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Titel: Robustness of quadratic hedging strategies in finance via Fourier transforms

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Abstract In this paper we investigate the consequences of the choice of the model to partial hedging in incomplete markets in finance. In fact we consider two models for the stock price process. The first model is a geometric Lévy process in which the small jumps might have infinite activity. The second model is a geometric Lévy process where the small jumps are truncated or replaced by a Brownian motion which is appropriately scaled. To prove the robustness of the quadratic hedging strategies we use pricing and hedging formulas based on Fourier transform techniques. We compute convergence rates and motivate the applicability of our results with examples.

Keywords Lévy processes · Options · Quadratic hedging · Fourier transforms · Robustness

1 Introduction

In financial markets, the hedging of derivatives is in general set in the non-arbitrage framework and can technically be performed under a related pricing measure that is a risk-neutral measure. Under this measure the discounted prices of the underlying primaries are martingales. In some markets, for example, in the context of energy derivatives, the underlying, electricity, cannot be stored. Hence hedging does not require that the pricing measure is a risk-neutral measure. See e.g., [3] for more details. In this case the discounted stock price process is a semimartingale under the pricing measure.

To model asset price dynamics we consider two geometric Lévy processes. This type of models describe well realistic asset price dynamics and are well established in the literature (see e.g. [8]). The first model $(S_t)_{t\in[0,T]}$ is driven by a Lévy process in which the small jumps might have infinite activity. The second model $(S_t^\varepsilon)_{t\in[0,T]}$ is driven by a Lévy process in which we truncate the jumps with absolute size smaller than $\varepsilon > 0$ or we replace them by an appropriately scaled Brownian motion. That is

$$S_t^\varepsilon = S_0 \exp\left(N_t^\varepsilon + s(\varepsilon)\widetilde{W}_t\right), \tag{1.1}$$

where $S_0 > 0$ is the initial price process, $(N_t^\varepsilon)_{t\in[0,T]}$ is a Lévy process with jumps bigger than ε, $(\widetilde{W}_t)_{t\in[0,T]}$ is an independent Brownian motion and $0 \le s(\varepsilon) \le 1$. The scaling factor $s(\varepsilon)$ should be either equal to zero or any sequence which vanishes when ε goes to 0. Notice that in this case, $(S_t^\varepsilon)_{t\in[0,T]}$ approximates $(S_t)_{t\in[0,T]}$. We do not discuss any preferences for the choice of $s(\varepsilon)$. We only present different possible choices and exploit the consequences of the approximation on the pricing and hedging formulas.

Because of the presence of jumps, the market is in general incomplete and there is no self-financing hedging strategy which allows to attain the contingent claim at maturity. In other words, one cannot eliminate the risk completely. However it is possible to find 'partial' hedging strategies which minimize some risk. One way to determine these 'partial' hedging strategies is to introduce a subjective criterion according to which strategies are optimized.
We consider two types of quadratic hedging. In the first approach, called *mean-variance hedging* (MVH), the strategy is self-financing and one minimizes the quadratic hedging error at maturity in mean square sense (see, e.g., [27]). The second approach is called *risk-minimization* (RM) in the martingale setting and *local risk-minimization* (LRM) in the semimartingale setting. These strategies replicate the option's payoff, but they are not self-financing (see, e.g., [27]). In the martingale setting the RM strategies minimize the risk process which is induced by the fact that the strategy is not self-financing. In the semimartingale setting the LRM strategies minimize the risk in a 'local' sense (see [23, 24]).

The approximation (1.1) is important from a simulation point of view. Indeed, the approximating process $(S_t^\varepsilon)_{t\in[0,T]}$ will contain a compound Poisson process and possibly a scaled Brownian motion which are both easy to simulate. Whereas it is not easy to simulate the infinite activity of the small jumps in the process $(S_t)_{t\in[0,T]}$ (see [8] for more about simulation of Lévy processes). We refer to [2] and [8] for a discussion about the choice of the scaling factor $s(\varepsilon)$.
This approximation is not only interesting from a simulation point of view but also from a modelling point of view. In fact, we may think of two financial agents who want to price and hedge

an option. One is considering $(S_t)_{t\in[0,T]}$ as a model for the price process and the other is considering $(S^\varepsilon_t)_{t\in[0,T]}$. In other words, the first agent chooses to consider infinitely small variations in a discontinuous way, i.e. in the form of infinitely small jumps of an infinite activity Lévy process. The second agent observes the small variations in a continuous way, i.e. coming from a Brownian motion. Hence the difference between both market models determines a type of model risk.

In the sequel, we intend by *robustness* of the model, the convergence results when ε goes to zero of $(S^\varepsilon_t)_{t\in[0,T]}$ and of its related pricing and hedging formulas. In the paper [18], it has been proved that the convergence of asset prices does not necessarily imply the convergence of option prices and found out that pricing and hedging are in general not robust. However, in [4] the approximation of the form (1.1) inspired by [2], was investigated and it was proved that the related option prices and the deltas are robust. In this paper we investigate whether the corresponding quadratic hedging strategies are also robust and we reconsider the conditions obtained in [4].
For the study of robustness, we first use Fourier transform techniques as in [11] and [28]. In these two papers, the authors considered the case where the market is observed under a martingale measure and wrote the option prices and hedging strategies for European options in terms of the Fourier transform of the *dampened* payoff function and the characteristic function of the underlying Lévy process. We use these formulas when the market is considered under each of the following equivalent martingale measures: the Esscher transform (ET), the minimal entropy martingale measure (MEMM), the minimal martingale measure (MMM), and the variance-optimal martingale measure (VOMM). In this context and under some integrability conditions on the Lévy process and the payoff function, we prove the robustness of the *optimal number* of risky assets in a RM strategy as well as in a MVH strategy. Moreover we compute convergence rates for our results.
Secondly, in case the market is observed under the world measure and thus the discounted stock price processes are modelled by semimartingales, the hedging strategies are written in [17] in terms of the cumulant generating function of the Lévy process and a complex measure which depends on the Fourier transform of the dampened payoff function. In this setting and under certain integrability conditions on the Lévy process and the payoff function, we also prove the robustness of the *optimal number* of risky assets in a LRM strategy. Moreover we prove the robustness of the *amount of wealth* in the risky asset in a MVH strategy and we compute convergence rates.
Notice that in [9] similar robustness results were studied for more general asset prices in the semimartingale case. Thereto the authors used backward-stochastic differential equations. Moreover their study mainly focused on the robustness of the amount of wealth in the quadratic hedging strategies.

The paper is organized as follows. In Section 2 we introduce the notations and recall some recent results about hedging in incomplete markets. In Section 3 we investigate the robustness of the quadratic hedging strategies where the stock price processes are observed under martingale measures. In Section 4 we prove the robustness of the quadratic hedging strategies where the discounted asset prices are modelled by semimartingales. In Section 5 we discuss the integrability conditions which allow the robustness results to hold true. Moreover we give examples of payoff functions and of driving Lévy processes to illustrate our computations. To finish, we conclude in Section 6.

2 Pricing and hedging in exponential Lévy setting, review of recent results

2.1 The exponential Lévy setting

Assume a finite time horizon $T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $L = (L_t)_{t\in[0,T]}$ denote a Lévy process in the given probability space. We work with the càdlàg version of the given Lévy process and we denote by $\Delta L_t := L_t - L_{t-}$ the jump size of the process L at time t. We introduce the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, defined by $\mathcal{F}_t = \mathcal{F}^0_t \vee \mathcal{N}$, for all t in $[0,T]$, where $(\mathcal{F}^0_t)_{t\in[0,T]}$ is the natural filtration of L and $\mathcal{N}$ contains the $\mathbb{P}$-null sets of $\mathcal{F}$. The filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual hypotheses, see [20]. From the Lévy–Itô decomposition (see, e.g., [8]) we know there exist a Brownian motion $W = (W_t)_{t\in[0,T]}$ and two constants $a, b \in \mathbb{R}$ such that the Lévy process L can be written as

$$L_t = at + bW_t + Z_t + \lim_{\varepsilon\to 0} \widetilde{Z}_t^\varepsilon, \tag{2.1}$$

where Z is a compound Poisson process including the jumps of L with $|\Delta L_t| > 1$ and for any $\varepsilon \in (0,1)$, $\widetilde{Z}^\varepsilon$ is a compensated compound Poisson process including the jumps of L with $\varepsilon \le |\Delta L_t| \le 1$. Moreover, the processes W, Z, and $\widetilde{Z}^\varepsilon$ are independent. We denote by ℓ the Lévy measure of L. The Lévy measure satisfies the following standard integrability conditions

$$\int_{|z|<1} z^2 \ell(dz) < \infty \quad \text{and} \quad \int_{|z|\ge 1} \ell(dz) < \infty. \tag{2.2}$$

The triplet (a, b^2, ℓ) is called the characteristic triplet of the Lévy process L. For $u \in \mathbb{R}$, we define

$$\psi(u) = iau - \frac{1}{2}b^2u^2 + \int_{\mathbb{R}_0} (\mathrm{e}^{iuz} - 1 - iuz1_{|z|<1})\ell(dz). \tag{2.3}$$

By the Lévy-Khintchine representation we know that the characteristic function of L_t is given by

$$\Phi_t(u) := \mathbb{E}[\mathrm{e}^{iuL_t}] = \mathrm{e}^{t\psi(u)}, \qquad \forall u \in \mathbb{R},\ 0 \le t \le T.$$

The moment generating function M_t and the cumulant generating function κ_t of L_t – when they exist – are respectively, for all $u \in \mathbb{R}$ and $0 \le t \le T$, given by

$$\begin{aligned} M_t(u) &:= \mathbb{E}[\mathrm{e}^{uL_t}] = \mathrm{e}^{t\psi(-iu)}, \\ \kappa_t(u) &:= \log \mathbb{E}[\mathrm{e}^{uL_t}] = t\psi(-iu). \end{aligned} \tag{2.4}$$

We assume that the stock price is modelled by a geometric Lévy process, i.e. the stock price is given by $S_t = S_0\mathrm{e}^{L_t}$, $\forall t \in [0,T]$, where $S_0 > 0$. Let $r > 0$ be the risk-free instantaneous interest rate. The value of the corresponding riskless asset equals e^{rt} for any time $t \in [0,T]$. We denote the discounted stock price process by $\hat{S}$. Hence at any time $t \in [0,T]$ it equals

$$\hat{S}_t = \mathrm{e}^{-rt}S_t = S_0\mathrm{e}^{-rt}\mathrm{e}^{L_t}.$$

Furthermore the case that $\hat{S}$ is deterministic is excluded by assuming that $\kappa_t(2) - 2\kappa_t(1) \neq 0$. By the fundamental theorem of asset pricing we know that the existence of an equivalent martingale measure excludes arbitrage opportunities. To make sure that our model does not allow arbitrage, we assume the results, considering exponential Lévy models, from [8] and [28]. The next subsection presents some recent results concerning the pricing and hedging of European options in the exponential Lévy setting.

2.2 Pricing and hedging in the exponential Lévy setting: the martingale case

In this paper we consider a European option with payoff $F(S_T)$ at time T and denote by f the function $f(x) := F(\mathrm{e}^x)$. To price such an option at any time previous to the maturity T, we rely on the Fourier-based approach introduced in [6] and [21] and extended later by several authors. Here we assume the setting is risk-neutral. Thereto we observe the stock price processes under a martingale measure $\widetilde{\mathbb{P}}$ which is equivalent to the historical measure $\mathbb{P}$. The Lévy triplet of the driving process L w.r.t. this martingale measure is denoted by $(\tilde{a}, b^2, \tilde{\ell})$. Similarly we denote the expectation and the characteristic function of L_t at $t \in [0,T]$ under $\widetilde{\mathbb{P}}$ by $\widetilde{\mathbb{E}}$ and $\widetilde{\Phi}_t$, respectively. We introduce the dampened payoff function g associated with the payoff function f as follows

$$g(x) := \mathrm{e}^{-Rx}f(x), \qquad x \in \mathbb{R},$$

for some $R \in \mathbb{R}_0$, which is called the *damping factor*. The *Fourier transform* of a function $g \in L^1(\mathbb{R})$ is denoted by $\hat{g}$, i.e.

$$\hat{g}(u) := \int_{\mathbb{R}} \mathrm{e}^{iux} g(x)dx, \qquad \forall u \in \mathbb{R},$$

Further on we use the notation $\hat{f}(u + iR) := \hat{g}(u)$ for all real numbers u and the damping factor R. In the next two propositions we present the formulas for the price and the delta of an option written in terms of the characteristic function of the underlying driving process and the Fourier transform of the dampened payoff function. The proofs of both propositions can be found in [11].

Proposition 2.1 (Option price) *Suppose there is a damping factor $R \neq 0$ such that*

$$\begin{cases} g \in L^1(\mathbb{R}) \text{ and} \\ \int_{|z|\geq 1} \mathrm{e}^{Rz} \tilde{\ell}(dz) < \infty. \end{cases} \tag{2.5}$$

Moreover assume that

$$u \mapsto \hat{f}(u + iR)\widetilde{\Phi}_{T-t}(-u - iR) \in L^1(\mathbb{R}). \tag{2.6}$$

Then the price at time t of the European option with payoff $F(S_T)$ equals

$$\begin{aligned} P(t, S_t) &= \mathrm{e}^{-r(T-t)} \widetilde{\mathbb{E}}[F(S_T)|\mathcal{F}_t] \\ &= \frac{\mathrm{e}^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \hat{f}(u + iR)\widetilde{\Phi}_{T-t}(-u - iR) S_t^{R-iu} du. \end{aligned} \tag{2.7}$$

The delta of the option price is defined as

$$\Delta_t = \Delta(t, S_t) = \frac{\partial P}{\partial S_t}(t, S_t).$$

From the price formula (2.7) the following Fourier transform formula for the delta can be derived.

Proposition 2.2 (Delta) *Let the damping factor $R \neq 0$ satisfy conditions* (2.5). *In addition assume*

$$u \mapsto (1 + |u|)\hat{f}(u + iR)\widetilde{\Phi}_{T-t}(-u - iR) \in L^1(\mathbb{R}). \tag{2.8}$$

Then the delta of a European option with payoff function f at time t is given by

$$\Delta_t = \frac{\mathrm{e}^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} (R - iu)\hat{f}(u + iR)\widetilde{\Phi}_{T-t}(-u - iR) S_t^{R-1-iu} du. \tag{2.9}$$

In case the discounted stock price process is a martingale, both quadratic hedging strategies, the mean-variance hedging (MVH) and the risk-minimization (RM) are related to the Galtchouk-Kunita-Watanabe (GKW) decomposition, see [14]. We introduce this decomposition in the following definition.

Definition 2.3 Let $\hat{S}$ be a local martingale. An $\mathcal{F}_T$-measurable and square integrable random variable H_T possesses the *GKW-decomposition* if there exist a constant H_0, a predictable process ξ for which we can determine the stochastic integral with respect to $\hat{S}$, and a square integrable $\widetilde{\mathbb{P}}$-martingale $\mathcal{L}$ with $\mathcal{L}_0 = 0$, such that $\mathcal{L}$ is $\widetilde{\mathbb{P}}$-orthogonal to $\hat{S}$ and

$$H_T = H_0 + \int_0^T \xi_t d\hat{S}_t + \mathcal{L}_T\,, \qquad \widetilde{\mathbb{P}}\text{-a.s.}$$

The quadratic hedging strategies are determined by the process ξ appearing in the GKW-decomposition of the discounted contingent claim $\mathrm{e}^{-rT}F(S_T)$. The process ξ indicates the number of discounted risky assets to hold in the portfolio in both, the MVH and the RM strategies. From now on we refer to this number as the *optimal number*. The amount invested in the riskless asset is different in both strategies and is determined by the self-financing property for the MVH strategy and by the replicating condition for the RM strategy. The following result is taken from Proposition 7 in [28]. We use it later in our analysis for the robustness study of the quadratic hedging strategies.

Proposition 2.4 (MVH and RM strategy, the martingale case) *Consider a European option written on an exponential Lévy process $S = S_0 e^L$. Assume the payoff function f and the driving Lévy process L allow the pricing formula* (2.7) *and the delta formula* (2.9) *with damping factor R, i.e. assume* (2.5) *and* (2.8). *Moreover assume that the damping factor R satisfies*

$$\int_{|z|\geq 1} e^{2(z\vee Rz)}\tilde{\ell}(dz) < \infty. \tag{2.10}$$

Then the position to take in the discounted underlying at time t in a quadratic hedging strategy is given by

$$\xi(t, S_{t-}) = \frac{b^2 \Delta(t, S_{t-}) + \frac{1}{S_{t-}} \int_{\mathbb{R}_0} (e^z - 1)[P(t, S_{t-}e^z) - P(t, S_{t-})]\tilde{\ell}(dz)}{b^2 + \int_{\mathbb{R}_0} (e^z - 1)^2 \tilde{\ell}(dz)}, \tag{2.11}$$

where P and Δ are given by (2.7) *and* (2.9), *respectively.*

The quadratic hedging approaches have been extended to the semimartingale case, we present the results in the following subsection.

2.3 Hedging in the exponential Lévy setting: the semimartingale case

In this section we work under the measure $\mathbb{P}$ which is not necessarily a martingale measure. In other words the discounted stock price process $\hat{S}$ is a semimartingale. $\hat{S}$ is supposed to be a special semimartingale, such that it possesses the unique Doob-Meyer decomposition, $\hat{S} = S_0 + M + A$, where S_0 is finite-valued and $\mathcal{F}_0$-measurable, M is a local martingale with $M_0 = 0$, and A is a predictable, finite variation process with $A_0 = 0$. We denote by $L(\hat{S})$ the class of predictable processes for which we can determine the stochastic integral with respect to $\hat{S}$. We define the space Ξ by

$$\Xi := \Big\{ \mathcal{X} \in L(\hat{S}) \,\Big|\, \mathbb{E}\Big[\int_0^T \mathcal{X}_s^2 d\langle M\rangle_s + \Big(\int_0^T |\mathcal{X}_s dA_s| \Big)^2 \Big] < \infty \Big\}.$$

With local risk-minimization (LRM), the strategies replicate the payoff at maturity, the cost process is a martingale (which means that the strategy is mean self-financing) and this cost process is orthogonal to the martingale part M. Finding a LRM strategy boils down to an extension of the GKW-decomposition to the semimartingale setting, known as the Föllmer-Schweizer (FS) decomposition. In the following we present the definition of the FS-decomposition.

Definition 2.5 Let $\hat{S}$ be a special semimartingale with Doob-Meyer decomposition $\hat{S} = S_0 + M + A$. An $\mathcal{F}_T$-measurable and square integrable random variable H_T admits a *FS-decomposition* if there exist a constant H_0, an $\hat{S}$-integrable process $\mathcal{X} \in \Xi$, and a square integrable martingale N with $N_0 = 0$, such that N is orthogonal to M and

$$H_T = H_0 + \int_0^T \mathcal{X}_t d\hat{S}_t + N_T\,, \qquad \mathbb{P}\text{-a.s.}$$

The LRM strategy is determined by taking $\mathcal{X}$ discounted risky assets in the hedging portfolio, where $\mathcal{X}$ is computed from the FS-decomposition of the discounted contingent claim by using predictable quadratic covariations under the world measure $\mathbb{P}$. The amount invested in the riskless asset at $t \in [0, T]$ can be obtained from the fact that the cost process in the non self-financing strategy equals $H_0 + N$. The existence of the FS-decomposition has been studied by many authors, see, e.g., [7] and [19]. In particular, it was shown that the decomposition exists in the case of exponential Lévy models. From the general formulas in [7] it is easy to derive that the LRM hedging number can be expressed as

$$\mathcal{X}(t, S_{t-}) = \frac{b^2 \Delta(t, S_{t-}) + \frac{1}{S_{t-}} \int_{\mathbb{R}_0} (e^z - 1)[P(t, S_{t-}e^z) - P(t, S_{t-})]\ell(dz)}{b^2 + \int_{\mathbb{R}_0} (e^z - 1)^2 \ell(dz)}, \tag{2.12}$$

in case the setting is observed under the historical measure $\mathbb{P}$ and where the prices P and the delta Δ are determined w.r.t. *the minimal martingale measure (MMM)*. The difference between the latter formula and (2.11) is that the Lévy measure ℓ is the original Lévy measure under $\mathbb{P}$ in contrast to $\widetilde{\ell}$ in formula (2.11) which is a Lévy measure under $\widetilde{\mathbb{P}}$. Moreover the prices and delta are specifically computed under the MMM.

Assume the contingent claim $F(S_T)$ is written as a function of the stock price S, with $F:(0,\infty)\to\mathbb{R}$ and satisfying the integral form

$$F(s)=\int_{\mathbb{C}} s^z \Pi(dz), \tag{2.13}$$

for some finite complex measure Π on a strip $\{z\in\mathbb{C}: B'\leq \mathrm{Re}(z)\leq B\}$, where $B,B'\in\mathbb{R}$ are chosen such that $\mathbb{E}[\mathrm{e}^{2B'L_1}]<\infty$ and $\mathbb{E}[\mathrm{e}^{2BL_1}]<\infty$. In [17] it was shown that several familiar payoff functions satisfy this integral representation. Moreover the strip on which Π is defined generally reduces to a single line. This means that B' and B both equal a number R, which plays again the role of a damping factor. Combining the conditions and including the assumed existence of the second moment of the stock price process we assume that the damping factor $R\neq 0$ guarantees

$$\begin{cases} g\in L^1(\mathbb{R}) \text{ and} \\ \int_{|z|\geq 1} \mathrm{e}^{2(z\vee Rz)}\ell(dz)<\infty. \end{cases} \tag{2.14}$$

When the complex measure Π exists, it equals

$$\Pi(dz)=\frac{1}{2\pi i}1_{\{R+i\mathbb{R}\}}(z)\hat{f}(iz)dz. \tag{2.15}$$

Note that the complex measure Π being finite is equivalent to the function $\hat{f}(\cdot+iR)$ being integrable. In the following proposition we present explicit formulas for the coefficients of the FS-decomposition for European options in terms of the cumulant generating function of the Lévy process L_t at time $t\in[0,T]$. The cumulant generating function κ_1 is defined in (2.4), from now on we drop the index 1 and let $\kappa=\kappa_1$. The following results and their proofs are presented in Lemma 3.3 and Proposition 3.1 in [17].

Proposition 2.6 (LRM strategy, explicit formulas for the FS-decomposition) *Any discounted contingent claim $\hat{H}_T=\mathrm{e}^{-rT}F(S_T)$, with F as described in* (2.13) *admits a Föllmer-Schweizer decomposition $\hat{H}_T=\hat{H}_0+\int_0^T \mathcal{X}_t d\hat{S}_t+N_T$. The processes $\hat{H}$, $\mathcal{X}$, and N are real-valued and given by*

$$\hat{H}_t=\mathrm{e}^{-rt}\int_{\mathbb{C}} \mathrm{e}^{\eta(z)(T-t)}S_t^z\Pi(dz), \tag{2.16}$$

$$\mathcal{X}_t=\int_{\mathbb{C}}\mu(z)\mathrm{e}^{\eta(z)(T-t)}S_{t-}^{z-1}\Pi(dz), \tag{2.17}$$

$$N_t=\hat{H}_t-\hat{H}_0-\int_0^t \mathcal{X}_u d\hat{S}_u,$$

where the functions μ and η are defined as

$$\mu(z)=\frac{\kappa(z+1)-\kappa(z)-\kappa(1)}{\kappa(2)-2\kappa(1)} \text{ and}$$
$$\eta(z)=\kappa(z)-\mu(z)(\kappa(1)-r)-r.$$

The determination of the LRM hedging strategy is related to the MMM (see [27] for more details). In particular, the process $\hat{H}$ equals the conditional expectation of the discounted payoff under the MMM. Let $\widetilde{\Phi}^{\gamma_0}_{T-t}$ denote the characteristic function of the Lévy process under the MMM (later in Section 3.3 we explain the choice of the notation). One can verify, by expression (2.15), that equations (2.16) and (2.17) are equivalent to

$$\hat{H}_t=\frac{\mathrm{e}^{-rT}}{2\pi}\int_{\mathbb{R}}\hat{f}(u+iR)\widetilde{\Phi}^{\gamma_0}_{T-t}(-u-iR)S_t^{R-iu}du,$$

$$\mathcal{X}_t = \frac{\mathrm{e}^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \mu(R-iu)\hat{f}(u+iR)\widetilde{\Phi}^{\gamma_0}_{T-t}(-u-iR)S_{t-}^{R-iu-1}du, \tag{2.18}$$

provided that conditions (2.5) hold for the Lévy measure w.r.t. the MMM, conditions (2.14) are valid and $\hat{f}(\cdot + iR) \in L^1(\mathbb{R})$.

The LRM strategy has the disadvantage of being harder to interpret economically compared to the MVH strategy. Since the MVH is self-financing, the optimal MVH number of discounted assets equals the non-discounted number which is not necessarily optimal, as shown in [29]. On the other hand, the LRM strategy is easy to find once we know the FS-decomposition, while finding the MVH strategy is more complicated. In the case the price process is modelled by exponential Lévy models the mean-variance tradeoff process is deterministic and thus the minimal martingale measure and variance-optimal martingale measure coincide (see [25]). Therefore the MVH hedging number can be obtained from the LRM hedging number as we present in the following proposition, see [17].

Proposition 2.7 (MVH strategy, the semimartingale case) *Let the discounted contingent claim equal $\hat{H}_T = \mathrm{e}^{-rT}F(S_T)$, with F as described in* (2.13)*. The variance-optimal initial capital V_0 and the variance-optimal hedging strategy ϕ are given by*

$$V_0 = \hat{H}_0 \qquad \text{and} \qquad \phi_t = \mathcal{X}_t + \frac{\lambda}{\hat{S}_{t-}}(\hat{H}_{t-} - V_0 - \hat{G}_{t-}(\phi)), \tag{2.19}$$

where the processes $\hat{H}$ and $\mathcal{X}$ are defined in Proposition 2.6, $\hat{G}(\phi)$ is the cumulative gain process corresponding to the strategy ϕ, i.e. $\hat{G}_t(\phi) = \int_0^t \phi_s d\hat{S}_s$ and the constant λ is given by

$$\lambda = \frac{\kappa(1) - r}{\kappa(2) - 2\kappa(1)}.$$

2.4 The approximating exponential Lévy setting

In this subsection we introduce an approximating Lévy process L^ε, for $0 < \varepsilon < 1$, which we obtain by truncating the jumps with absolute size smaller than ε and possibly replacing them by an independent Brownian motion which is appropriately scaled. Recalling the Lévy-Ito decomposition (2.1), we define

$$L_t^\varepsilon = at + bW_t + Z_t + \widetilde{Z}_t^\varepsilon + s(\varepsilon)\widetilde{W}_t, \qquad \forall \varepsilon \in (0,1) \tag{2.20}$$

where $\widetilde{W}$ is a Brownian motion independent of L and the *scaling factor* $s(\varepsilon)$ vanishes when ε goes to 0. Moreover, it has to satisfy $0 \leq s(\varepsilon) \leq s(1) < \infty$ and

$$s^2(\varepsilon) + \int_{|z|\geq\varepsilon} (\mathrm{e}^z - 1)^2 \ell(dz) \geq K > 0, \qquad \forall \varepsilon \in (0,1). \tag{2.21}$$

Paragraph 2.4.1 is devoted to a discussion about some interesting choices for $s(\varepsilon)$. From now on, we assume that the filtration $\mathbb{F}$ is enlarged with the information of the Brownian motion $\widetilde{W}$ and denote the new filtration by $\widetilde{\mathbb{F}}$. Besides, the state of absence of arbitrage is preserved.
It is clear that the process L^ε has the Lévy characteristic triplet $(a, b^2 + s^2(\varepsilon), 1_{|.|\geq\varepsilon}\ell)$ under the measure $\mathbb{P}$. Let $\sigma(\varepsilon)$ be the standard deviation of the jumps of L with size smaller than ε, i.e.

$$\sigma^2(\varepsilon) := \int_{|z|<\varepsilon} z^2 \ell(dz). \tag{2.22}$$

$\sigma(\varepsilon)$ clearly vanishes when ε goes to zero and it will turn out to be an interesting choice for the scaling factor $s(\varepsilon)$ (see Paragraph 2.4.1). Moreover it plays an important role in the robustness study. More specifically, it has an impact on the convergence rates that are determined in this paper.

2.4.1 Choices for the scaling factor

We present in this section different possible choices for the scaling factor $s(\varepsilon)$ introduced in (2.20). Notice that the assumption that $s(\varepsilon)$ vanishes when ε goes to 0 implies the a.s. convergence of L^ε to L (see Lemma 2.8 below). One may consider an approximating Lévy process in which the small jumps are truncated, resulting in $\sigma(\varepsilon)$ being zero for all $0 < \varepsilon < 1$. If one prefers to keep the same variance in the approximating Lévy process as in the original one, then an interesting choice would be determined by

$$s^2(\varepsilon) = \sigma^2(\varepsilon) = \int_{|z|<\varepsilon} z^2 \ell(dz)\,.$$

This latter choice is motivated in the paper [2], in which the authors showed that the compensated small jumps of a Lévy process behaves very similar in distribution to a Brownian motion scaled with the standard deviation of the small jumps. Another choice would be to keep the same variance in the approximating price process S^ε as in the original process S. In this case (see equation (2.26) for $\mathcal{R} = 1$) we put

$$s^2(\varepsilon) = \int_{|z|<\varepsilon} (\mathrm{e}^z - 1)^2 \ell(dz)\,.$$

Notice that considering a first order Taylor approximation at zero of the function $\mathrm{e}^z - 1$ in the integrand gives $\int_{|z|<\varepsilon}(\mathrm{e}^z-1)^2\ell(dz) \simeq \sigma^2(\varepsilon)$, resulting again in the choice of $s(\varepsilon)$ being the standard deviation of the small jumps.

In the book [8], it was shown that considering $s(\varepsilon) = \sigma(\varepsilon)$ gives better convergence rates, compared to the choice $s(\varepsilon) = 0$, when convergence is studied in the weak sense. In Sections 3 and 4, the convergence rate that we derived for the robustness of quadratic hedging strategies is expressed in terms of

$$\widetilde{s}(\varepsilon) := \max(s(\varepsilon), \sigma(\varepsilon))\,. \tag{2.23}$$

Thus choosing any $s(\varepsilon) \leq \sigma(\varepsilon)$, $\varepsilon \in (0,1)$, including the choice $s(\varepsilon) = 0$, gives the same convergence rate $\sigma(\varepsilon)$. Though we point out here that the appearance of a Brownian motion in the (approximating) Lévy process can lead to more convenient properties. Indeed such a Lévy process has a smooth density (see Lemma 5.2 and the discussion thereafter). Notice that by adding an independent Brownian motion in L^ε, we have to enlarge the filtration $\mathbb{F}$ with the information of $\widetilde{W}$. In this context we mention the paper [9] in which the authors investigated the role of the filtration corresponding to the approximation. They chose an approximation in which the factor b in (2.20) is rescaled, to obtain an approximation that has the same variance as the original process and that allows to work under the same filtration $\mathbb{F}$.

2.4.2 Robustness results for the Lévy processes and the stock price processes

The following result, which is an obvious extension of Proposition 2.2 in [4], focuses on the convergence of the approximating Lévy process to the original process for ε converging to zero.

Lemma 2.8 (Robustness Lévy process) *Let the processes L and L^ε be defined as in equation (2.1) and (2.20), respectively. Then, for every $t \in [0,T]$,*

$$\lim_{\varepsilon\to 0} L_t^\varepsilon = L_t, \qquad \mathbb{P}\text{-}a.s.$$

Moreover, if we assume that L admits a second moment, the limit above also holds in $L^2(\Omega,\mathcal{F},\mathbb{P})$, with for all t in $[0,T]$

$$\mathbb{E}[|L_t^\varepsilon - L_t|^2] \leq K(T)\tilde{s}^2(\varepsilon),$$

where $K(T)$ is a constant depending on T and $\tilde{s}(\varepsilon)$ is defined in (2.23).

From now on we assume that the Lévy process admits a second moment. Based on the approximating Lévy process, we consider the stock price process S^ε and its corresponding discounted price process $\hat{S}^\varepsilon$ defined respectively by

$$\begin{aligned} S^\varepsilon &:= S_0 \mathrm{e}^{L^\varepsilon} \text{ and} \\ \hat{S}_t^\varepsilon &:= \mathrm{e}^{-rt} S_t^\varepsilon = S_0 \mathrm{e}^{-rt+L_t^\varepsilon}, \quad \forall t \in [0,T]. \end{aligned} \tag{2.24}$$

We show the convergence of complex powers of the approximating stock price process to the underlying stock process which follows as a consequence of Lemma 2.8.

Lemma 2.9 (Robustness and boundedness of complex powers of the stock price process) *Observe the stock price processes $S = S_0\mathrm{e}^L$ and $S^\varepsilon = S_0 \mathrm{e}^{L^\varepsilon}$ where the Lévy processes L and L^ε are defined in (2.1) and (2.20), respectively. Then for any fixed time $t \in [0,T]$ and complex number $\zeta = \mathcal{R} + i\mathcal{I}$, we have*

$$\lim_{\varepsilon \to 0} (S_t^\varepsilon)^\zeta = S_t^\zeta, \qquad \mathbb{P}\text{-}a.s.$$

Assume $\int_{|z|\geq 1} \mathrm{e}^{2\mathcal{R}z}\ell(dz) < \infty$. Then for all $t \in [0,T]$ it holds that

$$\max\left(\mathbb{E}[|S_t^\zeta|^2], \mathbb{E}[|(S_t^\varepsilon)^\zeta|^2]\right) \leq K'(T,\mathcal{R}), \tag{2.25}$$

where $K'(T,\mathcal{R})$ is a positive constant depending on T and $\mathcal{R}$. Moreover, it turns out that for all t in $[0,T]$

$$\mathbb{E}\big[|(S_t^\varepsilon)^\zeta - S_t^\zeta|\big] \leq K(T,\mathcal{R})(1+|\mathcal{I}|)\tilde{s}(\varepsilon),$$

where $K(T,\mathcal{R})$ is a positive constant depending on T and $\mathcal{R}$.

Proof The $\mathbb{P}$-a.s. convergence follows immediately from Lemma 2.8. Let $U = S^\mathcal{R}$ and $U^\varepsilon = (S^\varepsilon)^\mathcal{R}$. From the Itô-formula we get the following SDE's for U_t and U_t^ε for any t in $[0,T]$

$$U_t = U_0 + \int_0^t U_{s-} a_1 ds + \int_0^t U_{s-} b\mathcal{R} dW_s + \int_0^t \int_{\mathbb{R}_0} U_{s-}(\mathrm{e}^{\mathcal{R}z} - 1)\widetilde{N}(ds,dz), \tag{2.26}$$

where $U_0 = S_0^\mathcal{R}$, $\widetilde{N}$ is the jump measure of the Lévy process L, and $a_1 = a\mathcal{R} + \frac{b^2\mathcal{R}^2}{2} + \int_{\mathbb{R}_0}(\mathrm{e}^{\mathcal{R}z} - 1 - \mathcal{R}z1_{|z|<1})\ell(dz)$ and

$$U_t^\varepsilon = U_0^\varepsilon + \int_0^t U_{s-}^\varepsilon a_1^\varepsilon ds + \int_0^t U_{s-}^\varepsilon b\mathcal{R} dW_s + s(\varepsilon)\int_0^t U_{s-}^\varepsilon \mathcal{R} d\widetilde{W}_s + \int_0^t \int_{|z|\geq\varepsilon} U_{s-}^\varepsilon (\mathrm{e}^{\mathcal{R}z} - 1)\widetilde{N}(ds,dz)\,,$$

where $U_0^\varepsilon = S_0^\mathcal{R}$ and $a_1^\varepsilon = a\mathcal{R} + \frac{b^2+s^2(\varepsilon)}{2}\mathcal{R}^2 + \int_{|z|\geq\varepsilon}(\mathrm{e}^{\mathcal{R}z} - 1 - \mathcal{R}z1_{|z|<1})\ell(dz)$. From Lemma 3.2 in [4] and assuming $\int_{|z|\geq 1}\mathrm{e}^{2\mathcal{R}z}\ell(dz) < \infty$, it follows there is a constant $K'(T,\mathcal{R})$, independent of ε, such that

$$\max\left(\mathbb{E}[U_t^2], \mathbb{E}[(U_t^\varepsilon)^2]\right) \leq K'(T,\mathcal{R}).$$

We have $|S_t^{\mathcal{R}+i\mathcal{I}}| = S_t^\mathcal{R}$ and a similar argument for $(S_t^\varepsilon)^\zeta$ shows statement (2.25) of the proposition. Based on properties of complex analysis, we have for $x, y \in \mathbb{R}$ that

$$|\mathrm{e}^{(\mathcal{R}+i\mathcal{I})x} - \mathrm{e}^{(\mathcal{R}+i\mathcal{I})y}| \leq |\mathrm{e}^{\mathcal{R}x}\cos(\mathcal{I}x) - \mathrm{e}^{\mathcal{R}y}\cos(\mathcal{I}y)| + |\mathrm{e}^{\mathcal{R}x}\sin(\mathcal{I}x) - \mathrm{e}^{\mathcal{R}y}\sin(\mathcal{I}y)|.$$

The real mean value theorem induces the existence of two numbers v and w on $L_{x,y}$, i.e. the line connecting x and y, such that

$$\begin{aligned} &|\mathrm{e}^{(\mathcal{R}+i\mathcal{I})x} - \mathrm{e}^{(\mathcal{R}+i\mathcal{I})y}| \\ &\leq |\mathcal{R}\mathrm{e}^{\mathcal{R}v}\cos(\mathcal{I}v) - \mathrm{e}^{\mathcal{R}v}\mathcal{I}\sin(\mathcal{I}v)||x-y| + |\mathcal{R}\mathrm{e}^{\mathcal{R}w}\sin(\mathcal{I}w) + \mathrm{e}^{\mathcal{R}w}\mathcal{I}\cos(\mathcal{I}w)||x-y| \\ &\leq (|\mathcal{R}| + |\mathcal{I}|)(\mathrm{e}^{\mathcal{R}v} + \mathrm{e}^{\mathcal{R}w})|x-y| \leq 2(|\mathcal{R}| + |\mathcal{I}|)\max(\mathrm{e}^{\mathcal{R}v}, \mathrm{e}^{\mathcal{R}w})|x-y|. \end{aligned} \tag{2.27}$$

Using this inequality for the case $(x,y)=(L_t^\varepsilon, L_t)$, we know there exists a random variable X_t^ε on $L_{L_t^\varepsilon, L_t}$ $\mathbb{P}$-a.s. such that

$$|\mathrm{e}^{(\mathcal{R}+i\mathcal{I})L_t^\varepsilon}-\mathrm{e}^{(\mathcal{R}+i\mathcal{I})L_t}|\le 2(|\mathcal{R}|+|\mathcal{I}|)\mathrm{e}^{\mathcal{R}X_t^\varepsilon}|L_t^\varepsilon-L_t|.$$

However for any $X_t^\varepsilon\in L_{L_t^\varepsilon,L_t}$, we know from (2.25) that $\mathbb{E}[\mathrm{e}^{2\mathcal{R}X_t^\varepsilon}]\le K_1(T,\mathcal{R})$. Hence applying the Cauchy-Schwarz inequality and Lemma 2.8, we get

$$\begin{aligned}\mathbb{E}\big[|(S_t^\varepsilon)^{\mathcal{R}+i\mathcal{I}}-S_t^{\mathcal{R}+i\mathcal{I}}|\big]&=\mathbb{E}[S_0^R|\mathrm{e}^{(\mathcal{R}+i\mathcal{I})L_t^\varepsilon}-\mathrm{e}^{(\mathcal{R}+i\mathcal{I})L_t}|]\\&\le K_2(T,\mathcal{R})(1+|\mathcal{I}|)(\mathbb{E}[|L_t^\varepsilon-L_t|^2])^{\frac{1}{2}}\\&\le K(T,\mathcal{R})(1+|\mathcal{I}|)\tilde{s}(\varepsilon)\end{aligned}$$

and we prove the statement. □

In the previous proof we could also derive robustness results for $(S_t^\varepsilon)^\zeta$ applying the Itô-formula and the SDE form as in [4]. However the upper bound we found using this method is not convenient for our robustness study in the next sections. For this reason we use the mean value theorem.

2.4.3 Pricing and hedging in the approximating exponential Lévy setting

In case we consider a stock price model with the approximating geometric Lévy process S^ε, we rely on Propositions 2.1, 2.2, and 2.4 for the pricing and hedging of a European option with payoff $F(S_T^\varepsilon)$ under a martingale measure $\widetilde{\mathbb{P}}^\varepsilon$ which is equivalent to $\mathbb{P}$. Denote the Lévy triplet of L^ε under $\widetilde{\mathbb{P}}^\varepsilon$ by $(\tilde{a}_\varepsilon, b^2+s^2(\varepsilon), \tilde{\ell}_\varepsilon)$ and assume the following

- The parameters of L^ε satisfy

$$\tilde{a}_\varepsilon,\quad \int_{|z|<1} z^2\tilde{\ell}_\varepsilon(dz),\quad\text{and}\quad \int_{|z|\ge1}\tilde{\ell}_\varepsilon(dz)\qquad\text{is bounded in } \varepsilon\in(0,1). \tag{2.28}$$

$$\int_{|z|\ge1}\mathrm{e}^{2(z\vee Rz)}\tilde{\ell}_\varepsilon(dz)\text{ is bounded in } \varepsilon\in(0,1). \tag{2.29}$$

- The damping factor $R\neq0$ satisfies

$$\begin{cases} g\in L^1(\mathbb{R})\text{ and}\\ \int_{|z|\ge1}\mathrm{e}^{Rz}\tilde{\ell}_\varepsilon(dz)\text{ is bounded in }\varepsilon\in(0,1).\end{cases} \tag{2.30}$$

- The approximating counterparts of (2.6) and (2.8) are in force, i.e.

$$u\mapsto u^k\hat{f}(u+iR)\widetilde{\Phi}^\varepsilon_{T-t}(-u-iR)\in L^1(\mathbb{R}),\qquad\text{for } k\in\{0,1\}. \tag{2.31}$$

Under the appropriate conditions (2.30) - (2.31), the following formulas hold for the option price, the delta, and the optimal number in a quadratic hedging strategy at any time $t\in[0,T]$

$$\begin{aligned}P^\varepsilon(t,S_t^\varepsilon)&=\mathrm{e}^{-r(T-t)}\mathbb{E}^{\widetilde{\mathbb{P}}^\varepsilon}[F(S_T^\varepsilon)|\mathcal{F}_t]\\&=\frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}\hat{f}(u+iR)\widetilde{\Phi}^\varepsilon_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu}du,\end{aligned} \tag{2.32}$$

$$\Delta^\varepsilon(t,S_t^\varepsilon)=\frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}(R-iu)\hat{f}(u+iR)\widetilde{\Phi}^\varepsilon_{T-t}(-u-iR)(S_t^\varepsilon)^{R-1-iu}du,\quad\text{and} \tag{2.33}$$

$$\xi^\varepsilon(t,S_{t-}^\varepsilon)=\frac{(b^2+s^2(\varepsilon))\Delta^\varepsilon(t,S_{t-}^\varepsilon)+\frac{1}{S_{t-}^\varepsilon}\int_{\mathbb{R}_0}(\mathrm{e}^z-1)[P^\varepsilon(t,S_{t-}^\varepsilon\mathrm{e}^z)-P^\varepsilon(t,S_{t-}^\varepsilon)]\tilde{\ell}_\varepsilon(dz)}{(b^2+s^2(\varepsilon))+\int_{\mathbb{R}_0}(\mathrm{e}^z-1)^2\tilde{\ell}_\varepsilon(dz)}. \tag{2.34}$$

Notice that the integrands in (2.34) are integrated over $\mathbb{R}_0$. However the support of the Lévy measure $\tilde{\ell}_\varepsilon$ is restricted to the set $\{|z|\ge\varepsilon\}$. This indicates the truncation of the small jumps.

Remark 2.10 (Connection between the optimal number and the delta) Formula (2.34) indicates a connection between the quadratic hedge and the delta. In case there are no jumps with absolute size larger than ε, the number ξ^ε equals Δ^ε. Numerical experiments in [28] showed that the delta and quadratic hedging strategy resulted in a similar hedging error in the absence of big jumps. This can be explained by the relation between ξ^ε and Δ^ε. In fact in case there are no jumps bigger than ε, the integrals in (2.34) are zero. On the other hand, the quadratic hedge and the delta hedge revealed a different hedging performance in the presence of large jumps. Formula (2.34) shows that the hedging numbers differ. Indeed the integrals w.r.t. the Lévy measure $\tilde{\ell}_\varepsilon(dz)$ are in this case different from zero.

Now we observe the approximating process under the historical measure $\mathbb{P}$. We compute the LRM strategy as follows. Either we use the results obtained in [7] in which the LRM hedging number is computed based on the option price and its delta computed under the MMM for the approximating model

$$\mathcal{X}^\varepsilon(t, S^\varepsilon_{t-}) = \frac{(b^2 + s^2(\varepsilon))\Delta^\varepsilon(t, S^\varepsilon_{t-}) + \frac{1}{S^\varepsilon_{t-}} \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)[P^\varepsilon(t, S^\varepsilon_{t-}\mathrm{e}^z) - P^\varepsilon(t, S^\varepsilon_{t-})]\ell_\varepsilon(dz)}{(b^2 + s^2(\varepsilon)) + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 \ell_\varepsilon(dz)}, \quad (2.35)$$

where $\ell_\varepsilon(dz) = 1_{|z|\geq\varepsilon}\ell(dz)$. Or one can apply the results of [17] in terms of the cumulant generating function κ^ε of L^ε_1, see Propositions 2.6 and 2.7. The latter imply that any discounted contingent claim $\hat{H}^\varepsilon = \mathrm{e}^{-rT} F(S^\varepsilon_T)$, with F as described in (2.13) admits a FS-decomposition $\hat{H}^\varepsilon = \hat{H}^\varepsilon_0 + \int_0^T \mathcal{X}^\varepsilon_t d\hat{S}^\varepsilon_t + N^\varepsilon_T$. The processes $\hat{H}^\varepsilon$, $\mathcal{X}^\varepsilon$, and N^ε are given by

$$\begin{aligned}
\hat{H}^\varepsilon_t &= \mathrm{e}^{-rt} \int_{\mathbb{C}} \mathrm{e}^{\eta^\varepsilon(z)(T-t)} (S^\varepsilon_t)^z \Pi(dz), \\
\mathcal{X}^\varepsilon_t &= \int_{\mathbb{C}} \mu^\varepsilon(z) \mathrm{e}^{\eta^\varepsilon(z)(T-t)} (S^\varepsilon_{t-})^{z-1} \Pi(dz), \\
N^\varepsilon_t &= \hat{H}^\varepsilon_t - \hat{H}^\varepsilon_0 - \int_0^t \mathcal{X}^\varepsilon_u d\hat{S}^\varepsilon_u.
\end{aligned} \quad (2.36)$$

Herein the functions μ^ε and η^ε are defined as

$$\begin{aligned}
\mu^\varepsilon(z) &= \frac{\kappa^\varepsilon(z+1) - \kappa^\varepsilon(z) - \kappa^\varepsilon(1)}{\kappa^\varepsilon(2) - 2\kappa^\varepsilon(1)} \text{ and} \\
\eta^\varepsilon(z) &= \kappa^\varepsilon(z) - \mu^\varepsilon(z)(\kappa^\varepsilon(1) - r) - r\,.
\end{aligned} \quad (2.37)$$

On the other hand, the variance-optimal initial capital V^ε_0 and the variance-optimal hedging strategy ϕ^ε, for the discounted claim $\hat{H}^\varepsilon = \mathrm{e}^{-rT} F(S^\varepsilon_T)$, with F as described in (2.13), are given by

$$V^\varepsilon_0 = \hat{H}^\varepsilon_0 \qquad \text{and} \qquad \phi^\varepsilon_t = \mathcal{X}^\varepsilon_t + \frac{\lambda^\varepsilon}{\hat{S}^\varepsilon_{t-}} (\hat{H}^\varepsilon_{t-} - V^\varepsilon_0 - \hat{G}^\varepsilon_{t-}(\phi^\varepsilon)), \quad (2.38)$$

where the processes $\hat{H}^\varepsilon$ and $\mathcal{X}^\varepsilon$ are as defined in (2.36). $\hat{G}^\varepsilon(\phi^\varepsilon)$ is the cumulative gain process corresponding to the strategy ϕ^ε, i.e. $\hat{G}^\varepsilon_t(\phi^\varepsilon) = \int_0^t \phi^\varepsilon_s d\hat{S}^\varepsilon_s$ and the constant λ^ε is given by $\lambda^\varepsilon = (\kappa^\varepsilon(1) - r)/(\kappa^\varepsilon(2) - 2\kappa^\varepsilon(1))$.

3 Robustness of the quadratic hedging strategies, the martingale case

In this section we assume that the stock price process is modelled by an exponential Lévy process and that the market is observed under a martingale measure. Due to the market incompleteness for this type of models, we know there exist infinitely many equivalent measures under which the discounted price process is a martingale. In our study, we consider the following martingale measures: the Esscher transform (ET), the minimal entropy martingale measure (MEMM), the minimal martingale measure (MMM), and the variance-optimal martingale measure (VOMM).

Notice that these martingale measures are structure preserving. This means that a Lévy process under $\mathbb{P}$ with characteristic triplet (a, b^2, ℓ) remains a Lévy process under each of the mentioned martingale measures with characteristic triplet $(\widetilde{a}, b^2, \widetilde{\ell})$. We refer to Theorems 33.1 and 33.2 in [22] for more about measure changes in the Lévy setting.
Now we denote the equivalent martingale measure under which the market is observed by $\widetilde{\mathbb{P}}_{\Theta_0}$, where Θ_0 is a parameter changing according to each specific martingale measure. Since the discounted stock price process $\hat{S}$ is a martingale under $\widetilde{\mathbb{P}}_{\Theta_0}$, the characteristic triplet $(\tilde{a}, b^2, \tilde{\ell})$ of the Lévy process L w.r.t. $\widetilde{\mathbb{P}}_{\Theta_0}$ satisfies

$$\tilde{a} + \frac{b^2}{2} + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1 - z1_{|z|<1})\tilde{\ell}(dz) = r.$$

Note that the approximating price process can also be observed under a martingale measure denoted by $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$ and the corresponding triplet equals $(\tilde{a}_\varepsilon, b^2 + s^2(\varepsilon), \tilde{\ell}_\varepsilon)$, for all $\varepsilon \in (0,1)$. The expectation under the martingale measure $\widetilde{\mathbb{P}}_{\Theta_0}$, $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$, is denoted by $\widetilde{\mathbb{E}}_{\Theta_0}$, $\widetilde{\mathbb{E}}_{\Theta_\varepsilon}$ respectively. Equivalently the notations $\widetilde{\Phi}^{\Theta_0}$ and $\widetilde{\Phi}^{\Theta_\varepsilon}$ are used for the characteristic functions. We explicitly choose to adapt the martingale measure according to the approximation. This results into a market model where the option price, delta, and quadratic hedging formulas given by (2.32)-(2.34) can be interpreted as real option prices and hedging strategies. Whereas e.g. the author of [10] considers an approximating value for the option prices, which is not a real option price. Moreover, for the formulas (2.32)-(2.34) to be tractable numerically, we do not need the characteristic function to be explicitly known, since one can easily simulate the driving process. This allows for the use of a broad class of Lévy processes. Indeed the family of Lévy processes is very rich (see [22]). However the use in finance is restricted to a small class.

In the previous section, the robustness results of the Lévy processes and stock price processes are expressed by $\mathbb{P}$-a.s. convergence and the rates are computed w.r.t. $\mathbb{P}$. Since the world measure and the martingale measures are equivalent, the robustness results hold w.r.t. the martingale measures too. Our goal now is to prove the robustness of the optimal quadratic hedging numbers computed under any martingale measure guaranteeing convergence properties $(\mathcal{A}1)$ and $(\mathcal{A}2)$ below for the corresponding Lévy triplets. Considering the drift coefficients, we assume

$(\mathcal{A}1)$ *There exists a constant $C(\Theta_0)$ depending on Θ_0 such that*

$$|\tilde{a}_\varepsilon - \tilde{a}| \leq C(\Theta_0)\tilde{s}^2(\varepsilon), \qquad \forall \varepsilon \in (0,1).$$

On the other hand, for the Lévy measures we introduce positive functions ρ_0 and ρ_ε, for $\varepsilon \in (0,1)$, which we define as

$$\tilde{\ell}(dz) = \rho_0(z)\ell(dz) \qquad \text{and} \qquad \tilde{\ell}_\varepsilon(dz) = 1_{|z|\geq\varepsilon}\rho_\varepsilon(z)\ell(dz). \tag{3.1}$$

Moreover we impose the assumption

$(\mathcal{A}2)$ *The functions ρ_0 and ρ_ε are bounded in z over the set $\{|z|<1\}$.*
There exists a function $\gamma : \mathbb{R}_0 \to \mathbb{R}^+$ such that $|\rho_\varepsilon(z) - \rho_0(z)| \leq \gamma(z)\tilde{s}^2(\varepsilon)$ for $\varepsilon \in (0,1)$ and $z \in \mathbb{R}_0$ and $\gamma(z) \leq K$ for $z \in \{|z|<1\}$.
Moreover the following integrals, with R being the damping factor, are finite

$$\int_{|z|\geq 1} h(z)\gamma(z)\ell(dz) \text{ for } h(z) \in \{1, \mathrm{e}^{2(z\vee Rz)}\}.$$

We observe that assumption $(\mathcal{A}2)$ implies that ρ_ε is bounded uniformly in ε since

$$\rho_\varepsilon(z) \leq |\rho_\varepsilon(z) - \rho_0(z)| + \rho_0(z) \leq \gamma(z)\tilde{s}^2(1) + \rho_0(z), \quad \forall \varepsilon \in (0,1). \tag{3.2}$$

In Sections 3.1-3.4 the characteristic triplets will be specified for each of the martingale measures we consider. Moreover, we show that assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ hold. Notice that combining assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$) together with (2.2), (2.5), or (2.10), leads to properties (2.29), (2.30), or (2.31), respectively.

As we see from formula (2.11), the optimal number is a weighted sum of two terms: the sensitivity of the option price to infinitesimal stock price movements, i.e. the delta, and the average sensitivity to infinitely sized jumps. To prove the robustness of the quadratic hedging strategies we prove the robustness of the terms appearing in this formula. Thereto we consider the robustness of the characteristic function, the option price process, and the delta. Similar results were discussed in [5]. Although one should note that the results in this present paper hold under less restrictive conditions concerning the payoff function. We present the proofs in the Appendix for the sake of completeness. First we mention the robustness properties of the characteristic functions appearing in the pricing and delta formulas.

Lemma 3.1 (Robustness characteristic function) *Assume that properties* ($\mathcal{A}1$) *and* ($\mathcal{A}2$) *hold for the characteristic triplets. For any real number u and damping factor R satisfying* (2.5)*, it holds that for all $t \in [0,T]$,*

$$\lim_{\varepsilon\to 0} \widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR) = \widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR).$$

Moreover, for all $t \in [0,T]$ and $\varepsilon \in (0,1)$, it holds that

$$\begin{aligned}&|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR) - \widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)| \\ &\quad \le K(T,R,\Theta_0)(1+|u|+u^2)\max(|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)|,|\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)|)\tilde{s}^2(\varepsilon),\end{aligned} \tag{3.3}$$

where $K(T,R,\Theta_0)$ is a positive constant depending on T, R, and a parameter Θ_0 corresponding to the specific martingale measure.

Note that the authors of [8] determined a difference in the convergence rate for certain types of Lévy processes when the approximation is obtained either by truncating or by substituting the small jumps. However since we choose to adapt the martingale measures according to the approximation, we cannot obtain a similar comparison.
From the robustness result of the characteristic function, we deduce the robustness of the option price process.

Proposition 3.2 (Robustness option price) *Assume* (2.5)*,* (2.6)*,* (2.31)*,* ($\mathcal{A}1$)*, and* ($\mathcal{A}2$)*. Let $\widetilde{\Psi}$ be a function satisfying*

$$|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)| \le \widetilde{\Psi}_{T-t}(u;R) \quad \text{and} \quad u \mapsto |\hat{f}(u+iR)|\widetilde{\Psi}_{T-t}(u;R) \in L^1(\mathbb{R}), \tag{3.4}$$

then we have for all $t \in [0,T]$ that

$$\lim_{\varepsilon\to 0} P^\varepsilon(t,S^\varepsilon_t) = P(t,S_t), \qquad \mathbb{P}\text{-a.s.}$$

Moreover if there is a function $\widehat{\Psi}$ such that

$$\begin{cases}\max(|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)|,|\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)|) \le \widehat{\Psi}_{T-t}(u;R), \text{ and} \\ u \mapsto (1+|u|+u^2)|\hat{f}(u+iR)|\widehat{\Psi}_{T-t}(u;R) \in L^1(\mathbb{R}),\end{cases} \tag{3.5}$$

then it holds for all $t \in [0,T]$ and $\varepsilon \in (0,1)$ that,

$$\mathbb{E}[|P^\varepsilon(t,S^\varepsilon_t) - P(t,S_t)|] \le C(T,r,R,\Theta_0)\tilde{s}(\varepsilon),$$

where $C(T,r,R,\Theta_0)$ is a positive constant depending on T, r, R, and a parameter Θ_0 corresponding to the specific martingale measure.

In the formulas determining the optimal numbers, see (2.11) and (2.34), the option price for an underlying stock with value $S_t e^z$ or $S^\varepsilon_t e^z$, $z \in \mathbb{R}_0$ appears. As a consequence of the previous proposition the following corollary can easily be deduced.

Corollary 3.3 *Under the assumptions of Proposition 3.2, it holds for all* $z \in \mathbb{R}_0$ *that*

$$\lim_{\varepsilon \to 0} P^\varepsilon(t, S_t^\varepsilon \mathrm{e}^z) = P(t, S_t \mathrm{e}^z), \qquad \mathbb{P}\text{-}a.s. \qquad \text{and}$$

$$\mathbb{E}[|P^\varepsilon(t, S_t^\varepsilon \mathrm{e}^z) - P(t, S_t \mathrm{e}^z)|] \leq C(T, r, R, \Theta_0)\mathrm{e}^{zR}\tilde{s}(\varepsilon).$$

Next we present the robustness results for the delta.

Proposition 3.4 (Robustness delta) *Assume that conditions* (2.5)*,* (2.8)*,* (2.31)*,* $(\mathcal{A}1)$*, and* $(\mathcal{A}2)$ *hold. Let* $\widetilde{\Psi}$ *be a function satisfying*

$$|\widetilde{\Phi}_{T-t}^{\Theta_\varepsilon}(-u - iR)| \leq \widetilde{\Psi}_{T-t}(u; R) \quad \text{and} \quad u \mapsto (1 + |u|)|\hat{f}(u + iR)|\widetilde{\Psi}_{T-t}(u; R) \in L^1(\mathbb{R}), \tag{3.6}$$

then we have for all $t \in [0, T]$ *that*

$$\lim_{\varepsilon \to 0} \Delta_t^\varepsilon = \Delta_t, \qquad \mathbb{P}\text{-}a.s.$$

Moreover the existence of a function $\widehat{\Psi}$ *guaranteeing*

$$\begin{cases} \max(|\widetilde{\Phi}_{T-t}^{\Theta_\varepsilon}(-u - iR)|, |\widetilde{\Phi}_{T-t}^{\Theta_0}(-u - iR)|) \leq \widehat{\Psi}_{T-t}(u; R), \text{ and} \\ u \mapsto (1 + |u| + u^2 + |u|^3)|\hat{f}(u + iR)|\widehat{\Psi}_{T-t}(u; R) \in L^1(\mathbb{R}), \end{cases} \tag{3.7}$$

implies that for all $\varepsilon \in (0, 1)$

$$\mathbb{E}\big[|\Delta_t^\varepsilon - \Delta_t|\big] \leq C(T, r, R, \Theta_0)\tilde{s}(\varepsilon),$$

where $C(T, r, R, \Theta_0)$ *is a positive constant depending on* T*,* r*,* R*, and a parameter* Θ_0 *corresponding to the specific martingale measure.*

In the following proposition we collect the previous robustness results to prove the robustness of the quadratic hedging strategy in case the market price is modelled by an exponential Lévy process and observed under a martingale measure.

Proposition 3.5 (Robustness optimal number) *Assume* (2.5)*,* (2.8)*,* (2.10)*,* (2.31)*,* $(\mathcal{A}1)$*, and* $(\mathcal{A}2)$*, in order that* ξ *and* ξ^ε *are given by* (2.11) *and* (2.34)*. Moreover assume there is a function* $\widetilde{\Psi}$ *satisfying* (3.6)*. Then it turns out that for all* t *in* $[0, T]$

$$\lim_{\varepsilon \to 0} \xi_t^\varepsilon = \xi_t, \qquad \mathbb{P}\text{-}a.s.$$

Proof Recall the expression of the optimal number (2.34). For the integral in the denominator we know

$$\int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 \tilde{\ell}_\varepsilon(dz) = \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 1_{|z| \geq \varepsilon} \rho_\varepsilon(z) \ell(dz)$$

and the function $(\mathrm{e}^z - 1)^2 1_{|z| \geq \varepsilon} \rho_\varepsilon(z)$ is bounded uniformly in ε (see (3.2)) by

$$(\mathrm{e}^z - 1)^2 \{\gamma(z)\tilde{s}^2(1) + \rho_0(z)\},$$

which is integrable w.r.t. ℓ using (2.2), (2.10) and $(\mathcal{A}2)$. Therefore the dominated convergence theorem implies that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 \tilde{\ell}_\varepsilon(dz) = \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 \tilde{\ell}(dz).$$

Consider the integral in the numerator in (2.34). Using price formula (2.32), definition (3.1), condition (3.6), and the process L^* defined by

$$L_t^* := |a|t + |b||W_t| + s(1)|\widetilde{W}_t| + \int_0^t \int_{|z| \geq 1} |z| N(ds, dz) + \int_0^t \int_{|z| < 1} |z| \widetilde{N}(ds, dz), \tag{3.8}$$

which is clearly an upper bound for the Lévy process L^ε, $\mathbb{P}$-a.s. for all $\varepsilon \in (0,1)$, we get that

$$
\begin{aligned}
&\left|(\mathrm{e}^z-1)[P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})]1_{|z|\geq\varepsilon}\rho_\varepsilon(z)\right| \\
&\leq |\mathrm{e}^z-1||\rho_\varepsilon(z)|\frac{\mathrm{e}^{-r(T-t)}}{2\pi}\left|\int_{\mathbb{R}}\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)\left[(S^\varepsilon_{t-}\mathrm{e}^z)^{R-iu}-(S^\varepsilon_{t-})^{R-iu}\right]du\right| \\
&\leq |\mathrm{e}^z-1||\rho_\varepsilon(z)|\frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}|\hat{f}(u+iR)|\left|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)\right|\left|(S^\varepsilon_{t-})^{R-iu}\right||\mathrm{e}^{z(R-iu)}-1|du \\
&\leq K_1(T,r,R,\Theta_0)|\mathrm{e}^z-1||\rho_\varepsilon(z)|(S^\varepsilon_{t-})^R \\
&\qquad\times\int_{\mathbb{R}}|\hat{f}(u+iR)|\widetilde{\Psi}_{T-t}(u;R)\{(\mathrm{e}^{Rz}+1)1_{|z|\geq 1}+2(|R|+|u|)\mathrm{e}^{|R||z|}|z|1_{|z|<1}\}du \\
&\leq K_2(T,r,R,\Theta_0)\mathrm{e}^{RL^*_{t-}}\{(\mathrm{e}^z+1)(\mathrm{e}^{Rz}+1)1_{|z|\geq 1}+z^2 1_{|z|<1}\}(\gamma(z)\tilde{s}^2(1)+\rho_0(z)) \\
&\qquad\times\int_{\mathbb{R}}(1+|u|)|\hat{f}(u+iR)|\widetilde{\Psi}_{T-t}(u;R)du, \qquad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

In the last line the integral with respect to u is finite by assumption (3.6). By (2.2), (2.10), and ($\mathcal{A}2$) the latter expression is integrable in z with respect to the Lévy measure ℓ. Thus we are allowed to take the limit inside the integral in the numerator of expression (2.34). Using Lemma 2.9, Propositions 3.2 and 3.4, and assumption ($\mathcal{A}2$), we prove the statement. □

In the following proposition we compute a convergence rate for the optimal number in the quadratic hedging strategy.

Proposition 3.6 (Convergence rate optimal number) *Assume* (2.5), (2.8), (2.10), (2.31), ($\mathcal{A}1$), *and* ($\mathcal{A}2$), *such that* ξ *and* ξ^ε *are given by* (2.11) *and* (2.34), *respectively. In addition impose the existence of a function* $\widehat{\Psi}$ *satisfying conditions* (3.7). *Then we have for all* $t \in [0,T]$ *and for all* $\varepsilon \in (0,1)$ *that*

$$\mathbb{E}[|\xi^\varepsilon_t-\xi_t|] \leq C(T,r,R,\Theta_0)\tilde{s}(\varepsilon),$$

where $C(T,r,R,\Theta_0)$ *is a positive constant depending on* T, r, R, *and a parameter* Θ_0 *corresponding to the specific martingale measure.*

Proof From the expressions (2.11) and (2.34) for ξ and ξ^ε, it is easily seen that the denominator of $\xi^\varepsilon_t-\xi_t$ is bounded from below by a positive constant since condition (2.21) is in force. Hence it remains to compute the convergence rate of the numerator. Grouping the terms with the factor $s^2(\varepsilon)$ and integrals over the area $|z|<\varepsilon$, we obtain that

$$
\begin{aligned}
\mathbb{E}[|\xi^\varepsilon_t-\xi_t|] &\leq K_1(T,r,R,\Theta_0)\Big(\tilde{s}^2(\varepsilon) \\
&+\mathbb{E}\Big[\Big|\Big\{b^2\Delta^\varepsilon_t+\int_{\mathbb{R}_0}(\mathrm{e}^z-1)\frac{P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})}{S^\varepsilon_{t-}}\tilde{\ell}_\varepsilon(dz)\Big\}\Big\{b^2+\int_{|z|\geq\varepsilon}(\mathrm{e}^z-1)^2\tilde{\ell}(dz)\Big\} \\
&\quad-\Big\{b^2\Delta_t+\int_{|z|\geq\varepsilon}(\mathrm{e}^z-1)\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\tilde{\ell}(dz)\Big\}\Big\{b^2+\int_{\mathbb{R}_0}(\mathrm{e}^z-1)^2\tilde{\ell}_\varepsilon(dz)\Big\}\Big|\Big]\Big)
\end{aligned}
$$

It is clear that the convergence rate is determined by the following three expressions

1. $\mathbb{E}[|\Delta^\varepsilon_t-\Delta_t|]$,
2. $\left|\int_{\mathbb{R}_0}(\mathrm{e}^z-1)^2\tilde{\ell}_\varepsilon(dz)-\int_{|z|\geq\varepsilon}(\mathrm{e}^z-1)^2\tilde{\ell}(dz)\right|$, and
3. $\int_{|z|\geq\varepsilon}|\mathrm{e}^z-1|\mathbb{E}\Big[\Big|\frac{P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})}{S^\varepsilon_{t-}}\rho_\varepsilon(z)-\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\rho_0(z)\Big|\Big]\ell(dz)$.

For the first we obtained the convergence rate in Proposition 3.4. For the second we derive, based on (2.2) and ($\mathcal{A}2$), that

$$\left|\int_{\mathbb{R}_0}(\mathrm{e}^z-1)^2\tilde{\ell}_\varepsilon(dz)-\int_{|z|\geq\varepsilon}(\mathrm{e}^z-1)^2\tilde{\ell}(dz)\right|$$

$$\leq \int_{|z|\geq\varepsilon} (\mathrm{e}^z-1)^2|\rho_\varepsilon(z)-\rho_0(z)|\ell(dz)$$
$$\leq \int_{\mathbb{R}_0} \{(\mathrm{e}^{2z}-2\mathrm{e}^z+1)1_{|z|\geq 1}+z^2\mathrm{e}^2 1_{|z|<1}\}\gamma(z)\tilde{s}^2(\varepsilon)\ell(dz)=K_2\tilde{s}^2(\varepsilon).$$

To compute the convergence rate of the third term we apply the triangle inequality

$$\mathbb{E}\Big[\Big|\frac{P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})}{S^\varepsilon_{t-}}\rho_\varepsilon(z)-\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\rho_0(z)\Big|\Big]$$
$$\leq \mathbb{E}\Big[\Big|\frac{P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})}{S^\varepsilon_{t-}}-\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\Big|\Big]|\rho_\varepsilon(z)|$$
$$+\mathbb{E}\Big[\Big|\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\Big|\Big]|\rho_\varepsilon(z)-\rho_0(z)|.$$

From the price formulas (2.7) and (2.32) it follows

$$\mathbb{E}\Big[\Big|\frac{P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})}{S^\varepsilon_{t-}}-\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\Big|\Big]$$
$$\leq \frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}\mathbb{E}\Big[\Big|\widetilde{\varPhi}^{\Theta_\varepsilon}_{T-t}(-u-iR)\left(S^\varepsilon_{t-}\right)^{R-iu-1}-\widetilde{\varPhi}^{\Theta_0}_{T-t}(-u-iR)S^{R-iu-1}_{t-}\Big|\Big]$$
$$\times|\hat{f}(u+iR)|\big|\mathrm{e}^{z(R-iu)}-1\big|du.$$

First we note that we obtain a similar result as in expression (.2) in the Appendix for the expectation in the integrand. Hence

$$\mathbb{E}\Big[\Big|\frac{P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})}{S^\varepsilon_{t-}}-\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\Big|\Big]$$
$$\leq K_3(T,r,R,\Theta_0)\int_{\mathbb{R}}(1+|u|+u^2)|\hat{f}(u+iR)|\widehat{\varPsi}_{T-t}(u;R)$$
$$\times\{(\mathrm{e}^{Rz}+1)1_{|z|\geq 1}+(1+|u|)|z|1_{|z|<1}\}du\ \ \tilde{s}(\varepsilon).$$

Second, by the integrability condition in (3.7), we get

$$\mathbb{E}\Big[\Big|\frac{P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})}{S^\varepsilon_{t-}}-\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\Big|\Big]$$
$$\leq K_4(T,r,R,\Theta_0)\{(\mathrm{e}^{Rz}+1)1_{|z|\geq 1}+|z|1_{|z|<1}\}\tilde{s}(\varepsilon).$$

In a similar way one obtains that

$$\mathbb{E}\Big[\Big|\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\Big|\Big]\leq K_5(T,r,R,\Theta_0)\{(\mathrm{e}^{Rz}+1)1_{|z|\geq 1}+|z|1_{|z|<1}\}.$$

Combining previous results leads to

$$\int_{|z|\geq\varepsilon}|\mathrm{e}^z-1|\mathbb{E}\Big[\Big|\frac{P^\varepsilon(t,S^\varepsilon_{t-}\mathrm{e}^z)-P^\varepsilon(t,S^\varepsilon_{t-})}{S^\varepsilon_{t-}}\rho_\varepsilon(z)-\frac{P(t,S_{t-}\mathrm{e}^z)-P(t,S_{t-})}{S_{t-}}\rho_0(z)\Big|\Big]\ell(dz)$$
$$\leq K_6(T,r,R,\Theta_0)\int_{\mathbb{R}_0}\{(\mathrm{e}^z+1)(\mathrm{e}^{Rz}+1)1_{|z|\geq 1}+z^2 1_{|z|<1}\}(\gamma(z)\tilde{s}^2(1)+\rho_0(z)+\gamma(z))\ell(dz)\ \tilde{s}(\varepsilon).$$

Hence the statement is proved by assumptions (2.2), (2.10), and ($\mathcal{A}2$). $\square$

Remark 3.7 (Robustness study resulting from assumptions on the changes of measure instead of on the characteristic triplets) To obtain the robustness results as discussed above we have imposed the assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$) on the characteristic triplets. This approach led to the robustness result (3.3) for the characteristic function and consequently the robustness properties as discussed in Propositions 3.2, 3.4, 3.5, and 3.6. It is possible to apply another approach, which is less interesting as we will see, based on making convergence assumptions on the equivalent martingale measures $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$ and $\widetilde{\mathbb{P}}_{\Theta_0}$. Assume

$$(\mathcal{M}1) \quad \mathbb{E}\Big[\Big|\frac{d\widetilde{\mathbb{P}}_{\Theta_\varepsilon}}{d\mathbb{P}} - \frac{d\widetilde{\mathbb{P}}_{\Theta_0}}{d\mathbb{P}}\Big|^2\Big] \leq K(T,\Theta_0)\hat{s}^2(\varepsilon), \text{ and}$$

$$(\mathcal{M}2) \quad \mathbb{E}\Big[\Big|\frac{d\widetilde{\mathbb{P}}_{\Theta_\varepsilon}}{d\mathbb{P}}\Big|^2\Big] \text{ is uniformly bounded in } \varepsilon,$$

where $\hat{s}(\varepsilon) := \max\big(s(\varepsilon), \sigma(\varepsilon), \sqrt{\varepsilon\sigma(\varepsilon)}\big)$. Then we have the following robustness result of the characteristic function as shown in the Appendix. It holds for any real number u, damping factor R, and all $t \in [0,T]$ that

$$|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR) - \widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)| \leq K(T,R,\Theta_0)(1+|u|)\hat{s}(\varepsilon). \tag{3.9}$$

Assume option price formulas (2.7) and (2.32) are in force and that ($\mathcal{M}1$) and ($\mathcal{M}2$) hold. Moreover if

$$u \mapsto (1+|u|)|\hat{f}(u+iR)| \in L^1(\mathbb{R}), \tag{3.10}$$

then it holds for all $t \in [0,T]$ and $\varepsilon \in (0,1)$ that,

$$\mathbb{E}[|P^\varepsilon(t, S^\varepsilon_t) - P(t, S_t)|] \leq C_1(T,r,R,\Theta_0)\hat{s}(\varepsilon),$$

where $C_1(T,r,R,\Theta_0)$ is a positive constant depending on T, r, R, and a parameter Θ_0 corresponding to the specific martingale measure. For the deltas given by (2.9) and (2.33) and the optimal numbers given by (2.11) and (2.34), conditions ($\mathcal{M}1$), ($\mathcal{M}2$), and the integrability property

$$u \mapsto (1+|u|+u^2)|\hat{f}(u+iR)| \in L^1(\mathbb{R}), \tag{3.11}$$

imply that for all $\varepsilon \in (0,1)$

$$\mathbb{E}\big[|\Delta^\varepsilon_t - \Delta_t|\big] \leq C_2(T,r,R,\Theta_0)\hat{s}(\varepsilon) \text{ and}$$

in case ($\mathcal{A}2$) is also in force,

$$\mathbb{E}[|\xi^\varepsilon_t - \xi_t|] \leq C_3(T,r,R,\Theta_0)\hat{s}(\varepsilon),$$

where $C_i(T,r,R,\Theta_0)$, $i \in \{2,3\}$ are positive constants depending on T, r, R, and a parameter Θ_0 corresponding to the specific martingale measure. One concludes that this approach might lead to larger convergence rates. On the other hand the integrability conditions (3.10) and (3.11) include a polynomial of a lower order than conditions (3.5) and (3.7). But the integrability properties of the characteristic function cannot be exploited anymore and popular payoff functions (such as call or put) do not satisfy (3.10) or (3.11). That is why we promote the previous approach.

We present different martingale measures in the following subsections, and show that assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$) hold for each of them. One could also verify that assumptions ($\mathcal{M}1$) and ($\mathcal{M}2$) hold true for each martingale measure. However we do not report on the computations because these are straightforward.

3.1 Esscher transform (ET)

For the definition and more details about the ET we refer to [16]. We assume that the moment generating function $M_t(\theta)$ of L_t exists for all $0 \leq t \leq T$ and $\theta \in \mathbb{R}$, which translates to the following condition

$$\int_{|z|\geq 1} \mathrm{e}^{\theta z}\ell(dz) < \infty, \qquad \forall \theta \in \mathbb{R}. \tag{3.12}$$

We define the measure $\widetilde{\mathbb{P}}_\theta \sim \mathbb{P}$, for all $\theta \in \mathbb{R}$, by

$$\frac{d\widetilde{\mathbb{P}}_\theta}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\exp(\theta L_t)}{M_t(\theta)} = \exp(X_t),$$

where

$$X_t = \theta b W_t - \frac{1}{2}b^2\theta^2 t + \theta \int_0^t\int_{\mathbb{R}_0} z\widetilde{N}(ds,dz) - t\int_{\mathbb{R}_0}(\mathrm{e}^{\theta z} - 1 - \theta z)\ell(dz).$$

We denote by θ_0 the parameter value for which $\widetilde{\mathbb{P}}_{\theta_0}$ is a martingale measure and call this measure the ET. In [16] the existence and uniqueness of this parameter $\theta_0 \in \mathbb{R}$ were shown. The Lévy triplet of L with respect to the ET $\widetilde{\mathbb{P}}_{\theta_0}$ is denoted by $(\tilde{a}, b^2, \tilde{\ell})$, where

$$\tilde{a} = a + b^2\theta_0 + \int_{|z|<1} z(\mathrm{e}^{\theta_0 z} - 1)\ell(dz) \quad \text{and} \quad \tilde{\ell}(dz) = \mathrm{e}^{\theta_0 z}\ell(dz). \tag{3.13}$$

We consider the ET $\widetilde{\mathbb{P}}_{\theta_\varepsilon}$ for the approximating model. In this case we have

$$\frac{d\widetilde{\mathbb{P}}_{\theta_\varepsilon}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\exp(\theta_\varepsilon L_t^\varepsilon)}{M_t^\varepsilon(\theta_\varepsilon)} = \exp(X_t^\varepsilon),$$

where

$$\begin{aligned} X_t^\varepsilon = \theta_\varepsilon b W_t + \theta_\varepsilon s(\varepsilon)\widetilde{W}_t - \frac{1}{2}b^2\theta_\varepsilon^2 t - \frac{1}{2}s^2(\varepsilon)\theta_\varepsilon^2 t + \theta_\varepsilon \int_0^t\int_{|z|\geq\varepsilon} z\widetilde{N}(ds,dz) \\ - t\int_{|z|\geq\varepsilon}(\mathrm{e}^{\theta_\varepsilon z} - 1 - \theta_\varepsilon z)\ell(dz). \end{aligned}$$

The process L^ε has the Lévy triplet $(\tilde{a}_\varepsilon, b^2 + s^2(\varepsilon), \tilde{\ell}_\varepsilon)$ under $\widetilde{\mathbb{P}}_{\theta_\varepsilon}$, where

$$\tilde{a}_\varepsilon = a + (b^2 + s^2(\varepsilon))\theta_\varepsilon + \int_{\varepsilon\leq|z|<1} z(\mathrm{e}^{\theta_\varepsilon z} - 1)\ell(dz) \quad \text{and} \quad \tilde{\ell}_\varepsilon(dz) = 1_{|z|\geq\varepsilon}\mathrm{e}^{\theta_\varepsilon z}\ell(dz). \tag{3.14}$$

In [5] it was shown that there is a positive constant $C(\theta_0)$ depending on θ_0 such that

$$|\theta_\varepsilon - \theta_0| \leq C(\theta_0)\tilde{s}^2(\varepsilon), \qquad \forall \varepsilon \in (0,1). \tag{3.15}$$

Therefore the parameter θ_ε is bounded uniformly in ε by

$$|\theta_\varepsilon| \leq |\theta_\varepsilon - \theta_0| + |\theta_0| \leq C(\theta_0)\tilde{s}^2(\varepsilon) + |\theta_0| \leq C(\theta_0)\tilde{s}^2(1) + |\theta_0|. \tag{3.16}$$

For the robustness results proved in this section to hold true, we have to show that assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$, concerning the convergence of the characteristic triplets, are verified when we consider the ET. This is the purpose of the next proposition.

Proposition 3.8 (Robustness of the ET) *Assume* (3.12). *Let the drift coefficients $\tilde{a}$ and $\tilde{a}_\varepsilon$ and the Lévy measures $\tilde{\ell}$ and $\tilde{\ell}_\varepsilon$ be as expressed in* (3.13) *and* (3.14). *Then conditions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ hold true.*

Proof Clearly it follows from (3.13) and (3.14) that

$$\tilde{a}_\varepsilon - \tilde{a} = b^2(\theta_\varepsilon - \theta_0) + s^2(\varepsilon)\theta_\varepsilon + \int_{\varepsilon\le|z|<1} z(\mathrm{e}^{\theta_\varepsilon z} - \mathrm{e}^{\theta_0 z})\ell(dz) - \int_{|z|<\varepsilon} z(\mathrm{e}^{\theta_0 z} - 1)\ell(dz).$$

Hence, by applying the mean value theorem (MVT) on the function $\theta \mapsto \mathrm{e}^{\theta z}$ twice, we obtain by expressions (2.2), (2.22), (3.15), and (3.16) that

$$\begin{aligned}|\tilde{a}_\varepsilon - \tilde{a}| &\le b^2|\theta_\varepsilon - \theta_0| + s^2(\varepsilon)|\theta_\varepsilon| + \int_{\varepsilon\le|z|<1} |z||\mathrm{e}^{\theta_\varepsilon z} - \mathrm{e}^{\theta_0 z}|\ell(dz) + \int_{|z|<\varepsilon} |z||\mathrm{e}^{\theta_0 z} - 1|\ell(dz)\\ &\le b^2C(\theta_0)\tilde{s}^2(\varepsilon) + s^2(\varepsilon)|\theta_\varepsilon| + |\theta_\varepsilon - \theta_0|\mathrm{e}^{C_1(\theta_0)}\int_{|z|<1}|z|^2\ell(dz) + |\theta_0|\mathrm{e}^{|\theta_0|}\int_{|z|<\varepsilon}|z|^2\ell(dz)\\ &\le C_2(\theta_0)\tilde{s}^2(\varepsilon), \quad \forall\varepsilon\in(0,1).\end{aligned}$$

We are left to check whether assumption $(\mathcal{A}2)$ is also satisfied. Recall the definition of the functions ρ_0 and ρ_ε in (3.1). Thus from (3.13) and (3.14) it turns out that

$$\rho_0(z) = \mathrm{e}^{\theta_0 z} \qquad \text{and} \qquad \rho_\varepsilon(z) = \mathrm{e}^{\theta_\varepsilon z}, \quad \forall\varepsilon\in(0,1).$$

Combining the MVT on the function $\theta \mapsto \mathrm{e}^{\theta z}$ and the property (3.15) leads to

$$|\rho_\varepsilon(z) - \rho_0(z)| = \mathrm{e}^{\theta_0 z}|\mathrm{e}^{(\theta_\varepsilon-\theta_0)z} - 1| \le \mathrm{e}^{\theta_0 z}|z|\mathrm{e}^{|\theta_\varepsilon-\theta_0||z|}|\theta_\varepsilon - \theta_0| \le \gamma(z)\tilde{s}^2(\varepsilon), \tag{3.17}$$

where $\gamma(z) = \mathrm{e}^{(1+|\theta_0|+C(\theta_0)\tilde{s}^2(1))|z|}C(\theta_0)$. Moreover, condition (3.12) ensures the integrability assumptions on γ and the statement is proved. □

3.2 Minimal entropy martingale measure (MEMM)

For the definition and more details about the MEMM, we refer to [15]. We introduce the Lévy process $\widehat{L}$

$$\begin{aligned}\widehat{L}_t &:= L_t + \frac{1}{2}b^2t + \int_0^t\int_{\mathbb{R}_0}(\mathrm{e}^z - 1 - z)N(ds,dz)\\ &= a_1t + bW_t + \int_0^t\int_{|z|\ge1}(\mathrm{e}^z-1)N(ds,dz) + \int_0^t\int_{|z|<1}(\mathrm{e}^z-1)\widetilde{N}(ds,dz),\end{aligned}$$

where $a_1 = a + \frac{1}{2}b^2 + \int_{|z|<1}(\mathrm{e}^z - 1 - z)\ell(dz)$. We assume that for all $\theta^*\in\mathbb{R}$ we have

$$\int_{|z|\ge1}\mathrm{e}^{\theta^*(\mathrm{e}^z-1)}\ell(dz) < \infty. \tag{3.18}$$

The latter condition implies that the moment generating function of $\widehat{L}$ exists. We introduce the measure $\widetilde{\mathbb{P}}_{\theta^*}\sim\mathbb{P}$, $\forall\theta^*\in\mathbb{R}$, by

$$\frac{d\widetilde{\mathbb{P}}_{\theta^*}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\exp(\theta^*\widehat{L}_t)}{\mathbb{E}[\exp(\theta^*\widehat{L}_t)]} = \exp(Y_t),$$

where

$$Y_t = \theta^* bW_t - \frac{1}{2}(\theta^*)^2b^2t + \theta^*\int_0^t\int_{\mathbb{R}_0}(\mathrm{e}^z-1)\widetilde{N}(ds,dz) - t\int_{\mathbb{R}_0}(\mathrm{e}^{\theta^*(\mathrm{e}^z-1)} - 1 - \theta^*(\mathrm{e}^z-1))\ell(dz).$$

We define $\widetilde{\mathbb{P}}_{\theta_0^*}$ to be the measure under which the discounted stock price process is a martingale. It is called the MEMM. The existence and uniqueness of the parameter $\theta_0^*\in\mathbb{R}$ are discussed in [15]. The Lévy triplet of L under the MEMM $\widetilde{\mathbb{P}}_{\theta_0^*}$ equals $(\tilde{a}, b^2, \tilde{\ell})$, where

$$\tilde{a} = a + b^2\theta_0^* + \int_{|z|<1}z(\mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)} - 1)\ell(dz) \quad \text{and} \quad \tilde{\ell}(dz) = \mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)}\ell(dz). \tag{3.19}$$

For any $\varepsilon \in (0,1)$ we define

$$\widehat{L}_t^\varepsilon = L_t^\varepsilon + \frac{1}{2}(b^2 + s^2(\varepsilon))t + \int_0^t \int_{|z|\geq\varepsilon} (\mathrm{e}^z - 1 - z)N(ds,dz).$$

The density of the MEMM for the approximating model is given by

$$\frac{d\widetilde{\mathbb{P}}_{\theta_\varepsilon^*}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\exp(\theta_\varepsilon^* \widehat{L}_t^\varepsilon)}{\mathbb{E}[\exp(\theta_\varepsilon^* \widehat{L}_t^\varepsilon)]} = \exp(Y_t^\varepsilon),$$

where

$$\begin{aligned} Y_t^\varepsilon = \theta_\varepsilon^*(bW_t + s(\varepsilon)\widetilde{W}_t) - \frac{1}{2}(\theta_\varepsilon^*)^2(b^2 + s^2(\varepsilon))t + \theta_\varepsilon^* \int_0^t \int_{|z|\geq\varepsilon} (\mathrm{e}^z - 1)\widetilde{N}(ds,dz) \\ - t\int_{|z|\geq\varepsilon} (\mathrm{e}^{\theta_\varepsilon^*(\mathrm{e}^z-1)} - 1 - \theta_\varepsilon^*(\mathrm{e}^z - 1))\ell(dz) \end{aligned}$$

and θ_ε^* is the parameter value ensuring the discounted stock price is martingale. The Lévy triplet of L^ε under the MEMM $\widetilde{\mathbb{P}}_{\theta_\varepsilon^*}$ is $(\tilde{a}_\varepsilon, b^2 + s^2(\varepsilon), \tilde{\ell}_\varepsilon)$, where

$$\tilde{a}_\varepsilon = a + (b^2 + s^2(\varepsilon))\theta_\varepsilon^* + \int_{\varepsilon\leq|z|<1} z(\mathrm{e}^{\theta_\varepsilon^*(\mathrm{e}^z-1)} - 1)\ell(dz) \quad \text{and} \quad \tilde{\ell}_\varepsilon(dz) = \mathrm{e}^{\theta_\varepsilon^*(\mathrm{e}^z-1)} 1_{|z|\geq\varepsilon}\ell(dz). \tag{3.20}$$

From [5] it follows that the parameter θ_ε^* converges to θ_0^* when ε goes to 0 and that

$$|\theta_\varepsilon^* - \theta_0^*| \leq C(\theta_0^*)\tilde{s}^2(\varepsilon), \qquad \forall \varepsilon \in (0,1), \tag{3.21}$$

for a positive constant $C(\theta_0^*)$ depending on θ_0^*. We state the following proposition in which we prove that assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ are verified for the MEMM.

Proposition 3.9 (Robustness of the MEMM) *Assume* (3.18). *Let the drift coefficients* $\tilde{a}$ *and* $\tilde{a}_\varepsilon$ *and the Lévy measures* $\tilde{\ell}$ *and* $\tilde{\ell}_\varepsilon$ *be as expressed in* (3.19) *and* (3.20), *respectively. Then conditions* $(\mathcal{A}1)$ *and* $(\mathcal{A}2)$ *hold true.*

Proof From (3.19) and (3.20) we compute

$$\begin{aligned} &|\tilde{a}_\varepsilon - \tilde{a}| \\ &= \Big| b^2(\theta_\varepsilon^* - \theta_0^*) + s^2(\varepsilon)\theta_\varepsilon^* + \int_{\varepsilon\leq|z|<1} z(\mathrm{e}^{\theta_\varepsilon^*(\mathrm{e}^z-1)} - \mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)})\ell(dz) - \int_{|z|<\varepsilon} z(\mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)} - 1)\ell(dz) \Big| \\ &\leq b^2|\theta_\varepsilon^* - \theta_0^*| + s^2(\varepsilon)|\theta_\varepsilon^*| + \int_{\varepsilon\leq|z|<1} |z| \big|\mathrm{e}^{\theta_\varepsilon^*(\mathrm{e}^z-1)} - \mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)}\big|\ell(dz) + \int_{|z|<\varepsilon} |z| \big|\mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)} - 1\big|\ell(dz). \end{aligned}$$

The MVT guarantees the existence of a number θ_1' on the line $L_{\theta_\varepsilon^*,\theta_0^*}$ such that for $|z| < 1$

$$\big|\mathrm{e}^{\theta_\varepsilon^*(\mathrm{e}^z-1)} - \mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)}\big| = \mathrm{e}^{\theta_1'(\mathrm{e}^z-1)}|\mathrm{e}^z - 1||\theta_\varepsilon^* - \theta_0^*| \leq C_1(\theta_0^*)|z|\tilde{s}^2(\varepsilon),$$

because of inequality (3.21). Hence by condition (2.2) it turns out that

$$\int_{\varepsilon\leq|z|<1} |z| \big|\mathrm{e}^{\theta_\varepsilon^*(\mathrm{e}^z-1)} - \mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)}\big|\ell(dz) \leq C_1(\theta_0^*) \int_{|z|<1} z^2\ell(dz)\tilde{s}^2(\varepsilon) = C_2(\theta_0^*)\tilde{s}^2(\varepsilon).$$

Analogously for some θ_2' on L_{0,θ_0^*}, it holds for $|z| < \varepsilon < 1$ that

$$\big|\mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)} - 1\big| = \mathrm{e}^{\theta_2'(\mathrm{e}^z-1)}|\mathrm{e}^z - 1||\theta_0^*| \leq C_3(\theta_0^*)|z|,$$

and therefore by definition (2.22) it follows that

$$\int_{|z|<\varepsilon} |z| \big|\mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)} - 1\big|\ell(dz) \leq C_3(\theta_0^*) \int_{|z|<\varepsilon} z^2\ell(dz) = C_3(\theta_0^*)\sigma^2(\varepsilon).$$

Collecting the obtained convergence properties of the different terms leads us to the conclusion that assumption ($\mathcal{A}1$) holds for the MEMM. We are left to check whether assumption ($\mathcal{A}2$) is also satisfied. Recall the functions ρ_0 and ρ_ε in (3.1), thus

$$\rho_0(z) = \mathrm{e}^{\theta_0^*(\mathrm{e}^z-1)} \qquad \text{and} \qquad \rho_\varepsilon(z) = \mathrm{e}^{\theta_\varepsilon^*(\mathrm{e}^z-1)}, \quad \forall \varepsilon \in (0,1).$$

Moreover by the MVT and property (3.21) one can obtain in a similar way as (3.17) that

$$|\rho_\varepsilon(z) - \rho_0(z)| \le \mathrm{e}^{(1+|\theta_0^*|+C(\theta_0^*)\tilde{s}^2(1))|\mathrm{e}^z-1|} C(\theta_0^*)\tilde{s}^2(\varepsilon).$$

This concludes the proof since the integrability conditions in ($\mathcal{A}2$) are satisfied by (3.18). □

3.3 Minimal martingale measure (MMM)

Studies about the MMM can be found in [1] and [13]. Let γ_0 be defined as

$$\gamma_0 = -\frac{a + \frac{1}{2}b^2 + \int_{\mathbb{R}_0}(\mathrm{e}^z - 1 - z1_{|z|<1})\ell(dz) - r}{b^2 + \int_{\mathbb{R}_0}(\mathrm{e}^z-1)^2\ell(dz)}.$$

We assume that $\gamma_0(\mathrm{e}^z - 1) + 1 > 0$, $\forall z \in \mathbb{R}$, hereto $\gamma_0 \in (0,1)$. This condition ensures that the MMM exists as a probability measure (see Proposition 3.1 in [1]). The MMM in this case is defined by means of γ_0 as

$$\frac{d\widetilde{\mathbb{P}}_{\gamma_0}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp(Z_t),$$

where

$$Z_t = \gamma_0 bW_t - \frac{1}{2}\gamma_0^2 b^2 t + \int_0^t\int_{\mathbb{R}_0} \log(\gamma_0(\mathrm{e}^z-1)+1)\widetilde{N}(ds,dz) - t\int_{\mathbb{R}_0}(\gamma_0(\mathrm{e}^z-1) - \log(\gamma_0(\mathrm{e}^z-1)+1))\ell(dz).$$

The Lévy triplet of L w.r.t. the MMM $\widetilde{\mathbb{P}}_{\gamma_0}$ equals $(\tilde{a}, b^2, \tilde{\ell})$, where

$$\tilde{a} = a + b^2\gamma_0 + \int_{|z|<1}\gamma_0 z(\mathrm{e}^z-1)\ell(dz) \quad \text{and} \quad \tilde{\ell}(dz) = \{\gamma_0(\mathrm{e}^z-1)+1\}\ell(dz). \tag{3.22}$$

We define for ε in $(0,1)$

$$\gamma_\varepsilon = -\frac{a + \frac{1}{2}(b^2 + s^2(\varepsilon)) + \int_{|z|\ge\varepsilon}(\mathrm{e}^z - 1 - z1_{|z|<1})\ell(dz) - r}{b^2 + s^2(\varepsilon) + \int_{|z|\ge\varepsilon}(\mathrm{e}^z-1)^2\ell(dz)}$$

and assume that $\gamma_\varepsilon(\mathrm{e}^z-1)+1>0$, $\forall z \in \mathbb{R}$, i.e. $\gamma_\varepsilon \in (0,1)$, which will ensure the existence of the MMM for the approximating process as a probability measure. The MMM for the approximating process is then given by

$$\frac{d\widetilde{\mathbb{P}}_{\gamma_\varepsilon}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp(Z_t^\varepsilon),$$

where

$$\begin{aligned} Z_t^\varepsilon = \gamma_\varepsilon(bW_t + s(\varepsilon)\widetilde{W}_t) - \frac{1}{2}\gamma_\varepsilon^2(b^2+s^2(\varepsilon))t + \int_0^t\int_{|z|\ge\varepsilon}\log(\gamma_\varepsilon(\mathrm{e}^z-1)+1)\widetilde{N}(ds,dz) \\ - t\int_{|z|\ge\varepsilon}(\gamma_\varepsilon(\mathrm{e}^z-1) - \log(\gamma_\varepsilon(\mathrm{e}^z-1)+1))\ell(dz). \end{aligned}$$

The Lévy triplet of L^ε under the MMM $\widetilde{\mathbb{P}}_{\gamma_\varepsilon}$ is given by $(\tilde{a}_\varepsilon, b^2+s^2(\varepsilon), \tilde{\ell}_\varepsilon)$, where

$$\tilde{a}_\varepsilon = a + (b^2+s^2(\varepsilon))\gamma_\varepsilon + \int_{\varepsilon\le|z|<1}\gamma_\varepsilon z(\mathrm{e}^z-1)\ell(dz) \quad \text{and} \quad \tilde{\ell}_\varepsilon(dz) = \{\gamma_\varepsilon(\mathrm{e}^z-1)+1\}1_{|z|\ge\varepsilon}\ell(dz). \tag{3.23}$$

It can easily be computed that the parameter characterizing the MMM owns a similar convergence result as those for the ET and MEMM in (3.15) and (3.21), namely

$$|\gamma_\varepsilon - \gamma_0| \leq C(\gamma_0)\tilde{s}^2(\varepsilon), \qquad \forall \varepsilon \in (0,1), \tag{3.24}$$

for $C(\gamma_0)$ a positive constant depending on γ_0. We state the following proposition in which we prove that assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ are verified for the MMM.

Proposition 3.10 (Robustness of the MMM) *Assume* (2.10)*. Let the drift coefficients $\tilde{a}$ and $\tilde{a}_\varepsilon$ and the Lévy measures $\tilde{\ell}$ and $\tilde{\ell}_\varepsilon$ be as expressed in* (3.22) *and* (3.23)*, respectively. Then conditions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ hold true.*

Proof One can easily compute that

$$\begin{aligned}
|\tilde{a}_\varepsilon - \tilde{a}| &= \Big| b^2(\gamma_\varepsilon - \gamma_0) + s^2(\varepsilon)\gamma_\varepsilon + \int_{\varepsilon \leq |z| < 1} (\gamma_\varepsilon - \gamma_0) z(\mathrm{e}^z - 1)\ell(dz) - \int_{|z|<\varepsilon} \gamma_0 z(\mathrm{e}^z - 1)\ell(dz) \Big| \\
&\leq b^2|\gamma_\varepsilon - \gamma_0| + s^2(\varepsilon)|\gamma_\varepsilon| + \int_{\varepsilon \leq |z| < 1} |\gamma_\varepsilon - \gamma_0||z||\mathrm{e}^z - 1|\ell(dz) + \int_{|z|<\varepsilon} |\gamma_0||z||\mathrm{e}^z - 1|\ell(dz) \\
&\leq b^2 C(\gamma_0)\tilde{s}^2(\varepsilon) + s^2(\varepsilon)C_1(\gamma_0) + C(\gamma_0)\tilde{s}^2(\varepsilon)\int_{|z|<1} z^2\ell(dz) + |\gamma_0|\int_{|z|<\varepsilon} z^2\ell(dz).
\end{aligned}$$

By expressions (2.2), (2.22), and (3.24) it turns out that $(\mathcal{A}1)$ is fulfilled. Recalling the terms of ρ_0 and ρ_ε in (3.1), we get

$$\rho_0(z) = \gamma_0(\mathrm{e}^z - 1) + 1 \qquad \text{and} \qquad \rho_\varepsilon(z) = \gamma_\varepsilon(\mathrm{e}^z - 1) + 1, \quad \forall \varepsilon \in (0,1).$$

Thus by (3.24) it appears that

$$|\rho_\varepsilon(z) - \rho_0(z)| \leq |\gamma_\varepsilon - \gamma_0||\mathrm{e}^z - 1| \leq \gamma(z)\tilde{s}^2(\varepsilon),$$

where $\gamma(z) = C(\gamma_0)|\mathrm{e}^z - 1|$. Hence assumption $(\mathcal{A}2)$ holds true by assuming (2.10) in addition to (2.2). □

3.4 Variance-optimal martingale measure (VOMM)

In Theorem 8 in [26] it was shown that for geometric Lévy processes, the VOMM and the MMM coincide. Thus the robustness of the MMM studied in the previous subsection is equivalent to the robustness of the mean-variance martingale measure.

4 Robustness of the quadratic hedging, the semimartingale case

In this section the market is observed under the historical measure. To prove the robustness of the LRM strategies, one approach would be to rely on formulas (2.12) and (2.35). These formulas are written in terms of the option price and the delta of the option computed w.r.t. the MMMs $\widetilde{\mathbb{P}}_{\gamma_0}, \widetilde{\mathbb{P}}_{\gamma_\varepsilon}$. The robustness of the LRM strategies using this approach will then follow immediately by applying similar computations as in Propositions 3.5 and 3.6. However, here we choose to discuss the robustness relying on the cumulant based formulas (2.17) and (2.36). We do this to avoid the use of explicit option prices and deltas. This approach can also be applied to the martingale case. Therefore it is expected to retrieve similar robustness results as in Section 3. We start by mentioning the following robustness results. The proof is presented in the Appendix.

Lemma 4.1 (Robustness of κ^ε and μ^ε) *Let κ, μ, and their approximating counterparts be as defined in Proposition 2.6 and equations* (2.37)*. Assume condition* (2.14)*. Then there exist constants K_1 and K_2 depending on the damping factor R such that for all real numbers u it holds that*

$$|\kappa^\varepsilon(R - iu) - \kappa(R - iu)| \leq K_1(R)(1 + |u| + u^2)\tilde{s}^2(\varepsilon) \ \textit{and} \tag{4.1}$$

$$|\mu^\varepsilon(R - iu) - \mu(R - iu)| \leq K_2(R)(1 + |u|)\tilde{s}^2(\varepsilon). \tag{4.2}$$

Note that the convergence of the function κ^ε to κ implies the convergence of the constant λ^ε to λ as defined in Proposition 2.7 and we have for K being a constant

$$|\lambda^\varepsilon - \lambda| \leq K\tilde{s}^2(\varepsilon). \tag{4.3}$$

In the following proposition, we consider the robustness of the LRM hedging number.

Proposition 4.2 (Robustness LRM hedging number) *Let $\mathcal{X}$ and $\mathcal{X}^\varepsilon$ be given by* (2.17) *and* (2.36)*, respectively. In addition assume* (2.14) *and* (2.15)*. Moreover if there is a function $\widetilde{\Psi}$ such that*

$$|\widetilde{\Phi}^{\gamma_\varepsilon}_{T-t}(-u-iR)| \leq \widetilde{\Psi}_{T-t}(u;R) \quad \text{and} \quad u \mapsto (1+|u|)|\hat{f}(u+iR)|\widetilde{\Psi}_{T-t}(u;R) \in L^1(\mathbb{R}), \tag{4.4}$$

then for all $t \in [0,T]$,

$$\lim_{\varepsilon\to 0} \mathcal{X}^\varepsilon_t = \mathcal{X}_t\,, \qquad \mathbb{P}\text{-}a.s.$$

Proof According to (2.15) and (2.36), the LRM hedging number for the approximating model is computed by the approximating counterpart of (2.18) which equals

$$\mathcal{X}^\varepsilon_t = \frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}} \mu^\varepsilon(R-iu)\hat{f}(u+iR)\widetilde{\Phi}^{\gamma_\varepsilon}_{T-t}(-u-iR)(S^\varepsilon_{t-})^{R-iu-1}du.$$

The function μ^ε is defined as

$$\mu^\varepsilon(z) = \frac{\kappa^\varepsilon(z+1)-\kappa^\varepsilon(z)-\kappa^\varepsilon(1)}{\kappa^\varepsilon(2)-2\kappa^\varepsilon(1)} = \frac{(b^2+s^2(\varepsilon))z + \int_{|x|\geq\varepsilon}(\mathrm{e}^{zx}-1)(\mathrm{e}^x-1)\ell(dx)}{b^2+s^2(\varepsilon)+\int_{|x|\geq\varepsilon}(\mathrm{e}^x-1)^2\ell(dx)}\,,$$

therefore it appears by condition (2.21) that $\mu^\varepsilon(z)$ is bounded uniformly in ε by

$$|\mu^\varepsilon(z)| \leq \frac{(b^2+s^2(1))|z| + \int_{\mathbb{R}_0}|\mathrm{e}^{zx}-1||\mathrm{e}^x-1|\ell(dx)}{b^2+K}\,.$$

Specifically for $z = R-iu$ and applying the MVT, we get that

$$\begin{aligned}
&\int_{\mathbb{R}_0}|\mathrm{e}^{zx}-1||\mathrm{e}^x-1|\ell(dx)\\
&= \int_{|x|\geq 1}|\mathrm{e}^{(R-iu)x}-1||\mathrm{e}^x-1|\ell(dx) + \int_{|x|<1}|\mathrm{e}^{(R-iu)x}-1||\mathrm{e}^x-1|\ell(dx)\\
&\leq \int_{|x|\geq 1}(\mathrm{e}^{Rx}+1)(\mathrm{e}^x+1)\ell(dx) + 2(|R|+|u|)\mathrm{e}^{|R|+1}\int_{|x|<1}x^2\ell(dx)\,.
\end{aligned}$$

Hence it turns out by (2.2) and (2.14) that $|\mu^\varepsilon(R-iu)| \leq K(R)(1+|u|)$. Finally, by definition (3.8) and condition (4.4) we can apply the dominated convergence theorem. By Lemmas 2.9, 3.1, and 4.1 we prove the statement. □

Proposition 4.3 (Convergence rate LRM hedging number) *Let $\mathcal{X}$ and $\mathcal{X}^\varepsilon$ be given by* (2.17) *and* (2.36)*, respectively. Moreover assume* (2.14) *and* (2.15)*. The existence of a function $\widehat{\Psi}$ guaranteeing*

$$\begin{cases}\max(|\widetilde{\Phi}^{\gamma_\varepsilon}_{T-t}(-u-iR)|, |\widetilde{\Phi}^{\gamma_0}_{T-t}(-u-iR)|) \leq \widehat{\Psi}_{T-t}(u;R), \ \text{and}\\ u \mapsto (1+|u|+u^2+|u|^3)|\hat{f}(u+iR)|\widehat{\Psi}_{T-t}(u;R) \in L^1(\mathbb{R}),\end{cases} \tag{4.5}$$

implies that for all $t \in [0,T]$ and all $\varepsilon \in (0,1)$, we have

$$\mathbb{E}[|\mathcal{X}^\varepsilon_t - \mathcal{X}_t|] \leq C(T,r,R,\gamma_0)\tilde{s}(\varepsilon),$$

where $C(T,r,R,\gamma_0)$ is a positive constant depending on T, r, R, and the parameter γ_0 corresponding to the MMM.

Proof Clearly it holds that

$$\mathbb{E}[|\mathcal{X}_t^\varepsilon - \mathcal{X}_t|] \leq \frac{\mathrm{e}^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \mathbb{E}[|\mu^\varepsilon(R-iu)\widetilde{\Phi}_{T-t}^{\gamma_\varepsilon}(-u-iR)(S_{t-}^\varepsilon)^{R-iu-1} \\ - \mu(R-iu)\widetilde{\Phi}_{T-t}^{\gamma_0}(-u-iR)S_{t-}^{R-iu-1}|]|\hat{f}(u+iR)|du.$$

Therein

$$\begin{aligned}
&\mathbb{E}[|\mu^\varepsilon(R-iu)\widetilde{\Phi}_{T-t}^{\gamma_\varepsilon}(-u-iR)(S_{t-}^\varepsilon)^{R-iu-1} - \mu(R-iu)\widetilde{\Phi}_{T-t}^{\gamma_0}(-u-iR)S_{t-}^{R-iu-1}|] \\
&\leq |\mu^\varepsilon(R-iu) - \mu(R-iu)||\widetilde{\Phi}_{T-t}^{\gamma_\varepsilon}(-u-iR)|\mathbb{E}[(S_{t-}^\varepsilon)^{R-1}] \\
&\quad + |\mu(R-iu)||\widetilde{\Phi}_{T-t}^{\gamma_\varepsilon}(-u-iR) - \widetilde{\Phi}_{T-t}^{\gamma_0}(-u-iR)|\mathbb{E}[(S_{t-}^\varepsilon)^{R-1}] \\
&\quad + |\mu(R-iu)||\widetilde{\Phi}_{T-t}^{\gamma_0}(-u-iR)|\mathbb{E}[|(S_{t-}^\varepsilon)^{R-iu-1} - (S_{t-})^{R-iu-1}|] \\
&\leq C(T,R,\gamma_0)(1+|u|+u^2+|u|^3)\widehat{\Psi}_{T-t}(u;R)\tilde{s}(\varepsilon)\,,
\end{aligned}$$

where the last expression is a consequence of Lemmas 2.9, 3.1, and 4.1, and condition (4.5). Therefore the statement is proved. □

Recall the expression of ϕ and ϕ^ε in (2.19) and (2.38), respectively. The *amounts of wealth* in the discounted risky asset of the MVH strategy for the original and the approximating models are defined by $\Upsilon_t := \phi_t\hat{S}_{t-}$ and $\Upsilon_t^\varepsilon := \phi_t^\varepsilon\hat{S}_{t-}^\varepsilon$ respectively for all t in $[0,T]$ and thus are given by the following equations

$$\Upsilon_t = \pi_t + \lambda(\hat{H}_{t-} - V_0 - \int_0^t \Upsilon_s d\widehat{L}_s)\,, \tag{4.6}$$

$$\Upsilon_t^\varepsilon = \pi_t^\varepsilon + \lambda^\varepsilon(\hat{H}_{t-}^\varepsilon - V_0^\varepsilon - \int_0^t \Upsilon_s^\varepsilon d\widehat{L}_s^\varepsilon)\,, \tag{4.7}$$

where $\pi_t = \mathcal{X}_t\hat{S}_{t-}$ and $\pi_t^\varepsilon = \mathcal{X}_t^\varepsilon\hat{S}_{t-}^\varepsilon$ are the amounts of wealth in the risky asset in the LRM strategies which can be computed by

$$\pi_t = \mathrm{e}^{-rt}\int_{\mathbb{C}} \mu(z)\mathrm{e}^{\eta(z)(T-t)}S_{t-}^z\Pi(dz), \tag{4.8}$$

$$\pi_t^\varepsilon = \mathrm{e}^{-rt}\int_{\mathbb{C}} \mu^\varepsilon(z)\mathrm{e}^{\eta^\varepsilon(z)(T-t)}(S_{t-}^\varepsilon)^z\Pi(dz), \tag{4.9}$$

and $\widehat{L}$ and $\widehat{L}^\varepsilon$ are given by (see proof of Lemma 2.9)

$$\begin{aligned}
d\widehat{L}_t &= \Big(a + \frac{b^2}{2} + \int_{\mathbb{R}_0}(\mathrm{e}^z - 1 - z1_{|z|<1})\ell(dz) - r\Big)dt + bdW_t + \int_{\mathbb{R}_0}(\mathrm{e}^z - 1)\widetilde{N}(dt,dz)\,, \\
d\widehat{L}_t^\varepsilon &= \Big(a + \frac{b^2 + s^2(\varepsilon)}{2} + \int_{|z|\geq\varepsilon}(\mathrm{e}^z - 1 - z1_{|z|<1})\ell(dz) - r\Big)dt + bdW_t + s(\varepsilon)d\widetilde{W}_t \\
&\quad + \int_{|z|\geq\varepsilon}(\mathrm{e}^z - 1)\widetilde{N}(dt,dz)\,.
\end{aligned}$$

In the following lemma, we prove the robustness of the amount of wealth in a LRM strategy.

Lemma 4.4 (Convergence rate LRM amount of wealth) *Assume* (2.14), (2.15), *and integrability properties* (4.5). *For the amounts of wealth π_t and π_t^ε expressed in* (4.8) *and* (4.9), *we have for all $t \in [0,T]$ and $\varepsilon \in (0,1)$ that*

$$\mathbb{E}[|\pi_t^\varepsilon - \pi_t|] \leq C(T,r,R,\gamma_0)\tilde{s}(\varepsilon),$$

where $C(T,r,R,\gamma_0)$ is a positive constant depending on $T, r,$ R, and γ_0.

Proof From (4.8) and (4.9) it appears that

$$\mathbb{E}[|\pi_t^\varepsilon - \pi_t|] \leq \frac{\mathrm{e}^{-rT}}{2\pi} \int_{\mathbb{R}} \mathbb{E}[|\mu^\varepsilon(R-iu)\widetilde{\Phi}^{\gamma_\varepsilon}_{T-t}(-u-iR)(S^\varepsilon_{t-})^{R-iu} - \mu(R-iu)\widetilde{\Phi}^{\gamma_0}_{T-t}(-u-iR)S^{R-iu}_{t-}|]|\hat{f}(u+iR)|du.$$

Proceeding by similar computations as in Proposition 4.3 proves the statement. □

We prove the robustness of the amount of wealth in a MVH strategy.

Proposition 4.5 (Convergence rate MVH amount of wealth) *Assume conditions* (2.14), (2.15), *and* (4.5) *are met. Consider the processes* Υ *and* Υ^ε *given by* (4.6) *and* (4.7). *For all* $t \in [0,T]$ *and* $\varepsilon \in (0,1)$, *we have that*

$$\mathbb{E}[|\Upsilon_t^\varepsilon - \Upsilon_t|] \leq C(T,r,R,\gamma_0)\tilde{s}(\varepsilon),$$

where $C(T,r,R,\gamma_0)$ *is a positive constant depending on* $T,r,R,$ *and* γ_0.

Proof Using the triangle inequality it turns out that

$$\begin{aligned}\mathbb{E}[|\Upsilon_t^\varepsilon - \Upsilon_t|] &\leq \mathbb{E}[|\pi_t^\varepsilon - \pi_t|] + |\lambda^\varepsilon - \lambda|\mathbb{E}\big[\big|\hat{H}_{t-} - V_0 - \int_0^t \Upsilon_s d\widehat{L}_s\big|\big] \\ &\quad + |\lambda^\varepsilon|\Big(\mathbb{E}[|\hat{H}_t^\varepsilon - \hat{H}_t|] + \mathbb{E}[|V_0^\varepsilon - V_0|] + \mathbb{E}\Big[\big|\int_0^t \Upsilon_s^\varepsilon d\widehat{L}_s^\varepsilon - \int_0^t \Upsilon_s d\widehat{L}_s\big|\Big]\Big). \end{aligned} \tag{4.10}$$

We have for $t \in [0,T]$,

$$\mathbb{E}[|\hat{H}_t^\varepsilon - \hat{H}_t|] \leq \frac{\mathrm{e}^{-rT}}{2\pi} \int_{\mathbb{R}} \mathbb{E}[|\widetilde{\Phi}^{\gamma_\varepsilon}_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu} - \widetilde{\Phi}^{\gamma_0}_{T-t}(-u-iR)S_t^{R-iu}|]|\hat{f}(u+iR)|du\,.$$

However, by Lemmas 2.9 and 3.1, condition (4.5), and following similar arguments as in the proof of Proposition 4.3 we know that

$$\begin{aligned}&\mathbb{E}[|\widetilde{\Phi}^{\gamma_\varepsilon}_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu} - \widetilde{\Phi}^{\gamma_0}_{T-t}(-u-iR)S_t^{R-iu}|] \\ &\leq K_1(T,R,\gamma_0)(1+|u|+u^2)\widehat{\Psi}_{T-t}(u;R)\tilde{s}(\varepsilon).\end{aligned}$$

The integrability assumption included in (4.5) implies that

$$\mathbb{E}[|\hat{H}_t^\varepsilon - \hat{H}_t|] \leq K_2(T,R,\gamma_0)\tilde{s}(\varepsilon) \quad \text{and} \quad \mathbb{E}[|V_0^\varepsilon - V_0|] \leq K_3(T,R,\gamma_0)\tilde{s}(\varepsilon). \tag{4.11}$$

From the expressions of $\widehat{L}$ and $\widehat{L}^\varepsilon$ it appears that

$$\begin{aligned}\mathbb{E}\Big[\big|\int_0^t \Upsilon_s^\varepsilon d\widehat{L}_s^\varepsilon - \int_0^t \Upsilon_s d\widehat{L}_s\big|\Big] &\leq \Big|a + \frac{b^2}{2} + \int_{|z|\geq\varepsilon} (\mathrm{e}^z - 1 - z1_{|z|<1})\ell(dz) - r\Big|\mathbb{E}\Big[\int_0^t |\Upsilon_s^\varepsilon - \Upsilon_s|ds\Big] \\ &\quad + \Big|\int_{|z|<\varepsilon} (\mathrm{e}^z - 1 - z)\ell(dz)\Big|\mathbb{E}\Big[\int_0^t |\Upsilon_s|ds\Big] + \frac{s^2(\varepsilon)}{2}\mathbb{E}\Big[\int_0^t |\Upsilon_s^\varepsilon|ds\Big] \\ &\leq \Big(|a| + \frac{b^2}{2} + \int_{\mathbb{R}_0} |\mathrm{e}^z - 1 - z1_{|z|<1}|\ell(dz) + r\Big)\int_0^t \mathbb{E}[|\Upsilon_s^\varepsilon - \Upsilon_s|]ds \\ &\quad + K_4\sigma^2(\varepsilon)\mathbb{E}\Big[\int_0^t |\Upsilon_s|ds\Big] + \frac{s^2(\varepsilon)}{2}\mathbb{E}\Big[\int_0^t |\Upsilon_s^\varepsilon|ds\Big].\end{aligned}$$

Moreover, using similar tools as above, one can prove that $\mathbb{E}[\int_0^t |\Upsilon_s|ds]$ and $\mathbb{E}[\int_0^t |\Upsilon_s^\varepsilon|ds]$ are bounded uniformly in ε by a positive constant. Thus collecting the terms in (4.10) and using equations (4.3), (4.11), and Lemma 4.4, we get

$$\mathbb{E}[|\Upsilon_t^\varepsilon - \Upsilon_t|] \leq K_5(T,R,\gamma_0)\tilde{s}(\varepsilon) + K_6(T,r)\int_0^t \mathbb{E}[|\Upsilon_s^\varepsilon - \Upsilon_s|]ds\,.$$

Finally, the result follows by applying Gronwall's lemma. □

5 Examples

For the robustness results studied in Sections 3 and 4 to hold true, we imposed integrability conditions on the driving Lévy process, on its characteristic function, and on the Fourier transform of the dampened payoff function of the contingent claim. The aim in this section is to summarize these integrability conditions and to illustrate our results with some examples.
An important assumption to guarantee the robustness results in our derivations is the existence of a function $\widehat{\Psi}$ satisfying

$$\begin{cases} \max(|\widetilde{\Phi}^{\varepsilon}_{T-t}(-u-iR)|,|\widetilde{\Phi}_{T-t}(-u-iR)|) \leq \widehat{\Psi}_{T-t}(u;R), \text{ and} \\ u \mapsto |u|^k |\hat{f}(u+iR)| \widehat{\Psi}_{T-t}(u;R) \in L^1(\mathbb{R}), \quad k \in \{0,1,2,3\}, \end{cases} \tag{5.1}$$

where $\widetilde{\Phi}$, $\widetilde{\Phi}^{\varepsilon}$ are the characteristic functions of the Lévy process L, L^{ε}, respectively under a related martingale measure. The function $\hat{f}(\cdot+iR)$ equals the Fourier transform of the dampened payoff function. One way to fulfill condition (5.1) for $k \in \{0,1,2,3\}$ is to check that

$$\begin{cases} u \mapsto |u|^l |\hat{f}(u+iR)| \in L^1(\mathbb{R}) \text{ and} \\ |u|^{k-l} \widehat{\Psi}_{T-t}(u;R) \text{ is bounded in } u, \end{cases} \quad \text{for some } l \in \{0,\dots,k\}. \tag{5.2}$$

Another way is to show that

$$\begin{cases} u \mapsto |u|^l |\widehat{\Psi}_{T-t}(u;R)| \in L^1(\mathbb{R}) \text{ and} \\ |u|^{k-l} |\hat{f}(u+iR)| \text{ is bounded in } u, \end{cases} \quad \text{for some } l \in \{0,\dots,k\}. \tag{5.3}$$

In the next two subsections we give examples of payoff functions and of driving Lévy processes which contribute to condition (5.2) or (5.3).

5.1 Examples of payoff functions

Let us consider a *power p call option*, with $p \geq 1$. The payoff function of such an option is given by

$$f(x) = [\max(\mathrm{e}^x - K, 0)]^p, \tag{5.4}$$

where $K \geq 0$ is the strike. Notice that the choice $p = 1$ corresponds to the standard call option. It holds that the dampened payoff function g is integrable for $R > p$ (see [11]). For the *power p put option*, $p \geq 1$, the payoff function equals

$$f(x) = [\max(K - \mathrm{e}^x, 0)]^p \tag{5.5}$$

and the dampened payoff function g is integrable for $R < 0$. In the following lemma we discuss for which powers p, the power call and put options contribute to (5.2) or (5.3).

Lemma 5.1 *Let f be as in* (5.4) *or* (5.5)*, with $p \geq 1$. For $l \in \{0,1,2,3\}$, f verifies* (5.2) *for $p \geq l+1$. For $k-l \in \{0,1,2,3\}$, f verifies* (5.3) *for $p \geq k-l-1$.*

Proof The Fourier transform of the dampened payoff function for a power $p \geq 1$ is given by

$$\hat{f}(u+iR) = \hat{g}(u) = \frac{p! K^{p-R+iu}}{(R-iu)(R-1-iu)\dots(R-p-iu)}. \tag{5.6}$$

The statement of the lemma follows easily from the latter equation. □

For the self-quanto call option the Fourier transform of the dampened payoff function has a similar form as (5.6) for the case $p = 1$. This means similar properties hold for the self-quanto call option as for the regular call and put options.
Unfortunately, payoff functions of digital options do not verify (5.2) and (5.3) is only fulfilled for $k - l \leq 1$. This follows directly from the expressions of their dampened function (see [11]).

5.2 Examples of driving Lévy processes

In the sequel we first give examples of Lévy processes allowing formulas (2.11) or (2.34) for the optimal hedging number. We state the following lemma in which we compute upper bounds for the characteristic functions of the Lévy processes L and L^ε. We present the proof in the Appendix.

Lemma 5.2 *Consider L, L^ε with characteristic triplet $(\tilde{a}, b^2, \tilde{\ell})$, $(\tilde{a}_\varepsilon, b^2 + s^2(\varepsilon), \tilde{\ell}_\varepsilon)$, respectively. Assume R guarantees that $\int_{|z|\geq 1} \mathrm{e}^{Rz}\tilde{\ell}(dz)$ and $\int_{|z|\geq 1} \mathrm{e}^{Rz}\tilde{\ell}_\varepsilon(dz)$ are finite, then it holds for all $u \in \mathbb{R}$ and all $t \in [0,T]$ that*

$$|\widetilde{\Phi}_{T-t}(-u - iR)| \leq K(T,R)\exp\{-\frac{1}{2}(T-t)b^2u^2\}\,, \tag{5.7}$$

$$|\widetilde{\Phi}^\varepsilon_{T-t}(-u - iR)| \leq \widetilde{K}(T,R)\exp\{-\frac{1}{2}(T-t)(b^2 + s^2(\varepsilon))u^2\}\,. \tag{5.8}$$

where $K(T,R)$ and $\widetilde{K}(T,R)$ are positive constants depending on T and R.

Notice that this lemma implies the following:

– Both characteristic functions are bounded in u.
– For $b \neq 0$, we have for all $k \in \mathbb{N}$,

$$|u|^k|\widetilde{\Phi}_{T-t}(-u - iR)| \in L^1(\mathbb{R}) \text{ and } |u|^k|\widetilde{\Phi}^\varepsilon_{T-t}(-u - iR)| \in L^1(\mathbb{R})\,, \tag{5.9}$$

  which are necessary for the hedging formulas (2.11) and (2.34) to hold.
– For $b = 0$ and considering the original Lévy process L, we cannot conclude (5.9) from the upper bound (5.7). However, we refer to Proposition 28.3 in [22], in which it is shown that under certain conditions on the Lévy measure around 0, the characteristic function of several Lévy processes such as Normal Inverse Gaussian (NIG), Carr, Geman, Madan, and Yor (CGMY), and symmetric stable processes verify (5.9).
– For $b = 0$ and considering the approximating process L^ε, we differentiate between two cases. When $s(\varepsilon) = 0$, then we cannot conclude (5.9) from the upper bound (5.8). We cannot follow the approach in Proposition 28.3 in [22] either, since we do not have small jumps in the approximating process. In the case $s(\varepsilon) \neq 0$, (5.9) is fulfilled and we can write the hedging formula (2.34).

In the following lemma we show that condition (5.1) is always fulfilled in the case $b \neq 0$.

Lemma 5.3 *For a Lévy process and its approximation with characteristic triplets $(\tilde{a}, b^2, \tilde{\ell})$ and $(\tilde{a}_\varepsilon, b^2 + s^2(\varepsilon), \tilde{\ell}_\varepsilon)$ respectively, where $b \neq 0$, it holds that there exists a function $\widehat{\Psi}$ satisfying* (5.1).

Proof Lemma 5.2 suggests to define $\widehat{\Psi}_{T-t}(u; R) = \widetilde{K}(T,R)\mathrm{e}^{-\frac{1}{2}(T-t)b^2u^2}$. This function clearly guarantees (5.2) and (5.3). □

Thus the robustness results hold true for all Lévy processes which have a diffusion term. Of course the model should also satisfy the existence of exponential moments. This is necessary for the pricing and hedging formulas and also for the existence of the ET and the MEMM. We claim that all the Lévy processes mentioned in this subsection have exponential moments (see e.g. [8]). Unfortunately, in the case there is no Brownian motion component in the original model we cannot conclude.

Combining this discussion about the driving Lévy processes with the discussion in Section 5.1 devoted to the payoff functions, leads to the following concluding remarks.

Remark 5.4 (Concluding remarks)

– It turns out, from Section 5.1, that the Fourier transform of the dampened payoff function can guarantee enough integrability properties by itself. However this is not always the case for most familiar payoff functions.

– All conditions guaranteeing the existence of the Fourier transform formulas and of the robustness results, are fulfilled when the original Lévy process has a non-trivial Brownian motion part, whether the small jumps are truncated or replaced in the approximation.
– In the absence of a Brownian motion in the original Lévy process, the approximating formula (2.34) exists when the small jumps are replaced by a scaled Brownian motion. However, we cannot confirm that (2.34) exists when we truncate the small jumps in the approximation. Although, as remarked above all conditions can be satisfied if the payoff function provides strong integrability conditions.

6 Conclusion

In this paper we considered an incomplete market where stock price dynamics are modelled at any time $t \in [0,T]$, by $S_t = S_0 e^{L_t}$, with L being a Lévy process under the physical measure. Considering the approximation (1.1), constructed either by truncating the small jumps or by substituting them by a scaled Brownian motion, we observed different models for the dynamics of the stock price process. In Sections 3 and 4 we showed that the quadratic hedging strategies for these models in the martingale as well as in the semimartingale setting are robust under certain integrability conditions. We discussed these integrability conditions and gave some examples to illustrate our results in Section 5.

As far as further investigations are concerned, we consider in another paper a time-discretization of the original stock price model and of its approximations and we study the robustness of the quadratic hedging strategies to the choice of the models. Moreover, we aim at investigating the formulas for pricing and hedging that involve Fourier transform techniques in case there are no small jumps and no Brownian component in the model and to relax the integrability conditions. Finally, we aim at extending our work to a multidimensional setting.

7 Appendix

Proof (of Lemma 3.1) Note that the robustness result follows directly from the existence of a convergence rate, therefore we only determine the convergence rate here. We compute for R satisfying $\int_{|z|\geq 1} e^{Rz}\widetilde{\ell}(dz) < \infty$ and $(\mathcal{A}2)$, $u \in \mathbb{R}$, and $t \in [0,T]$ that

$$|\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR) - \widetilde{\Phi}_t^{\Theta_0}(-u-iR)| = |\widetilde{\Phi}_t^{\Theta_0}(-u-iR)|\Big|\frac{\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR)}{\widetilde{\Phi}_t^{\Theta_0}(-u-iR)} - 1\Big|.$$

We introduce the real numbers $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{I}}$ such that

$$\exp(\widetilde{\mathcal{R}} + i\widetilde{\mathcal{I}}) := \frac{\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR)}{\widetilde{\Phi}_t^{\Theta_0}(-u-iR)},$$

i.e.

$$\widetilde{\mathcal{R}} + i\widetilde{\mathcal{I}} = \log\Big(\frac{\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR)}{\widetilde{\Phi}_t^{\Theta_0}(-u-iR)}\Big) = \log(\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR)) - \log(\widetilde{\Phi}_t^{\Theta_0}(-u-iR)).$$

Therefore it turns out, according to the result (2.27) in the proof of Lemma 2.9, that

$$\begin{aligned}|\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR) - \widetilde{\Phi}_t^{\Theta_0}(-u-iR)| &= |\widetilde{\Phi}_t^{\Theta_0}(-u-iR)||\exp(\widetilde{\mathcal{R}} + i\widetilde{\mathcal{I}})1 - \exp(\widetilde{\mathcal{R}} + i\widetilde{\mathcal{I}})0| \\ &\leq |\widetilde{\Phi}_t^{\Theta_0}(-u-iR)|2(|\widetilde{\mathcal{R}}| + |\widetilde{\mathcal{I}}|)\max(e^{\widetilde{\mathcal{R}}}, 1),\end{aligned}$$

where we used the fact that for two numbers v, w on the line $L_{0,1}$ it holds that $\max(e^{\widetilde{\mathcal{R}}v}, e^{\widetilde{\mathcal{R}}w}) \leq 1$, for $\widetilde{\mathcal{R}} < 0$ and $\max(e^{\widetilde{\mathcal{R}}v}, e^{\widetilde{\mathcal{R}}w}) \leq e^{\widetilde{\mathcal{R}}}$, for $\widetilde{\mathcal{R}} > 0$. Since

$$|\widetilde{\Phi}_t^{\Theta_0}(-u-iR)|e^{\widetilde{\mathcal{R}}} = |\widetilde{\Phi}_t^{\Theta_0}(-u-iR)|\Big|\frac{\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR)}{\widetilde{\Phi}_t^{\Theta_0}(-u-iR)}\Big| = |\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR)|,$$

we obtain that

$$|\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR)-\widetilde{\Phi}_t^{\Theta_0}(-u-iR)|\leq 2(|\widetilde{\mathcal{R}}|+|\widetilde{\mathcal{I}}|)\max(|\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR)|,|\widetilde{\Phi}_t^{\Theta_0}(-u-iR)|).$$

Next we determine the real numbers $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{I}}$. By (2.3) it turns out that

$$\begin{aligned}\widetilde{\mathcal{R}}+i\widetilde{\mathcal{I}}&=\log(\widetilde{\Phi}_t^{\Theta_\varepsilon}(-u-iR))-\log(\widetilde{\Phi}_t^{\Theta_0}(-u-iR))\\&=t[i\tilde{a}_\varepsilon(-u-iR)-\frac{1}{2}(b^2+s^2(\varepsilon))(-u-iR)^2+\int_{\mathbb{R}_0}(\mathrm{e}^{i(-u-iR)z}-1-i(-u-iR)z1_{|z|<1})\tilde{\ell}_\varepsilon(dz)]\\&\qquad-t[i\tilde{a}(-u-iR)-\frac{1}{2}b^2(-u-iR)^2+\int_{\mathbb{R}_0}(\mathrm{e}^{i(-u-iR)z}-1-i(-u-iR)z1_{|z|<1})\tilde{\ell}(dz)]\\&=t[(\tilde{a}_\varepsilon-\tilde{a})R-\frac{1}{2}s^2(\varepsilon)(u^2-R^2)+\int_{\mathbb{R}_0}(\mathrm{e}^{Rz}\cos(uz)-1-Rz1_{|z|<1})(\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz))]\\&\qquad+it[(\tilde{a}_\varepsilon-\tilde{a})(-u)-s^2(\varepsilon)uR+\int_{\mathbb{R}_0}(\mathrm{e}^{Rz}\sin(-uz)+uz1_{|z|<1})(\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz))].\end{aligned}$$

Hence for the real part we compute that

$$|\widetilde{\mathcal{R}}|\leq t\Big[|\tilde{a}_\varepsilon-\tilde{a}||R|+\frac{1}{2}s^2(\varepsilon)(u^2+R^2)+\int_{\mathbb{R}_0}|\mathrm{e}^{Rz}\cos(uz)-1-Rz1_{|z|<1}||\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)|\Big],$$

for the imaginary part we obtain that

$$|\widetilde{\mathcal{I}}|\leq t\Big[|\tilde{a}_\varepsilon-\tilde{a}||u|+s^2(\varepsilon)|u||R|+\int_{\mathbb{R}_0}|\mathrm{e}^{Rz}\sin(-uz)+uz1_{|z|<1}||\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)|\Big].$$

Now we focus on the integrals w.r.t. $|\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)|$. For the real part $\widetilde{\mathcal{R}}$ and for $|z|<1$, the mean value theorem (MVT) applied to the function $z\mapsto \mathrm{e}^{Rz}\cos(uz)-Rz$ guarantees the existence of $z^*\in L_{0,z}$ and another application of the MVT to $z\mapsto R\mathrm{e}^{Rz}\cos(uz)-u\mathrm{e}^{Rz}\sin(uz)$ leads to $z^{**}\in L_{0,z^*}$, such that

$$\begin{aligned}|\mathrm{e}^{Rz}\cos(uz)-1-Rz|&=|R\mathrm{e}^{Rz^*}\cos(uz^*)-u\mathrm{e}^{Rz^*}\sin(uz^*)-R||z|\\&=|R^2\mathrm{e}^{Rz^{**}}\cos(uz^{**})-2uR\mathrm{e}^{Rz^{**}}\sin(uz^{**})+u^2\mathrm{e}^{Rz^{**}}\cos(uz^{**})||z^*||z|\\&\leq C_1(R)(1+|u|+u^2)z^2.\end{aligned}\tag{.1}$$

Hence it appears by definition (2.22) of $\sigma^2(\varepsilon)$ and assumption ($\mathcal{A}2$) that

$$\begin{aligned}&\int_{\mathbb{R}_0}|\mathrm{e}^{Rz}\cos(uz)-1-Rz1_{|z|<1}||\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)|\\&=\int_{|z|<\varepsilon}|\mathrm{e}^{Rz}\cos(uz)-1-Rz|\tilde{\ell}(dz)+\int_{\varepsilon\leq|z|<1}|\mathrm{e}^{Rz}\cos(uz)-1-Rz||\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)|\\&\qquad+\int_{|z|\geq1}|\mathrm{e}^{Rz}\cos(uz)-1||\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)|\\&\leq C_2(R,\Theta_0)(1+|u|+u^2)\int_{|z|<\varepsilon}z^2\ell(dz)+C_1(R)(1+|u|+u^2)\int_{\varepsilon\leq|z|<1}z^2\gamma(z)\tilde{s}^2(\varepsilon)\ell(dz)\\&\qquad+\int_{|z|\geq1}(\mathrm{e}^{Rz}+1)\gamma(z)\tilde{s}^2(\varepsilon)\ell(dz)\\&\leq C_3(R,\Theta_0)(1+|u|+u^2)\tilde{s}^2(\varepsilon).\end{aligned}$$

For the imaginary part, again for $|z|<1$ we obtain by similar applications of the MVT as in (.1) that

$$|\mathrm{e}^{Rz}\sin(-uz)+uz|\leq C_4(R)(1+|u|+u^2)z^2.$$

Combining this result with (2.22) and ($\mathcal{A}2$) shows that

$$\begin{aligned}
&\int_{\mathbb{R}_0} |\mathrm{e}^{Rz}\sin(-uz)+uz1_{|z|<1}||\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)| \\
&= \int_{|z|<\varepsilon} |\mathrm{e}^{Rz}\sin(uz)-uz|\tilde{\ell}(dz) + \int_{\varepsilon\le|z|<1} |\mathrm{e}^{Rz}\sin(uz)-uz||\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)| \\
&\qquad + \int_{|z|\ge1} |\mathrm{e}^{Rz}\sin(uz)||\tilde{\ell}_\varepsilon(dz)-\tilde{\ell}(dz)| \\
&\le C_5(R,\Theta_0)(1+|u|+u^2)\int_{|z|<\varepsilon} z^2\ell(dz) + C_4(R)(1+|u|+u^2)\int_{\varepsilon\le|z|<1} z^2\gamma(z)\tilde{s}^2(\varepsilon)\ell(dz) \\
&\qquad + \int_{|z|\ge1} \mathrm{e}^{Rz}\gamma(z)\tilde{s}^2(\varepsilon)\ell(dz) \\
&\le C_6(R,\Theta_0)(1+|u|+u^2)\tilde{s}^2(\varepsilon).
\end{aligned}$$

Reminding that assumption ($\mathcal{A}1$) is also in force, concludes the proof of result (3.3). □

Proof (of Proposition 3.2) Recall from (2.7) and (2.32) the expressions for the option prices of both models we are considering in this paper. As a consequence of the robustness of the stock price process and the characteristic function, see Lemmas 2.9 and 3.1, we know that

$$\begin{aligned}
\lim_{\varepsilon\to0} P^\varepsilon(t,S_t^\varepsilon) &= \lim_{\varepsilon\to0}\frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu}du \\
&= \frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)S_t^{R-iu}du = P(t,S_t), \qquad \mathbb{P}\text{-a.s.}
\end{aligned}$$

It is allowed to interchange limit and integral in the computation above using the dominated convergence theorem (see e.g. [12]). Indeed, from the fact that $|(S_t^\varepsilon)^{R-iu}| = (S_t^\varepsilon)^R$ and using condition (3.4) and equation (3.8), we conclude

$$|\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu}| \le K_1(T,R,\Theta_0)|\hat{f}(u+iR)|\widetilde{\Psi}_{T-t}(u;R), \qquad \mathbb{P}\text{-a.s.}$$

The right hand side is integrable w.r.t. u by assumption (3.4).

In the following we compute a rate of the convergence for the approximating option price process to the underlying option process. We have

$$\begin{aligned}
&\mathbb{E}[|P^\varepsilon(t,S_t^\varepsilon)-P(t,S_t)|] \\
&\quad= \mathbb{E}\Big[\Big|\frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu}du \\
&\qquad\quad - \frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)S_t^{R-iu}du\Big|\Big] \\
&\quad\le \frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}|\hat{f}(u+iR)|\mathbb{E}\Big[\Big|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu}-\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)S_t^{R-iu}\Big|\Big]du.
\end{aligned}$$

Applying the triangle inequality on the second factor in the integrand of the latter expression, we get that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu}-\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)S_t^{R-iu}\Big|\Big] \\
&\le \big|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)\big|\mathbb{E}\Big[\Big|(S_t^\varepsilon)^{R-iu}-S_t^{R-iu}\Big|\Big] + \big|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)-\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)\big|\mathbb{E}\big[|S_t^{R-iu}|\big].
\end{aligned}$$

By applying Lemmas 2.9 and 3.1, and assumption (3.5) it follows

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S_t^\varepsilon)^{R-iu}-\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)S_t^{R-iu}\Big|\Big] \\
&\le K_2(T,R,\Theta_0)(1+|u|+u^2)\widehat{\Psi}_{T-t}(u;R)\tilde{s}(\varepsilon).
\end{aligned} \tag{.2}$$

Finally the result follows by the integrability condition (3.5). □

Proof (of Proposition 3.4) The robustness of the delta can be proved in the same way as for the option price. For the integrand of expression (2.33), we know by condition (3.6) and (3.8) that

$$|(R-iu)\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S^\varepsilon_t)^{R-iu-1}| \leq K_1(T,R,\Theta_0)(1+|u|)|\hat{f}(u+iR)|\widetilde{\Psi}_{T-t}(u;R),$$

$\mathbb{P}$-a.s., for a random variable $K_1(T,R,\Theta_0)$, which is independent of u. Thus we take the limit inside the integral and the result follows. Moreover, we have that

$$\begin{aligned}\mathbb{E}[|\Delta^\varepsilon_t-\Delta_t|] &= \frac{\mathrm{e}^{-r(T-t)}}{2\pi}\mathbb{E}\Big[\Big|\int_{\mathbb{R}}(R-iu)\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S^\varepsilon_t)^{R-iu-1}du \\ &\qquad -\int_{\mathbb{R}}(R-iu)\hat{f}(u+iR)\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)S_t^{R-iu-1}du\Big|\Big] \\ &\leq \frac{\mathrm{e}^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}|R-iu||\hat{f}(u+iR)| \\ &\qquad \times\mathbb{E}\Big[\Big|\widetilde{\Phi}^{\Theta_\varepsilon}_{T-t}(-u-iR)(S^\varepsilon_t)^{R-iu-1}-\widetilde{\Phi}^{\Theta_0}_{T-t}(-u-iR)S_t^{R-iu-1}\Big|\Big]du.\end{aligned}$$

Using similar arguments as in the proof of Proposition 3.2 we obtain that

$$\mathbb{E}[|\Delta^\varepsilon_t-\Delta_t|] \leq \frac{\mathrm{e}^{-r(T-t)}}{2\pi}K_2(T,R,\Theta_0)\int_{\mathbb{R}}(1+|u|+u^2+|u|^3)|\hat{f}(u+iR)|\widehat{\Psi}_{T-t}(u;R)du\;\tilde{s}(\varepsilon)$$

and the result follows by assumption (3.7). □

Proof (of robustness property (3.9)*)* Based on a change of measure, the triangle and Cauchy-Schwarz inequality, we derive that

$$\begin{aligned}&|\widetilde{\Phi}^{\Theta_\varepsilon}_t(-u-iR)-\widetilde{\Phi}^{\Theta_0}_t(-u-iR)| \\ &=\Big|\widetilde{\mathbb{E}}_{\Theta_\varepsilon}[\mathrm{e}^{(R-iu)L^\varepsilon_t}]-\widetilde{\mathbb{E}}_{\Theta_0}[\mathrm{e}^{(R-iu)L_t}]\Big| \\ &=\Big|\mathbb{E}\Big[\mathrm{e}^{(R-iu)L^\varepsilon_t}\frac{d\widetilde{\mathbb{P}}_{\Theta_\varepsilon}}{d\mathbb{P}}-\mathrm{e}^{(R-iu)L_t}\frac{d\widetilde{\mathbb{P}}_{\Theta_0}}{d\mathbb{P}}\Big]\Big| \\ &\leq \mathbb{E}\Big[|\mathrm{e}^{(R-iu)L^\varepsilon_t}-\mathrm{e}^{(R-iu)L_t}|\Big|\frac{d\widetilde{\mathbb{P}}_{\Theta_\varepsilon}}{d\mathbb{P}}\Big|\Big]+\mathbb{E}\Big[\mathrm{e}^{RL_t}\Big|\frac{d\widetilde{\mathbb{P}}_{\Theta_\varepsilon}}{d\mathbb{P}}-\frac{d\widetilde{\mathbb{P}}_{\Theta_0}}{d\mathbb{P}}\Big|\Big] \\ &\leq \mathbb{E}[|\mathrm{e}^{(R-iu)L^\varepsilon_t}-\mathrm{e}^{(R-iu)L_t}|^2]^{\frac{1}{2}}\mathbb{E}\Big[\Big|\frac{d\widetilde{\mathbb{P}}_{\Theta_\varepsilon}}{d\mathbb{P}}\Big|^2\Big]^{\frac{1}{2}}+\mathbb{E}[\mathrm{e}^{2RL_t}]^{\frac{1}{2}}\mathbb{E}\Big[\Big|\frac{d\widetilde{\mathbb{P}}_{\Theta_\varepsilon}}{d\mathbb{P}}-\frac{d\widetilde{\mathbb{P}}_{\Theta_0}}{d\mathbb{P}}\Big|^2\Big]^{\frac{1}{2}}.\end{aligned}$$

Moreover using similar arguments as in Lemma 2.9 and considering the robustness of the driving Lévy process L in $L^4(\Omega,\mathcal{F},\mathbb{P})$, we get

$$\mathbb{E}[|\mathrm{e}^{(R-iu)L^\varepsilon_t}-\mathrm{e}^{(R-iu)L_t}|^2] \leq K(T,R)(1+|u|)^2\hat{s}^2(\varepsilon).$$

Therefore we conclude that expression (3.9) holds by assumptions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$. □

Proof (of Lemma 4.1) For $z=\mathcal{R}+i\mathcal{I}\in\mathbb{C}$ and $(x,y)\in\mathbb{R}^2$ and applying the MVT, we know there exist $\{v,w\}\subset L_{x,y}$, $v'\in L_{0,v}$, and $w'\in L_{0,w}$ satisfying

$$\begin{aligned}&|\mathrm{e}^{zx}-zx-(\mathrm{e}^{zy}-zy)| \\ &\leq |\mathrm{e}^{\mathcal{R}x}\cos(\mathcal{I}x)-\mathcal{R}x-(\mathrm{e}^{\mathcal{R}y}\cos(\mathcal{I}y)-\mathcal{R}y)|+|\mathrm{e}^{\mathcal{R}x}\sin(\mathcal{I}x)-\mathcal{I}x-(\mathrm{e}^{\mathcal{R}y}\sin(\mathcal{I}y)-\mathcal{I}y)| \\ &\leq |\mathcal{R}\mathrm{e}^{\mathcal{R}v}\cos(\mathcal{I}v)-\mathcal{I}\mathrm{e}^{\mathcal{R}v}\sin(\mathcal{I}v)-\mathcal{R}||x-y|+|\mathcal{R}\mathrm{e}^{\mathcal{R}w}\sin(\mathcal{I}w)+\mathcal{I}\mathrm{e}^{\mathcal{R}w}\cos(\mathcal{I}w)-\mathcal{I}||x-y| \\ &\leq |\mathcal{R}^2\mathrm{e}^{\mathcal{R}v'}\cos(\mathcal{I}v')-2\mathcal{R}\mathcal{I}\mathrm{e}^{\mathcal{R}v'}\sin(\mathcal{I}v')-\mathcal{I}^2\mathrm{e}^{\mathcal{R}v'}\cos(\mathcal{I}v')||v||x-y| \\ &\quad +|\mathcal{R}^2\mathrm{e}^{\mathcal{R}w'}\sin(\mathcal{I}w')+2\mathcal{R}\mathcal{I}\mathrm{e}^{\mathcal{R}w'}\cos(\mathcal{I}w')-\mathcal{I}^2\mathrm{e}^{\mathcal{R}w'}\sin(\mathcal{I}w')||w||x-y| \\ &\leq (\mathcal{R}^2+2|\mathcal{R}||\mathcal{I}|+\mathcal{I}^2)(\mathrm{e}^{\mathcal{R}v'}|v|+\mathrm{e}^{\mathcal{R}w'}|w|)|x-y|\,.\end{aligned}$$

Specifically for $y = 0$ this implies that

$$|\mathrm{e}^{zx} - 1 - zx| \leq (\mathcal{R}^2 + 2|\mathcal{R}||\mathcal{I}| + \mathcal{I}^2)2\mathrm{e}^{|\mathcal{R}||x|}x^2\,. \tag{.3}$$

We use the latter result to prove the robustness of the cumulant generating function. Indeed, we know that, to have that,

$$\kappa^\varepsilon(z) = \log \mathbb{E}[\mathrm{e}^{zL_1^\varepsilon}] = az + \frac{1}{2}(b^2 + s^2(\varepsilon))z^2 + \int_{|x|\geq\varepsilon}(\mathrm{e}^{zx} - 1 - zx1_{|x|<1})\ell(dx)\,,$$
$$\kappa(z) = az + \frac{1}{2}b^2z^2 + \int_{\mathbb{R}_0}(\mathrm{e}^{zx} - 1 - zx1_{|x|<1})\ell(dx)\,.$$

Clearly it holds that $\lim_{\varepsilon\to 0}\kappa^\varepsilon(z) = \kappa(z)$, $\forall z \in \mathbb{C}$. Next we compute the convergence rate for $z = \mathcal{R} + i\mathcal{I}$, thereto we compute by using (.3),

$$\begin{aligned}
|\kappa^\varepsilon(z) - \kappa(z)| &= \Big|\frac{1}{2}s^2(\varepsilon)z^2 - \int_{|x|<\varepsilon}(\mathrm{e}^{zx} - 1 - zx)\ell(dx)\Big| \\
&\leq \frac{1}{2}s^2(\varepsilon)|z|^2 + \int_{|x|<\varepsilon}(\mathcal{R}^2 + 2|\mathcal{R}||\mathcal{I}| + \mathcal{I}^2)2\mathrm{e}^{|\mathcal{R}||x|}x^2\ell(dx) \\
&\leq \frac{1}{2}s^2(\varepsilon)(\mathcal{R}^2 + \mathcal{I}^2) + (\mathcal{R}^2 + 2|\mathcal{R}||\mathcal{I}| + \mathcal{I}^2)2\mathrm{e}^{|\mathcal{R}|}\sigma^2(\varepsilon).
\end{aligned}$$

Thus for $z = R - iu$, $u \in \mathbb{R}$, this results into (4.1). For the robustness of the function μ, we recall condition (2.21) and obtain for $z \in \mathbb{C}$

$$\begin{aligned}
&|\mu^\varepsilon(z) - \mu(z)| \\
&= \Big|\frac{(b^2 + s^2(\varepsilon))z + \int_{|x|\geq\varepsilon}(\mathrm{e}^{zx} - 1)(\mathrm{e}^x - 1)\ell(dx)}{b^2 + s^2(\varepsilon) + \int_{|x|\geq\varepsilon}(\mathrm{e}^x - 1)^2\ell(dx)} - \frac{b^2z + \int_{\mathbb{R}_0}(\mathrm{e}^{zx} - 1)(\mathrm{e}^x - 1)\ell(dx)}{b^2 + \int_{\mathbb{R}_0}(\mathrm{e}^x - 1)^2\ell(dx)}\Big| \\
&\leq K\Big|\Big[(b^2 + s^2(\varepsilon))z + \int_{|x|\geq\varepsilon}(\mathrm{e}^{zx} - 1)(\mathrm{e}^x - 1)\ell(dx)\Big]\Big[b^2 + \int_{\mathbb{R}_0}(\mathrm{e}^x - 1)^2\ell(dx)\Big] \\
&\qquad - \Big[b^2z + \int_{\mathbb{R}_0}(\mathrm{e}^{zx} - 1)(\mathrm{e}^x - 1)\ell(dx)\Big]\Big[b^2 + s^2(\varepsilon) + \int_{|x|\geq\varepsilon}(\mathrm{e}^x - 1)^2\ell(dx)\Big]\Big| \\
&= K\Big|(b^2 + s^2(\varepsilon))zb^2 + (b^2 + s^2(\varepsilon))z\int_{\mathbb{R}_0}(\mathrm{e}^x - 1)^2\ell(dx) + b^2\int_{|x|\geq\varepsilon}(\mathrm{e}^{zx} - 1)(\mathrm{e}^x - 1)\ell(dx) \\
&\qquad + \int_{|x|\geq\varepsilon}(\mathrm{e}^{zx} - 1)(\mathrm{e}^x - 1)\ell(dx)\int_{\mathbb{R}_0}(\mathrm{e}^x - 1)^2\ell(dx) \\
&\qquad - b^2z(b^2 + s^2(\varepsilon)) - b^2z\int_{|x|\geq\varepsilon}(\mathrm{e}^x - 1)^2\ell(dx) - (b^2 + s^2(\varepsilon))\int_{\mathbb{R}_0}(\mathrm{e}^{zx} - 1)(\mathrm{e}^z - 1)\ell(dx) \\
&\qquad - \int_{\mathbb{R}_0}(\mathrm{e}^{zx} - 1)(\mathrm{e}^x - 1)\ell(dx)\int_{|x|\geq\varepsilon}(\mathrm{e}^x - 1)^2\ell(dx)\Big| \\
&\leq K\Big[b^2|z|\int_{|x|<\varepsilon}(\mathrm{e}^x - 1)^2\ell(dx) + s^2(\varepsilon)|z|\int_{\mathbb{R}_0}(\mathrm{e}^x - 1)^2\ell(dx) + b^2\int_{|x|<\varepsilon}|\mathrm{e}^{zx} - 1||\mathrm{e}^x - 1|\ell(dx) \\
&\qquad + s^2(\varepsilon)\int_{\mathbb{R}_0}|\mathrm{e}^{zx} - 1||\mathrm{e}^x - 1|\ell(dx) + \int_{|x|\geq\varepsilon}|\mathrm{e}^{zx} - 1||\mathrm{e}^x - 1|\ell(dx)\int_{|x|<\varepsilon}(\mathrm{e}^x - 1)^2\ell(dx) \\
&\qquad + \int_{|x|<\varepsilon}|\mathrm{e}^{zx} - 1||\mathrm{e}^x - 1|\ell(dx)\int_{|x|\geq\varepsilon}(\mathrm{e}^x - 1)^2\ell(dx)\Big]
\end{aligned}$$

In the latter we know by the MVT that there exists an $x' \in L_{0,x}$ such that for $z = \mathcal{R} + i\mathcal{I}$

$$\int_{|x|<\varepsilon}(\mathrm{e}^x - 1)^2\ell(dx) = \int_{|x|<\varepsilon}x^2\mathrm{e}^{2x'}\ell(dx) \leq \mathrm{e}^2\sigma^2(\varepsilon),\text{ and}$$

$$\int_{|x|<\varepsilon} |\mathrm{e}^{zx} - 1||\mathrm{e}^{x} - 1|\ell(dx) \leq \int_{|x|<\varepsilon} 2(|\mathcal{R}| + |\mathcal{I}|)\mathrm{e}^{|\mathcal{R}||x|}|x||\mathrm{e}^{x'}x|\ell(dx) \leq 2(|\mathcal{R}| + |\mathcal{I}|)\mathrm{e}^{|\mathcal{R}|+1}\sigma^2(\varepsilon).$$

On the other hand we obtain by similar arguments that

$$\int_{|x|\geq\varepsilon} (\mathrm{e}^{x} - 1)^2\ell(dx) \leq \int_{\mathbb{R}_0} (\mathrm{e}^{x} - 1)^2\ell(dx) \leq \mathrm{e}^2 \int_{|x|<1} x^2\ell(dx) + \int_{|x|\geq 1} (\mathrm{e}^{2x} + 2\mathrm{e}^{x} + 1)\ell(dx)$$

and

$$\begin{aligned}\int_{|x|\geq\varepsilon} |\mathrm{e}^{zx} - 1||\mathrm{e}^{x} - 1|\ell(dx) &\leq \int_{\mathbb{R}_0} |\mathrm{e}^{zx} - 1||\mathrm{e}^{x} - 1|\ell(dx)\\ &\leq 2(|\mathcal{R}| + |\mathcal{I}|)\mathrm{e}^{|\mathcal{R}|+1} \int_{|x|<1} x^2\ell(dx) + \int_{|x|\geq 1} (\mathrm{e}^{\mathcal{R}x} + 1)(\mathrm{e}^{x} + 1)\ell(dx).\end{aligned}$$

Both right hand sides are finite by conditions (2.2) and (2.14). Thus for $z = R - iu$, $u \in \mathbb{R}$, this proofs (4.2). □

Proof (of Lemma 5.2) We compute by (2.2) and (2.3) that

$$\begin{aligned}|\widetilde{\Phi}_t(-u - iR)| &= \Big| \exp\Big(t\Big\{i\tilde{a}(-u - iR) - \frac{1}{2}b^2(-u - iR)^2\\ &\qquad + \int_{\mathbb{R}_0} (\mathrm{e}^{i(-u-iR)z} - 1 - i(-u - iR)z1_{|z|<1})\tilde{\ell}(dz)\Big\}\Big)\Big|\\ &= \exp\Big(t\Big\{\tilde{a}R - \frac{1}{2}b^2(u^2 - R^2) + \int_{\mathbb{R}_0} (\mathrm{e}^{Rz}\cos(uz) - 1 - Rz1_{|z|<1})\tilde{\ell}(dz)\Big\}\Big)\\ &\leq C_1(T, R)\mathrm{e}^{-\frac{1}{2}tb^2u^2} \exp\Big(t\int_{\mathbb{R}_0} (\mathrm{e}^{Rz}\cos(uz) - 1 - Rz1_{|z|<1})\tilde{\ell}(dz)\Big)\\ &\leq C_1(T, R)\mathrm{e}^{-\frac{1}{2}tb^2u^2} \exp\Big(t\int_{\mathbb{R}_0} (\mathrm{e}^{Rz} - 1 - Rz1_{|z|<1})\tilde{\ell}(dz)\Big) \leq C_2(T, R)\mathrm{e}^{-\frac{1}{2}tb^2u^2}.\end{aligned}$$

Upper bound (5.8) can be obtained through similar computations as above. □

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