Bayesian model updating of a tunnel in soft soil with settlement measurements

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ABSTRACT: Monitoring is an essential element of modern tunneling construction. The most common monitoring method is measuring displacements, for example convergence of the tunnel opening or surface settlements. Measurement outcomes can be used to update the knowledge on material properties of the soil or other parameters that enter numerical models of the structural behavior of the tunnel. In probability theory, this process can be formalized in the concept of Bayesian updating. In this paper, we apply the Bayesian concept to update the numerical model of a tunnel in soft soil conditional on settlement measurements. The tunnel is constructed by means of the conventional tunneling method and modeled with 2D finite elements applying the stress reduction method. We assume that settlement measurements are taken at full excavation and utilize the measurements to update the material properties of the soil as well as the the relaxation factor of the stress reduction method. Updating is performed by means of BUS, a recently proposed method for Bayesian updating of mechanical models with structural reliability methods.

1 INTRODUCTION

In tunneling design, engineers establish numerical models of the tunnel excavation and conduct structural analyses to predict the stresses and deformations for the considered designs. However, there is significant uncertainty in the choice of the model parameters. Uncertainties may be related to the inherent spatial variability of the mechanical properties of the soil but also to the application of dimensionally reduced models to represent complex phenomena. A proper assessment of the safety and serviceability of the structural design involves the modeling of the uncertainties by use of probabilistic models and the evaluation of the structural reliability against the respective design requirements.

During the tunnel construction process, measurements of physical quantities such as deformations and stresses are typically conducted. Measurements can be used to compare predictions of the numerical model with the actual structural behavior, to verify the reliability of the structural design as well as to update the probabilistic description of the parameters of the numerical model. The latter is formalized in the concept of Bayesian updating. Thereby, a prior probabilistic model is updated with new data and information to a posterior probabilistic model.

Bayesian updating requires the solution of a potentially high-dimensional integral to obtain the posterior distribution of the model parameters. Commonly, Markov Chain Monte Carlo (MCMC) sampling is used to sample directly from the posterior distribution, thus bypassing the solution of the aforementioned integral (Gelman 2004). An alternative approach is based on interpreting the updating problem as a structural reliability problem (Straub & Papaioannou 2013). This approach, termed BUS, applies methods originally developed for structural reliability analysis to obtain samples from the posterior distribution.

In this paper, we apply BUS to learn the model parameters of a tunnel in soft soil using settlement measurements. The tunnel is constructed by the conventional tunneling method. We model the tunnel in 2D using nonlinear plain-strain finite elements and the 3D arching effect is approximated by application of the stress reduction method. Using assumed settlement measurements at full excavation, we update the material properties of the soil as well as the relaxation factor of the stress reduction method.

2 MODEL DESCRIPTION

2.1 *Mechanical model*

A conventional driven tunnel with a horse-shoe shaped profile is considered in this study (see Fig. 1).

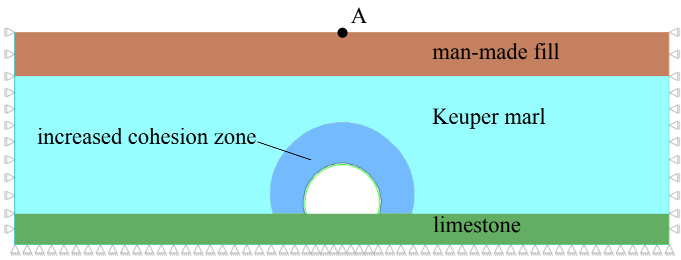


Figure 1: Ground layers considered in the model.

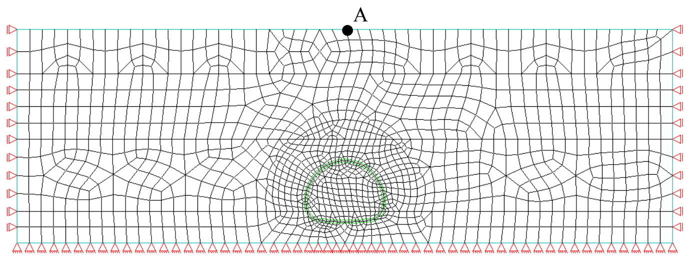


Figure 2: Finite element mesh.

The problem is modeled in the SOFiSTiK finite element (FE) software package (SOFiSTiK AG 2012), using 2D plain strain finite elements. The numerical model has a width of $80\mathrm{m}$ and a height of $26\mathrm{m}$. The FE mesh is shown in Figure 2. In this study, we are interested in surface settlements over the tunnel center line (point A in Fig. 1). The excavation process is modeled by application of the stress reduction method (Panet & Guenot 1982). In this method, a prescribed fraction β of the initial stress is left inside the tunnel as a support pressure to approximately account for the three-dimensional arching effect. This support pressure is then removed after installation of the lining. The parameter β is termed relaxation factor.

The model consists of three different ground layers; the layers are illustrated in Figure 1. The cover layer is a man-made fill and has a depth of $5.4\mathrm{m}$. Heavily weathered soft rock known as Keuper marl forms the second layer. The thickness of this layer is $16.8\mathrm{m}$. We adopt a hardening plasticity soil model (SOFiSTiK AG 2012) to describe the material behavior of the first two layers. This material model allows for a realistic description of the stiffness and hardening behavior of soft soil in settlement analysis (Möller 2006). The material properties of the cover layer are as follows: elastic modulus for unloading-reloading: $30\mathrm{MPa}$, Poisson's ratio: 0.2, specific weight: $20\mathrm{kN/m^3}$, friction angle: $25°$, cohesion: $10\mathrm{kPa}$, oedometric stiffness modulus: $10\mathrm{MPa}$, stiffness modulus for primary loading: $10\mathrm{MPa}$. The exponent in the hardening law is selected as 0.5 for the first and the second layer. The angle of dilatancy is assumed as zero, corresponding to a non-associated flow rule. The soil parameters of the Keuper marl layer are modeled as random and their prior probabilistic description is discussed in Section 2.2. Strong limestone constitutes the bottom layer. The Mohr-Coulomb law is applied for this layer. The material properties are: Young's modulus: $575\mathrm{MPa}$, Poisson's ratio: 0.2, specific weight: $23\mathrm{kN/m^3}$, friction angle: $35°$, cohesion: $200\mathrm{kPa}$. Due to the much larger stiffness of the limestone compared to the stiffness of the overlaying materials, only $3.8\mathrm{m}$ of this layer are modeled.

The height of the tunnel above the limestone layer is $6.2\mathrm{m}$. Consequently, the tunnel is located in a depth of $16\mathrm{m}$ below the ground surface. At the intersection of the second and the third layer, the tunnel has a width of $9.16\mathrm{m}$. In the vicinity of the tunnel the Keuper marl is reinforced with nails. This is modeled by increasing the cohesion in the affected region (see Fig. 1) by $25\mathrm{kPa}$. Moreover, the tunnel is located above the groundwater level. The shotcrete lining is modeled using linear beam elements with a normal stiffness of $10.5\mathrm{GN}$ and a flexural rigidity of $26.78\mathrm{MNm^2}$.

2.2 *Prior probabilistic model*

The cover layer and the limestone layer are considered as deterministic in the analysis. Since the cover layer is a man-made fill, we assume that its soil properties are well-known, and the associated uncertainties are small compared to the uncertainties in the material description of the Keuper marl layer and hence can be neglected. The limestone layer is also modeled as deterministic, because, due to its large stiffness, the contribution of this layer to the surface settlements is negligible. The probability distributions describing the uncertainties in the material parameters of the Keuper marl layer are listed in Table 1. We assume that the stiffness modulus for primary loading E_{50} equals the oedometric stiffness modulus E_{oed}. We also consider a correlation of 0.7 between E_{oed} and the elastic modulus E_{ur}. The friction angle and the cohesion are assumed to have a negative correlation of -0.5.

In conventionally driven tunnels, there is usually a large uncertainty in the choice of the relaxation factor $\beta \in [0,1]$ of the stress reduction method (Möller 2006). In this study β is modeled as a beta-distributed random variable (see Table 1).

A reliability assessment of the tunnel was presented in Ranjan et al. (2013). Therein, a two-step approach is adopted. In a first step, the reliability analysis was performed applying the first order reliability method (FORM) [e.g. see (Der Kiureghian 2005)]. As a byproduct of the FORM, the influence coefficients provide information on the sensitivity of the reliability in terms of the random variables. This information was used to identify the random variables with the highest influence that, in a second step, are modeled as random fields.

Figure 3 depicts in a pie graph the squared influence coefficients obtained by the FORM. It is observed that the variable with the largest influence is the oedometric stiffness modulus E_{oed} followed by the relaxation factor β. Based on this result, we account

Table 1: Prior distribution of the parameters of the Keuper marl layer.

Parameter	Distribution	Mean	CV
Relaxation factor β	Beta(0.0,1.0)	0.5	10%
Elast. mod. E_{ur} [MPa]	Lognormal	80.0	32%
Oedometr. mod. E_{oed} [MPa]	Lognormal	30.0	32%
Poisson's ratio ν	Beta(0.0,0.5)	0.2	15%
Friction angle φ [°]	Beta(0.0,45.0)	20.0	15%
Cohesion c [kPa]	Lognormal	25.0	30%
Specific weight γ [kN/m^3]	Lognormal	24.0	5%

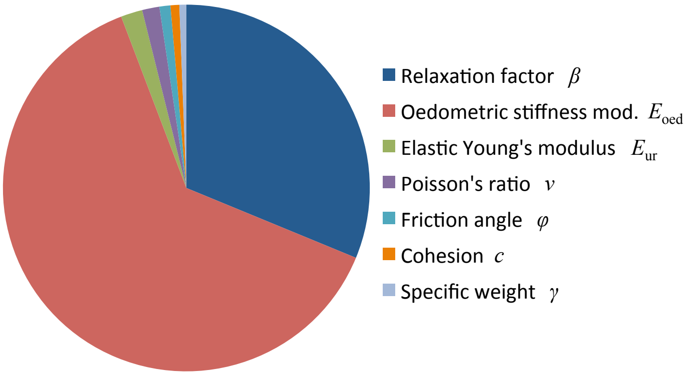


Figure 3: Squared influence coefficients obtained by FORM.

for the inherent spatial variability of E_{oed}. Since E_{oed} is strongly correlated with the elastic modulus E_{ur}, the spatial variability of the latter parameter is also modeled. This is achieved by modeling the two parameters as cross-correlated homogeneous random fields. The joint distribution of the two fields at each pair of locations is modeled by the Nataf distribution (Der Kiureghian & Liu 1986) with lognormal marginals according to Table 1. The spatial variability depends only on the separation in horizontal and vertical direction between two locations, Δx and Δy. The following exponential autocorrelation coefficient function is chosen for both random fields:

$$\rho(\Delta x, \Delta y) = \exp\left(-\frac{\Delta x}{l_x} - \frac{\Delta y}{l_y}\right) \quad (1)$$

where $l_x = 20$m and $l_y = 5$m denote the correlation lengths in horizontal and vertical direction, respectively. The cross-correlation coefficient function is:

$$\rho_{\mathrm{cross}}(\Delta x, \Delta y) = \rho_{\mathrm{c}} \cdot \rho(\Delta x, \Delta y) \quad (2)$$

where $\rho_{\mathrm{c}} = 0.7$ denotes the correlation of E_{oed} and E_{ur} at the same location.

Since the random fields have the Nataf distribution, they can be expressed as functions of correlated Gaussian fields. Due to the form of their cross-correlation function, the latter fields can be transformed to independent Gaussian fields by performing the Cholesky decomposition of the 2×2 correlation matrix, whose off-diagonal terms express the correlation of the two fields at the same location. The underlying independent Gaussian fields are discretized by application of the Karhunen-Loéve expansion (Ghanem & Spanos 1991). That is, each field is represented as a truncated series of products of the eigenfunctions of its autocorrelation function and independent random variables. Each random field is dicretized with 100 random variables. Therefore the total number of random variables in the problem is 205.

3 BAYESIAN UPDATING WITH STRUCTURAL RELIABILITY METHODS (BUS)

Let $\mathbf{X}$ denote the n-dimensional random vector representing the uncertain model parameters discussed in Section 2.2. Also, let $f(\mathbf{x})$ be the prior joint probability density function (PDF) of $\mathbf{X}$. Assume that a measurement $u_{\mathrm{A,m}}$ of the surface settlement (point A in Fig. 1) is made at full excavation. The measurement is subjected to an additive error ϵ which is described by a normal PDF f_ϵ with zero mean and standard deviation σ_ϵ. The measurement information can be described by the event $Z = \{u_{\mathrm{A,m}} - u_{\mathrm{A}}(\mathbf{x}) = \epsilon\}$, where $u_{\mathrm{A}}(\mathbf{x})$ is the surface settlement evaluated by the FE program for a realization $\mathbf{x}$ of the random vector $\mathbf{X}$. The corresponding likelihood function can be expressed as follows:

$$\begin{aligned} L(\mathbf{x}) &= f_\epsilon\left[u_{\mathrm{A,m}} - u_{\mathrm{A}}(\mathbf{x})\right] \\ &= \frac{1}{\sigma_\epsilon\sqrt{2\pi}} \exp\left(-\frac{(u_{\mathrm{A,m}} - u_{\mathrm{A}})^2}{2\sigma_\epsilon^2}\right) \end{aligned} \quad (3)$$

The posterior joint PDF of $\mathbf{X}$ conditional on the measurement event Z can be obtained by application of Bayes' rule:

$$f(\mathbf{x}|Z) = \frac{L(\mathbf{x})f(\mathbf{x})}{\int_{\mathbb{R}^n} L(\mathbf{x})f(\mathbf{x})d\mathbf{x}} \quad (4)$$

The evaluation of the n-fold integral in the denominator of Equation (4) is computationally demanding; this has motivated the application of MCMC algorithms for sampling directly from the posterior $f(\mathbf{x}|Z)$ (Gilks et al. 1998, Gelman 2004). Here, we apply an alternative approach, termed BUS, which uses methods originally developed for structural reliability analysis to obtain samples from the posterior distribution (Straub & Papaioannou 2013). The method is based on the algorithm for reliability updating developed in Straub (2011) and applied in Papaioannou & Straub (2012) to the reliability updating of geotechnical structures.

The BUS approach introduces the following limit state function:

$$h(\mathbf{x}, u_0) = u_0 - \Phi^{-1}\left[cL(\mathbf{x})\right] \quad (5)$$

where u_0 is the outcome of a standard normal random variable U_0, $\Phi^{-1}(.)$ is the inverse of the standard normal cumulative distribution function and c is a positive constant chosen to ensure that $cL(\mathbf{x}) \leq 1$. It is shown in Straub & Papaioannou (2013) that the posterior PDF $f(\mathbf{x}|Z)$ is proportional to the prior PDF $f(\mathbf{x})$ conditional on the event $Z_e = \{h(\mathbf{x}, u_0) \leq 0\}$. Hence, solving the updating problem becomes equivalent to solving the structural reliability problem of estimating the probability $\Pr(Z_e)$.

It should be noted that the constant c has considerable influence on the efficiency of the BUS approach, since its value is directly proportional to the probability $\Pr(Z_e)$ (Straub & Papaioannou 2013). A large value of $\Pr(Z_e)$ is beneficial for most structural reliability methods. Therefore, c should be chosen a large as possible, while still ensuring that $cL(\mathbf{x}) \leq 1$. For the likelihood function of Equation (3), the optimal choice is $c = [\max f_\epsilon(\epsilon)]^{-1} = \sigma_\epsilon \sqrt{2\pi}$.

For most reliability methods, it is convenient to transform the problem from the original random variable space to a space of independent standard normal random variables. Since the distribution of the random vector $\mathbf{X}$ is described by the Nataf model, such a transformation $[U_1; \ldots; U_n] = \mathbf{T}(\mathbf{X})$ is straightforward (Der Kiureghian & Liu 1986). The limit-state function $h(\mathbf{x}, u_0)$ can be expressed in the transformed space as $H(\mathbf{u}) = h[\mathbf{T}^{-1}(u_1; \ldots; u_n), u_0]$, where $\mathbf{T}^{-1}$ denotes the inverse transformation and $\mathbf{u} \in \mathbb{R}^{n+1}$ is the outcome of $\mathbf{U} = [U_0; U_1; \ldots; U_n]$.

The formulation of the updating problem in terms of the limit-state function of Equation (5) allows for the application of a variety of structural reliability methods for estimation of the posterior PDF. Application of crude Monte Carlo method will lead to a rejection-acceptance scheme, with $\Pr(Z_e)$ being the acceptance probability. However, this approach becomes very inefficient for small $\Pr(Z_e)$, which correspond to cases where the posterior distribution differs considerably from the prior. In the following section, we discuss the application of BUS in conjunction with subset simulation (SubS), which is an adaptive Monte Carlo method for estimating small probabilities. The SubS is especially efficient in high dimensional problems, as is the case in the present application where a large number of random variables is used for the random field representation of the soil properties.

3.1 *SubS-based BUS*

The SubS method, originally developed in Au & Beck (2001), evaluates the probability $\Pr(Z_e)$ of the event $Z_e = \{H(\mathbf{u}) \leq 0\}$ as a product of larger conditional probabilities. This is achieved by expressing the event Z_e as the intersection of M intermediate events that are nested, i.e. it holds $Z_1 \supset Z_2 \supset \cdots \supset Z_M = Z_e$. The events $\{Z_i, i = 1, \ldots, M\}$ are defined as $Z_i = \{H(\mathbf{u}) \leq b_i\}$, where $b_1 > b_2 > \cdots > b_M = 0$. The probability $\Pr(Z_e)$ is then expressed as

$$\Pr(Z_e) = \prod_{i=1}^{M} \Pr(Z_i|Z_{i-1}) \tag{6}$$

where Z_0 denotes the certain event and $\Pr(Z_i|Z_{i-1})$ is the probability of the event Z_i conditional on the occurrence of the event Z_{i-1}. The values b_i can be chosen adaptively, such that the estimates of the conditional probabilities correspond to a given value p_0.

The probability $\Pr(Z_1|Z_0) = \Pr(Z_1)$ is computed by applying crude Monte Carlo simulation. To estimate the conditional probabilities $\{\Pr(Z_i|Z_{i-1}), j = 2, \ldots, M\}$, we need to obtain samples of $\mathbf{U}$ conditional on the occurrence of the events $\{Z_{i-1}, j = 2, \ldots, M\}$. Assume that at each subset level i, J samples $\{\mathbf{u}^{(j)}, j = 1, \ldots, J\}$ of $\mathbf{U}$ conditional on Z_{i-1} are available. The threshold b_i is set as the $(1-p_0)$-percentile of the samples; the samples $\mathbf{u}^{(j)}$ for which $H(\mathbf{u}^{(j)}) \leq b_i$ are then used as seeds for the simulation of samples conditional on Z_i by application of an MCMC algorithm (Papaioannou et al. 2013).

For Bayesian updating, we are interested in obtaining samples conditional on Z_e. Therefore, we add one final step, which is to obtain K such samples trough MCMC starting from the samples generated at the last subset level M for which $H(\mathbf{u}^{(j)}) \leq 0$. These samples are then transformed to the original space as $\{\mathbf{x}^{(k)} = \mathbf{T}^{-1}(u_1^{(k)}; \ldots; u_n^{(k)}), k = 1, \ldots, K\}$ in order to obtain samples from the posterior distribution $f(\mathbf{x}|Z)$. In this study, the parameters of the SubS algorithm for Bayesian updating are set as follows: $p_0 = 0.1$; number of samples per level $J = 1000$; number of target samples $K = 1500$.

4 RESULTS AND DISCUSSION

We consider a measurement outcome of $u_{\mathrm{A,m}} = 20$mm. The prior mean of $u_{\mathrm{A}}(\mathbf{X})$ is 10.5mm, which indicates that the prior model underestimates the measured surface settlement. The updating was performed for two different values of the standard deviation σ_ϵ of the measurement error: 1mm and 2mm. Figure 4 and Figure 5 show the posterior sample means of the oedometric stiffness modulus E_{oed} and the elastic stiffness modulus E_{ur}, respectively. The samples statistics of the remaining material parameters of the Keuper marl layer are shown in Table 2.

Looking at the results for $\sigma_\epsilon = 2$mm, one can observe that the posterior mean of E_{oed} at the elements within and around the tunnel is smaller than the prior. Note that E_{oed} is the parameter with the highest influence on the tunnel's reliability (see Fig. 3). Away from the tunnel, the posterior mean of E_{oed} increases and the value of its prior mean is reached at the upper left and right corners of the computational domain. Similar results are obtained for E_{ud}, which is highly correlated with E_{oed}. Due to the symmetry of the problem, it is expected that the spatial distribution of the posterior means of both E_{oed} and E_{ud} be symmetric about the vertical central axis. This result is evident in the area close to the tunnel, however away from the

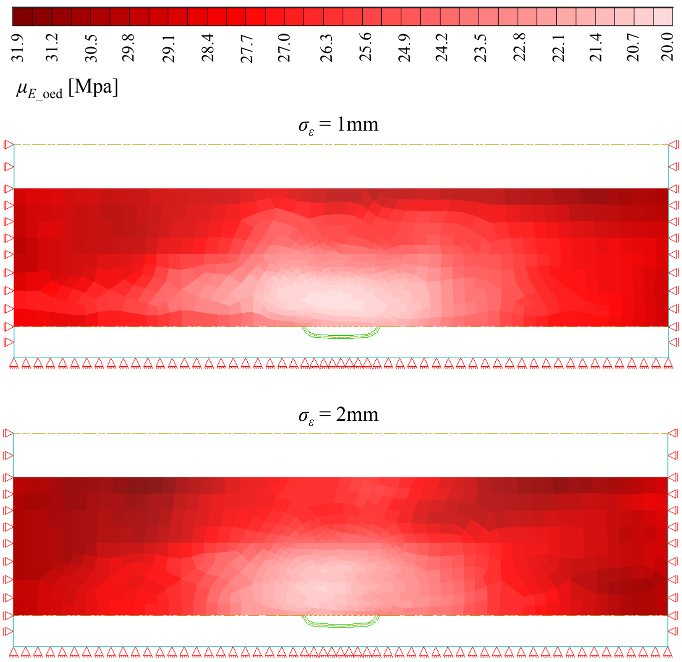


Figure 4: Posterior mean of the oedometric stiffness modulus E_{oed} of the Keuper marl layer.

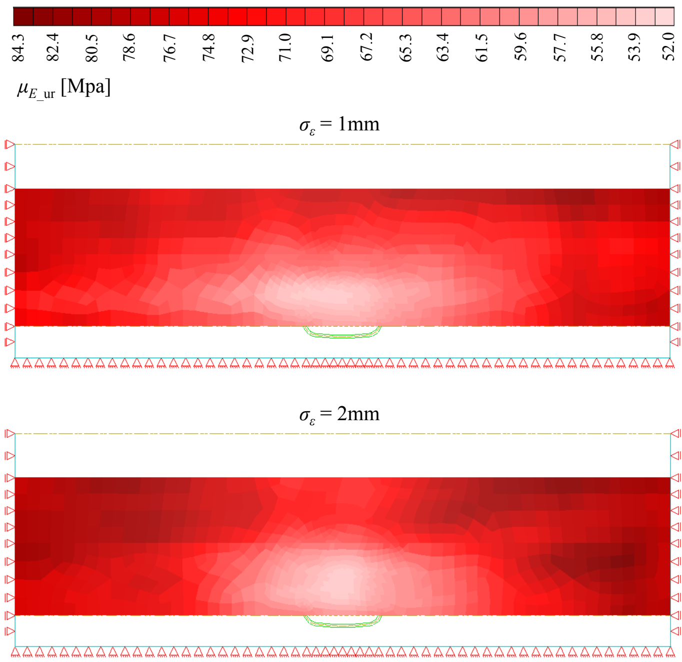


Figure 5: Posterior mean of the elastic stiffness modulus E_{ur} of the Keuper marl layer.

tunnel one can observe what seems like local random fluctuations from the expected result. This effect is attributed to sampling error and is related to the fact that in the areas away from the tunnel the influence of the values of E_{oed} and E_{ud} on the surface settlements is minor. That is, a large number of combinations of material values in these areas can justify the measurement outcome, which requires a large number of samples for the SubS algorithm to account for all the possible combinations.

The effect of the measurement is also evident in the posterior mean of the relaxation factor β of the stress reduction method, which is decreased compared to its prior. Its posterior coefficient of variation (CV) is also smaller than its prior, which reflects the impact of the measurement on the variable. Moreover, the initial stress for the stress reduction is computed based on the elements corresponding to the tunnel whose posterior means are much lower than their priors. This reveals the influence of the 2D modeling of the arching effect by the stress reduction method, which is consistent with the fact that most settlements will take place in the excavation phase, i.e. before the installation of the lining.

The friction angle ϕ is slightly influenced by the measurement; its posterior mean is decreased compared to the prior however its coefficient of variation is somewhat increased which indicates that the impact of the measurement on ϕ is small. The mean of the cohesion c is increased reflecting its negative correlation with the friction angle, while the influence of the measurement on the rest of the parameters is negligible.

The results for the case where the standard deviation of the measurement is decreased, i.e. when $\sigma_\epsilon = 1\text{mm}$, confirm the high impact of the measurement outcome on the stiffness variables E_{oed}, E_{ud} and the relaxation factor β. In this case, which implies higher information content of the measurement, the posterior mean of β is further decreased and the weak zone around the tunnel with low values of E_{oed} and E_{ud} is increased.

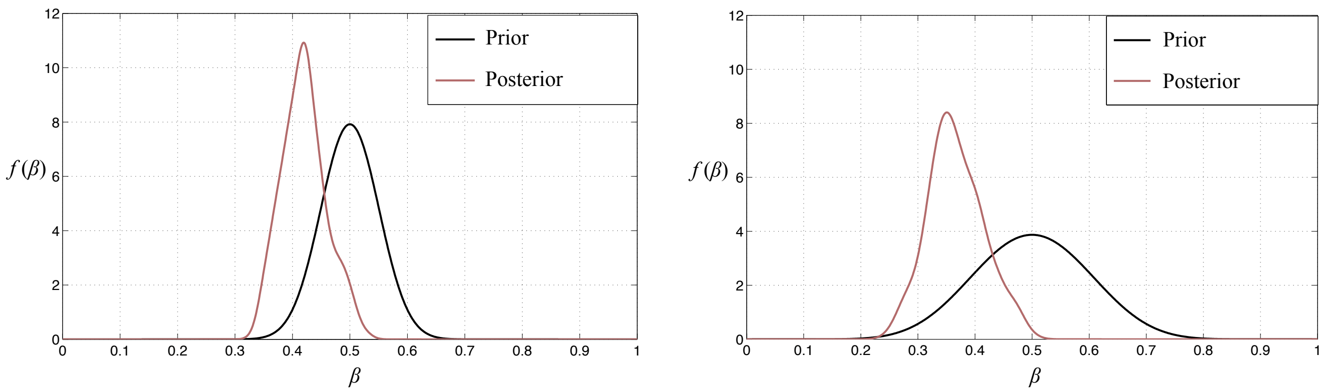
Figure 6 demonstrates the influence of the prior knowledge on the relaxation factor β. Therein, the prior and posterior PDFs of β are plotted for two different assumed prior coefficients of variation (10% and 20%) and for $\sigma_\epsilon = 1\text{mm}$. It is observed that as the prior knowledge on beta decreases, i.e. as its prior coefficient of variation increases, the influence of the measurement becomes higher. Comparing the posterior PDFs for the two cases, we see that the same measurement information leads to much lower values of β when a larger prior coefficient of variation is assumed. This result further highlights the influence of the 2D modeling of the arching effect on the surface settlements. Moreover, it shows how the confidence on the prior assumption can influence the updating results that may provide a basis for further risk and reliability assessment.

5 CONCLUSION

In this paper, we performed Bayesian updating of the parameters of a 2D numerical model of a tunnel in soft soil, conditional on settlement measurements. We applied BUS, a recently proposed method for Bayesian updating with structural reliability methods, combined with subset simulation, an adaptive Monte Carlo method that is able to handle efficiently problems with a large number of random variables. The results demonstrate the influence of the accuracy

Table 2: Statistics of the posterior distribution of the random variables of the Keuper marl layer.

Parameter	$\sigma_\epsilon = 1$mm		$\sigma_\epsilon = 2$mm	
	Mean	CV	Mean	CV
Relaxation factor β	0.42	8.3%	0.44	8.4%
Poisson's ratio ν	0.19	14.5%	0.19	15%
Friction angle φ [°]	18.5	17.2%	18.9	17.2%
Cohesion c [kPa]	29.04	30.5%	26.3	29.3%
Specific weight γ [kN/m^3]	24.2	5%	24.4	5.2%



(a) Prior CV of β: 10%.

(b) Prior CV of β: 20%.

Figure 6: Posterior PDF of the relaxation factor β for $\sigma_\epsilon = 1$mm.

of the measurement device as well as the prior knowledge of the uncertain parameters on their posterior distributions. It was shown that the highest impact of the measurement fell on the stiffness moduli and the relaxation parameter of the stress reduction method that models the 3D arching effect of the stress distribution.

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