TECHNISCHE UNIVERSITÄT MÜNCHEN
Lehrstuhl für Informationstechnische Regelung

Optimal Event-triggered Control with Communication Constraints

Adam Molin

Vollständiger Abdruck der von der Fakultät für Elektrotechnik und Informationstechnik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktor-Ingenieurs (Dr.-Ing.)

genehmigten Dissertation.

Vorsitzender:		Univ.-Prof. Dr. sc. Samarjit Chakraborty
Prüfer der Dissertation:		
	1.	Univ.-Prof. Dr.-Ing. Sandra Hirche
	2.	Prof. Karl Henrik Johansson, Ph.D.,
		KTH Stockholm, Schweden

Die Dissertation wurde am 16. Januar 2014 bei der Technischen Universität München eingereicht und durch die Fakultät für Elektrotechnik und Informationstechnik am 12. Mai 2014 angenommen.

Abstract

The increased interest in networked control systems has led to various paradigm shifts in the design of digital controllers. The systems under consideration usually consist of a multitude of small-scale integrated entities that share resources represented through common communication and computation capabilities. The efficient usage of these resources is a prerequisite for the successful operation of the control system. This fact has stimulated scientists to look for advanced sampling schemes beyond the conventional periodic sampling scheme for digital control to reduce resource consumption. Event-triggered sampling has impressively demonstrated its superiority for control applications with communication constraints in the feedback loop. Intuitively, the event-trigger only provides new data to the controller when truly needed. This leads to a natural reduction in the resource consumption. Despite of its evident benefits arising in many control applications, only a limited number of theoretical results is available for the systematic analysis and synthesis of event-triggered control. This urges the need for the development of novel methods that go beyond the classical theory developed for periodically triggered control systems.

The present thesis addresses the endeavour for a better understanding of event-triggered control in the context of optimal stochastic control. The design problem is formulated as a team decision problem related to linear quadratic regulation, where the event-trigger and the controller are regarded as individual decision makers. The co-design of event-trigger and controller aims at the joint optimization of a common objective that takes into account the communication constraints. Team decision problems are known to be challenging, as they commonly do not allow to apply the standard mathematical tools from stochastic optimal control, nor do they permit the development of efficient solution algorithms in general. This fact necessitates a careful study of the problem structure to obtain innovative solution methods.

The main contributions of this thesis can be divided into two parts. First, structural properties are found that arise in the underlying team decision problem for a single resource-constrained feedback loop. Aside from being fundamental for conceiving the basic principles of optimal event-triggered control, the obtained characterization is crucial in the second part of the thesis, as it enables the tractable design of optimal event-triggered controllers. The second part develops efficient design methods for multiple event-triggered controllers sharing a common communication network. Besides developing systematic design methods for control systems with communication constraints, the event-triggered system is analyzed with regard to aspects related to stability, performance, adaptability, and the decentralization of the synthesis procedure.

Zusammenfassung

Das wachsende Interesse an vernetzten Regelungssystemen hat zu mehreren Paradigmenwechseln im Entwurf von digitalen Regelungssystemen geführt. Vernetzte Regelungssysteme bestehen aus einer Vielzahl an kleineren, eigenständigen Funktionseinheiten, welche üblicherweise auf gemeinsame Kommunikations- und Rechenkapazitäten zugreifen müssen. Die effiziente Nutzung der beschränkten Ressourcen ist eine Grundvoraussetzung für den erfolgreichen Einsatz dieser Regelungssysteme.

Diese Tatsache hat Wissenschaftler dazu motiviert, alternative Abtastschemata jenseits konventioneller digitaler Regelungstechnik, in der Signale periodisch abgetastet werden, zu erforschen. Die Überlegenheit ereignisbasierter Abtastsysteme gegenüber konventioneller Methoden konnte dabei in einer Mehrzahl an Arbeiten empirisch belegt werden. Durch die ereignisbasierte Abtastung wird neue Information erst dann bereitgestellt, wenn sie wirklich benötigt wird. Dies führt zu einer natürlichen Senkung des Ressourcenverbrauches bei gleichbleibender Regelgüte. Trotz der erwiesenen Vorteile in vielen Regelungsanwendungen existieren nur vereinzelte theoretische Ergebnisse für eine systematische Analyse und Synthese ereignisbasierter Regelungssysteme. Deshalb besteht ein großes Interesse an der Entwicklung neuartiger Methoden, die über die klassische Theorie der periodisch abgetasteten Regelungssysteme hinausgehen. Die vorliegende Dissertation hat sich zum Ziel gesetzt, zu einem besseren Verständniss ereignisbasierter Regelungssysteme im Kontext von mathematischer Optimierung beizutragen. Das Entwurfsproblem wird dabei als optimaler Entscheidungsprozess in Gruppen formuliert, bei dem der Regler und der Ereignisgenerator als individuelle Entscheidungseinheiten definiert sind.

Diese legen ihre Entscheidungsregeln so aus, dass eine gemeinsame Kostenfunktion, welche die Ressourceneinschränkungen beinhaltet, minimiert wird. Das entstehende Optimierungsproblem wird im Zusammenhang linear-quadratischer Regelung studiert. Selbst in diesem Kontext ist bekannt, dass sich im Allgemeinen das Lösen von optimalen Entscheidungsprozessen in Gruppen als schwierig heraustellt, da die mathematischen Methoden für den optimalen Regelungsentwurf nicht anwendbar sind. Diese Tatsache erfordert eine innovative Untersuchung der Problemstruktur, um effiziente Lösungsansätze zu entwickeln.

Die zentralen wissenschaftlichen Beiträge dieser Disseration gliedern sich in zwei Teile. Im ersten Teil werden strukturelle Eigenschaften des zugrunde liegenden optimalen Entscheidungsprozesses für einschleifige Regelungssysteme mit Ressourcenbeschränkungen in der Rückkopplung aufgezeigt. Abgesehen von den gewonnenen Einsichten in die grundlegenden Arbeitsprinzipien optimaler ereignisbasierter Regelungssysteme, nimmt die Charakterisierung eine entscheidende Rolle im zweiten Teil ein, da sie den Entwurf optimaler ereignisbasierter Regelungen realisierbar macht. Im zweiten Teil werden effiziente Entwurfsmethoden für mehrschleifige ereignisbasierte Regelungskreise über ein gemeinsames Kommunikationsnetz entwickelt. Darüberhinaus wird das ereignisbasierte Regelungssystem in Bezug auf Stabilität, Regelgüte, Adaptivität und Dezentralisierung der Entwurfsmethodik analysiert.

Contents

Notations

Abbreviations

i.i.d.	independent identically distributed
CE	certainty equivalence
ET	event-triggered
TT	time-triggered
LQ	linear quadratic
LQR	linear quadratic regulation
LQG	linear-quadratic-Gaussian
NCS	networked control system
ODE	ordinary differential equation
P-a.s.	P-almost surely
TCP	Transmission Control Protocol
UDP	User Datagram Protocol

Symbols

Subscripts and Superscripts

x_k	time index k
$\{x_k\}_k$	time sequence of x_k, i.e., $\{x_k\}_k = \{x_0, x_1, \dots\}$
X^k	truncated sequence up to time k, i.e. $X^k = \{x_0, \dots, x_k\}$
X_l^k	truncated sequence from l to time k, i.e. $X_l^k = \{x_l, \dots, x_k\}$
x^i	index i indicating corresponding subsystem (Chapter 7-9)

Main Variables

x_k	system state
y_k	measurement at event-trigger (Chapter 5), sum rate (Chapter 8)
z_k	received signal at controller
w_k	system noise
v_k	measurement noise
u_k	control input
δ_k	event-trigger output
q_k	dropout variable of the communication network
A	system matrix
B	input matrix
C	output matrix
N	time horizon (Chapter 2-6), number of subsystems (Chapter 7-9)
T	time horizon (Chapter 7-9)

Spaces

$\mathbb{R}$	real line
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^{n\times m}$	space of $n \times m$-dimensional matrices
$\mathbb{R}_{\geq 0}$	non-negative real line
$\mathbb{Z}$	set of integers
$\mathbb{Z}^+$	set of non-negative integers

Functions

$\gamma(\cdot)$	control law
$\mathrm{f}(\cdot)$	triggering law
$\mathbb{1}_A(\cdot)$	indicator function
$[\cdot]^+$	projection onto $\mathbb{R}_{\geq 0}$, i.e., $[\cdot]^+ = \max\{0, \cdot\}$

Operators

A^{T}	transpose of matrix A
A^{-1}	inverse of square matrix A
$\mathrm{tr}[\cdot]$	trace of matrix A
I_n	identity matrix in $\mathbb{R}^{n\times n}$
0_n	matrix in $\mathbb{R}^{n\times n}$ with all entries 0
$\|\|v\|\|_2$	Euclidean norm of vector v
$\|\|v\|\|_\infty$	maximum norm of vector v
$\|\|A\|\|_2$	matrix norm of matrix A induced by $\|\|v\|\|_2$
$f * g$	convolution of two real-valued functions f and g
A^{c}	complement of set A
$\wedge$	logical and-operation

Probability Theory

Ω	abstract sample space
$\mathscr{F}$	σ-algebra on Ω
P	probability measure on $\mathscr{F}$
$P(\cdot,\cdot)$	Markov transition kernel
$\mathsf{E}[\cdot]$	expected value
$\mathsf{E}[\cdot\vert\cdot]$	conditional expectation
$\mathsf{E}^{\mathrm{f}}[\cdot\vert\cdot]$	conditional expectation given law f
$\mathsf{E}_x[\cdot\vert\cdot]$	conditional expectation assuming initial state x
$x \sim \mathcal{N}(\mu, C_x)$	Gaussian random variable x with mean μ and covariance matrix C_x

List of Figures

1

Introduction

This chapter introduces the basic concept of event-triggered control. On the one hand, it aims at illustrating the benefits of event-triggered sampling for control applications in the presence of communication constraints in the feedback loop. On the other hand, it identifies the major challenges and open questions that arise from the use of event-triggered sampling with a particular emphasis on optimality. These two aspects are discussed in Section 1.1 and 1.2, respectively. In Section 1.3, the literature related to event-triggered control is reviewed and its links to this thesis are highlighted. Eventually, an outline of this thesis and its main contributions are described in Section 1.4.

1.1 The Principle of Event-Triggered Sampling in Control

Computer-controlled systems are commonly sampled periodically. By considering the evolution of the system at its equidistant sampling times, the system can be described by difference equations with constant coefficients provided the initial continuous-time system is time-invariant. For such discrete-time systems, a well-established theory exists that has been used extensively in many application of digital control systems [ÅW11]. The advent of networked control systems has however caused a rethinking in the design of sampling strategies for such systems. Networked control systems are composed of a multitude of discrete embedded systems that exchange information through common communication and computation resources. These resources have commonly limited capabilities that may arise from bandwidth limitations, restricted computational power, or energy constraints. Throughout this thesis, we will abstract these limitations by communication constraints in the feedback loop. Examples for networked control systems range from flexible manufacturing, multi-robot systems, and automated traffic systems to power grids and other distribution systems like HVAC

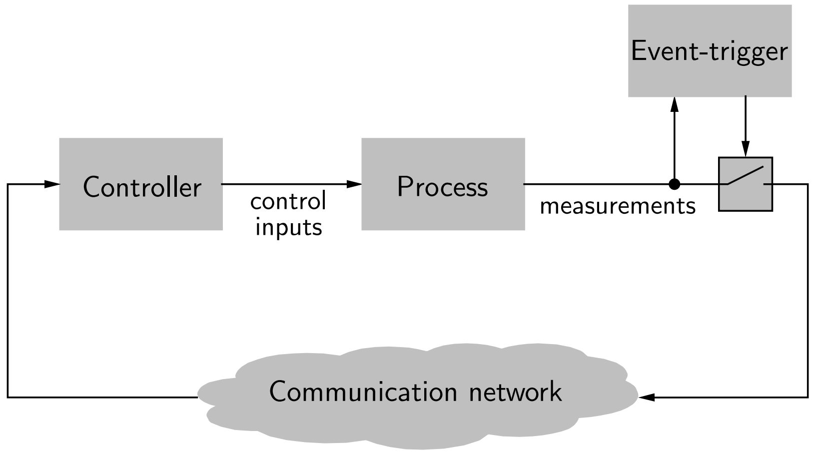


Figure 1.1: Principle of event-triggered sampling for feedback control over a resource-constrained communication network.

control (heating, ventilation, and air conditioning) for building automation and water flow control. The increasing complexity of such systems with the ability to acquire and process an almost unlimited amount of data from a multitude of networked sensors, urges the need for an efficient usage of resources for real-time optimal decision making in these time-critical systems.

With regard to control systems having resource constraints in their feedback loops, event-triggered sampling constitutes a promising alternative compared to periodic sampling. By only sampling when needed, event-triggered sampling naturally reduces the amount of data to be processed. This is illustrated via the following example of a simple event-triggered control system sketched in Fig. 1.1.

Consider the task of controlling a process through measurement feedback in order to counteract the impact of disturbances. The event-trigger situated at the sensor node monitors the current measurement output of the process and decides upon its evolution, whether or not to update the controller by transmitting the current measurement over the resource-constrained feedback link to the controller. The most common way of implementing an event-triggered sampling strategy is to provide a new measurement to the controller whenever a tolerable threshold in the (possibly transformed) measurement space is surpassed. By tolerating a certain amount of deviation when regulating the process, which is also inevitable in periodically sampled systems, the control input is then adjusted accordingly only when necessary. In that way, it can be imagined that the event-triggered sampling mechanism leads to a reduction of sampling instances while maintaining a certain level of performance.

In the course of this thesis, we will formulate the design problem of event-triggered control as an abstract optimization problem over a very general space of possible solutions that does not take the above notion of event-triggered sampling into account. However, it will turn out that the optimal solutions will allow very natural interpretations that resemble the intuitive idea of event-triggered control just described.

It should be noted that a persistent monitoring of the current measurement is a substantial requirement for the implementation of event-triggered controllers. This monitoring can be performed continuously or in discrete-time. Throughout the thesis, we will focus on event-triggered control in discrete-time. Aside from being more realistic in applications, where sensor polling is performed periodically, several issues that need to be taken care of in continuous-time, e.g. the potential occurrence of the Zeno-behavior, are directly avoided. This lets us focus on the main issues arising in the optimal design of event-triggered control under communication constraints.

1.2 Challenges in Optimal Event-Triggered Control

Along with the benefits of event-triggered sampling in resource-constrained feedback control systems described in the previous section, several issues emerge in the analysis and design of such systems that are not present in time-triggered control. As this thesis studies the optimal design of event-triggered control under communication constraints, it is subsequently focused on the challenges arising in the context of optimality.

The optimal design problem for single-loop control systems with an information-constrained feedback loop can be regarded as a two-person team decision problem, in which the controller and the event-trigger take the role of individual decision makers aiming at the optimization of a common objective [Rad62]. The distinguishing feature compared to common optimal control problems is that there is no central coordination of decisions, but both decision makers need to take actions based on their individual information. The available information differs between both decision makers, as the event-trigger has a continuous access to current measurements, while the controller only receives data when an event is triggered. What makes such class of problems challenging is that standard techniques of stochastic optimal control theory like dynamic programming and the separation principle are in general not directly applicable. Even for a simple linear system with quadratic cost, it has been shown in [Wit68] that the optimal solution is hard to find when having two decision makers with a distributed information pattern.

In particular, there are two phenomena that complicate the analysis of optimal event-triggered control, which do not appear for the optimal control problems that are time-triggered. These arise from the fact that one decision maker may signal the other decision maker through its actions either in order to improve the information of the other decision maker or in order to improve its own knowledge on the overall system state. For the underlying event-triggered control system, this can occur in two ways. First, signaling is naturally present through the event-trigger that decides upon its own observed data whether or not the controller shall obtain new state information. It should also be noted that by not sending information to the controller, the event-trigger implicitly signals also information to the controller depending on the choice of the event-triggering law. On the other hand, assuming that the law of the event-trigger is fixed, the controller may invoke the event-trigger to send another state update by choosing control inputs accordingly. This phenomenon is also referred to as the dual effect of control. The term dual comes from the dual role of the controller: (i) the controller affects the state evolution, and (ii) it can probe the system to reduce its state uncertainty.
It should be remarked that both effects - signaling and dual control - do not occur in the equivalent formulation for the design of optimal time-triggered controllers in the linear quadratic framework. Signaling can not occur, as the sampling times are chosen beforehand in the time-triggered sampling strategy, and therefore no additional information is

contained in the timing variables. The dual effect of control is also not present, as the resulting time-triggered system is a possibly time-varying linear system, in which the sampling times can not be altered by the control inputs.
Though signalling and the dual effect can improve the system performance, they are in general undesired, as they do not admit the development of efficient numerical algorithms for optimal control, even not in the linear quadratic framework. The solutions are generally non-linear and depend on the complete observation history. This usually forces the system designer to resort to heuristic methods in order to tame the degree of freedom with regard to the event-triggered control design.

The situation gets even more challenging when multiple control subsystems share a common communication network in their feedback loops. In addition to the issues emerging for single-loop event-triggered systems, the shared resource which needs commonly to be displayed as a sample-path constraint complicates the analysis and design of event-triggered control algorithms for multi-loop systems considerably.
Since data transmission is event-driven and therefore is influenced by the physical process, the communication system and the control system can not be analyzed independently from each other, but there is a tight interaction between them. On the one hand, the fact that the event-triggered control system operates in a decentralized fashion, in which events are triggered based on local measurements, results in unpredictable delays and packet collisions during data transmission in the feedback loop. On the other hand, the sample-path constraints reflecting the scarce communication resource pose a major challenge with regard to the optimal design, because the resource limitations need to be satisfied at each time step. These complications demand for suitable problem relaxations in the optimal design formulation taking into account the difficulties arising from the loss of predictability and the interaction between communication and control.

1.3 State of the Art

There are contact points to two main lines of research that are related to Part I and Part II of this thesis. Besides having a tight connection to the study of the analysis and design of event-triggered control, focusing in particular on the joint optimization of event-trigger and control, this thesis shares also links to results of event-triggered scheduling algorithms in networked control systems discussed at the end of this section.

In the following, results in event-triggered control are reviewed. Initiated by Karl Åström, Bo Bernhardsson in [ÅB02] and by Karl-Erik Årzén in [Årz99], there is an increased interest in event-triggered control to reduce the information flow in the feedback-loop for more than a decade. Several research groups have demonstrated empirically that event-triggered control is capable to reduce data processing significantly compared with traditional periodically sampled control methods, while maintaining the same level of system performance [LL10; AMA+14; WL11; MT11; Tab07; HSVDB08; DH12; DFJ12; SDJ13; ZC10; WL09; WL10; MUA12]. Event-triggering has proven to be successful in various domains, such as control over communications [ÅB02; LL10; AMA+14; WL11; MT11], embedded real-time control design [Årz99; Tab07; HSVDB08], distributed optimization algorithms [ZC10; WL10; WL09; MUA12], and multi-agent systems [DFJ12; SDJ13].

The synthesis of event-triggered controllers often takes an emulation-based approach, which presumes a stabilizing continuous-time controller. The control inputs are chosen according to this control law at triggering times and are kept constant in between triggering times. The event-trigger is commonly defined as a threshold function depending on the current state value and the discrepancy between the last sampled state and the current state value. By using concepts from Lyapunov theory for hybird systems and input-to-state stability, the event-trigger thresholds are chosen such that practical stability in terms of uniform boundedness [LL10; HSVDB08], or even global asymptotic stability of the event-triggered control system can be guaranteed [Tab07; WL11; MT11]. Similar to the emulation-based concept for event-triggered control, existing algorithms for distributed optimization algorithms and for multi-agent systems can be extended to use event-triggered message passing between agents [ZC10; WL10; WL09; MUA12; DFJ12; SDJ13]. The events are commonly triggered locally whenever the discrepancy between current and the sampled localized value at an agent exceeds a threshold.

It can be summarized that the work on event-triggered control mentioned so far focuses on maintaining convergence and stability when implementing an event-triggered sampling strategy. As the results do not explicitly take into account the scarce resource in their design, it is difficult for these works to assess the required computational and communicational requirements beforehand that guarantee a certain level of performance. The system model is also not capable of considering blocking and delaying effects due to the asynchronous exchange of data when multiple event-triggers are used. Apart from these issues, a critical point motivating this thesis is the open question, which controller will eventually be most suitable within the event-triggered sampling framework. This question can not be answered by the emulation-based approach as the controller needs to be chosen a priori.

A different approach for the event-trigger design, that is more closely related to this thesis since it incorporates the limited resource, formulates the synthesis problem in the framework of optimal stochastic control and estimation [XH04; Cog09b; RJJ08; RJ09a; RMB12; SL12; WJJS13; RSJ13; IB10; LM11]. These works propose several ways for taking into account the limited resource. There is either a constraint on the total number of transmission over a finite time interval [RJJ08; RJ09a; RMB12; IB10], or an average sending rate constraint [WJJS13], or the limited resource is incorporated as an additional communication penalty in the cost function [XH04; Cog09b; LM11].

In [XH04; RMB12; IB10; LM11; SL12; WJJS13], the problem of event-triggered estimation for linear systems is studied. Similar to the optimal event-triggered control problem, this problem can be viewed as a two-person team problem with the decision makers being the event-trigger and the estimator that obtains new information based on the event-trigger's decision. Several attempts have been made to by-pass the problems arising in the joint optimization of the event-trigger and the estimator which have been mentioned in the previous section. On the one hand, the work in [XH04] circumvents the problem of jointly optimizing both decision makers by fixing the estimator to be the optimal linear state predictor. This state predictor omits the additional information that arises from not sending information. Not sending information can however be valuable information as it tells the estimator that the state is within a certain region, where no event is triggered. On the other hand, the work in [SL12; WJJS13] takes into account this information in order to design optimal estimators, while fixing the event-trigger. By assuming that the event-trigger is an even function of the network-induced estimation error, it turns out that the optimal state estimator coincides with the optimal linear predictor that does not take into account the information of not having transmitted [RMB12; IB10]. Based on this symmetry assumption, the design of the event-triggered estimator can be facilitated considerably, as it reduces to an optimal stopping problem [RMB12] or an optimal control problem that can be solved by dynamic programming [IB10]. By using result in majorization theory that have previously been applied in the joint optimization of paging and registration [HMY08], the authors in [LM11] adapt this problem to the optimal event-triggered estimation problem for first-order systems and show that symmetric event-triggers are indeed optimal. Following, a different approach that generalizes the optimal design of event-triggered estimators, a similar result is derived in this thesis in Chapter 3 that also underlines the importance of symmetric event-triggers.

In [RSJ13], it has been shown that there is in general a dual effect of control for a networked system with a fixed event-triggered scheduler. This makes it hard to find the optimal control strategy, even in the linear quadratic framework. However, the authors identify a special class of event-triggers, in which the dual effect of control is not present and standard techniques of stochastic control like the separation principle and certainty equivalence hold. The class of event-triggers coincides with the symmetric event-triggering laws depending only on the networked-induced error which have been introduced in the last paragraph for event-triggered estimation.
In previously mentioned work, the control inputs are allowed to vary between sampling times. Contrary to this, the work in [RJJ08; RJ09a] considers the case, in which the control input is kept constant between sampling times. The problem is shown to be related to optimal stopping time problems, which enable an analytical solution in certain cases. It should be however remarked that by assuming a given control waveform between transmissions, the optimal design of event-triggered controllers reduces to an optimal control problem with a classical information pattern, see Section 2.3.1.

In summary, it can be observed that the joint optimization of event-trigger and control or estimation has been addressed by several authors. Except of the results in [LM11], the analysis of the underlying two-person team decision problem is accompanied by various assumptions that by-pass the inherent complications arising through the dual effect of control and signaling between decision makers. In contrast to the previous work, one of the objectives of this thesis is to keep the restrictions on the design space as little as possible.

While the majority of results studies event-triggered sampling for single-loop control systems, systems with multiple control loops over a shared network have attained only little attention. Exceptions can be found in the works [CH08; HC10; BA11a; BA11b; BA11c] that analyze event-triggered sampling in multi-loop control systems. Depending on the model that represents the resource-constrained communication medium the authors draw different conclusions. Using carrier sense multiple access schemes with priority or randomized arbitration as proposed in [CH08; HC10], event-triggered sampling for data transmission enhances the control performance significantly compared with periodic transmission schemes. On the other hand, the results in [BA11a; BA11b] suggest that time-triggered sampling outperforms event-triggered sampling for slotted and unslotted ALOHA transmission schemes. By a specific choice of the triggering rule inspired by [RJ09b], the sampling process can be modelled as a renewal process. This fact is then used extensively to analyze the performance of the event-triggered scheduling mechanism. A more comprehensive work of the same authors in [BA11c] that analyzes different protocols with event-triggered sampling, shows that event-triggered schemes outperform time-triggered schemes for certain protocols. In summary, these works indicate that there is a correlation of the performance of the event-triggered sampling strategy with the sophistication of the communication protocol. It should be noted that the analysis of the previously mentioned works is restricted to scalar integrator dynamics driven by Brownian noise process, in which the state of the subsystem is reset after a successful transmission. In this thesis, we are however interested in developing design guidelines for event-triggered scheduling for the control of general linear systems.

1.4 Outline and Contributions

This thesis targets towards a fundamental understanding of optimal event-triggered control under communication constraints. It studies the scientific questions of *"what characteristics do optimal event-triggered controllers have?"*, *"are there potential principles that allow a problem decomposition of the co-design of event-trigger and controller?" "how can these principles be used for the efficient design of (near-)optimal event-triggered control algorithms?"*. All these questions will be addressed in the control design framework related to the well-known linear quadratic regulation (LQR) problem. In order to analyze these issues, the thesis is divided into two parts. The first part (Chapters 2-6) is concerned with structural properties of optimal event-triggered control systems with a single resource-constrained feedback loop. Based on these results, Part II (Chapters 7-9) is devoted to the analysis and design of multiple control loops sharing a common constrained resource.

Outline

Chapter 2 develops one of the core results in this thesis stating that the optimal control law resulting in the joint optimization of event-trigger and controller has the certainty equivalence property. This structural property of the controller makes the optimal design of event-triggered controllers tractable with regard to numerical methods and with regard to a further analysis. The certainty equivalence property reduces the problem into two subproblems - the solution of a standard LQR problem, and the joint optimization of event-trigger and estimator. The latter problem is discussed in Chapter 3 by introducing an iterative algorithm alternating between the optimization of the event-trigger and the estimator. Built upon the results obtained in Chapter 2, Chapter 4 and 5 analyze the structural properties of the optimal event-triggered controller with regard to network degradations, such as time delay and packet loss, and with regard to partial state information at the event-trigger, respectively. Part I of this thesis is concluded with a stability analysis of the event-triggered control system using results for stochastic stability of Markov chains in Chapter 6.

The asymptotic behavior of the multi-loop event-triggered control system over a shared communication network is analyzed in Chapter 7 following the guidelines developed in Chapter 6. In Chapter 8, the structural results for single-loop event-triggered control systems developed in Chapter 2-4 are used to develop decentralized event-triggered scheduling algorithms for multiple control loops sharing a common network. By introducing a dual price exchange mechanism, the design method developed in Chapter 8 is adapted in Chapter 9 so that the optimal event-triggered scheduler can be determined in a decentralized fashion. Chapter 10 eventually gives a concluding discussion of the results in this thesis and an outlook on open problems.

The links between the individual chapters are summarized in Fig. 1.2. In the following, the major contributions within each chapter are outlined in more detail.

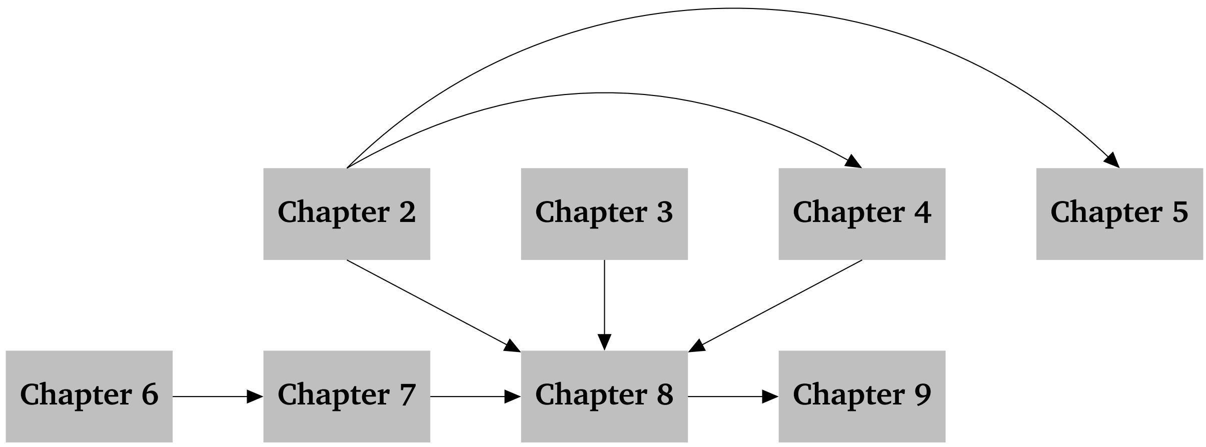


Figure 1.2: Overview of relations between chapters.

Chapter 2: Structural Properties of Optimal Event-Triggered Control

This chapter investigates structural properties of optimal event-triggered controllers. The design problem is cast in the framework of stochastic linear quadratic regulation extended by an additional communication penalty for closing the feedback loop. Due to the additional communication penalty, a trade-off between control performance and the communication rate must be found, which prevents trivial solutions of the joint optimization of controller and event-trigger. In this chapter, we will observe that the special information pattern of the decision makers being nestedness of the information available at the event-trigger and the controller enables us to prove that one can *not* take advantage of the dual effect of control in such problem setting. As a consequence, the optimal control law will have the structure of a certainty equivalence controller. A controller is said to be a certainty equivalence controller, if it takes the solution of the problem without communication penalty implying a continuous availability of state information at the controller and by replacing the state by its least-squares estimate. Besides its implications for the numerical computation of optimal event-triggered controllers, this result can be regarded as an approval for the usage of emulation-based approaches in the design of event-triggered controllers introduced in the first paragraph of Section 1.3. This is simply because the certainty equivalence controller reflects exactly the emulation-based approach, as the control law is computed without taking into account the event-triggering mechanism. The results in this chapter are partly based on the work in [MH09; MH13a], and section 5.7 of [GHJ+13]. A continuous-time version of the main structural result can be found in [MH10a].

Chapter 3: Structural Properties of Optimal Event-Triggered Estimation

In this chapter, we focus on the joint design of event-trigger and estimator. An iterative method is developed for the joint optimization of event-trigger and estimator for linear systems. The algorithm iteratively alternates between optimizing one decision maker while fixing the other one. When the densities of the initial state and the noise variables are symmetric and unimodal functions, it is shown for scalar systems that the solution of the algorithm converges to a symmetric event-trigger and a Kalman-like estimator taking only the latest received state information into account. This result coincides with results obtained in [LM11], which uses majorization theory and rearrangement inequalities to show that there always exists a symmetric threshold policy that outperforms an arbitrary event-triggering law. Our approach gives an alternative line of proof for this result by analyzing the convergence properties of the proposed iterative algorithm. When the noise distribution is assumed to be multimodal, it also turns out that the proposed iterative method can yield a remarkable decrease of the cost compared to an emulation-based approach, in which the estimator takes the form of a linear predictor that assumes that transmission instants are statistically independent of the state. The contributions presented in this chapter are based on the work in [MH12b; MH12d].

Chapter 4: Event-Triggered Control under Communication Delays and Packet Dropouts

In this chapter, the design of optimal event-triggered controllers is studied in the presence of both time-delay and packet-dropouts in the feedback loop. As the information pattern of the decision makers needs not to be nested, the optimal control law will generally not have the certainty equivalence property found in Chapter 2. This motivates us to identify different conditions for the communication model, where the nestedness property can be recovered. One of the prerequisites for the communication model is an error-free acknowledgement channel. It turns out that the certainty equivalence property can be assured, if either (i) the acknowledgement channel is delay-free or (ii) the feedback link is error-free or (iii) intervals between subsequent transmission times are restricted to be equal or greater than the round-trip time. Inspired by these conditions, two suboptimal design approaches are developed. The notion of *suboptimality* refers to the introduction of certain assumptions that enable the calculation of optimal event-triggered controllers. The first approach assumes that the event-trigger is idle for the duration of a round-trip time after transmitting information. The second approach assumes that the controller is a certainty equivalence controller. In contrast to the first strategy, there are no restrictions on the duration between subsequent transmissions. The optimal event-trigger is shown to have finite memory, where the number of past state values to be taken into account scales linearly with the round-trip time. The results in this chapter are based on [MH10c; MH13b].

Chapter 5: Structural Characterization of Event-Triggered Control with Partial State Information

This chapter extends the structural results obtained in Chapter 2 to systems with partial state information, in which state information can not be accessed directly, but only through noisy measurements. It turns out that the certainty equivalence property is still valid in this problem setup. Based on the certainty equivalence property, the least-squares estimators at the controller and at the event-trigger are characterized. By fixing the control law, it turns out that the event-trigger can not make use of the dual effect of control and the optimal estimator is given by the Kalman filter. The least-squares estimator at the controller takes the form of a biased linear predictor of the Kalman estimate resulting at the event-trigger. Similar as in Chapter 2, the estimation bias can be determined beforehand and depends on the choice of the event-triggering law. The structure of the optimal estimators allows us to state that it suffices to transmit the Kalman estimate to controller in order to maintain optimality. Based on these results, the optimal event-triggering law can be characterized as a policy depending on the discrepancy of the least-squares state estimate at the controller and at the event-trigger. The contribution of this chapter is partly based on the work in [MH10b].

Chapter 6: Optimal Event-Triggered Control for Long-Run Average-Cost Problems

Here, we study the design of optimal event-triggered controllers over an infinite horizon. Among other formulations for infinite horizon costs, we focus in this chapter on the long-run average-cost criterion, as our main interest lies in the behavior of the overall system in the stationary regime and it gives us a direct interpretation of the communication penalty as the average transmission rate. The average-cost formulation is particularly challenging, as the dynamics of the underlying Markov chain takes a crucial role in the solution of the average-cost criterion. In order to guarantee that the event-triggering law can be solved by means of dynamic programming, we need to assert certain ergodicity conditions on the Markov chain. By including a technical assumption on the event-triggering law, it is possible to guarantee this condition, and we can conclude that the average-cost problem for the optimal event-trigger design can be computed via value iteration. Furthermore, we analyze the stability properties of the proposed event-triggered controller in terms of drift criteria for Markov chains. It turns out that this notion offers appropriate mathematical tools to address the issue of closed-loop stability of the event-triggered control system. Assuming a stabilizing control law in the case of continuous transmission and a uniform bound on the triggering threshold, we prove stochastic stability for ideal communication and we derive sufficient conditions to guarantee stochastic stability in the presence of packet loss. The stability analysis will give us key insights in order to study the asymptotic behavior of multi-loop control systems sharing a common communication network in subsequent chapters. The contribution of this chapter is partly based on [MTH11; MH13b].

Chapter 7: Stochastic Stability of Multiple Event-Triggered Control Systems

This chapter analyzes the stability properties of multiple event-triggered control systems whose feedback loops are closed over a common communication network. The system under consideration consists of several individual subsystems whose sensor information needs to be sent over a shared communication link to the controller. An event-triggered scheduler situated at the sensor node of each subsystem decides upon its local information whether to transmit information. Due to the limited number of transmissions per time step, there is the chance for contention among subsystems. In order to counteract potential collisions, we assume a probabilistic collision resolution scheme, in which an arbitration mechanism selects randomly which subsystem is permitted to transmit its sensor information to the controller. What makes the analysis of such multi-loop system challenging is the tight interaction between the individual control loops and the communication system due to the event-triggered nature of the scheduling mechanism. By making use of results in Chapter 6, sufficient conditions for stability are derived. These conditions will relate the ratio between the availability of the resource and the number of control loops with the open-loop system dynamics of each control system. The results of this chapter are partly based on the work in [MH11; MH14].

Chapter 8: Optimal Event-Triggered Control over a Shared Network

The focus of this chapter is to develop an efficient algorithm for the design of decentralized event-triggered scheduling of multiple control systems whose feedback loops are closed over a common communication network. The design procedure is formulated as an average-cost problem that aims at the minimization of a social cost criterion. By proposing a relaxed formulation of the average-cost problem that allows us to circumvent the coupling of control and communication in the design, the optimization problem becomes tractable as it can be split into two levels: a local optimal control problem and a global resource allocation problem. While the results of Chapter 2-4 on optimal event-triggered control apply in the local optimization problem, the global resource allocation problem can be studied by techniques of convex analysis. Based on the stability analysis for the multi-loop system in Chapter 7, it is further shown that the proposed bi-level approach is asymptotically optimal, when the number of users approaches infinity. The contribution of this chapter is based on the work in [MH11; MH12c; MH14].

Chapter 9: Price Exchange Mechanism for Event-Triggered Control

The design method developed in Chapter 8 needs to solve a global resource allocation problem that must incorporate information of every subsystem sharing the common communication resource. This might be inconvenient due to its difficulty of implementation when the number of control loops is large, and it lacks of flexibility, as it needs to be rerun completely whenever changes in the system occur. This motivates us to design a decentralized version of the resource allocation algorithm, which is pursued in this chapter. By applying techniques from distributed optimization and adaptive Markov decision processes, we develop a dual price exchange mechanism, in which the distributed self-regulating event-triggers adapt their average communication rate to accommodate the global resource constraint. Assuming the absence of contention, Almost-sure convergence properties of the distributed event-triggered scheme are established by using a time-scale separation approach that decouples the process dynamics from the communication rate adaptation. In the case of contention, stochastic stability in terms of Harris recurrence is verified based on the stability conditions from Chapter 7. Aside from the development of a decentralized resource allocation algorithm, this chapter clearly demonstrates the benefits of event-triggered sampling with regard to the ability of adaptation which is crucial for the implementation of distributed mechanisms. The contribution of this chapter is based on the work in [MH12a].

Part I

Single-Loop Control

2

Structural Properties of Optimal Event-Triggered Control

In this chapter, we focus on finding structural properties of optimal event-triggered controllers. The design problem is cast in the framework of stochastic linear quadratic regulation extended by an additional communication penalty for closing the feedback loop. Due to the additional communication penalty, a trade-off between control performance and the communication rate must be found, which prevents trivial solutions of the joint optimization of controller and event-trigger. The key result of this chapter is to show that one can not benefit of the dual effect of control in such problem setting. As a consequence, the optimal control law can be characterized by having the certainty equivalence property. This result is enabled by the special information pattern of the decision makers being nestedness of the information available at the event-trigger and the controller.

This chapter is structured as follows. First, the optimal design of event-triggered control is formulated as a team decision problem related to linear quadratic regulation in Section 2.1. In Section 2.2, it is then shown that the problem can be separated into standard optimal control problems for the case of time-triggered sampling. Section 2.3 considers the design of optimal event-triggered controllers for two different cases. In 2.3.1, we assume that the control input is constant between transmissions. For this case, it turns out that the joint optimization of controller and event-trigger can be cast as a standard optimal control. The main result of this chapter is obtained in 2.3.2, which shows that the optimal control law is a certainty equivalence controller, in the case of having no restrictions on the design of controller and event-trigger.

2.1 Linear Quadratic Control under Communication Constraints

The system under consideration is illustrated in Fig. 2.1 and consists of a process $\mathscr{P}$, an event-trigger $\mathscr{E}$, a controller $\mathscr{C}$, and a resource-constrained communication channel $\mathscr{N}$. In the following, we will pose the design problem in the framework of linear quadratic control extended by a resource constraint that will be modeled as an additional communication penalty in the cost function. The stochastic discrete-time process $\mathscr{P}$ to be controlled is described by the time-invariant difference equation

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times d}$. The variables x_k and u_k denote the state and the control input and are taking values in $\mathbb{R}^n$ and $\mathbb{R}^d$, respectively. The initial state x_0 is a random variable with finite mean and covariance C_{x_0}. The system noise process $\{w_k\}$ is i.i.d. (independent identically distributed) and w_k takes values in $\mathbb{R}^n$ and is zero-mean and has finite covariance matrix C_w. The random variables x_0 and w_k are statistically independent for each k. Let $(\Omega, \mathscr{F}, \mathsf{P})$ denote the probability space generated by the initial state x_0 and noise sequence W^{N-1}, where N is the horizon of the problem considered. We call x_0 and w_k the primitive random variables of the system. The statistics of the process $\mathscr{P}$ are known a-priori to both decision makers, the event-trigger $\mathscr{E}$ and the controller $\mathscr{C}$.

The event-trigger $\mathscr{E}$ situated at the sensor station has access to the complete state information and decides, whether the controller $\mathscr{C}$ should receive an update over the feedback channel $\mathscr{N}$. The controller calculates inputs u_k to regulate the process $\mathscr{P}$.

Concerning our system model, it is needed to define the amount of information available at the control station at each time step k. The output signal of the event-trigger, δ_k, takes values in $\{0,1\}$ deciding whether information is transmitted at time k, i.e.,

$$\delta_k = \begin{cases} 1, & \text{measurement } x_k \text{ sent,} \\ 0, & \text{no measurement transmitted.} \end{cases}$$

Therefore, the signal z_k is defined as

$$z_k = \begin{cases} x_k, & \delta_k = 1, \\ \varnothing, & \delta_k = 0. \end{cases} \tag{2.2}$$

As various steps of decisions are made within one time period k, a causal ordering is specified by the following sequence in which the events within the system occur.

$$\cdots \to x_k \to \delta_k \to z_k \to u_k \to x_{k+1} \to \cdots$$

We allow both decision makers – the controller and the event-trigger – to select their actions upon their complete past history. Let the event-triggering law $\mathrm{f} = [\mathrm{f}_0, \mathrm{f}_1, \ldots, \mathrm{f}_{N-1}]$ and the control law $\gamma = [\gamma_0, \gamma_1, \ldots, \gamma_{N-1}]$ denote admissible policies for the finite horizon N with

$$\delta_k = \mathrm{f}_k(X^k), \qquad u_k = \gamma_k(Z^k).$$

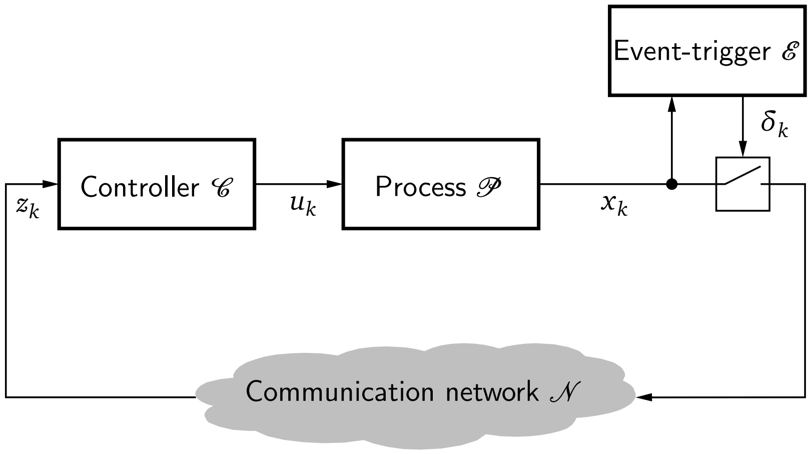


Figure 2.1: System model of the resource-constrained control system with process $\mathscr{P}$, event-trigger $\mathscr{E}$, controller $\mathscr{C}$, and communication channel $\mathscr{N}$.

We assume that the mappings f_k and γ_k are measurable mappings of their available information X^k and Z^k, respectively.

We define the information available at the event-trigger and the controller at time step k as the σ-algebra generated by X^k and Z^k, respectively. These are denoted by $\sigma(X^k)$ and $\sigma(Z^k)$. We make the following crucial observation. For an arbitrary choice of laws γ and f, we have $\sigma(Z^{k-1}) \subset \sigma(X^k)$ because Z^{k-1} can be expressed as a measurable function of X^k implying that the information available at the controller can be recovered by the event-trigger. Since we assume the control law to be deterministic, it can therefore be concluded that the control inputs U^{k-1} are known by the event-trigger at time k.

Let $J_{\mathscr{C}}$ be the control objective related to linear quadratic regulation over the finite horizon N, i.e.,

$$J_{\mathscr{C}} = x_N^{\mathrm{T}} Q_N x_N + \sum_{k=0}^{N-1} x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k. \tag{2.3}$$

The communication channel $\mathscr{N}$ takes the role of restricting or penalizing transmissions in the feedback loop. This will be reflected in the optimization problem as follows. Define $J_{\mathscr{E}}$ to be the communication cost given by the number of transmissions, i.e.,

$$J_{\mathscr{E}} = \sum_{k=0}^{N-1} \delta_k. \tag{2.4}$$

We formulate the design problem of event-triggering law f and control law γ as the following two-person team decision problem, where both decision makers have to find a trade-off between the expected control cost and the expected communication cost. In other words, we aim at finding the optimal f^* and γ^* that

$$\inf_{(\mathrm{f},\gamma)\in\mathscr{U}} \mathsf{E}\left[J_{\mathscr{C}} + \lambda J_{\mathscr{E}}\right] \tag{2.5}$$

with communication penalty $\lambda > 0$ and $\mathscr{U}$ being the set of all admissible policy pairs (f, γ). For notational convenience, we define the cost function $J(\mathrm{f}, \gamma)$ for $(\mathrm{f}, \gamma) \in \mathscr{U}$ to be

$$J(\mathrm{f}, \gamma) = \mathsf{E}\left[J_{\mathscr{C}} + \lambda J_{\mathscr{E}}\right]. \tag{2.6}$$

2.2 Optimal Time-Triggered Control

In this section, we are concerned with the optimal design of the time-triggered controller. Time-triggered sampling implies that the decision whether to transmit a state update to the controller must be made beforehand. Therefore, time-triggered sampling can be regarded as a special case of event-triggered sampling, in which the sampling decisions must not depend on the observation history. Let us denote the admissible time-triggered control policies as $\mathscr{U}_{TT}$ defined by

$$\mathscr{U}_{TT} = \{(\mathrm{f}, \gamma) \in \mathscr{U} \,|\, \mathrm{f}_k = \text{const.}, k \in \{0, \dots, N-1\}\}. \tag{2.7}$$

The corresponding design problem for determining the optimal time-triggered controller can then be formulated as the following optimization problem.

$$\inf_{(\mathrm{f},\gamma)\in\mathscr{U}_{TT}} \mathsf{E}\left[J_{\mathscr{C}} + \lambda J_{\mathscr{E}}\right]. \tag{2.8}$$

Let us fix an arbitrary triggering sequence $\mathrm{f}_{TT} = [\delta_0, \dots, \delta_{N-1}] \in \{0,1\}^N$ and investigate the corresponding optimal control law γ^* stated by the following problem using Eq. (2.6).

$$\inf_{\gamma} J(\mathrm{f}_{TT}, \gamma) \tag{2.9}$$

The communication cost $J_{\mathscr{E}}$ is constant in this case, and it can therefore be omitted from the optimization. What remains is the expected quadratic cost term $\mathsf{E}[J_{\mathscr{C}}]$. Furthermore, Eq. (2.2) can be written as a linear time-varying measurement equation

$$z_k = C_k x_k, \quad C_k = \begin{cases} I_n, & \delta_k = 1, \\ 0_n, & \delta_k = 0. \end{cases} \tag{2.10}$$

As the process evolves according to a linear difference equation given by Eq. (2.1) and the measurement equation given by Eq. (2.10) is also linear in the state x_k and the costs are quadratic in the state and the control input, we can conclude that the problem of finding the optimal control law in Eq. (2.9) reduces to a standard linear quadratic control problem without measurement noise [Ber05]. For such problem, the mathematical tools of stochastic optimal control, such as dynamic programming and the separation principle can be applied. In fact, the solution is given by a certainty equivalence controller consisting of a linear gain and a Kalman estimator [Ber05]. A *certainty equivalence controller* is given by solving a related deterministic optimal control problem, where all primitive random variables are set to their means, and by replacing the state variable by its least-squares estimate within the deterministic solution. We also say the optimal solution has the certainty equivalence property, if the optimal controller is a certainty equivalence controller. As the corresponding deterministic optimal control problem coincides with the deterministic LQR problem, the optimal controller is linear and the control gains can be calculated recursively by a Riccati difference equation. The least-squares estimate is given by the state itself, if a transmission occurs, or is determined by the linear prediction, if no state update is received. We summarize these results in the following proposition.

Proposition 2.1. *Let an arbitrary time-triggered transmission sequence* f_{TT} *be fixed. Then, the optimal solution of the optimization problem defined in Eq. (2.9) is given by the certainty equivalence controller*

$$u_k = \gamma_k^*(Z^k) = -L_k \mathsf{E}[x_k|Z^k], \quad k \in \{0, \ldots, N-1\} \tag{2.11}$$

with

$$\begin{aligned} L_k &= \left(R + B^{\mathrm{T}} P_{k+1} B\right)^{-1} B^{\mathrm{T}} P_{k+1} A, \\ P_k &= A^{\mathrm{T}} P_{k+1} A + Q - A^{\mathrm{T}} P_{k+1} B \left(R + B^{\mathrm{T}} P_{k+1} B\right)^{-1} B^{\mathrm{T}} P_{k+1} A, \end{aligned} \tag{2.12}$$

where $P_N = Q_N$ *and* $P_k \in \mathbb{R}^{n\times n}$ *is non-negative definite for* $k \in \{0, \ldots, N-1\}$*. The estimator* $\mathsf{E}[x_k|Z^k]$ *is given by the following recursive form*

$$\mathsf{E}[x_k|Z^k] = \begin{cases} x_k, & \delta_k = 1, \\ (A - BL_k)\mathsf{E}[x_{k-1}|Z^{k-1}], & \delta_k = 0, \end{cases} \tag{2.13}$$

with $\mathsf{E}[x_0|Z^0] = \mathsf{E}[x_0]$ *for* $\delta_0 = 0$.

Having obtained the optimal controller for a given transmission sequence, we focus now on the calculation of the optimal time-triggered transmission sequence f_{TT}^*. For that reason, let ϵ_k be the estimation error at time k defined as

$$\epsilon_k = x_k - \mathsf{E}[x_k|Z^k]. \tag{2.14}$$

By using Lemma 6.1 in Chapter 8 of [Åst06] and a couple of reformulations whose details can be found in the proof of Lemma 2.1, the cost function can be rewritten in the following form.

$$\begin{aligned} J =& \lambda \mathsf{E}[\sum_{k=0}^{N} \delta_k] + \mathsf{E}[x_0^{\mathrm{T}} P_0 x_0] + \mathsf{E}[\sum_{k=0}^{N-1} w_k^{\mathrm{T}} P_{k+1} w_k] + \mathsf{E}[\sum_{k=0}^{N-1} \epsilon_k^{\mathrm{T}} L_k^{\mathrm{T}} \Gamma_k L_k \epsilon_k] \\ &+ \mathsf{E}[\sum_{k=0}^{N-1} (u_k + L_k \mathsf{E}[x_k|Z^k])^{\mathrm{T}} \Gamma_k (u_k + L_k \mathsf{E}[x_k|Z^k)], \end{aligned} \tag{2.15}$$

where Γ_k is defined as

$$\Gamma_k = B^{\mathrm{T}} P_{k+1} B + R, \qquad k \in \{0, \ldots, N-1\}. \tag{2.16}$$

Obviously, the second and the third term on the right-hand side of Eq. (2.15) are constant. For a fixed time-triggered transmission sequence, we observe that the first term is constant. In addition, it can be shown that ϵ_k is a random variable that is independent of the policy γ chosen, see for example lemma in [Ber05]. This gives the optimal control law γ^* in Eq. (2.11) for a fixed transmission sequence as already stated in Proposition 2.1. On the other hand, only the first and the forth term are varying with different transmission sequences when assuming that the control law is given by Eq. (2.11). In order to calculate the optimal time-triggered transmission sequence, we define the one-step ahead estimation error e_k by

$$e_k = x_k - \mathsf{E}[x_k|Z^{k-1}]. \tag{2.17}$$

From this definition, we have the following connection to the estimation error.

$$\epsilon_k = \begin{cases} 0, & \delta_k = 1, \\ e_k, & \delta_k = 0. \end{cases}$$

The evolution of e_k can be derived by

$$\begin{aligned} e_{k+1} &= x_{k+1} - \mathsf{E}[x_{k+1}|Z^k] \\ &= Ax_k + Bu_k + w_k - \mathsf{E}[Ax_k + Bu_k + w_k|Z^k] \\ &= A(x_k - \mathsf{E}[x_k|Z^k]) + w_k \\ &= (1-\delta_k)Ae_k + w_k. \end{aligned}$$

The remaining optimization problem has then the following form

$$\begin{aligned} \mathrm{f}_{TT} = \arg \inf_{\delta_0,\ldots,\delta_{N-1}} & \mathsf{E}[\sum_{k=0}^{N-1}(1-\delta_k)e_k^{\mathrm{T}}L_k^{\mathrm{T}}\Gamma_k L_k e_k + \lambda\delta_k], \\ \text{s.t.} \quad & e_{k+1} = (1-\delta_k)Ae_k + w_k. \end{aligned} \tag{2.18}$$

Since the triggering variable δ_k is chosen before execution, i.e., it is independent of e_k, it is possible to rewrite above optimization problem in order to apply dynamic programming. For that reason, we define the error covariance

$$\Phi_k = \mathsf{E}[e_k e_k^{\mathrm{T}}].$$

The evolution of Φ_k is given by

$$\Phi_{k+1} = (1-\delta_k)A\Phi_k A^{\mathrm{T}} + C_w, \quad \Phi_0 = C_{x_0}.$$

Then, the optimization problem in Eq. (2.18) can be written as

$$\begin{aligned} \mathrm{f}_{TT} = \arg \inf_{\delta_0,\ldots,\delta_{N-1}} & \sum_{k=0}^{N-1}(1-\delta_k)\mathrm{tr}[\Phi_k L_k^{\mathrm{T}}\Gamma_k L_k] + \lambda\delta_k, \\ \text{s.t.} \quad & \Phi_{k+1} = (1-\delta_k)A\Phi_k A^{\mathrm{T}} + C_w, \quad \Phi_0 = C_{x_0}. \end{aligned} \tag{2.19}$$

We observe that the initially stochastic optimization problem reduces to a deterministic optimal control problem with state variable Φ_k. This implies that the calculation of the optimal time-triggered transmission sequence, f_{TT}^*, can be performed by deterministic dynamic programming [Ber05]. In summary, the optimal time-triggered controller within the set $\mathcal{U}_{TT}$ can be calculated in two steps:

1. Obtain the optimal control gain L_k from the discrete-time Riccati equation in Eq. (2.12).

2. Solve optimization problem Eq. (2.19) that yields the optimal transmission timings.

Inspired by this design approach, the more challenging problem of event-triggered transmission strategies is studied in the next section.

2.3 Optimal Event-Triggered Control

What makes the derivation in the previous section appealing relies on the fact that the cost function J defined in Eq. (2.6) is separable with respect to the control design and the choice of the transmission times. This becomes evident when regarding Eq. (2.15) where the summands either depend on the choice of the transmission times or on the choice of the control input. When allowing the transmissions to be triggered by events rather than by a priori fixed timings, the separation does not hold in the same way as for the time-triggered mechanism. This is due to the fact that the estimation error ϵ_k is generally not independent of the control law anymore when assuming a fixed event-triggering law f. In other words, the controller is able to signal through the plant to the event-trigger that it may want to receive another state update. Such signalling is called the dual effect of control, [BST74], and refers to the dual role of control: (i) influencing the state evolution and (ii) decreasing the estimation error. When the second phenomenon is not present, which is also referred to as the absence of the dual effect, then the optimal control law is given by Eq. (2.11). However, in the event-triggered case the dual effect is present in general, which implies that the optimal control law will be a nonlinear function of the complete history Z^k, which highly depends on the choice of the fixed event-trigger.

Another approach that might be taken is the joint optimization of both the control law and the event-triggering law as a team decision problem. But as the information available at the controller and at the event-trigger differ, the optimization problem has a non-classical information pattern, whose solution is very hard to find and no systematic algorithms are available, even for simple cases [Wit68]. In fact, the joint optimization problem under consideration falls into the category of sequential stochastic control problems, for which a dynamic programming formulation is possible [Wit73]. But the value function must be parameterized by the distribution of the state, which implies an infinite dimensional state space, and the minimum is taken over all control laws rather than over the inputs. Obviously, this formulation will unlikely reveal new insights either.

The aforementioned arguments suggest that only little can be said about the optimal event-triggered controller that solves the optimization problem described in Eq. (2.5). Nevertheless, we will observe in the following that it is possible to give a characterization of the optimal event-triggered controllers. Before analyzing the general case, the following subsection considers the optimization over a special class of event-triggered controllers in which the control inputs need to be constant between transmissions. It turns out that this special class allows a formulation as a standard optimal control problem.

2.3.1 Event-Triggered Sampling with Zero-Order Hold Control

An interesting special case of event-triggered control is the class of zero-order hold controllers that are constant between event-triggered transmissions. By extending the state space accordingly, it will turn out that the optimization problem given in Eq. (2.5) can be recast as a standard optimal control problem in the case of zero-order hold control. In order to incorporate the fact that the control input remains constant between transmission times, we extend the system dynamics by

$$u_k = \begin{cases} u_{k-1}, & \delta_k = 0, \\ u'_k, & \delta_k = 1, \end{cases} \tag{2.20}$$

where $u'_k \in \mathbb{R}^d$ is the new control input. Due to the zero-order hold element, the state space needs to be extended by u_{k-1}. The initial value of the control input, u_{-1}, element can either be part of the optimization problem or is given beforehand.

By considering the process dynamics defined in Eq. (2.1) and the zero-order hold element defined in Eq. (2.20), we observe that the overall system can be regarded as a (δ_k, u'_k)-controlled Markov chain with state (x_k, u_{k-1}) driven by the i.i.d. noise process given by w_k. The ensuing question that needs to be addressed is whether the event-trigger and controller have full state information. By assuming that u'_k is a measurable function of its observation history Z^k, we can follow that past control inputs prior to time step k can be recovered at the event-trigger. This implies that the event-trigger has full state information. However, this is not the case for the controller, which does not have access to the process state x_k when $\delta_k = 0$. But for $\delta_k = 0$, the action u'_k has no influence on the evolution of (x_k, u_{k-1}). This implies that permitting the controller to have full state information also at $\delta_k = 0$ will not change the optimization problem.

Hence, we can conclude that the problem of optimal event-triggered control design defined in Eq. (2.5) reduces to a standard optimal control problem that can be solved by dynamic programming, when control inputs are restrained to be constant between transmissions.

Rather than further investigating the explicit calculation of the optimal solution, we want to point out by this example that the information pattern of the decision makers can be crucial for the analysis of the underlying team decision problem. What made the formulation as a standard optimal control problem possible was the fact that the problem could be posed such that both decision makers were having the same information available reducing it to a problem with a single decision maker.

2.3.2 The Certainty Equivalence Property of Event-Triggered Control

In this subsection, we return to the initial team decision problem formulated in Eq. (2.5). It is shown that the control law of the optimal event-triggered controller in the set of admissible laws $\mathcal{U}$ will be a certainty equivalence controller given by Eq. (2.11), i.e., it takes the same form as in the time-triggered case. This result can be regarded as one of the central new insights of this thesis concerning the design of optimal event-triggered controllers. The key property that enables such result is the nested structure of the information pattern, since the information available at the controller is a subset of the information available at the event-trigger. The importance of nested information patterns has been demonstrated in [HC72] for a different design problem, where it has been shown that the design of decentralized controllers in the LQG framework yields linear solutions when the information structure is *partially nested*. As pointed out in [HC72], it is also crucial to assert that the admissible laws are deterministic, which is also assumed in this thesis by the definition of $\mathcal{U}$. Intuitively speaking, the reason for such assumption is to be able to recover the control inputs applied by the decision maker whose information is a subset of the information of the other decision maker.

In order to establish the structural result, we introduce the formal concept of dominating strategies in optimal control.

Definition 2.1 (Dominating policies). A set of policy pairs $\mathscr{U}' \subset \mathscr{U}$ is called a *dominating class of policies* for optimization problem (2.5), if for any feasible $(\mathrm{f}, \gamma) \in \mathscr{U}$, there exists a feasible $(\mathrm{f}', \gamma') \in \mathscr{U}'$, such that

$$J(\mathrm{f}', \gamma') \leq J(\mathrm{f}, \gamma),$$

where J is the cost function defined by Eq. (2.6) for the corresponding problem.

Once a dominating class of policies is found, the above definition implies that we can restrict the solutions of the optimization problem to such policies. Subsequently, we show that the set of policy pairs where the controller is a certainty equivalence controller denoted by γ^* is a dominating class of policies. A certainty equivalence controller is given by solving a related deterministic control problem, where all primitive random variables are set to their means, and by replacing the state variable by its least-squares estimate within the deterministic solution. The remaining goal is to prove that for any pair (f, γ), we can find a pair (f', γ^*) whose costs are at most that of (f, γ).

In order to achieve this, we introduce a suitable reparametrization of the triggering law. Given a policy (f, γ), we define another policy (g, γ) where $\mathrm{g} = \{\mathrm{g}_0, \dots, \mathrm{g}_{N-1}\}$ is the triggering law, and g_k is a function of $\{x_0, W^{k-1}\}$, such that

$$\mathrm{g}_k(x_0, W^{k-1}) = \mathrm{f}_k(X^k), \quad k \in \{0, \dots, N-1\}, \omega \in \Omega, \tag{2.21}$$

when both systems use the control law γ. As the control inputs U^{k-1} are known at the event-trigger at time k by the law γ due to $\sigma(Z^{k-1}) \subset \sigma(X^k)$, the variables $\{x_0, W^{k-1}\}$ can be fully recovered by the state sequence X^k and vice versa. Therefore, the triggering law g satisfying Eq. (2.21) always exists. On the other hand, this also implies that given (g, γ), there is always a (f, γ) satisfying Eq. (2.21).

The next intermediate result states on the optimal control law for fixed g.

Lemma 2.1. *Let the triggering law* g *be a function of primitive variables given by*

$$\delta_k = \mathrm{g}_k(x_0, W^{k-1}), \quad k \in \{0, \dots, N-1\}. \tag{2.22}$$

If the triggering law g *is fixed, then the optimal control law* γ^* *minimizing* $J(\mathrm{g}, \gamma)$ *defined in Eq. (2.6) is a certainty equivalence controller given by*

$$u_k = \gamma_k^*(Z^k) = -L_k \,\mathsf{E}[x_k | Z^k], \quad k \in \{0, \dots, N-1\} \tag{2.23}$$

with L_k *being the solution of the Riccati equation defined in Eq. (2.12).*

Proof. Since g is fixed, the output δ_k is a random variable described by a function of primitive random variables that is independent of the choice of the control law γ. This implies that $\mathsf{E}[J_{\mathscr{E}}]$ is a constant for a fixed g. Thus, solving the optimization problem (2.5) for a fixed g reduces to minimizing $\mathsf{E}[J_{\mathscr{C}}]$ over all admissible control laws γ. The resulting objective function is purely quadratic, and tools from stochastic control can be applied [Ber05].

Similarly to [Ber05], we first show that the estimation error, ϵ_k, at the controller defined by Eq. (2.14) is a random variable that can be described as a function of primitive random variables x_0 and W^{k-1} which is independent of the control law γ. Let us fix a control law γ and consider two types of systems: a forced and an un-forced system. In the first system, control inputs are determined by the law γ and the system evolves by Eq. (2.1) and Eq. (2.2), whereas the second system has zero-input and is given by

$$\tilde{x}_{k+1} = A\tilde{x}_k + \tilde{w}_k,$$
$$\tilde{z}_k = \begin{cases} \tilde{x}_k, & \tilde{\delta}_k = 1, \\ \varnothing, & \tilde{\delta}_k = 0. \end{cases}$$

We assume the primitive random variables are identical for both systems, i.e.,

$$\tilde{x}_0 = x_0, \quad \tilde{w}_k = w_k, \quad k = 0, \ldots, N-1. \tag{2.24}$$

Since the triggering output δ_k is a function of primitive random variables defined by Eq. (2.22) that is independent of γ, we have

$$\tilde{\delta}_k = \delta_k, \quad k = 0, \ldots, N-1.$$

Because of linearity, we can rewrite the systems into the following matrix-vector notation

$$x_k = F_k x_0 + G_k U^{k-1} + H_k W^{k-1},$$
$$\tilde{x}_k = F_k x_0 + H_k W^{k-1},$$

where U^{k-1}, W^{k-1} are the augmented signal vectors and F_k, G_k, and H_k are appropriate matrices constructed from A and B. As U^{k-1} is measurable with respect to the information pattern Z^k, the conditional expectations are given by

$$\mathsf{E}[x_k|Z^k] = F_k \mathsf{E}[x_0|Z^k] + G_k U^{k-1} + H_k \mathsf{E}[W^{k-1}|Z^k],$$
$$\mathsf{E}[\tilde{x}_k|Z^k] = F_k \mathsf{E}[x_0|Z^k] + H_k \mathsf{E}[W^{k-1}|Z^k].$$

Hence, we obtain

$$\epsilon_k = x_k - \mathsf{E}[x_k|Z^k] = \tilde{x}_k - \mathsf{E}[\tilde{x}_k|Z^k].$$

Given the laws γ and g, it is trivial to show that there exists a bijective mapping between Z^k and $\tilde{Z}^k$. This implies that the σ-algebra generated by $\tilde{Z}^k$ is identical to the σ-algebra generated by Z^k. This is because the vectors δ^k and $\tilde{\delta}^k$ are identical random variables, and

$$\tilde{z}_k = \begin{cases} z_k - G_k U^{k-1}, & \delta_k = 1, \\ \varnothing, & \delta_k = 0, \end{cases}$$

while

$$z_0 = \tilde{z}_0, \qquad u_0 = \gamma_0(z_0) = \gamma_0(\tilde{z}_0),$$
$$z_1 = \begin{cases} \tilde{z}_0 + G_1\gamma_0(\tilde{z}_0), & \tilde{\delta}_1 = 1, \\ \varnothing, & \tilde{\delta}_1 = 0, \end{cases}$$
$$u_1 = \begin{cases} \gamma_1(\tilde{z}_0, \tilde{z}_0 + G_1\gamma_0(\tilde{z}_0)), & \tilde{\delta}_1 = 1, \\ \gamma_1(\tilde{z}_0), & \tilde{\delta}_1 = 0, \end{cases}$$
$$\vdots$$

Therefore, we can write

$$\epsilon_k = \tilde{x}_k - \mathsf{E}[\tilde{x}_k|\tilde{Z}^k]. \tag{2.25}$$

Since $\tilde{x}_k - \mathsf{E}[\tilde{x}_k|\tilde{Z}^k]$ in Eq. (2.25) can be expressed in terms of primitive random variables and is independent of the control law γ, we have showed that the estimation error ϵ_k is given by a function of primitive random variables which is independent of γ.

Next, we use the identity to reformulate $J_{\mathscr{C}}$ defined by Eq. (2.3), see Lemma 6.1 of Chapter 8 in [Åst06] that is given by

$$\begin{aligned} J_{\mathscr{C}} =& x_0^{\mathrm{T}} P_0 x_0 + \sum_{k=0}^{N-1} (u_k + L_k x_k)^{\mathrm{T}} (B^{\mathrm{T}} P_{k+1} B + R)(u_k + L_k x_k) \\ &+ \sum_{k=0}^{N-1} w_k^{\mathrm{T}} P_{k+1} (A x_k + B u_k) + (A x_k + B u_k)^{\mathrm{T}} P_{k+1} w_k \\ &+ \sum_{k=0}^{N-1} w_k^{\mathrm{T}} P_{k+1} w_k, \end{aligned}$$

where L_k and P_k are given by Eq. (2.12). By taking expectation and incorporating independence of w_k with respect to x_k and u_k, we have

$$\begin{aligned} \mathsf{E}[J_{\mathscr{C}}] =& \mathsf{E}[x_0^{\mathrm{T}} P_0 x_0] + \mathsf{E}[\sum_{k=0}^{N-1} w_k^{\mathrm{T}} P_{k+1} w_k] \\ &+ \mathsf{E}[\sum_{k=0}^{N-1} (u_k + L_k x_k)^{\mathrm{T}} \Gamma_k (u_k + L_k x_k)], \end{aligned}$$

where Γ_k is defined in Eq. (2.16). The first two terms are constant and can be omitted from the optimization. After replacing x_k with $\mathsf{E}[x_k|Z^k] + \epsilon_k$, we have

$$\begin{aligned} &(u_k + L_k x_k)^{\mathrm{T}} \Gamma_k (u_k + L_k x_k) = \\ =&(u_k + L_k \mathsf{E}[x_k|Z^k] + L_k \epsilon_k)^{\mathrm{T}} \Gamma_k (u_k + L_k \mathsf{E}[x_k|Z^k] + L_k \epsilon_k) \\ =&(u_k + L_k \mathsf{E}[x_k|Z^k])^{\mathrm{T}} \Gamma_k (u_k + L_k \mathsf{E}[x_k|Z^k]) \\ &+ (u_k + L_k \mathsf{E}[x_k|Z^k])^{\mathrm{T}} \Gamma_k L_k \epsilon_k + \epsilon_k^{\mathrm{T}} L_k^{\mathrm{T}} \Gamma_k (u_k + L_k \mathsf{E}[x_k|Z^k]) \\ &+ \epsilon_k^{\mathrm{T}} L_k^{\mathrm{T}} \Gamma_k L_k \epsilon_k. \end{aligned} \tag{2.26}$$

By applying the tower property of conditional expectations, we obtain

$$\begin{aligned} &\mathsf{E}[(u_k + L_k \mathsf{E}[x_k|Z^k])^{\mathrm{T}} \Gamma_k L_k \epsilon_k] = \\ &= \mathsf{E}[\mathsf{E}[(u_k + L_k \mathsf{E}[x_k|Z^k])^{\mathrm{T}} \Gamma_k L_k \epsilon_k | Z^k]] \\ &= \mathsf{E}[(u_k + L_k \mathsf{E}[x_k|Z^k])^{\mathrm{T}} \Gamma_k L_k \mathsf{E}[\epsilon_k|Z^k]]. \end{aligned}$$

The second equality is because $u_k = \gamma_k(Z^k)$ and $\mathsf{E}[x_k|Z^k]$ are measurable functions with respect to Z^k. In fact,

$$\begin{aligned} \mathsf{E}[\epsilon_k|Z^k] &= \mathsf{E}[x_k|Z^k] - \mathsf{E}[\mathsf{E}[x_k|Z^k]|Z^k] \\ &= \mathsf{E}[x_k|Z^k] - \mathsf{E}[x_k|Z^k] = 0. \end{aligned}$$

Thus, the cross terms in Eq. (2.26) vanish and we obtain

$$\begin{aligned}\mathsf{E}[J_{\mathscr{C}}] &= \mathsf{E}[x_0^{\mathrm{T}} P_0 x_0] + \mathsf{E}[\sum_{k=0}^{N-1} w_k^{\mathrm{T}} P_{k+1} w_k] + \mathsf{E}[\sum_{k=0}^{N-1} \epsilon_k^{\mathrm{T}} L_k^{\mathrm{T}} \Gamma_k L_k \epsilon_k] \\ &\quad + \mathsf{E}[\sum_{k=0}^{N-1} (u_k + L_k \mathsf{E}[x_k|Z^k])^{\mathrm{T}} \Gamma_k (u_k + L_k \mathsf{E}[x_k|Z^k])]. \qquad (2.27)\end{aligned}$$

As the first three terms are constant, we observe that $\mathsf{E}[J_{\mathscr{C}}]$ attains its minimum for γ^* given by Eq. (2.23). This concludes the proof. □

Built upon this intermediate result, we obtain the following key theorem which is the main result of this chapter.

Theorem 2.1. *Let the system be given by Eq. (2.1) and Eq. (2.2). Then the class of policies $\mathscr{U}_{CE} \subset \mathscr{U}$ defined by*

$$\mathscr{U}_{CE} = \{(\mathrm{f}, \gamma^*) \,|\, \gamma_k^* = -L_k \mathsf{E}[x_k|Z^k],\ L_k \textit{ given by (2.12)}\}$$

is a dominating class of policies for the optimization problem (2.5).

Proof. According to Definition 2.1, it suffices to show that for any feasible pair $(\mathrm{f}, \gamma) \in \mathscr{U}$, there is a feasible policy $(\mathrm{f}', \gamma^*) \in \mathscr{U}_{\mathrm{CE}}$ whose costs are at most that of (f, γ).

Given a feasible pair (f, γ), there exists a feasible pair (g, γ) with g_k being a function of primitive variables that satisfies Eq. (2.21). This is because of the nestedness property $\sigma(Z^{k-1}) \subset \sigma(X^k)$. Condition Eq. (2.21) implies that for (f, γ) and (g, γ), we have identical random variables u_k and δ_k for $k \in \{0, \ldots, N-1\}$ and therefore identical costs. In the same way for the pair (g, γ^*), we can find a triggering law f' being a function of X^k, such that both (g, γ^*) and (f', γ^*) output identical random variables u_k and δ_k for $k \in \{0, \ldots, N-1\}$. Due to Lemma 2.1, we obtain

$$J(\mathrm{f}, \gamma) = J(\mathrm{g}, \gamma) \geq \min_{\gamma} J(\mathrm{g}, \gamma) = J(\mathrm{g}, \gamma^*) = J(\mathrm{f}', \gamma^*).$$

Since (g, γ^*) is feasible, the pair (f', γ^*) is also feasible. This concludes the proof. □

Theorem 2.1 implies that one can not benefit from the dual effect of control in the joint optimization problem (2.5). Intuitively, this can be reasoned by the observation that probing the system by the controller to reduce its uncertainty about the state becomes redundant, since the event-trigger takes complete control in providing information to the controller. Due to the nested information pattern, any initiative for probing the system by the controller can be accommodated by an appropriate adaptation of the event-trigger.

Aside from this fact, Theorem 2.1 facilitates the two-person decision problem defined in Eq. (2.5) significantly. By considering the cost formulation in Eq. (2.27) and substituting the control law (2.23) into the problem (2.5), we obtain the following optimization problem.

$$\inf_{\mathrm{f}} \mathsf{E}\left[\sum_{k=0}^{N-1} (x_k - \mathsf{E}[x_k|Z^k])^{\mathrm{T}} L_k^{\mathrm{T}} \Gamma_k L_k (x_k - \mathsf{E}[x_k|Z^k]) + \lambda \delta_k\right]. \qquad (2.28)$$

This problem remains challenging, as the least-squares estimate $\mathsf{E}[x_k|Z^k]$ in the cost function depends on the choice of the event-triggering law f. This implies that the mathematical tools from stochastic optimal control are still not directly applicable. Above optimization problem will therefore be studied in more detail in the subsequent chapter.

2.4 Summary

We have considered the joint design of controller and event-triggered formulated in the framework of team decision theory. By analyzing an extended version of the LQ problem, we have been interested in structural properties of the optimal control law. The key result of this chapter is that one can not benefit of the dual effect of control in such problem setting. As a consequence, the optimal control law can be characterized by having the certainty equivalence property. This result was enabled by the special information pattern of the decision makers being nestedness of the information available at the event-trigger and the controller. Besides its advantages for the analysis and computation of optimal event-triggered controllers that will also be crucial in the following chapters, this structural result underlines the importance for a careful incorporation of the information pattern in the analysis of team decision problems.

2.5 Bibliographical Notes

The results in this chapter are partly based on the work in [MH09; MH13a], and Section 5.7 of [GHJ+13]. A continuous-time version of the main structural result can be found in [MH10a]. The certainty equivalence property for the optimization problems also holds for related control problems with communication constraints that distinguish themselves from the previously considered problem in Eq. (2.5) by the way the communication constraint is incorporated. Besides the results in Section 2.3.2, it has been showed in [MH13a] that Theorem 2.1 is valid either when constraining the number of transmissions over the horizon N or the expected number of transmissions.

In [RSJ13], it has been demonstrated that there is in general a dual effect of control for a networked system with a fixed event-triggered scheduler. The authors show that the dual effect of control is absent for symmetric event-triggering rules, in which the decision function is even in the estimation error. This implies that standard techniques of stochastic control like the separation principle and certainty equivalence hold in this special class of event-triggers.

Similarities can be also found for optimal control under bit-rate limitations in the feedback loop. It is shown in [BM97; TSM04; NFZE07] that the certainty equivalence controller is optimal within the LQG framework under such communication constraints for different communication models. Finally, it should be stated that the case of joint optimization of the time-triggered sampling sequence and control studied in Section 2.2 has also been addressed in [MPD67; WA08] for optimal sensor querying and control. Within the LQG framework with costly queries, it is found that the optimal control law has the certainty equivalence property and the timings for queries can be determined offline by dynamic programming [MPD67; WA08].

3

Structural Properties of Optimal Event-Triggered Estimation

We have observed in the last chapter that the optimal design of event-triggered controllers in the framework of linear quadratic control can be split up into two subproblems – the calculation of a certainty equivalence controller and the joint optimization of event-trigger and estimator. The latter subproblem is the subject of this chapter. Similar to the previous chapter, the estimator and the event-trigger are regarded as two decision makers of a team decision problem, in which the cost function is composed of the mean squared error and a communication penalty.

After having derived basic properties of the optimal estimator and event-trigger, an iterative method is developed for the joint optimization of event-trigger and estimator for linear systems. The algorithm iteratively alternates between optimizing one decision maker while fixing the other one. Then, we restrict the analysis to first-order systems and analyze the convergence properties of the iterative method. This lets us draw conclusions on the optimal design of event-triggered estimators. On the one hand, when the densities of the initial state and the noise variables are symmetric and unimodal functions, it is shown for scalar systems that the solution of the algorithm converges to a symmetric event-trigger and a linear predictor that does not take the triggering law into account. This implies that one can not take advantage of signaling information from the event-trigger to the estimator through not triggering in this case. On the other hand, when the noise distribution is assumed to be bimodal, it turns out that the proposed iterative method can yield a remarkable decrease of the cost compared to an emulation-based approach, in which the estimator takes the form of a linear predictor that does not take the triggering law into account. In the case of bimodal distributions, the event-triggering rule resulting from the iterative algorithm is commonly asymmetric and the estimator can improve its state uncertainty considerably by incorporating the information of not having triggered.

This chapter can be outlined as follows. Section 3.1 introduces the optimal design problem of event-triggered estimation as a least-squares estimation problem under communication constraints. A basic characterization with respect to the optimal solution of the underlying optimization problem is discussed in Section 3.2. Based on these properties, an iterative algorithm is developed in Section 3.3 whose convergence properties are analyzed in Section 3.4.

3.1 Least-Squares Estimation under Communication Constraints

We consider the following linear process $\mathscr{P}$ driven by noise w_k

$$x_{k+1} = Ax_k + w_k, \tag{3.1}$$

where x_k takes values in $\mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. The system noise w_k takes values in $\mathbb{R}^n$ and is an i.i.d. random variable described by the probability density function ϕ_w, which is zero-mean and has a covariance matrix C_w. The initial state, x_0, is statistically independent of w_k and is described by density function ϕ_{x_0}, which has a finite mean $\mathsf{E}[x_0]$ and a covariance matrix C_{x_0} System parameters and statistics are known to both the event-trigger and estimator.

The system model is illustrated in Fig. 3.1. The process $\mathscr{P}$ outputs the state x_k. The event-trigger $\mathscr{E}$ decides upon its available information whether or not to transmit the current state to the remote state estimator $\mathscr{S}$. We define the output of the event-trigger as

$$\delta_k = \begin{cases} 1, & \text{update } x_k \text{ sent}, \\ 0, & \text{otherwise}. \end{cases}$$

The communication channel between the process $\mathscr{P}$ and the state estimator $\mathscr{S}$ can be viewed as a δ_k-controlled erasure channel whose outputs are described by

$$z_k = \begin{cases} x_k, & \delta_k = 1, \\ \varnothing, & \delta_k = 0, \end{cases} \tag{3.2}$$

where $\varnothing$ is the erasure symbol. As it will be useful for subsequent analysis, we define the last update time τ_k as

$$\tau_k = \max\{\kappa | \delta_\kappa = 1,\ \kappa < k\} \tag{3.3}$$

with $\tau_k = -1$, if no transmissions have occurred prior to k. The variable τ_k can be described by the following δ_k-controlled difference equation

$$\tau_{k+1} = \begin{cases} k, & \delta_k = 1, \\ \tau_k, & \delta_k = 0, \end{cases} \qquad \tau_0 = -1. \tag{3.4}$$

Admissible event-triggers are given by measurable mappings of their past history to

$$\delta_k = \mathrm{f}_k(X^k), \quad k = 0, \ldots, N-1.$$

The estimator $\mathscr{S}$ outputs the state estimate $\hat{x}_k$ and is given by measurable mappings g_k defined by

$$\hat{x}_k = g_k(Z^k), \quad k = 0, \ldots, N-1.$$

The design objective is to jointly design the event-trigger $\mathrm{f} = [\mathrm{f}_0, \ldots, \mathrm{f}_{N-1}]$ and the estimator $g = [g_0, \ldots, g_{N-1}]$ that minimize cost J defined by

$$J = \mathsf{E}^{\mathrm{f},g}\left[\sum_{k=0}^{N-1} \|x_k - \hat{x}_k\|^2 + \lambda\delta_k\right]. \tag{3.5}$$

The per-stage cost of J is composed of the squared estimation error $\|x_k - \hat{x}_k\|_2^2$ and a communication penalty $\lambda\delta_k$. The weight $\lambda > 0$ determines the amount of penalizing transmissions over the channel $\mathscr{N}$.

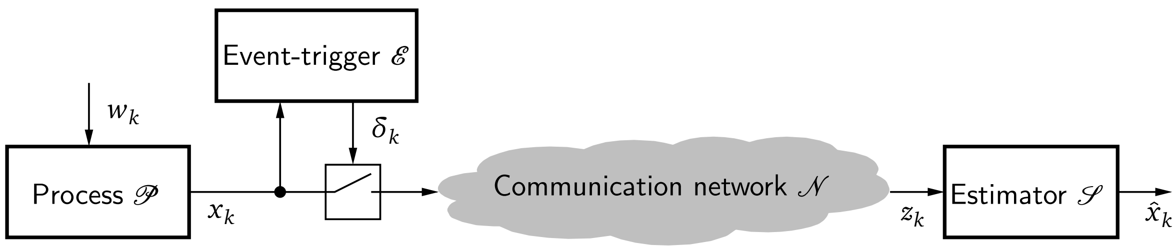


Figure 3.1: System model of the networked estimation system with plant $\mathscr{P}$, event-trigger $\mathscr{E}$, state estimator $\mathscr{S}$ and communication channel $\mathscr{N}$.

3.2 Basic Properties

In this section, we are concerned with finding basic properties of optimal solutions of the joint optimization defined in Eq. (3.5) that will facilitate the description of optimal event-triggered estimators. We begin with a characterization of the optimal estimator given an arbitrary event-trigger.

Proposition 3.1. *For any event-trigger* f*, the optimal state estimator* g^* *is given by the least-squares estimator*

$$\hat{x}_k = g_k^*(Z^k) = \mathsf{E}^{\mathrm{f}}[x_k | Z^k], \quad k = 0, \ldots, N-1.$$

Proof. Fix an arbitrary event-trigger f. The communication penalty term $\mathsf{E}^{\mathrm{f}}\left[\sum_{k=0}^{N-1} \lambda\delta_k\right]$ is then constant and can be omitted from the optimization. In the remaining estimation problem the mean squared error $\mathsf{E}^{\mathrm{f}}\left[\sum_{k=0}^{N-1} \|x_k - \hat{x}_k\|^2\right]$ is to be minimized. The optimal solution for this problem is given by the least-squares estimator $\mathsf{E}^{\mathrm{f}}[x_k|Z^k]$, [Ber05]. This completes the proof. □

In the following, we introduce a time-variant translatory change of coordinates of the state space evolution that will enable us to focus on the main issues involved in the joint optimization of event-trigger and estimator. As the coordinate transformation at each time k can be computed at the event-trigger and the estimator, the optimization problem remains unchanged. Let us define the linear predictor $\hat{x}_k^{\mathrm{LP}}$ by the following recursion

$$\hat{x}_k^{\mathrm{LP}} = \begin{cases} x_k, & \delta_k = 1, \\ A\hat{x}_{k-1}^{\mathrm{LP}}, & \delta_k = 0, \end{cases} \tag{3.6}$$

for $k \in \{1, \ldots, N-1\}$ and $\hat{x}_0^{\mathrm{LP}} = \mathsf{E}[x_0]$. The linear predictor can be regarded as the optimal estimator, when having no information about the choice of the event-trigger f and assuming that transmission instances are statistically independent of the state evolution. This also implies that the linear predictor is optimal in the case, when transmission instances are selected in advance.

Let us rewrite the optimization problem by defining the one-step ahead estimation error of the linear predictor as

$$e_k = x_k - A\hat{x}_{k-1}^{\mathrm{LP}}, \quad k = 1, \ldots, N-1 \tag{3.7}$$

and $e_0 = w_{-1}$, where we define $w_{-1} = x_0 - \mathsf{E}[x_0]$. The variable e_k defines our new state to be estimated and follows the recursion

$$e_{k+1} = h_k(e_k, \delta_k, w_k) = (1 - \delta_k)Ae_k + w_k. \tag{3.8}$$

Further, we define $\hat{e}_k$ to be the least-squares estimate $\mathsf{E}[e_k | \tilde{Z}^k]$, where $\tilde{z}_k$ is defined accordingly as

$$\tilde{z}_k = \begin{cases} e_k, & \delta_k = 1, \\ \varnothing, & \delta_k = 0. \end{cases}$$

The next proposition gives us further insights into the structure of $\hat{e}_k$.

Proposition 3.2. *Let the event-trigger* f *be fixed. Then, the least-squares estimate of e_k is given by*

$$\hat{e}_k = \begin{cases} e_k, & \delta_k = 1, \\ \alpha_k(\tau_k), & \delta_k = 0, \end{cases} \tag{3.9}$$

where τ_k is defined by Eq. (3.3) and $\alpha_k(\tau_k)$ is defined by

$$\alpha_k(\tau_k) = \mathsf{E}^{\mathrm{f}}\left[\sum_{l=\tau_k}^{k-1} A^{k-l-1} w_l | \delta_{\tau_k+1} = 0, \ldots, \delta_k = 0\right]. \tag{3.10}$$

Proof. Clearly, we have $\hat{e}_k = e_k$ for $\delta_k = 1$, as $e_k \in \tilde{Z}^k$. For $\delta_k = 0$, τ_k is a sufficient statistics for $\hat{e}_k$. The mapping α_k is determined by applying recursively Eq. (3.8) with $e_{\tau_k+1} = w_{\tau_k}$. This completes the proof. □

The function α in Proposition 3.2 can be interpreted as a bias term to improve the state estimate by incorporating additional information $\delta_{\tau_k+1} = \cdots = \delta_k = 0$ at time k.

It is straightforward to see that the estimation error $e_k - \hat{e}_k$ and $x_k - \hat{x}_k$ are identical random variables for a fixed event-trigger f, as e_k corresponds to a translatory coordinate transformation of x_k shifted by $-A\hat{x}_{k-1}^{\mathrm{LP}}$ which is known since the sequence δ^{k-1} is measurable with respect to Z^k. Therefore, our initial optimization problem with cost function J can be rewritten as

$$\inf_{\mathrm{f}} \mathsf{E}^{\mathrm{f}} \left[\sum_{k=0}^{N-1} (1-\delta_k)\|e_k - \alpha_k(\tau_k)\|^2 + \lambda \delta_k \right]. \tag{3.11}$$

It can be observed that the running cost reduces to λ and is therefore independent of the current α_k in the case $\delta_k = 1$. Because of the introduction of the state e_k, the event-trigger f is given by a mapping from E^k to $\{0,1\}$. Since there always exists a bijection from X^k to E^k given the variables $\delta_0, \dots, \delta_{k-1}$, this change of variables does not put any restrictions on the further analysis keeping in mind that any policy expressed in E^k can also be written as a function in X^k.

3.3 An Iterative Algorithm

What prevents a further study of the optimization problem (3.11) is the fact that the estimation bias $\alpha_k(\tau_k)$ depends on the particular policy f chosen up to time k. Therefore, methods like dynamic programming are not directly applicable to solve (3.11). In order to overcome this burden, we broaden the optimization problem (3.11) by considering the variable α_k as a new decision variable being a function of τ_k. Then, the optimization problem is given by

$$\inf_{\mathrm{f},\alpha} J \tag{3.12}$$

with

$$J(\mathrm{f},\alpha) = \mathsf{E}^{\mathrm{f}} \left[\sum_{k=0}^{N-1} (1-\delta_k)\|e_k - \alpha_k(\tau_k)\|^2 + \lambda \delta_k \right]. \tag{3.13}$$

The optimization problem (3.12) enlarges the set of possible solutions compared to optimization problem (3.11), because it omits the constraint for α given by Eq. (3.10). Because of this fact, we can conclude that any optimal solution of problem (3.12) will also be optimal for Eq. (3.11). By considering optimization problem (3.12), we are however able to specify the structure of the optimal event-trigger, which is given by the following proposition.

Proposition 3.3. *Let α be fixed. Then, for all $k \in \{0, \dots, N-1\}$ the variables e_k and τ_k are a sufficient statistics for the optimal event-trigger* f_k.

Proof. The evolution of the pair (e_k, τ_k) can be regarded as a δ_k-controlled Markov process defined by Eq. (3.4) and Eq. (3.8). The running cost of J at time k is a function of the pair (e_k, τ_k), input δ_k and noise w_k. By [Ber05], this problem can be solved by dynamic programming with (e_k, τ_k) being the state, which is a sufficient statistics of the optimal solution f_k. This completes the proof. □

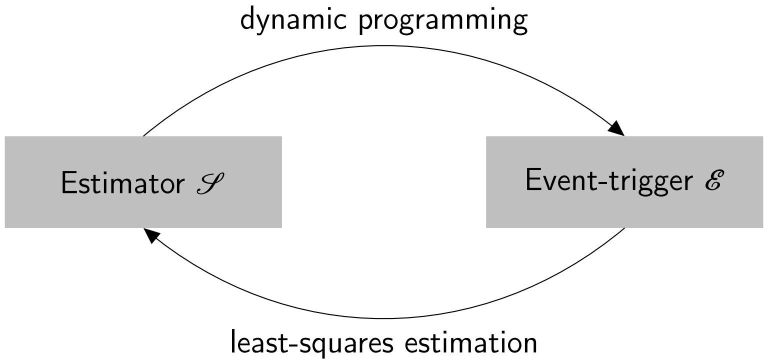


Figure 3.2: Iterative scheme to calculate event-trigger $\mathscr{E}$ and estimator $\mathscr{S}$.

Proposition 3.3 implies that the optimal event-trigger is a function of e_k and τ_k. It can be observed that for a fixed event-trigger f, the optimal map α can be calculated by Eq. (3.10). On the other hand, for any fixed map α, the optimal event-trigger f can be calculated by dynamic programming. We therefore define the running cost as

$$c_k^{\alpha_k}(e_k, \tau_k, \delta_k) = (1-\delta_k)|e_k - \alpha_k(\tau_k)|^2 + \lambda\delta_k,$$

and the Bellman operator as

$$\mathscr{T}_k^{\alpha_k} J_{k+1}(\cdot) = \min_{\delta_k \in \{0,1\}} c_k^{\alpha_k}(\cdot, \delta_k) + \mathsf{E}\left[J_{k+1}(e_{k+1}, \tau_{k+1})|\cdot, \delta_k\right].$$

The value function J_k being a function of the augmented state (e_k, τ_k) is determined by recursive application of the Bellman equation given by

$$J_k = \mathscr{T}_k^{\alpha_k^i} J_{k+1}$$

with $J_N \equiv 0$, where the argument in the minimization yields the optimal event-trigger f and we have

$$J(\mathrm{f}, \alpha) = \mathsf{E}^{\mathrm{f}}[J_0(e_0, -1)].$$

This observation motivates us to propose the following iterative procedure sketched in Fig. 3.2, which alternates between optimizing f while fixing policy α and vice versa. Algorithm 1 describes the iterative procedure. With slight abuse of notation, we declared τ_k as a second subscript instead of an argument of α_k.

As the cost J decreases or is at least kept constant in each step of the iteration, the sequence $[(\mathrm{f}^0, \alpha^0), (\mathrm{f}^1, \alpha^1), \ldots]$ produces a non-increasing succession of costs J.

3.4 Convergence Properties

In the following, we are interested in the convergence properties of the proposed iterative algorithm for scalar systems. We will therefore restrict our analysis to linear first-order processes $\mathscr{P}$ defined as

$$x_{k+1} = ax_k + w_k, \tag{3.14}$$

Algorithm 1 Iterative procedure to calculate (f, α)

Require: $\alpha^0_{k,\tau_k} \in \mathbb{R}, \quad k = 0, \ldots, N-1, \tau_k = -1, \ldots, k-1$

$i \leftarrow 0$

repeat

  $k = N, J_N \equiv 0$

  repeat

    $k \leftarrow k - 1$

    $J_k \leftarrow \mathcal{T}_k^{\alpha_k^i} J_{k+1}$

    $\mathrm{f}_k^i(e_k, \tau_k) \in \arg\min_{\delta_k \in 0,1} c_k^{\alpha_k^i}(e_k, \tau_k, \delta_k) + \mathsf{E}\left[J_{k+1}(e_{k+1}, \tau_{k+1}) | e_k, \tau_k, \delta_k\right]$

  until $k = 0$

  $\alpha^{i+1}_{k,\tau_k} \leftarrow \mathsf{E}^{\mathrm{f}^i}\left[\sum_{l=\tau_k}^{k-1} a^{k-l-1} w_l | \delta^k_{\tau_k+1} = 0\right]$

  $i \leftarrow i + 1$

until convergence

where $a \in \mathbb{R} - \{0\}$. The system noise w_k takes values in $\mathbb{R}$ and is an i.i.d. random variable described by the probability density function ϕ_w, which is zero-mean and has finite variance. The initial state, x_0 is statistically independent of w_k and is described by density function ϕ_{x_0}, which has a finite mean $\mathsf{E}[x_0]$ and a finite variance. As in the previous sections, we will study the transformed system with state variable e_k defined in Eq. (3.7). Additionally, it is assumed that the density functions are symmetric around their means, i.e.,

$$\phi_w(w) = \phi_w(-w),$$
$$\phi_{e_0}(e) = \phi_{e_0}(-e)$$

for all $w, e \in \mathbb{R}$. Rather than regarding α as a function of k and τ_k, we will interpret α as a vector in $\mathbb{R}^{\frac{1}{2}N(N+1)}$ by reindexing its entries appropriately.

The following definition of person-by-person optimality is an adaptation of the definition for person-by-person maximization in [Rad62] to optimization problem (3.12).

Definition 3.1 (Person-by-person optimality). A solution (α^*, f^*) is called *person-by-person optimal*, if

$$J(\alpha^*, \mathrm{f}^*) \leq J(\alpha, \mathrm{f}^*),$$
$$J(\alpha^*, \mathrm{f}^*) \leq J(\alpha^*, \mathrm{f})$$

for all $\alpha \in \mathbb{R}^{\frac{1}{2}N(N+1)}$ and all admissible policies f.

The above definition means to say that the cost of a person-by-person optimal solution can not be decreased by either changing the estimation bias α or by changing the event-trigger f while fixing the other. Person-by-person optimality is a necessary condition for optimality, since it would be otherwise possible to improve the solution by the iterative algorithm defined by Algorithm 1. It can therefore be concluded that every fixpoint (f^*, α^*) in the iterative algorithm is a person-by-person optimal solution of optimization problem (3.12).

The following proposition shows that $\alpha^* = 0$ together with its resulting optimal event-trigger denoted by f^* is a person-by-person optimal solution.

Proposition 3.4. *Let the initial state e_0 and the noise process $\{w_k\}$ have symmetric distributions. Then $\alpha^* = 0$ is a fixpoint of the Algorithm 1. The policy of the event-trigger f^* that corresponds to α^* is an even mapping of e_k and is independent of τ_k for every $k = 0, \ldots, N-1$.*

Proof. Let us choose the map α^0 to be 0 for all k and all τ in the initialization of Algorithm 1. The cost function J reduces then to

$$J(\mathrm{f}, \alpha^0) = \mathsf{E}^{\mathrm{f}}\left[\sum_{k=0}^{N-1}(1-\delta_k)|e_k|^2 + \lambda\delta_k\right]$$

where e_k evolves by the recursion (3.8). Therefore, the resulting optimal f_k^0 is only a function of e_k for all $k = 0, \ldots, N-1$. In the following, we first show that the application of the Bellman operator $\mathcal{T}_k^0$ preserves symmetry of the value function J_{k+1} for any k. Given an even value function J_{k+1}, the conditional expectation $\mathsf{E}\left[J_{k+1}(e_{k+1}, \tau_{k+1})|\cdot, \delta_k\right]$ preserves symmetry for both $\delta_k = 0$ and $\delta_k = 1$. Adding the cost $c_k^0(\cdot, \delta_k)$ also preserves symmetry, because the sum of two even functions is again even. Taking the pointwise minimum of two even functions yields an even function. Therefore, an even function remains even after application of the Bellman operator. As $J_N \equiv 0$ is an even function, it follows by induction that every value function J_k is even for $k \in \{0, \ldots, N-1\}$. This implies that the f_k^0 resulting in the first iteration step from Algorithm 1 is an even mapping of e_k, if $\alpha^0 = 0$.
Next, we calculate α^1 assuming f_k^0 being an even function of e_k for $k \in \{0, \ldots, N-1\}$. Let $\phi_{e_k|\tau}$ be defined as the density function of the conditional probability distribution of e_k given τ_k and $\delta_k = 0$, when using event-trigger f^0. The definition of $\phi_{e_k|\tau}$ yields the following calculation of $\alpha_{k,\tau}^1$.

$$\alpha_{k,\tau}^1 = \int_{e\in\mathbb{R}} e \cdot \phi_{e_k|\tau}(e) de.$$

For $k = 0$, $\phi_{e_k|\tau}$ is determined by truncating the density function ϕ_{e_0} of the initial state e_0 at all (e, τ), where f_0^1 takes a value of 1 and by normalizing the resulting function, i.e.

$$\phi_{e_0|\tau}(e) = \frac{\phi_{e_0}(e) \cdot (1 - \mathrm{f}_0^0(e, \tau))}{\int_{e\in\mathbb{R}} \phi_{e_0}(e) \cdot (1 - \mathrm{f}_0^0(e, \tau)) de}. \tag{3.15}$$

Since ϕ_{e_0} and f_0^0 are even functions, we conclude that $\phi_{e_0|\tau}$ is even and therefore we have $\alpha_{0,-1}^1 = 0$. Along the same lines, we can show that $\phi_{e_k|k-1}$ is even and $\alpha_{k,k-1}^1 = 0$ for $k \in \{1, \ldots, N-1\}$ by replacing ϕ_{e_0} with ϕ_w in Eq. (3.15). For a constant τ, the conditional density function $\phi_{e_k|\tau}$ evolves by the recursion

$$\phi_{e_{k+1}|\tau}(e) = \frac{(\frac{1}{|a|}\phi_{e_k|\tau}(\frac{(\cdot)}{a}) * \phi_w)(e) \cdot (1 - \mathrm{f}_k^0(e, \tau))}{\int_{e\in\mathbb{R}}(\frac{1}{|a|}\phi_{e_k|\tau}(\frac{(\cdot)}{a}) * \phi_w)(e) \cdot (1 - \mathrm{f}_k^0(e, \tau)) de}.$$

It can be observed that this recursion preserves symmetry of the conditional density function $\phi_{e_k|\tau}$, as f_k^0 is an even function. Therefore, we have shown that $\alpha^* = 0$ is a fixpoint of Algorithm 1, which completes the proof. □

3.4.1 Symmetric Unimodal Distributions

A natural question arising from Proposition 3.4 is whether the person-by-person optimal solution $\alpha^* = 0$ with its optimal event-trigger f^* is also the globally optimal solution. We partly answer this question in the following by assuming that the distributions are unimodal.

Definition 3.2 (Unimodality). A distribution in $\mathbb{R}$ is called *unimodal*, if there exists $w_0 \in \mathbb{R}$ such that the density function of the distribution $\phi(w)$ is a non-increasing function for $w \geq w_0$ and a non-decreasing function for $w \leq w_0$.

This additional assumptions enables us to state the following useful convergence property of Algorithm 1.

Theorem 3.1. *Let the initial state e_0 and the noise process $\{w_k\}$ have symmetric and unimodal distributions. Then, $\alpha^* = 0$ is a globally asymptotically stable fixpoint of Algorithm 1.*

Proof. By considering the evolution of α^i as a dynamical system evolving over variable i, the asymptotic behavior of the iterative Algorithm can be analyzed by means of Lyapunov stability theory and it is shown that $\alpha^* = 0$ is a globally asymptotically stable equilibrium point. The details of the proof can be found in Section 3.7. □

As the iterative Algorithm 1 produces a sequence of pairs (f^i, α^i) whose costs are non-increasing with increasing i, we conclude that 0 is the optimal choice for α, when noise distributions are symmetric and unimodal according to Theorem 3.1. The optimal state estimator of x_k is then given by the linear predictor in Eq. (3.6) and is therefore independent of the choice of the event-trigger f. The distribution of the initial state x_0 must be also symmetric and unimodal, but its mean $\mathsf{E}[x_0]$ can be chosen arbitrarily. Hence, the symmetry axis of the distribution of x_0 need not to be at zero. In order to determine the optimal f^*, dynamic programming must only be applied once with $\alpha = 0$. Therefore, the joint design approach in the case of symmetric densities can be considered as an independent design of event-trigger and estimator.
This result is in accordance with [LM11] and constitutes an alternative way by analyzing the asymptotic behavior of Algorithm 1 to prove that symmetric event-triggering laws are optimal in the presence of symmetric unimodal distributions. Moreover, the iterative algorithm may be applied to arbitrary distributions that are absolute continuous. Although $\alpha = 0$ is a fix point of the Algorithm 1 by Proposition 3.4 assuming symmetric density functions, the next section shows that an independent approach given by $\alpha = 0$ can be outperformed by Algorithm 1 by almost 50%. Hence, we can conclude that symmetry of the densities is not sufficient to show that the independent design is optimal. Therefore, additional assumptions are required to show that the independent design is optimal. In the case of Theorem 3.1 such requirement is given by the unimodality assumption of the density functions.

It is an open question whether Theorem 3.1 is also valid for higher-order systems. Neither the use of majorization theory in [LM11] nor our iterative approach allow a direct extension to the case of multi-dimensional systems. Therefore, we will henceforth assume that Theorem 3.1 is also true for higher-order systems, which implies that the linear predictor defined in Eq. (3.6) is the optimal state estimator. This will give us the opportunity to derive further results for optimal event-triggered control systems in the follow-up chapters without restricting our attention to first-order systems.

3.4.2 Symmetric Bimodal Distributions

In the following, we study the solutions obtained from the iterative methods for the case of bimodal noise distributions. This subsection intends to outline the benefits of the proposed iterative algorithm by numerical examples. It will demonstrate how the event-trigger and the estimator can benefit from signaling through the absence of triggering.

Besides, it validates the obtained results for unimodal noise distributions. We compare the solutions obtained from the iterative algorithm with the optimal symmetric event-trigger having a linear predictor, i.e. assuming $\alpha = 0$. Suppose the process defined by Eq. (3.14) with $a = 1$, a communication penalty $\lambda = 0.5$ and the distribution of the initial state and the system noise are identical and defined by the density functions ϕ_{e_0} and ϕ_w as

$$\phi_{e_0} = (\mu, \sigma) = \phi_w(\mu, \sigma) = \frac{1}{2}\phi_{\mathcal{N}}(\mu, \sigma) + \frac{1}{2}\phi_{\mathcal{N}}(-\mu, \sigma)$$

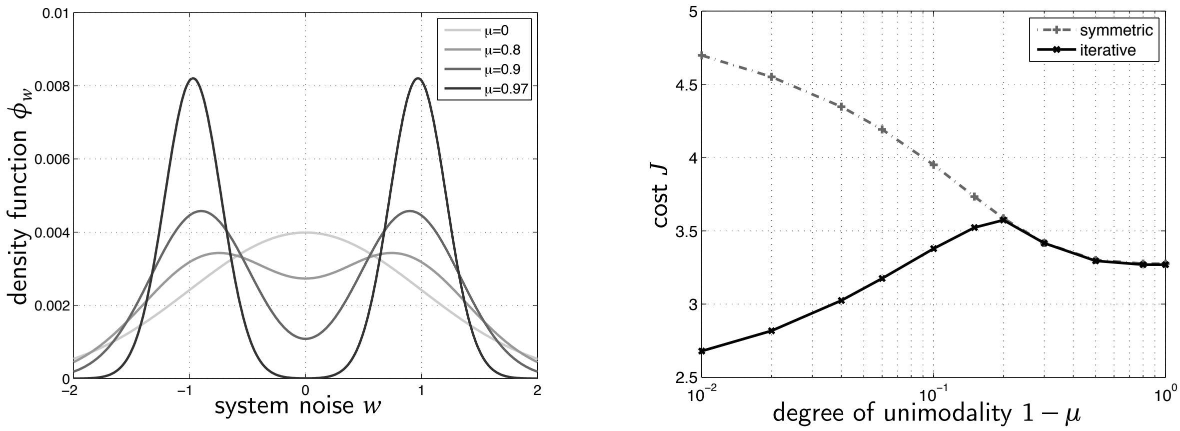
with

$$\phi_{\mathcal{N}}(\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

In the special case of $\mu = 0$, we retrieve the normal distribution. In order to facilitate comparability between different distributions, we choose $\mu \in [0, 1)$ and set

$$\sigma = \sqrt{1 - \mu^2}$$

that yields an identical variance of 1 for all $\mu \in [0, 1)$. In the limit $\mu \to 1$, the noise process degrades to a Bernoulli process taking discrete values $\{-1, 1\}$ with probability $\frac{1}{2}$. Various density functions for different μ are sketched in Fig. 3.3a.



(a) Density functions

(b) Performance analysis

Figure 3.3: (a) Various bimodal/unimodal density functions with zero-mean and identical variance of 1 composed of two Gaussian kernels shifted by $\pm\mu$. (b) Performance comparison for a horizon $N = 10$. The degree of unimodality $1 - \mu$ (1 for zero-mean Gaussian and 0 for Bernoulli process with discrete parameters in {-1,1}) is drawn on a logarithmic scale.

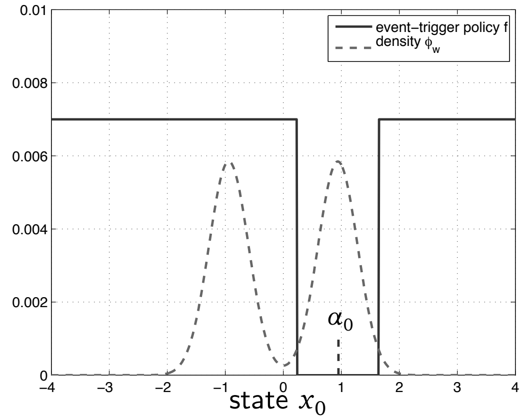


Figure 3.4: Event-trigger policy f (scaled by 0.007) resulting from the iterative Algorithm 1 with initial noise distribution ϕ_w, $\mu = 0.95$, horizon $N = 1$ and initial choice $\alpha_0^0 = 0.1$. The algorithm converges to $\alpha_0 = 0.95$ and an asymmetric event-trigger $\mathrm{f}(x_0) = \mathbb{1}_{\{[0.25,0.65]\}^c}(x_0)$.

We observe that for $\mu < 0.8$ the peaks of the bimodal density function are less distinctive. Therefore, we can not expect that large gains of the iterative procedure can be attained compared with the optimal symmetric solution for $\mu < 0.8$. A performance comparison of the iterative procedure and the optimal symmetric event-trigger is drawn in Fig. 3.3b for a horizon $N = 10$ and various μ. The initialization for the iterative procedure is chosen to be $\alpha^0 \equiv 0.1$. As expected the costs are almost identical for $\mu \in [0, 0.8]$. This also validates Theorem 3.1, since ϕ_w is unimodal for sufficient small choice of μ. For $\mu > 0.8$ a rapid performance improvement can be observed. In the limit $\mu \to 1$, the costs are reduced by a factor of 45% by the iterative procedure compared with the optimal symmetric event-trigger. This may seem surprising, because the cost function as well as the noise distribution are all even functions. Fig. 3.4 gives an illustrative explanation of such significant performance improvement for $N = 1$ and $\mu = 0.95$. With an initial value $\alpha_0^0 = 0.1$, the iterative algorithm converges to $\alpha_0 = 0.95$ and an asymmetric event-trigger policy $\mathrm{f}(x_0) = \mathbb{1}_{\{[0.25,1.65]\}^c}(x_0)$, whereas the optimal symmetric event-trigger is given by $\mathrm{f}(x_0) = \mathbb{1}_{\{[-0.7,0.7]\}^c}(x_0)$. The event-trigger and estimator resulting from the iterative procedure have therefore an implicit agreement, if no state update is sent over the resource-constrained channel. In that case, no transmission indicates the estimator that the state x_0 is situated at the right peak resulting in the estimate α_0. In contrast to that, the linear predictor defined in Eq. (3.6), which is optimal for the symmetric event-trigger, is independent to the choice of the threshold of the symmetric event-trigger and the noise-distribution.

3.5 Summary

By considering the joint optimal design of state estimator and event-trigger as a two-player problem, we were able to develop an efficient iterative algorithm, which alternates between optimizing the estimator while fixing the event-trigger and vice versa. The iterative method shows special properties in the case of unimodal and symmetric distributions of the uncertainty. In this situation it is shown that the optimal event-triggered estimator can be obtained by a separate design and is given by a linear predictor and a symmetric threshold policy. This result is along previous results and offers an alternative line of proof for showing that such separate design is optimal in case of symmetric unimodal distributions.

In the case of symmetric and bimodal distributions, the iterative procedure offers a systematic method, which leads to asymmetric event-triggers and biased estimators that outperform symmetric threshold policies.

Similar properties of the iterative method are likely to hold as well in the case of multi-dimensional systems, but a conclusive derivation for higher-order systems is still an open issue.

3.6 Bibliographical Notes

The contributions presented in this chapter are based on the work in [MH12b; MH12d].

The implication of Theorem 3.1, that the estimation bias is zero implying that symmetric event-triggering laws are optimal in the presence of symmetric unimodal distributions, has also been proven in [LM11]. The authors in [LM11] make use of majorization theory and the Riesz rearrangement inequality [HLP52] in order to arrive at this result. In fact, the proof follows a similar guideline used for a closely related problem that studies the joint optimization of paging and registration policies in mobile networks [HMY08]. The consideration of the asymptotic behavior of Algorithm 1 constitutes an alternative way to prove that symmetric event-triggering laws are optimal for first-order systems. Besides the work in [HMY08], iterative methods for the solution of team decision problems, in which one policy is optimized while the others are fixed, has also been applied for the study of optimal solutions of the Witsenhausen's counter-example [KGOS11] and for the joint design of source-channel-relay mappings [KS10] and is also the basic idea of the Lloyd–Max algorithm for vector quantization [GG92].

3.7 Proof of Theorem 3.1

Proof. First, we define the following time-variant transformations of e_k and α_{k,τ_k} by

$$\begin{aligned} y_k &= \frac{1}{a^k} e_k, \quad k = 0, \dots, N-1, \\ \beta_{k,\tau_k} &= \frac{1}{a^k} \alpha_{k,\tau_k}, \quad k = 0, \dots, N-1, \\ &\qquad\qquad \tau_k = -1, \dots, k-1. \end{aligned}$$

By this transformation, the running cost and the Bellman operator are defined by

$$\begin{aligned} \hat{c}_k^{\beta_k}(y_k, \tau_k, \delta_k) &= (1-\delta_k) a^{2k} |y_k - \beta_{k,\tau_k}|^2 + \lambda \delta_k, \\ \hat{\mathcal{T}}_k^{\beta_k} \hat{J}_{k+1}(\cdot) &= \min_{\delta_k \in \{0,1\}} \hat{c}_k^{\beta_k}(\cdot, \delta_k) + \mathsf{E}\left[\hat{J}_{k+1}(y_{k+1}, \tau_{k+1}) | \cdot, \delta_k\right]. \end{aligned}$$

The optimization problem (3.12) can then be restated by replacing J with $\hat{J}$ defined by

$$\hat{J}(\mathrm{f},\beta) = \mathsf{E}^{\mathrm{f}}\left[\sum_{k=0}^{N-1} \hat{c}_k^{\beta_k}(y_k,\tau_k,\delta_k)\right].$$

The event-trigger f_k is a function of y_k and τ_k, where y_k evolves by

$$y_{k+1} = (1-\delta_k)y_k + v_k, \quad y_0 = e_0.$$

with $v_k = \frac{1}{a^k}w_k$ and the evolution of τ_k is given by Eq. (3.4). It is easy to see that the distribution of v_k is again unimodal and symmetric. In the following, we adapt Algorithm 1 to the transformed system. We consider β^i as a vector in $\mathbb{R}^{\frac{1}{2}N(N+1)}$ that evolves by the procedure defined by Eq. (3.16). By this view, β^i is the state of a non-linear time-invariant discrete-time system described by

$$\begin{aligned} \mathrm{f}^i &= \arg\min_{\mathrm{f}} \hat{J}(\mathrm{f},\beta^i), \\ \beta_{k,\tau}^{i+1} &= \mathsf{E}^{\mathrm{f}^i}\left[\sum_{l=\tau}^{k-1} v_l \,|\, \delta_{\tau+1}=0,\dots,\delta_k=0\right]. \end{aligned} \tag{3.16}$$

In order to analyze the asymptotic behavior with increasing i, we introduce the following Lyapunov candidate $V(\beta^i)$ defined by

$$V(\beta^i) = \|\beta^i\|_\infty.$$

In order to show that $V(\beta^i)$ is decreasing with respect to i, we establish several auxiliary results. For notational convenience, let β_∞^i be defined as

$$\beta_\infty^i = \|\beta^i\|_\infty.$$

What we want to show first is that for every event-trigger f^i resulting from Eq. (3.16) for a given β^i, we have

$$\begin{aligned} \mathrm{f}_k^i(\beta_\infty^i+\Delta,\tau) = 0 \quad &\Longrightarrow \quad \mathrm{f}_k^i(\beta_\infty^i-\Delta,\tau) = 0, \\ \forall \Delta \geq 0, k = 0,\dots,&N-1, \tau=-1,\dots,k-1. \end{aligned} \tag{3.17}$$

The validity of above implication is shown by induction starting with $k = N-1$. We fix a β^i and apply dynamic programming to obtain f^i. Because of $\hat{J}_N \equiv 0$, the value function $\hat{J}_{N-1}$ is then given by

$$\hat{J}_{N-1}(y,\tau) = \min_{\delta\in\{0,1\}} \hat{c}_{N-1}^{\beta_{N-1}^i}(y,\tau,\delta).$$

Note that the running cost exhibits the symmetry property

$$\hat{c}_k^{\beta_k^i}(\beta_{k,\tau}^i+\Delta,\tau,\delta) = \hat{c}_k^{\beta_k^i}(\beta_{k,\tau}^i-\Delta,\tau,\delta), \qquad \forall \Delta\in\mathbb{R}, \delta\in\{0,1\}$$

with $\tau = -1,\dots,k-1$ and the monotonicity property

$$0 \leq \Delta_1 \leq \Delta_2 \implies \hat{c}_k^{\beta_k^i}(\beta_{k,\tau}^i+\Delta_1,\tau,\delta) \leq \hat{c}_k^{\beta_k^i}(\beta_{k,\tau}^i+\Delta_2,\tau,\delta)$$

for $\delta \in \{0,1\}$ and $\tau = -1,\ldots,k-1$. Both properties are preserved after taking the minimum over δ implying that they are also valid for $\hat{J}_{N-1}$. Therefore, we obtain

$$\hat{J}_{N-1}(\beta^i_\infty + \Delta, \tau) \geq \hat{J}_{N-1}(\beta^i_\infty - \Delta, \tau), \quad \forall \Delta \geq 0 \tag{3.18}$$

with $\tau = -1,\ldots,N-1$. For $\Delta \leq \beta^i_\infty - \beta^i_{k,\tau}$, inequality (3.18) is valid due to the monotonicity property of $\hat{J}_{N-1}$. In case of $\Delta > \beta^i_\infty - \beta^i_{k,\tau}$, we have

$$\begin{aligned}
&\hat{J}_{N-1}(\beta^i_\infty - \Delta, \tau)\\
&= \hat{J}_{N-1}(\beta^i_\infty - \beta^i_{k,\tau} + \beta^i_{k,\tau} - \Delta, \tau)\\
&= \hat{J}_{N-1}(\beta^i_{k,\tau} + (\beta^i_{k,\tau} - \beta^i_\infty + \Delta, \tau)\\
&\leq \hat{J}_{N-1}(\beta^i_\infty + \Delta, \tau).
\end{aligned}$$

The second equality is due to the symmetry property and the inequality is due to the monotonicity property as

$$\beta^i_{k,\tau} \leq \beta^i_{k,\tau} + (\beta^i_{k,\tau} - \beta^i_\infty + \Delta) \leq \beta^i_\infty + \Delta.$$

By knowing that the value function $\hat{J}_{N-1} = \lambda$ is constant for all pairs (y,τ), when $\delta_{N-1} = 1$, we have

$$\begin{aligned}
\mathrm{f}^i_{N-1}(\beta^i_\infty - \Delta, \tau) = 1 &\implies \lambda = \hat{J}_{N-1}(\beta^i_\infty - \Delta, \tau) \leq \hat{J}_{N-1}(\beta^i_\infty + \Delta, \tau)\\
&\implies J_{N-1}(\beta^i_\infty + \Delta, \tau) = \lambda\\
&\implies \mathrm{f}^i_{N-1}(\beta^i_\infty + \Delta, \tau) = 1.
\end{aligned}$$

Next, we show that by applying the Bellman operator will preserve the inequality given by Eq. (3.18). Assume, we have

$$\hat{J}_{k+1}(\beta^i_\infty + \Delta, \tau) \geq \hat{J}_{k+1}(\beta^i_\infty - \Delta, \tau), \quad \forall \Delta \geq 0 \tag{3.19}$$

with $\tau = -1,\ldots,k-1$. We want to show statement (3.19) implies

$$\hat{J}_{k}(\beta^i_\infty + \Delta, \tau) \geq \hat{J}_{k}(\beta^i_\infty - \Delta, \tau), \quad \forall \Delta \geq 0 \tag{3.20}$$

with $\tau = -1,\ldots,k-1$. The Bellman equation is

$$\hat{J}_k = \hat{\mathscr{T}}^{\beta^i_k}_k \hat{J}_{k+1}.$$

For all pairs (y,τ), where the argument of the minimization in $\hat{\mathscr{T}}^{\beta^i_k}_k$ yields $\delta_k = 1$, $\hat{J}_k$ is constant. This also implies that $\hat{J}_k$ takes its maximum for these pairs. In the following, we are interested in outcomes for $\hat{J}_k$ in case of $\delta_k = 0$. Along the same lines as for $\hat{J}_{N-1}$, we obtain for the running cost $\hat{c}^{\beta^i_k}_k$

$$\hat{c}^{\beta^i_k}_k(\beta^i_\infty + \Delta, \tau, \delta) \geq \hat{c}^{\beta^i_k}_k(\beta^i_\infty - \Delta, \tau, \delta), \quad \forall \Delta \in \mathbb{R}, \delta \in \{0,1\} \tag{3.21}$$

with $\tau = -1,\ldots,k-1$. We rewrite $\hat{J}_{k+1}$ to

$$\hat{J}_{k+1} = \hat{J}^{\mathrm{SYM}}_{k+1} + \hat{J}^{\mathrm{REM}}_{k+1}$$

with

$$\hat{J}^{\mathrm{SYM}}_{k+1}(y,\tau)=\begin{cases}\hat{J}_{k+1}(y,\tau), & y\le\beta^i_\infty,\\ \hat{J}_{k+1}(\beta^i_\infty+(\beta^i_\infty-y),\tau), & y>\beta^i_\infty,\end{cases} \tag{3.22}$$

$$\hat{J}^{\mathrm{REM}}_{k+1}(y,\tau)=J_{k+1}(y,\tau)-\hat{J}^{\mathrm{SYM}}_{k+1}(y,\tau). \tag{3.23}$$

By the assumption (3.19), we have

$$\hat{J}^{\mathrm{REM}}_{k+1}(y,\tau)\begin{cases}=0, & y\le\beta^i_\infty,\\ \ge 0, & y>\beta^i_\infty.\end{cases} \tag{3.24}$$

Taking the expectation of $\hat{J}_{k+1}$ given δ_k, y_k and τ_k, gives either a constant function over (y_k,τ_k) for $\delta_k=1$ or is given by convolution with the density function of v_k for $\delta_k=0$ denoted by ϕ. By assumption the density function ϕ is symmetric and unimodal. By linearity of the convolution operator, we follow

$$\mathsf{E}\left[\hat{J}_{k+1}|\cdot,\tau_k,\delta_k=1\right]=J^{\mathrm{SYM}}_{k+1}(\cdot,\tau_k)*\phi+J^{\mathrm{REM}}_{k+1}(\cdot,\tau_k)*\phi. \tag{3.25}$$

For the first term of Eq. (3.25), we observe that symmetry is preserved, i.e.,

$$(J^{\mathrm{SYM}}_{k+1}(\cdot,\tau_k)*\phi)(\beta^i_\infty+\Delta)=(J^{\mathrm{SYM}}_{k+1}(\cdot,\tau_k)*\phi)(\beta^i_\infty-\Delta) \tag{3.26}$$

for $\Delta\in\mathbb{R}$. On the other hand due to Eq. (3.24) and

$$\phi(y-(\beta^i_\infty+\Delta))\ge\phi(y-(\beta^i_\infty-\Delta)),\Delta\ge 0,y\ge\beta^i_\infty,$$

we have for any $\Delta\ge 0$

$$(J^{\mathrm{REM}}_{k+1}(\cdot,\tau_k)*\phi)(\beta^i_\infty+\Delta)\ge(J^{\mathrm{REM}}_{k+1}(\cdot,\tau_k)*\phi)(\beta^i_\infty-\Delta). \tag{3.27}$$

Summing up the terms and taking the minimum to obtain $\hat{J}_k$, we obtain statement (3.20) by using Eqs. (3.21), (3.26) and (3.27). By induction, statement (3.20) is valid for all $k=0,\dots,N-1$. Along the same lines as for $N-1$, we follow Eq. (3.17) from the statement in Eq. (3.20). Equivalently to Eq. (3.17), it can be showed that

$$\begin{aligned}&\mathrm{f}^i(-\beta^i_\infty-\Delta,\tau)=0 \quad\Longrightarrow\quad \mathrm{f}^i(-\beta^i_\infty+\Delta,\tau)=0,\\ &\forall\Delta\ge 0,k=0,\dots,N-1,\tau=-1,\dots,k-1.\end{aligned}$$

Let $\phi^i_{y_k|\tau}$ be defined as the density function of the conditional probability distribution of y_k given τ_k and $\delta_k=0$, when using event-trigger f^i. The definition of $\phi^i_{y_k|\tau}$ yields the following calculation of $\beta^{i+1}_{k,\tau}$

$$\beta^{i+1}_{k,\tau}=\int_{y\in\mathbb{R}} y\cdot\phi^i_{y_k|\tau}(y)dy.$$

By assuming an event-trigger f^i that satisfies statement (3.17), we show inductively that

$$\begin{aligned}\phi^i_{y_k|\tau}(\beta^i_\infty+\Delta)\le\phi^i_{y_k|\tau}(\beta^i_\infty-\Delta),\quad\forall\Delta\ge 0,\\ k=0,\dots,N-1,\tau=-1,\dots,k-1.\end{aligned} \tag{3.28}$$

For $k = 0$, $\phi^i_{y_k|\tau}$ is calculated by truncating the density function ϕ_{y_0} of the initial state y_0 at all (y, τ), where f^i_k takes a value of 1 and by normalizing the resulting function, i.e.

$$\phi^i_{y_0|\tau}(y) = \frac{\phi_{y_0}(y) \cdot (1 - \mathrm{f}^i_0(y, \tau))}{\int_{y\in\mathbb{R}} \phi_{y_0}(y) \cdot (1 - \mathrm{f}^i_0(y, \tau))dy}.$$

As ϕ_{y_0} is an even and unimodal function, we have

$$\phi^i_{y_0|\tau}(\beta^i_\infty + \Delta) \leq \phi^i_{y_0|\tau}(\beta^i_\infty - \Delta), \qquad \Delta \geq 0, \mathrm{f}^i_k(\beta^i_\infty + \Delta, \tau) = 0.$$

For all (y, τ) with $\mathrm{f}^i_k(\beta^i_\infty + \Delta, \tau) = 1$, we have

$$\phi^i_{y_0|\tau}(\beta^i_\infty + \Delta) = 0,$$

which trivially validates inequality (3.28). Similarly as for $k = 0$ and $\tau = -1$, we can prove the validity of Eq. (3.28) for $k \in \{1, N-1\}$ and $\tau = k-1$ by replacing the density function ϕ_{y_0} by the density function $\phi_{v_{k-1}}$ of the noise variable v_{k-1}. By assuming that inequality (3.28) is satisfied for time step k, we prove that Eq. (3.28) holds for $k + 1$ for an arbitrary $k \in \{0, \ldots, N-2\}$ and fixed $\tau \in \{-1, \ldots, k-1\}$. For a fixed τ, the density function $\phi^i_{y_k|\tau}(y)$ can be calculated by the recursion

$$\phi^i_{y_{k+1}|\tau}(y) = \frac{(\phi^i_{y_k|\tau} * \phi_{v_k})(y) \cdot (1 - \mathrm{f}^i_k(y, \tau))}{\int_{y\in\mathbb{R}} (\phi^i_{y_k|\tau} * \phi_{v_k})(y) \cdot (1 - \mathrm{f}^i_k(y, \tau))dy}. \tag{3.29}$$

As having already been observed for $\hat{J}_{k+1}$, the convolution of $\phi^i_{y_k|\tau}$ with ϕ_{v_k} preserves the inequality (3.28). With the same arguments as for $k = 0$, we follow that

$$\phi^i_{y_k|\tau}(\beta^i_\infty + \Delta) \leq \phi^i_{y_k|\tau}(\beta^i_\infty - \Delta), \qquad \Delta \geq 0$$

implies

$$\phi^i_{y_{k+1}|\tau}(\beta^i_\infty + \Delta) \leq \phi^i_{y_{k+1}|\tau}(\beta^i_\infty - \Delta), \qquad \Delta \geq 0,$$

which concludes the induction. Inequality (3.28) implies that $\beta^{i+1}_{k,\tau} \leq \beta^i_\infty$. Similarly, it can be showed that $\beta^{i+1}_{k,\tau} \geq -\beta^i_\infty$. In fact, it is straight forward to see that the inequalities are strict for all $\beta^i_\infty \neq 0$ and therefore the Lyapunov candidate V decreases with increasing i for all $\beta \neq 0$. Hence, the iterative procedure defined in Eq. (3.16) converges to 0 for any initial condition of β. By transforming β back into the initial state space system, we can conclude the proof. □

4

Event-Triggered Control under Communication Delays and Packet Dropouts

Until now, the communication network in the feedback loop has been primarily considered as a resource-constrained entity. Once, the event-trigger decides to transmit data to the controller, it is assumed that the communication network guarantees to convey the data instantaneously. However, such assertion can not be supposed in many practical communication systems, where data messages are delayed or even get lost. This motivates us to study the design of optimal event-triggered controllers in the presence of both time-delay and packet-dropouts in this chapter.

Our first concern addresses the question whether the structural properties obtained in previous chapters carry over to the case of imperfect communication over the feedback loop. As the information pattern of the decision makers needs not to be nested anymore, the optimal control law will generally not have the certainty equivalence property found in Chapter 2. This encourages us to identify different conditions for the communication model, where the nestedness property can be recovered. One of the prerequisite for the communication model is an error-free acknowledgement channel. It turns out that the certainty equivalence property can be assured, if either (i) the acknowledgement channel is delay-free or (ii) the feedback link is error-free or (iii) intervals between subsequent transmission times are restricted to be equal or greater than the round-trip time.

Inspired by these conditions, two suboptimal design approaches are developed. The notion of *suboptimality* refers to the introduction of certain assumptions that enable the calculation of optimal event-triggered controllers. The first approach assumes that the event-trigger is idle for the duration of a round-trip time after transmitting information. The second approach assumes the controller to be a certainty equivalence controller. The design of the event-triggering rule can be cast in the framework of optimal control problems with partial state information. It turns out that this problem implicitly needs to estimate potential packet dropouts. Furthermore, the optimal event-trigger is shown to have finite memory, where the number of past state values to be taken into account scales linearly with the round-trip time.

This chapter can be outlined as follows. In Section 4.1, the design problem and the communication model under consideration is introduced. Section 4.2 identifies conditions for the communication model under which the certainty equivalence property of the optimal controller can be recovered, and it develops two suboptimal design approaches. The efficacy of the suboptimal solutions for the event-triggered controller is evaluated numerically in Section 4.3.

4.1 Linear Quadratic Control over Delayed and Intermittent Feedback Loops

The design problem can be regarded as an extension of the LQ problem under communication constraints introduced in Section 2.1. Besides the resource-constraint that is taken care of in the optimization problem, the communication network $\mathcal{N}$ in the feedback loop may delay or intermit transmitted data. The complete model of the networked control system (NCS) is illustrated in Fig. 4.1. The control system consists of a process $\mathscr{P}$, an event-trigger $\mathscr{E}$ and a controller $\mathscr{C}$. The stochastic discrete-time process $\mathscr{P}$ to be controlled is described by the following time-invariant difference equation

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times d}$. The variables, x_k and u_k denote the state and the control input. They are taking values in $\mathbb{R}^n$ and $\mathbb{R}^d$, respectively. The system noise w_k takes values in $\mathbb{R}^n$ and is an i.i.d. zero-mean Gaussian distributed sequence with covariance matrix C_w. The initial state, x_0 is Gaussian with finite mean $\mathsf{E}[x_0]$ and covariance C_{x_0}.

The event-trigger output $\delta_k \in \{0,1\}$ is defined as follows.

$$\delta_k = \begin{cases} 1, & \text{update is sent,} \\ 0, & \text{nothing transmitted.} \end{cases}$$

The system model for the communication system is given by an erasure channel in the forward link. When $\delta_k = 1$, packet dropouts are modeled as a Bernoulli process $\{q_k\}_k$ defined as

$$q_k = \begin{cases} 1, & \text{update successfully transmitted,} \\ 0, & \text{packet dropout occurred,} \end{cases}$$

with packet dropout probability $\beta = \mathsf{P}[q_k = 0|\delta_k = 1]$ and q_k takes a value of 0, if $\delta_k = 0$. We assume a TCP-like communication protocol. The main feature of TCP-like communication protocols is that a binary acknowledgement is sent over the reverse link to the event-trigger, whenever a packet has been transmitted successfully. It is assumed that the reverse link is error-free. Most point-to-point protocols for wired connections fulfill this assumption. For example, the CAN-Bus protocol achieves such behavior by letting each transmitting node compare its priority with the other nodes that want to access the bus. Forward and reverse link delay packets by the duration of T_1 and T_2, respectively. Both, $T_1 \geq 0$ and $T_2 \geq 1$ are assumed to be known apriori. If only upper bounds on these delays are known, a buffering approach can be used to obtain constant time-delays equal to the bounds and the subsequent analysis can still be carried out.

Let $(\Omega, \mathscr{F}, \mathsf{P})$ denote the probability space generated by the random variables x_0, W^{N-1} and Q^{N-1} over the horizon N. These variables are the primitive random variables of our system under consideration. System parameters and statistics are known to the event-trigger and controller. Besides having the information $Q^{k-T_1-T_2}$ from the acknowledgement channel, the event-trigger $\mathscr{E}$ situated at the sensor side has access to the complete observation history X^k and decides, whether the controller $\mathscr{C}$ should receive an update.

If the event-trigger decides to update the controller, it transmits the current state over the erasure channel with delay T_1 to the controller. The received signal at the controller can be defined as

$$z_{k+T_1} = \begin{cases} x_k, & \delta_k = 1 \wedge q_k = 1 \\ \varnothing, & \text{otherwise} \end{cases} \tag{4.2}$$

with $z_0 = \cdots = z_{T_1-1} = \varnothing$. The admissible policies for the event-trigger and the controller at time k are defined as Borel-measurable functions of their past available data, i.e.,

$$\begin{aligned} \delta_k &= \mathrm{f}_k(X^k, Q^{k-T_1-T_2}), \\ u_k &= \gamma_k(Z^k). \end{aligned}$$

Let $\mathscr{U}$ be the set of all admissible policy pairs (f, γ) over the horizon N, where the event-triggering policies is given by $\mathrm{f} = \{f_1, f_2, \ldots, \mathrm{f}_{N-1}\}$ and control policies is given by $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_{N-1}]$. The cost function is defined as

$$J(\mathrm{f}, \gamma) = \mathsf{E}\left[x_N^{\mathrm{T}} Q_N x_N + \sum_{k=0}^{N-1} x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k + \lambda \delta_k\right], \tag{4.3}$$

where the weighting matrices $Q_n, Q \in \mathbb{R}^{n \times n}$ are positive semi-definite and $R \in \mathbb{R}^{d \times d}$ is positive definite. The positive factor λ can be regarded as the weight of penalizing information exchange between sensor and controller.

Then, the design objective is to find the pair $(\mathrm{f}, \gamma) \in \mathscr{U}$ that minimizes the cost function J

$$\inf_{(\mathrm{f}, \gamma) \in \mathscr{U}} J(\mathrm{f}, \gamma), \tag{4.4}$$

where J is defined in (4.3).

4.2 Design Approach

This section is divided into three parts. In Section 4.2.1, conditions for the communication network $\mathscr{N}$ are derived that enable a structural characterization of the optimal solution with regard to the certainty equivalence property introduced in Chapter 2. These structural properties allow an efficient calculation of the optimal event-trigger. As the conditions for the communication system are generally not satisfied, we develop two different suboptimal design procedures, a waiting strategy and a dropout estimation strategy, which are discussed in Section 4.2.2 and 4.2.3

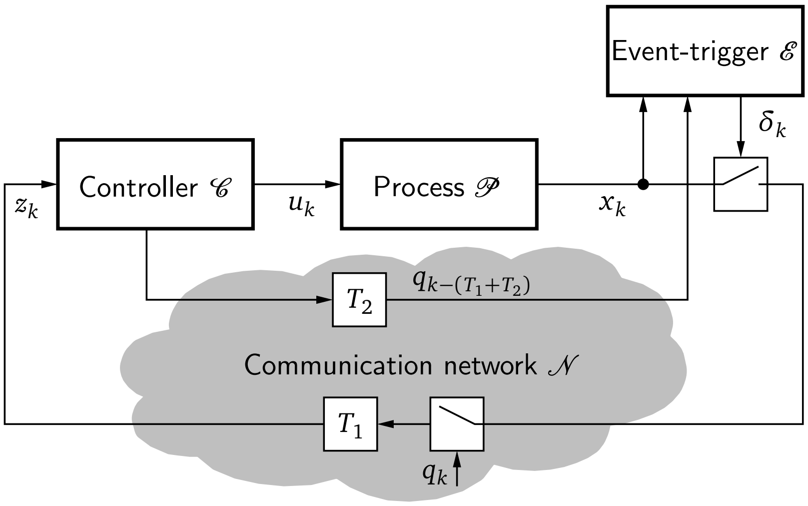


Figure 4.1: System model of the NCS over delayed and intermittent feedback loops with plant $\mathscr{P}$, event-trigger $\mathscr{E}$, controller $\mathscr{C}$ and a resource-constrained communication system $\mathscr{N}$. The forward link is an erasure channel and the reverse link carries the acknowledgement. The links have transmission delays T_1 and T_2, respectively.

4.2.1 Certainty-Equivalence-Preserving Communication Models

The main property that has allowed us to characterize the optimal solution of the LQR problem under communication constraints in Chapter 2 was the nested information pattern of the event-trigger and the controller. Because data transmission has been assumed to have no delays and packet dropouts, the information available at the controller is a subset of the information available at the event-trigger. This implies that it is possible to recover the applied control inputs and include these explicitly in the decision of the event-trigger. This property has enabled us to prove that the certainty equivalence controller is optimal for the considered problem in Chapter 2.

However, the assertion of the nestedness property does generally not hold anymore for the communication model introduced in the previous section. The possibility of packet dropouts introduces another source of information that is first perceived at the controller, and becomes available at the event-trigger with delay. Therefore, the controller has additional information that is not available at the event-trigger.

In the following, we seek for a generalization of Theorem 2.1 that includes degradations of the data transmission in the feedback loop. Let us formally revise the nestedness property. The information pattern is said to be nested when the information at the controller represented by the σ-algebra $\sigma(Z^{k-1})$ generated by the information available at the controller at time $k-1$ is embedded in the σ-algebra $\sigma(X^k, Q^{k-T_1-T_2})$ generated by the information available at the event-trigger at time k. Together with the concept of dominating strategies introduced in Definition 2.1, we state the following key theorem that will enable us to identify conditions for the communication network such that the certainty equivalence controller is optimal.

Theorem 4.1. *If the information pattern is nested, i.e.*

$$\sigma(Z^k) \subset \sigma(X^k, Q^{k-T_1-T_2}),$$

then the set $\mathscr{U}^{CE} = \{(\mathrm{f}, \gamma^{CE}) | \,\mathrm{f}$ *is an admissible event-triggering policy*$\}$ *is a dominating class of policies, where* γ^{CE} *is given by*

$$u_k = \gamma^{CE}(\mathscr{I}_k^{\mathscr{C}}) = -L_k \,\mathsf{E}[x_k | Z^k] \tag{4.5}$$

with L_k *being the solution of the Riccati equation defined in (2.12).*

Proof. Suppose that we are given an admissible pair $(\mathrm{f}, \gamma) \in \mathscr{U}$. Because of the hypothesis $\sigma(Z^{k-1}) \subset \sigma(X^k, Q^{k-T_1-T_2})$, we can define a reparameterized event-triggering law $\mathrm{g} = \{\mathrm{g}_1, \mathrm{g}_2, \ldots, \mathrm{g}_{N-1}\}$ with g_k being a function of the primitive random variables $\{x_0, W^{k-1}, Q^k - T_1 - T_2\}$ such that

$$\mathrm{g}_k(x_0, W^{k-1}, Q^{k-T_1-T_2}) = \mathrm{f}_k(X^k, Q^{k-T_1-T_2}), \quad k \in \{0, \ldots, N-1\} \tag{4.6}$$

when both systems are using a control law γ. The function g always exists, since the sequence of control inputs U^{k-1} can be recovered by the information $X^k, Q^{k-T_1-T_2}$ due to the nestedness property. This implies also that x_0, W^{k-1} can be fully recovered from X^k given $Q^{k-T_1-T_2}$ and vice versa. Therefore, the event-triggering law f can always be replaced by g and vice versa because of the hypothesis of the theorem. First note that both the pair (f, γ) and (g, γ) produce identical random variables u_k and δ_k and therefore yield the same cost J.

In the following, we fix the event-trigger g and aim at finding the optimal law γ^* minimizing J. Lemma 2.1 has already addressed this issue in Chapter 2. Because of Lemma 2.1, we attain the minimum by choosing γ to be γ^{CE} defined in Equation (4.5).

Summarizing these results, we obtain

$$J(\mathrm{f}, \gamma) = J(\mathrm{g}, \gamma) \geq \inf_{\gamma} J(\mathrm{g}, \gamma) = J(\mathrm{g}, \gamma^{\mathrm{CE}}) = J(\mathrm{f}', \gamma^{\mathrm{CE}}), \tag{4.7}$$

where f' satisfies Eq. (4.6) when assuming the control law γ^{CE}. The statement in Eq. (4.7) states that for any given pair $(\mathrm{f}, \gamma) \in \mathscr{U}$, we find another pair $(\mathrm{f}', \gamma^{\mathrm{CE}})$ where $J(\mathrm{f}, \gamma) \geq J(\mathrm{f}', \gamma^{\mathrm{CE}})$. Therefore, we have shown that the set of solutions given by $(\mathrm{f}', \gamma^{\mathrm{CE}})$ is a dominating class of policies. This concludes the proof. □

Based on the above theorem, we are able to obtain conditions for the communication system, where the set of pairs $\mathscr{U}^{\mathrm{CE}}$ is a dominating class of policies. These conditions are given by the following three propositions.

Proposition 4.1. *Let* $T_1 \geq 0$ *and* $T_2 \geq 1$. *If the packet dropout probability* β *is* 0*, then* $\mathscr{U}^{CE}$ *is a dominating class of policies.*

Proof. In case of $\beta = 0$, we observe that $\{q_k\}_k$ is deterministic and does not carry any information. As the data Z^{k-1} can be reconstructed by X^k through Eq. (4.2), the information pattern is nested. By applying Theorem 4.1, the proof is complete. □

Proposition 4.2. *Let* $\beta \in [0, 1]$ *and* $T_1 \geq 0$. *If the delay* T_2 *is equal to* 1*, then* $\mathscr{U}^{CE}$ *is a dominating class of policies.*

Proof. In case of $T_2 = 1$, we are able to reconstruct Z^k by Eq. (4.2), as the event-trigger has the data Q^{k-T_2-1} and X^k are available at time k. Applying Theorem 4.1 concludes the proof. □

In order to formulate the next proposition, we define the number of unacknowledged packets in the communication system denoted by M_k as

$$M_k = \sum_{l=k-T_1+T_2}^{k-1} \delta_l$$

In TCP-like networks it is common to bound M_k by a so-called TCP window size [Tan02]. The next proposition shows that setting the TCP window size to 1 enables separation.

Proposition 4.3. *Let $\beta \in [0,1]$, $T_1 \geq 0$ and $T_2 \geq 1$. If the communication system only allows one unacknowledged packet, i.e. $M_k \leq 1$ for all $k \geq 1$, then $\mathscr{U}^{\mathrm{CE}}$ is a dominating class of policies.*

Proof. Suppose the event-trigger decides to transmit information at time k_1, i.e. $\delta_{k_1} = 1$. The subsequent sequence of δ_k is predefined as $[\delta_{k_1}, \ldots, \delta_{k_1+T_1+T_2-1}] = [0, \ldots, 0]$. Therefore, no decision are taken at the event-trigger during this period. At time step $k_1+T_1+T_2$, the event-trigger may again decide to transmit information. But as $\delta_k = 0$ for $k \in \{k_1, \ldots, k_1 + T_1 + T_2 - 1\}$, the event-trigger knows the history $Q^{k_1+T_1+T_2-1}$ and is able to reconstruct $Z^{k_1+T_1+T_2-1}$ by the information available at the event-trigger. In case no transmissions occurred prior to time k the same arguments hold, as $\delta_l = 0$ for $l \in \{0, \ldots, k-1\}$ By using Theorem 4.1, we find that $\mathscr{U}^{\mathrm{CE}}$ is a dominating class of policies. □

The results in Proposition 4.3 motivate us to propose a special class of event-triggered controllers, which is studied in the following section.

4.2.2 Waiting Strategy

The main idea of the waiting strategy is to wait for the acknowledgement before sending the next message to the controller. Setting the TCP window size to 1 enforces the event-trigger to wait for the length of a round trip time $T_1 + T_2$, before transmitting again information. The benefits of such approach are two-fold. Waiting for the acknowledgment before sending the next packet enhances predictability of the event-trigger for communication and diminishes the possibility of congestion in the communication network. Second, such restriction facilitates the solution of the optimization problem by reducing it to numerically tractable subproblems. Besides the structural property due to Proposition 4.3, it turns out that the optimal event-trigger is described by a decision function in $\mathbb{R}^n$ that can be found by value iteration.

Based on Proposition 4.3, we restrict our attention to find the optimal solution in $\mathscr{U}^{\mathrm{CE}}$ that satisfies $M_k \leq 1$ for all $k \geq 0$. Therefore, the remaining problem reduces to finding

$$\inf_{\mathrm{f}} J^{\mathscr{E}}, \quad \text{s.t. } M_k \leq 1, \tag{4.8}$$

where

$$J^{\mathscr{E}} = \mathsf{E}\left[\sum_{k=0}^{N-1}(x_k - \hat{x}_k^{\mathscr{C}})^{\mathrm{T}}\Gamma_k(x_k - \hat{x}_k^{\mathscr{C}}) + \lambda\delta_k\right] \tag{4.9}$$

with $\hat{x}_k^{\mathscr{C}} = \mathsf{E}[x_k|Z^k]$ and Γ_k as defined in Eq. (2.16). In order to embed the constraint $M_k \leq 1$ into the system evolution, we define the following variable s_k representing the state of the communication network

$$s_{k+1} = \begin{cases} T_1 + T_2 - 1, & \delta_k = 1 \wedge s_k = 0, \\ s_k - 1, & \delta_k = 0 \wedge s_k > 0, \\ 0, & \delta_k = 0 \wedge s_k = 0, \end{cases} \tag{4.10}$$

with $s_0 = 0$ and the following modified interconnection relation which differs from Eq. (4.2)

$$z_{k+T_1} = \begin{cases} x_k, & \delta_k = 1 \wedge q_k = 1 \wedge s_k = 0, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{4.11}$$

Equation (4.11) implies that even if $\delta_k = 1$, the update will be blocked, when $s_k > 1$. This reflects exactly the behavior of the waiting strategy, as choosing $\delta_k = 1$ when $s_k > 0$, will have no effect on the system evolution, although the communication penalty λ is paid. Therefore, the optimal event-triggering law always selects $\delta_k = 0$ for $s_k > 0$.

Since the least-squares estimate $\hat{x}_k^{\mathscr{C}} = \mathsf{E}[x_k|Z^k]$ depends explicitly on the choice of f as shown in Chapter 2, the optimization problem (4.8) can not be formulated in the framework of dynamic programming directly and remains hard to solve. Nevertheless, it has been shown in Theorem 3.1 of Chapter 3 that symmetric event-triggering laws are optimal in the presence of symmetric unimodal distributions. However, a rigorous proof in the general case for higher-order systems does not exist. Nevertheless, we assume in the following that this symmetry property is also present for any arbitrary higher-order system. When using symmetric event-triggering laws, it then turns out that the least-squares estimate $\hat{x}_k^{\mathscr{C}}$ is given by a linear predictor defined by

$$\hat{x}_{k+T_1}^{\mathscr{C}} = \begin{cases} A^{T_1}x_k - \sum_{m=0}^{T_1-1} A^{T_1-m-1}BL_{k+m}\hat{x}_{k+m}^{\mathscr{C}}, & \delta_k = 1 \wedge\ q_k = 1 \wedge s_k = 0, \\ (A - BL_{k+T_1-1})\hat{x}_{k+T_1-1}^{\mathscr{C}}, & \text{otherwise,} \end{cases} \tag{4.12}$$

with initial condition

$$\hat{x}_k^{\mathscr{C}} = (A - BL_{k-1})\cdots(A - BL_0)\,\mathsf{E}[x_0],\ k = 0,\ldots,T_1 - 1. \tag{4.13}$$

Given the above state estimator at the controller and the definition of s_k, the optimization problem (4.8) can be written as

$$\begin{aligned} &\inf_f \mathsf{E}\left[\sum_{k=0}^{N-1}(1 - \mathbb{1}_{\{0\}}(s_k)q_k\delta_k)e_k^{\mathrm{T}}(A^{T_1})^{\mathrm{T}}\Gamma_k A^{T_1}e_k + \lambda\delta_k\right] \\ &e_{k+1} = \big(1 - \mathbb{1}_{\{0\}}(s_k)q_k\delta_k\big)Ae_k + w_k, \quad e_0 = x_0 - \mathsf{E}[x_0], \end{aligned} \tag{4.14}$$

where the evolution of s_k is given by Eq. (4.10). The variable e_k can be considered as the estimation of a one-step ahead prediction at the controller assuming a time-delay $T_1 = 0$. It is interesting to see that this variable is crucial for the event-trigger for arbitrary time-delay T_1. Another important property that is attributed to the waiting strategy is that the signal e_k can be calculated at the event-trigger, whenever $s_k = 0$. For $s_k \neq 0$, it is easy to see that $\delta_k = 0$. Therefore, the optimization problem can be viewed as an optimal control problem with full state information $[e_k, s_k]$. Such problem can be solved numerically in the framework of dynamic programming by applying value iteration.

In summary, we have developed a numerically tractable algorithm for determining the optimal event-triggered controller. By restricting the optimal policies to a waiting strategy, the initial optimization problem with distributed information pattern reduces to the calculation of control gain L_k given by (2.12), a least-squares estimator defined by Eq. (4.12) and the solution of a dynamic program stated in Eq. (4.14).

4.2.3 Dropout Estimation Strategy

The waiting strategy discussed in the last section is certainly suboptimal, as there are circumstances, where another update should be sent, although the outstanding acknowledgement has not been received yet. For example, this would be the case, if a significant state disturbance is observed, while the event-trigger has to wait. This fact motivates us to relax the waiting strategy and allow update transmissions before an outstanding acknowledgement has been received.

In the following, we restrict our attention to policies in $\mathscr{U}^{\mathrm{CE}}$. In other words, the control law is assumed to be the certainty equivalence controller γ^{CE} given by Eq. (4.5). We assume a linear predictor as the suboptimal state estimator which is similar to (4.12) that does not take into account the event-triggering law, i.e.,

$$\hat{x}^{\mathscr{C}}_{k+T_1} = \begin{cases} A^{T_1} x_k - \sum_{m=0}^{T_1-1} A^{T_1-m-1} B L_{k+m} \hat{x}^{\mathscr{C}}_{k+m}, & \delta_k = 1 \wedge q_k = 1, \\ (A - BL_{k+T_1-1}) \hat{x}^{\mathscr{C}}_{k+T_1-1}, & \text{otherwise}, \end{cases} \tag{4.15}$$

with initial conditions given by Eq. (4.13). The above state estimator differs from (4.12) merely by its independence of s_k.

Based on the least-squares estimator (4.15), finding the optimal event-trigger is the solution of the following optimization problem

$$\begin{aligned} &\inf_f \mathsf{E}\left[\sum_{k=0}^{N-1} (1 - q_k\delta_k) e_k^{\mathrm{T}} (A^{T_1})^{\mathrm{T}} \Gamma_k A^{T_1} e_k + \lambda \delta_k\right] \\ &e_{k+1} = (1 - q_k \delta_k) A e_k + w_k, \quad e_0 = x_0 - \mathsf{E}[x_0]. \end{aligned} \tag{4.16}$$

Although the above optimization problem differs only slightly from the dynamic program for solving the optimal waiting strategy given by (4.14), there is a major burden in solving problem (4.16), as the variable e_k is generally not perfectly known at the event-trigger.

It should be noted that the cases of no packet dropouts, i.e. $\beta = 0$, or one-step delayed acknowledgement channel, i.e. $T_2 = 1$ and $T_1 = 0$, constitute special situations, where e_k can be fully recovered at the event-trigger. Due to Proposition 4.1 and 4.2, respectively, the optimal event-triggered controller (f^*, γ^*) is given by $\gamma^* = \gamma^{\mathrm{CE}}$ with a state estimator given by Eq. (4.15) and f^* is the solution of the dynamic program stated in Eq. (4.16).

Problems with partial state information can be restated as problems with perfect state information by considering the information available as the current state as described in chapter 5 in [Ber05]. As the information state $\mathscr{I}_k^{\mathscr{E}}$ increases with time, such approach is not suitable for the above infinite horizon problem. Due to this fact, we need to reduce the data in $\mathscr{I}_k^{\mathscr{E}}$ to its essential quantities, which are known as sufficient statistics. The main feature of a sufficient statistic $\mathscr{I}_k^{\mathrm{red}}$ is that the optimal policy f^* can be rewritten as

$$\mathrm{f}_k^*(\mathscr{I}_k^{\mathscr{E}}) = \tilde{\mathrm{f}}_k(\mathscr{I}_k^{\mathrm{red}}).$$

A well known sufficient statistic is given by the conditional distribution $\mathsf{P}_{e_k|\mathscr{I}_k^{\mathscr{E}}}$ for the above problem with partial state information, see also section 5.4 in [Ber05].

The subsequent paragraph is concerned with the calculation of $\mathsf{P}_{e_k|\mathscr{I}_k^{\mathscr{E}}}$. Such problem can be posed in the framework of optimal filtering of discrete-time Markov jump linear systems, which is studied in [CFM05], with unknown discrete mode q_k and observation $\mathsf{P}_{e_k|\mathscr{I}_k^{\mathscr{E}}}$. In this case it is well known that the optimal nonlinear filter can be described by a bank of Kalman filters, which requires exponentially increasing memory and computation with time [BSL93]. However, the event-trigger receives a delayed version of the discrete mode q_k through the acknowledgement channel. Based on results by [MB11], such additional information leads to nonlinear filters that require merely finite memory and their computation has a polynomial complexity that does not increase over time. This fact is stated formally in the subsequent lemma.

Lemma 4.1. *A sufficient statistics for the optimal event-trigger f^* solving the optimization problem (4.16) is given by the information*

$$\mathscr{I}_k^{red} = \{e_{k-T_1-T_2+1}, X_{k-T_1-T_2+1}^k, \delta_{k-T_1-T_2+1}^k\},$$

where $\mathscr{I}_k^{red} = \{e_0, X^k\}$ for $k < T_1 + T_2$. The information $\mathscr{I}_k^{red}$ is measurable with respect to $\{X^k, Q^{k-T_1-T_2}\}$.

Proof. The initial value of e_k is given by $x_0 - \mathsf{E}[x_0]$ and is therefore known by the event-trigger, i.e. $\sigma(\mathscr{I}_k^{\mathrm{red}}) \subset \sigma(X^k)$ for $k \leq T_1 + T_2 - 1$. On the other hand, $\mathscr{I}_k^{\mathrm{red}} = \{e_0, X^k\}$ is the complete information available at the event-trigger at time $k \leq T_1 + T_2 - 1$. This implies $\sigma(\mathscr{I}_k^{\mathrm{red}}) = \sigma(X^k)$ for $k \leq T_1 + T_2 - 1$.
For $k \geq T_1 + T_2$, the event-trigger obtains additional information $Q^{k-T_1-T_2}$ that enables us to determine $e_{k-T_1-T_2+1}$ from the difference equation of e_k defined in the optimization problem (4.16), where the noise sequence $W^{k-T_1-T_2}$ can be recovered from knowledge of $X^{k-T_1-T_2+1}$ and the control inputs $U^{k-T_1-T_2}$.
As the sequence δ^{k-1} is known at the event-trigger, the state e_k conditioned on δ^{k-1} has the Markov property , i.e. given the current state e_k and the sequence δ^{k-1}, the future evolution of e_k is statistically independent of past observations. Therefore, we obtain

$$\mathsf{P}_{e_k|\mathscr{I}_k^{\mathrm{red}}} = \mathsf{P}_{e_k|\{X^k, Q^{k-T_1-T_2}\}} \tag{4.17}$$

As $\mathsf{P}_{e_k|\{X^k, Q^{k-T_1-T_2}\}}$ is in general a sufficient statistics for problems with partial state information $\{X^k, Q^{k-T_1-T_2}\}$, Eq. (4.17) implies that $\mathscr{I}_k^{\mathrm{red}}$ is also a sufficient statistics. This completes the proof. □

The resulting event-trigger is called dropout estimation strategy as it internally estimates the unknown discrete modes that have not been acknowledged at the event-trigger in order to determine the conditional distribution of e_k. This suggests that $\mathsf{P}_{e_k|\mathscr{I}_k^{\mathscr{E}}}$ is given by a finite set of point masses increasing with T_2. For the calculation of the conditional distribution, we define the discrete mode i_k at time step k as

$$i_k = q_k \delta_k = \begin{cases} 1, & \text{successful state update,} \\ 0, & \text{no update.} \end{cases}$$

We also define the probability matrix of i_k conditioned on δ_k as

$$\mathscr{T} = \begin{bmatrix} 1 & 0 \\ 1-\beta & \beta \end{bmatrix}$$

which satisfies

$$\mathsf{P}[i_k = j|\delta_k = l] = \mathscr{T}_{l+1,j+1}, \quad j,l \in \{0,1\}.$$

In the following, we assume that $T_1 = 1$ for illustrative purposes, but the main principles for computing $\mathsf{P}_{e_k|\mathscr{I}_k^{\mathrm{red}}}$ also carry over to arbitrary forward delays T_1. We further define $\hat{x}_k^{\mathscr{C},i_{k-T_2}^{k-1}}$ as the state estimate at time k given the sequence $i_{k-T_2}^{k-1}$ with initial condition $\hat{x}_{k-T_2}^{\mathscr{C}}$. The term $\hat{x}_k^{\mathscr{C},i_{k-T_2}^{k-1}}$ can be calculated recursively by Eq. (4.15). By defining the estimator as

$$\hat{x}_{k+1}^{\mathscr{C}} = g(\hat{x}_k^{\mathscr{C}}, x_k, i_k) = \begin{cases} Ax_k - BL\hat{x}_k^{\mathscr{C}}, & i_k = 1, \\ (A-BL)\hat{x}_k^{\mathscr{C}}, & i_k = 0, \end{cases}$$

we then obtain for $T_1 = 1$

$$\hat{x}_k^{\mathscr{C},i_{k-T_2}^{k-1}} = g(\cdot, x_{k-1}, i_{k-1}) \circ \cdots \circ g(\hat{x}_{k-T_2}^{\mathscr{C}}, x_{k-T_2}, i_{k-T_2})$$

According to [MB11], we then have

$$\mathsf{P}_{i_{k-T_2}^{k-1}|\mathscr{I}_k^{\mathrm{red}}} = \frac{\alpha(i_{k-T_2}^{k-1}, \mathscr{I}_k^{\mathrm{red}})}{\sum_{i_{k-T_2}^{k-1}} \alpha(i_{k-T_2}^{k-1}, \mathscr{I}_k^{\mathrm{red}})}$$

where $\sum_{i_{k-T_2}^{k-1}}$ denotes the sum over all possible permutations of $i_{k-T_2}^{k-1}$, i.e. $\sum_{i_{k-T_2}^{k-1}} = \sum_{i_{k-T_2}=0}^{1} \cdots \sum_{i_{k-1}=0}^{1}$ and

$$\alpha(i_{k-T_2}^{k-1}, \mathscr{I}_k^{\mathrm{red}}) = \prod_{t=0}^{T_2-1} \mathscr{T}_{\delta_{k-t-1}+1, i_{k-t-1}+1} p_{k-t}(x_{k-t})$$

where $p_{k-t}(\cdot)$ is probability density function of the conditional probability distribution of x_{k-t} given $\{x_{k-t-1}, e_{k-T_2}, i_{k-T_2}^{k-t-1}\}$, which is described by the multivariate normal distribution $\mathscr{N}(Ax_{k-t-1} - BL\hat{x}_{k-t-1}^{\mathscr{C},i_{k-T_2}^{k-t-2}}, C_w)$. Finally, we obtain $\mathsf{P}_{e_k|\mathscr{I}_k^{\mathrm{red}}}$ by computing the points

$$e_k = x_k - \hat{x}_k^{\mathscr{C},i_{k-T_2}^{k-1}}$$

which have a probability $\mathsf{P}_{i_{k-T_2}^{k-1}|\mathscr{I}_k^{\mathrm{red}}}$. The number of point masses can be reduced by taking into account that $\alpha(i_{k-T_2}^{k-1}, \mathscr{I}_k^{\mathrm{red}})$ is zero whenever $i_{k-t-1} = 1$ and $\delta_{k-t-1} = 0$, which corresponds to $\mathscr{T}_{1,2} = 0$. Hence, the number of point masses depends on the number of transmission attempts during a round trip time. In case no transmissions occurred during this period, we recover the waiting strategy as there is no ambiguity for e_k with probability one.

In contrast to the initial problem stated by (4.16), the information state $\mathscr{I}_k^{\mathrm{red}}$ does not increase in time, but is bounded by the round-trip time $T_1 + T_2$. Therefore, finding the optimal event-trigger f is feasible for practical implementation in the case of infinite horizon problems with a moderate round-trip time.

4.3 Numerical Performance Comparison

In this section, we conduct several Monte Carlo simulations in order to evaluate the quality of the proposed suboptimal algorithms numerically. In the following we consider the following four triggering mechanisms:

1. the waiting strategy derived in Section 4.2.2
2. the dropout strategy derived in Section 4.2.3
3. the optimal time-triggered policy
4. the optimal event-triggered policy with instantaneous dropout information Q^{k-1}.

It should be noted that the last mechanism listed in the above list serves as a lower bound for the performance analysis.

Suppose a scalar process $\mathscr{P}$ defined by (4.1) with $A = 1$, $B = 1$ and variances $C_w = 1$, $C_{x_0} = 1$ and mean $\mathsf{E}[x_0] = 0$. The cost function J is defined by (4.3), where Q and R are selected apriori to $Q = 1$ and $R = 5$. Subsequently, we analyze the performance with respect to diverse settings of the communication network. We consider three different packet dropout probabilities $\beta \in \{0,\ 0.25,\ 0.5\}$, a forward delay $T_1 = 1$ and two different reverse delays $T_2 \in \{1, 2\}$.

The time-triggered strategy does not need an acknowledgement channel and is therefore independent of T_2. The optimal transmission timings of the time-triggered controller are calculated by the deterministic dynamic programming algorithm described in Section 2.2. It is straight forward to show that Proposition 2.1 also holds for the case of Bernoulli-distributed packet dropouts. This implies that the optimal time-triggered control law is a certainty equivalence controller.

For the calculation of the optimal event-triggered policy with instantaneous dropout information Q^{k-1}, Proposition 4.2 applies and we can therefore assume that the set of policies given by $\mathscr{U}^{\mathrm{CE}}$ is a dominating class of policies. As in the previous sections, we also assume that the estimator at the controller will be given by the linear predictor defined in Eq. (4.15). Furthermore, the variable e_k defined in Eq. (4.16) is known at the event-trigger at time k. Therefore, the solution of optimization problem (4.16) can be solved directly by dynamic

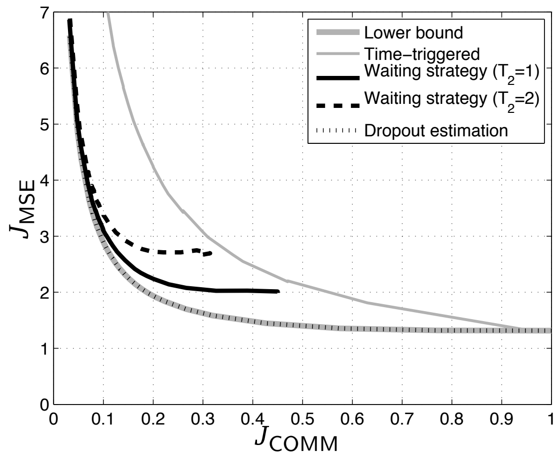


Figure 4.2: Performance comparison in the absence of packet dropouts.

programming, which will then yield a lower bound. It should be noticed that this bound is not tight for non-zero packet dropout probability β and $T_1 + T_2 \geq 2$. It can not be attained by any event-triggered controller, as it imposes that the event-trigger guesses q_k at any time step $k+1$ correctly with probability 1.

Due to these observations, we can conclude that all four considered triggering mechanisms will have the same form of controller that is given by Eq. (4.5) with identical linear gains L_k computed by Eq. (2.12). The difference with regard to performance among the approaches is reflected in $J^{\mathscr{E}}$ defined by Eq. (4.9). The objective function $J^{\mathscr{E}}$ consists of a weighted mean squared error J_{MSE} per time step given by

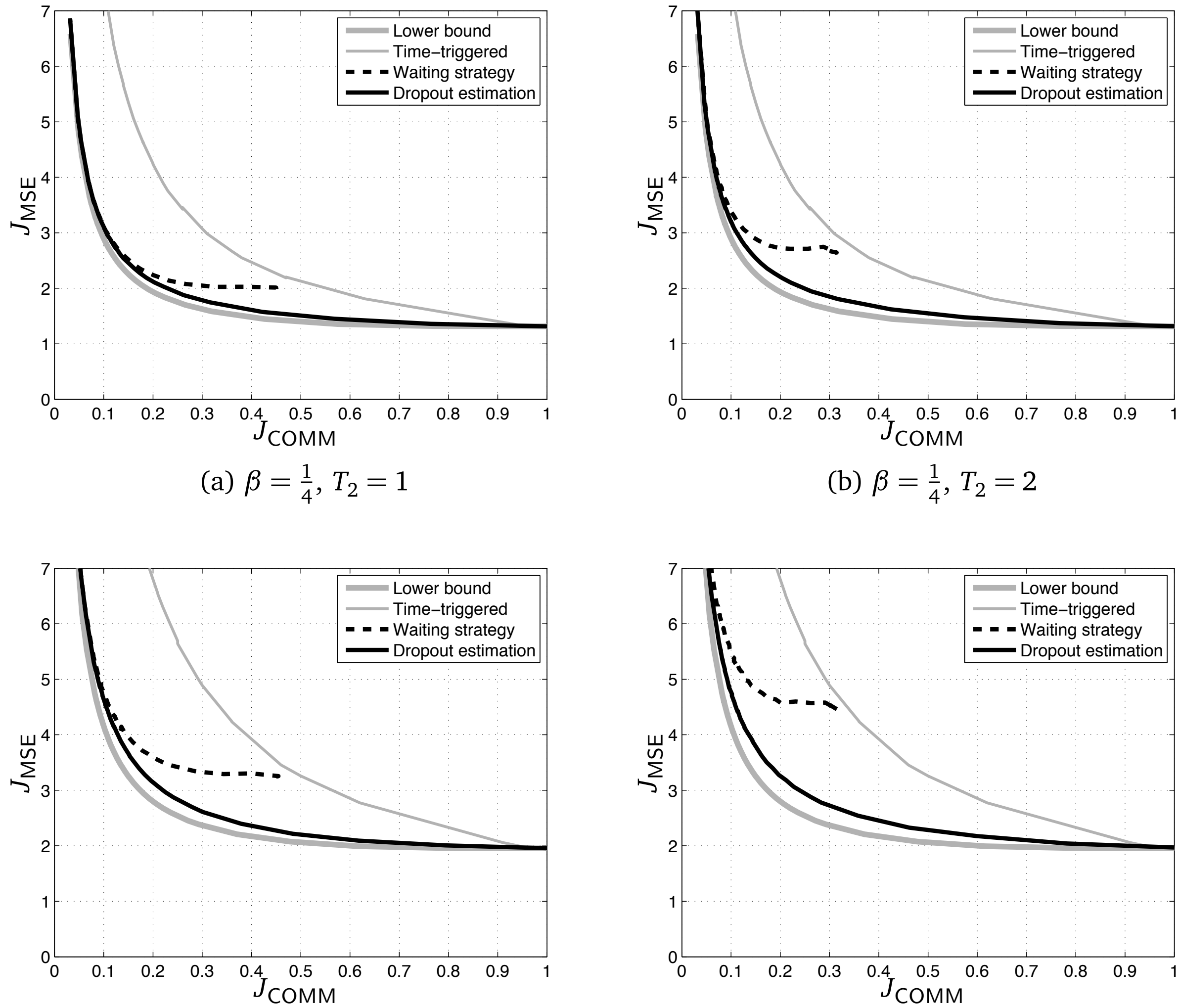
$$J_{\text{MSE}} = \frac{1}{N}\,\mathsf{E}\left[\sum_{k=0}^{N-1}(x_k - \hat{x}_k^{\mathscr{C}})^{\mathrm{T}}\Gamma_k(x_k - \hat{x}_k^{\mathscr{C}})\right]$$

and the communication penalty J_{COMM} per time step which is defined as

$$J_{\text{COMM}} = \frac{1}{N}\,\mathsf{E}\left[\sum_{k=0}^{N-1}\delta_k\right].$$

Subsequently, we are interested in the trade-off curves between J_{MSE} and J_{COMM} that are determined through varying the weight λ. The numerical results are illustrated in Fig. 4.2 for the case without packet dropouts and in Fig. 4.3 for different packet dropout probabilities and different reverse delays in the communication system. Both figures draw the achievable cost pairs $[J_{\text{MSE}}, J_{\text{COMM}}]$ for varying communication penalty λ, where cost pairs with J_{COMM} close to 0 corresponds to large λ and pairs with a transmission rate J_{COMM} close to 1 corresponds to a vanishing λ.

In all depicted scenarios in Fig. 4.2 and 4.3, the dropout estimation strategy outperforms the optimal time-triggered scheme and approaches the lower bound very closely. For the case of no packet dropouts depicted in Fig. 4.2, i.e. $\beta = 0$, the dropout estimation strategy is equal to the lower bound, as both optimal solutions coincide, because q_k is deterministic in this case. The waiting strategy also outperforms the optimal time-triggered scheme and



(a) $\beta = \frac{1}{4}$, $T_2 = 1$

(b) $\beta = \frac{1}{4}$, $T_2 = 2$

(c) $\beta = \frac{1}{2}$, $T_2 = 1$

(d) $\beta = \frac{1}{2}$, $T_2 = 2$

Figure 4.3: Performance comparison with different packet dropout probabilities β and reverse delays T_2.

deviates only slightly from the lower bound for low transmission rates. Figure 4.2 and 4.3 also reveal the natural upper bound on the communication cost J_{COMM} given by $\frac{1}{T_1+T_2}$ for the waiting strategy. Evidently, the dropout estimation strategy shows better performance than the waiting strategy at the cost of additional computations due to the filtering procedure.

4.4 Summary

In this chapter, we have attested that the optimal design of event-triggered control is a challenging problem, if having both time delays and packet dropouts in the feedback loop. This becomes evident when taking into account that these communication degradations lead to a fully distributed information structure of the decision makers. Opposed to previous chapters, such information structure prohibits the assumption of a nested information pattern, in which the information at the controller is a subset of the information available at the event-trigger. However, by using an acknowledgement channel, we were able to identify conditions for the communication system that enable a structural characterization of the optimal triggering and control policies. The structural properties allow an efficient optimal design of the event-triggered controller. The conditions may not hold in general for the communication system, but restrictions on the communication protocol can recover these conditions, as observed in the proposed waiting strategy. Despite of these restrictions, which facilitate the design procedure significantly, numerical simulations indicate that both suboptimal algorithms – the waiting strategy and the dropout estimation strategy – outperform time-triggered control systems, while marginally deviating from a lower bound on the system performance.

Furthermore, the observations made in this chapter will be taken into account in Chapter 8 when choosing the model of the communication network that is shared among the feedback loops of the multiple control systems.

4.5 Bibliographical Notes

The results in this chapter are based on [MH10c; MH13b]. The usage of a TCP-like communication protocol for data transmission in an unreliable feedback control loop has been studied in [SSF+07]. This work additionally considers an unreliable link from the controller to the process. Rather than designing triggering rules for data transmission over resource-constrained feedback channels, the focus is on the compensation of packet dropouts in this link and the analysis of UDP-like communication protocols for NCSs.

The work in [XH04] studies the design optimal event-triggered estimation with communication constraints and time delays. Similar to the optimization problem formulated in Eq. (4.16), it also shows that the event-triggering rule is a function of the one-step ahead estimation error irrespective of the forward delay T_1.

Finally, the estimation of the dropouts studied in Section 4.2.3 is related to optimal state estimation for Markovian jump linear systems with delayed mode observations studied in [MB11].

5

Structural Characterization of Event-Triggered Control with Partial State Information

In previous chapters, it has been assumed that the event-trigger has complete state information. In many situations, the complete state can however not be accessed directly and only noisy measurements correlated to the state variable are available. This chapter studies the optimal design of event-triggered controllers related to the framework of linear quadratic Gaussian (LQG) control in the presence of noisy state measurements. It turns out that the structural results obtained in Chapter 2 carry over to the case of noisy state measurements. In particular, it is shown that the certainty equivalence property is still valid in this problem setup. Based on the certainty equivalence property, the least-squares estimators at the controller and at the event-trigger are characterized. By fixing the control law, it turns out that the event-trigger can not make use of the dual effect of control and the optimal estimator is given by the Kalman filter. The least-squares estimator at the controller takes the form of a biased linear predictor of the Kalman estimate resulting at the event-trigger. Similar as in Chapter 2, the estimation bias can be determined beforehand and depends on the choice of the event-triggering law. The structure of the optimal estimators allow us to state that it suffices to transmit the Kalman estimate to controller in order to maintain optimality. Based on these results, the optimal event-triggering law can be characterized as a policy depending on the discrepancy of the least-squares state estimate at the controller and at the event-trigger.

This chapter is organized as follows. The design problem posed as an extended version of the linear-quadratic-Gaussian control problem is introduced in Section 5.1. Section 5.2 gives a characterization of the optimal solutions and studies the form of the optimal control law, the optimal state estimators, and the optimal event-triggering rule. The results are summarized in Section 5.3

5.1 Linear-Quadratic-Gaussian Control under Communication Constraints

We consider the following stochastic time-invariant discrete-time system $\mathscr{P}$

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \\ y_k &= Cx_k + v_k, \end{aligned} \tag{5.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times d}$ and $C \in \mathbb{R}^{m\times n}$. The variables, x_k, u_k and y_k denote the state, the control input and measurement output and are taking values in $\mathbb{R}^n$, $\mathbb{R}^d$ and $\mathbb{R}^m$, respectively. The system noise w_k and measurement noise v_k take values in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and are i.i.d. zero-mean Gaussian distributed sequences with covariance matrices C_w and C_v, respectively. The initial state, x_0 is Gaussian with mean $\mathsf{E}[x_0]$ and covariance matrix C_{x_0}. Let $(\Omega, \mathscr{F}, \mathsf{P})$ denote the probability space generated by the primitive random variables defined by the initial state x_0 and noise sequences W^{N-1} and V^{N-1} over the horizon N. We assume that system parameters and statistics are known to both the event-trigger and the controller.

The event-trigger $\mathscr{E}$ situated at the sensor side has access to the complete observation history and decides, whether the controller $\mathscr{C}$ should receive an update over the network denoted by $\mathscr{N}$. The information send over the network is determined by the filter $\mathscr{H}$. The controller calculates inputs u_k to regulate process $\mathscr{P}$. The system model is illustrated in Fig. 5.1.

The event-trigger output $\delta_k \in \{0,1\}$ is defined as follows:

$$\delta_k = \begin{cases} 1, & \text{update is sent,} \\ 0, & \text{otherwise.} \end{cases}$$

The received data, z_k, at the controller is defined as

$$z_k = \begin{cases} H_k \tilde{Y}^k, & \delta_k = 1 \\ \emptyset, & \delta_k = 0 \end{cases} \tag{5.2}$$

with H_k being a linear mapping with respect to the observation history Y^k defining the linear filter $\mathscr{H}$. The linear filter $\mathscr{H}$ can take many different forms. It can for example output only the current measurement y_k, or it can also output the complete observation history Y^k. It should also be noted that filter $\mathscr{H}$ can also be the Kalman filter, as this filter is linear, where we assume that the influence of the applied control inputs u_k have been compensated for notational convenience. Let

$$\tau_k = \max\{l \mid \delta_l = 1, l < k\}$$

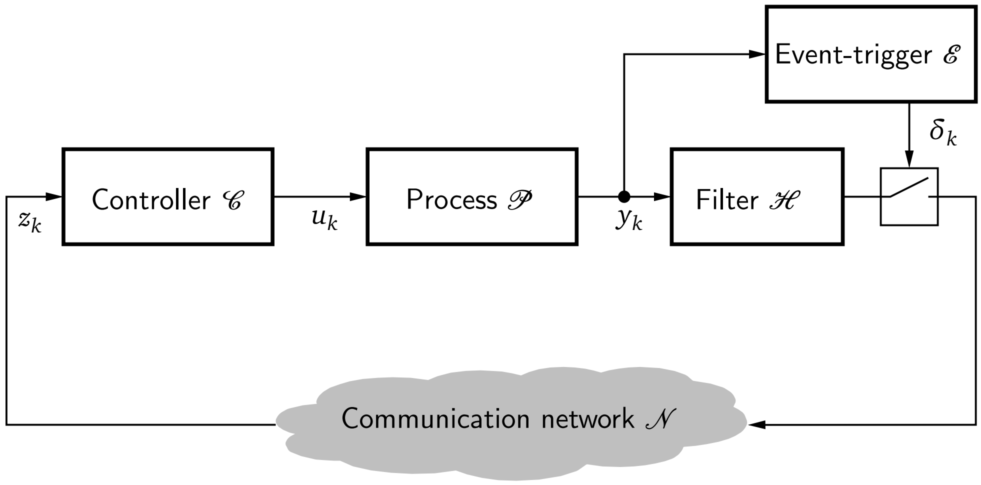


Figure 5.1: System model of the NCS under noisy state measurements with plant $\mathscr{P}$, filter $\mathscr{H}$, event-trigger $\mathscr{E}$, controller $\mathscr{C}$ and communication network $\mathscr{N}$.

be the last time step, where an update has been transmitted. In case no transmission has occurred, we define $\tau_k = -1$. The admissible policies for the event-trigger and the controller at time k are defined as Borel-measurable functions of their past available data

$$\delta_k = \mathrm{f}_k(Y^k),$$
$$u_k = \gamma_k(Z^k).$$

The design objective is to find admissible event-triggering policies f and control policies γ that minimize the finite-horizon criterion

$$J(\mathrm{f}, \gamma) = \mathsf{E}\left[x_N^{\mathrm{T}} Q_N x_N + \sum_{k=0}^{N-1} x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k + \lambda \delta_k \right]. \tag{5.3}$$

The weighting matrices Q, Q_N are positive definite and R is positive semi-definite. The positive factor λ can be regarded as the weight of penalizing information exchange between sensor and controller.

5.2 Structural Characterization

This section is divided into three subsections. First, we investigate the form of the optimal controller in Section 5.2.1. This urges us to characterize the least-squares state estimators at the event-trigger and controller, which is studied in Section 5.2.2. It is also found that the least-squares estimate at the event-trigger contains all important information to be transmitted to the controller. This fact enables a characterization of the optimal filter $\mathscr{H}$ at the sensor side. Finally, the obtained results allow us to specify the form of optimal event-triggers in Section 5.2.3.

5.2.1 Certainty Equivalence Property

We will heavily rely on the key fact that the σ-algebra of Z^{k-1} is a subset of the σ-algebra of Y^k given any admissible mappings f and γ. Similar as in 2.3.2, this nestedness property of the information structure allows us once more to compensate the applied control inputs in order to express the event-trigger as a function of the virtual measurements that are produced for the corresponding unforced system.

Let us define the measurements $\tilde{y}_k$ obtained at the event-trigger for the unforced system as

$$\tilde{y}_k = y_k - \sum_{m=0}^{k-1} CA^{k-m-1}Bu_m. \tag{5.4}$$

Given any sequence of control inputs U^{k-1}, the information $\tilde{Y}^k$ is a measurable bijective mapping of Y^k. As past control inputs can be recovered at the event-trigger due to the nestedness property, an event-trigger f that takes actions upon the information Y^k can as well be expressed by another admissible mapping denoted as g depending on $\tilde{Y}^k$ for all $k \in \{0,\ldots,N-1\}$. The next lemma gives a statement on the optimal control law for fixed g.

Lemma 5.1. *Let δ_k be given as a function*

$$\delta_k = \mathrm{g}_k(\tilde{Y}^k), \quad k \in \{0,\ldots,N-1\},$$

where $\tilde{y}_k$ is the measurement of the unforced system defined in Eq. (5.4). If the triggering law g is fixed, then the optimal control law γ^* minimizing $J(\mathrm{g},\gamma)$ defined in Eq. (5.3) is a certainty equivalence controller given by

$$u_k = \gamma_k^*(Z^k) = -L_k\,\mathsf{E}[x_k|Z^k], \quad k \in \{0,\ldots,N-1\} \tag{5.5}$$

with L_k being the solution of the Riccati equation defined in Eq. (2.12).

Proof. Due to the fact that the sequence of δ_k is independent of the sequence of control inputs for each sample path $\omega \in \Omega$ when g is fixed, the term $\mathsf{E}\left[\sum_{k=0}^{N-1} \lambda\delta_k\right]$ in Eq. (5.3) is constant and can be omitted from the optimization. Similar to the proof of Lemma 2.1, we show that the estimation error $\epsilon_k = x_k - \mathsf{E}[x_k|Z^k]$ can be expressed as a function of primitive random variables which does not depend on the policy γ being used. We fix a policy γ and consider two types of systems: The forced system model is given by Eq. (5.1), whereas the unforced system model with zero inputs is related to the definition in Eq. (5.4) and reads

$$\tilde{x}_{k+1} = A\tilde{x}_k + \tilde{w}_k, \quad \tilde{y}_k = C\tilde{x}_k + \tilde{v}_k.$$

We assume both have the same evolution of primitive random variable, i.e.

$$x_0 = \tilde{x}_0, \quad w_k = \tilde{w}_k, \quad v_k = \tilde{v}_k, \quad k = 0,\ldots,N-1.$$

The received signal at the controller is given by Eq. (5.2) for the forced system and

$$\tilde{z}_k = \begin{cases} H_k\tilde{Y}^k, & \tilde{\delta}_k = 1 \\ \emptyset, & \tilde{\delta}_k = 0 \end{cases} \tag{5.6}$$

for the unforced system. Due to linearity, we can rewrite the forced and the unforced to the following form

$$\begin{aligned} x_k &= F_k x_0 + G_k U^{k-1} + M_k W^{k-1} \\ \tilde{x}_k &= F_k x_0 + M_k W^{k-1} \end{aligned}$$

where F_k, G_k and M_k are appropriate matrices constructed from A, B and k. As U^{k-1} is measurable with respect to the information pattern Z^k, the conditional expectations are

$$\begin{aligned} \mathsf{E}[x_k|Z^k] &= F_k \, \mathsf{E}[x_0|Z^k] + G_k U^{k-1} + M_k \, \mathsf{E}[W^{k-1}|Z^k] \\ \mathsf{E}[\tilde{x}_k|Z^k] &= F_k \, \mathsf{E}[x_0|Z^k] + M_k \, \mathsf{E}[W^{k-1}|Z^k] \end{aligned}$$

Therefore, we have

$$\epsilon_k = x_k - \mathsf{E}[x_k|Z^k] = \tilde{x}_k - \mathsf{E}[\tilde{x}_k|Z^k] \tag{5.7}$$

The output vector $\tilde{Y}^k$ of the unforced system can be expressed by

$$\tilde{Y}^k = Y^k - R_k U^{k-1} = C F_k x_0 + S_k W^{k-1} + T_k V^{k-1} \tag{5.8}$$

where R_k, S_k and T_k are appropriate matrices. As the outputs of the event-trigger g are a function of primitive variables that is independent of the control law chosen, we state

$$\tilde{\delta}_k(\omega) = \delta_k(\omega) \quad \text{and} \quad \tilde{\tau}_k(\omega) = \tau_k(\omega), \quad k = 0, \ldots, N-1. \tag{5.9}$$

Hence,

$$\tilde{Z}^k = [H^{\tau_{k+1}} \tilde{Y}^{\tau_{k+1}}, \varnothing, \ldots, \varnothing] = [H^{\tau_{k+1}} Y^{\tau_{k+1}} - H^{\tau_{k+1}} R_{\tau_{k+1}} U^{\tau_{k+1}-1}, \varnothing, \ldots, \varnothing]. \tag{5.10}$$

Given the control sequence U^{k-1}, we observe that the information $\tilde{Z}^k$ can be regarded as a bijective mapping with respect to Z^k. Therefore, Eq. (5.10) implies that $\mathsf{E}[\tilde{x}_k|Z^k] = \mathsf{E}[\tilde{x}_k|\tilde{Z}^k]$ and we conclude from Eq. (5.7) that

$$\epsilon_k = \tilde{x}_k - \mathsf{E}[\tilde{x}_k|\tilde{Z}^k].$$

Accordingly, the estimation error ϵ_k is a function of the primitive random variables that is independent on the control policy used. By rewriting the cost function in Eq. (5.3) in the same way as in Eq. (2.15), we have

$$\begin{aligned} J =& \lambda \mathsf{E}[\sum_{k=0}^{N} \delta_k] + \mathsf{E}[x_0^{\mathrm{T}} P_0 x_0] + \mathsf{E}[\sum_{k=0}^{N-1} w_k^{\mathrm{T}} P_{k+1} w_k] + \mathsf{E}[\sum_{k=0}^{N-1} \epsilon_k^{\mathrm{T}} L_k^{\mathrm{T}} \Gamma_k L_k \epsilon_k] \\ &+ \mathsf{E}[\sum_{k=0}^{N-1} (u_k + L_k \mathsf{E}[x_k|Z^k])^{\mathrm{T}} \Gamma_k (u_k + L_k \mathsf{E}[x_k|Z^k])], \end{aligned} \tag{5.11}$$

where P_k is the solution of a Riccati equation, L_k is the linear control gain, both defined in Eq. (2.12), and $\Gamma_k = B^{\mathrm{T}} P_{k+1} B + R$. Due to the fact that the first four terms take constant values with respect to the admissible control laws, we observe that the controller γ^* defined in Eq. (5.5) minimizes J. This completes the proof. □

The result in Lemma 5.1 for the event-trigger g can be used to characterize the optimal control law of the original problem in terms of dominating class of strategies, see Definition 2.1. This is summarized in Theorem 5.1.

Theorem 5.1. *Let γ^* be the certainty equivalence controller defined in Eq. (5.5). The set of policies (f, γ^*) with* f *being an admissible event-triggering rule is a dominating class of policies for minimizing the cost function J defined by Eq. (5.3).*

Proof. Given a feasible pair (f, γ), there exists a feasible pair (g, γ) with g_k being a function of primitive variables which outputs the identical random variable δ_k as the event-triggering rule f_k for all $k \in \{0, \dots, N-1\}$. The existence of g is guaranteed because of the nestedness property of the information pattern. This implies that for (f, γ) and (g, γ), we also have identical random variables u_k for $k \in \{0, \dots, N-1\}$ and therefore identical costs. In the same way for the pair (g, γ^*), we can find a triggering law f' being a function of X^k, such that both (g, γ^*) and (f', γ^*) output identical random variables u_k and δ_k for $k \in \{0, \dots, N-1\}$. Assuming γ^* to be the certainty equivalence controller defined in Eq. (5.5), we obtain the following relation due to due to Lemma 5.1.

$$J(\mathrm{f}, \gamma) = J(\mathrm{g}, \gamma) \geq \min_{\gamma} J(\mathrm{g}, \gamma) = J(\mathrm{g}, \gamma^*) = J(\mathrm{f}', \gamma^*).$$

Since (g, γ^*) is an admissible law, the pair (f', γ^*) is also an admissible policy. This concludes the proof. □

As the laws (f, γ^*) with γ^* being the certainty equivalence controller establish a dominating class of policies due to Theorem 5.1, we can restrict our analysis to such form of event-triggered controllers in the following.

5.2.2 Optimal Filter Design

It has been shown in the last section that the optimal controller is a certainty equivalence controller consisting of linear gains L_k that can be computed in advance and a state estimator $\mathsf{E}[x_k | Z^k]$. The aim of this subsection is two-fold. On the one hand, we study the form of the estimator at the controller given by $\mathsf{E}[x_k | Z^k]$. On the other hand, we aim at a compressed form of the information that needs to be transmitted to controller without omitting valuable data. This goal is pursued by an appropriate design of the filter $\mathscr{H}$ at the sensor side.

For notational convenience, we define

$$\begin{aligned}
\hat{x}^{\mathscr{E}}_{k|k} &= \mathsf{E}[x_k | \mathscr{I}^{\mathscr{E}}_k], \\
\hat{x}^{\mathscr{E}}_{k|k-1} &= \mathsf{E}[x_k | \mathscr{I}^{\mathscr{E}}_{k-1}], \\
\hat{x}^{\mathscr{C}}_{k|k} &= \mathsf{E}[x_k | Z^k]
\end{aligned}$$

with $\hat{x}^{\mathscr{E}}_{0|-1} = \mathsf{E}[x_0]$ and the error covariance matrices

$$\begin{aligned}
\Sigma^{\mathscr{E}}_{k|k} &= \mathsf{E}[(x_k - \hat{x}^{\mathscr{E}}_{k|k})(x_k - \hat{x}^{\mathscr{E}}_{k|k})^{\mathrm{T}}], \\
\Sigma^{\mathscr{E}}_{k|k-1} &= \mathsf{E}[(x_k - \hat{x}^{\mathscr{E}}_{k|k-1})(x_k - \hat{x}^{\mathscr{E}}_{k|k-1})^{\mathrm{T}}].
\end{aligned}$$

Lemma 5.2 (State estimation at the event-trigger). *The least-squares state estimator at the event-trigger is linear and can be computed recursively by the Kalman filter*

$$\hat{x}^{\mathscr{E}}_{k|k} = A\hat{x}^{\mathscr{E}}_{k-1|k-1} + Bu_{k-1} + \Sigma^{\mathscr{E}}_{k|k} C^{\mathrm{T}} C_v^{-1} (y_k - CA\hat{x}^{\mathscr{E}}_{k-1|k-1} - CBu_{k-1}) \tag{5.12}$$

independently of the scheduling law chosen.

Proof. Fix a scheduling law f. Due to the nestedness property, it can be assumed that the control inputs U^{k-1} are known at the event-trigger. Hence, we have

$$\hat{x}^{\mathscr{E}}_{k|k} = \mathsf{E}[x_k | U^{k-1}, Y^k].$$

Given the control inputs U^{k-1}, the state x_k and past observations Y^k are jointly Gaussian. This is due to the fact that the primitive random variable are Gaussian and x_k and y_k are linear functions of them. Therefore, the least-squares estimator $\mathsf{E}[x_k | U^{k-1}, Y^k]$ is linear and can be computed by the Kalman filter given by Eq. (5.12), see [Ber05]. □

As a consequence of Lemma 5.2, it is interesting to note that the event-trigger can not improve its state estimate by probing the system through its variables δ_k. Therefore, we can state that there is no dual effect of control for the event-trigger.

As it will be useful for further study, the one-step ahead prediction can also be computed recursively by

$$\hat{x}^{\mathscr{E}}_{k|k-1} = A\hat{x}^{\mathscr{E}}_{k-1|k-1} + Bu_{k-1}$$

with $\hat{x}^{\mathscr{E}}_{0|-1} = \mathsf{E}[x_0]$. The covariance matrix $\Sigma^{\mathscr{E}}_{k|k}$ appearing in Eq. (5.12) is computed in advance by

$$\begin{aligned} \Sigma^{\mathscr{E}}_{k|k} &= \Sigma^{\mathscr{E}}_{k|k-1} - \Sigma^{\mathscr{E}}_{k|k-1} C^{\mathrm{T}} (C\Sigma^{\mathscr{E}}_{k|k-1} C^{\mathrm{T}} + C_v)^{-1} C\Sigma^{\mathscr{E}}_{k|k-1} \\ \Sigma^{\mathscr{E}}_{k+1|k} &= A\Sigma^{\mathscr{E}}_{k|k} A^{\mathrm{T}} + C_w, \quad \Sigma^{\mathscr{E}}_{0|-1} = C_{x_0}. \end{aligned}$$

In the following, our analysis is devoted to both the structural properties of the least-squares estimator at the controller and the amount of valuable information that is to be transmitted in order to obtain the best least-squares estimate at the controller. This valuable information is determined in terms of sufficient statistics and is determined by the choice for the filter $\mathscr{H}$ at sensor side. The maximal amount of information that can be sent is given by the complete observation history at the event-trigger, Y^k. In fact, it suffices to send only the observations that have not been transmitted yet. In the following, it is assumed that the sequence $Y^k_{\tau_k+1}$ is transmitted to the controller whenever $\delta_k = 1$. However, we will eventually observe that this information can be condensed to the least-squares estimate at the event-trigger.

Determining the optimal state estimator at the controller $\hat{x}^{\mathscr{C}}_{k|k}$ turns out to be a more difficult task, as (i) it is dependent on the scheduling law and (ii) non-Gaussian observations are to be incorporated. This is because sending no update is still valuable information for the controller. Despite these difficulties, it is possible to give a characterization of $\hat{x}^{\mathscr{C}}_{k|k}$ stated in the following lemma.

Proposition 5.1 (State estimation at the controller). *Let the event-triggering law* f *be fixed, let γ^* defined in Eq. (5.5) be the control law and let the filter $\mathscr{H}$ output the sequence $Y^k_{\tau_k+1}$ at time k. Then, the optimal state estimator at the controller takes the recursive form*

$$\hat{x}^{\mathscr{C}}_{k|k} = \begin{cases} \hat{x}^{\mathscr{E}}_{k|k}, & \delta_k = 1, \\ (A - BL_k)\hat{x}^{\mathscr{C}}_{k-1|k-1} + \alpha_{\tau_k,k}, & \delta_k = 0, \end{cases} \tag{5.13}$$

where the bias term $\alpha_{\tau_k,k} \in \mathbb{R}^n$ depends on the event-triggering law f.

Proof. In case of $\delta_k = 1$, we have controller and event-trigger have the same information, i.e. the sigma algebra of Z^k is identical to that of Y^k. This implies that $\hat{x}^{\mathscr{C}}_{k|k} = \hat{x}^{\mathscr{E}}_{k|k}$ for $\delta_k = 1$. Let r_k denote the innovation process at the event-trigger defined as

$$r_k = y_k - C\hat{x}^{\mathscr{E}}_{k|k-1}.$$

It is well-known that the innovation is zero-mean and r_k is uncorrelated with respect to past observations Y^{k-1}, see [KSH00]. As r_k and Y^{k-1} are jointly Gaussian, this also implies that these variables are statistically independent.
Assume that $\tau_k \geq 0$ and $\delta_k = 0$. By Eq. (5.12), the least-squares estimate at the event-trigger can expressed as follows

$$\hat{x}^{\mathscr{E}}_{k|k} = A^{k-\tau_k}\hat{x}^{\mathscr{E}}_{\tau_k|\tau_k} + F_{k,\tau_k}U^{k-1}_{\tau_k} + G_{k,\tau_k}R^k_{\tau_k+1}.$$

where F_{k,τ_k} and G_{k,τ_k} are appropriate matrices. We can use the above expression of $\hat{x}^{\mathscr{E}}_{k|k}$ in order to compute $\mathsf{E}[x_k|Z^k]$ by

$$\begin{aligned} \mathsf{E}[x_k|Z^k] &= \mathsf{E}[\hat{x}^{\mathscr{E}}_{k|k}|Z^k] \\ &= \mathsf{E}[A^{k-\tau_k}\hat{x}^{\mathscr{E}}_{\tau_k|\tau_k} + F_{k,\tau_k}U^{k-1}_{\tau_k} + G_{k,\tau_k}R^k_{\tau_k+1}|Z^k] \\ &= \mathsf{E}[A^{k-\tau_k}\hat{x}^{\mathscr{E}}_{\tau_k|\tau_k}|Y^{\tau_k},\tau_k] + \mathsf{E}[F_{k,\tau_k}U^{k-1}_{\tau_k}|Y^{\tau_k},\tau_k] + \mathsf{E}[G_{k,\tau_k}R^k_{\tau_k+1}|Z^k,\tau_k] \\ &= A^{k-\tau_k}\hat{x}^{\mathscr{E}}_{\tau_k|\tau_k} + F_{k,\tau_k}U^{k-1}_{\tau_k} + \mathsf{E}[G_{k,\tau_k}R^k_{\tau_k+1}|\tau_k] \end{aligned} \tag{5.14}$$

The third equality follows from linearity of the conditional expectation and the fact that the random variables Y^{τ_k} and τ_k are measurable with respect to Z^k. The last equality is due to the fact that the estimate $\hat{x}^{\mathscr{E}}_{\tau_k|\tau_k}$ is measurable with respect to Y^{τ_k}, the fact that $U^{k-1}_{\tau_k}$ is measurable with respect to Z^k, and the fact that the innovation r_j is statistically independent of past observations Y^{j-1} with $j \in \{\tau_k + 1, \ldots, k\}$. By defining

$$\begin{aligned} \alpha_{i,i+1} &= \mathsf{E}[G_{i+1,i}r_{i+1}|\tau_{i+1} = i], \\ \alpha_{i,i+j} &= -(A - BL_k)\alpha_{i,i+j-1} + \mathsf{E}[G_{i+j,i}R^{i+j}_{i+1}|\tau_{i+j} = i], \end{aligned} \tag{5.15}$$

for $i \in \{0, \ldots, N-2\}$ and $j \in \{2, \ldots, N-i-1\}$, we can write Eq. (5.14) in the recursive form defined by Eq. (5.13). The same result can also be proven for $\tau = -1$. This completes the proof. □

As in Chapter 3, the estimation bias $\alpha_{\tau_k,k}$ can be interpreted as a correction term to enhance the state estimate at the controller, when incorporating additional information $\delta_{\tau_k} = \cdots = \delta_k = 0$ that depends on the choice of the event-triggering rule f. In that sense, Proposition 5.1 can be regarded as the extension of Proposition 3.2 to systems, in which only noisy measurements are available.

As a side result of Proposition 5.1, it can be seen from Eq. (5.13) that it suffices to transmit the state estimate $\hat{x}^{\mathscr{E}}_{k|k}$ instead of the complete sequence $Y^{k}_{\tau_k+1}$. We summarize this fact in the following proposition.

Proposition 5.2 (Optimal design of filter $\mathscr{H}$). *The least-squares estimate $\hat{x}^{\mathscr{E}}_{\tau_k|\tau_k}$ and the last transmission time τ_k are a sufficient statistics for the least-squares estimation of x_k at the controller.*

Hence, we can conclude from Proposition 5.2 that the filter $\mathscr{H}$ with linear filter gains H_k can be implemented recursively by the Kalman filter defined in Eq. (5.12). This is advantageous from a technological point of view, because it roughly states that the amount of information to be transmitted is independent of the time elapsed between two subsequent transmissions. Furthermore, only $\hat{x}^{\mathscr{E}}_{k|k}$ needs to be stored at the filter $\mathscr{H}$. In the following, we will assume that the filter $\mathscr{H}$ outputs the least-squares estimate $\hat{x}^{\mathscr{E}}_{k|k}$.

5.2.3 Structure of the Optimal Event-Trigger

Based on the preceding results with regard to the structure of the optimal controller, the state estimators, and filters, we investigate the form of the optimal event-triggering law in this subsection.

By taking the optimal control law γ^* given by Eq. (5.5) and Theorem 5.1 into account, we obtain the following optimization problem that determines the optimal event-triggering law for optimization problem (5.3), when considering Eq. (5.11).

$$\inf_{f} \mathsf{E}\Big[\sum_{k=0}^{N-1}(x_k - \hat{x}^{\mathscr{C}}_{k|k})^{\mathrm{T}}\Gamma_k(x_k - \hat{x}^{\mathscr{C}}_{k|k}) + \lambda\delta_k\Big], \tag{5.16}$$

The fact that $\hat{x}^{\mathscr{C}}_{k|k}$ is dependent on the event-triggering law f restrains us from applying the dynamic programming algorithm to problem (5.16). Proposition 5.1 states that this dependence appears within the parameters $\alpha_{\tau_k,k}$. Let us denote the cost function as

$$J^{\mathscr{E}}(\mathrm{f}) = \mathsf{E}\Big[\sum_{k=0}^{N-1}(x_k - \hat{x}^{\mathscr{C}}_{k|k})^{\mathrm{T}}\Gamma_k(x_k - \hat{x}^{\mathscr{C}}_{k|k}) + \lambda\delta_k\Big].$$

When fixing an event-triggering policy f, we are able to calculate bias terms $\alpha_{\tau_k,k}$ by Eq. (5.15). Subsequently, we proceed the other way round by fixing arbitrary bias terms $\alpha_{\tau_k,k}$ and calculating the optimal f for such configuration. We will obtain properties for the resulting event-triggering law f that are independent of the specific configuration of $\alpha_{\tau_k,k}$. Through this approach, we obtain further insights into the structure of the optimal event-triggering law. It should be noted that the optimal control policy γ^* is fully determined, when fixing parameters $\alpha_{\tau_k,k}$. This implies that the problem reduces to a classical optimal control problem with partial state information. The following theorem is the key observation for the structural characterization of the optimal event-trigger.

Theorem 5.2. *Let $\alpha_{\tau_k,k}$ be fixed for all $k, \tau_k \in \{0,\dots,N-1\}$. Then, the estimation discrepancy defined as*

$$e_k = \hat{x}^{\mathscr{E}}_{k|k} - \mathsf{E}[x_k|Z^{k-1}, \delta_k = 0] \tag{5.17}$$

and the last transmission time τ_k are a sufficient statistics for the optimal event-triggering law f_k *for all* $k \in \{0, \ldots, N-1\}$ *with regard to optimization problem (5.16).*

Proof. For notational convenience, we define

$$\|x_k - \hat{x}^{\mathscr{C}}_{k|k}\|^2_{\Gamma_k} = (x_k - \hat{x}^{\mathscr{C}}_{k|k})^{\mathrm{T}} \Gamma_k (x_k - \hat{x}^{\mathscr{C}}_{k|k}).$$

Assume $\hat{x}^{\mathscr{C}}_{k|k}$ be given by Proposition 5.1 with fixed $\alpha_{\tau_k,k}$ for all k and τ_k. Fix an arbitrary event-triggering law f and apply the tower property to the expectation in Eq. (5.16).

$$J^{\mathscr{E}}(\mathrm{f}) = \mathsf{E}\Big[\sum_{k=0}^{N-1} \mathsf{E}[\|x_k - \hat{x}^{\mathscr{C}}_{k|k}\|^2_{\Gamma_k} + \lambda\delta_k | Y^k]\Big]$$

Let us inspect the expected running cost defined by $c(Y^k) = \mathsf{E}[\|x_k - \hat{x}^{\mathscr{C}}_{k|k}\|^2_{\Gamma_k} + \lambda\delta_k | Y^k]$. By adding and subtracting $\hat{x}^{\mathscr{E}}_{k|k}$ inside the norm of

$$\begin{aligned} c(Y^k) &= \mathsf{E}\left[\|x_k - \hat{x}^{\mathscr{E}}_{k|k} + \hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k}\|^2_{\Gamma_k} + \lambda\delta_k | Y^k\right] \\ &= \mathsf{E}\left[\|x_k - \hat{x}^{\mathscr{E}}_{k|k}\|^2_{\Gamma_k} + 2(\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})^{\mathrm{T}}(x_k - \hat{x}^{\mathscr{E}}_{k|k}) + \|\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})\|^2_{\Gamma_k} + \lambda\delta_k | Y^k\right]. \end{aligned}$$

Since the estimators $x^{\mathscr{C}}_{k|k}$ and $x^{\mathscr{E}}_{k|k}$ are measurable with respect to Y^k and δ_k and due to the fact that the state x_k and the estimator $x^{\mathscr{E}}_{k|k}$ do no depend on δ_k, we have

$$\begin{aligned} \mathsf{E}\left[2(\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})^{\mathrm{T}}(x_k - \hat{x}^{\mathscr{E}}_{k|k}) | Y^k\right] &= 2(\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})^{\mathrm{T}}\, \mathsf{E}\left[x_k - \hat{x}^{\mathscr{E}}_{k|k} | Y^k\right] \\ &= 2(\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})^{\mathrm{T}}\, \mathsf{E}\left[x_k - \hat{x}^{\mathscr{E}}_{k|k} | Y^k\right] \\ &= 2(\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})^{\mathrm{T}} \left(\mathsf{E}\left[x_k | Y^k\right] - \hat{x}^{\mathscr{E}}_{k|k}\right) \\ &= 0. \end{aligned}$$

Then, we arrive at the following expression of for the expected running cost.

$$c(Y^k) = \mathsf{E}\left[\|x_k - \hat{x}^{\mathscr{E}}_{k|k}\|^2_{\Gamma_k} | Y^k\right] + \mathsf{E}\left[\|\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})\|^2_{\Gamma_k} + \lambda\delta_k | Y^k\right].$$

Substituting this expression into the cost function and applying the tower property reversely, we obtain

$$\begin{aligned} J^{\mathscr{E}}(\mathrm{f}) &= \mathsf{E}\Big[\sum_{k=0}^{N-1} \|x_k - \hat{x}^{\mathscr{E}}_{k|k}\|^2_{\Gamma_k} + \|\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})\|^2_{\Gamma_k} + \lambda\delta_k\Big] \\ &= \sum_{k=0}^{N-1} \mathsf{E}\left[\|x_k - \hat{x}^{\mathscr{E}}_{k|k}\|^2_{\Gamma_k}\right] + \mathsf{E}\Big[\sum_{k=0}^{N-1} \|\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})\|^2_{\Gamma_k} + \lambda\delta_k\Big] \\ &= \sum_{k=0}^{N-1} \mathrm{tr}[\Gamma_k \Sigma^{\mathscr{E}}_{k|k}] + \mathsf{E}\Big[\sum_{k=0}^{N-1} \|\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k})\|^2_{\Gamma_k} + \lambda\delta_k\Big]. \end{aligned}$$

where we observe that the first summation is a constant that is independent of f and can therefore be omitted from the optimization. Hence, the optimization problem defined in Eq. (5.16) can be formulated as follows

$$\inf_{\mathrm{f}} \mathsf{E}\Big[\sum_{k=0}^{N-1} \|\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k}\|^2_{\Gamma_k} + \lambda\delta_k\Big] \tag{5.18}$$

Next, we observe that the $\hat{x}^{\mathscr{E}}_{k|k} - \hat{x}^{\mathscr{C}}_{k|k} = 0$ whenever $\delta_k = 1$. Consider the estimation error e_k defined in Eq. (5.17). Because of its definition, the optimization problem can be rewritten as

$$\inf_{\mathrm{f}} \mathsf{E}\Big[\sum_{k=0}^{N-1}(1-\delta_k)\|e_k\|^2_{\Gamma_k} + \lambda\delta_k\Big] \tag{5.19}$$

Subsequently, the evolution of the error signal e_k is derived. For $\delta_k = 1$, we have

$$\begin{aligned} e_{k+1} &= \hat{x}^{\mathscr{E}}_{k+1|k+1} - \mathsf{E}[x_{k+1}|Z^k, \delta_{k+1} = 0] \\ &= A\hat{x}^{\mathscr{E}}_{k|k} + Bu_k + \Sigma^{\mathscr{E}}_{k+1|k+1}C^{\mathrm{T}}(y_{k+1} - CA\hat{x}^{\mathscr{E}}_{k|k} - CBu_k) - A\hat{x}^{\mathscr{E}}_{k|k} - Bu_k - \alpha_{\tau_k,k} \end{aligned} \tag{5.20}$$

For $\delta_k = 0$, we have

$$\begin{aligned} e_{k+1} &= \hat{x}^{\mathscr{E}}_{k+1|k+1} - \mathsf{E}[x_{k+1}|Z^k, \delta_{k+1} = 0] \\ &= A\hat{x}^{\mathscr{E}}_{k|k} + Bu_k + \Sigma^{\mathscr{E}}_{k+1|k+1}C^{\mathrm{T}}(y_{k+1} - CA\hat{x}^{\mathscr{E}}_{k|k} - CBu_k) - A\hat{x}^{\mathscr{C}}_{k|k} - Bu_k - \alpha_{\tau_k,k} \end{aligned} \tag{5.21}$$

Summarizing Eqs. (5.20) and (5.21) yields

$$e_{k+1} = (1-\delta_k)Ae_k - \alpha_{\tau_k,k} + \Sigma^{\mathscr{E}}_{k|k}C^{\mathrm{T}}(y_{k+1} - CA\hat{x}^{\mathscr{E}}_{k|k} - CBu_k) \tag{5.22}$$

The state estimate $\hat{x}^{\mathscr{E}}_{k|k}$ is obtained by Eq. (5.12) and the evolution of τ_k is given by

$$\tau_{k+1} = \begin{cases} \tau_k, & \delta_k = 0, \\ k, & \delta_k = 1. \end{cases}$$

As the innovation process $y_{k+1} - CA\hat{x}^{\mathscr{E}}_{k|k} - CBu_k$ of the Kalman filter defined in Lemma 5.2 is a white noise process [KSH00], it can be observed that the process (e_k, τ_k) is a δ_k-controlled Markov chain. Because of the Markov property and the form of the optimization problem defined in Eq. (5.19), the optimal event-trigger f^* can be solved by dynamic programming with (e_k, τ_k) being the state of the system. This concludes the proof. □

Having fixed the estimation bias $\alpha_{\tau,k}$ for all τ and k, Theorem 5.2 suggests that the optimal control problem defined by Eq. (5.16) with partial state information given the information state Y^k can be converted to an optimal control with full state information (e_k, τ_k). This has a significant impact on the numerical complexity for finding the optimal event-triggering law for a fixed estimation bias, as the complexity of the dynamic programming algorithm does not grow with horizon N, but primarily depends on the state dimension n of the process.

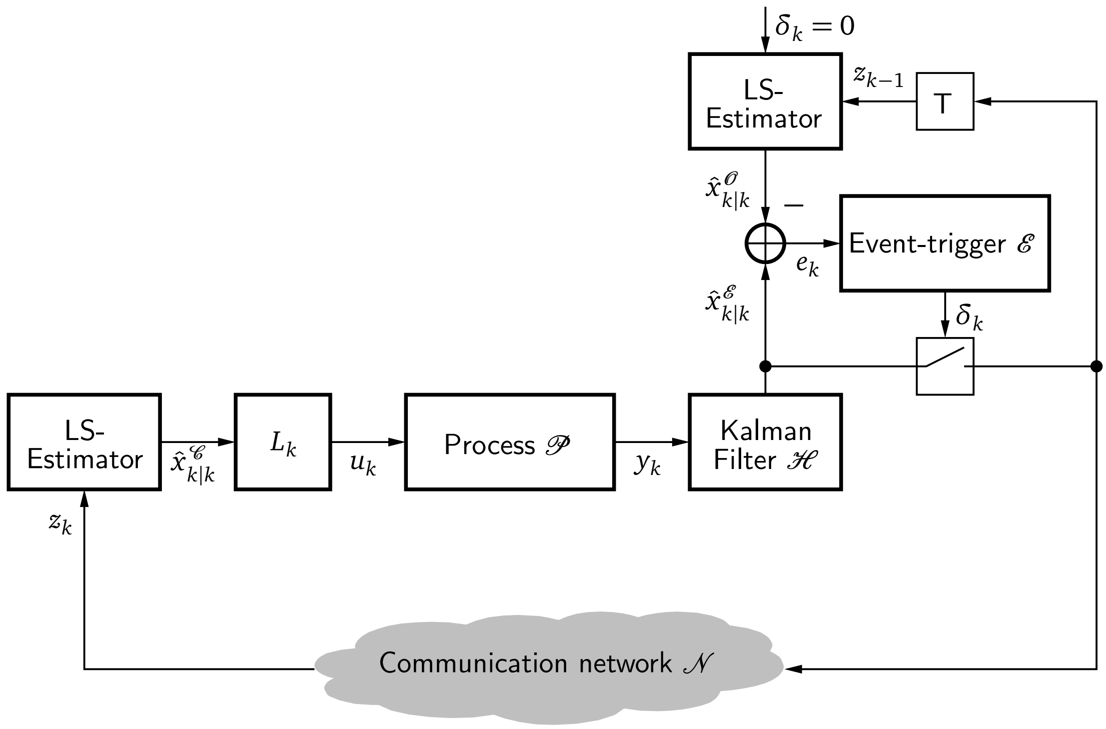


Figure 5.2: System structure of the optimal event-triggered controller for process $\mathcal{P}$. The certainty equivalence controller $\mathcal{C}$ is given by linear gains L_k and least-squares (LS) estimator generating $\hat{x}^{\mathcal{C}}_{k|k}$. Kalman filter $\mathcal{H}$ generates the least-squares state estimate $\hat{x}^{\mathcal{E}}_{k|k}$. The element denoted by T corresponds to a one-step delay element, whereas the least-squares estimate that outputs $\hat{x}^{\mathcal{O}}_{k|k} = \mathsf{E}[x_k|Z^{k-1}, \delta_k = 0]$ is a copy of the least-squares estimator at the controller.

Because of the above theorem and the form of the evolution of e_k and optimization problem (5.19), it also follows from Theorem 3.1 in the case of first-order single-input single-output systems that the optimal choice for the estimation bias $\alpha_{\tau_k,k}$ is zero. This is due to the fact that the noise distributions have density functions as Gaussian kernel functions that are symmetric and unimodal. This implies that the optimal event-trigger f_k merely depends on e_k at time k and is given by a symmetric threshold function in this case. For higher-order systems, the iterative method developed in Chapter 3 can be used to determine the optimal optimal event-trigger f and the optimal estimation bias α. Although numerical simulations suggest that the optimal estimation bias α is also identical to 0, a mathematical proof for higher-order systems remains an open issue.

5.3 Summary

This chapter has addressed the problem of characterizing optimal event-triggered controllers in the framework of LQG control under costly observations. It has turned out that many properties obtained in previous chapters for perfect state observations carry over to systems with partial state information. Figure 5.2 summarizes the obtained structural results for the optimal event-triggered control system minimizing cost function defined in Eq. (5.3). Due to Theorem 5.1, the optimal control law has the certainty equivalence property. Therefore, it consists of the linear control gains L_k, $k \in \{0, \ldots, N-1\}$ that can be computed by Eq. (2.12) and a least-squares estimator $\hat{x}^{\mathscr{C}}_{k|k}$. The least-squares estimator $\hat{x}^{\mathscr{C}}_{k|k}$ characterized by Proposition 5.1 takes the form of a biased linear predictor whose bias can be determined beforehand and depends on the event-triggering rule. Based on the structure of the least-squares estimator, Proposition 5.2 shows that the Kalman estimate $\hat{x}^{\mathscr{C}}_{k|k}$ contains all valuable information to be transmitted to the controller. Furthermore, the optimal event-trigger depends on the difference between the state estimates at controller and event-trigger defined by e_k in Eq. (5.17), which is shown in Theorem 5.2. This characterization enables the systematic design of optimal event-triggered controllers. For the case of first-order single-input single-output systems, Theorem 3.1 derived in Chapter 3 can be applied, which implies that the optimal choice for the estimation bias $\alpha_{\tau_k,k}$ is zero. For higher-order systems, the iterative method developed in Chapter 3 can be used to determine the optimal optimal event-trigger f and the optimal estimation bias α.

As already mentioned in Chapter 3, a detailed characterization of the optimal bias term α with regard to higher-order systems is however an open issue that needs to be considered in future investigations.

5.4 Bibliographical Notes

The contribution of this chapter is partly based on the work in [MH10b]. There exists merely a limited number of results that are concerned with performance-related results for event-triggered output feedback systems. The analysis methods developed in [HDT13] are capable to derive stability properties and $\mathscr{L}_2$-gain performance guarantees for output-based event-triggered controllers. The work in [LL14] studies the dynamic output feedback systems with event-triggered communication for weakly coupled sensor-to-controller and controller-to-actuator links. The obtained near-optimal event-triggered control scheme resembles the obtained optimal structure in this chapter.

6

Optimal Event-Triggered Control for Long-Run Average-Cost Problems

In this chapter, we consider the design of optimal event-triggered controllers over an infinite horizon. The focus is on the minimization of the average per-stage cost in the long-run, where the running cost is given by a quadratic control cost term and a communication penalty similarly as in Chapter 2. Though being underselective [HL89], the long-run average cost is the favored criterion in many applications, in particular in communication and queuing networks [EV89; ABFG+93]. Underselective means that there may exist two policies, which differ with respect to their performance in the first k time steps, but eventually converge to the same stationary behavior and therefore yield the same average cost. Hence, the optimization problem does not distinguish between these two policies and is therefore underselective. Among other formulations for infinite horizon costs, we also prefer the long-run average-cost criterion, since we are primarily interested in the behavior of the overall system in the stationary regime and it gives us a direct interpretation of the communication penalty as the average transmission rate. Furthermore, the underselective nature of the average cost problem will turn out to be an advantage from the point of view of adaptive control in Chapter 9. This is because the event-trigger can adjust its law according to a price whose optimal value is learned over time during execution of the scheduling mechanism.

The average-cost problem raises challenges that have not appeared in either previous chapters focussing on finite horizon problems or other optimal control problems over an infinite horizon such as problems with discounted cost criterion. Opposed to these problems, the dynamics of the underlying Markov chain takes a crucial role in the solution of the average-cost criterion. In order to guarantee that the event-triggering law can be solved by means of dynamic programming, we need to assert certain ergodicity conditions on the

Markov chain. By including a technical assumption on the event-triggering law, it is possible to guarantee this condition, and we can conclude that the average-cost problem for the optimal event-trigger design can be computed via value iteration. Furthermore, we analyze the stability properties of the proposed event-triggered controller in terms of drift criteria for Markov chains, which are defined in Appendix B. It turns out that this notion offers appropriate mathematical tools to address the issue of closed-loop stability of the event-triggered control system. Assuming a stabilizing control law in the case of continuous transmission and a uniform bound on the triggering threshold, we prove stochastic stability for ideal communication and we derive sufficient conditions to guarantee stochastic stability in the presence of packet loss. The stability analysis will give us key insights in order to analyze the asymptotic behavior of multi-loop control systems sharing a common communication network studied in subsequent chapters.

This chapter can be outlined as follows. The average-cost problem for the optimal event-triggered control design is introduced in Section 6.1. Sections 6.2 and 6.3 address the structural properties and the optimal solution of the event-trigger, respectively. A stability analysis for the case of perfect communication and the case of packet dropouts is conducted in Section 6.4.

6.1 Long-Run Average-Cost Problem Formulation

The following problem formulation for the optimal event-triggered design over an infinite horizon is based on the problem statement introduced in Chapter 2 for finite horizon problems. The long-run average cost can be regarded as the limiting case of the finite horizon problem, where the horizon N goes to infinity while the costs are normalized by N.

Let us consider the following stochastic discrete-time process to be controlled defined by the time-invariant difference equation

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{6.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times d}$. The variables x_k and u_k denote the state and the control input and are taking values in $\mathbb{R}^n$ and $\mathbb{R}^d$, respectively. The initial state x_0 is a random variable with finite mean and covariance C_{x_0}. The system noise process $\{w_k\}$ is i.i.d. (independent identically distributed) and w_k takes values in $\mathbb{R}^n$ and is zero-mean Gaussian distributed with covariance matrix C_w that has full rank. The random variables x_0 and w_k are statistically independent for each k. Let $(\Omega, \mathscr{F}, \mathsf{P})$ denote the probability space generated by the primitive random variables x_0 and noise sequence $\{w_k\}$. The output signal of the event-trigger, δ_k, takes values in $\{0, 1\}$ deciding whether information is transmitted at time k, i.e.,

$$\delta_k = \begin{cases} 1, & \text{measurement } x_k \text{ sent,} \\ 0, & \text{no measurement transmitted.} \end{cases}$$

Therefore, the signal received at the controller, that is denoted by z_k, is defined as

$$z_k = \begin{cases} x_k, & \delta_k = 1, \\ \emptyset, & \delta_k = 0. \end{cases} \tag{6.2}$$

We allow both decision makers to choose their actions upon their complete past history. Let the event-triggering law $\mathrm{f} = \{\mathrm{f}_0, \mathrm{f}_1, \dots\}$ and the control law $\gamma = \{\gamma_0, \gamma_1, \dots\}$ denote admissible policies. Hence, we have

$$\delta_k = \mathrm{f}_k(X^k), \qquad u_k = \gamma_k(Z^k).$$

We assume that the mappings f_k and γ_k are measurable mappings of their available information X^k and Z^k, respectively. Let $\mathscr{U}$ be set of the admissible policy pairs (f, γ). A policy f is said to be *stationary*, if it is time-homogeneous, i.e., $\mathrm{f} = [\mathrm{f}', \mathrm{f}', \dots]$.

The design objective is to find optimal $(\mathrm{f}^*, \gamma^*) \in \mathscr{U}$ that minimize the long-run average cost criterion

$$J = \limsup_{N \to \infty} \frac{1}{N} \mathsf{E}\left[\sum_{k=0}^{N-1} x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k + \lambda \delta_k\right], \tag{6.3}$$

whose per-stage cost is composed of a quadratic control cost term $x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k$ and a communication cost term $\lambda \delta_k$. The weighting matrix Q is positive definite and R is positive semi-definite. The positive factor λ penalizes the information exchange between sensor and controller. Opposed to previous chapters that have considered finite horizon problems, we additionally assume that the pair (A, B) is controllable and the pair $(A, Q^{\frac{1}{2}})$ is observable with $Q = (Q^{\frac{1}{2}})^{\mathrm{T}} Q^{\frac{1}{2}}$. These assumptions will guarantee that an optimal solution with finite costs exists.

6.2 Structural Properties

The techniques used in Section 2.3.2 to prove that the optimal control law for the considered finite horizon problem has the certainty equivalence property do not rely on the fact of a finite horizon. The only issue that formally needs to be taken into account in the average-cost case is the possibility of infinite costs for certain choice of $(\mathrm{f}, \gamma) \in \mathscr{U}$. These kind of laws can be however be excluded from our consideration due to the controllablilty and observability assumptions. As we know that the pair $(\mathrm{f}', \gamma^{\mathrm{CE}})$ with $\mathrm{f}' \equiv 1$ and γ^{CE} being the certainty equivalence controller defined in Eq. (6.4) yields finite costs, the set of policies $\mathscr{U}_{\mathrm{CE}} = \{(\mathrm{f}, \gamma^{\mathrm{CE}}) | \, \mathrm{f} \text{ admissible}\}$ will always have a member yielding finite costs. Therefore, the results for the finite horizon problem discussed in Section 2.3.2 carry over to the case of infinite horizon. This fact is summarized in the following theorem.

Theorem 6.1. *Let the system be given by Eq. (6.1) and Eq. (6.2). Let the certainty equivalence control law be defined by*

$$\gamma_k^{CE}(Z^k) = -L\, \mathsf{E}[x_k | Z^k], \qquad k \in \{0, 1, \dots\} \tag{6.4}$$

with

$$\begin{aligned} L &= -(B^{\mathrm{T}} P B + R)^{-1} B^{\mathrm{T}} P A, \\ P &= A^{\mathrm{T}} (P - P B (B^{\mathrm{T}} P B + R)^{-1} B^{\mathrm{T}} P) A + Q. \end{aligned} \tag{6.5}$$

Then the set of policies $\mathscr{U}_{CE} \subset \mathscr{U}$ defined by

$$\mathscr{U}_{CE} = \{(\mathrm{f}, \gamma^{CE}) \in \mathscr{U}\}$$

is a dominating class of policies for the minimization of the average cost defined in Eq. (6.3).

Because of the above theorem, we can restrict our attention to control policies that have the certainty equivalence property. This implies that the problem of minimizing J defined in Eq. (6.3) reduces to the optimization of the event-trigger f. As already observed at the end of Chapter 2, the least-squares estimate $\mathsf{E}[x_k|Z^k]$ depends on the particular choice for the event-trigger f. Therefore, a direct application of dynamic programming is still prohibited, although there is only one decision maker. We circumvent this problem by assuming that the estimator $\mathsf{E}[x_k|Z^k]$ does not take the side information into account that arises from not sending data to the controller. By this assumption, we can conclude from Section (3.2) that the estimation bias defined in Eq. (3.10) vanishes. Therefore, the event-trigger f is symmetric and the estimator can be written as

$$\mathsf{E}[x_k|Z^k] = \begin{cases} x_k, & \delta_k = 1, \\ (A - BL)\,\mathsf{E}[x_{k-1}|Z^{k-1}], & \delta_k = 0, \end{cases} \tag{6.6}$$

According to Theorem 6.1 and by defining the one-step ahead estimation error e_k as

$$e_k = x_k - (A - BL)\,\mathsf{E}[x_{k-1}|Z^{k-1}] \tag{6.7}$$

with $e_0 = x_0 - \mathsf{E}[x_0]$, the optimization problem that minimizes cost (6.3) reduces to

$$\inf_{\mathrm{f}} \limsup_{N\to\infty} \frac{1}{N}\,\mathsf{E}\left[\sum_{k=0}^{N-1} (1 - \delta_k) e_k^{\mathrm{T}} \Gamma e_k + \lambda \delta_k\right], \tag{6.8}$$

where

$$\Gamma = L^{\mathrm{T}}(R + B^{\mathrm{T}}PB)L. \tag{6.9}$$

As derived in Section 3.2, the one-step ahead estimation error e_k evolves by the following difference equation

$$e_{k+1} = (1 - \delta_k)Ae_k + w_k, \qquad k \geq 0. \tag{6.10}$$

Therefore, it can be observed that the process $\{e_k\}$ evolves according to a δ_k-controlled time-homogeneous Markov chain. This implies that the optimization problem defined in Eq. (6.8) is an average-cost optimal control problem where known techniques from stochastic optimal control can be applied.

6.3 Optimal Event-Trigger Design

In this section, we are interested in finding the optimal event-trigger that is given by solving the optimization problem defined in Eq. (6.8). We define the running cost as

$$c(e_k, \delta_k) = (1 - \delta_k) e_k^{\mathrm{T}} \Gamma e_k + \lambda \delta_k$$

and introduce the so-called dynamic programming operator $\mathscr{T}$ as

$$\mathscr{T} h(e_k) = \min_{\delta_k \in \{0,1\}} \left[c(e_k, \delta_k) + \mathsf{E}[h(e_{k+1})|e_k, \delta_k] \right]. \tag{6.11}$$

In order to simplify the following analysis, we introduce the following technical assumption on the event-triggering laws f.

(A1) The event-triggering law f satisfies

$$\mathrm{f}_k(e_k) = 1, \qquad \|e_k\|_2 > M,$$

with $M > 0$ for all $k \geq 0$.

Assumption **(A1)** does not impose a severe restriction on the scheduling laws, as the bound M can be chosen arbitrarily large. What can be noted first from the above assumption is that the running cost c_k is uniformly bounded with regard to the inputs δ_k and the error e_k.

The following *average-cost Bellman equation* gives us an optimality criterion for the solution of the AC problem defined in Eq. (6.8).

$$h^*(e_k) + J^* = \mathscr{T}h(e_k) \quad \text{for all } e_k \in \mathbb{R}^n. \tag{6.12}$$

Suppose that Assumption **(A1)** holds true and assume that there exists a bounded measurable function h^* and constant J^*, such that the AC Bellman equation (6.12) is satisfied. According to [HL89], we then have

$$\inf_{\mathrm{f}} J^{\mathscr{E}}(\mathrm{f}) \geq J^*$$

and if f^* is a stationary solution which is the solution of the minimization of the right-hand side of (6.11), then f^* is optimal and $J^{\mathscr{E}}(\mathrm{f}^*) = J^*$.

There arise two questions for the solution of the AC Bellman equation (6.12). The existence of a solution for the AC Bellman equation (6.12) and the convergence of the value iteration. The value iteration applies the Bellman operator recursively on the function $h(e_k)$ in order to converge to the optimal $h^*(e_k)$. After each step, the function needs shifted accordingly in order to yield bounded solutions.

The questions can be answered by establishing certain ergodicity conditions on the dynamics of the δ_k-controlled Markov chain. A specific requirement defined in the condition 3.1(4) in [HL89] related to the total variation norm has been proven in [XH04] to hold, which implies the existence of a solution and the convergence of value iteration. Therefore, we can conclude that the optimal event-trigger can be computed by value iteration based on Assumption **(A1)**.

6.4 Stability Analysis

In this section, we aim at analyzing the asymptotic behavior of the event-triggered closed-loop system. The controller is assumed to be given by Eqs. (6.4) and (6.6), while the event-trigger f is assumed to be stationary and satisfies Assumption **(A1)** introduced in the previous section. We distinguish between two scenarios: (i) perfect communication and (ii) packet loss in the feedback loop. The first case will discussed in the following and continues the preceding analysis, whereas the presence of packet loss needs us to alter the definition of the received signal in Eq. (6.2) in Section 6.4.2 accordingly.

With regard to the notion of stability of the closed-loop system, we focus on concept of ergodicity of Markov chains introduced in Appendix B.

6.4.1 Perfect Communication

The closed-loop system evolves according to the following difference equation with the augmented state $(x_k, e_k) \in \mathbb{R}^{2n}$.

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A - BL & (1 - \mathrm{f}(e_k))BL \\ 0 & (1 - \mathrm{f}(e_k))A \end{bmatrix} \cdot \begin{bmatrix} x_k \\ e_k \end{bmatrix} + \begin{bmatrix} w_k \\ w_k \end{bmatrix}$$

with appropriate initial condition x_0 and $e_0 = x_0 - \mathsf{E}[x_0]$. We directly observe that the overall closed-loop system has a triangular structure, in which the state x_k does not affect the evolution of e_k. Furthermore, it can be stated that the stochastic process $(1 - \mathrm{f}(e_k))BLe_k$ has a uniform bounded support for every k because of Assumption **(A1)**. As the matrix $(A - BL)$ is Hurwitz [Åst06], because of the controllablilty and detectability assumption in Section 6.1, it suffices to analyze the asymptotic behavior of the estimation error e_k, from whose properties we can infer the behavior of the overall closed-loop system. We also observe that the control gain L may not be the optimal gain resulting from the solution of Eq.(6.5), but it suffices to assume that L is a stabilizing feedback gain.

The next theorem shows how drift criteria can be applied to prove closed-loop stability of the evolution of the estimation error e_k.

Theorem 6.2. *Suppose e_k evolves according to the Markov chain defined by Eq. (6.10) with $\delta_k = f(e_k)$, and suppose that* f *satisfies Assumption* **(A1)**. *Then, the Markov chain e_k is ergodic.*

Proof. In the following, we will make use of the Foster's criterion for stochastic stability stated in Theorem B.3 of the appendix and Aperiodic Ergodic Theorem stated in Theorem B.4. Due to the fact that the noise process w_k is Gaussian with $C_w > 0$, we can conclude that the Markov chain e_k is φ-irreducible with φ being the standard Lebesgue measure. Moreover, it can be observed from section 5.3.5 in [MT93] that any compact set in $\mathbb{R}^n$ is also small. Furthermore, we can also conclude from the absolute continuity of the distribution of w_k that the Markov chain is strongly aperiodic. What remains to be shown is that we can find a small set $\mathscr{D}$ and Lyapunov candidate V where V is a real-valued non-negative function in $\mathbb{R}^n$, such that the drift ΔV defined in B.10 satisfies Foster's criterion defined in Theorem B.3, which is restated here.

$$\Delta V(e_k) \leq -1, \qquad \text{for all} \quad e_k \in \mathbb{R}^n \backslash \mathscr{D}. \tag{6.13}$$

By supposing that $V(e_k) = \|e_k\|_2^2$ and $\mathscr{O} = \{e_k \in \mathscr{R}^n \,|\, \|e_k\|_2 \leq M\}$, we have

$$\begin{aligned} \Delta V(e_k) &= \mathsf{E}[V(e_{k+1})|e_k] - V(e_k) \\ &= \mathsf{E}[\|(1 - \mathrm{f}(e_k))Ae_k + w_k\|_2^2|e_k] - \|e_k\|_2^2. \end{aligned}$$

Due to Assumption **(A1)**, we conclude for $e_k \notin \mathscr{O}$

$$\begin{aligned} \Delta V(e_k) &= \mathsf{E}[\|w_k\|_2^2|e_k] - \|e_k\|_2^2 \\ &= \mathrm{tr}[C_w] - \|e_k\|_2^2. \end{aligned}$$

By choosing $\mathscr{D} = \mathscr{O} \cup \{e_k \in \mathscr{R}^n \,|\, \|e_k\|_2 \leq \sqrt{\mathrm{tr}[C_w] + 1}\}$, the drift condition defined in Eq. (6.13) holds and we can conclude that the Markov chain is positive Harris recurrent. As the chain is also strongly aperiodic, we conclude that it is ergodic due to the Aperiodic Ergodic Theorem stated in Theorem B.4. This completes the proof. □

6.4.2 Presence of Packet Loss

In this section, we aim at deriving sufficient conditions in order to prove stability of the closed-loop system in the presence of packet loss. Therefore, we need to extend the definition for the received signal by the following equation.

$$z_k = \begin{cases} x_k, & \delta_k = 1 \wedge q_k = 1 \\ \emptyset, & \text{otherwise} \end{cases} \tag{6.14}$$

where the packet dropouts are modeled as a Bernoulli process $\{q_k\}_k$ defined as

$$q_k = \begin{cases} 1, & \text{update successfully transmitted,} \\ 0, & \text{packet dropout occurred,} \end{cases}$$

with packet dropout probability $\beta = \mathsf{P}[q_k = 0 | \delta_k = 1]$ and q_k takes a value of 0, if $\delta_k = 0$. Similar as in Chapter 4, the event-trigger receives an acknowledgement, whenever a packet has been transmitted successfully. We assume that the acknowledgement is transmitted instantaneously and error-free. Accordingly, we adapt the estimator defined in Eq. (6.6) by

$$\mathsf{E}[x_k|Z^k] = \begin{cases} x_k, & \delta_k = 1 \wedge q_k = 1, \\ (A - BL)\,\mathsf{E}[x_{k-1}|Z^{k-1}], & \text{otherwise.} \end{cases} \tag{6.15}$$

The evolution of the one-step ahead estimation error can then be defined as

$$e_{k+1} = (1 - q_k\delta_k)Ae_k + w_k, \qquad k \geq 0. \tag{6.16}$$

By the same arguments as in Section 6.4.1, we can restrict our analysis to the evolution of e_k. Then, we obtain the following stability condition summarized in the subsequent theorem.

Theorem 6.3. *Suppose e_k evolves according to the Markov chain defined by Eq. (6.16) with $\delta_k = f(e_k)$, and suppose that* f *is stationary and satisfies Assumption* **(A1)**. *If the packet dropout probability β satisfies*

$$\beta < \frac{1}{\|A\|_2^2}, \tag{6.17}$$

then, the Markov chain e_k is ergodic.

Proof. Along the same lines as in the proof of Theorem 6.2, it can be shown that the Markov chain is φ-irreducible and aperiodic. Furthermore, it can also be assumed that compact sets are small. Hence, it remains to show that the drift criterion (Foster's criterion in Theorem B.3) in Eq. (6.13) can be satisfied by an appropriate choice for V and $\mathscr{D}$. Define $\mathscr{O} = \{e_k \in \mathscr{R}^n \,|\, \|e_k\|_2 \leq M\}$ and assume that $e_k \notin \mathscr{O}$. Because of Assumption **(A1)**, the drift for the Lyapunov candidate $V(e_k) = \|e_k\|_2^2$ can be written as

$$\begin{aligned} \Delta V(e_k) &= \mathsf{E}[V(e_{k+1})|e_k] - V(e_k) \\ &= \mathsf{E}[\|(1-q_k)Ae_k + w_k\|_2^2|e_k] - \|e_k\|_2^2 \\ &= \mathsf{E}[\|(1-q_k)Ae_k\|_2^2|e_k] + \mathsf{E}[\|w_k\|_2^2] - \|e_k\|_2^2 \\ &= \mathsf{E}[\|(1-q_k)Ae_k\|_2^2|e_k] + \mathrm{tr}[C_w] - \|e_k\|_2^2, \end{aligned}$$

where the second last equality is due to the statistical independence of w_k with respect to e_k and q_k. The statistical independence of q_k and e_k implies

$$\begin{aligned}\Delta V(e_k) &= \mathsf{E}[(1-q_k)]\|Ae_k\|_2^2 + \operatorname{tr}[C_w] - \|e_k\|_2^2 \\ &\leq \mathsf{E}[(1-q_k)]\|A\|_2^2\|e_k\|_2^2 + \operatorname{tr}[C_w] - \|e_k\|_2^2\end{aligned}$$

As we assume that $\delta_k = 1$, the expectation on the packet dropout yields

$$\mathsf{E}[(1-q_k)] = \mathsf{P}[q_k = 0|\delta_k = 1] = \beta.$$

Therefore, we obtain the following inequality for the drift

$$\Delta V(e_k) \leq (\beta\|A\|_2^2 - 1)\|e_k\|_2^2 + \operatorname{tr}[C_w].$$

By taking the hypothesis of Theorem 6.3 defined by Eq. (6.17) into account, we define the compact set

$$\mathscr{O}' = \left\{ e_k \in \mathbb{R}^n \,\middle|\, \|e_k\|_2 \leq \sqrt{\frac{\operatorname{tr}[C_w]+1}{1-\beta\|A\|_2^2}} \right\}.$$

By defining $\mathscr{D} = \mathscr{O} \cup \mathscr{O}'$, the drift criterion in Eq (6.13) is satisfied. Due to the strong aperiodicity of the Markov chain, the chain is also ergodic by Theorem B.4. This completes the proof. □

6.5 Summary

In this chapter, we have seen that the structural properties developed for finite horizon problem in previous chapters carry over to the case of infinite horizon with the average-cost criterion. The introduction of the technical assumption **(A1)** that puts a uniform bound on the triggering threshold has enabled us to compute the optimal event-triggering policy by value iteration. Furthermore, this condition has allowed to prove stability of the closed-loop system in a straight-forward manner by the application of drift criteria for Markov chains.

The ideas developed in this chapter will be crucial in the design and analysis of event-triggered mechanisms in the subsequent chapters. The techniques developed in the stability proof in Sections 6.4.1 and 6.4.2 will be in particular useful in the stability analysis of the multi-loop system studied in the follow-up chapter. Furthermore, we will also rely heavily on the properties obtained in Sections 6.2 and 6.3 for the average-cost problem with regard to the optimal event-triggered control design for multiple control loops sharing a common communication resource studied in Chapters 8 and 9.

6.6 Bibliographical Notes

The contribution of this chapter is partly based on [MTH11; MH13b]. The analysis of the average-cost Bellman equation for the underlying optimization problem in Section 6.3 is partly based on the results in [XH04]. Inspired by this work, we have adapted their assumption on the admissible event-triggering laws having bounded triggering thresholds. In order to overcome the curse of dimensionality of dynamic programming, several authors have proposed suboptimal algorithms [Cog09a; LWL13]. In these works, the event-triggering law is computed by approximate dynamic programming through approximating the value function by either quadratic or polynomial functions.

Part II

Multi-Loop Control

7

Stochastic Stability of Multiple Event-Triggered Control Systems

This chapter analyzes the stability properties of multiple event-triggered control systems whose feedback loops are closed over a common communication network. The system under consideration consists of several individual subsystems whose sensor information needs to be sent over a shared communication link to the controller. An event-triggered scheduler situated at the sensor node of each subsystem decides upon its local information whether to transmit information. Due to the contention-based communication scheme that only allows a limited number of transmissions per time step, there is the chance for collisions among subsystems. In order to counteract potential collisions, we assume a probabilistic collision resolution scheme, in which an arbitration mechanism selects randomly which subsystem is permitted to transmit its sensor information to the controller. What makes the analysis of such multi-loop system challenging is the tight interaction between the individual control loops and the communication system due to the event-triggered nature of the scheduling mechanism.

By making use of results in Chapter 6, sufficient conditions for stability are derived. These conditions will relate the ratio between the availability of the resource and the number of control loops with the open-loop system dynamics of each control system.

This chapter is organized as follows. In Section 7.1, the mathematical system model is introduced, which describes the subsystem control model and the communication model. The stability analysis of the event-triggered multi-loop system is addressed in Section 7.2.

7.1 System Model of the Multi-loop Control System

Figure 7.1 shows the structure of the considered networked control system (NCS). It comprises of N independent subsystems whose feedback loops are closed over a shared communication network. The i-th subsystem consists of a process $\mathscr{P}^i$, a controller $\mathscr{C}^i$, which is implemented at the actuator, and a sensor $\mathscr{S}^i$. In the following, we first describe the control model of the subsystems and introduce then the considered communication model.

7.1.1 Subsystem Control Model

The process $\mathscr{P}^i$ to be controlled by the ith subsystem is described by the following difference equation.

$$x^i_{k+1} = A^i x^i_k + B^i u^i_k + w^i_k \tag{7.1}$$

with $A^i \in \mathbb{R}^{n_i \times n_i}, B^i \in \mathbb{R}^{n_i \times d_i}$ for $i \in \{1, \ldots, N\}$. The state x^i_k and the control input u^i_k are taking values in $\mathbb{R}^{n_i}$ and in $\mathbb{R}^{d_i}$, respectively. The noise process w^i_k takes values in $\mathbb{R}^{n_i}$ and is independent and identically distributed with $w^i_k \sim \mathscr{N}(0, C^i)$ where C^i has full rank. The initial state, x^i_0, $i \in \{1, \ldots, N\}$ is a random variable with a symmetric distribution around its mean and has a finite second order moment. The statistics of the random variables and the system parameters within a subsystem are known to the controller as well as to sensor station.

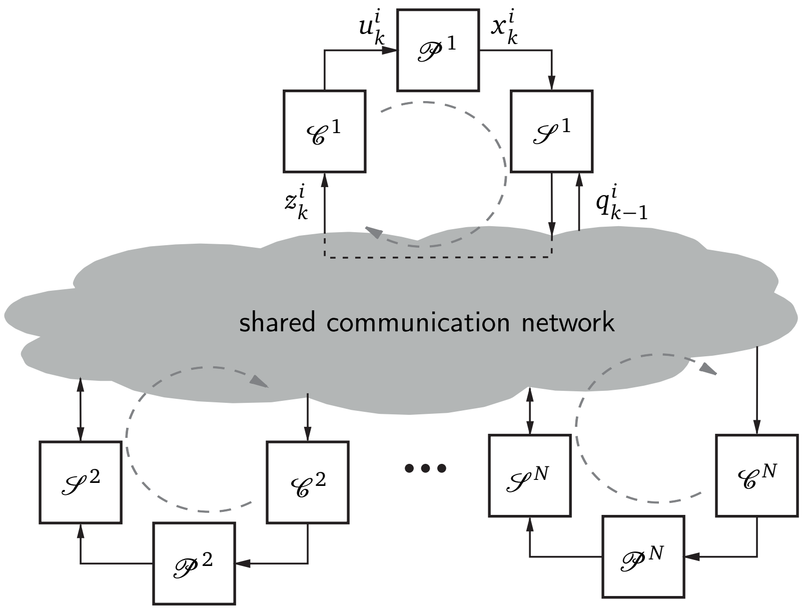


Figure 7.1: System model of the NCS with N control systems closed over a shared communication network with processes $\mathscr{P}^1, \ldots, \mathscr{P}^N$, sensors $\mathscr{S}^1, \ldots, \mathscr{S}^N$ and controllers $\mathscr{C}^1, \ldots, \mathscr{C}^N$.

At any time k the scheduler $\mathscr{S}^i$ situated at the sensor decides, whether a transmission slot should be requested to transmit the current state of subsystem i to the controller $\mathscr{C}^i$. Therefore, an event occurs within a subsystem i at time k when a transmission slot is requested.

Further, it is assumed that control inputs may not be constant in between of successful transmissions and the controller $\mathscr{C}^i$ may adjust the control inputs based on past updates.

Similar to Part I of this thesis, the request for a transmission of the ith subsystem at time k is defined by the triggering variable δ_k^i which takes the following values.

$$\delta_k^i = \begin{cases} 1, & \text{request for transmission,} \\ 0, & \text{idle.} \end{cases}$$

We represent the arbitration mechanism of the communication network by the random variable q_k^i taking values

$$q_k^i = \begin{cases} 1, & \text{allow transmission,} \\ 0, & \text{block transmission.} \end{cases}$$

Then, the received data at the controller $\mathscr{C}^i$ denoted as z_k^i is defined by

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1 \wedge q_k^i = 1 \\ \varnothing, & \text{otherwise} \end{cases} \tag{7.2}$$

It is assumed that the controller takes the following form.

$$u_k = -L^i \, \mathsf{E}\left[x_k^i | Z_k^i\right]. \tag{7.3}$$

We assume that each loop is stabilized with the state feedback controller in Eq. (7.3) in case of ideal communication. Therefore, we assume that each L^i is chosen such that the closed-loop matrix $\left(A^i - B^i L^i\right)$ is Hurwitz for $i \in \{1, \ldots, N\}$. Similar as in Eq. (6.6), the controllers are updated by a linear predictor, in case of a blocked data transmission request, i.e.,

$$\mathsf{E}\left[x_k^i | Z_k^i\right] = \begin{cases} x_k^i, & \delta_k^i = 1 \wedge q_k^i = 1, \\ (A^i - B^i L^i)\, \mathsf{E}\left[x_{k-1}^i | Z_{k-1}^i\right], & \text{otherwise,} \end{cases} \tag{7.4}$$

with the initial condition $\mathsf{E}\left[x_0^i | Z_0^i\right] = \mathsf{E}[x_0]$ for $\delta_k^i = 0$ or $q_k^i = 0$.

7.1.2 Communication Model

What has been left open to be defined in the introduction of the system model in Section 7.1.1 is the choice for the triggering variable δ_k^i and the arbitration variable q_k^i at each subsystem i, $i \in \{1, \ldots, N\}$. Inspired by the results in Part I of this thesis, we assume that the event-trigger depends on the networked-induced estimation error e_k^i defined as

$$e_k^i = x_k^i - (A^i - B^i L^i)\, \mathsf{E}\left[x_{k-1}^i | Z_{k-1}^i\right].$$

The estimation error e_k^i evolves according to the following difference equation.

$$e_{k+1}^i = \left(1 - q_k^i \delta_k^i\right) A^i e_k^i + w_k^i. \tag{7.5}$$

with $e_0^i = x_0^i - \mathsf{E}[x_0]$. We consider a TCP-like communication network that instantaneously acknowledges the event-trigger at the sensor whether a transmission has been successful. Therefore, the estimation error e_k^i is known at the event-trigger at time k. This implies that its decision whether to transmit can directly depend on e_k^i. The event-triggering law is assumed to be stationary and defined by the measurable mapping f^i, i.e.,

$$\delta_k^i = \mathrm{f}^i(e_k^i), \qquad , i \in \{1, \ldots, N\}.$$

We adopt Assumption **(A1)** introduced in Section 6.3 for the case of multiple event-triggers as follows.

(A2) The event-triggering laws f^i satisfies

$$\mathrm{f}^i(e_k^i) = 1, \qquad \|e_k\|_2 > M^i,$$

with $M^i > 0$ for all $i \in \{1, \ldots, N\}$.

Due to bandwidth limitations the number of transmission slots per time step denoted by c is constrained If there are more requests than available transmission slots at time k, then the arbitration mechanism within the communication system selects c subsystems that may transmit information. All other subsystems are blocked and are informed instantaneously that their request has been rejected. The arbitration mechanism does not prioritize subsystems, i.e., in case of arbitration, the subsystems are chosen with identical probability. Therefore, the probability distribution of $[q_k^1, \ldots, q_k^N]$ conditioned on the requests δ_k^i, $i \in \{1, \ldots, N\}$ is time-invariant and has the following property.

$$\mathsf{P}[q_k^i = 1 | \delta_k^i, i \in \{1, \ldots, N\}] = \begin{cases} 1, & \sum_{i=1}^N \delta_k^i \leq c, \\ \frac{c}{\sum_{i=1}^N \delta_k^i}, & \text{otherwise}, \end{cases} \tag{7.6}$$

for subsystem i with $\delta_k^i = 1$ and

$$q_k^1(\omega) + \cdots + q_k^N(\omega) = c$$

for all sample paths $\omega \in \Omega$ for which $\delta_k^1 + \cdots + \delta_k^N \geq c$. From the definition of q_k^i, we can observe that this model differs significantly from the model of q_k used to model packet loss in Chapter 4 and Section 6.4.2, where q_k has been assumed to be i.i.d. and Bernoulli distributed. However, the stability analysis conducted in Section 6.4.2 will serve as a starting point for proving stability of the multi-loop event-triggered control system.

7.2 Stochastic Stability

In this section, we study the asymptotic analysis of the aggregated multi-loop system. After establishing the Markov chain of our interest, Theorem 7.1 will state a sufficient condition for stochastic stability in terms of ergodicity.

The closed-loop system evolves according to the following difference equation with the augmented state $(x_k, e_k) \in \mathbb{R}^{2n}$.

$$\begin{bmatrix} x_{k+1}^i \\ e_{k+1}^i \end{bmatrix} = \begin{bmatrix} A^i - B^i L^i & (1 - q_k^i \, \mathrm{f}^i(e_k^i)) B^i L^i \\ 0 & (1 - q_k^i \, \mathrm{f}^i(e_k^i)) A^i \end{bmatrix} \cdot \begin{bmatrix} x_k^i \\ e_k^i \end{bmatrix} + \begin{bmatrix} w_k^i \\ w_k^i \end{bmatrix}, \qquad i \in \{1, \ldots, N\}$$

with appropriate initial condition x_0^i and $e_0^i = x_0^i - \mathsf{E}[x_0^i]$. Within each subsystem, we observe that the dynamics has a triangular structure, in which the state x_k^i does not affect the evolution of e_k^i. Furthermore, it can be stated that the stochastic process $(1 - q_k^i \, \mathrm{f}^i(e_k)) B^i L^i e_k^i$ has a uniform bounded support for every k because of Assumption **(A2)**. Because of the assumption that the matrix $(A^i - B^i L^i)$ is Hurwitz, it suffices to analyze the asymptotic behavior of the augmented estimation error $e_k = [e_k^1, \ldots, e_k^N]^{\mathrm{T}}$, from whose properties we can infer the behavior of the overall closed-loop system.

By incorporating the evolution of e_k^i defined in Eq. (7.5), the definition of the event-triggering rule f^i and the definition of the random variable q_k^i for $i \in \{1, \ldots, N\}$, it can be concluded that e_k is a time-homogeneous Markov chain. Therefore, the tools for the stochastic stability of Markov chains can applied to study its asymptotic behavior.

The following theorem states a sufficient condition that guarantees stochastic stability for the multi-loop control system. It shall be noted that this condition is established separately for each subsystem and does not put requirements on the scheduling behavior of the other subsystems. Therefore, even if a malicious subsystem is continuously requesting for transmission, stochastic stability can still be guaranteed.

Theorem 7.1. *Suppose $e_k = [e_k^1, \ldots, e_k^N]$ evolves according to the Markov chain defined by Eq. (7.5) with $\delta_k^i = f^i(e_k^i)$ satisfying Assumption* **(A2)**. *If the ratio between the number of available slots and the number of loops, $\frac{c}{N}$ satisfies*

$$\frac{c}{N} > 1 - \frac{1}{\|A^i\|_2^2}, \qquad \text{for all} \quad i \in \{1, \ldots, N\}, \tag{7.7}$$

then the Markov chain e_k is ergodic.

Proof. As the noise w_k^i is Gaussian distributed with positive definite covariance matrix C^i, we observe that the transition kernel yields an absolute continuous distribution for any state e irrespective of the choice of f^i, $i \in \{1, \ldots, N\}$. Similar as in the proof of Theorem 6.3, it can be concluded that the Markov chain e_k is φ-irreducible and strongly aperiodic, and that all compact sets are small.

We will use Foster's criterion defined in Theorem B.3 and the Aperiodic Ergodic Theorem defined in B.4 in the appendix.
Consider the Lyapunov candidate

$$V(e_k) = \sum_{i=1}^{N} \|e_k^i\|_2^2. \tag{7.8}$$

Define the compact set $\mathscr{O}$ as

$$\mathscr{O} = \left\{ e_k \in \mathbb{R}^{n_1 + \cdots + n_N} \mid \|e_k^i\|_2 \le M^i, i \in \{1, \ldots, N\} \right\}$$

Further, we define the drift within a subsystem as

$$\Delta_i V(e_k) = \mathsf{E}\left[\|e^i_{k+1}\|_2^2 \middle| e^i_k\right] - \|e^i_k\|_2^2$$

Due to the choice of V and the linearity of conditional expectation, the above definition implies

$$\Delta V(e_k) = \sum_{i=1}^{N} \Delta_i V(e_k).$$

Consider two cases for calculating an upper bound on the drift $\Delta_i V(e_k)$ of subsystem i. First, let $\|e^i_k > M^i\|$. The statistical independence of w^i_k with respect to q^i_k and e_k, the fact that w^i_k is zero-mean Gaussian with covariance matrix C^i, and Assumption **(A2)** allows us to simplify $\Delta_i V(e_k)$ as follows.

$$\Delta_i V(e_k) = \mathsf{E}\left[1 - q^i_k \middle| e^i_k\right] \|A^i e^i_k\|_2^2 + \mathrm{tr}[C^i] - \|e^i_k\|_2^2$$

The expression $\mathsf{E}\left[1 - q^i_k \middle| e^i_k\right]$ describes the average probability that a request of subsystem i is blocked. Because of the definition of q^i_k, we have the following upper bound by Eq. (7.6)

$$\mathsf{E}\left[1 - q^i_k \middle| e^i_k\right] \leq 1 - \frac{c}{N}.$$

Therefore, the drift can be bounded by

$$\Delta_i V(e_k) \leq \left\{ \left(1 - \frac{c}{N}\right) \|A^i\|_2^2 - 1 \right\} \|e^i_k\|_2^2 + \mathrm{tr}[C^i] \tag{7.9}$$

for $\|e^i_k\|_2 > M^i$. In the case of $\|e^i_k\|_2 \leq M^i$, we have the following uniform bound

$$\Delta_i V(e_k) \leq |\|A^i\|_2^2 - 1| \left(M^i\right)^2 + \mathrm{tr}[C^i]. \tag{7.10}$$

From Eqs. (7.9) and (7.10), it follows immediately that the drift is uniformly bounded within any bounded set in $\mathbb{R}^{n_1+\cdots+n_N}$.

Let the compact set $\mathscr{D}$ take the form

$$\mathscr{D} = \left\{ e_k \in \mathbb{R}^{n_1+\cdots+n_N} \,\middle|\, \|e^i_k\|_2 \leq d^i, i \in \{1, \ldots, N\} \right\}.$$

The appropriate choice for the values of d^i, $i \in \{1, \ldots, N\}$ is discussed in the following. By the hypothesis of Theorem 7.1, we see that the first term of the bound given in Eq. (7.9) is negative. Therefore, the bound obtained by Eq. (7.10) can be considered as a uniform bound for the drift corresponding to a subsystem. By restricting the compact set $\mathscr{D}$ such that $\mathscr{D} \supset \mathscr{O}$, we ensure that at least one of the subsystems tries to send information over the communication network. This implies that there is at least one subsystem for which we have a Δ_i satisfying the upper bound given by Eq. (7.9). Assume that the ith system requests for transmission implying that inequality (7.9) holds. Then, by choosing d^i as

$$d^i = \sqrt{\frac{1 + \mathrm{tr}[C^i] + \sum_{j \neq i} \left(|\|A^j\|_2^2 - 1| \left(M^j\right)^2 + \mathrm{tr}[C^j] \right)}{\left(1 - \frac{c}{N}\right) \|A^i\|_2^2 - 1}} \qquad \text{for all} \quad i \in \{1, \ldots, N\},$$

we can guarantee that the drift $\Delta V(e_k)$ will not exceed -1 for all $e_k \in \mathscr{D}$. Hence, all assertions of the Foster's criterion are satisfied, which implies that the Markov chain is positive Harris recurrent. As the Markov chain is strongly aperiodic, we conclude that it is also ergodic due to Theorem B.4. This completes the proof. □

7.3 Summary

Here, we have introduced the system model of multiple independent control systems whose feedback loops are closed over a shared communication network. Our focus has been on the stability analysis of the overall system that uses a decentralized event-triggered scheduling mechanism. What has made the analysis of such multi-loop system challenging is the tight interaction between the individual control loops and the communication system due to the event-triggered nature of the considered scheduling mechanism. By making use of results in Chapter 6, we were able to derive sufficient conditions for stability. These conditions relate the ratio between the availability of the resource and the number of control loops with the open-loop system dynamics of each control system. It can be observed that the stability condition in Theorem 7.1 does not put any constraints on the scheduling behavior of the other subsystems. This implies that stochastic stability can still be guaranteed, even if a malicious subsystem is continuously requesting for transmission.

7.4 Bibliographical Notes

The results of this chapter are partly based on the work in [MH11; MH14]. The stability analysis is built upon drift criteria developed for Markov chains in uncountable spaces that can be found in [MT93]. By strenghtening the drift criterion used in this chapter, it is additionally shown in [MH11; MH14] that the second-order moment of the network-induced estimation error is bounded in its stationary regime.

8

Optimal Event-Triggered Control over a Shared Network

In this chapter, we develop an efficient algorithm for the design of multiple event-triggered controllers whose feedback loops are closed over a common communication network. By assuming the system model introduced in Chapter 7, the design procedure is formulated as an average-cost problem that aims at the minimization of a social cost criterion. The considered cost function is composed of the summation of the LQ costs of each subsystem within the communication network.

We propose a relaxed formulation of the average-cost problem that allows us to circumvent the coupling of control and communication in the design. The optimization problem becomes tractable as it can be split into two levels: a local optimal control problem and a global resource allocation problem. While the results of Chapter 2-4 on optimal event-triggered control apply in the local optimization problem, the global resource allocation problem can be analyzed by techniques of convex analysis. The local optimal control problem computes a Pareto frontier of operation points within each subsystem that represents the trade-off between control performance and resource consumption in terms of the transmission rate. The trade-off curves are taken into account by the global resource allocation in order to assign the transmission rate to each subsystem. The rate eventually determines the optimal operation point of the event-triggered controller within each subsystem. Based on the results on stochastic stability for the multi-loop system in Chapter 7, we show that the proposed bi-level approach is asymptotically optimal, when the number of users approaches infinity.

This chapter is organized as follows. In Section 8.1, we review the system model for the multi-loop system sharing a common communication network. The bi-level design approach is discussed in Section 8.2. Section 8.3 evaluates the numerical efficacy of the bi-level approach.

8.1 Problem Formulation

In the following, we briefly review the system model that has already been introduced in the previous chapter. This will then lead us to the design problem for the event-triggered controllers that is formulated as a social cost minimization.

The system is composed of N independent control subsystems that share a common communication network. Each subsystem i needs to regulate a process described by the difference equation

$$x_{k+1}^i = A^i x_k^i + B^i u_k^i + w_k^i, \quad i \in \{1, \dots, N\}, \tag{8.1}$$

with $A^i \in \mathbb{R}^{n_i \times n_i}, B^i \in \mathbb{R}^{n_i \times d_i}$. The state x_k^i and the control input u_k^i are taking values in $\mathbb{R}^{n_i}$ and in $\mathbb{R}^{d_i}$, respectively. The noise process w_k^i takes values in $\mathbb{R}^{n_i}$ and is independent and identically distributed with $w_k^i \sim \mathcal{N}(0, C^i)$. The initial state, x_0^i, $i \in \{1, \dots, N\}$ is a random variable with a symmetric distribution around its mean and has a finite second order moment. The statistics of the random variables and the system parameters within a subsystem are known to the controller as well as to sensor station.

At any time k the scheduler situated at the sensor decides based on its state observations, whether a transmission slot should be requested to transmit the current state of subsystem i to the controller. Therefore, an event occurs within a subsystem i at time k when a transmission slot is requested. The request for a transmission of the ith subsystem at time k is defined by the variable δ_k^i which takes the following values.

$$\delta_k^i = \begin{cases} 1, & \text{request for transmission,} \\ 0, & \text{idle.} \end{cases}$$

We represent the arbitration mechanism by the random variable q_k^i taking values

$$q_k^i = \begin{cases} 1, & \text{allow transmission,} \\ 0, & \text{block transmission.} \end{cases}$$

The probability distribution of $[q_k^1, \dots, q_k^N]$ conditioned on the requests δ_k^i, $i \in \{1, \dots, N\}$ is time-invariant and has the following property.

$$\mathsf{P}[q_k^i = 1 | \delta_k^i, i \in \{1, \dots, N\}] = \begin{cases} 1, & \sum_{i=1}^N \delta_k^i \leq N_{\text{slot}} \\ \frac{N_{\text{slot}}}{\sum_{i=1}^N \delta_k^i}, & \text{otherwise,} \end{cases} \tag{8.2}$$

for subsystem i with $\delta_k^i = 1$ and

$$q_k^1(\omega) + \cdots + q_k^N(\omega) = N_{\text{slot}}$$

for all sample paths $\omega \in \Omega$ for which $\delta_k^1 + \cdots + \delta_k^N \geq c$. The received data at the ith controller at time k is denoted by z_k^i and is defined by

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1 \wedge q_k^i = 1 \\ \emptyset, & \text{otherwise} \end{cases} \tag{8.3}$$

Every subsystem i, $i \in \{1, \ldots, N\}$, possesses an individual cost function J^i which is given by the average-cost

$$J^i = \limsup_{T \to \infty} \frac{1}{T} \mathsf{E}\left[\sum_{k=0}^{T-1} x_k^{i,\mathrm{T}} Q_x^i x_k^i + u_k^{i,\mathrm{T}} Q_u^i u_k^i\right]. \tag{8.4}$$

The weighting matrix Q_x^i is positive definite and Q_u^i is positive semi-definite for each $i \in \{1, \ldots, N\}$. We assume that the pair (A^i, B^i) is stabilizable and the pair $(A^i, Q_x^{i,\frac{1}{2}})$ is detectable with $Q_x^i = (Q_x^{i,\frac{1}{2}})^{\mathrm{T}} Q_x^{i,\frac{1}{2}}$. We define the individual average transmission rate of the ith subsystem by

$$r^i = \limsup_{T \to \infty} \frac{1}{T} \mathsf{E}[\sum_{k=0}^{T-1} \delta_k^i], \tag{8.5}$$

which also takes the form of an average-cost criterion with values in the closed interval $[0, 1]$.

The design objective is to design control and scheduling laws that minimize the social cost V that is given by the average of the individual costs , i.e.,

$$J = \frac{1}{N} \sum_{i=1}^{N} J^i. \tag{8.6}$$

The control law $\gamma^i = \{\gamma_0^i, \gamma_1^i, \ldots\}$ that reflects the behavior of the controller $\mathscr{C}^i$ at subsystem i is described by the mappings γ_k^i, $k \in \{0, 1, \ldots\}$. Admissible laws are measurable, causal maps of the available observations, i.e.,

$$u_k^i = \gamma_k^i(Z^{k,i}),$$

where $Z^{k,i}$ is the observation history until time k of subsystem i. The policy of the scheduler is given by $\mathrm{f}^i = \{\mathrm{f}_1^i, \mathrm{f}_2^i, \ldots\}$. The map f_k^i is defined as

$$\delta_k^i = \mathrm{f}_k^i(X^{k,i}, Q^{k-1,i}),$$

where f_k^i is measurable with respect to the observation history $\{X^{k,i}, Q^{k-1,i}\}$. The set of admissible policies (f^i, γ^i) is denoted by $\mathscr{U}^{\mathrm{ET}}$. The scheduling laws in $\mathscr{U}^{\mathrm{ET}}$ are only using local information $\{X^{k,i}, Q^{k-1,i}\}$ in order to determine whether a slot for transmission is to be requested. Time-triggered schedulers constitute a special case within $\mathscr{U}^{\mathrm{ET}}$, as the map f_k^i is independent of $\{X^{k,i}, Q^{k-1,i}\}$ for any k, i.e., time-triggered schedulers are elements of $\{0, 1\}^{\infty}$.

Therefore, the set of admissible time-triggered scheduling laws denoted by $\mathscr{U}^{\mathrm{TT}}$ can be considered as a subset of $\mathscr{U}^{\mathrm{ET}}$

Furthermore, by allowing that the decisions of the schedulers may depend on all measurements $\{X^{k,1}, Q^{k-1,1}, \ldots, X^{k,N}, Q^{k-1,N}\}$, we obtain a centralized scheduling structure. Let $\mathscr{U}^{\mathrm{CEN}}$ denote the set of admissible centralized schedulers. It can be observed that the decentralized event-trigger policies in $\mathscr{U}^{\mathrm{ET}}$ is also contained in $\mathscr{U}^{\mathrm{CEN}}$. Hence, we obtain the following relationship among the three scheduling structures.

$$\mathscr{U}^{\mathrm{TT}} \subset \mathscr{U}^{\mathrm{ET}} \subset \mathscr{U}^{\mathrm{CEN}}.$$

It implies that the cost J of the optimal decentralized event-triggered law in $\mathscr{U}^{\mathrm{ET}}$ is lower bounded by the optimal centralized scheduler and upper bounded by the minimal costs of the optimal time-triggered scheduler. We will return to the analysis of the optimal time-triggered and the optimal centralized scheduling laws when evaluating the performance of the developed scheduling mechanism in Section 8.3.

In the following, we are interested in finding the optimal decentralized scheduling law f^i and control law γ^i that minimizes the social cost J, i.e.,

$$\inf_{\substack{(\mathrm{f}^i, \gamma^i) \in \mathscr{U}^{\mathrm{ET}} \\ i \in \{1, \ldots, N\}}} J. \tag{8.7}$$

8.2 Design Approach

In this section, we develop an approximative design method for the decentralized event-triggered controller related to the optimization problem (8.7). This section is divided into five subsections. In Section 8.2.1, the approximative bi-level formulation that divides the original optimization problem into a local and global optimization problem is introduced. The structural properties of the solution of the local problem are studied in Section 8.2.2. In Section 8.2.3, the resulting global optimization problem is discussed. The issue of computational complexity of the approximative bi-level approach is addressed in Section 8.2.4, and the optimality properties are analyzed in Section 8.2.5.

8.2.1 Approximative Bi-Level Formulation

Although the coupling between subsystems is solely caused by the resource limitation, determining the optimal event-based control system that solves (8.7) is a hard problem. The reason for this is secondarily given by the fact that the number of subsystems might be large, but is rather due to the distributed information pattern. Besides the distributed information pattern, the impact of the bandwidth limitation is another complicating factor.

In contrast to time-triggered scheduling schemes, it is in general not possible to guarantee that a transmission request will be approved or not for decentralized event-triggered scheduling with resource constraints. Despite of the non-determinism due to the contention-based communication network, we will observe a significant performance improvement of the event-triggered scheme compared with a time-triggered scheme in Section 8.3.

However, in order to still obtain a systematic approach to find the event-triggered controllers that minimize J in Eq. (8.7), we introduce the following approximation. Thereby, the hard constraint that c transmissions are allowed at maximum at each time k is weakened and we require merely that the average number of transmissions per time step is upper bounded by c. Then, the average rate constraint can be stated as

$$\sum_{i=1}^{N} r^i \leq c.$$

With this rate constraint, the approximative optimization problem can be stated as a bi-level optimization problem. Both levels are coupled through the average transmission rates r^i, $i \in \{1, \ldots, N\}$. After assigning an upper bound, $\bar{r}^i$, on the transmission rate to each subsystem i, the first level of the optimization problem is given by

$$J^{i,*}(\bar{r}^i) = \inf_{\substack{(\mathrm{f}^i, \gamma^i) \in \mathscr{U}^{\mathrm{ET}} \\ r^i \leq \bar{r}^i}} J^i(\gamma^i, \mathrm{f}^i) \tag{8.8}$$

where x_k^i is the state evolving by (8.1) and it is assumed that in contrast to (8.3) every request is permitted, i.e.,

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It should be noted that above optimization problem can be solved locally in each subsystem for a given $\bar{r}^i$, i.e., assuming a given rate distribution, the optimization problems in (8.8) for $i \in \{1, \ldots, N\}$ are completely decoupled from each other. The second level of the optimization problem determines finally the optimal transmission rate distribution among the subsystems and is given by

$$\bar{J}^* = \inf_{\substack{\bar{r}^1, \ldots, \bar{r}^N \\ \sum_{i=1}^N \bar{r}^i \leq c}} \frac{1}{N} \sum_{i=1}^{N} J^{i,*}(\bar{r}^i) \tag{8.9}$$

Thus, the resulting bi-level approach has a hierarchical structure, where the second level can be considered as the global coordinating layer assigning resources to the first layer, where the optimization problem is solved locally in each subsystem, see Fig. 8.1.

8.2.2 Local Design Problem

In this section, we focus on the solution of the local design problem that is stated in the optimization problem (8.8). It can be immediately observed that this problem is closely related to the problem of optimal event-triggered control design studied extensively in Part I. Instead of having an additional cost penalty, the communication constraint is reflected by the average rate constraint $r^i \leq \bar{r}^i$. We will see in the following that this difference will play a minor role in the form of the optimal solution of optimization problem (8.8) as many of the results of Part I carry over to such case.

The subsequent theorem sheds light on the form of the optimal control law $\gamma^{i,*}$.

Theorem 8.1 (Certainty Equivalence Property). *Let (A^i, B^i) be stabilizable and $(A^i, Q_x^{i,\frac{1}{2}})$ detectable. Then, the form of the optimal control law $\gamma^{i,*}$ of optimization problem (8.8) for subsystem i is given by*

$$u_k^i = \gamma_k^{i,*}(Z^{k,i}) = -L^i \,\mathsf{E}[x_k^i | Z^{k,i}], \tag{8.10}$$

where the linear gain L^i can be calculated equivalently to Eq. (6.5)

The stabilizability and detectability assumptions in the above theorem ensure that the stabilizing gain L^i resulting from Eq. (6.5) exists. Similar to Theorem 2.1 in Chapter 2 and Theorem 6.1 in Chapter 6, the above theorem shows that the certainty equivalence property holds for the optimal control law $\gamma^{i,*}$ of each subsystem i. The proof follows the same line of argumentation as in the proof of Theorem 2.1, where the details for the case with the average transmission constraint can be found in [MH13a]. Theorem 8.1 enables a number of further simplifications.

When taking into account that the distributions of the noise variables are symmetric, then it is shown in Theorem 3.1 in Chapter 3 for first-order systems that the optimal scheduling law $\mathrm{f}^{i,*}$ is a symmetric threshold function of the estimation error. Subsequently, it is also assumed that this also valid for higher-order systems for the sake of convenience. The optimal estimator can then be stated similarly to Chapter 3 as

$$\mathsf{E}[x_k^i | Z^{k,i}] = \begin{cases} x_k^i, & \delta_k^i = 1, \\ (A^i - B^i L^i)\,\mathsf{E}[x_{k-1}^i | Z^{k,i}], & \text{otherwise,} \end{cases} \tag{8.11}$$

with $\mathsf{E}[x_0^i | z_0^i] = 0$ for $z_0^i = \varnothing$. As in Chapter 4, we remark that the first condition in above distinction of cases is extended to $\delta_k^i = 1 \wedge q_k^i = 1$ for the original communication network.

By defining the estimation error e_k^i by

$$e_k^i = x_k^i - \mathsf{E}[x_k^i | Z^{k-1,i}],$$

the determination of the optimal scheduling law can be regarded as a constrained Markov decision process [Alt99]. The Markov state $e_k \in \mathbb{R}^{n_i}$ evolves by the time-invariant difference equation model

$$e_{k+1}^i = g^i(e_k^i, \delta_k^i, w_k^i) = (1 - \delta_k^i) A^i e_k^i + w_k^i \tag{8.12}$$

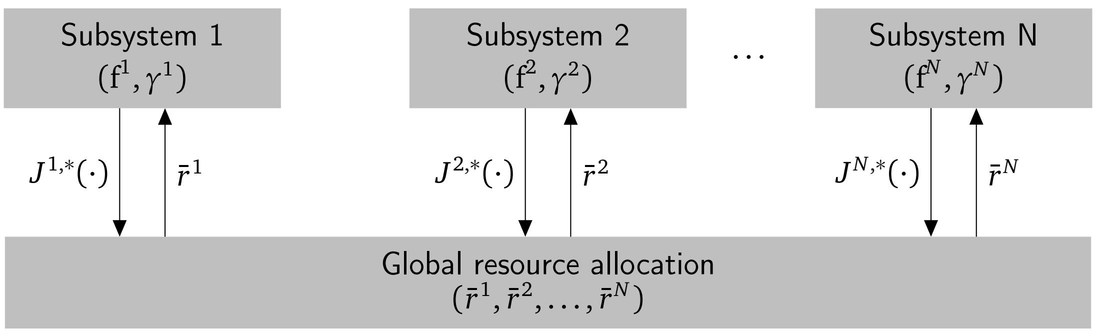


Figure 8.1: Bi-level design of the multi-loop event-triggered control system with a local event-triggered control design in each subsystem (Level 1) given by Eq. (8.8) and a global resource allocation (Level 2) given by Eq. (8.9).

with initial condition $e_0^i = x_0^i - \mathsf{E}[x_0^i]$.

Substituting the optimal control law $\gamma^{i,*}$ of Theorem 8.1 into the costs J^i, we obtain the following stochastic optimal control problem.

$$\inf_{\substack{\mathrm{f}^i \in \mathscr{U}^{\mathrm{M}} \\ r^i \leq \bar{r}^i}} \quad J^{i,\mathscr{S}} \tag{8.13}$$

with

$$J^{i,\mathscr{S}} = \lim_{T\to\infty} \frac{1}{T}\,\mathsf{E}\left[\sum_{k=0}^{T-1}(1-\delta_k^i)e_k^{i,\mathrm{T}}Q_e^i e_k^i\right]$$

where $Q_e^i = L^{i,\mathrm{T}}(Q_u^i + B^{i,\mathrm{T}}P^iB^i)L^i$. The set $\mathscr{U}^{\mathrm{M}}$ denotes the set of all Markov policies, which is defined as the set of all measurable maps from $\mathbb{R}^{n_i}$ to $\{0,1\}$. Considering the optimal control law $\gamma^{i,*}$ with the optimal estimator given by (8.11), we have the following relationship between the original optimization problem (8.8).

$$J^i(\gamma^{i,*},\cdot) = J^{i,\mathscr{S}}(\cdot) + \mathrm{tr}[P^iC^i].$$

The expression $\mathrm{tr}[P^iC^i]$ is constant and can therefore be omitted in the optimization problem (8.13).

In order to simplify the analysis of solving (8.13), we restate the technical assumption **(A2)** introduced in the previous chapter.

(A3) The event-triggering laws $\mathrm{f}^i \in \mathscr{U}^{\mathrm{M}}$ satisfies

$$\mathrm{f}^i(e_k^i) = 1, \qquad \|e_k^i\|_2 > M^i,$$

with $M^i > 0$ for all $i \in \{1,\dots,N\}$.

Assumption **(A3)** does not impose a severe restriction on the scheduling laws, as M^i can be chosen arbitrarily large for each subsystem. On the one hand, it follows from above assumption that the running cost $(1-\delta_k^i)e_k^{i,\mathrm{T}}Q_e^i e_k^i$ is uniformly bounded. On the other hand, it has been shown in [XH04] that the resulting Markov chain satisfies the ergodicity condition 3.1(4) in the textbook [HL89] that ensures the existence of a solution for the Average Cost Bellman Equation and the convergence of the value iteration, see also Appendix C. The constrained Markov decision process (8.13) under Assumption **(A3)** is formulated as an optimization problem without constraints by taking a Lagrangian approach that results in

$$\inf_{\mathrm{f}^i \in \mathscr{U}^{\mathrm{M}}} J^{i,\mathscr{S}} + \lambda r^i. \tag{8.14}$$

The non-negative weight λ can be regarded as the Lagrange multiplier of the constrained Markov decision process (8.13), [Alt99]. If λ is fixed, the optimization problem (8.14) becomes a standard stochastic optimal control problem that can be solved by value iteration as for example shown in [Ber07]. The resulting optimal scheduling policy is a threshold policy of the estimation error e_k^i.

Instead of the direct determination of the optimal solution together with the optimal Lagrange multiplier, we regard above optimization problem as the scalarization approach as in [BV04] of the corresponding multi-objective optimization problem with cost vector $[J^{i,\mathscr{S}}, r^i]$.

For any $\lambda \in [0,\infty)$, we obtain a Pareto optimal point in the $J^{i,\mathscr{S}}$-r^i-plane. It is easy to show that the coordinates $[J^{i,\mathscr{S},*}, r^{i,*}]$ are monotone in λ, i.e., $J^{i,\mathscr{S},*}$ is monotonically increasing in λ and $r^{i,*}$ is monotonically decreasing in λ.

From the continuity of the difference value function in λ that follows from chapter 3.5 in [HL89], and the absolute continuity of the stationary distribution of the f^i-controlled Markov chain we can conclude that $[J^{i,\mathscr{S},*}, r^{i,*}]$ is continuous in λ. Therefore, the scalarization approach in (8.14) yields the desired function $J^{i,*}(\bar{r}^i)$ that results from the set of Pareto optimal points.

8.2.3 Global Resource Allocation

The function $J^{i,*}(\bar{r}^i)$ is convex and monotonically decreasing in $\bar{r}^i$. This implies that the global optimization problem in the second level defined by (8.9) is a convex resource allocation problem. This is a well-studied optimization problem in the literature on convex optimization, for which many efficient solution algorithms exist, such as in [BV04; SS07].

8.2.4 Computational Complexity

For the calculation of the Pareto frontier in the first level of the bi-level approach as discussed in Section 8.2.2, we need to solve a dynamic program for different values of λ. This can be accomplished by a sequence of value iterations for subsystems with a moderate state dimension. For higher-order systems, there exist approximative methods, e.g., as developed in [Cog09a] that reduces the problem to a sequence of semidefinite programs. The solution of the algebraic Riccati equation for the optimal control law in (8.10) can be computed in polynomial-time. Therefore, it does not represent a computational burden for higher-dimensional systems. It should be noted that the Pareto frontier can be determined offline for each subsystem without having to take the system parameters of the communication network into account.

The solution of the global resource allocation problem with N design parameter in the second level discussed in Section 8.2.3 is a convex optimization problem, which can be solved efficiently. Hence, the optimal solution can also be computed for a large number of subsystems. However, if we encounter the situation in which subsystems are frequently attached and detached to the shared communication network, a global solution algorithm may not be desirable. Apart from other benefits, this will be the motivation for the introduction of a decentralized adaptive mechanism in order to determine the decentralized event-triggers in Chapter 9.

8.2.5 Asymptotic Optimality

In the following, we focus on the analysis of the approximative bi-level approach developed in subsection 8.2.1, when the number of subsystems, N, approaches infinity. A design approach is said to be *asymptotically optimal*, when the costs of the solution approaches the optimal costs arbitrarily close for a sufficiently large N. The relevant system parameters of subsystem i are summarized in the 4-tuple $\mathcal{K}^i = \{A^i, B^i, Q_x^i, Q_u^i\}$. In order to compare the control performance between the aggregated systems with increasing N, we scale the communication network accordingly such that the ratio c/N stays constant. It is also assumed that there is a finite number of subsystem classes denoted by $\bar{N}$, i.e., $\mathcal{K}^i \in \{\mathcal{K}^1, \dots \mathcal{K}^{\bar{N}}\}$ for all $i \in \{1, \dots, N\}$. The number of subsystems in a subsystem class is scaled with increasing N, such that their ratio between each other remains constant. This assumption implies that it suffices to consider subsystems of a subsystem class $\mathcal{K}^j$ separately with a fixed slot assignment of $c^j \leq c$. We also assume that we have *chaoticity in equilibrium* [Gra00] which corresponds to the exchange of the limits of time and the number of subsystems, i.e., $\lim_{k\to\infty} \lim_{N\to\infty} = \lim_{N\to\infty} \lim_{k\to\infty}$. In the following analysis, we therefore consider that the system is in its stationary regime and consider the limit when N approaches ∞. The subsequent theorem gives a statement about the optimality properties of the approximative design approach.

Theorem 8.2. *Let Assumption* **(A3)** *hold and let the stability condition in Eq. (7.7) be satisfied. Then, the solution of the bi-level optimization problem defined by (8.8) and (8.9) is asymptotically optimal with respect to the optimization problem given by (8.7).*

Proof. First, note that the optimal cost $\bar{J}^*$ resulting from the second level optimization given by (8.9) are a lower bound of the original optimization problem. This is because the hard rate constraint to be satisfied at each time step is relaxed by merely restraining the average total transmission rate. In the following, we show that the deviation from the optimal cost $\bar{J}^*$ because of the actual hard rate constraint becomes arbitrarily small for sufficiently large N. As already mention above, it suffices to restrict ourselves to a multi-loop control system composed of identical subsystems. It follows from the convexity of the function $J^{i,*}(\bar{r}^i)$ resulting from (8.8) that the assigned individual average transmission rates are identical. Therefore, the transmission rate is given by $r^i = c/N$ for each subsystem i, $N \in \{1, \dots, N\}$.

Next, we observe that the event of a request of a subsystem relatively to its most recent successful transmission can be regarded as a renewal process. This process is identical with a system without hard rate constraint as the blocking behavior of the arbitration mechanism is removed. Due to Theorem 7.1 from Chapter 7, it follows from Assumption **(A3)** and condition (7.7) that the resulting Markov chain characterizing the aggregate behavior is ergodic for any N. This implies that there exists a stationary distribution of the Markov state which implies that the renewal process is aperiodic and recurrent.

Therefore, we have

$$r^i = \lim_{k\to\infty} \mathsf{P}[\delta_k^i = 1 | \text{last transmission successful}].$$

Because of the law of large numbers, we conclude that for $N \to \infty$ and $k \to \infty$ with constant ratio $r^i = c/N$, we have

$$\lim_{N\to\infty} \lim_{k\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_k^i = r^i.$$

Hence, the ratio of subsystems that are reset converges weakly to 1. Therefore, by letting $\epsilon \to 0$ and $N \to \infty$, we converge to the optimal cost $\bar{J}^*$ resulting from (8.9) by continuity of $\bar{r}^i$ with respect to $J^{*,i}$. This implies that the design method described by (8.8) and (8.9) is asymptotically optimal. $\square$

8.3 Numerical Results

In this section, the proposed event-triggered approach is evaluated and compared with the time-triggered and the centralized approach. In order to facilitate the presentation, we restrict our attention to scalar subsystems.

First, suppose we have identical subsystems with parameters

$$\mathscr{K}^i = \mathscr{K} = (1, 1, 1, 0).$$

The communication network has a ratio between available transmission slots and the number of subsystems of $c/N = 0.2$. The Pareto optimal cost region $[J^i, r^i]$ for a subsystem with parameters $\mathscr{K}$ including the rate constraint is drawn in Fig. 8.2. We observe that J^i is a decreasing and convex function with respect to r^i. For identical subsystems, there is a substantial simplification in the global resource allocation problem performed in the second level as all subsystems attain the same transmission rate, i.e., the optimal transmission rate is given by $r^{i,*} = 0.2$. The optimal cost point is attained at $[J^{i,*}, r^{i,*}] = [1.54, 0.2]$ by an event-triggered scheduling policy $\mathrm{f}^{i,*}$ that is given by $\delta_k^i = \mathbb{1}_{\{[-1.7,1.7]\}^c}(e_k^i)$. The optimal control law gain L^i is given by 1.

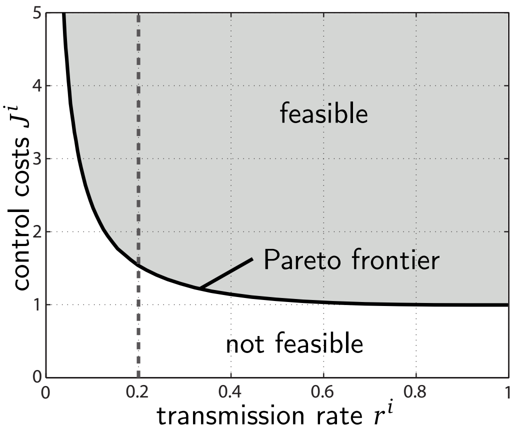


Figure 8.2: Pareto frontier of a subsystem with system parameters $\mathscr{K} = (1, 1, 1, 0)$. The vertical line indicates the rate constraint.

Figure 8.3 compares the cost of the decentralized event-triggered scheme with the optimal time-triggered scheme and the optimal centralized scheduling shows the cost per subsystem for various numbers of identical subsystems N with c/N being constantly at 0.2. The costs for $N \in \{5, 25, 100, 250, 500\}$ are determined through Monte Carlo simulations with a time horizon of $T = 10\,000$. The optimal control law for both the optimal time-triggered scheme and the optimal centralized scheme are given by $u_k^i = -L^i \,\mathsf{E}[x_k^i | Z^{k,i}]$ with $L^i = 1$. In the optimal time-triggered scheme, time slots for transmission are assigned successively. Subsystems transmit information periodically with transmission period $\frac{N}{c}$, where we assume that N is a multiple of 5. In the case of identical subsystems, the optimal centralized scheduler selects at each time step k the c subsystems with maximum magnitude $|e_k|$ whose feedback loop are then closed. Such kind of protocol can be realized by prioritize the medium access through $|e_k|$ which has also been done in [WYB02]. It should be noted that this scheduler can be regarded as a lower bound on the performance that can be achieved over the communication networks. In the case of heterogeneous multi-dimensional systems, it remains an open problem how to realize centralized schedulers without gathering the state information of all subsystems.

We observe in Fig. 8.3 that the cost of the optimal decentralized scheduling algorithm approximates this lower bound very closely and outperforms the optimal time-triggered scheme significantly. On the other hand, it can be seen that the costs converge to the asymptotic costs for $N \to \infty$ very rapidly. Already for $N \geq 100$, the performance gap is less than 10%.

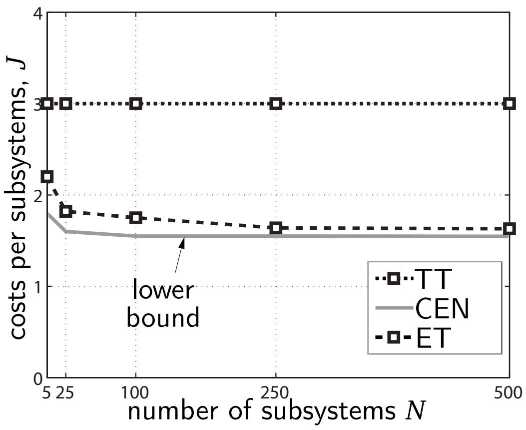


Figure 8.3: Numerical comparison of time-triggered (TT), event-triggered (ET) and centralized schemes for a multi-loop control system with homogeneous subsystems $\mathscr{K} = (1,\ 1,\ 1,\ 0)$ and $c/N = 0.2$.

Next, we consider a heterogeneous system, where we have two different kinds of subsystems occurring at the same amount. The system parameters are $\mathscr{K}^1 = (1.25, 1, 1, 0)$ and $\mathscr{K}^2 = (0.75, 1, 1, 0)$ and the communication network has a ratio of $c/N = 0.5$. We note that the stability condition (7.7) in Theorem 7.1 is satisfied for the underlying subsystems.

Having obtained the Pareto curves from the first level optimization for both subsystems sketched in Fig. 8.4, the resource allocation problem given by (8.9) determines the optimal rate pair. The dashed line in Fig. 8.4 depicts the mean cost per subsystem J as a function of r^1 for $N = 2$ without collisions. It can be seen that the total cost J is convex with respect to r^1 and it is minimized at the rate pair $[r^1,\ r^2] = [0.6,\ 0.4]$ taking a value of 1.07. The optimal control gain is given by $L^i = A^i$ for both subsystems and the scheduling laws are threshold policies, where $\delta_k^1 = \mathbb{1}_{\{[-0.5,0.5]\}^c}(e_k^1)$ for $\mathscr{K}^1$ and $\delta_k^2 = \mathbb{1}_{\{[-0.95,0.95]\}^c}(e_k^2)$ for $\mathscr{K}^2$.

Concerning the performance in the presence of the shared network, we consider the mean costs J depicted in Fig. 8.5 for $N \in \{2, 10, 50, 100, 250, 500\}$. The optimal time-triggered scheme involves a brute-force search over all possible combinations of transmission times. To

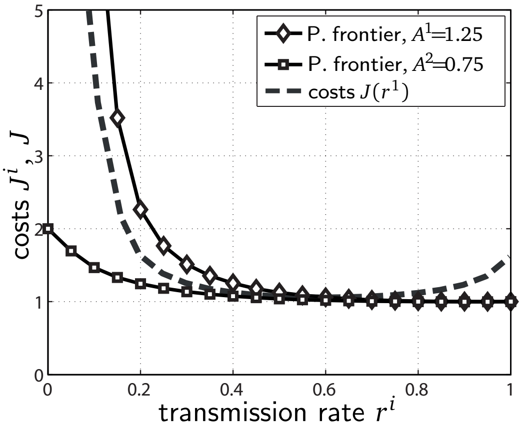


Figure 8.4: Solid lines: Pareto frontiers of two different subsystems with system parameters $\mathscr{K}^1 = (1.25, 1, 1, 0)$ and $\mathscr{K}^2 = (0.75, 1, 1, 0)$. Dashed line: Total cost $J(r^1) = \frac{1}{2}(J^1(r^1)+J^2(r^2))$ and constraint $\frac{1}{N}(r^1 + r^2) \leq 0.5$. The optimal rate pair is given at $[r^1,\ r^2] = [0.6,\ 0.4]$ with total cost $J = 1.07$ for the two subsystems without collisions.

keep this combinatorial problem numerically tractable, we restricted the admissible transmission scheme to be periodical for subsystems $\mathscr{K}^2$. The optimal periodical transmission scheme is then given by $[\delta_0^1, \delta_1^1, \delta_2^1, \ldots] = [1, 1, 0, \ldots]$ and $[\delta_0^2, \delta_1^2, \delta_2^2, \ldots] = [0, 0, 1, \ldots]$ with period 3. A lower bound is given by $J = 1.07$ from the relaxed optimization problem that assumes no contention. As can be regarded from Fig. 8.5, this lower bound is approached with a gap of less than 3% for $N \geq 100$ and the time-triggered scheme is outperformed for every N.

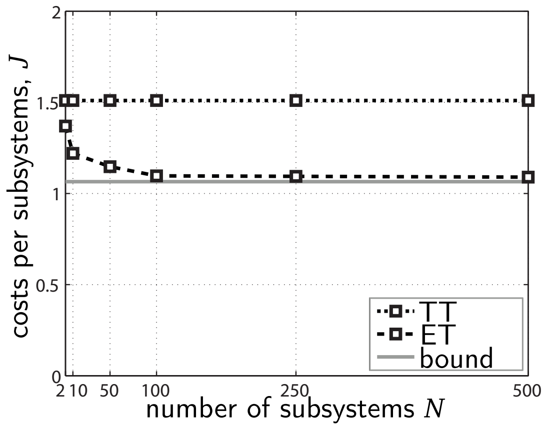


Figure 8.5: Numerical comparison of time-triggered (TT) and event-triggered (ET) schemes for a multi-loop control system with heterogeneous subsystems of two classes $\mathscr{K}^1$ and $\mathscr{K}^2$ and $c/N = 0.5$.

8.4 Summary

This chapter has shown that decentralized event-triggered control constitutes an attractive design approach for multiple feedback loop systems over a common communication network. The proposed design method manages to establish a compromise between computational complexity and overall performance that circumvents to take the complex behavior of the contention-based network into account. It also demonstrates how the theoretical results from Part I of this thesis can be applied for the development of multiple event-triggered controllers taking into account a shared resource.

Despite the decreased predictability and a close interaction between control and communication in contrast to time-triggered control schemes, the gain from the proposed event-triggered control scheme is an increased level of robustness and a significant improvement on the control performance for the shown examples.

8.5 Bibliographical Notes

The contribution of this chapter is based on the work in [MH11; MH12c; MH14]. The fact that the certainty equivalence property holds for the optimal solution of the local optimization problem (8.8) with an average rate constraint is shown in [MH13a].

9

Price Exchange Mechanism for Event-Triggered Control

The bi-level design method developed in Chapter 8 needs to solve a global resource allocation problem that must incorporate information of every subsystem sharing the common communication resource. This might be inconvenient due to its difficulty of implementation when the number of control loops is large. It also lacks of flexibility in case of frequent changes in the system as the global resource allocation needs to be rerun completely whenever subsystems are attached or detached from the communication network. This motivates us to design a decentralized version of the resource allocation algorithm in this chapter. We develop a dual price exchange mechanism, in which the distributed self-regulating event-triggers adapt their average communication rate to accommodate the global resource constraint. This will be achieved by analyzing the dual problem of the constrained average-cost problem stated in the previous chapter. As already indicated in the introduction of Chapter 6, we will benefit from the underselective nature of the average-cost criterion, since it allows us to learn the optimal pricing variable during execution.

The introduction of the pricing mechanism yields however another source of dynamics intertwined with the overall multi-loop control system that requires a careful treatment. By using a time-scale separation approach that decouples the process dynamics from the communication rate adaptation, we will be able to derive stability conditions and we will establish almost-sure convergence properties of the distributed event-triggered scheme. The stability and convergence issues are investigated by using the theory of stochastic stability for Markov chains and methods from stochastic approximation.

Aside from the development of a decentralized resource allocation algorithm for the multi-loop system, this chapter clearly demonstrates the benefits of event-triggered sampling with regard to the ability of adaptation that is crucial for the implementation of distributed mechanisms.

This chapter is organized as follows. In Section 9.1, we review the multi-loop control system and introduce the extended architecture of the communication network that enables an adaptation mechanism for the schedulers. In Section 9.2, the pricing mechanism for the decentralized event-triggered control system is developed and its stability, convergence, and numerical properties are analyzed in Section 9.3. Numerical simulations conducted in Section 9.4 show the effectiveness of the approach and illustrate the convergence properties.

9.1 Problem Statement

The aim of this section is two-fold. First, we review the multi-loop control system and the design problem introduced in the previous section. Second, we introduce an extended architecture of the communication network that enables an adaptation mechanism of the event-triggered schedulers.

Figure 9.1 depicts the networked control system (NCS) under consideration that shows N independent control subsystems whose feedback loops are connected through a shared communication network. A control subsystem i consists of a process $\mathscr{P}^i$, a controller $\mathscr{C}^i$ that is implemented at the actuator and a sensor $\mathscr{S}^i$. The process $\mathscr{P}^i$ is given by a controlled time-homogeneous Markov chain with state x_k taking values in $\mathbb{R}^{n_i}$ and evolving by the following difference equation

$$x^i_{k+1} = A^i x^i_k + B^i u^i_k + w^i_k, \tag{9.1}$$

where $A^i \in \mathbb{R}^{n_i \times n_i}, B^i \in \mathbb{R}^{n_i \times m_i}$. The control input u^i_k is taking values in $\mathbb{R}^{m_i}$. The system noise w^i_k takes values in $\mathbb{R}^{n_i}$ at each k and is i.i.d. with $w^i_k \sim \mathscr{N}(0, C^i)$ being zero-mean Gaussian distributed with covariance matrix C^i. The initial states x^i_0, $i \in \{1, \dots, N\}$, have a distribution whose density function is symmetric around its mean value $\mathsf{E}[x^i_0]$ and has finite second moment.

At any time k the scheduler $\mathscr{S}^i$ situated at the sensor decides, whether a transmission slot should be requested to transmit the current state of subsystem i to the controller $\mathscr{C}^i$. Therefore, an event occurs within a subsystem i at time k when a transmission slot is requested. Further, it is assumed that control inputs may not be constant in between of successful transmissions and the controller $\mathscr{C}^i$ may adjust the control inputs based on past updates. Due to bandwidth limitations the number of transmission slots denoted by c is constrained and event-triggers must be designed at the sensors that judge the importance of transmitting an update to the corresponding controller. In order to make the problem setting non-trivial we assume that

$$1 \leq c \leq N.$$

If there are more requests than available transmission slots at time k, then the arbitration mechanism within the communication system selects c subsystems that may transmit information. All other subsystems are blocked and are informed instantaneously that their request has been rejected. The arbitration mechanism does not prioritize subsystems, i.e., in case of arbitration, the subsystems are chosen with identical probability. The request for a transmission of the ith subsystem at time k is defined by the variable δ_k^i which takes the following values.

$$\delta_k^i = \begin{cases} 1, & \text{request for transmission,} \\ 0, & \text{idle.} \end{cases}$$

We represent the arbitration mechanism to resolve contention by the random variable q_k^i taking values

$$q_k^i = \begin{cases} 1, & \text{allow transmission,} \\ 0, & \text{block transmission.} \end{cases}$$

The probability distribution of $[q_k^1, \ldots, q_k^N]$ conditioned on the requests δ_k^i with $i \in \{1, \ldots, N\}$ is time-invariant and has the following property.

$$\mathsf{P}[q_k^i = 1 | \delta_k^i, i \in \{1, \ldots, N\}] = \begin{cases} 1, & \sum_{i=1}^N \delta_k^i \leq c, \\ \frac{c}{\sum_{i=1}^N \delta_k^i}, & \text{otherwise,} \end{cases} \tag{9.2}$$

for subsystem i with $\delta_k^i = 1$ and

$$q_k^1(\omega) + \cdots + q_k^N(\omega) = c$$

for all sample paths $\omega \in \Omega$ for which $\delta_k^1 + \cdots + \delta_k^N \geq c$. The received data at the controller $\mathscr{C}^i$ at time k is denoted by z_k^i and is defined by

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1 \wedge q_k^i = 1 \\ \varnothing, & \text{otherwise} \end{cases} \tag{9.3}$$

Each subsystem $i \in \{1, \ldots, N\}$ has an individual cost function J^i given by the linear quadratic average-cost criterion

$$J^i = \limsup_{T \to \infty} \frac{1}{T} \mathsf{E}\left[\sum_{k=0}^{T-1} x_k^{i,\mathrm{T}} Q_x^i x_k^i + u_k^{i,\mathrm{T}} Q_u^i u_k^i\right]. \tag{9.4}$$

The weighting matrix Q_x^i is positive definite and Q_u^i is positive semi-definite for each $i \in \{1, \ldots, N\}$. We assume that the pair (A^i, B^i) is stabilizable and the pair $(A^i, Q_x^{i,\frac{1}{2}})$ is detectable with $Q_x^i = (Q_x^{i,\frac{1}{2}})^{\mathrm{T}} Q_x^{i,\frac{1}{2}}$.

As it will take a central role in the subsequent analysis, we define the individual request rate r^i of the ith subsystem by

$$r^i = \limsup_{T \to \infty} \frac{1}{T} \mathsf{E}[\sum_{k=0}^{T-1} \delta_k^i], \tag{9.5}$$

which also has the form of an average-cost criterion.

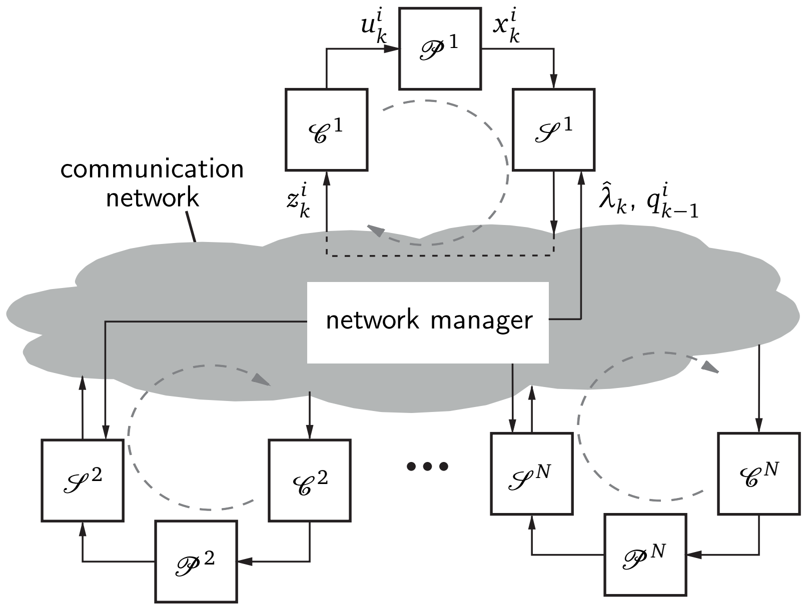


Figure 9.1: System model of the NCS with N control systems closed over a shared communication network with processes $\mathscr{P}^1,\ldots,\mathscr{P}^N$, sensors $\mathscr{S}^1,\ldots,\mathscr{S}^N$ and controllers $\mathscr{C}^1,\ldots,\mathscr{C}^N$. The network manager sends an acknowledgement q_{k-1}^i to each subsystem i and broadcasts the variable $\hat{\lambda}_k$.

It is assumed that the sensor and the controller of the ith subsystem merely have knowledge of the local system parameters. These are A^i, B^i, C^i, the distribution of x_0, and Q_x^i, Q_u^i of Eq. (9.4).

The control law $\gamma^i = \{\gamma_0^i, \gamma_1^i, \ldots\}$ reflecting the behavior of controller $\mathscr{C}^i$ is described by causal mappings γ_k^i of the past observations for each time step k, i.e.,

$$u_k^i = \gamma_k^i(Z^{k,i}), \tag{9.6}$$

where $Z^{k,i}$ is the observation history until time k of subsystem i. We distinguish between two classes of schedulers $\mathrm{f}^i = \{\mathrm{f}_0^i, \mathrm{f}_1^i, \ldots\}$ resulting from two types of network managers. In the first case, the network manager broadcasts a fixed parameter λ initially and the *static event-triggered* scheduler is then given by

$$\delta_k^i = \mathrm{f}_k^{i,\lambda}(X^{k,i}, Q^{k-1,i}), \tag{9.7}$$

where $X^{k,i}$ is the state history of subsystem i. It should be remarked that we will usually omit λ for notational convenience. In the second case, the network parameter $\hat{\lambda}_k$ changes over time k and the scheduler adapts its law w.r.t. to $\hat{\lambda}_k$, i.e.,

$$\delta_k^i = \mathrm{f}_k^{i,\hat{\lambda}_k}(X^{k,i}, Q^{k-1,i}). \tag{9.8}$$

The parameter $\hat{\lambda}_k$ itself is given by a causal mapping f_k of the past transmission history, i.e.,

$$\hat{\lambda}_k = \pi_k(\delta_0^1, \ldots, \delta_0^N, \ldots, \delta_k^1, \ldots, \delta_k^N).$$

The mapping π_k represents the adaptation mechanism of the network manager to the current traffic and will be specified later in more detail. The laws described by Eq. (9.8) are referred to as *adaptive event-triggered* schedulers.

The design objective is to find the optimal control laws γ^i and optimal scheduling laws f^i, $i \in \{1,\ldots,N\}$ that minimize the social cost given as an average-cost criterion. The social cost is defined by the sum of the individual costs J^i of each subsystem. Therefore, the optimization problem can be summarized as follows.

$$\inf_{\substack{(\mathrm{f}^i,\gamma^i)\in\mathscr{U}^{\mathrm{ET}}\\ i\in\{1,\ldots,N\}}} \sum_{i=1}^{N} J^i, \tag{9.9}$$

where $\mathscr{U}^{\mathrm{ET}}$ is the set of admissible policy pairs that are defined either by Eqs. (9.6), (9.7) or Eqs. (9.6), (9.8).

9.2 Design via Distributed Optimization

This section focuses on the synthesis of the distributed event-triggered control system that addresses the solution of the optimization problem (9.9). Inspired by the bi-level design approach introduced in the previous chapter, we will study the synthesis problem with a relaxed communication constraint stated in Section 9.2.1. But instead of analyzing the primal problem, we investigate the relaxed problem from the perspective of its dual problem. This is the aim of Section 9.2.2 that applies ideas from dual decomposition and adaptive MDPs in order to develop a distributed approach of the overall problem. This is in contrast to the approach developed in Chapter 8 that needed to solve a global resource allocation problem. A Lagrange approach is taken to formulate the dual problem of the relaxed problem. It enables us to derive a dual price exchange mechanism that broadcasts a price for the resource to each subsystem. An adaptive sample-path algorithm is proposed in Section 9.2.3 that estimates the average total transmission rate to approximate the pricing gradient.

9.2.1 Problem Relaxation

We have already observed in Chapter 8 that determining the optimal event-based control system that solves (9.9) is a hard problem though the coupling between subsystems appears solely in the shared resource. In order to still obtain a systematic approach to find the event-triggered controllers that minimize the social cost in (9.9), we follow the bi-level approach introduced in the previous chapter. Thereby, the hard constraint that c transmissions are allowed at maximum at each time k is weakened and we require merely that the total average request rate is upper bounded by c. With the definition of the individual request rate in Eq. (9.5), the total average request rate is defined as

$$y = \sum_{i=1}^{N} r^i. \tag{9.10}$$

The relaxed optimization problem is given by the following constrained MDP.

$$\inf_{\substack{(\mathrm{f}^i,\gamma^i)\in\mathscr{U}^{\mathrm{ET}}\\ i\in\{1,\ldots,N\}}} \sum_{i=1}^{N} J^i \qquad \text{s.t.} \quad y \le c. \tag{9.11}$$

As in Chapter 6, it should be noted that optimization problems with an average-cost criterion are underselective [HL89]. Therefore, there may exist two policies, which differ completely with respect to their performance in the first k time steps, but eventually converge to the same stationary behavior and therefore yield the same average cost in the long-run. Hence, the optimization problem does not distinguish between these two policies. For the purpose of this chapter, we will consider this feature as an advantage, as it allows us to design an adaptation mechanism for each subsystem that learns the appropriate transmission rate that achieves the optimal performance.

9.2.2 Lagrange Approach

In the following, we define the Lagrangian function, introduced in [Alt99] for constrained MDPs, by

$$\mathscr{L}(\mathrm{f}^1,\dots,\mathrm{f}^N,\gamma^1,\dots,\gamma^N,\lambda)=\sum_{i=1}^{N}J^i+\lambda(y-c)$$

With this, we can rewrite the optimization problem given by (9.11) into the corresponding dual problem

$$\sup_{\lambda\geq 0}\ \inf_{\substack{(\mathrm{f}^i,\gamma^i)\in\mathscr{U}^{\mathrm{ET}}\\ i\in\{1,\dots,N\}}}\ \mathscr{L}(\mathrm{f}^1,\dots,\mathrm{f}^N,\gamma^1,\dots,\gamma^N,\lambda). \tag{9.12}$$

The Lagrange multiplier λ can be interpreted as a penalty or price for the transmission rate. Therefore, we sometimes refer to λ as the communication penalty or price. It should be remarked that for the underlying problem strong duality holds, as it is shown in Chapter 8 that the primal problem is a convex optimization problem and Slater's condition holds for $c>0$. By reordering the terms in $\mathscr{L}$ and using the definition of y in Eq. (9.10), we obtain

$$\mathscr{L}(\cdot,\lambda)=\sum_{i=1}^{N}(J^i+\lambda r^i)-\lambda c.$$

For a fixed $\lambda\geq 0$ and each subsystem i, the values of J^i and r^i only depend on the choice of the local control law γ^i and the local scheduling law f^i. Therefore, the minimization in Eq. (9.12) can be stated as a separate local minimization within each subsystem for a fixed λ, i.e.,

$$\inf_{(\mathrm{f}^1,\gamma^1)\in\mathscr{U}^{\mathrm{ET}}}(J^1+\lambda r^1)+\cdots+\inf_{(\mathrm{f}^N,\gamma^N)\in\mathscr{U}^{\mathrm{ET}}}(J^N+\lambda r^N).$$

From the above formulation, we observe that the task of the network manager is to broadcast the price λ in order to coordinate the local optimization problems.

The local optimization problems in the above expression are identical with the joint design of control and scheduling for a single-loop system with a communication penalty in the feedback loop studied in Part I. Therefore, the results from Chapters 2-6 can be used to obtain a precise characterization of optimal local solutions for a fixed λ. In the following, we summarize the results. It is shown in Chapters 2, 4, and 6 that the certainty equivalence controller is optimal. Therefore, the control law is given by

$$u_k^i = \gamma_k^{i,*}(Z^{k,i}) = -L^i\,\mathsf{E}[x_k^i|Z^{k,i}], \tag{9.13}$$

where the linear gain L^i can be calculated equivalently to Eq. (6.5). The stabilizability and detectability assumptions introduced in Section 9.1 guarantee that the stabilizing gain L^i resulting from Eq. (6.5) exists. When taking into account that the distributions of the noise variables are symmetric, it is shown in Theorem 3.1 in Chapter 3 for first-order systems that the optimal scheduling law $\mathrm{f}^{i,*}$ is a symmetric threshold function of the estimation error. Subsequently, it is also assumed that this also valid for higher-order systems for the sake of convenience. The optimal estimator can be written as

$$\hat{x}_{k|k}^{\mathscr{C},i} = \mathsf{E}[x_k^i|Z^{k,i}] = \begin{cases} x_k^i, & \delta_k^i = 1 \wedge q_k^i = 1, \\ (A^i - B^iL^i)\,\mathsf{E}[x_{k-1}^i|Z^{k-1,i}], & \text{otherwise,} \end{cases}$$

with $\mathsf{E}[x_0^i|z_0^i] = \mathsf{E}[x_0^i]$ for $\delta_0^i = 0$. It should be noted that in the absence of contention, which applies in the design stage, the first case in above estimator must be replaced by $\delta_k^i = 1$. It should be noted that the optimal control law is independent of λ. Therefore, the control law can be fully implemented prior to execution without additional knowledge. It also justifies the fact that the network manager need not broadcast the price λ to the controllers $\mathscr{C}^i$, $i \in \{1,\dots,N\}$.

By defining the estimation error

$$e_k^i = x_k^i - \mathsf{E}[x_k^i|Z_{k-1}^i],$$

the remaining problem to determine the optimal scheduling law $\mathrm{f}^{i,\lambda}$ can be cast as the following MDP with state e_k^i, whose solution has been discussed in Section 6.3.

$$\inf_{\mathrm{f}^{i,\lambda}} \lim_{T\to\infty} \frac{1}{T}\,\mathsf{E}\left[\sum_{k=0}^{T-1}(1-\delta_k^i)e_k^{i,\mathrm{T}}Q_e^i e_k^i + \lambda\delta_k^i\right] + \mathrm{tr}[P^iC^i], \tag{9.14}$$

where $Q_e^i = L^{i,\mathrm{T}}(Q_u^i + B^{i,\mathrm{T}}P^iB^i)L^i$ and e_k^i is described as a δ_k^i-controlled Markov chain evolving by the following difference equation

$$e_{k+1}^i = (1-\delta_k^i)Ae_k^i + w_k \tag{9.15}$$

with initial condition $e_0 = x_0 - \mathsf{E}[x_0]$. The additional term $\mathrm{tr}[P^iC^i]$ is constant and can be omitted from the optimization in (9.14). Optimal policies are therefore stationary mappings of the estimation error e_k^i. Under Assumption **(A3)** introduced in Section 8.2.2 on the admissible policies $\mathrm{f}^{i,\lambda}$, the above optimization problem can be solved by value iteration for each λ.

In the remainder of this chapter, we restrict our attention to the aggregate error state given by the individual errors e_k^i, which is defined as

$$e_k = [e_k^1,\dots,e_k^N]^{\mathrm{T}}.$$

This is because the state x_k^i in subsystem i can be viewed as an isolated stable process controlled by e_k^i, when considering the closed-loop system. This is discussed in more detail in Section 6.4.

The resulting optimal scheduling law is a symmetric threshold policy and takes the following form for a first-order subsystem

$$\delta_k^i = \mathrm{f}^{i,\lambda}(e_k^i) = \mathbb{1}_{\{[-d^i(\lambda), d^i(\lambda)]\}^c}(e_k^i) \tag{9.16}$$

parameterized by the threshold d^i that depends on the price λ. This is because the value iteration outputs a sequence of even and radially non-decreasing value functions [HMY08]. By varying $\lambda \in (0,\infty)$ in Eq. (9.14) different pairs of individual costs J^i and transmission rates r^i are attained by the optimal $\mathrm{f}^{i,\lambda}$ for each subsystem $i \in \{1,\dots,N\}$. In fact, it has been shown in the previous chapter that the relation between optimal J^i and $r^i \in (0,1)$ is described by a decreasing and strictly convex function, which is denoted by $J^i(r^i)$ in the following. Subsequently, we also assume that $J^i(r^i)$ is twice continuously differentiable and its curvature is bounded away from zero on $(0,1)$. Then, the optimization in (9.12) can be rewritten as the dual formulation of a network utility maximization problem [SS07] with a single link, i.e.,

$$\max_{\lambda \geq 0} \min_{r^i, i\in\{1,\dots,N\}} \sum_{i=1}^{N} J^i(r^i) + \lambda(y - c). \tag{9.17}$$

It is well known that this problem has a unique solution for assigning the optimal transmission rates r^i, $i \in \{1,\dots,N\}$. As there is only one link, it is also clear that there is a unique λ^* that solves (9.17). In the following, we aim at developing a distributed gradient method that finds the optimal λ^*. Let $g(\lambda)$ be defined as

$$g(\lambda) = \min_{r^i, i\in\{1,\dots,N\}} \sum_{i=1}^{N} J^i(r^i) + \lambda(y - c).$$

In [SS07], it is shown that the derivative of $g(\lambda)$ with respect to λ is obtained by

$$\frac{\partial g(\lambda)}{\partial \lambda} = y - c.$$

Therefore, the continuous-time gradient algorithm to solve the dual problem is given by the following ordinary differential equation (ODE).

$$\dot{\lambda}(t) = [y(t) - c]_\lambda^+ \tag{9.18}$$

with an arbitrary initial value $\lambda(0)$. The projection guarantees that the penalty $\lambda(t)$ remains non-negative at all times t and is defined as

$$[\xi]_\lambda^+ = \begin{cases} \xi, & \lambda > 0, \\ \max(\xi, 0), & \text{otherwise.} \end{cases}$$

The total request rate $y(t)$ is defined as

$$y(t) = \sum_{i=1}^{N} r^i(t) \tag{9.19}$$

where $r^i(t)$ is the average request rate defined in Eq. (9.5) assuming that the price $\lambda(t)$, the controller $\gamma^{i,*}$ given by Eq. (9.13) and the scheduling law $\mathrm{f}^{i,\lambda(t)}$ obtained from (9.14) are used. Taking into account the uniqueness of the optimal λ^* for the considered problem, it is shown in [SS07] that the differential equation (9.18) converges to λ^* for any initial condition $\lambda(0)$.

In the following, we focus on the discrete-time version of Eq. (9.18) given by

$$\lambda_{k+1} = [\lambda_k + \beta_k(y_k - c)]^+ \tag{9.20}$$

with an arbitrary initial value λ_0 and step size $\beta_k > 0$ for all k. Similar to the continuous-time case, the total request rate is defined as

$$y_k = \sum_{i=1}^{N} r_k^i \tag{9.21}$$

where r_k^i is the average request rate defined in Eq. (9.5) assuming the controller $\gamma^{i,*}$ and the scheduling law f^{i,λ_k} are used. It should be noted that the discrete-time algorithm (9.20) can be viewed as a numerical approximation of the ODE given by Eq. (9.18)

The complete algorithm can be summarized as a dual price exchange mechanism: after broadcasting the price λ_k by a central network manager to all subsystems at time step k, each subsystem adjusts its scheduling policy according to the local optimization problem (9.14) with λ_k as the dual price. It is shown in [LL99] that the algorithm converges to the optimal price λ^* for sufficiently small β_k.

The above gradient method is completely decoupled from the actual dynamics of the subsystems. Hence, the optimal price λ^* can be calculated prior to the execution of the control process and is spread to the subsystems that use the stationary event-trigger $\delta_k^i = \mathrm{f}^{i,\lambda^*}(e_k^i)$ for $k \geq 0$.

9.2.3 An Adaptive Sample-Based Algorithm

The drawback of the gradient method in Eq. (9.20) is obvious. The total average transmission rate y_k is not exactly known at time step k, as it is neither efficient to gather information about every individual transmission rate r^i from each subsystem at the central network manager, nor it is feasible to determine y_k through its empirical mean by letting $T \to \infty$. Instead, we consider an estimate $\hat{y}_k$ of the total request rate over a window length $T_{0,k}$ to approximate the gradient in (9.20) at time k. While estimating $\hat{y}_k$, the price remains constant. Hence, updates of the estimated price $\hat{\lambda}_k$ occur after an estimation period $T_{0,k}$ which may not be uniform. Therefore, the mapping π_k representing the network manager introduced in Section 9.1 is determined by

$$\hat{\lambda}_{k+T_{0,k}} = [\hat{\lambda}_k + \beta_k(\hat{y}_k - c)]^+ \tag{9.22}$$

with an initial value $\hat{\lambda}_0 \in \mathbb{R}_{\geq 0}$, and $k \in \mathbb{T}$, where the index set $\mathbb{T}$ defines the set of update times, i.e., $\mathbb{T} = \{\sum_{\ell=0}^{l} T_{0,\ell}\}_l$. In between updates, while $\hat{\lambda}_k$ remains constant, the total request rate y_k is estimated by its empirical mean during that period, i.e.,

$$\hat{y}_k = \frac{1}{T_{0,k}} \sum_{\ell=k}^{k+T_{0,k}-1} \sum_{i=1}^{N} \delta_\ell^i, \quad k \in \mathbb{T}.$$

We will consider two different choices for the step size β_k and the window length $T_{0,k}$ in the following. The next subsection assumes that the step size and window length are constant, whereas we assume in Section 9.3.2 that the step size decreases, while the window length increases in time.

Figure 9.2 summarizes the mechanism of the complete adaptive event-triggered control system by illustrating one particular subsystem and its interplay with the network manager. In contrast to the design mechanism described at the end of Section 9.2.2, the price is not determined prior to execution, but is continuously estimated within the network manager for every time step k.

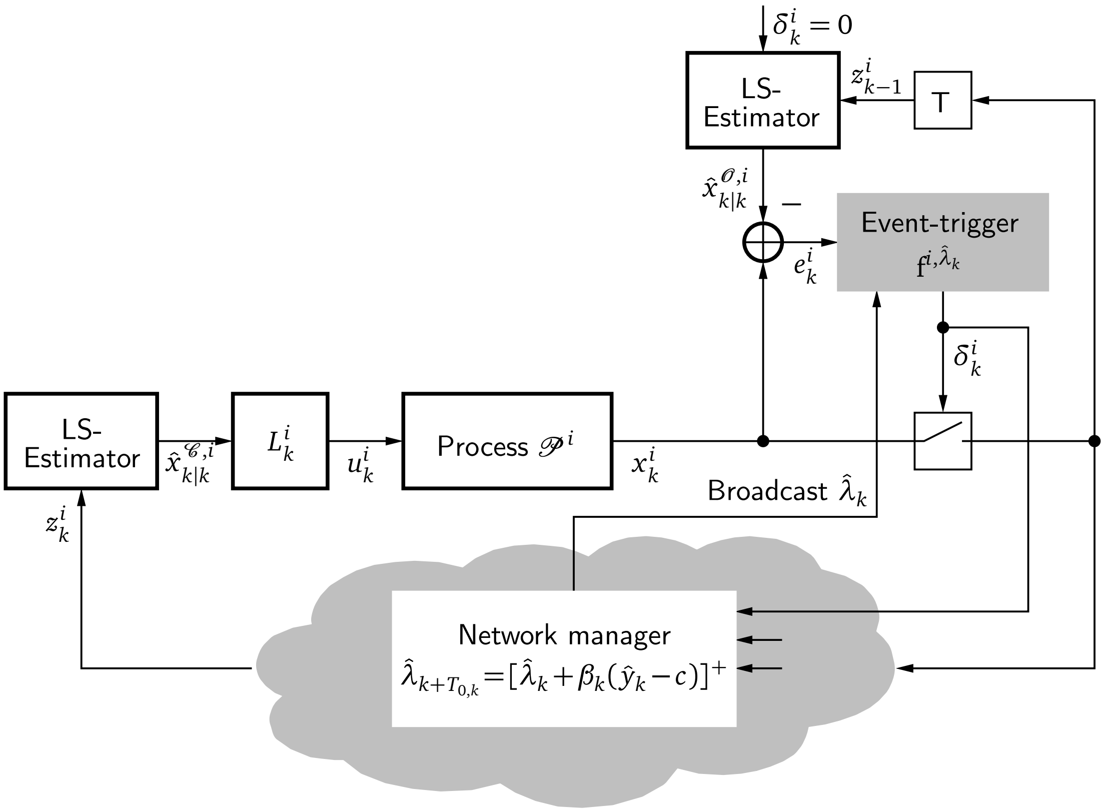


Figure 9.2: Complete structure of the adaptive event-triggered control system for subsystem i. The event-triggered scheduler $\mathrm{f}^{i,\hat{\lambda}_k}$ adapts its law according to the price $\hat{\lambda}_k$ which is broadcasted to each subsystem by a network manager. The element denoted by T corresponds to a one-step delay element, whereas the least-squares estimate that outputs $\hat{x}^{\mathscr{O},i}_{k|k} = \mathrm{E}[x_k|Z^{k-1}, \delta_k = 0]$ is a copy of the least-squares estimator at the controller.

9.3 Analysis of the Adaptive Event-Triggered Control System

Section 9.3.1 and9.3.2 address stability and convergence properties of the adaptive event-triggered scheme, respectively. The implementation of the algorithm is discussed in Section 9.3.3.

9.3.1 Stability

Until now, stability was a minor issue, as no contention among subsystems has been considered due to the relaxed problem formulation that made the problem tractable. In this section, we focus on stability properties of the aggregate system with an adaptive sample-based event-triggered system in the presence of contention represented by the variable q_k^i for each subsystem i. We consider the adaptive sample-based algorithm in Eq. (9.22) with constant step size β and constant window length T_0. Under this assumption, the resulting process can be viewed as a T_0-sampled time-homogeneous Markov chain. The stability notion used here is given by positive Harris recurrence, see Definition B.9 in the appendix. To prove positive Harris recurrence for Markov chains with an uncountable state space, we use Foster's criterion defined in Theorem B.3.

In order to simplify the following analysis, we introduce the following two technical assumptions. The first assumption is an adaptation of Assumption **(A3)** introduced in Chapter 8 for a varying price λ.

(A4) For any $\lambda \in \mathbb{R}_{\geq 0}$, the scheduling policy of subsystem i satisfies $\mathrm{f}^{i,\lambda}(e_k^i) = 1$ for $\|e_k^i\|_2 > M^i$ for some arbitrary M^i, $i \in \{1, \ldots, N\}$, where M^i may depend on λ.

(A5) The function $M^i(\lambda)$ grows asymptotically at most linear in λ, i.e., $M_i \in \mathcal{O}(\lambda)$.

These assumptions do not put severe restrictions on the admissible scheduling policies. On the one hand, the bound M^i in Assumption **(A4)** may be chosen arbitrarily large as in Chapter 6 and 7. On the other hand, it is possible to weaken Assumption **(A5)** to higher growth rates.

The next statement gives us a means to analyze the stability of the stochastic process $\{e_k, \hat{\lambda}_k\}_k$ in terms of positive Harris recurrence. We will make use of Theorem 7.1 derived in Chapter 7. The result in Theorem 7.1 gives a sufficient condition that guarantees ergodicity for the aggregated system for a fixed price λ. In the case of a varying price λ, the main idea is to find a sufficiently large window length T_0, such that a time-scale separation can be established between the dynamics within the subsystems and the dynamics of $\hat{\lambda}_k$.

Theorem 9.1. *Let $\beta_k = \beta$ and $T_{0,k} = T_0$ be constant. If Assumptions* **(A4)**, **(A5)**, *and Eq. (7.7) hold, then there exists a sufficiently large $\bar{T}_0$, such that the T_0-sampled Markov chain $\{e_k, \hat{\lambda}_k\}_{k\in\mathbb{T}}$ with $T_0 \geq \bar{T}_0$ is a positive Harris recurrent Markov chain.*

Proof. The sampled Markov chain evolves by Eq. (9.22) and the evolution of the estimation error e_k^i of subsystem i is given by

$$e_{k+T_0}^i = (1 - q_k^i \mathrm{f}^{i,\hat{\lambda}_k}(e_k^i)) \prod_{l=k+1}^{k+T_0-1} (1 - q_l^i \mathrm{f}^{i,\hat{\lambda}_k}(e_l^i)) A^i e_{k+1}^i + \\ + \sum_{l=k}^{k+T_0-1} \prod_{n=k+1}^{k+T_0-1-l} (q_n^i \mathrm{f}^{i,\hat{\lambda}_k}(e_n^i)) A^{k+T_0-1-l} w_l^i.$$

The subsequent stability analysis for Markov chains is mainly based on the tools introduced in Appendix B. First, it can be seen that the chain is φ-irreducible. This is because of the absolute continuity of the Gaussian noise process w_k^i and the fact that Eq. (9.22) can be viewed as a random walk on the half line with a probability of a negative drift greater than zero, see Proposition 4.3.1 in [MT93].

We consider the following Lyapunov candidate

$$V(e_k, \hat{\lambda}_k) = b_1 \sum_{i=1}^{N} \|e_k^i\|_2^2 + b_2 \hat{\lambda}_k^3 \tag{9.23}$$

with suitable $b_1, b_2 > 0$. It follows immediately that the drift for this choice of V is bounded within any compact set. Based on the assertion of φ-irreducibility, we rely on the drift criterion given by Theorem B.3 in the appendix. By the definition of the drift Δ as in B.10, we take the following form of the drift criterion given by Eq. (B.5): If the condition

$$\Delta V(e_k, \hat{\lambda}_k) \leq -\epsilon, \quad (e_k, \lambda_k) \in \mathbb{R}^{n_1+\cdots+n_N} \backslash \mathscr{D}, \tag{9.24}$$

where $\epsilon > 0$ and $\mathscr{D}$ is a compact set, is satisfied, then the T_0-sampled Markov chain $\{e_k, \hat{\lambda}_k\}_{\{k \in \{0, T_0, 2T_0, \ldots\}\}}$ is positive Harris recurrent.

Due to linearity of the conditional expectation, see Theorem A.1, we can split the drift into $N+1$ contributing terms given by

$$\Delta^i = \mathsf{E}[\|e_{k+T_0}^i\|_2^2 | e_k, \hat{\lambda}_k] - \|e_k^i\|_2^2, i \in \{1, \ldots, N\}$$
$$\Delta^\lambda = \mathsf{E}[\hat{\lambda}_{k+T_0}^2 | e_k, \hat{\lambda}_k] - \hat{\lambda}_k^3$$

In a first step, fix a price $\hat{\lambda}_k$ and with it fix the individual bounds M^i in Assumption **(A4)**. Subsequently, we focus on determining upper bounds on Δ^i.

The conditional expectation in Δ^i can be rewritten as

$$\begin{aligned}\mathsf{E}[\|e_{k+T_0}^i\|_2^2 | e_k, \hat{\lambda}_k] &= \mathsf{E}[\mathsf{E}[\|e_{k+T_0}^i\|_2^2 | e_{k+T_0-1}, e_k, \hat{\lambda}_k] | e_k, \hat{\lambda}_k] \\ &= \mathsf{E}[\mathsf{E}[\|e_{k+T_0}^i\|_2^2 | e_{k+T_0-1}^i, \hat{\lambda}_k] | e_k, \hat{\lambda}_k],\end{aligned}$$

where the first equality is due to the tower property of the conditional expectation stated in Theorem A.1 and the second because of the Markov property of e_k between the updates of $\hat{\lambda}_k$. The statistical independence of w_k^i with respect to q_k^i and e_k^i and the fact that $w_k^i \sim \mathscr{N}(0, C^i)$ allows the following simplification.

$$\begin{aligned}&\mathsf{E}[\|e_{k+T_0}^i\|_2^2 | e_{k+T_0-1}^i, \hat{\lambda}_k] \leq \\ &\leq \mathsf{E}[1 - q_k^i | e_k, \hat{\lambda}_k] \|A^i\|_2^2 \|e_{k+T_0-1}^i\|_2^2 + \mathrm{tr}[C^i]\end{aligned}$$

Therefore, we have the following upper bounds

$$\mathsf{E}[\|e_{k+T_0}^i\|_2^2 | e_k, \hat{\lambda}_k] \leq \begin{cases} |||A^i\|_2^2 |(M^i)^2 + \mathrm{tr}[C^i], & \|e_{k+T_0-1}^i\|_2 \leq M^i \\ (1 - \frac{c}{N}) \|A^i\|_2^2 \, \mathsf{E}[\|e_{k+T_0-1}^i\|_2^2 | e_k, \hat{\lambda}_k] + \mathrm{tr}[C^i], & \|e_{k+T_0-1}^i\|_2 > M^i. \end{cases}$$

In the second bound, we have used that $\mathsf{E}[1-q_k^i|e_k,\hat{\lambda}_k]$ describes the probability that a request of subsystem i is blocked and is upper bounded by $1-\frac{c}{N}$ because of Eq. (9.2). For notational convenience, let $\alpha=(1-\frac{c}{N})\|A^i\|_2^2$. Proceeding inductively, we obtain the following T_0+1 bounds on the drift Δ^i

$$\Delta^i \leq \alpha^{t_0}\| \|A^i\|_2^2 (M^i)^2 + \left(\sum_{n=0}^{t_0}\alpha^n\right)\mathrm{tr}[C^i] - \|e_k^i\|_2^2, \quad 0\leq t_0\leq T_0-1, e_k^i\in\mathbb{R}_i^n \tag{9.25}$$

$$\Delta^i \leq (\alpha^{T_0}-1)\|e_k^i\|_2^2 + \left(\sum_{n=0}^{T_0-1}\alpha^n\right)\mathrm{tr}[C^i], \qquad \|e_k^i\|_2 > M^i \tag{9.26}$$

The condition in Eq. (7.7) guarantees that we can find an appropriate ϵ and a sufficiently large compact set $\mathscr{E}$ such that the drift term $\sum_i\Delta^i\leq -h$ all $e_k\notin\mathscr{E}$, where h can be made arbitrarily large.

For the subsequent analysis, it should be noted that the drift term $\sum_i\Delta^i$ can be uniformly bounded from above as a function of M^i for a fixed $\hat{\lambda}_k$. As a next step, we aim at finding an upper bound on the drift Δ^λ. We have from Eq. (9.22)

$$\begin{aligned}
\hat{\lambda}_{k+T_0}^3 &= \left([\hat{\lambda}_k+\beta(\hat{y}_k-c)]_+\right)^3 \\
&\leq \left|\hat{\lambda}_k+\beta(\hat{y}_k-c)\right|^3 \\
&= \hat{\lambda}_k^3 + 3\hat{\lambda}_k^2\beta(\hat{y}_k-c)+3\hat{\lambda}_k\beta^2(\hat{y}_k-c)^2+\beta^3|\hat{y}_k-c|^3 \\
&\leq \hat{\lambda}_k^3 + 3\hat{\lambda}_k^2\beta(\hat{y}_k-c)+3\hat{\lambda}_k\beta^2N^2+\beta^3N^3
\end{aligned}$$

The first inequality is because $[\cdot]_+$ is non-expansive [Ber99]. Therefore, the drift term Δ^λ can be bounded by

$$\Delta^\lambda \leq 3\hat{\lambda}_k^2\beta\,\mathsf{E}[(\hat{y}_k-c)|e_k,\hat{\lambda}_k]+3\hat{\lambda}_k\beta^2N^2+\beta^3N^3 \tag{9.27}$$

What remains to be analyzed is the estimation of the gradient given by $\mathsf{E}[(\hat{y}_k-c)|e_k,\hat{\lambda}_k]$. Because of the ergodicity of the process $\{e_k\}$ for a fixed $\hat{\lambda}_k$ due to Theorem 7.1, the empirical mean of the total request rate converges to a biased estimate of y_k, i.e, the empirical request rate is given by

$$\lim_{T_0\to\infty}\frac{1}{T_0}\mathsf{E}[\sum_{l=k}^{k+T_0-1}\delta_l^i|e_k,\hat{\lambda}_k]=\eta^i r_k^i, \quad \mathsf{P}\text{-a.s.},$$

where $\eta^i\geq 0$ denotes the deviation from the request rate r_k^i resulting in the absence of contention among subsystems. In the following, we give an upper bound on η^i. For that reason we define ρ_n^i that counts the number of subsequent requests of subsystem i until successful transmission and is then reset to 0 again. Formally, we define the increasing sequences

$$\begin{aligned}
\{t_{1,n}\}_n &= \{k|q_k^i\delta_k^i=1\}, \\
t_{2,n} &= \min(k|\delta_k^i=1\wedge k\in(t_n,t_{n+1}]), n\geq 1
\end{aligned}$$

with $t_{2,0} = 0$. Then by setting $\rho_n^i = \sum_{k=t_{2,n}}^{t_{1,n}} \delta_k^i$, we have

$$\begin{aligned}\lim_{T_0\to\infty} \frac{1}{T_0} \mathsf{E}[\sum_{l=k}^{k+T_0-1} \delta_l^i | e_k, \hat{\lambda}_k] &= \lim_{T_0\to\infty} \frac{1}{T_0} \mathsf{E}[\sum_n \rho_n^i | e_k, \hat{\lambda}_k] \\ &= \lim_{T_0\to\infty} \frac{1}{T_0} \mathsf{E}[\sum_n \mathsf{E}[\rho_n^i | e_k, \hat{\lambda}_k] | e_k, \hat{\lambda}_k] \\ &\le \lim_{T_0\to\infty} \frac{1}{T_0} \mathsf{E}[\sum_n \frac{N}{c} | e_k, \hat{\lambda}_k] \\ &= r_k^i \frac{N}{c}\end{aligned}$$

where the last inequality is due to the fact that the probability that a request of subsystem i is granted is lower bounded by $\frac{c}{N}$ because of Eq. (9.2). The last equality is because the number of elements of the sum is related to the delayed renewal process given by the requests assuming no contention whose rate is r_k^i. Therefore, we have $\eta^i \le \frac{N}{c}$. This implies for the empirical total request rate $\hat{y}_k$

$$\lim_{T_0\to\infty} \mathsf{E}[\hat{y}_k | e_k, \hat{\lambda}_k] = \eta y_k, \quad \eta \le \frac{N}{c}, \quad \mathsf{P}\text{-a.s.},$$

As $\hat{y}_k \le N$ for every $\omega \in \Omega$, almost sure convergence implies $\mathscr{L}_1$ convergence [Res98], i.e.,

$$\lim_{T_0\to\infty} \mathsf{E}[| \mathsf{E}[\hat{y}_k | e_k, \hat{\lambda}_k] - \eta y_k | | e_k, \hat{\lambda}_k] = 0.$$

Continuing our analysis of Δ^λ from Eq. (9.27), we obtain

$$\begin{aligned}\Delta^\lambda \le & 3\hat{\lambda}_k^2 \beta \left((\eta y_k - c) + \mathsf{E}[| \mathsf{E}[\hat{y}_k | e_k, \hat{\lambda}_k] - \eta y_k | | e_k, \hat{\lambda}_k] \right) + \\ & + 3\hat{\lambda}_k \beta^2 N^2 + \beta^3 N^3.\end{aligned}$$

By choosing $\hat{\lambda}_k$ and T_0 sufficiently large, y_k and $\mathsf{E}[| \mathsf{E}[\hat{y}_k | e_k, \hat{\lambda}_k] - \eta y_k | | e_k, \hat{\lambda}_k]$ become arbitrarily small. Due to Assumption **(A5)** and $\Delta^i \le \mathscr{O}((M^i)^2)$, we conclude that by choosing b_1, b_2 accordingly, we can show that there exists a $\bar{\lambda} > 0$ such that for any $\hat{\lambda}_k > \bar{\lambda}$ and any e_k, we have $\Delta V(e_k, \hat{\lambda}_k) \le -\epsilon$, where $\epsilon > 0$. On the other hand, for $\hat{\lambda}_k \le \bar{\lambda}$, we can find a sufficiently large compact set $\mathscr{B}$ such that for every $e_k \notin \mathscr{B}$, we also have $\Delta V(e_k, \hat{\lambda}_k) \le -\epsilon$. By setting $\mathscr{D} = \mathscr{B} \times [0, \bar{\lambda}]$, the drift condition in Eq. (9.24) is satisfied which concludes the proof. □

An explicit determination of a sufficiently large $\bar{T}_0$ that yields stability can be circumvented by letting $T_{0,k}$ grow towards infinity for $k \to \infty$, as done in the following. This is possible because the upper bounds on the drifts determined in the above proof can be chosen to be uniform in $T_0 \ge \bar{T}_0$ when taking into account the hypothesis of Theorem 9.1.

9.3.2 Convergence

Opposed to the previous analysis, diminishing step sizes β_k and increasing window lengths $T_{0,k}$ for the adaptive sample-based algorithm are considered in this subsection. With this, we are able to show almost sure convergence of the process $\hat{\lambda}_k$ to the optimal solution of the relaxed problem (9.11) in the case without contention. The absence of contention implies that no transmissions are blocked and only an average resource constraint is considered. The main idea to show convergence comes from stochastic approximation [KY03] and relates the limiting behavior of the stochastic process $\{\hat{\lambda}_k\}_k$ to the ordinary differential equation given by Eq. (9.18). In the case with contention, we can not expect to obtain the same results, as there is no immediate deterministic description that relates to $\{\hat{\lambda}_k\}_k$, when transmissions are blocked. However, the results in the absence of contention will serve as an indicator for the convergence of the contention-based case for an increasing number of subsystems N.

The next assumption gives a condition on the step size β_k.

(A6) Assume that $\beta_k \to 0$ as $k \to \infty$ with $\sum_{k\in\mathbb{T}} \beta_k = \infty$ and $\sum_{k\in\mathbb{T}} \beta_k^2 < \infty$.

In order to show almost sure convergence $\{\hat{\lambda}_k\}_k$ in the absence of contention, the analysis is split into two parts. In the first part, we show stability of the process by establishing a recurrence property of $\{\hat{\lambda}_k\}_k$. In the second part given by Theorem 9.2, a local analysis takes over and ODE methods from [KY03] are used. In the local analysis, the process $\{\hat{\lambda}_k\}_k$ starts from a neighborhood set around the optimal solution λ^* and it is shown that while it enters the set infinitely many times due to the recurrence property, it leaves such neighborhood only finitely many times. This establishes the almost sure convergence result.

The first part is summarized by the following lemma.

Lemma 9.1. *Let λ^* be the solution of the relaxed problem (9.11) and let β_k satisfy Assumption* **(A6)** *and let $T_{0,k} \to \infty$. Under the absence of contention and the Assumption* **(A4)***, the stochastic process $\{\hat{\lambda}_k\}_k$ evolving by Eq. (9.22) visits any small neighborhood of λ^* infinitely many times* P*-almost surely.*

Proof. Consider the Lyapunov function

$$V(\hat{\lambda}_k) = (\hat{\lambda}_k - \lambda^*)^2.$$

By Eq. (9.22), we have for $k \in \mathbb{T}$

$$(\hat{\lambda}_{k+T_{0,k}} - \lambda^*)^2 \leq (\hat{\lambda}_k - \lambda^*)^2 + 2\beta(\hat{\lambda}_k - \lambda^*)(\hat{y}_k - c) + \beta^2 N^2$$

where the inequality follows from the fact that $[\cdot]_+$ is non-expansive [Ber99].

Note that the T_0-sampled stochastic process $\{e_k, \hat{\lambda}_k\}_{k\in\mathbb{T}}$ is a time-inhomogeneous Markov chain. We define for $k \in \mathbb{T}$, the time-variant drift $\Delta_k V$ as

$$\Delta_k V = \mathsf{E}[V(\hat{\lambda}_{k+T_{0,k}})|e_k, \hat{\lambda}_k] - V(\hat{\lambda}_k)$$

With this definition, we have

$$\begin{aligned}\Delta_k V_k &\leq 2\beta(\hat{\lambda}_k - \lambda^*)\,\mathsf{E}[(\hat{y}_k - c)|e_k, \hat{\lambda}_k] + \beta^2 N^2 \\ &= 2\beta(\hat{\lambda}_k - \lambda^*)((y_k - c) + \\ &\quad + \mathsf{E}[(\mathsf{E}[\hat{y}_k|e_k, \hat{\lambda}_k] - y_k)|e_k, \hat{\lambda}_k]) + \beta^2 N^2,\end{aligned} \tag{9.28}$$

where y_k is defined in Eq. (9.21). Therefore, $y_k - c$ is the gradient $\frac{\partial g(\lambda)}{\partial \lambda}$ at $\lambda = \hat{\lambda}_k$. Based on this upper bound on Δ_k, we take the following super-Martingale lemma as a condition for Harris recurrence from [ZZC08].

Lemma 9.2 ([ZZC08]). *Suppose that there exists a set $\mathscr{D} \in \mathbb{R}_{\geq 0}$ such that for all $k \in \mathbb{T}$*

$$\Delta_k V \leq -\beta_k \epsilon + \nu_k, \quad \lambda_k \notin \mathscr{D}, \tag{9.29}$$

where $\epsilon > 0$ β_k satisfies Assumption **(A6)** *and $\sum_k |\nu_k| < \infty$ P-almost surely. Then $\mathscr{D}$ is Harris recurrent with respect to the process $\{\hat{\lambda}_k\}_{k\in\mathbb{T}}$.*

By identifying $\nu_k = \beta_k^2 N^2$ and taking Assumption **(A6)** into account, we observe that $\sum_k \in \mathbb{T}\beta^2 N^2 < \infty$. Due to Assumption **(A4)**, the Markov chain $\{e_k^i\}$ is ergodic assuming a fixed $\hat{\lambda}_k$. This implies that we have almost sure convergence of $\mathsf{E}[\hat{y}_k|e_k, \hat{\lambda}_k]$ to y_k as $T_{0,k} \to \infty$ due to $0 \leq \hat{y}_k \leq N$. Therefore, we have

$$\lim_{k\to\infty} \mathsf{E}[|\,\mathsf{E}[\hat{y}_k|e_k, \hat{\lambda}_k] - y_k)|\;|e_k, \hat{\lambda}_k] = 0. \tag{9.30}$$

By fixing the set $\mathscr{D}$, where λ^* is in the interior of $\mathscr{D}$, we have an appropriate $\epsilon_1 > 0$ such that $(\hat{\lambda}_k - \lambda^*)((y_k - c) < -\epsilon_1$ for each $\hat{\lambda}_k \notin \mathscr{D}$ due to the properties of the gradient $\frac{\partial g(\lambda)}{\partial \lambda}$. This implies together with Eq. (9.30) that an ϵ can be found that satisfies Eq. (9.29) for every neighborhood $\mathscr{D}$ of λ^*. This concludes the proof. □

The subsequent theorem gives a statement on the limiting behavior of $\{\hat{\lambda}_k\}_k$ in terms of almost sure convergence.

Theorem 9.2 (Absence of contention). *Let λ^* be the solution of the relaxed problem (9.11) and let β_k satisfy Assumption* **(A6)** *and let $T_{0,k} \to \infty$. Under the absence of contention and the Assumption* **(A4)***, the stochastic process $\{\hat{\lambda}_k\}_k$ evolving by Eq. (9.22) converges to λ^* P-almost surely for $k \to \infty$.*

The proof of Theorem 9.2 can be summarized as follows. With the recurrence result of Lemma 9.1, we can suppose that the process $\{\hat{\lambda}_k\}_k$ visits any neighborhood of λ^* infinitely many times. By constructing a continuous-time interpolation of $\{\hat{\lambda}_k\}_k$ that approximates the ODE in Eq. (9.18), it is shown that the process can exit any neighborhood of λ^* only finitely many times due to the asymptotic stability of ODE in Eq. (9.18). Hence, we can conclude almost sure convergence to λ^*. The details can be found in Section 9.7.

We have already observed in Section 8.2.5 that the asymptotic behavior of the contention-based system under the stability condition (7.7) resembles that one without contention as the number of subsystems grows. This suggests that the stochastic process $\{\hat{\lambda}_k\}_k$ converges arbitrarily close to λ^* for an increasing number of subsystems.

9.3.3 Implementation and Discussion

The implementation of the overall system is accomplished in two phases. In the first phase, which is performed before execution and locally in each subsystem, the optimal controller $\gamma^{i,*}$ given by Eq. (9.13) is calculated and the mapping from price λ to the optimal event-trigger $\mathrm{f}^{i,\lambda}$ by solving (9.14) through value iteration. In the second phase, the network manager adjusts the price accordingly to the total average transmission rate estimated given by $\hat{y}_k$, which serves as an estimate to approximate the gradient $\frac{\partial g}{\partial \lambda}$. Unlike broadcasting the price $\hat{\lambda}_k$ by a network manager, it is important to note that a complete decentralized adaptation mechanism can be realized when each subsystem is able to sense the amount of requests directly. Then, the calculation of $\hat{\lambda}_k$ can be performed locally in each subsystem.

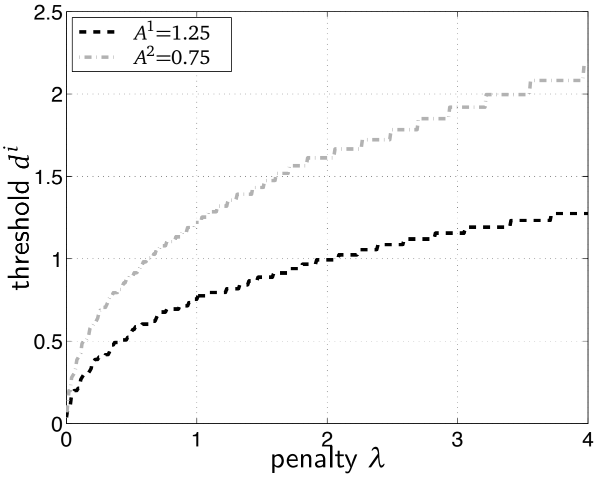


Figure 9.3: Mapping from price λ to the optimal symmetric scheduling law described by the threshold d^i for subsystem class 1 with $A^1 = 1.25$ and 2 with $A^2 = 0.75$.

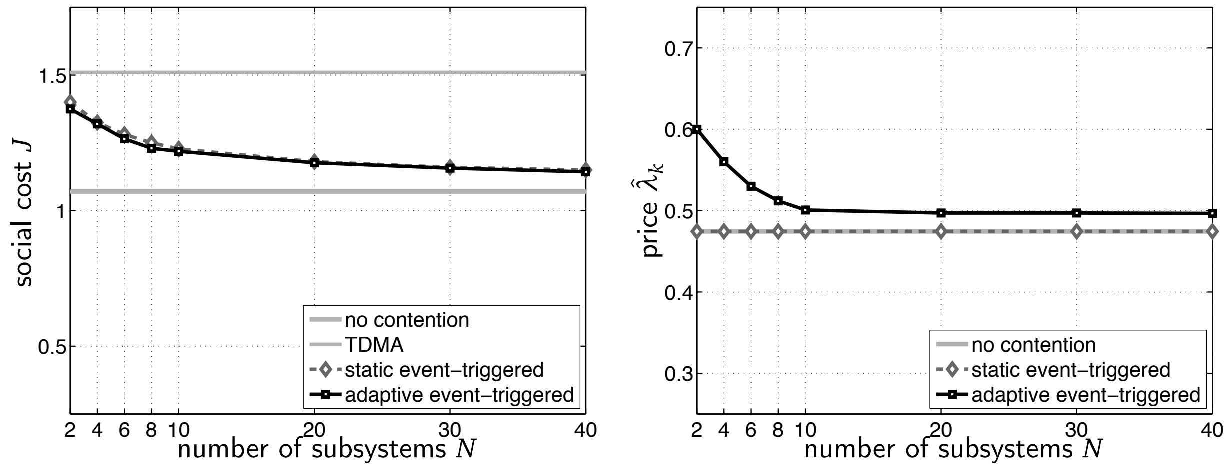
In contrast to a time-triggered scheduling mechanism, which needs a global combinatorial search at runtime to find the optimal scheduling sequence, the event-triggered scheme allows therefore a tractable implementation. Apart from the fact that the adaptation mechanism enables the distributed architecture, the local event-triggers are capable to adjust their thresholds according to runtime changes that are often found in real applications. These are for example given by adding or removing control loops during runtime, changes in the resource constraint, or changes in the local system parameters.

9.4 Numerical Results

In the following, we revisit the numerical example introduced in Section 8.3. Suppose the system comprises of N subsystems with two differing system parameters $A^1 = 1.25$, $B^1 = 1$, $Q_x^1 = 1$, $Q_u^1 = 0$ and $A^2 = 0.75$, $B^2 = 1$, $Q_x^2 = 1$, $Q_u^2 = 0$. The initial state of the process $\mathscr{C}^i$ is given by $x_0^i = 0$ and the system noise is given by $w_k^i \sim \mathscr{N}(0,1)$, $i \in \{1,\dots,N\}$. The communication constraint is set to $c = 1$. The optimal control gain in Eq. (9.13) is given by $L^i = A^i$ for each subsystem i, $i \in \{1,\dots,N\}$. The optimal scheduling laws for a fixed λ of each subsystem is a symmetric threshold policy defined in Eq. (9.16) with threshold d^i, $i \in \{1,\dots,N\}$. The optimal threshold for various fixed Lagrange multipliers λ is obtained by value iteration for the average-cost problem in Eq. (9.14).

Figure 9.3 shows the mapping from λ to the optimal thresholds of both subsystems. It should be noted that the determination of the mapping shown in Fig. 9.3 can be performed offline and locally for each subsystem. Therefore, the computationally intensive part for the determination of the optimal threshold can be accomplished before runtime. It can be seen that the growth rate of d^i, $i \in \{1,2\}$ is at most linear in Fig. 9.3 which supports Assumption **(A5)**.

We suppose a decreasing step size β_k and an increasing window length $T_{0,k}$ for the adaptive event-triggered scheduler with the adaptation mechanism for $\hat{\lambda}_k$ given by Eq. 9.22). In the lth update step, we have $T_{0,\cdot} = \{2,4,6,\dots\}$ and $\{\beta_k\}_{k\in\mathbb{T}} = \{2,1,\frac{2}{3},\frac{2}{4},\frac{2}{5},\dots\}$. Note that β_k



(a) Social cost (b) Limit price

Figure 9.4: Performance comparison of social cost J and limit price $\hat{\lambda}_k$ for $k \to \infty$ for various number of subsystems and $c/N = 0.5$.

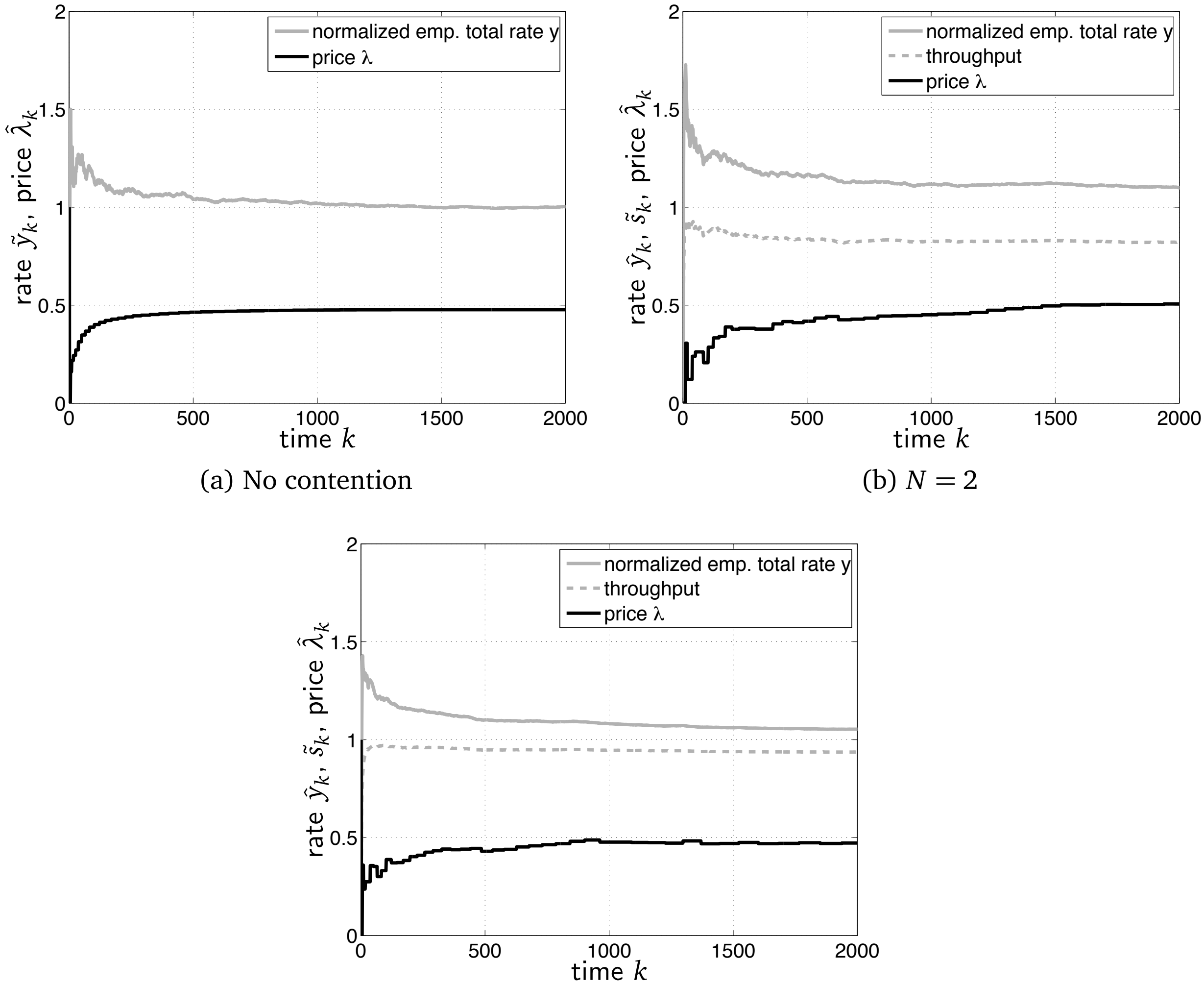
satisfies Assumption **(A6)**.

Figure 9.4a compares the social cost for the adaptive event-triggered scheduler, the static event-triggered scheduler, the time-triggered case, and the optimal solution without contention, which corresponds to the minimum of the relaxed problem in Eq. (9.11). Different N are considered, where it is assumed that the ratio between subsystem classes remains equal, and the capacity c grows with the number of subsystems, i.e., $c/N = 0.5$.

The static event-triggered scheme determines the optimal λ for the relaxed problem setting beforehand and takes the stationary event-triggered schedulers $\mathrm{f}^{i,\lambda}$, $i = \{1,2\}$ that coincides with the solution of the bi-level design in Chapter 8. The time-triggered scheduler has already been determined in Section 8.3 and is given by $\{\delta_k^1\}_k = \{1,1,0,1,1,0,\ldots\}$ and $\{\delta_k^2\}_k = \{0,0,1,0,0,1,\ldots\}$. The event-triggered schemes outperform the time-triggered scheme. The social cost of the adaptive scheme is slightly below the static event-triggered scheme. The cost obtained without contention takes a value of 1.07 and can be considered as a lower bound for the optimal scheduling policy with contention. This lower bound is slowly approached by the event-triggered schemes for increasing N. The fact that the social cost of the event-triggered scheme approaches the relaxed problem setting is also observed in Section 8.3 and can be reasoned by its convergence to a deterministic flow limit of the aggregate system when N approaches infinity.

Figure 9.4b compares the obtained price $\hat{\lambda}_k$ in the limit in the presence and in the absence of contention for the adaptive event-triggered scheduler, and with the optimal Lagrange multiplier λ^* for the relaxed problem setting in Eq. (9.11). Obviously, the optimal λ^* taking a value of 0.48 remains constant for different N, as $c/N = 0.5$. We observe that in the case of the adaptive event-triggered scheduler without contention, the price $\hat{\lambda}_k$ converges to the optimal λ^*. In the presence of contention, the price $\hat{\lambda}_k$ converges to a value strictly above the optimal price λ^*. It can also be observed that its deviation from λ^* becomes smaller when N increases.

Figures 9.5a, 9.5b, and 9.5c show the sample-path behavior of the adaptive algorithm



(a) No contention

(b) $N = 2$

(c) $N = 20$

Figure 9.5: Sample path of the normalized empirical total request rate $\hat{y}_k$, the normalized empirical throughput $\tilde{s}_k$, and the price $\hat{\lambda}_k$ for different scenarios.

over a time-horizon of 2000 for the communication penalty $\hat{\lambda}_k$ and the normalized empirical total request rate $\tilde{y}_k$ for the case without contention, $N = 2$, and $N = 20$, respectively. The normalized request rate $\tilde{y}_k$ is defined as

$$\tilde{y}_k = \frac{1}{kc} \sum_{\ell=0}^{k-1} \sum_{i=1}^{N} \delta_\ell^i.$$

Figures 9.5b and 9.5c show in addition the normalized empirical throughput $\tilde{s}_k$ defined as

$$\tilde{s}_k = \frac{1}{kc} \sum_{\ell=0}^{k-1} \sum_{i=1}^{N} q_\ell^i \delta_\ell^i.$$

In the absence of contention, we observe that the price $\hat{\lambda}_k$ converges to $\lambda^* = 0.48$, while the normalized total request rate converges to 1 which is in accordance with Theorem 9.2. In the presence of contention, the price $\hat{\lambda}_k$ takes in the limit a value of 0.6 and 0.497 for

$N = 2$ and $N = 20$, respectively. It should be noted that the price $\hat{\lambda}_k$ converges slowly a stationary variable for $N = 2$ in Fig. 9.5b. The normalized request rate and throughput converge to 1.05 and 0.8 for $N = 2$ and they converge towards 1.01 and 0.92 for $N = 20$.

As a concluding remark, it is interesting to observe that although the adaptive event-triggered scheduling schemes have a significant gap to the maximal throughput, they outperform the time-triggered scheme (which is throughput optimal) significantly with respect to the control performance as shown in Fig. 9.4a.

9.5 Summary

This chapter has demonstrated the capability of adaptive event-triggered scheduling for the distributed design in resource-constrained multi-loop control systems. Based on a dual price exchange mechanism, we have developed a framework for the synthesis of distributed event-triggered control systems that share a common resource. By considering the relaxation approach for the event-trigger design introduced in Chapter 8, it is shown that convexity properties of the relaxed problem enable the application of dual formulations related to network utility maximization. The distributed design is realized by local adaptive event-triggers that adjust their thresholds according to the estimated price for the resource. The use of a time-scale separation technique allows us to establish stability in terms of Harris recurrence in the case of contention and almost-sure convergence of the aggregate adaptive event-triggered scheme in the case without contention.

9.6 Bibliographical Notes

The contribution of this chapter is partly based on the work in [MH12a]. There are several links to recent results in the analysis and design of data networks. However, these works need not take into account the dynamical properties of the information sources as well as real-time requirements for controlling the processes over the network. On the one hand, the decentralization of the overall optimization problem via dual decomposition in Section 9.2.2 is closely related to congestion control in data networks [SS07]. On the other hand, the distributed scheduler design developed in this chapter is also related to the analysis of throughput and utility maximization of random access algorithms in [ZZC08; JW10; JSSW10; BMP08]. Similar as in this chapter, these works consider the joint design of scheduler and congestion control by optimizing a common objective function. Eventually, the proof of Theorem 9.2 relies on the ODE approach developed in the field of stochastic approximation and is built upon ideas from Section 5.4 of [KY03].

9.7 Proof of Theorem 9.2

Before proving Theorem 9.2, we give the following definitions for the local analysis of the process $\{\hat{\lambda}_k\}_{k\in\mathbb{T}}$. With slight abuse of notation, we redefine the index in the evolution of $\{\hat{\lambda}_k\}_{k\in\mathbb{T}}$ by

$$\hat{\lambda}_{n+1} = [\hat{\lambda}_n + \beta_n(\hat{y}_n - c)]^+, \tag{9.31}$$

where at each time n corresponds to a update time $k \in \mathbb{T}$.

Because of the recurrence property due to Lemma 9.1, the ϵ_1-ball with center λ^* defined as $\mathscr{B}_{\epsilon_1}(\lambda^*)$ is visited infinitely many times with probability 1. Moreover, we consider another neighborhood $\mathscr{B}_{\epsilon_2}(\lambda^*)$ with $\epsilon_2 > 2\epsilon_1$. As $\lambda^* > 0$, we further assume that the closed ball $\mathscr{B}_{\epsilon_2}(\lambda^*)$ does not intersect with the origin. Subsequently, we consider the family of sequences that start within $\mathscr{B}_{\epsilon_1}(\lambda^*)$ until they leave $\mathscr{B}_{\epsilon_2}(\lambda^*)$. As the gradient $\hat{y}_n - c$ is bounded within $[-N, N]$ and we are interested in the limiting behavior for large n, the projection operator in Eq. (9.22) can be omitted. Furthermore, due to the fact that $\mathscr{B}_{\epsilon_2}(\lambda^*)$ is a positively invariant set of the ODE (9.18), the projection operator can be discarded from the subsequent analysis. Then, we rewrite the difference equation in Eq. (9.22) as

$$\hat{\lambda}_{n+1} = \hat{\lambda}_n + \beta_n(y_n - c) + \beta_n\xi_n \tag{9.32}$$

where ξ_n is the error when estimating y_n by the empirical mean $\hat{y}_n$, i.e.,

$$\xi_n = \hat{y}_n - y_n. \tag{9.33}$$

Because of ergodicity of the overall system without contention under Assumption **(A4)**, we have $\lim_{n\to\infty}\xi_n = 0$ P-almost surely. In order to give a continuous-time interpolation of the discrete-time process (9.32) evolving within $\mathscr{B}_{\epsilon_2}(\lambda^*)$, we define the data points of the interpolation at times

$$t_n = \sum_{\ell=1}^{n-1}\beta_n.$$

with $t_0 = 0$. For $t \geq 0$, define $m(t)$ to be the unique index $n = m(t)$ such that $t_n \leq t < t_{n+1}$. Define the continuous-time interpolation $\hat{\lambda}^0$ of $\{\lambda_n\}_n$ by

$$\hat{\lambda}^0(t) = \hat{\lambda}_n, \quad t \in (t_n, t_{n+1})$$

and a shifted version of the interpolation by

$$\hat{\lambda}^n(t) = \hat{\lambda}^0(t_n + t).$$

We are now ready to prove Theorem 9.2 by the so-called ODE approach [KY03].

Proof of Theorem 9.2. Let Ω_0 be the null set on which ξ_n defined in Eq. (9.33) does not converge to zero or on which the process does not return to $\mathscr{B}_{\epsilon_1}(\lambda^*)$ infinitely often. Fix $\omega \in \Omega\backslash\Omega_0$.

Suppose that there are infinitely many escapes of $\{\hat{\lambda}_n(\omega)\}_n$ from $\mathscr{B}_{\epsilon_1}(\lambda^*)$ to $\mathscr{B}^{\mathrm{c}}_{\epsilon_2}(\lambda^*)$. Then, there is an increasing sequence $n_k(\omega) \to \infty$ such that n_k is the last index at which $\hat{\lambda}_{n_k} \in \mathscr{B}^{\mathrm{c}}_{2\epsilon_1}(\lambda^*)$ before leaving $\mathscr{B}^{\mathrm{c}}_{\epsilon_2}(\lambda^*)$.

For $s \geq 0$, we have the following piecewise constant interpolation of the discrete-time process given by Eq. (9.32)

$$\hat{\lambda}^{n_k}(s) = \hat{\lambda}_{n_k}(\omega) + \sum_{l=n_k}^{m(t_{n_k}+s)-1} \beta_l(y_l - c) + B^{n_k}(s)$$

with $B^{n_k}(s) = \sum_{l=n_k}^{m(t_{n_k}+s)-1} \beta_l \xi_l$. Due to the boundedness of the stochastic gradient $\hat{y}_n - c$, we have $\beta_n(\hat{y}_n - c) \to 0$, which implies that the sequence $\{\hat{\lambda}_{n_k}\}_k$ converges to a point on the boundary $\partial \mathscr{B}^{c}_{2\epsilon_1}(\lambda^*)$. It also implies that there is a $T > 0$ such that we have $\hat{\lambda}^{n_k}(\omega, t) \in \mathscr{B}_{\epsilon_2}(\lambda^*)$, $t \leq T$ for sufficiently large k. As $\xi_n(\omega) \to 0$, we also have $B^{n_k}(\omega, t) \to 0$ for all $t \leq T$. Due to these results, it can be stated that the family of sequences $\{\hat{\lambda}^{n_k}(\omega, \cdot), 0 \leq t \leq T\}$ is equicontinuous and bounded, and we have $\hat{\lambda}^{n_k}(\omega, \cdot) 0 \leq t \leq T\}$. By the Arzelà-Ascoli theorem, we conclude that there exists a convergent subsequence of $\{\hat{\lambda}^{n_k}(\omega, \cdot), 0 \leq t \leq T\}$.

Let $\lambda(\omega, \cdot)$ be the limit of such convergent subsequence and let

$$\rho^{n_k}(\omega, t) = \int_0^t (y(s) - c)ds - \sum_{l=n_k}^{m(t_{n_k}+t)-1} \beta_l(y_l - c), \quad t \leq T.$$

Because the sequences $\rho^n(\omega, t)$ and $B^n(\omega, t)$ vanish over finite time intervals for $n \to \infty$, we conclude that the convergent sequence $\lambda(\omega, \cdot)$ satisfies the ODE given by Eq. (9.18) for $t \leq T$ with initial condition $\lambda(\omega, 0) \in \partial \mathscr{B}^{c}_{2\epsilon_1}(\lambda^*)$ and $|\hat{\lambda}(\omega, t) - \lambda^*| \geq 2\epsilon_1$ for $t \leq T$.

However, the right-hand side of Eq. (9.18) points in the interior of $\partial \mathscr{B}^{c}_{2\epsilon_1})$ at $\partial \mathscr{B}^{c}_{2\epsilon_1}$, and $\lambda(\omega, \cdot)$ converges towards λ^*, which contradicts with the previous statement.

Therefore, our initial assertion does not hold and that there are only finitely many excursions from $\mathscr{B}_{\epsilon_1}(\lambda^*)$. As ϵ_1 can be chosen arbitrarily small, almost sure convergence to λ^* is guaranteed. This concludes the proof.

□

10

Conclusions and Outlook

The increasing complexity of modern control systems with the ability to acquire an almost unlimited amount of data from a multitude of networked sensors, urges the need for an efficient usage of communication and computational resources. Event-triggered control can be understood as an enabling technology to cope with the abundance of information in these real-time decision making problems. In order to benefit from the event-triggered mechanism at its full potential, it is necessary to conceive its fundamental working principles within the feedback control system as much as they are known for time-triggered systems. This thesis contributes to this endeavor by studying event-triggered control systems from the aspect of stochastic systems and optimal control. The analysis is divided into two parts: single-loop control systems with communication constraints in the feedback loop and multi-loop control systems sharing a common resource.

The main achievement of the first part is a detailed characterization of the optimal co-design of event-trigger and controller in the framework of linear quadratic control taking communication constraints into account. What makes the optimal design challenging is the non-standard information pattern of the decision makers given by the event-trigger and the controller. The differing information available at the decision makers does not allow a direct application of mathematical tools from stochastic optimal control, such as the separation principle and dynamic programming. Despite of the complications arising from the possibility of signalling between event-trigger and controller in this two-person team decision problem, we were able to show that many standard results obtained for the time-triggered version of the problem carry over to the case of event-triggered control. This fact is rather surprising as issues such as signaling do not appear in the optimal design of time-triggered control systems.

The core results of the structural characterization of optimal event-triggered controllers can be summarized as follows. As a starting point for the study of optimal event-triggered control, it has been proved that the optimal solution of the controller takes the form of a certainty equivalence controller. This implied that the resulting optimal control law is linear in its least-squares state estimate. The key property that allows such assertion is nestedness of the information pattern, i.e., the information available at the controller is a subset of the information available at the event-trigger. As an implication of this result, it can be observed that one can not benefit of the dual effect of control in such problem setting. Furthermore, it is shown that the certainty equivalence property of the optimal control law does not only hold in the case of perfect state feedback, but it is also valid in the case of intermittent and delayed state information or noisy state measurements. Nevertheless, the nestedness condition remains a prerequisite in all extensions of the problem setup. The certainty equivalence property enables the reduction of the event-triggered control problem into the optimal co-design of event-trigger and state estimator. When studying the joint design of event-trigger and estimator, there emerges another complicating feature that is not present in time-triggered estimation: Not triggering an event can be valuable information for the least-squares estimation of the process state. This fact implies that the measurement information at the controller becomes non-Gaussian and – more importantly – depends on the triggering rule. However, it turns out that the optimal state estimator takes the form of a biased linear predictor that resembles the optimal time-triggered estimator. The estimation bias in the linear predictor reflects the additional information obtained from not receiving a new update from the event-trigger. Furthermore, it is shown that the optimal event-trigger can be represented as a function of the network-induced estimation error and the time of the last transmission. By analyzing the convergence properties of an iterative algorithm, which alternates between optimizing the estimator while fixing the event-trigger and vice versa, we were able to study the joint optimal event-triggered estimator in more detail. The iterative method shows special properties in the case of unimodal and symmetric distributions of the uncertainty. In the case of symmetric distributions, the event-triggered estimator given by the unbiased linear predictor and the corresponding symmetric threshold policy is proven to be person-by-person optimal. Person-by-person optimality can be considered as a necessary condition for optimality. We have showed that this design is indeed optimal for first-order systems in the case of unimodal and symmetric distributions. When having bimodal distributions, the iterative procedure offers a systematic method, which leads to asymmetric event-triggers and biased estimators that outperform symmetric threshold policies. Similar properties of the iterative method are likely to hold as well in the case of multi-dimensional systems, but a conclusive derivation for higher-order systems is still an open issue. Eventually, the first part is completed with a stability analysis of the obtained event-triggered control system, where we have observed that drift criteria are suitable tools for studying the asymptotic behavior of the event-triggered system.

In summary, the structural characterization derived in the first part of this work admits the development of tractable algorithms to compute the optimal event-triggered controller. When assuming a first-order system with a symmetric unimodal distribution of the noise process and the initial state, the co-design of event-trigger and controller can be fully characterized and the synthesis problem decomposes into three standard subproblems of optimal control and estimation: (i) the solution of an (algebraic) Riccati equation in order to obtain the linear control gains, (ii) the determination of the linear least-squares estimator at the controller, and (iii) the solution of a dynamic program that yields the optimal triggering thresholds for the event-trigger. In the general case, the coupling between (ii) and (iii) remains unresolved, but the iterative algorithm developed in Chapter 3 serves as an adequate method to seek for the optimal event-triggered estimator. Moreover, the existing literature provides efficient approximative solutions of the dynamic program in (iii) for higher dimensional systems.

Based on the results obtained in the first part, the aim of the second part of this thesis is the analysis and the design of multiple control loops sharing a common communication network. The main contribution is the development of efficient design guidelines for the optimal co-design of control and communication. By introducing an approximative problem setting, in which the path constraints modeling the limited capabilities of the communication network are relaxed by an average rate constraint, we were able to formulate the co-design problem as a constrained Markov decision process. Such problem relaxation allows the application of the structural results derived in the first part for single-loop control systems with communication constraints. It turns out that the resulting optimization problem splits into two levels of optimization that are numerically feasible: a local optimal event-triggered control design related to the first part of this thesis and a global resource allocation problem. The global resource allocation determines the optimal transmission rates of the individual local control loops. The rates define the operation point of the corresponding event-triggered scheduler that is determined locally in each control loop. On the one hand, it is shown under some mild assumptions that the proposed bi-level design approach converges to the optimal solution for an increasing number of loops and transmission slots. On the other hand, numerical results indicate that the aggregate performance of the bi-level method deviates only slightly from a lower bound already for a moderate number of control loops.
Furthermore, we were able to derive sufficient conditions for the stability of the resulting multi-loop control system. These conditions relate the ratio between the availability of the resource and the number of control loops with the open-loop system dynamics of each control system. Moreover, it has been observed that this condition can be checked locally when knowing the the ratio between the number of transmission slots and the number of subsystems. This adds robustness to the aggregate control system, as stability can still be guaranteed, even if a malicious subsystem is continuously requesting for transmission.
Finally, the second part of this thesis demonstrates the capability of adaptive event-triggered scheduling for the distributed design in resource-constrained multi-loop control systems. Based on a dual price exchange mechanism, we have developed a framework for the synthesis of distributed event-triggered control systems. The distributed design is realized by local adaptive event-triggers that adjust their thresholds according to the estimated price for the resource. The use of a time-scale separation technique allows us to prove stability in the case of contention and almost-sure convergence of the aggregate adaptive event-triggered scheme in the case without contention.

The second part can be summarized as follows. Despite the decreased predictability and the close interaction between control and communication in contrast to time-triggered control schemes, we have shown that the proposed event-triggered control scheme yields simple stability conditions, an increased level of robustness, and a significant improvement on the overall control performance. Furthermore, the ability of adaptation of the event-triggered scheduling scheme enables a straight-forward implementation of distributed scheduling algorithms.

10.1 Outlook

The research area of event-triggered control has attained increasing attention for more than one decade from both practical and theoretical aspects. Several contributions towards a fundamental understanding of event-triggered sampling in the framework of linear quadratic control have been made in this work. These results support the believe that event-triggered control carries great promises for mastering the complexity of modern real-time decision making problems. However, various open research questions in the area of event-triggered control must be addressed in the future in order to comply with the expectation for its potential.

Event-triggered control for general optimal control problems

This work has primarily focussed on optimal control problems related to linear quadratic regulation. We were able to give a detailed description of the optimal solution when jointly designing event-trigger and controller. This raises the question whether a similar characterization can be obtained for general optimal control problems with communication constraints. Although it is unlikely that the majority of the results directly carries over to the case of general optimal control problems, it is crucial to identify subclasses of optimal control problems and additional assumptions that enable an intuitive description of the optimal event-triggered controller. Model predictive control methods that use online optimization techniques seem to be an attractive approach that needs to be analyzed in more detail under the paradigm of optimal event-triggered sampling.

Event-triggered sampling for distributed optimization and control

The initial motivation for event-triggered sampling has its roots in taming the complexity of real-time decision making. Distributed optimization and control are another enabling means that address the solution of complex large-scale problems. In order to develop efficient algorithms for these problems in the future, tight links between these domains need to be established. One reason for this comes from the fact that distributed algorithms commonly suffer from slow convergence rates which makes them less attractive for application. However, it is believed that event-triggered sampling is capable to enhance convergence of distributed systems, as it promotes the distribution of essential information while suppressing redundant data.

Physical coupling among subsystems

Another open issue concerns the introduction of physical coupling among subsystems that naturally arises in distributed control systems. Coupling between subsystems carries changing implications for the distributed event-triggered control system. The conventional analysis does not apply as measurement signals are not independent from with each other anymore. The signal correlation implies that the local event-triggers must be designed carefully as the triggering times may be highly correlated leading to a higher chance of collisions. This also increases the demands on the stability analysis that need to be resolved.

Implementation in available communication protocols/processor schedulers

In this thesis, the model for the communication network was chosen to be generic. While this made it possible to focus on the main issues of event-triggered scheduling and control, the results are applicable to a wide range of resource constrained entities that are not limited to communication systems. On the other hand, there have already been made several attempts to implement event-triggered control methods in real communication protocols in an ad-hoc fashion. What needs to be done in the future is to bridge the gap between both approaches by appropriately modeling the communication system. The model needs to abstract the main features of the resource constraint while allowing for the development of efficient algorithms for the joint design of event-triggered scheduler and controller. This modeling is not restricted to communication protocols but also applies to processor scheduling.

Part III

Appendices

A

Probability Theory

This appendix is intended to review some of the basic notions of probability theory. Rather than being exhaustive, it is meant to familiarize the reader with the main principles used throughout this thesis. The presentation follows the textbooks [Res98; Øks03; MT93].

A.1 Preliminaries on Measure Theory

In this section, we give a minimal set of measure-theoretic definitions needed for the construction of a probability space.

Definition A.1 (Measurable Space). A tuple $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ with $\mathscr{X}$ being an abstract set of points is said to be a *measurable space* if $\mathscr{B}(\mathscr{X})$ is a σ-algebra of $\mathscr{X}$, i.e.,

1. $\mathscr{X} \in \mathscr{B}(\mathscr{X})$,
2. if $A \in \mathscr{B}(\mathscr{X})$ then $A^c \in \mathscr{B}(\mathscr{X})$,
3. if $A_k \in \mathscr{B}(\mathscr{X})$, $k \in \{1, 2, \dots\}$ then $\bigcup_{k=1}^{\infty} A_k \in \mathscr{B}(\mathscr{X})$.

A topological space $\mathscr{X}$ is always endowed with the Borel σ-algebra $\mathscr{B}(\mathscr{X})$. A Borel σ-algebra is the smallest σ-algebra of subsets of $\mathscr{X}$ that contains all open subsets of $\mathscr{X}$.

Definition A.2 (Measurable Function). Let $(\mathscr{X}_1, \mathscr{B}(\mathscr{X}_1))$ and $(\mathscr{X}_2, \mathscr{B}(\mathscr{X}_2))$ be two measurable spaces. Then, a mapping $h : \mathscr{X}_1 \to \mathscr{X}_2$ is said to be a *measurable function* if the preimage $h^{-1}\{B\} = \{x : h(x) \in B\}$ satisfies

$$h^{-1}\{B\} \in \mathscr{B}(\mathscr{X}_1)$$

for all sets $B \in \mathscr{B}(\mathscr{X}_2)$.

A measure on a measurable space is defined as follows.

Definition A.3 (Measure). A set function $\mu : \mathscr{B}(\mathscr{X}) \to [0,\infty]$ on the space $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ is said to be a *measure* if it is countably additive, i.e., if $A_k \in \mathscr{B}(\mathscr{X})$, $k \in \{1,2,\ldots\}$, and $A_i \cap A_j = \emptyset$, $i \neq j$ then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

The measure μ is called a *probability measure* if $\mu(\mathscr{X}) = 1$.

A.2 Random Variables

A probability space is defined by the triple $(\Omega, \mathscr{F}, \mathsf{P})$ where Ω is an abstract set of points, $\mathscr{F}$ is an σ-algebra of Ω, and P is a probability measure on $(\Omega, \mathscr{F})$.

Definition A.4 (Random Variable). Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability measure and $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ be a measurable space. A mapping $x : \Omega \to \mathscr{X}$ is said to be a *random variable* if x is a measurable function from Ω to $\mathscr{X}$, i.e.,

$$X^{-1}(B) \in \mathscr{F}$$

for all sets $B \in \mathscr{B}(\mathscr{X})$.

Given a random variable x in the probability space $(\Omega, \mathscr{F}, \mathsf{P})$, we define the σ-algebra *generated* by the random variable x, denoted as $\sigma(x)$, to be the smallest σ-algebra on which x is measurable.

Suppose x is a random variable from $(\Omega, \mathscr{F}, \mathsf{P})$ to $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$, and suppose h is a real-valued measurable mapping from $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Then, $h(x)$ is a real-valued random variable on $(\Omega, \mathscr{F}, \mathsf{P})$, for which we define the *expectation* as

$$\mathsf{E}[h(x)] = \int_{\Omega} h(x(\omega))\,\mathsf{P}(d\omega).$$

A.3 Conditional Expectation

In this section, we give a formal definition for the conditional expectation in the notion of the σ-algebras and state several useful properties that we refer to throughout the thesis.

Let the triple $(\Omega, \mathscr{F}, \mathsf{P})$ define a probability space. Let $x : \Omega \to \mathscr{R}^n$ be an integrable random variable, i.e., $\mathsf{E}[|x|] \leq \infty$. Suppose z be another integrable random variable defined within the above probability space. Let $\mathscr{Z} = \sigma(z) \subset \mathscr{F}$ denote the σ-algebra generated by the random variable z. Then, the *conditional expectation* of x given z denoted by $\mathsf{E}[x|z]$ is defined as follows:

Definition A.5 (Conditional Expectation). The conditional expectation $\mathsf{E}[x|z]$ is the (a.s. unique) mapping from Ω to $\mathbb{R}^n$ satisfying

1. $\mathsf{E}[x|z]$ is measurable with respect to z,
2. $\int_Z \mathsf{E}[x|z] d\,\mathsf{P} = \int_Z x d\,\mathsf{P} \quad \text{for all} \quad Z \in \mathscr{Z}$.

The existence and uniqueness of the conditional expectation $\mathsf{E}[x|z]$ can be shown by applying the Radon-Nikodym theorem [Øks03]. Throughout the thesis, we make extensively use of several calculation rules for the conditional expectation summarized in the following theorem.

Theorem A.1. *Suppose $y : \Omega \to \mathbb{R}^n$ be another integrable random variable and let $a, b \in \mathbb{R}$. Then,*

1. $\mathsf{E}[ax + by|z] = a\,\mathsf{E}[x|z] + b\,\mathsf{E}[y|z]$, *(Linearity)*
2. $\mathsf{E}[\mathsf{E}[x|z]] = \mathsf{E}[x]$, *(Total expected value)*
3. $\mathsf{E}[\mathsf{E}[x|y]|z] = \mathsf{E}[x|z]$ *if z is measurable with respect to y,* *(Tower property)*
4. $\mathsf{E}[x|z] = E[x]$ *if x is independent of z,*
5. $\mathsf{E}[y^{\mathrm{T}} x|z] = y^{\mathrm{T}}\,\mathsf{E}[x|z]$ *if y is measurable with respect to z.*

The proofs of the above properties for the conditional expectation can be found in [Øks03].

B

Markov Chains

In this appendix, we give a brief summary of the notions and tools for the analysis of Markov chains in uncountable state spaces, which appear throughout this thesis. A comprehensive study of Markov chains in uncountable state spaces can be found in the seminal book of Meyn and Tweedie [MT93]. The subsequent definitions and theorems rely to a great extend on the results of this book.

The study of Markov chains in uncountable state spaces is particular challenging as the results developed for countable Markov chains can not directly be used if the state space is uncountable. Nevertheless, by extending the definitions accordingly, it is shown in [MT93] that most of the results regarding the asymptotic analysis of Markov chains can be recovered for uncountable state spaces.

B.1 Preliminaries

We consider a stochastic process $\{x_k\}$ evolving in the state space $\mathcal{X} \subset \mathbb{R}^n$ equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{X})$. First, we indicate how to construct the Markov chain $\{x_k\}$ as a stochastic process in the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ given an initial probability measure μ, and a probability transition kernel $P(\cdot,\cdot)$ defined as follows.

Definition B.1 (Transition Probability Kernel). The function $P = \{P(x,A), x \in \mathcal{X}, A \in \mathcal{B}(\mathcal{X})\}$ is said to be a transition probability kernel if

1. $P(\cdot,A)$ is a non-negative measurable function on $\mathcal{X}$ with respect to $\mathcal{B}(\mathcal{X})$ for each $A \in \mathcal{B}(\mathcal{X})$,

2. $P(x,\cdot)$ is a probability measure on $\mathcal{B}(\mathcal{X})$.

Both properties in the above definition are essential in order to define the probability measure on $\mathscr{F}$ properly. It can be observed that there is a substantial difference when moving from the countable to the uncountable sate space. The transition kernel $P(\cdot,\cdot)$ operates on quite different entities with respect to the first and the second argument and can therefore no longer be viewed as symmetric in its arguments in contrast to countable state spaces.

The following theorem shows how the probability space $(\Omega, \mathscr{F}, \mathsf{P})$ of the stochastic process $\{x_k\}$ can be constructed.

Theorem B.1. *Let μ be any initial measure on $\mathscr{B}(\mathscr{X})$ and $P(\cdot,\cdot)$ any transition probability kernel as given in Definition B.1. Then, there exists a stochastic process $\{x_k\}$ on the probability space $(\Omega, \mathscr{F}, \mathsf{P}_\mu)$, where $\Omega = \mathscr{X}^\infty$, $\mathscr{F} = \bigotimes_{i=0}^\infty \mathscr{B}(\mathscr{X}_i)$. P_μ is the probability distribution on $\mathscr{F}$ such that*

$$\mathsf{P}_\mu(x_0 \in A_0, x_1 \in A_1, \ldots, x_n \in A_n) = \int_{y_0 \in A_0} \cdots \int_{y_{n-1} \in A_{n-1}} \mu(dy_0) P(y_0, dy_1) \cdots P(y_{n-1}, A_n) \tag{B.1}$$

for any n and sets $A_i \in \mathscr{B}(\mathscr{X}_i)$, $i = 0, \ldots, n$.

Proof. The existence of $\{x_k\}$ is a consequence of the Kolmogorov Extension Theorem for construction of probabilities on topological spaces, see [Øks03]. □

Thus, we can now give a formal definition of a time-homogeneous Markov chain, see also Chapter 3 in [MT93].

Definition B.2 (Markov Chain). A stochastic process on $(\Omega, \mathscr{F})$ is said to be a *time-homogeneous Markov chain* with transition probability kernel $P(\cdot,\cdot)$ and initial distribution μ if the finite dimensional distributions of $\{x_k\}$ satisfy (B.1) for every $n \in \mathbb{Z}^+$.

Having shown that a Markov chain is uniquely defined given its initial distribution and transition probability kernel, we seek another expression than Eq. (B.1) that gives us a succinct expression of the Markovian characteristic: The future evolution of the Markov chain is independent of its given its present value. This is given by the following theorem, see also Proposition 3.4.3 in [MT93].

Theorem B.2 (Markov Property). *Let $h : \Omega \to \mathbb{R}$ be a bounded, measurable function and $\{x_k\}$ be a Markov chain on $(\Omega, \mathscr{F})$ with initial distribution μ. Then, for $n \geq 0$*

$$\mathsf{E}[h(x_{n+1}, x_{n+2}, \ldots) \,|\, x_0, \ldots, x_n; x_n = x] = \mathsf{E}_x[h(x_1, x_2, \ldots)]. \tag{B.2}$$

Our subsequent characterization of Markov chains is often phrased in properties of the (first) return time τ_A defined as follows.

Definition B.3 (Return Time). For any $A \in \mathscr{B}(\mathscr{X})$, the *return time* τ_A on the set A is defined as

$$\tau_A := \min\{n \geq 1 : x_n \in A\}.$$

We will also refer to the n-step transition kernel that is defined as

$$P^n(x, A) = \mathsf{P}_x(x_n \in A), \qquad n \in \mathbb{Z}^+, x \in \mathscr{X}, A \in \mathscr{B}(\mathscr{X}).$$

B.2 φ-Irreducibility and Small Sets

Roughly speaking, a Markov chain is said to be *irreducible*, if every part of the state space can be reached in finite time independently of the initial state. This seemingly simple prerequisite for a chain has wide-ranging consequence for the analysis of the asymptotic behavior of the Markov chain. Not until we can ensure irreducibility for a chain it will be reasonable to discuss global stability properties of the chain.

The idea of irreducibility is much simplified in countable state spaces. The reason is due to the symmetry of the transition kernel with respect to its arguments. Hence, one can define the concept of communicating states, meaning that two distinct states can reach each other in finite time with positive probability. Eventually, a Markov chain on a countable state space is said to be irreducible if every pair of states communicates with each other.

Adopted from Chapter 4 in [MT93], an analogous notion of irreducibility for Markov chains on uncountable state spaces is defined as follows.

Definition B.4 (φ-Irreducibility). Let φ be a measure on $\mathscr{B}(\mathscr{X})$. Then, a Markov chain $\{x_k\}$ is said to be *φ-irreducible*, if for all $x \in \mathscr{X}$

$$\varphi(A) > 0 \Rightarrow \mathsf{P}_x(\tau_A < \infty) > 0$$

and φ is called irreducibility measure of $\{x_k\}$.

We exclude the trivial measure, i.e., the measure which is zero for any $A \in \mathscr{B}(\mathscr{X})$, from possible irreducible measures on $\mathscr{B}(\mathscr{X})$, since Definition B.4 would be a trivial statement for this measure. Notice that the choice of an appropriate measures is left open. In fact, there will be a lot of different possibilities. Intuitively, Definition B.4 guarantees that sets B that are in some sense big enough (meaning that $\varphi(B) > 0$), are always reached from any initial state with positive probability. This will be the sets of interest.

When specifying the stability criteria for general Markov chains in the next section, an appropriate notion of so-called *small sets* introduced in Chapter 5 of [MT93] is needed that are the counterpart of finite sets in Markov chains with a countable state space.

Definition B.5 (Small Sets). A set $C \in \mathscr{B}(\mathscr{X})$ is called a *small set* if there exists an $m > 0$, and a non-trivial measure ν_m on $\mathscr{B}(\mathscr{X})$, such that

$$P^m(x, B) \geq \nu_m(B) \tag{B.3}$$

for all $x \in C$, $B \in \mathscr{B}(\mathscr{X})$.

In the following, we give a definition of aperiodicity as in Chapter 5 of [MT93].

Definition B.6 (Strong Aperiodicity). A Markov chain is said to be *strongly aperiodic* if there exist a small set A with $\nu_1(A) > 0$ and ν_1 satisfying (B.3) for $m = 1$.

Aperiodicity prevents the chain from having a cyclic behavior, i.e. the chain is admitted to return to certain sets only at specific time points which occur periodically. A periodic behavior is particularly undesirable when proving ergodicity properties of the Markov chain.

It should be note that there exists a generalization of small sets called *petite sets*, which are introduced in Chapter 5 of [MT93]. However, if the Markov chain is φ-irreducible and (strongly) aperiodic, then every petite set is also small, see Theorem 5.5.7 in [MT93]. For the purposes of this thesis, it is therefore sufficient to restrict our attention to the collection of small sets within the state space $\mathscr{X}$, as the Markov chains considered can be assumed to be strongly aperiodic throughout this thesis.

B.3 Stochastic Stability

Here, we consider three different notions of stability: Harris recurrence, positive Harris recurrence, and ergodicity. We take the following definition of Harris recurrence in terms of the first return time τ_A, see [MT94].

Definition B.7 (Harris Recurrence). Let $\{x_k\}$ be a φ-irreducible Markov chain. Then, $\{x_k\}$ is said to be *Harris recurrent* if

$$\mathsf{P}_x(\tau_k \leq \infty) = 1$$

for every $x \in \mathscr{X}$ and every $A \in \mathscr{B}(\mathscr{X})$ with $\varphi(A) > 0$.

An alternative definition of Harris recurrence that is equivalent to that above is given as follows, see also Chapter 9 of [MT93]: A Markov chain is Harris recurrent if every set A with $\varphi(A) > 0$ is visited infinitely many times P-almost surely.

Adopted from Chapter 10 of [MT93], we define the *invariant measure* of a Markov chain as follows.

Definition B.8 (Invariant Measure). Let π be a σ-finite measure on $\mathscr{B}(\mathscr{X})$. Then, π is said to be *invariant* if it has the property

$$\pi(A) = \int_{\mathscr{X}} \pi(dx)P(x,A), \quad A \in \mathscr{B}(\mathscr{X}). \tag{B.4}$$

It is well-known that an invariant measure π exists if the Markov chain $\{x_k\}$ is Harris recurrent. This leads us to a stronger notion of stability given by the next definition.

Definition B.9 (Positive Harris Recurrence). Let $\{x_k\}$ be Harris recurrent. Then, $\{x_k\}$ is said to be *positive Harris recurrent* if the invariant measure π defined in Eq. (B.4) is finite.

Subsequently, we introduce the notion of *drift criteria* which resembles Lyapunov stability theory for deterministic dynamical systems. The following definition can be found in Chapter 8 of [MT93].

Definition B.10 (Drift). The drift operator Δ for any non-negative measurable function V on $\mathscr{X}$ is defined by

$$\Delta V(x) := \int_{\mathscr{X}} P(x,dy)V(y) - V(x), \quad x \in \mathscr{X}.$$

With slight abuse of notation with regard of the conditional expectation, we often compute the drift $\Delta V(x)$ by $\Delta V(x_k) = \mathsf{E}[V(x_{k+1})|x_k] - V(x_k)$ with $x_k \in \mathscr{X}$. The following drift criterion related to Theorem 11.0.1 in [MT93] is a useful means to prove stochastic stability.

Theorem B.3 (Foster's Criterion). *Let $\{x_k\}$ be φ-irreducible. Then, $\{x_k\}$ is positive Harris recurrent if there exists a small set $C \in \mathscr{B}(\mathscr{X})$ and a non-negative function V on $\mathscr{X}$ such that*

$$\Delta V(x) \leq -1, \quad x \in C^c, \tag{B.5}$$

and ΔV is uniformly bounded on C.

The following theorem shows the correspondence of positive Harris recurrence and ergodicity by assuming that the chain is aperiodic, see also Theorem 13.0.1 in [MT93].

Theorem B.4 (Aperiodic Ergodic Theorem). *Let $\{x_k\}$ be φ-irreducible and strongly aperiodic. If $\{x_k\}$ is an positive Harris recurrent chain with invariant probability measure π, then the chain is ergodic, i.e.,*

$$\|P^n(x, \cdot) - \pi\| \to 0 \tag{B.6}$$

as $n \to \infty$ for any initial state $x \in \mathscr{X}$.

In the above theorem, $\|\cdot\|$ denotes the total variation norm defined as

$$\|\mu\| = \sup_{A \in \mathscr{B}(\mathscr{X})} \mu(A) - \inf_{A \in \mathscr{B}(\mathscr{X})} \mu(A),$$

where μ is a signed measure on $\mathscr{B}(\mathscr{X})$. In Theorem 13.0.1 of [MT93], a weaker notion than strongly aperiodicity is used. However, the Markov chains considered in this thesis turn out to be strongly aperiodic, which lets us resort to the above theorem.

C

Dynamic Programming

Here, we review the basic principles of optimal stochastic control – the theory of optimal sequential decision making under stochastic disturbances. The presentation of the subsequent sections follows the textbooks [Ber05; Ber07; HL89; HLL96].

C.1 Finite Horizon Problems

In the following, we consider a u_k-controlled Markov chain described by the following difference equation model

$$x_{k+1} = f_k(x_k, u_k, w_k), \qquad k \in \{0, 1, \ldots\} \tag{C.1}$$

where the state x_k takes values in $\mathscr{X} \subset \mathbb{R}^n$, the input takes values in $\mathscr{A}(x)$ and the noise process $\{w_k\}$ is i.i.d. and independent of the initial condition x_0. The noise w_k takes values in some Borel space $\mathscr{W}$ with a common distribution ϕ. The function $f_k : \mathscr{X} \times \mathscr{A} \times \mathscr{W} \to \mathscr{X}$ is assumed to be a Borel measurable mapping for all $k \in \mathbb{Z}^+$.

The control problem we are interested in is the minimization of the *finite-horizon* performance criterion

$$J(\pi, x) = \mathsf{E}_x^\pi[c_T(x_T) + \sum_{k=0}^{T-1} c_k(x_k, u_k)]$$

with c_k as the running cost and c_T as the terminal cost. Both are measurable non-negative functions on their corresponding spaces. In general, it can be distinguished between randomized and deterministic (pure) policies π. We are primarily focussed on deterministic policies defined by the admissible set of policies $\mathscr{U}$. The policy π can be chosen out of the set of admissible policies denoted by $\mathscr{U}$ that covers all measurable mapping of the past observation history $\mathscr{I}_k = \{X^k, U^{k-1}\}$, i.e.,

$$u_k = \pi_k(\mathscr{I}_k) \tag{C.2}$$

By defining the value function $J^*(x)$ as

$$J^*(x) = \inf_{\pi \in \mathscr{U}} J(\pi, x), \qquad x \in \mathscr{X}, \tag{C.3}$$

we want to find a policy $\pi^* \in \mathscr{U}$, s.t.

$$J(\pi^*, x) = J^*(x), \qquad \text{for all} \quad x \in \mathscr{X}. \tag{C.4}$$

The subsequent class of policies plays a crucial role in optimal control problems with full state information.

Definition C.1 (Markov Policy). A policy $\pi \in \mathscr{U}$ is said to be a *Markov policy* if the law π_k is a measurable mapping with regard to the state x_k for every $k \in \{0, \dots, N-1\}$.

The following theorem adopted from Chapter 3 of [HLL96] states that the optimal solution of problem can be solved by the dynamic programming algorithm defined in Eq. (C.5).

Theorem C.1 (Dynamic Programming Theorem). *Let J_k with $k \in \{0, \dots, T-1\}$ be real-valued functions on $\mathscr{X}$ defined recursively (backwards in time) as*

$$\begin{aligned} J_T(x_N) &= c_T(x_N), \\ J_k(x_k) &= \min_{u_k \in \mathscr{A}(x)} \left[c_k(x_k, u_k) + \mathsf{E}[J_{k+1}(f_k(x_k, u_k, w_k)) | x_k, u_k] \right] \end{aligned} \tag{C.5}$$

Suppose that these functions are measurable and suppose that there exists a deterministic Markov policy $\pi_k^*(x_k)$ that attains the minimum in (C.5) for $k \in \{0, \dots, N-1\}$.
Then, the deterministic Markov policy $\pi* = [\pi_0^*, \dots, \pi_{N-1}^*]$ is optimal and the value function J^* is given by

$$J^*(x) = J_0(x) = J(\pi^*, x), \qquad \textit{for all} \quad x \in \mathscr{X}.$$

The *measurable selection condition* discussed in Section 3.3 of [HLL96] guarantees that the resulting policy yields a measurable selection of the optimal control inputs. This basically implies that the measurability assertions in the above theorem hold and that the “inf”-operation can be replaced by the “min”-operation in the dynamic programming algorithm. Throughout the thesis, we are mainly concerned with a finite number of choices for the input that is reflected by the triggering variable taking values 0 and 1. Therefore, the measurability issue will be a minor problem and we can assume that a solution exists and conveniently take the min-operator in the dynamic programming equation as given in Eq. (C.5)

C.2 Problems with Partial State Information

Suppose the u_k-controlled Markov chain defined in Eq. (C.1). However, instead of assuming that the complete state x_k is available to the decision maker, we suppose in the following that the controller has merely access to the observations z_k defined as

$$z_k = h_k(x_k, v_k), \qquad k \in \{0, \dots, N-1\},$$

where the observation noise process $\{v_k\}$ is i.i.d. and takes values in a Borel measurable space $\mathscr{V}$ and is independent of w_k and x_0.

The information available at the controller is now given by the observation history $\mathscr{I}_k = \{Z^k, U^{k-1}\}$. Subsequently, we aim at finding the optimal solution of Eq. (C.4) by reducing the problem to an optimal control problem with full state information. We assume that the admissible set of control inputs, $\mathscr{A}$, does not depend on the state. We define the *information state* recursively by the difference equation model

$$I_{k+1} = (I_k, z_{k+1}, u_k), \quad k \in \{0, \dots, N-2\}$$

with $I_0 = z_0$. Considering I_k as our new state, we obtain the following expression for the *cost-to-functions* J_k from Chapter 5 of [Ber05]:

$$\begin{aligned} J_k(I_k) &= \min_{u_k \in \mathscr{A}} \left[\mathsf{E}[c_k(x_k, u_k) + J_{k+1}(f_k(x_k, u_k, w_k)) | I_k, u_k] \right], \quad k \in \{0, \dots, N-2\}, \\ J_{T-1}(I_{T-1}) &= \min_{u_{T-1} \in \mathscr{A}} \left[\mathsf{E}[c_{T-1}(x_{T-1}, u_{T-1}) + c_T(f_{T-1}(x_{T-1}, u_{T-1}, w_{T-1})) | I_{N-1}, u_{N-1}] \right] \end{aligned} \tag{C.6}$$

In order to reduce the data that is truly necessary for obtaining the optimal control law, we seek for so-called *sufficient statistics* which comprise all essential information needed for control in the following sense. Suppose that there is a function $S_k(I_k)$ of the information state, such that the minimization of the right-hand side in Eq. (C.6) can be reformulated as

$$J_k(I_k) = \min_{u_k \in \mathscr{A}} Q_k(S_k(I_k)) = J'(S_k(I_k)). \tag{C.7}$$

Then, the function S_k is said to be a *sufficient statistic* and the corresponding optimal control law has the form $u_k = \pi'_k(S_k(I_k))$ for $k \in \{0, \dots, N-1\}$. Certainly, the identity function is a sufficient statistic. Another important sufficient statistic for optimal control problems with partial state information under the above assumptions on the noise models is the conditional state distribution $\mathsf{P}_{x_k|I_k}$ of the state x_k given the information I_k, see [BS78].

A stronger notion of sufficient statistics is the concept of certainty equivalence control. A *certainty equivalence controller* is given by solving first the related deterministic optimal control problem, where all primitive random variables are set to their "typical " values. Within the solution of the deterministic optimal control problem, the state variable is then replaced by its "typical " value based on the available information. The common choice for "typical" values of the disturbances and estimate that is also assumed here is given by their mean values $\mathsf{E}[w_k]$ and the least-squares estimate $\mathsf{E}[x_k|I_k]$, respectively.

The solution of an optimal control problem is said to have the *certainty equivalence property* if the it takes the form of the certainty equivalence controller.

In the context of sufficient statistics, it can be concluded that the certainty equivalence property implies that the conditional mean $\mathsf{E}[x_k|I_k]$ is sufficient statistic of the optimal control problem.

C.3 Long-Run Average-Cost Problems

Suppose a time-homogeneous version of the u_k-controlled Markov chain defined in Eq. (C.1) by dropping the index k of the function f_k, and suppose the control law defined as in Eq. (C.2) for $k \geq 0$. In this section, we consider the problem of minimizing the long-run average expected cost per time step, abbrev. average-cost (AC) criterion that is defined as

$$J(\pi, x) = \limsup_{T\to\infty} \frac{1}{T} \mathsf{E}_x^\pi[\sum_{k=0}^{T-1} c(x_k, u_k)]$$

with the per-stage cost $c(\cdot,\cdot)$ being a measurable non-negative bounded function. A policy π^* is said to be *(AC) optimal* if it satisfies $J(\pi^*, x) = J^*(x)$ for all $x \in \mathscr{X}$, where the value function $J^*(x)$ is defined in Eq. (C.3).

The set of stationary policies within the class of Markov policies takes a crucial role for AC optimal control problems.

Definition C.2 (Stationary Policy). A Markov policy $\pi \in \mathscr{U}$ is said to be a *stationary (Markov) policy* if the laws do not depend on the time k, i.e., $\pi = [\pi', \pi', \ldots]$.

The following theorem adopted from Chapter 3 of [HL89] optimality conditions in order to find the AC optimal policy π^*.

Theorem C.2 (AC Bellman Equation). *Suppose there exists a constant j^* and a real-valued bounded measurable function h^* such that*

$$h^*(x) + j^* = \min_{u\in\mathscr{A}(x)} \left[c(x,u) + \mathsf{E}[h^*(f(x,u,w))]\right], \quad \text{for all} \quad x \in \mathscr{X} \tag{C.8}$$

with $w \in \mathscr{W}$ having the distribution ϕ. Then, we have

1. $\inf_\pi J(\pi, x) \geq j^*$, *for all* $x \in \mathscr{X}$
2. *If $\pi^* = [\pi', \pi', \ldots]$ is a stationary policy that minimizes the right-hand side of the AC Bellman Equation given by Eq. (C.8), i.e.,*

   $$h^*(x) + j^* = c(x, \pi'(x)) + \mathsf{E}[h^*(f(x, \pi'(x), w_k))], \quad \text{for all} \quad x \in \mathscr{X},$$

   then π^* is AC optimal and $J(\pi^*, x) = j^*$ for all $x \in \mathscr{X}$.

In order to ensure the existence of the solution for the AC Bellman Equation defined by Eq. (C.8), several ergodicity conditions are introduced in Section 3.3 of [HL89]. The majority of these conditions also guarantee the convergence of the value iteration

$$h_{i+1} = \mathscr{T} h_i, \qquad i \in \{0, 1, \ldots\}, \tag{C.9}$$

with h_0 being an arbitrary real-valued bounded measurable function on $\mathscr{X}$ and $\mathscr{T}$ being the Bellman operator defined as the right-hand side of Eq. (C.8), i.e.,

$$\mathscr{T} h(x) = \min_{u\in\mathscr{A}(x)} \left[c(x,u) + \mathsf{E}[h^*(f(x,u,w))]\right], \quad \text{for all} \quad x \in \mathscr{X}.$$

An appropriate transformation that shifts h_i vertically after each step may be necessary in order to attain convergence.

Bibliography

[Alt99] E. Altman, *Constrained Markov decision processes*. CRC Press, 1999, vol. 7.

[ABFG+93] A. Arapostathis, V. S. Borkar, E. Fernández-Gaucherand, M. K. Ghosh, and S. I. Marcus, "Discrete-time controlled markov processes with average cost criterion: a survey," *SIAM Journal on Control and Optimization*, vol. 31, no. 2, pp. 282–344, 1993.

[AMA+14] J. Araujo, M. Mazo, A. Anta, P. Tabuada, and K. H. Johansson, "System architectures, protocols and algorithms for aperiodic wireless control systems," *Industrial Informatics, IEEE Transactions on*, vol. 10, no. 1, pp. 175–184, 2014.

[Årz99] K.-E. Årzén, "A simple event-based pid controller," in *Proc. 14th IFAC World Congress*, vol. 18, 1999, pp. 423–428.

[Åst06] K. J. Åström, *Introduction to stochastic control theory*. Dover Publications, 2006.

[ÅB02] K. J. Åström and B. M. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems," in *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, vol. 2, Dec. 2002, pp. 2011–2016.

[ÅW11] K. J. Åström and B. Wittenmark, *Computer-controlled systems: theory and design*, 3rd. Courier Dover Publications, 2011.

[BSL93] Y. Bar-Shalom and X. R. Li, "Estimation and tracking- principles, techniques, and software," *Norwood, MA: Artech House, Inc, 1993.*, 1993.

[BST74] Y. Bar-Shalom and E. Tse, "Dual effect, certainty equivalence, and separation in stochastic control," *Automatic Control, IEEE Transactions on*, vol. 19, no. 5, pp. 494–500, 1974.

[Ber99] D. P. Bertsekas, *Nonlinear programming*. Athena Scientific, 1999.

[Ber05] ——, *Dynamic programming and optimal control. Vol. I*, 3rd ed. Athena Scientific, Belmont, MA, 2005.

[Ber07] ——, *Dynamic programming and optimal control. Vol. II*, 3rd ed. Athena Scientific, Belmont, MA, 2007.

[BS78] D. P. Bertsekas and S. E. Shreve, *Stochastic optimal control: The discrete time case*. Academic Press New York, 1978, vol. 139.

[BA11a] R. Blind and F. Allgöwer, "Analysis of networked event-based control with a shared communication medium: Part I - Pure ALOHA," in *IFAC World Congress*, 2011.

[BA11b] ——, "Analysis of networked event-based control with a shared communication medium: Part II - Slotted ALOHA," in *IFAC World Congress*, 2011.

[BA11c] R. Blind and F Allgöwer, "On the optimal sending rate for networked control systems with a shared communication medium," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, IEEE, 2011, pp. 4704–4709.

[BMP08] C. Bordenave, D. McDonald, and A. Proutiere, "Performance of random medium access control, an asymptotic approach," *ACM SIGMETRICS Performance Evaluation Review*, vol. 36, no. 1, pp. 1–12, 2008.

[BM97] V. S. Borkar and S. K. Mitter, "LQG control with communication constraints," *Communications, Computation, Control, and Signal Processing: A Tribute to Thomas Kailath*, pp. 365–373, 1997.

[BV04] S. P. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge Univ. Press, 2004.

[CH08] A. Cervin and T. Henningsson, "Scheduling of event-triggered controllers on a shared network," in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, IEEE, 2008, pp. 3601–3606.

[Cog09a] R. Cogill, "Event-based control using quadratic approximate value functions," in *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, Dec. 2009, pp. 5883 –5888.

[Cog09b] R. Cogill, "Event-based control using quadratic approximate value functions," in *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, IEEE, 2009, pp. 5883–5888.

[CFM05] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*, 1st. Springer London, 2005.

[DFJ12] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *Automatic Control, IEEE Transactions on*, vol. 57, no. 5, pp. 1291–1297, 2012.

[DH12] M. C. F. Donkers and W. P. M. H. Heemels, "Output-based event-triggered control with guaranteed linf-gain and improved and decentralized event-triggering," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1362–1376, 2012.

[EV89] A. Ephremides and S. Verdu, "Control and optimization methods in communication network problems," *Automatic Control, IEEE Transactions on*, vol. 34, no. 9, pp. 930–942, 1989.

[GG92] A. Gersho and R. M. Gray, *Vector quantization and signal compression*. Springer, 1992.

[Gra00] C. Graham, "Chaoticity on path space for a queueing network with selection of the shortest queue among several," *Journal of Applied Probability*, vol. 37, no. 1, pp. 198–211, 2000.

[GHJ+13] L. Grüne, S. Hirche, O. Junge, P. Koltai, D. Lehmann, J. Lunze, A. Molin, R. Sailer, M. Sigurani, S. Stöcker, and F. Wirth, "Event-based control," in *Control Theory of Digitally Networked Dynamic Systems*, J. Lunze, Ed., Springer, 2013, pp. 169–263.

[HMY08] B. Hajek, K. Mitzel, and S. Yang, "Paging and registration in cellular networks: jointly optimal policies and an iterative algorithm," *Information Theory, IEEE Transactions on*, vol. 54, no. 2, pp. 608–622, 2008.

[HLP52] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*. Cambridge university press, 1952.

[HDT13] W. P. M. H. Heemels, M. C. F. Donkers, and A. R. Teel, "Periodic event-triggered control for linear systems," *Automatic Control, IEEE Transactions on*, vol. 58, no. 4, pp. 847–861, 2013.

[HSVDB08] W. P. M. H. Heemels, J. H. Sandee, and P. P. J. Van Den Bosch, "Analysis of event-driven controllers for linear systems," *International journal of control*, vol. 81, no. 4, pp. 571–590, 2008.

[HC10] T. Henningsson and A. Cervin, "A simple model for the interference between event-based control loops using a shared medium," in *Decision and Control (CDC), 2010 49th IEEE Conference on*, Dec. 2010, pp. 3240 –3245.

[HL89] O. Hernández-Lerma, *Adaptive Markov control processes*. Springer New York, 1989, vol. 79.

[HLL96] O. Hernández-Lerma and J. B. Lasserre, *Discrete-time Markov control processes: Basic optimality criteria*. Springer New York, 1996.

[HC72] Y.-C. Ho and K.-C. Chu, "Team decision theory and information structures in optimal control problems–Part I," *Automatic Control, IEEE Transactions on*, vol. 17, no. 1, pp. 15 –22, Feb. 1972.

[IB10] O. C. Imer and T. Basar, "Optimal estimation with limited measurements," *International Journal of Systems, Control and Communications*, vol. 2, no. 1, pp. 5–29, 2010.

[JSSW10] L. Jiang, D. Shah, J. Shin, and J. Walrand, "Distributed random access algorithm: scheduling and congestion control," *Information Theory, IEEE Transactions on*, vol. 56, no. 12, pp. 6182–6207, 2010.

[JW10] L. Jiang and J. Walrand, "A distributed csma algorithm for throughput and utility maximization in wireless networks," *IEEE/ACM Transactions on Networking (TON)*, vol. 18, no. 3, pp. 960–972, 2010.

[KSH00] T. Kailath, A. H. Sayed, and B. Hassibi, *Linear Estimation*. Prentice Hall New Jersey, 2000.

[KGOS11] J. Karlsson, A. Gattami, T. J. Oechtering, and M. Skoglund, "Iterative source-channel coding approach to Witsenhausen's counterexample," in *American Control Conference (ACC'11)*, 2011, pp. 5348 –5353.

[KS10] J. Karlsson and M. Skoglund, "Optimized low-delay source-channel-relay mappings," *Communications, IEEE Transactions on*, vol. 58, no. 5, pp. 1397–1404, 2010.

[KY03] H. J. Kushner and G. Yin, *Stochastic approximation and recursive algorithms and applications*. Springer Verlag, 2003, vol. 35.

[LL14] L. Li and M. D. Lemmon, "Weakly coupled event triggered output feedback system in wireless networked control systems," *Discrete Event Dynamic Systems*, vol. 24, no. 2, pp. 247–260, 2014.

[LWL13] L. Li, Z. Wang, and M. D. Lemmon, "Polynomial approximation of optimal event triggers for state estimation problems using sostools," in *American Control Conference (ACC), 2013*, IEEE, 2013, pp. 2699–2704.

[LM11] G. Lipsa and N. Martins, "Remote state estimation with communication costs for first-order LTI systems," *Automatic Control, IEEE Transactions on*, vol. 56, no. 9, pp. 2013–2025, 2011.

[LL99] S. H. Low and D. E. Lapsley, "Optimization flow control—i: basic algorithm and convergence," *Networking, IEEE/ACM Transactions on*, vol. 7, no. 6, pp. 861–874, 1999.

[LL10] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, 2010.

[MB11] I. Matei and J. S. Baras, "Optimal state estimation for discrete-time markovian jump linear systems, in the presence of delayed output observations," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2235–2240, 2011.

[MT11] M. Mazo and P. Tabuada, "Decentralized event-triggered control over wireless sensor/actuator networks," *Automatic Control, IEEE Transactions on*, vol. 56, no. 10, pp. 2456–2461, 2011.

[MPD67] L. Meier III, J. Peschon, and R. Dressler, "Optimal control of measurement subsystems," *Automatic Control, IEEE Transactions on*, vol. 12, no. 5, pp. 528 –536, Oct. 1967.

[MUA12] M. Meinel, M. Ulbrich, and S. Albrecht, "A class of distributed optimization methods with event-triggered communication," *Computational Optimization and Applications*, 2012.

[MT93] S. P. Meyn and R. L. Tweedie, *Markov chains and stochastic stability*, 1993.

[MT94] S. P. Meyn and R. L. Tweedie, “State-dependent criteria for convergence of markov chains,” *The Annals of Applied Probability*, pp. 149–168, 1994.

[MH09] A. Molin and S. Hirche, “On LQG joint optimal scheduling and control under communication constraints,” in *Decision and Control (CDC), 48th IEEE Conference on*, 2009, pp. 5832–5838.

[MH10a] ——, “Optimal event-triggered control under costly observations,” in *19th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, 2010.

[MH10b] ——, “Structural characterization of optimal event-based controllers for linear stochastic systems,” in *Decision and Control (CDC), 49th IEEE Conference on*, 2010, pp. 3227–3233.

[MH10c] ——, “Suboptimal Event-based Control of Linear Systems over Lossy Channels,” in *2nd IFAC Workshop on Estimation and Control of Networked Systems (NecSys)*, 2010, pp. 55–60.

[MH11] ——, “Optimal design of decentralized event-triggered controllers for large-scale systems with contention-based communication,” in *Decision and Control and European Control Conference (CDC-ECC), 50th IEEE Conference on*, 2011, pp. 4710–4716.

[MH12a] ——, “Adaptive event-triggered control over a shared network,” in *Decision and Control (CDC), 51st IEEE Conference on*, 2012, pp. 6591–6596.

[MH12b] ——, “An iterative algorithm for optimal event-triggered estimation,” in *IFAC Conference on Analysis and Design of Hybrid Systems*, 2012, pp. 64–69.

[MH12c] ——, “Entwurf vernetzter ereignisbasierter regelungssysteme mit kommunikationsbeschränkungen,” *at-Automatisierungstechnik*, vol. 60, no. 8, pp. 476–486, 2012.

[MH12d] ——, “Event-triggered estimation of linear systems: an iterative algorithm and optimality properties,” *Arxiv preprint arXiv:1203.4980*, 2012.

[MH13a] ——, “On the optimality of certainty equivalence for event-triggered control systems,” *Automatic Control, IEEE Transactions on*, vol. 58, no. 2, pp. 470–474, 2013.

[MH13b] ——, “Suboptimal event-triggered control for networked control systems,” *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 2013. DOI: 10.1002/zamm.201100139.

[MH14] ——, “A bi-level approach for the design of event-triggered control systems over a shared network,” *Discrete Event Dynamic Systems*, vol. 24, no. 2, pp. 153–171, 2014.

[MTH11] A. Molin, H. Tischer, and S. Hirche, “Order reduction in optimal event-triggered control design for linear stochastic systems,” in *American Control Conference (ACC)*, 2011, pp. 2222–2227.

[NFZE07] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: an overview," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 108–137, 2007.

[Øks03] B. Øksendal, *Stochastic Differential Equations - An Introduction with Applications*, 6th. Springer, 2003.

[RJ09a] M. Rabi and K. H. Johansson, "Optimal stopping for updating controls," in *Proccedings of the second International workshop on sequential methods, UTT, Troyes, France*, 2009.

[RMB12] M. Rabi, G. V. Moustakides, and J. S. Baras, "Adaptive sampling for linear state estimation," *SIAM Journal on Control and Optimization*, vol. 50, no. 2, pp. 672–702, 2012.

[RJ09b] M. Rabi and K. H. Johansson, "Scheduling packets for event-triggered control," in *Proceedings of 10th European Control Conference*, 2009, pp. 3779–3784.

[RJJ08] M. Rabi, K. H. Johansson, and M. Johansson, "Optimal stopping for event-triggered sensing and actuation," in *Decision and Control, 2008, Proceedings of the 47th IEEE Conference on*, Dec. 2008, pp. 3607–3612.

[Rad62] R. Radner, "Team decision problems," *The Annals of Mathematical Statistics*, vol. 33, no. 3, pp. 857–881, 1962.

[RSJ13] C. Ramesh, H. Sandberg, and K. H. Johansson, "Design of state-based schedulers for a network of control loops," *Automatic Control, IEEE Transactions on*, vol. 58, no. 8, pp. 1962–1975, 2013.

[Res98] S. Resnick, *A probability path*. Birkhäuser Boston, 1998.

[SSF+07] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163 –187, Jan. 2007.

[SDJ13] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, vol. 49, pp. 245–252, 2013.

[SS07] S. Shakkottai and R. Srikant, "Network optimization and control," *Foundations and Trends in Networking*, vol. 2, no. 3, pp. 271–379, 2007.

[SL12] J. Sijs and M. Lazar, "Event based state estimation with time synchronous updates," *Automatic Control, IEEE Transactions on*, vol. 57, no. 10, pp. 2650–2655, 2012.

[Tab07] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *Automatic Control, IEEE Transactions on*, vol. 52, no. 9, pp. 1680–1685, 2007.

[Tan02] A. S. Tanenbaum, *Computer Networks*. Prentice Hall, 2002.

[TSM04] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *Automatic Control, IEEE Transactions on*, vol. 49, no. 9, pp. 1549–1561, Sep. 2004.

[WYB02] G. C. Walsh, H. Ye, and L. G. Bushnell, "Stability analysis of networked control systems," *Control Systems Technology, IEEE Transactions on*, vol. 10, no. 3, pp. 438 –446, May 2002.

[WL10] P. Wan and M. D. Lemmon, "Optimal power flow in microgrids using event-triggered optimization," in *American Control Conference (ACC), 2010*, IEEE, 2010, pp. 2521–2526.

[WL09] P. Wan and M. D. Lemmon, "An event-triggered distributed primal-dual algorithm for network utility maximization," in *Decision and Control, 2009, Proceedings of the 48th IEEE Conference on*, Dec. 2009, pp. 5863 –5868.

[WL11] X. Wang and M. D. Lemmon, "Event-triggering in distributed networked control systems," *Automatic Control, IEEE Transactions on*, vol. 56, no. 3, pp. 586–601, 2011.

[Wit68] H. S. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM Journal on Control*, vol. 6, no. 1, pp. 131–147, 1968.

[Wit73] ——, "A standard form for sequential stochastic control," *Theory of Computing Systems*, vol. 7, pp. 5–11, 1 1973.

[WJJS13] J. F. Wu, Q. S. Jia, K. H. Johansson, and L. Shi, "Event-based sensor data scheduling: trade-off between communication rate and estimation quality," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 1041–1046, 2013.

[WA08] W. Wu and A. Arapostathis, "Optimal sensor querying: general markovian and LQG models with controlled observations," *Automatic Control, IEEE Transactions on*, vol. 53, no. 6, pp. 1392 –1405, Jul. 2008.

[XH04] Y. Xu and J. P. Hespanha, "Optimal communication logics in networked control systems," in *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, IEEE, vol. 4, 2004, pp. 3527–3532.

[ZZC08] J. Zhang, D. Zheng, and M. Chiang, "The impact of stochastic noisy feedback on distributed network utility maximization," *Information Theory, IEEE Transactions on*, vol. 54, no. 2, pp. 645–665, 2008.

[ZC10] M. Zhong and C. G. Cassandras, "Asynchronous distributed optimization with event-driven communication," *Automatic Control, IEEE Transactions on*, vol. 55, no. 12, pp. 2735–2750, 2010.