^n I(X_{1i} X_{2i}; Y_i | U_i) \end{aligned}$$

$$\begin{aligned} &+ \sum_{i=1}^n I(X_{1i}; U_i | X_{2i}) + \sum_{i=1}^n I(X_{2i}; U_i | X_{1i}) \\ &\leq nC_1 + nC_2 + nI(X_{1,Q}, X_{2,Q}; Y_Q | U_Q) \\ &\quad + nI(X_{1,Q}; U_Q | X_{2,Q}) + nI(X_{2,Q}; U_Q | X_{1,Q}). \end{aligned}$$

Step (a) follows because we have the following two Markov chains:

$$(X_1^n X_2^n U^n Y^{i-1}) - (X_{1i} X_{2i} U_i) - Y_i \quad (8)$$

$$(X_1^n X_2^n U^{i-1}) - (X_{1i} X_{2i}) - U_i. \quad (9)$$

We thus have the following upper bound.

Theorem 1. *The rate R is achievable only if there exists a joint distribution $p(x_1, x_2)$ for which inequalities (10)-(14) hold for every auxiliary channel $p(u|x_1, x_2, y)$.*

$$R \leq C_1 + C_2 \quad (10)$$

$$R \leq C_2 + I(X_1; Y|X_2) \quad (11)$$

$$R \leq C_1 + I(X_2; Y|X_1) \quad (12)$$

$$R \leq I(X_1 X_2; Y) \quad (13)$$

$$\begin{aligned} 2R &\leq C_1 + C_2 + I(X_1 X_2; Y|U) \\ &\quad + I(X_1; U|X_2) + I(X_2; U|X_1) \end{aligned} \quad (14)$$

Remark 1. *The bound of Theorem 1 is in the form of a max-min problem, where the maximization is over $p(x_1, x_2)$ and the minimization is over $p(u|x_1, x_2, y)$.*

Remark 2. *For a fixed auxiliary channel $p(u|x_1, x_2, y)$ and a fixed MAC $p(y|x_1, x_2)$, the upper bound in Theorem 1 is concave in $p(x_1, x_2)$. See [6].*

Remark 3. *The right hand side (RHS) of (14) may be written as*

$$C_1 + C_2 + I(X_1 X_2; YU) - I(X_1; X_2) + I(X_1; X_2|U). \quad (15)$$

We study the upper bound of Theorem 1 with a Gaussian MAC and a binary adder MAC in Sections V and VI respectively.

IV. A LOWER BOUND

The achievable scheme that we present here is based on [3], but we further allow a common message at both relaying nodes. This is done by rate splitting, superposition coding and Marton's coding and is summarized in the following theorem¹.

Theorem 2. *The rate R is achievable if it satisfies the following conditions for some pmf $p(u, x_1, x_2, y)$, where $p(u, x_1, x_2, y) = p(u, x_1, x_2)p(y|x_1, x_2)$, and $U \in \mathcal{U}$ with $|\mathcal{U}| \leq \min\{|\mathcal{X}_1||\mathcal{X}_2| + 2, |\mathcal{Y}| + 4\}$.*

$$R \leq \min \left\{ \begin{array}{l} C_1 + C_2 - I(X_1; X_2|U) \\ C_2 + I(X_1; Y|X_2 U) \\ C_1 + I(X_2; Y|X_1 U) \\ \frac{1}{2}(C_1 + C_2 + I(X_1 X_2; Y|U) - I(X_1; X_2|U)) \\ I(X_1 X_2; Y) \end{array} \right\} \quad (16)$$

¹The method to derive this theorem was discussed in general terms by G. Kramer and S. Shamai after G. Kramer presented the paper [3] at ITW in 2007.

Sketch of proof: The inner bound in Theorem 2 is found using the following achievable scheme.

a) *Codebook construction:* Fix the joint pmf $P_{U X_1 X_2}(u, x_1, x_2)$. Let $R = R_{12} + R'$ and let

$$R' = R_1 + R_2 - I(X_1; X_2|U) - \delta_\epsilon. \quad (17)$$

Generate $2^{nR_{12}}$ sequences $u^n(m_{12})$ independently, each in an i.i.d manner according to $\prod_l P_U(u_l)$. For each sequence $u^n(m_{12})$, generate (i) 2^{nR_1} sequences $x_1^n(m_{12}, m_1)$ (conditionally) independently, each in an i.i.d manner according to $\prod_l P_{X_1|U}(x_{1,l}|u_l)$ and (ii) 2^{nR_2} sequences $x_2^n(m_{12}, m_2)$ (conditionally) independently, each in an i.i.d manner according to $\prod_l P_{X_2|U}(x_{2,l}|u_l)$. For each sequence $u^n(m_{12})$, pick $2^{nR'}$ sequence pairs $(x_1^n(m_{12}, m_1), x_2^n(m_{12}, m_2))$ that are jointly typical. Index such pairs by $m' \in \{1, \dots, 2^{nR'}\}$.

b) *Encoding:* To communicate message $W = (m_{12}, m')$, communicate (m_{12}, m_1) with relay 1 and (m_{12}, m_2) with relay 2 (we assume $(x_1^n(m_{12}, m_1), x_2^n(m_{12}, m_2))$ is the m' th jointly typical pair for m_{12}). Relays 1 and 2 then send $x_1^n(m_{12}, m_1)$ and $x_2^n(m_{12}, m_2)$ over the MAC, respectively.

c) *Decoding:* Upon receiving y^n , the receiver looks for (\hat{m}_{12}, \hat{m}') for which the following tuple is jointly typical:

$$(u^n(\hat{m}_{12}), x_1^n(\hat{m}_{12}, \hat{m}_1), x_2^n(\hat{m}_{12}, \hat{m}_2), y^n) \in \mathcal{T}_\epsilon^n.$$

d) *Error Analysis:* The error analysis is straightforward and we omit it for the sake of brevity. The above scheme performs reliably if

$$R_{12} + R_1 < C_1 \quad (18)$$

$$R_{12} + R_2 < C_2 \quad (19)$$

$$R < I(U X_1 X_2; Y) \quad (20)$$

$$R_1 + R_2 - I(X_1; X_2|U) < I(X_1 X_2; Y|U) \quad (21)$$

$$R_2 < I(X_2; Y|X_1, U) + I(X_1; X_2|U) \quad (22)$$

$$R_1 < I(X_1; Y|X_2, U) + I(X_1; X_2|U). \quad (23)$$

Conditions (17)-(23), together with $R_1 \geq 0$, $R_2 \geq 0$, $R_{12} \geq 0$, characterize an achievable rate. Eliminating R_{12} , R_1 , R_2 , we arrive at Theorem 2. Cardinality bounds are derived using the standard method via Caratheodory's theorem [7]. ■

Proposition 1. *The lower bound of Theorem 2 is concave in C_1, C_2 .*

Proof: We prove the statement for the case of $C_1 = C_2 = C$. The same argument holds for the more general case. We express the lower bound of Theorem 2 in terms of the maximization problem $\max_{p(u, x_1, x_2)} f_\ell^p(C, p(u, x_1, x_2))$ and we denote it by $f_\ell(C)$. In this formulation, $f_\ell^p(C, p(u, x_1, x_2))$ is just the minimum of the five terms on the RHS of (16).

The proof of the theorem is by contradiction. Suppose that the lower bound is not concave in C ; i.e., there exist values $C^{(1)}, C^{(2)}$, and α , $0 \leq \alpha \leq 1$, such that $C^* = \alpha C^{(1)} + (1 - \alpha)C^{(2)}$ and $f_\ell(C^*) < \alpha f_\ell(C^{(1)}) + (1 - \alpha)f_\ell(C^{(2)})$. Let $p^{(1)}(u, x_1, x_2)$ (resp. $p^{(2)}(u, x_1, x_2)$) be the probability distribution that maximizes $f_\ell^p(C^{(1)}, p(u, x_1, x_2))$ (resp. $f_\ell^p(C^{(2)}, p(u, x_1, x_2))$). Let $p_Q(1) = \alpha$ and $p_Q(2) =$

$1 - \alpha$, and define $p_{U X_1 X_2|Q}(u, x_1, x_2|1) = p^{(1)}(u, x_1, x_2)$ and $p_{U X_1 X_2|Q}(u, x_1, x_2|2) = p^{(2)}(u, x_1, x_2)$. Then we have the following chain of inequalities.

$$\begin{aligned} & f_\ell(C^*) \\ & < \alpha f_\ell(C^{(1)}) + (1 - \alpha)f_\ell(C^{(2)}) \\ & \leq \min \left\{ \begin{array}{l} 2C^* - I(X_1; X_2|UQ) \\ C^* + I(X_1; Y|X_2 UQ) \\ C^* + I(X_2; Y|X_1 UQ) \\ \frac{1}{2}(2C^* + I(X_1 X_2; Y|UQ) - I(X_1; X_2|UQ)) \\ I(U X_1 X_2; Y|Q) \end{array} \right\} \\ & \leq \min \left\{ \begin{array}{l} 2C^* - I(X_1; X_2|UQ) \\ C^* + I(X_1; Y|X_2 UQ) \\ C^* + I(X_2; Y|X_1 UQ) \\ \frac{1}{2}(2C^* + I(X_1 X_2; Y|UQ) - I(X_1; X_2|UQ)) \\ I(U X_1 X_2; Y) \end{array} \right\} \\ & \stackrel{(a)}{\leq} f_\ell(C^*). \end{aligned}$$

Step (a) follows by renaming (U, Q) as U and comparing the left hand with the lower bound characterization of Theorem 2. So we have a contradiction, and the proposition is proved. ■

V. THE GAUSSIAN MAC

The output of the Gaussian MAC is given by

$$Y = X_1 + X_2 + Z$$

where $Z \sim \mathcal{N}(0, 1)$, and the transmitters have average power constraints P_1, P_2 ; i.e., $\frac{1}{n} \sum_{i=1}^n \mathbb{E}(X_{1,i}^2) \leq P_1$ and $\frac{1}{n} \sum_{i=1}^n \mathbb{E}(X_{2,i}^2) \leq P_2$.

We specialize Theorem 1 to obtain an upper bound on the achievable rate R . To simplify the bound, pick U to be a noisy version of Y in the form of $U = Y + Z_N$, where Z_N is a Gaussian noise with zero mean and variance N (to be optimized later). Inequalities (10)-(14) are upper bounded as follows using maximum entropy lemmas:

$$R \leq C_1 + C_2 \quad (24)$$

$$R \leq C_2 + \frac{1}{2} \log(1 + P_1(1 - \rho^2)) \quad (25)$$

$$R \leq C_1 + \frac{1}{2} \log(1 + P_2(1 - \rho^2)) \quad (26)$$

$$R \leq \frac{1}{2} \log\left(1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}\right) \quad (27)$$

$$2R \stackrel{(a)}{\leq} \left(C_1 + C_2 + \frac{1}{2} \log(1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}) \right) + \frac{1}{2} \log\left(\frac{(1+N+P_1(1-\rho^2))(1+N+P_2(1-\rho^2))}{(1+N+P_1+P_2+2\rho\sqrt{P_1 P_2})(1+N)}\right). \quad (28)$$

To obtain inequality (a) above, write the RHS of (14) as

$$\begin{aligned} & C_1 + C_2 + h(Y|U) - h(YU|X_1 X_2) + h(U|X_1) \\ & + h(U|X_2) - h(U|X_1 X_2). \end{aligned}$$

The negative terms are easy to calculate because of the Gaussian nature of the channel and the choice of U . The positive terms are bounded from above using the conditional version of the maximum entropy lemma [8]. It remains to solve a max-min problem (max over ρ and min over N). So

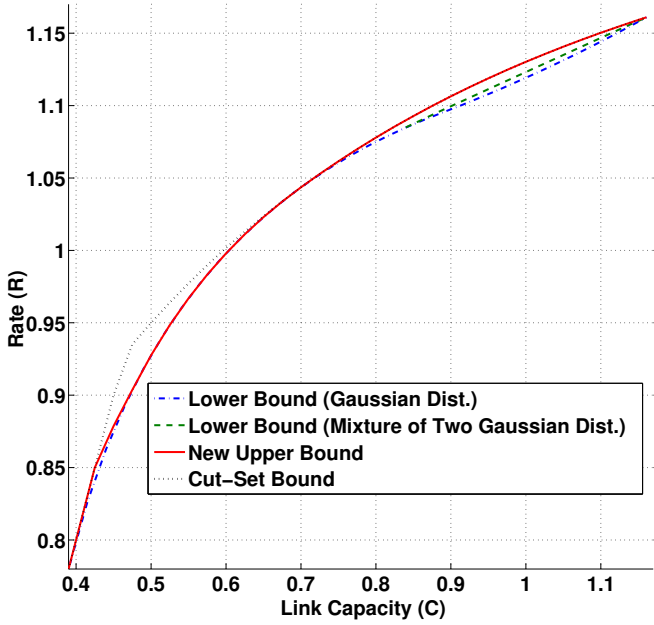


Fig. 2: Upper and lower bounds on R as functions of C for a Gaussian MAC with $P_1 = P_2 = 1$.

the rate R is achievable only if there exists some $\rho > 0$ for which for every $N > 0$ inequalities (24)-(28) hold.

We choose N to be (see [4, eqn. (21)])

$$N = \left(\sqrt{P_1 P_2} \left(\frac{1}{\rho} - \rho \right) - 1 \right)^+. \quad (29)$$

Let us first motivate this choice. It is easy to see that inequality (28) is simply

$$I(X_1 X_2; Y) - I(X_1; X_2) + I(X_1; X_2 | U) \quad (30)$$

evaluated for the joint Gaussian distribution $p(x_1, x_2)$ with covariance matrix

$$\begin{bmatrix} P_1 & \rho \sqrt{P_1 P_2} \\ \rho \sqrt{P_1 P_2} & P_2 \end{bmatrix}. \quad (31)$$

The choice (29) makes U satisfy the Markov chain $X_1 - U - X_2$ for the regime where

$$\sqrt{P_1 P_2} \left(\frac{1}{\rho} - \rho \right) - 1 \geq 0 \quad (32)$$

and thus minimizes the RHS of (28). Otherwise, $U = Y$ is chosen which results in a redundant bound. The upper bound is summarized in Corollary 1.

Corollary 1. Rate R is achievable only if there exists $\rho \geq 0$

such that

$$\begin{aligned} \rho &\leq \sqrt{1 + \frac{1}{4P_1 P_2}} - \sqrt{\frac{1}{4P_1 P_2}}, \\ R &\leq C_1 + C_2 \\ R &\leq C_2 + \frac{1}{2} \log(1 + P_1(1 - \rho^2)) \\ R &\leq C_1 + \frac{1}{2} \log(1 + P_2(1 - \rho^2)) \\ R &\leq \frac{1}{2} \log(1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}) \\ 2R &\leq C_1 + C_2 + \frac{1}{2} \log(1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}) \\ &\quad - \frac{1}{2} \log\left(\frac{1}{1 - \rho^2}\right), \end{aligned}$$

or

$$\begin{aligned} \sqrt{1 + \frac{1}{4P_1 P_2}} - \sqrt{\frac{1}{4P_1 P_2}} &\leq \rho \leq 1, \\ R &\leq C_1 + C_2 \\ R &\leq C_2 + \frac{1}{2} \log(1 + P_1(1 - \rho^2)) \\ R &\leq C_1 + \frac{1}{2} \log(1 + P_2(1 - \rho^2)) \\ R &\leq \frac{1}{2} \log(1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}). \end{aligned}$$

For a lower bound, we use Theorem 2 and choose (U, X_1, X_2) to be jointly Gaussian with mean 0 and covariance matrix $\mathbf{K}_{UX_1 X_2}$. Fig. 2 shows the upper and lower bound as a function of C for a symmetric network with $C_1 = C_2 = C$ and $P_1 = P_2 = 1$. The dash-dotted curve in Fig. 2 shows the rates achievable using the scheme of Section IV with a joint Gaussian distribution. It is interesting to see that the obtained lower bound is not concave in C . This does not contradict Proposition 1 because Gaussian distributions are sub-optimal. The improved dashed curve shows rates that are achievable using a mixture of two Gaussian distributions.

For symmetric diamond networks where $C_1 = C_2 = C$ and $P_1 = P_2 = P$, we specify a regime of C for which the lower and upper bounds meet and thus characterize the capacity. This is summarized in Theorem 3. See [6] for the proof.

Theorem 3. For a symmetric Gaussian diamond network, the upper bound in Theorem 1 meets the lower bound if $C \leq \frac{1}{4} \log(1 + 2P)$, $C \geq \frac{1}{2} \log(1 + 4P)$, or

$$\frac{1}{4} \log \frac{1 + 2P(1 + \rho^{(1)})}{1 - \rho^{(1)2}} \leq C \leq \frac{1}{4} \log \frac{1 + 2P(1 + \rho^{(2)})}{1 - \rho^{(2)2}} \quad (33)$$

where

$$\rho^{(1)} = \frac{-(1 + 2P) + \sqrt{12P^2 + (1 + 2P)^2}}{6P} \quad (34)$$

$$\rho^{(2)} = \sqrt{1 + \frac{1}{4P^2}} - \frac{1}{2P}. \quad (35)$$

For example, the upper and lower bounds match in Fig. 2 (where $P = 1$) for $C \leq 0.3962$, $0.4807 \leq C \leq 0.6942$, and $C \geq 1.1610$.

Remark 4. There is a close connection between the bound of Corollary 1 and that of [4]. The upper bound in [4] is tighter than Corollary 1 in certain regimes of operation. For example in Fig. 2, the upper bound of [4] matches the lower bound for all $C \leq 0.6942$. The reason seems to be that the methods used in [4] provide a better single letterization of the upper bound in (5) for the Gaussian MAC.

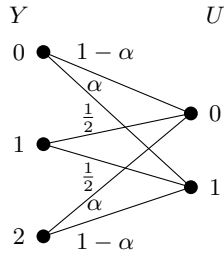


Fig. 3: Channel $p(u|x_1, x_2, y)$

VI. THE BINARY ADDER CHANNEL

Consider the binary adder channel defined by $\mathcal{X}_1 = \{0, 1\}$, $\mathcal{X}_2 = \{0, 1\}$, $\mathcal{Y} = \{0, 1, 2\}$, and $Y = X_1 + X_2$. The best known upper bound for this channel is the cut-set bound. We specialize Theorem 1 to derive a new upper bound on the achievable rate. For simplicity, we assume $C_1 = C_2 = C$.

Define $p(u|x_1, x_2, y)$ to be a symmetric channel as shown in Fig. 3, with parameter α to be optimized. From Theorem 1, the rate R is achievable only if for some pmf $p(x_1, x_2)$ inequalities (10)-(14) hold for the aforementioned choice of U and for every α ; i.e., we have to solve a max-min problem (max over $p(x_1, x_2)$, min over α).

In the rest of this Section we manipulate the above bound to formulate a simplified upper bound. First, loosen the upper bound by looking at the min-max problem (rather than the max-min problem). Fix α . For a fixed channel $p(u|x_1, x_2, y)$, the upper bound is concave in $p(x_1, x_2)$ (see Remark 2). The concavity, together with the symmetry of the problem and the auxiliary channel imply the following lemma.

Lemma 1. *For a fixed α , the optimizing pmf is that of a doubly symmetric binary source.*

Using Lemma 1, $p(x_1, x_2)$ may be assumed to be a doubly symmetric binary source with cross-over probability p . The bounds in (10)-(14) reduce to

$$\begin{aligned} R &\leq 2C \\ R &\leq C + h_2(p) \\ R &\leq h_2(p) + 1 - p \\ 2R &\stackrel{(a)}{\leq} 2C + 2h_2(p) - p + I(X_1; X_2|U). \end{aligned} \quad (36)$$

The inequality in (a) follows from (15) and the fact that $(X_1 X_2) - Y - U$ forms a Markov chain.

To obtain the best bound, we choose U such that $X_1 - U - X_2$ forms a Markov chain. The optimal α satisfies

$$\alpha(1 - \alpha) = \left(\frac{p^*}{2(1 - p^*)} \right)^2 \quad (37)$$

where p^* is the p which maximizes

$$\min \{2C, C + h_2(p), h_2(p) + 1 - p, 2C + 2h_2(p) - p\}. \quad (38)$$

Since we have $p^* \leq \frac{1}{2}$, the choice (37) is valid and we therefore obtain Corollary 2. We refer the reader to [6] for a detailed proof.

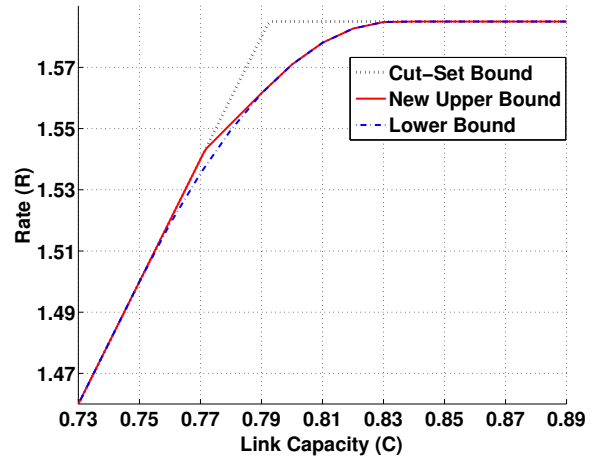


Fig. 4: Upper and lower bounds on R as functions of C for the symmetric binary adder MAC.

Corollary 2. *Rate R is achievable only if there exists some p , $0 \leq p \leq \frac{1}{2}$, such that*

$$\begin{aligned} R &\leq 2C \\ R &\leq C + h_2(p) \\ R &\leq h_2(p) + 1 - p \\ 2R &\leq 2C + 2h_2(p) - p. \end{aligned} \quad (39)$$

Fig. 4 plots this bound (Corollary 2) and compares it with the cut-set bound and the lower bound of Theorem 2 for different values of C . It turns out that the lower and upper bounds meet for $C \leq .75$ and $C \geq .7929$. See [6] for the proof.

VII. ACKNOWLEDGMENT

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