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Algorithm for Brune's Synthesis of Multiport Circuits

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Doctoral Dissertation

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Date: December 10, 2013

Ich versichere, dass ich diese Doktorarbeit selbständig verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet habe.

München, December 10, 2013 Farooq Mukhtar

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Abstract

An algorithm is presented for the synthesis of lumped element equivalent network model for linear passive reciprocal multiports. The synthesized equivalent multiport circuit contains only resistors, inductors and capacitors with positive values, and ideal transformers. The algorithm is formulated for a positive real symmetric impedance or admittance matrix, representing the muliport and is composed of rational functions.

The developed iterative algorithm exhibits seven cases distinguished by the location of the poles and the zeros of the given matrix of functions in the complex frequency plane and a stop criterion. In each iteration, a part of the matrix is extracted and a corresponding subcircuit is synthesized according to the procedure associated with the respective case. The extraction procedure associated with the seventh case is based on the Tellegen's extension of Brune's method described for multiport impedance matrices. Accordingly, a dual Brune's method for admittance multiport circuits is established in this work. The topology of the equivalent network for the subcircuit extracted from impedance matrix during the Brune's method is already established, according to Tellegen's extension of Brune's method. New equivalent topologies of partial circuits to replace the extracted circuits from admittance matrix are introduced in this work.

Implementation of the algorithm is done in MATLAB to generate SPICE netlist for the equivalent circuit. Equivalent lumped element circuit models of microwave structures are developed as examples to demonstrate the application of the algorithm.

Contents

1 Introduction

The expanding market of modern electronic devices, especially that of the mobile communication devices, requires fast and efficient designing of digital integrated circuits (IC). Furthermore the requirement of fast, compact and multi-tasking electronic devices has always raised the clock rates and the frequency bandwidth of digital IC's, and pushed their design domain from microwave to millimeter-wave region.

Distributed microwave and millimeter-wave circuits require full wave electromagnetic analysis in order to account for the electromagnetic field propagation and to generate precise response. Different from the network oriented description which is used at lower frequencies, electromagnetic full wave analysis requires a computational effort which is by order of magnitude higher than that required for topological network analysis. Moreover, the design process usually requires iterative optimization which analyzes the given structure many times, consequently slowing the design process and making it much more difficult. To overcome the huge requirements of computational effort and to make the design process efficient, hybridization of different computational techniques and full-wave based modeling are applied. Network methods offer versatile and efficient way for hybridization of different computational techniques [1], while equivalent lumped element circuits provide compact models in the area of modeling [2–7].

Equivalent lumped element circuit models are used in design process in number of ways. One of the ways is to directly integrate them in the full-wave analysis [2–4]. This facility is commonly provided in modern electromagnetic simulation softwares like Cadence®, SONNET®, MEFiSTo, CST and Agilent ADS. The other most common use of such models is in the computer-aided design (CAD) tools. Libraries of circuits models provided in these tools allow the designer to choose the model, adjust few parameters and use them in system level design. The wide use of circuit models is due to their compactness, very small simulation time and due to preservation of physical properties of the original structure like energy and power properties, passivity, stability and reciprocity.

The modeling of distributed microwave and millimeter-wave structures is evermore required as new and sophisticated structures are emerging. The software libraries, however, cannot account for all types and kinds of structures. Moreover, a model designed for microwave region may not work for millimeter-wave region. In order to find a suitable model, usually the parameter adjustment of predefined topology of network is used. This method cannot be regarded as a general method, because finding a suitable topology is based on heuristic method and as the structure becomes more and more complicated the topology becomes more and more difficult to find. This necessitates the need of a general and systematic procedure starting from the tabular response (S, Z or Y-parameters) obtained from full wave simulation or measurement of the given structure, following step after step down to an equivalent lumped element circuit model.

The generalized procedure can be solved in two parts: first is to find the rational function (or rational function matrix) for a given tabular data of impedance or admittance which does not violate the physical conditions of passivity, stability and reciprocity and, secondly, to devise general algorithm for synthesizing equivalent lumped element circuit from the given matrix of rational functions. The first part is the problem of the *system identification* and the second part is a *synthesis* problem. For the system identification, well established procedures like *vector fitting* (V.F.) method [8–10] or *prony's* method [11] can be used. This thesis is focused on synthesis part of the whole procedure, i.e. *to develop and implement an algorithm for synthesis of a multiport lumped element equivalent circuit given a matrix of rational functions representing an impedance or an admittance matrix of a linear, reciprocal passive multiport electromagnetic structure.*

1.1 State of the Art

The research on network methods and modeling is versatile exhibiting variety of ways of application. Some researchers have developed equivalent circuit models in connection with electromagnetic fields (e.g. [12, pp. 276-279]), or in connection with certain electromagnetic method (e.g. [6, 7]). A brief survey of important works would be discussed under.

A general and systematic method for linear, reciprocal and lossless structures based on transmission line matrix (TLM) method is described in [6] and its extension to lossy structures in [7]. The elegant procedure in [7] follows a certain topology (extendable to any number of ports). Following a certain topology puts extra constrains on the rational function matrices of admittance to be synthesized, apart from the constraints put by physical properties such as passivity, stability and reciprocity. It limits the class of the functions or matrices of the functions to be synthesized. An admittance matrix of rational functions representing generally any passive and reciprocal multiport may not be synthesized by this method if it does not fulfill topology constraints.

The earliest work on the application of circuit theory concepts on electromagnetic fields was done by L. J. Chu in 1948 [13], [12, pp. 276-279]. In investigating the physical limits of the antennas, he expanded the electric and magnetic fields on an imaginary spherical surface embedding the antenna, into spherical modes and for each mode the impedance was found. Due to the spherical modes, the impedance associated with every mode have the Hankel function of second kind in their expressions. The mathematical expansion of the impedance expressions through the use of the continued fraction makes its possible to develop a equivalent lumped element network. The impedance expressions for TE and TM modes are given as [12, 14–16]:

$$Z_{mn}^{TM} = j\eta \frac{H_n^{(2)'}(kr)}{H_n^{(2)}(kr)} = \eta \left\{ \frac{n}{jkr} + \cfrac{1}{\cfrac{2n-1}{jkr} + \cfrac{1}{\cfrac{2n-3}{jkr} + \ddots + \cfrac{1}{\cfrac{3}{jkr} + \cfrac{1}{\cfrac{1}{jkr} + 1}}}} \right\}, \quad (1.1)$$

$$Z_{mn}^{TE} = -j\eta \frac{H_n^{(2)}(kr)}{H_n^{(2)'}(kr)} = \eta \left\{ \cfrac{1}{\cfrac{n}{jkr} + \cfrac{1}{\cfrac{2n-1}{jkr} + \cfrac{1}{\cfrac{2n-3}{jkr} + \ddots + \cfrac{1}{\cfrac{3}{jkr} + \cfrac{1}{\cfrac{1}{jkr} + 1}}}}} \right\}, \quad (1.2)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the wave impedance of the material surrounding the antenna with permitivity ϵ and permeability μ, and $H_n^{(2)}(kr) = krh_n^{(2)}(kr)$, where $h_n^{(2)}(kr)$ is Hankel function of second kind. The prime $'$ denotes the derivative of $H_n^{(2)}(kr)$ with respect to kr ($k = \omega\sqrt{\mu\epsilon}$). The continued fraction expansion of the impedance forms a ladder network with reactive elements terminated by a resistance of value equal to the wave impedance η. The circuit realizations for both TE and TM modes are shown in Fig. 1.1 (below). The advantage of this equivalent circuit is having direct connection between the value of circuit elements with the dimension of the sphere and the material properties surrounding the sphere. On the other hand the drawback of such approach is lack of generality as the problem is only confined to radiating structures in infinite space. If some reflecting surface is placed outside the imaginary sphere, the circuit becomes invalid as it does not incorporate reaction of the reflecting surface.

Petr Lorenz used the Chu's work in hybridization of circuit model and transmission line matrix method (TLM) [15, 16]. The radiating structures were simulated using TLM with spherical boundary terminated by spherical modes circuit models. The choice of TLM method for hybridization is excellent because the TLM cell has twelve ports (two on each of the six faces) and the faces of the TLM cells at the boundary of the sphere can easily be connected to the equivalent network of outer space. The concept of Lorenz's work is shown in Fig. 1.2 (below).

It is well known that in the case of distributed circuits generalized voltages and generalized currents can be be introduced at the boundaries. Their description is based on the

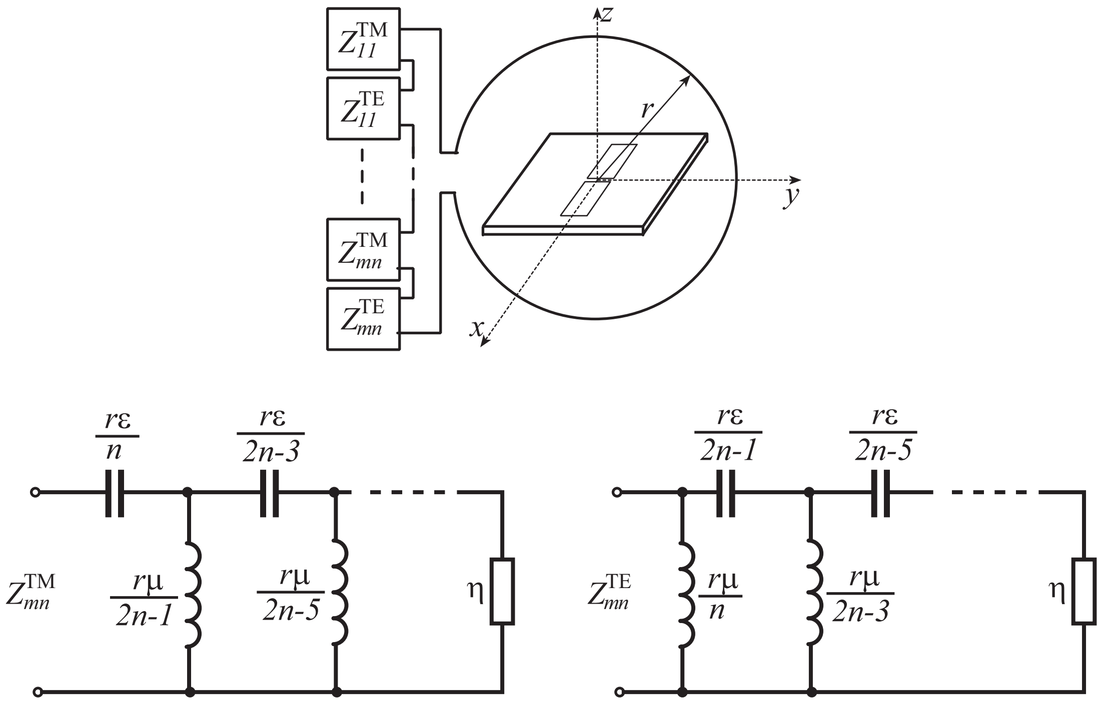


Figure 1.1: Circuit representation of the infinite free space outside imaginary sphere around any radiating electromagnetic structure. Each spherical mode (TE or TM) is represented by one of the ladder networks [13], [12, pp. 276-279].

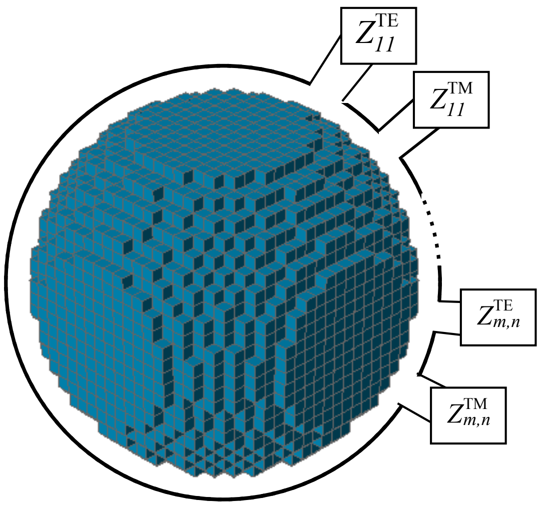


Figure 1.2: Hybridization of transmission line matrix method (TLM) and equivalent circuit [15, 16]; The structure inside the sphere is simulated using TLM and outer space with equivalent circuit according to Chu's work.

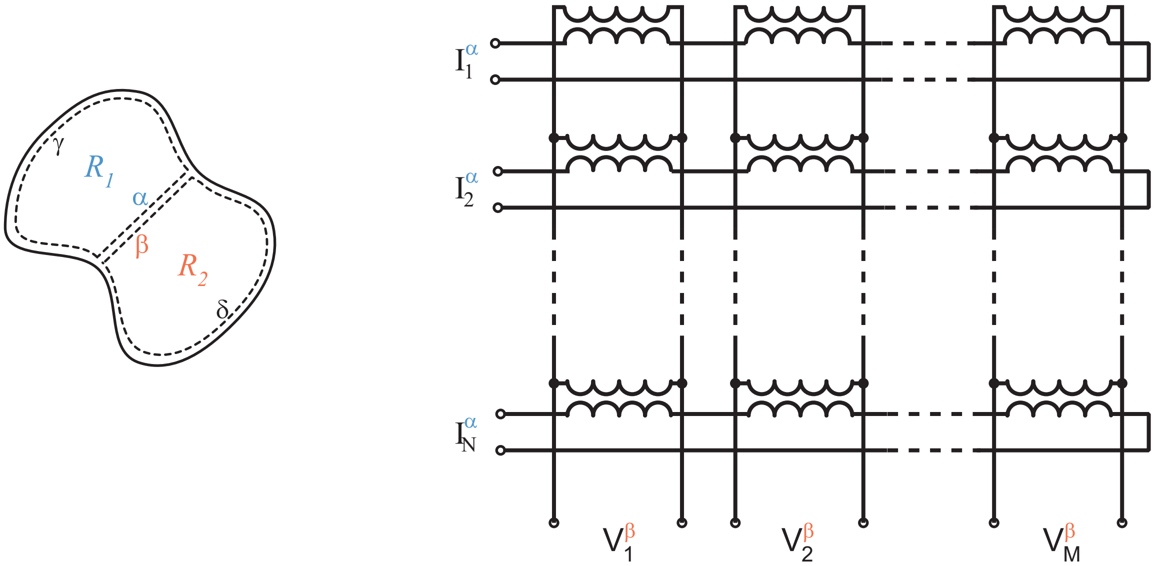


Figure 1.3: Generalized hybridization of different electromagnetic computation methods [1]. Different regions (R_1 and R_2) of computational domain being solved through different EM computation techniques can be connected through network of transformers at the boundaries α and β.

definitions of impedance and power flow in the electromagnetic field [14, Ch. 8] [1, Ch. 3 §IV]. For any boundary in the lossless structure, with local coordinate system (u, v, n), transverse electric field, E_t, and transverse magnetic field, H_t, can be expanded into a series of orthogonal modes (Transverse Electric (TE) and Transverse Magnetic (TM) modes) and each mode can further be represented as a product of *structure functions* $\boldsymbol{e}(u, v)$ and $\boldsymbol{h}(u, v)$ depending only on transverse coordinates (u and v) and *scalar amplitudes* $V(n)$ and $I(n)$ which are generalized voltages and currents respectively [14], depending only on longitudinal coordinate (n):

$$E_t(u,v,n) = \sum_{p,q}[E^{TE}_{t,pq} + E^{TM}_{t,pq}] = \sum_{p,q}[V^{TE}_{pq}(n)\boldsymbol{e}^{TE}_{pq}(u,v) + V^{TM}_{pq}(n)\boldsymbol{e}^{TM}_{pq}(u,v)]\,,$$
$$H_t(u,v,n) = \sum_{p,q}[H^{TE}_{t,pq} + H^{TM}_{t,pq}] = \sum_{p,q}[I^{TE}_{pq}(n)\boldsymbol{h}^{TE}_{pq}(u,v) + I^{TM}_{pq}(n)\boldsymbol{h}^{TM}_{pq}(u,v)]\,.$$

Each pair of the generalized voltage, $V_{pq}(n)$, and the generalized current, $I_{pq}(n)$, belonging to one mode (p, q) can be used to define port voltage and port current. Their ratio defines an impedance or an admittance which can be synthesized into an equivalent lumped element circuit model.

Generalized hybridization of electromagnetic computation methods was presented by Felsen, Mongiardo and Russer using circuit network techniques [1]. The Fig. 1.3 shows the basic concept of hybridization. A computational domain can be sub divided into

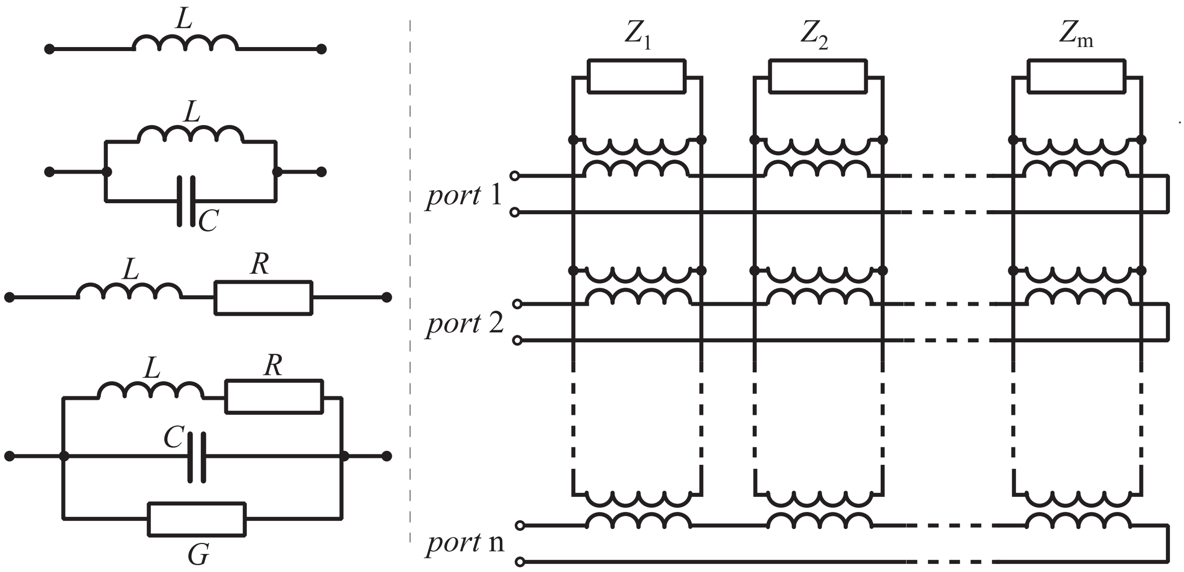


Figure 1.4: Generalized foster synthesis for impedance matrices and its possible circuits representing one pole or conjugate pole pair. For lossy circuits it may give elements with negative values.

different regions. Each region can be simulated using different techniques and the connection over the mutual boundary would be provided by a network of transformers. Each mode at the boundary of one region is connected to each mode of the other boundary. The use of ideal transformers makes it possible to realize zero-volume region between the mutual boundaries as well as preserve the energy and power properties.

Lumped element networks have advantage over digital filters of preserving energy and power properties of the original structure, and can also be translated easily to wave-digital filters [17, 18].

Apart from variety of ways network methods are used in electromagnetics, synthesis problem has been addressed in the area of circuit theory extensively. Foster's work [19] is popularly regarded as the beginning of circuit synthesis. It results in a series or a parallel connection of resonant sub-circuits, forming a network for pure reactive impedance or admittance. Foster synthesis for lossless one-port circuits, can be generalized to multiport lossless circuits by synthesizing each pole or conjugate pole pair separately, as done in [14]. It can further be generalized for lossy multiport circuits by including resistances as shown in Fig. 1.4. This simple and straight forward method does not guarantee occurrence of only positive values of the elements, because some pole pairs along with their residues may result in elements with negative values. Contrary to that, the Brune's algorithm only synthesizes equivalent circuits with positive element values, thus ensuring stability and passivity at all frequencies.

Ladder networks involving only two elements (either RL or RC) were introduced by

Table 1.1: Summary of Synthesis Procedures (adopted from [20])

Items	Author	Reference	Ports	Notes
1	Foster	[19]	1	Lossless circuits
2	Cauer	[21, 22]	1	Involves two elements
3	Brune	[23]	1	First lossy synthesis
4	Darlington	[24]	1	Lossy, only one resistance
5	Bott-Duffin	[25]	1	Transformer-less
6	Gewertz	[26]	2	Extension of [24]
7	Tellegen	[27]	n	For Impedance Matrices
8	McMillan	[28]	n	Realizability Theory
9	Baum	[29–31]	-	Positive functions (PF)
10	Belevitch	[32]	1	Brune's alternative using PF
11	Belevitch	[33]	n	Generalized for positive matrices
12	Oono	[34]	n	
13	Oono	[35, 36]	n	Synthesis from S-parameters

Cauer [21]. Although losses were considered in these networks but the question arose that do they model all impedances or admittances, described as a function of complex frequency, which was investigated by Brune.

Brune under the supervision of Guillemin defined the class of complex functions which he named *positive real* functions and described the generalized procedure for synthesizing a 1-port network from a positive real function [23]. Brune's work has such a fundamental importance that for almost next four decades the area of circuit synthesis was flourished based on his work.

Brune's work was extended by Tellegen [27] to n-ports. He defined the poles and zeros for positive real symmetric matrices (chapter 2) and described Brune's cycle for generalized impedance matrices. Extention of the work of Tellegen from impedance matrices to admittance matrices is presented in this work. Tellegen introduced a network topology (referred to as Brune's circuit of Type-I in this writing) which made it possible to avoid negative value elements formed in extraction procedure. Analogous to Brune's circuit of Type-I, three new topologies (Type-II to Type-IV) are also introduced here for admittance matrices.

Realizability theory by McMillan is an extensive mathematical treatment of theorems and procedures related to circuit synthesis [28]. Details on the properties of the degree of the positive real matrix and its operations are also described by Duffin in [37, 38]

Baum generalized the positive real function to *positive function* (PF) in which the real part of the function on the imaginary axis of complex frequency s must be positive but the condition of having real value of the function for real s is not required [29–31]. Positive functions can have complex coefficients unlike positive real functions in which the coefficients must be real numbers. Thus PF form a super-set of the set of positive real functions. Belevitch then made use of PF in alternative derivation of Brunes process for one-port in [33]. While describing the alternative process he made use of hypothetical resistance with imaginary value.

Belevitch combined Tellegen's and McMillan's work together in [33]. In this work he also extended the concept of positive function to *positive matrix* (PM). Positive matrices describe all types of networks, lossy, lossless, reciprocal or non-reciprocal and even real and imaginary networks. Not only the description of PM, Belevitch has also pointed out useful procedures and theorems to synthesize positive matrices by introducing ideal transformers with complex turns ratios.

Completely different from Brune's approach, Darlington developed a technique which results in a lossless two-port terminated at one-port by a resistance [24]. The detailed discussion is given in [39, Ch. 9]. Darlington synthesis was extended to two port network using two resistances by Gewertz [26].

Other works have been done by Bott, Duffin, Hazony and Oono. Synthesis of networks through hybrid matrices is described in [40], transformer-less synthesis in [25] and [39, Ch. 10]. Oono has extended Brune's work in [34,41]. It is also well mentioning and interesting that Oono also developed a method for synthesizing passive reciprocal network from S-parameters [35,36]. His other work on synthesis is [42].

1.2 Contribution of this Work

This work develops a general algorithm for the equivalent circuit synthesis of linear, passive and reciprocal multiports. Passivity and reciprocity properties of the multiport impose conditions on the impedance and admittance matrices of complex functions representing the multiport, and these conditions are satisfied by a class of matrices of complex functions called *positive real symmetric matrices* (PRSM). The algorithm takes a positive real symmetric impedance or admittance matrix of rational functions, representing a multiport, and synthesizes an equivalent network containing resistors, capacitors, inductors (with positive values, unlike generalized foster synthesis) and ideal transformers.

The developed algorithm is iterative and in each iteration a part of the positive real symmetric matrix (PRSM) is extracted and synthesized into a canonical sub-circuit leaving reduced positive real symmetric matrix of smaller order for the next iteration. The iterations continue until a stop criterion is met. There are seven cases and a stop criterion, distinguished according to the location of poles and zeros of PRSM on the complex frequency plane and each case has an extraction procedure and a canonical sub-circuit associated with it.

The extraction procedure associated with the seventh case is based on the multi-port Brune method description by Tellegen [27]. As the Tellegen's description is confined to impedance matrices, a dual Brune method for the admittance matrices is established in this work and included in the algorithm.

Brune's method is a four step extraction procedure which ensures that positive real character of the given positive real symmetric matrix is maintained during each step. The reactive element (capacitor or inductor) involved in the second step is extracted in a way that it always has a negative value. This is done to maintain positive real character of PRSM. Moreover, the Brune procedure yields one of the four different sub-circuit topologies (referred to as *Extracted Brune's circuits of Type-I to IV*) and for each topology there exists a special relation between the parameters of the topology. To avoid the obtained negative valued element, the extracted topologies have to be replaced by an equivalent topologies (referred to as *Equivalent Brune's circuits of Type-I to IV*).

Tellegen in [27] has derived the relationship between the parameters of the extracted Brune circuit of Type-I only and has given its equivalent circuit. A direct and straight forward derivation of the relationships of the parameters of all four extracted sub-circuits is described in this work. Moreover, three additional equivalent topologies (*Extracted Brune's circuits of Type-II to IV*) are introduced and their equivalence is shown with their corresponding extracted circuit topologies.

The implementation of the algorithm is done in MATLAB using "*Symbolic toolbox*". Some of the implementation issues are discussed along with the description of the algorithm. Equivalent network models of passive microwave circuits are developed as an example and presented.

In the following, chapter 2, "*Classes of Synthesizable Functions and Matrices*", lays the mathematical foundation for the algorithm. Positive real symmetric matrices along with their properties are discussed. Its super-class *positive matrices*(PM) is also discussed to some extent in this chapter. Basic terms, for example the pole, the zero and the order or the degree of a matrix of functions, used throughout the description of algorithm are also defined.

Chapter 3, "*Brune Multiport Algorithm and its Implementation*", describes the algorithm in detail. Description of all cases, associated extraction procedures, canonical sub-circuits and some of the implementation issues are given in it.

Chapter 4, "*Parameter Relationships and Equivalence of Circuits*", derives the relationships between the parameters of the extracted Brune's circuits and establishes their equivalence with corresponding equivalent Brune's circuits. Examples are presented in chapter 5.

Appendix A, summarizes canonical circuits repeatedly appearing in Brune's algorithm and are given certain symbols. While reading Brune's algorithm (chapter 3), referring to flow diagram given in Fig. 3.2, table of the cases given in 3.1, and a demonstrative example given in section 5.1 would be very helpful in understanding the algorithm itself.

2 Classes of Synthesizable Functions and Matrices

It is well known that an impedance or an admittance of a linear one-port circuit can be represented as a function of complex frequency. For a linear multiport circuit, the impedance or the admittance can be represented by a matrix of complex functions. As far as voltage and current at the ports are concerned, these complex functions or matrix of functions represent the complete behaviour of the circuit. Inherent physical properties of the circuit must impose conditions and restrictions on the forms, these representing complex functions can take.

The complex functions describing an impedance or an admittance of linear passive one-port circuits belongs to the class of *Positive Real* (P.R.) functions [23]. For multi-port linear passive reciprocal circuits the concept extends to the *Positive Real Symmetric Matrices* (PRSM) [33]. Positive real symmetric matrices form a subset of even bigger class of matrices called *Positive Matrices* (P.M.) which is the most general class of matrices of functions in the area of synthesis and realizability theory [28, 33]. These classes will be discussed briefly in this chapter.

The research for the class of synthesizable functions was initiated by Wilhelm Cauer [43]. The necessary and sufficient conditions for the function to be positive real were provided in 1931 by Otto Brune [23], [44, Ch. 9], [39, Ch. 1]. In 1958, Richard Baum introduced the concept of the *resistance with imaginary value* and the concept of *positive functions* [30]. Later, in 1960, Vitold Belevitch introduced not only the concept of *ideal transformers with imaginary turns ratios* but also used it to define the positive matrices [33].

Some researchers described the properties of the positive matrices and worked for their proofs. The *degree* of a positive matrix is discussed in detail by R. Duffin and D. Hazony [37] and by B. McMillan [28]. The definitions of poles and zeros of these matrices are given by Tellegen [27].

This chapter describes the classification and properties of positive matrices briefly. The proofs of theorems and properties of positive matrices are out of the scope of this thesis. The interested reader is encouraged to read the referenced literature. As the Brune's algorithm is only applied to positive real symmetric matrices, most of the discussion would be according to the conditions and properties of positive real symmetric matrices.

2.1 Positive Matrices and their Classification

An impedance or an admittance of a passive and stable multiport can be represented by a positive matrix. It can describe both reciprocal as well as non-reciprocal multiports. On top of that, positive matrices also allow functions with imaginary coefficients, thus making it possible to include imaginary components, like imaginary resistance and transformers with imaginary turns ratios, in the network [33]. A comprehensive description of positive matrices is given in [33] and of positive function (PF) in [30], which is only for one-port circuits.

A square matrix $\mathbf{W}(s)$ of functions of complex variable s is positive matrix if the real part of the matrix $\Re[\mathbf{W}(s)]$, calculated as:

$$\Re[\mathbf{W}(s)] = \frac{1}{2}[\mathbf{W}(s) + \mathbf{W}^{\dagger}(s)] \,, \tag{2.1}$$

where $\mathbf{W}^{\dagger}(s)$ is the hermitian of $\mathbf{W}(s)$; is positive semi-definite in right half plane of complex variable plane. The test for positive semi-definiteness is given in [39, 45]. The condition of positive semi-definiteness includes the stability criterion that there is no pole in right half of the complex variable plane [39, 46]. The positiveness in right half plane ensures positiveness at the boundary (i.e. imaginary axis in s-plane), according to the *maximum modulus principle* of analytic functions [47, Ch. 20], [45, Ch. 6].

Generally, positive matrices may have irrational functions as well as rational functions. An impedance or admittance matrix which is also positive matrix and having irrational functions, as is in the case of transmission line impedance matrix, is of little interest from synthesis point of view. This is because the irrational functions require suitable series expansion to be synthesized as a network. Therefore, the following discussions would be confined to the positive matrices with rational functions.

2.1.1 Classification of Positive Matrices

The classification of the positive matrices is based on three independent features of the matrices or of the functions of these matrices:

- **Type of coefficients in the functions of the matrix:** The coefficients can be either pure real or complex [33]. If the coefficients are real then the poles occur in conjugate pairs with conjugate residues and the poles on axis have real residues [39, Ch. 1]. On the other hand the complex coefficients indicate that either the poles do not occur as conjugate pairs or if the poles are conjugate pairs then their residues are not conjugate of each other. The impedance or admittance represented by a positive matrix with real coefficient in the rational functions would be physically realizable and if the coefficients of the rational functions are complex then the synthesized model (with imaginary resistances and transformers with complex turns ratios) would not be a physical model .

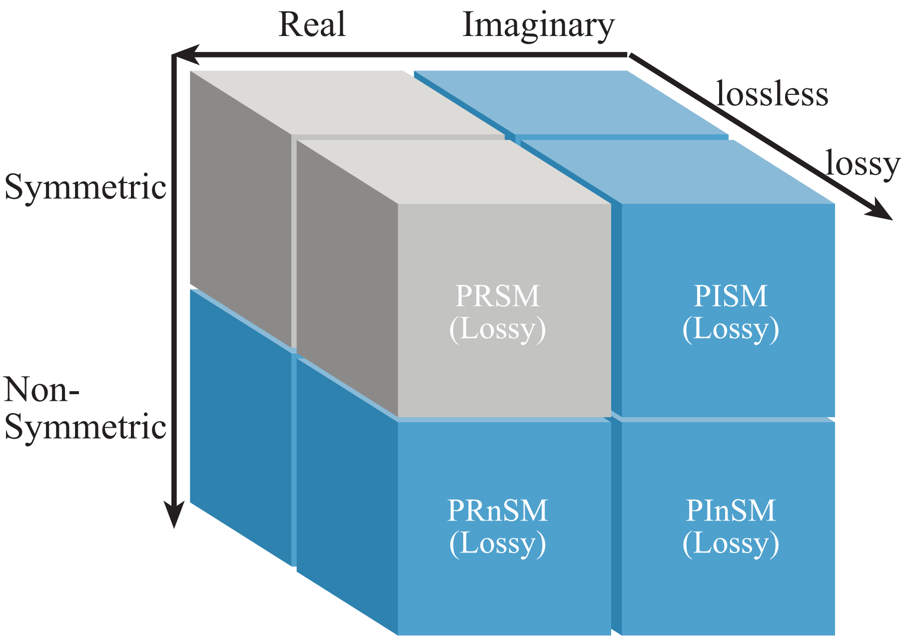


Figure 2.1: Classification of positive matrices on three properties. 'P' stands for positive, 'R' and 'I' for real and imaginary, and 'S' for symmetric [33].

- **Symmetry of matrix:** The impedance or admittance represented by a positive matrix can either be symmetric or non-symmetric. Consequently, the corresponding equivalent circuit would either be reciprocal or a non-reciprocal network respectively [14, §10.5], [48].

- **Types of poles:** The poles of the impedance or the admittance represented by a positive matrix can either be pure imaginary axis poles (with positive real coefficients) or can be located in left half plane of the complex frequency. If all poles are located on the imaginary axis the corresponding network would be loss-less and if any of the poles is located on left half plane then the corresponding network is lossy network [33].

Above properties classify the positive matrices into eight different sub-classes described in Fig. 2.1. Positive real symmetric matrices (PRSM) and positive real non-symmetrical matrices (PRnSM) are physically realizable into an equivalent network while others are non-physical models. As the Brune algorithm (chapter 3) is described for positive real symmetric matrices, following discussion would be confined to the conditions and the properties of it PRSM.

2.2 Positive Real Symmetric Matrices

Positive real symmetric matrices (PRSM) are a sub-class of positive matrices satisfying the first two conditions of classification (i.e. real coefficients and symmetry). The term *'positive real'* was first used by Brune for the functions of a complex variable s with certain properties [39, Ch. 1]. A complex rational function with real coefficients is positive real if:

- The real part of the function is *positive* in the right half plane of variable s. This property enforces analyticity of the function in right half plane.
- The function is *real* on the *real* axis of variable s. This property is the difference between real and imaginary functions.

The term was later adopted by Belevitch [33] with the addition of term 'symmetric' to emphasize the reciprocity of circuits.

2.2.1 Conditions of Positive Real functions

A function in complex variable s, which is real for real values of s is positive real if and only if following conditions are fulfilled [39, Ch. 1, §3]:

- **Stability:** No pole of the function is located on right half plane of complex variable, i.e. analytic in right half plane.
- **Passivity-I:** The real part of the function is not negative on imaginary axis of complex frequency s. As imaginary axis is also the boundary of the right half plane, maximum and minimum values should lie on it (maximum modulus principle). This way it can be make sure that the real part is also non-negative in right half plane.
- **Passivity-II:** Any imaginary axis pole of the function must be simple and with positive real residue.

2.2.2 Conditions for PRSM

Apart from a rigorous derivation of the conditions for positive real symmetric matrices, Guillemin has also given a physical and elegant interpretation of the conditions [39, Ch. 1, §2] using a physical passive network. Consider an N-port network represented with impedance matrix $\mathbf{Z}(s)$, connected with ideal transformers having turns ratios $p_1, p_2, p_3, \cdots, p_n$ as shown in Fig. 2.2, then it can be derived that the impedance function $Z(s)$ is:

$$Z(s) = \boldsymbol{p}^T \mathbf{Z}(s) \boldsymbol{p},$$

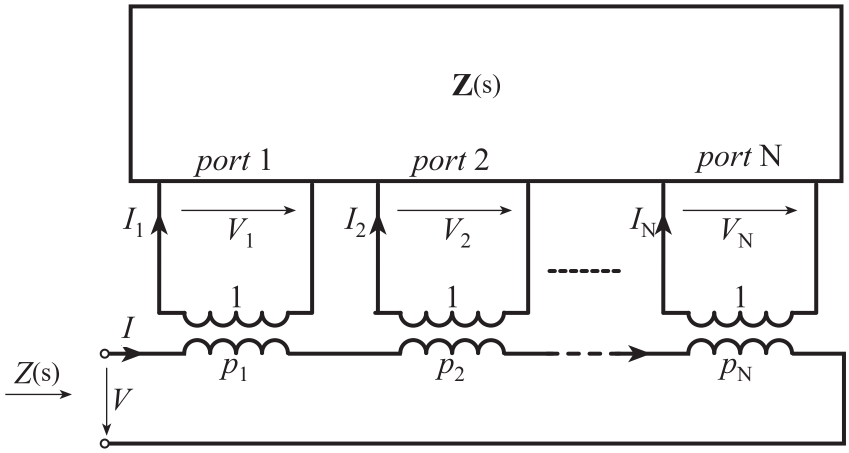


Figure 2.2: Physical interpretation of PRSM conditions; adopted from [39, Ch.1]

where $\boldsymbol{p}$ is a vector containing all turns ratios i.e.

$$\boldsymbol{p} = [p_1, p_2, p_3, \cdots, p_N]^T.$$

The impedance $Z(s)$ has to be positive real for all real combinations of the values of transformer turns ratios p_i's. This is the basis for the set of necessary and sufficient conditions which an impedance or admittance matrix has to satisfy to be able to synthesize into passive lumped element network [27, §2] and [39, Ch. 1, §6]. The conditions for positive real symmetric matrix are analogous to the conditions described for positive real functions described above. A matrix of rational functions in complex variable s, which is real for real values of s, is positive real symmetric matrix if following conditions are fulfilled:

1. **Stability:** No pole of the matrix functions is located on right half plane of complex variable s.

2. **Passivity-I:** The real part of the matrix at imaginary axis of the complex variable plane is positive semi-definite. Let $\mathbf{A}(s)$ be the real part of the positive real symmetric matrix,

$$\begin{aligned} \mathbf{A}(s) &= \Re\{\mathbf{W}(s)\} \\ &= \begin{pmatrix} A_{11}(s) & A_{12}(s) & \cdots & A_{1N}(s) \\ A_{12}(s) & A_{22}(s) & \cdots & A_{2N}(s) \\ \vdots & \vdots & \ddots & \vdots \\ A_{1N}(s) & A_{2N}(s) & \cdots & A_{NN}(s) \end{pmatrix}, \end{aligned}$$

then $\mathbf{A}(s = j\omega)$ is positive semi-definite. The matrix $\mathbf{A}(s = j\omega)$ is positive semi-definite if $A_{ii}(s = j\omega) \geq 0$ for all $i = 1, 2, \cdots, N$ and all principal minors and the determinant of $\mathbf{A}(s = j\omega)$ are positive. It should be noted that off-diagonal entries can be negative.

3. **Passivity-II:** If there are poles on imaginary axis then for each such pole ω_p the corresponding matrix of residues,

$$\mathbf{K}^{\omega_p} = \begin{pmatrix} k_{11}^{\omega_p} & k_{12}^{\omega_p} & \cdots & k_{1N}^{\omega_p} \\ k_{12}^{\omega_p} & k_{22}^{\omega_p} & \cdots & k_{2N}^{\omega_p} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1N}^{\omega_p} & k_{2N}^{\omega_p} & \cdots & k_{NN}^{\omega_p} \end{pmatrix} ,$$

has real elements and the matrix $\mathbf{K}^{\omega_p}$ is positive semi-definite. Again, diagonal elements in the matrix $\mathbf{K}^{\omega_p}$ as well as the principal minors and the determinant are positive or equal to zero.

2.3 Order of Positive Real Symmetric Matrix

The order or the degree of the given $N \times N$ positive real symmetric matrix of rational functions $\mathbf{W}(s)$, denoted as $\mathcal{O}(\mathbf{W})$, is the degree of the rational function formed by the determinant

$$|\mathbf{W}(s) + \mathbf{A}| ,$$

where $\mathbf{A}$ is an $N \times N$ constant real symmetric matrix [27]. As the degree of the rational function developed by the above determinant is dependent on the choice of the matrix $\mathbf{A}$, the degree of the positive real symmetric matrix is taken to be the maximum degree of the rational function (developed by the determinant) for all possible choices of matrix $\mathbf{A}$ [37].

The order $\mathcal{O}(\mathbf{W})$, is zero if the matrix $\mathbf{W}$ is a constant matrix otherwise it depends on the degree of the rational functions in $\mathbf{W}(s)$ and their dependency on each other. If positive real symmetric matrix $\mathbf{W}(s)$ is realized into an equivalent multiport circuit, the minimum number of reactive elements (capacitors or inductors) required are equal to the order of the matrix, $\mathcal{O}(\mathbf{W})$ [27]. The most comprehensive discussion on the order or degree of matrix of functions and its properties are given in [37] and in [28].

The above definition of the order of the matrix is analogous to the definition of the degree of the polynomial in one variable. Consider a polynomial in one variable s:

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 ,$$

where the coefficients $a_n, a_{n-1}, \cdots, a_1, a_0$ are complex numbers. If a_n is non-zero, then fundamental theorem of algebra [49] ensures that the equation

$$P(s) = c ;$$

where c is also a complex number; has n solutions and n would be the degree of polynomial $P(s)$ [49].

Physically the order of a positive real symmetric matrix can be interpreted as follows: If $\mathbf{W}(s)$ is a representing an impedance or an admittance matrix of an N-port network, then the matrix $\mathbf{A}$ can be considered as an N-port resistance network, connected port-by-port to the original network. After perturbation of the network, the maximum number of free-oscillations obtained in this closed system is the order of the network [27].

2.3.1 Properties of Order of Positive Real Symmetric Matrix

Some properties of the order of the positive real symmetric matrix $\mathcal{O}(.)$ would be discussed here. The proofs of the properties can be found in the referred literature.

- For any given matrix $\mathbf{W}(s)$, the order of the inverse of the matrix (if exists) is same as the order of the matrix itself [27, 28, 37]:

  $$\mathcal{O}(\mathbf{W}) = \mathcal{O}(\mathbf{W}^{-1})\,.$$

  As impedance and admittance matrix representations represent the same network, both should have same order.

- If two matrices $\mathbf{W}_1$ and $\mathbf{W}_2$ have no pole in common, then

  $$\mathcal{O}(\mathbf{W}_1 + \mathbf{W}_2) = \mathcal{O}(\mathbf{W}_1) + \mathcal{O}(\mathbf{W}_2)\,.$$

  If they have some poles in common, then the combined order would be less. Thus it can also be written as [28, 37]

  $$\mathcal{O}(\mathbf{W}_1 + \mathbf{W}_2) \leq \mathcal{O}(\mathbf{W}_1) + \mathcal{O}(\mathbf{W}_2)\,.$$

- For matrices of form $\mathbf{W} = \mathbf{C}w(s)$, where $\mathbf{C}$ is a constant matrix and $w(s)$ is a function in s, the degree of the whole matrix $\mathbf{W}$ is the product of the degree of the function $w(s)$ and the rank of the matrix $\mathbf{C}$ [28]. This property is elaborated in sections A.2 and A.3

2.4 Poles and Zeros

Pole and zero of a complex function of single variable are well defined in literature [47], but the poles and zeros of a matrix of complex functions are not discussed in detail. Along with their definitions there is also a concept of *rank* of pole or zero of matrix [27, 28].

The pole of any element of the matrix is also the *pole* of that matrix. The *rank of the pole* of the matrix is the rank of the matrix of residues associated with that pole in original given matrix [27, 28]. It must be noted that a pole in off-diagonal elements can only exist if it is also present in the corresponding diagonal elements [39, Ch. 1].

A *zero* of the matrix is the zero of the determinant of that matrix [27, 28]. Zeros of a matrix is the pole of its inverse. The rank of a zero is the rank of the matrix of residues associated with the corresponding pole in inverse matrix [27, 28]. A matrix is *singular* if it is singular for all values of complex variable s.

Above definitions fit well for matrices of rational functions. In such case, even the poles of the determinant are also the poles of the matrix, and the multiplicity of the pole in determinant is the rank of the pole in matrix. The only exception to the definitions is in two port transmission line. The admittance and impedance matrices of a transmission line are given in 2.2 and 2.3. It can be easily verified that the determinants are $1/Z_0^2$ and Z_0^2 respectively. So both are sharing same number of poles but there are no zeros according to above definitions.

$$\mathbf{Y}_{TL} = \frac{1}{Z_0}\begin{pmatrix} \coth\gamma l & -\frac{1}{\sinh\gamma l} \\ -\frac{1}{\sinh\gamma l} & \coth\gamma l \end{pmatrix}, \tag{2.2}$$

$$\mathbf{Z}_{TL} = Z_0\begin{pmatrix} \coth\gamma l & \frac{1}{\sinh\gamma l} \\ \frac{1}{\sinh\gamma l} & \coth\gamma l \end{pmatrix}. \tag{2.3}$$

3 Brune Multiport Algorithm and its Implementation

Otto Brune (1901- 1982) gave a groundbreaking contribution to the theory of positive real functions. In 1931, he presented an algorithm for equivalent one-port passive circuit synthesis from an impedance represented as positive real function of complex frequency. The synthesized circuit consist of positive resistors, capacitors and inductors as the circuit elements and ideal transformers as connection elements [23, 39]. Tellegen [27] presented an extension of Brune's method for multiports whose impedance is represented as positive real symmetric matrix. Other well known extensions are done by McMillan [28] and Belevitch [33].

This chapter presents a general algorithm (referred as *Brune's multiport algorithm*)for equivalent lumped element circuit synthesis and discusses implementation issues in MATLAB. The algorithm takes an impedance or an admittance matrix represented by a positive real symmetric matrix of rational functions and synthesizes an equivalent passive multiport network. The main contribution of this work are:

- Division of positive real symmetric matrices into seven cases according to the location of poles and zeros on complex frequency plane, description of extraction procedure for each case addressing the implementation issues and minimizing the numerical errors, and describing canonical sub-circuits for each case.
- Description of the dual Brune's method for the admittance matrices represented as positive real symmetric matrices.
- Description of new equivalent sub-circuit topologies to replace the extracted sub-circuit topologies during the Brune's process in order to avoid negative valued elements.

3.1 Overview of the Algorithm

Brune's algorithm is an iterative algorithm. In each iteration a part of the given positive real symmetric matrix of rational functions $\mathbf{W}(s)$, is extracted and synthesized into a canonical sub-circuit, leaving a reduced matrix $\mathbf{W}'(s)$ of smaller order (section 2.3). The number of iterations depends on the order of $\mathbf{W}(s)$. Generalized output of Brune's

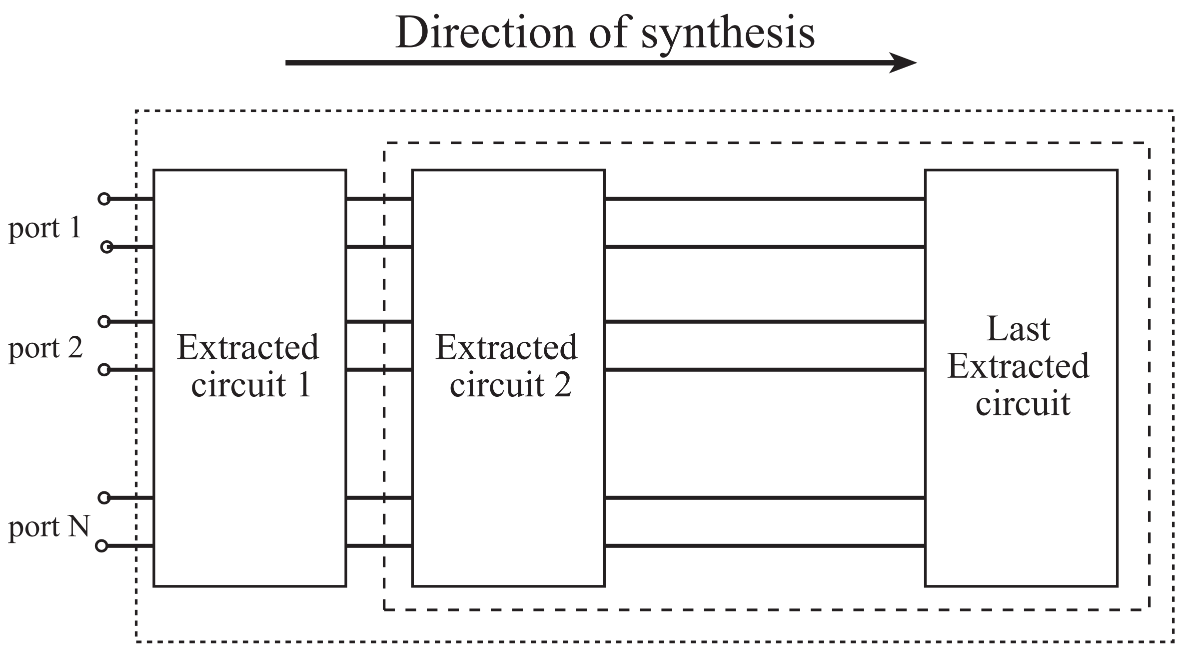


Figure 3.1: General out-put of the Brune's algorithm comprises cascaded sub-circuits. In each Iteration a sub-circuit is extracted leaving behind a reduced matrix to be synthesized.

algorithm is given in Fig. 3.1. It consists of cascaded sub-circuits. The last sub-circuit is usually a resistance multi-port circuit.

There are seven cases (Cases 1 to 7) of a positive real symmetric matrix ($\mathbf{W}(s)$), including a stop criterion (Case 0) distinguished according to the location of poles and zeros of the matrix on complex frequency plane. These are analogous to the cases given for one-port impedance positive real functions in [50]. For each case the extraction and the synthesis procedures are different. In each iteration, the case of $\mathbf{W}(s)$ is determined and the corresponding extraction procedure is applied. A small extracted part forms a canonical sub-circuit and the rest of the matrix is another reduced positive real symmetric matrix. Fig. 3.2 shows the flow diagram of the Brune's algorithm and table 3.1 represents all cases for quick reference.

Canonical sub-circuits formed during the iterations are mostly of the form $\mathbf{W}(s) = \mathbf{C}w(s)$, where $\mathbf{C}$ is a constant symmetric matrix and $w(s)$ is a function of complex frequency s. Sub-circuits of this kind are synthesized in cases 1 to 6. Case 7 sub-circuits are different and would be discussed during the description of case 7. Appendix A details the synthesis of the sub-circuits (of matrix form $\mathbf{W}(s) = \mathbf{C}w(s)$) and their corresponding symbols. The symbols would be used through out the description of the algorithm.

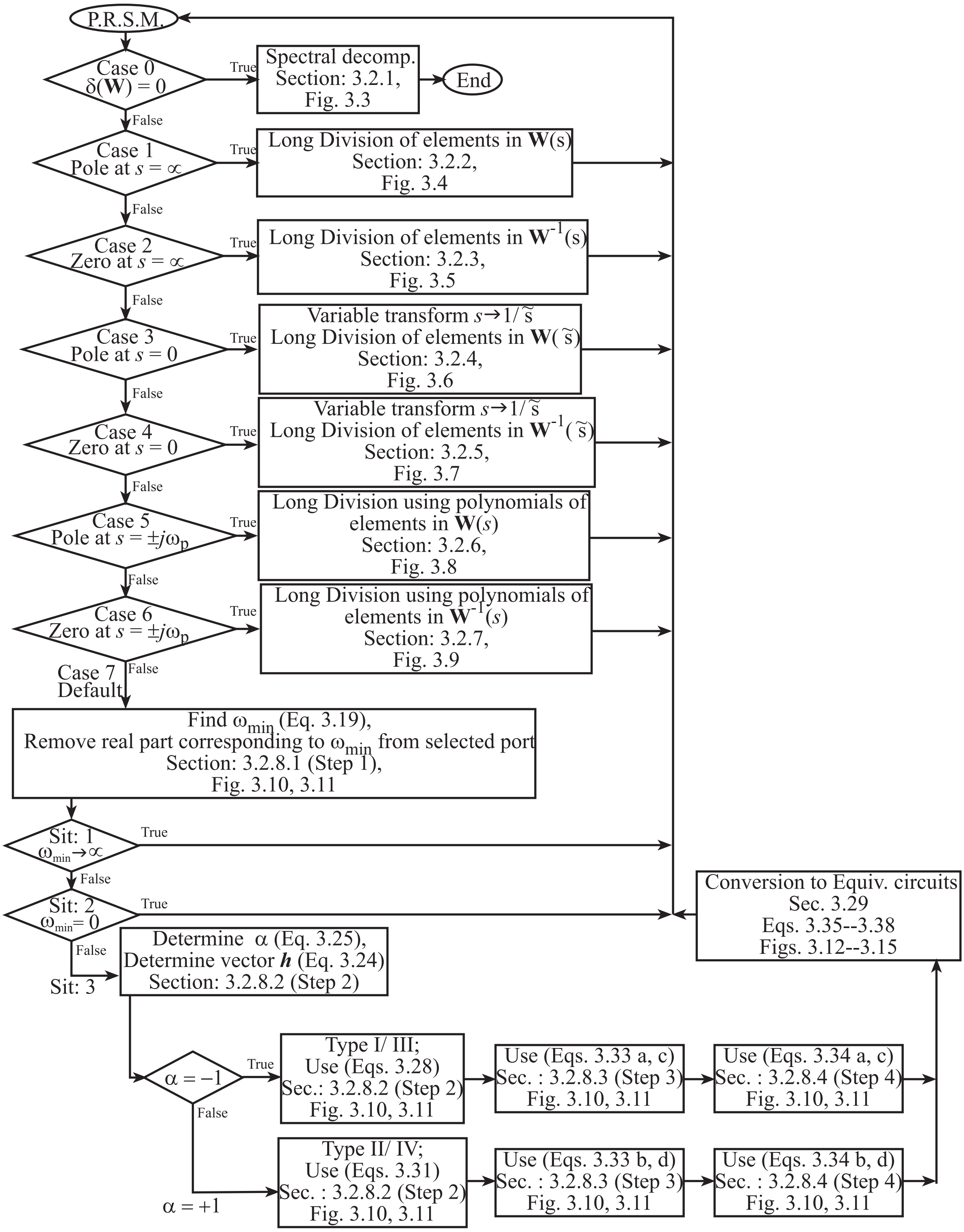


Figure 3.2: Flow Diagram of Brune's Algorithm

Table 3.1: Cases of Brune's Algorithm and their Realizations

Cases			Effect	Circuit
0			Pure resistive/conductive	$\mathbf{W}=\mathbf{Z}$, Fig. 3.3(a)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.3(b)
1			$\lim\limits_{s\to\infty} W_{ij}=\infty$ Pole at $s=\infty$	$\mathbf{W}=\mathbf{Z}$, Fig. 3.4(a)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.4(b)
2			$\lim\limits_{s\to\infty} \lvert\mathbf{W}\rvert=0$ Zero at $s=\infty$	$\mathbf{W}=\mathbf{Z}$, Fig. 3.5(a)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.5(b)
3			$\lim\limits_{s\to 0} W_{ij}=\infty$ Pole at $s=0$	$\mathbf{W}=\mathbf{Z}$, Fig. 3.6(a)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.6(b)
4			$\lim\limits_{s\to 0} \lvert\mathbf{W}\rvert=0$ Zero at $s=0$	$\mathbf{W}=\mathbf{Z}$, Fig. 3.7(a)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.7(b)
5			$\lim\limits_{s\to\pm j\omega_p} W_{ij}=\infty$ Poles at $\pm j\omega_p$	$\mathbf{W}=\mathbf{Z}$, Fig. 3.8(a)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.8(b)
6			$\lim\limits_{s\to\pm j\omega_p} \lvert\mathbf{W}\rvert=0$ Zeros at $\pm j\omega_p$	$\mathbf{W}=\mathbf{Z}$, Fig. 3.9(a)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.9(g)
7	Situation 1		$\omega_{min}=\infty$, Case 2 would follow	$\mathbf{W}=\mathbf{Z}$, Fig. 3.10 (Only Step 1)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.11 (Only Step 1)
	Situation 2		$\omega_{min}=0$, Case 4 would follow	$\mathbf{W}=\mathbf{Z}$, Fig. 3.10 (Only Step 1)
				$\mathbf{W}=\mathbf{Y}$, Fig. 3.11 (Only Step 1)
	Situation 3	Type I	$\omega_{min}\neq 0$ & finite, $\alpha=-1$	$\mathbf{W}=\mathbf{Z}$, Fig. 3.10 (All Steps)
		Type II	$\omega_{min}\neq 0$ & finite, $\alpha=+1$	$\mathbf{W}=\mathbf{Z}$, Fig. 3.10 (All Steps)
		Type III	$\omega_{min}\neq 0$ & finite, $\alpha=-1$	$\mathbf{W}=\mathbf{Y}$, Fig. 3.11 (All Steps)
		Type IV	$\omega_{min}\neq 0$ & finite, $\alpha=+1$	$\mathbf{W}=\mathbf{Y}$, Fig. 3.11 (All Steps)

3.2 Cases of Brune's Algorithm

The cases of the algorithm are divided according to the location of the poles or zeros of the given positive real symmetric matrix. The extraction procedure for each case is also different. The stop criterion (Case 0) is reached when, after the iterative reduction, the matrix is reduced to frequency independent real matrix. Such matrix can be readily synthesized to a pure resistance network.

The cases are not all mutually exclusive, meaning that certain cases may combine in one matrix simultaneously. This combination of cases does not effect the algorithm. In consecutive iterations, cases combined in one matrix can be processed. The sequence in which the cases are placed has some significance from theoretical point of view but for implementation side it gives an advantage. For example case 7 can only occur when all other cases are not present, thus forming a default case. In cases 1 to 4, the extraction procedures are easier than those in cases 5 and 6. If two cases (say 1 and 5) occur simultaneously, then processing case 5 would be easier if case 1 is done first as case 5 would have a less complex matrix to handle.

The processes involved in the determination of cases and the corresponding extraction procedures are numerical and cannot differentiate if the given matrix is an impedance or an admittance matrix. Similarly the differentiation that the extracted values are representing a resistance, conductance, capacitance or inductance are immaterial to these procedures. Thus for the sake of brevity and to avoid confusions, common symbols would be used. For impedance matrix $\mathbf{Z}(s)$ and admittance matrix $\mathbf{Y}(s)$ a common symbol $\mathbf{W}(s)$ would be used. For resistance R and conductance G, symbol A is taken. Lastly, for capacitance C and inductance L, symbol D is chosen. Capital bold-face letters, for example $\mathbf{A}$, would represent a matrix, while italic letters (capital or small) represent scalar quantity. Small bold face letters, for example $\boldsymbol{\beta}$, are specific for column vectors. For real and imaginary part of $\mathbf{W}(s)$, symbols $\mathbf{A}(s)$ and $\mathbf{I}(s)$ are used. The table 3.2 shows the common symbols used in the following text. Reader may also refer to the list of symbols given in table B.

Table 3.2: Common used in the description of Algorithm cases

Common symbol	Used for	Description
$\mathbf{W}(s)$	$\mathbf{Z}(s)$, $\mathbf{Y}(s)$	impedance or admittance matrix
$\mathbf{A}$/ A	$\mathbf{R}$, $\mathbf{G}$/ R, G	resistance or conductance matrix /scalar
D	L, C	inductance or capacitance
$\mathbf{A}(s)$/ $A(s)$	$\Re[\mathbf{W}](s)$/ $\Re[W](s)$	real part of given matrix/ function
$\mathbf{C}_r^W$, $\mathbf{C}_r^Z$, $\mathbf{C}_r^Y$		real symmetric matrix of rank r, representing the connection network of ideal transformers (section A.2 and A.3)

3.2.1 Case 0: Stop Criterion

When the given positive real symmetric impedance or admittance matrix $\mathbf{W}$ is independent of complex frequency, the last stage would be reached. Such matrix represents a pure resistance or conductance circuit network. The order of the network would be zero:

$$\mathcal{O}(\mathbf{W}) = 0\,.$$

The synthesis of such network is straight forward and described in sections A.2 and A.3. Let the matrix $\mathbf{W}$ be of rank r, then after spectral decomposition [51, §5.5], it can be written as

$$\mathbf{W} = \sum_{i=1}^{r} \mathbf{C}_{1,i}^{W} A\,, \tag{3.1}$$

where $\mathbf{C}_{1,i}^{W}$ are rank-one real symmetric matrices. Depending on the given positive real symmetric matrix $\mathbf{W}$ representing an impedance or an admittance matrix the equations would be

$$\mathbf{Z} = \sum_{i=1}^{r} \mathbf{C}_{1,i}^{Z} R\,, \qquad \mathbf{Y} = \sum_{i=1}^{r} \mathbf{C}_{1,i}^{Y} G\,. \tag{3.2}$$

Fig. 3.3 (a) and (b) shows circuit realizations for impedance and admittance matrices in case 0, respectively.

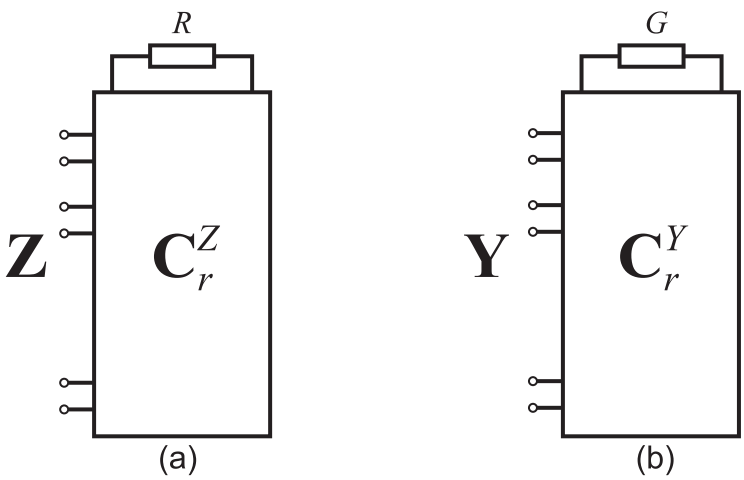


Figure 3.3: Sub-circuit realization networks of case 0; (a) for impedance matrix, and (b) for admittance matrix, corresponding to Eqs. 3.2. The symbol used are explained in Figs. A.2 and A.4. As these are the last stages, there are no reduced matrices after them.

3.2.2 Case 1

If any element of the given matrix $\mathbf{W}(s)$ has a pole at infinity then $\mathbf{W}(s)$ is in case 1. In rational functions, if the numerator is one degree higher than the denominator, then it shows the presence of a pole at infinity. It must be noted that, the off diagonal elements can have a certain pole only if corresponding diagonal elements have that pole (section 2.4). Thus it would suffice to check the degrees of the numerators and the denominators of the rational functions in the diagonal of the given matrix in order to identify the presence of the pole at infinity. Mathematically, if:

$$\lim_{s\to\infty} W_{ii}(s) = \infty \,, \qquad \forall \quad i = 1, 2, \cdots N \,,$$

where N is number of ports, then the given matrix is in case 1.

The extraction procedure involves separation of the pole and its residue from each element of the matrix where it is present. The best way to separate the pole and its residue from the rational function is through polynomial long division. The problem with long division is that it accumulates numerical errors over each step of division, especially in the division of polynomials of very high degree. This problem is of least concern in given scenario because the numerator is only one degree higher than the denominator and only one step of division is required. Moreover, in modern computer algebra systems (like Symbolic toolbox of MATLAB, Maxim etc) variable precision arithmetic (VPA) is supported where user can specify very large number of digits in the numbers. Specifying sufficient number of digits greatly reduce the errors caused by very high degree polynomials. The MATLAB function '*quorem*' is a useful function available in symbolic toolbox. This function can handle VPA. Usually the available functions try to make the rational function a proper rational function i.e. making the degree of the numerator smaller than that of the denominator. This functionality is not required, rather after the separation of the pole and its residue (first step of division), the function should stop. To make the polynomial division stop after first step, one can multiply and divide each rational function of the matrix by the complex frequency variable s [52]. Let $W(s)$ be any rational function of the positive real symmetric matrix having a pole at infinity:

$$\begin{aligned}
W(s) &= \frac{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0} \\
&= s\left[\frac{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}{b_{n-1}s^{n} + b_{n-2}s^{n-1} + \cdots + b_0 s}\right] \\
&= s\left[\frac{a'_{n-1}s^{n-1} + a'_{n-2}s^{n-2} + \cdots + a'_0}{b_{n-1}s^{n} + b_{n-2}s^{n-1} + \cdots + b_0 s} + \frac{a_n}{b_{n-1}}\right] \\
&= \frac{a'_{n-1}s^{n-1} + a'_{n-2}s^{n-2} + \cdots + a'_0}{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0} + Ds \\
&= W'(s) + Ds \,,
\end{aligned}$$

where $D = \frac{a_n}{b_{n-1}}$.

For whole matrix it can be written as

$$\mathbf{W}(s) = \mathbf{C}_r^W Ds + \mathbf{W}^{'}(s)\,, \tag{3.3}$$

and for impedance and admittance matrix representations, as:

$$\mathbf{Z}(s) = \mathbf{C}_r^Z Ls + \mathbf{Z}^{'}(s)\,, \qquad \mathbf{Y}(s) = \mathbf{C}_r^Y Cs + \mathbf{Y}^{'}(s)\,. \tag{3.4}$$

Fig. 3.4 (a) and (b) show the extracted sub-circuits for impedance and admittance matrices after extracted procedure of case 1, respectively.

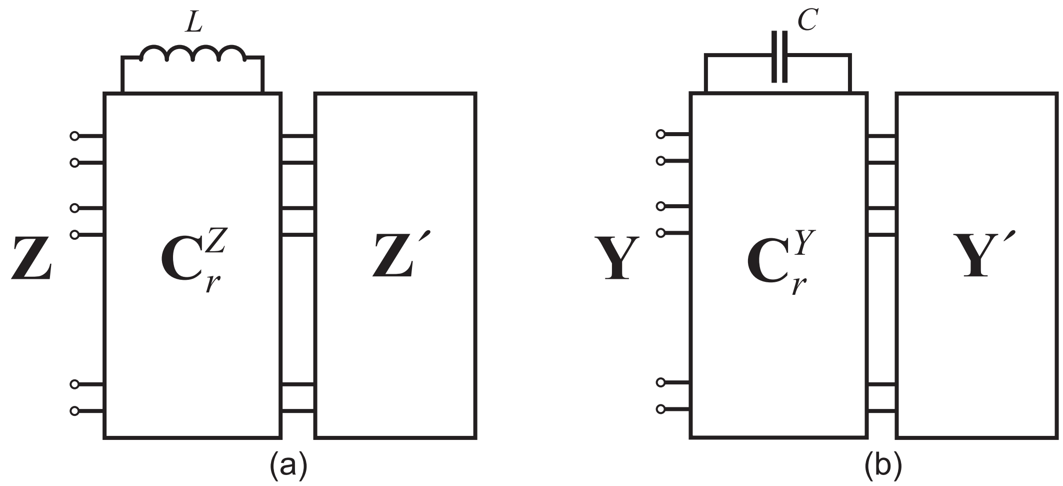


Figure 3.4: Sub-circuit realization of case 1; (a) for impedance matrix and (b) for admittance matrix; the symbol used are explained in Figs. A.2 and A.4. $\mathbf{Z}$ and $\mathbf{Y}$ are given matrices while $\mathbf{Z}^{'}$ and $\mathbf{Y}^{'}$ are corresponding reduced matrices; according to Eqs. 3.4

3.2.3 Case 2

Case 2 is complementary to case 1 discussed above. Instead of a pole, this case occurs when there is a zero at infinity in the given positive real symmetric matrix. According to the definition of zero of matrix (section 2.4), the condition for this case is

$$\lim_{s\to\infty} |\mathbf{W}(s)| = 0\,.$$

If the determinant $|\mathbf{W}(s)|$ is a rational function, then the degree of its numerator would be less than the degree of its denominator. As a consequence, the inverse matrix $\mathbf{W}^{-1}(s)$ must have a pole at infinity. Either by checking the determinant of the matrix for the degree of the numerator less than the degree of the denominator, or the elements of the inverse matrix for a pole at infinity, the existence of the case 2 can be determined.

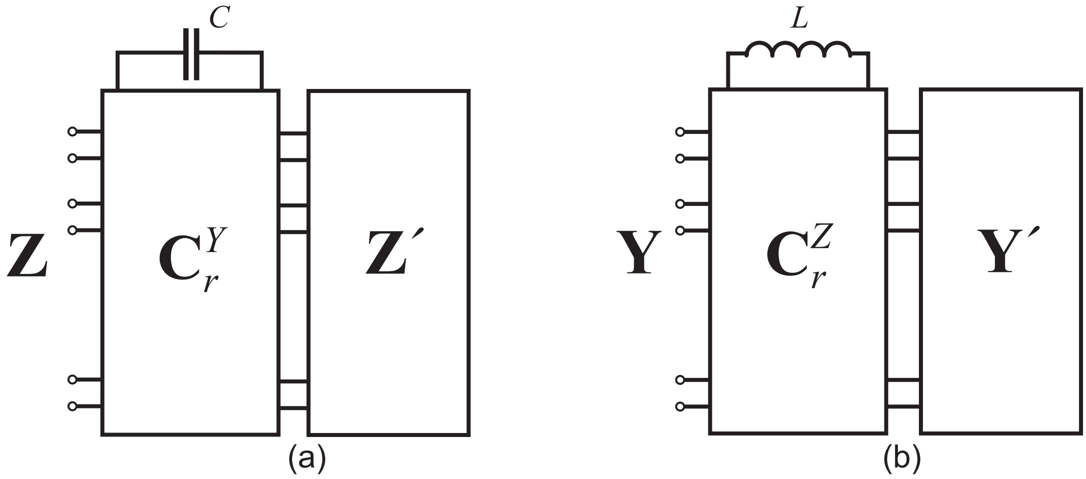


Figure 3.5: Sub-circuit realizations of case 2; (a) for impedance matrix and (b) for admittance matrix; the symbol used are explained in Figs. A.2 and A.4. The figures correspond to the Eqs. 3.6

The extraction procedure for the zero at infinity, is application of long division on the elements of inverse matrix as discussed in case 1 above.

$$\mathbf{W}(s) = \left[\mathbf{C}_r^W Ds + \mathbf{W}'^{-1}(s)\right]^{-1} . \tag{3.5}$$

For impedance and admittance matrix representations, it would be

$$\mathbf{Z}(s) = \left[\mathbf{C}_r^Y Cs + \mathbf{Z}'^{-1}(s)\right]^{-1} , \qquad \mathbf{Y}(s) = \left[\mathbf{C}_r^Z Ls + \mathbf{Y}'^{-1}(s)\right]^{-1} . \tag{3.6}$$

Fig. 3.5 (a) and (b) show the extracted sub-circuits for impedance and admittance, respectively, after the case 2 extraction procedure is applied.

3.2.4 Case 3

Positive real symmetric matrix is in case 3 if it has a pole at $s = 0$:

$$\lim_{s \to 0} W_{ij}(s) = \infty , \qquad \forall \quad i, j = 1, 2, \cdots N ,$$

where N is number of ports. The constant term in the denominator polynomial of the rational function is zero in this case. As cases 1 and 2 precede this case and any possible pole or zero at infinity would be extracted before, thus all polynomials in the matrix would be of same degree.

This pole can be extracted from individual element through long division. For that the transformation of variable from s to $\tilde{s} = 1/s$ is necessary, before applying the long

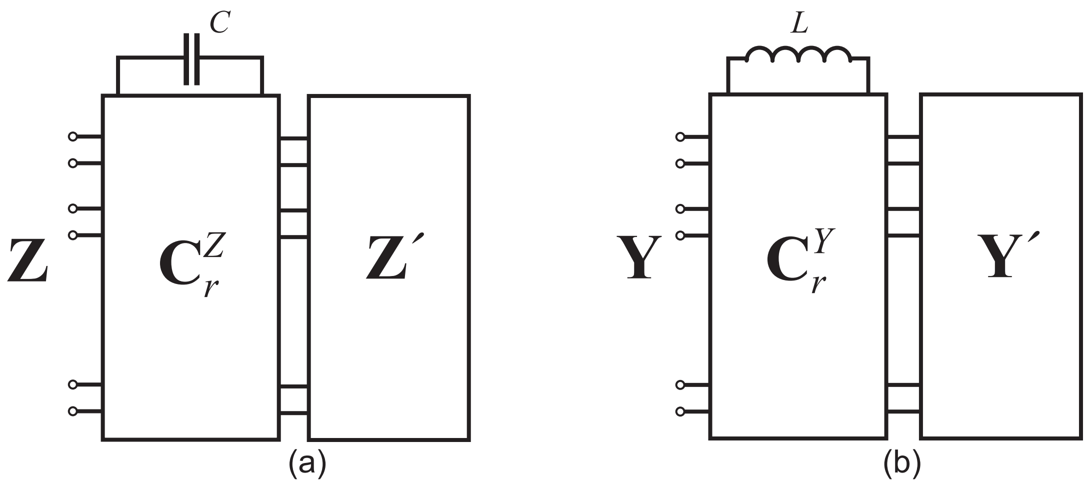


Figure 3.6: Sub-circuit realizations of case 3; (a) for impedance matrix and (b) for admittance matrix, corresponding to Eqs. 3.8; the symbol used are explained in Figs. A.2 and A.4.

division. After the transformation of the variable, the function turns into a function similar to the functions in case 1. After long division, the transformation can again be applied. The steps for one rational function $W(s)$ of the matrix are given as [52]:

$$\begin{aligned} W(s) &= \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s} \\ W\left(s = \frac{1}{\tilde{s}}\right) &= \frac{a_n + a_{n-1}\tilde{s} + \cdots + a_0\tilde{s}^n}{b_n + b_{n-1}\tilde{s} + \cdots + b_1\tilde{s}^{n-1}} \\ &= \frac{a'_n + a'_{n-1}\tilde{s} + \cdots + a'_1\tilde{s}^{n-1}}{b_n + b_{n-1}\tilde{s} + \cdots + b_1\tilde{s}^{n-1}} + \frac{a_0}{b_1}\tilde{s} \\ W(s) &= \frac{a'_n s^{n-1} + a'_{n-1}s^{n-2} + \cdots + a'_1}{b_n s^{n-1} + b_{n-1}s^{n-2} + \cdots + b_1} + \frac{1}{Ds}, \end{aligned}$$

where $D = \frac{b_1}{a_0}$. For whole matrix it can be written as

$$\mathbf{W}(s) = \mathbf{C}_r^W \frac{1}{Ds} + \mathbf{W}'(s), \tag{3.7}$$

and individually for impedance and admittance matrices

$$\mathbf{Z}(s) = \mathbf{C}_r^Z \frac{1}{Cs} + \mathbf{Z}'(s), \qquad \mathbf{Y}(s) = \mathbf{C}_r^Y \frac{1}{Ls} + \mathbf{Y}'(s). \tag{3.8}$$

The elements C and L will have value D i.e. reciprocal of the residue of the pole. Figs. 3.6 (a) and (b) show the sub-circuits after case 3 extraction from impedance and admittance matrices respectively.

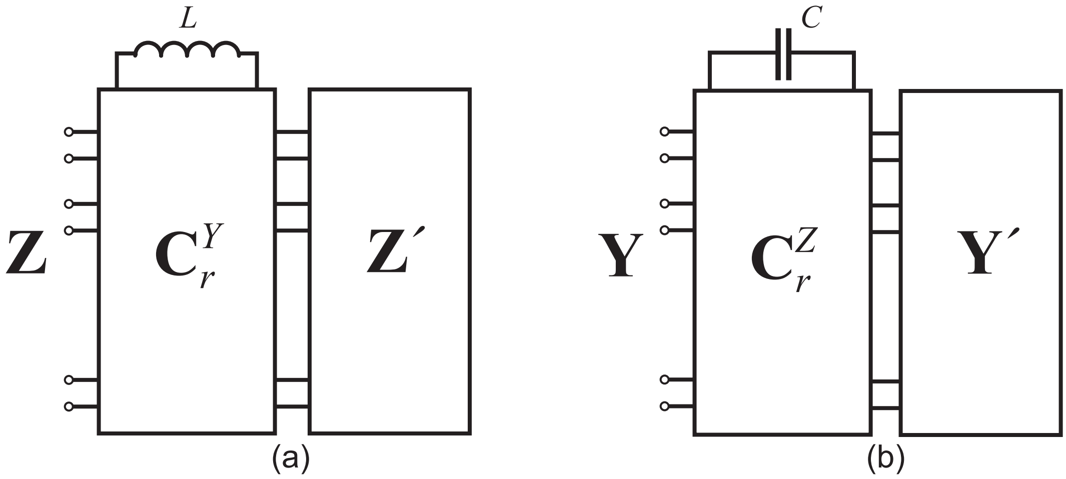


Figure 3.7: Sub-circuit realizations of case 4; (a) for impedance matrix and (b) for admittance matrix, corresponding to Eqs. 3.10; the symbol used are explained in Figs. A.2 and A.4.

3.2.5 Case 4

Case 4 is complementary to case 3. Instead of a pole, if the given positive real symmetric matrix has a zero at $s = 0$:

$$\lim_{s \to 0} |\mathbf{W}(s)| = 0\,,$$

then it is in case 3. In the numerator of the determinant, the constant term would be equal to zero and the inverse matrix $\mathbf{W}^{-1}(s)$ would have a pole.

The extraction procedures are same as discussed in case 3 but are applied to the elements of inverse matrix. The general extraction equation is

$$\mathbf{W}(s) = \left[\mathbf{C}_r^W \frac{1}{Ds} + \mathbf{W}'^{-1}(s)\right]^{-1}, \tag{3.9}$$

and for impedance and admittance PRSM,

$$\mathbf{Z}(s) = \left[\mathbf{C}_r^Y \frac{1}{Ls} + \mathbf{Z}'^{-1}(s)\right]^{-1}, \qquad \mathbf{Y}(s) = \left[\mathbf{C}_r^Z \frac{1}{Cs} + \mathbf{Y}'^{-1}(s)\right]^{-1}. \tag{3.10}$$

Figs. 3.7 (a) and (b) show the sub-circuits after the extraction procedures are applied for impedance and admittance matrices respectively.

3.2.6 Case 5

A given positive real symmetric matrix lies in the case 5 if it has a pole pair at finite frequency on imaginary axis. Positive real character of the matrix makes sure that the

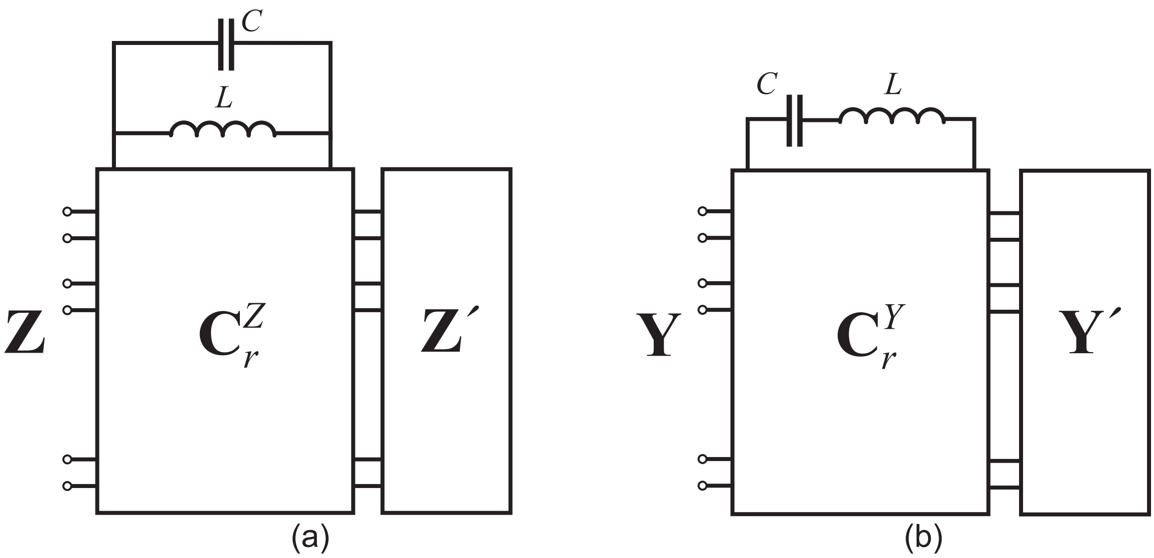


Figure 3.8: Sub-circuit realizations of case 5; (a) for impedance matrix and (b) for admittance matrix, according to Eqs. 3.15; the symbol used are explained in Figs. A.2 and A.4.

poles always occur in complex conjugate pairs (section 2.2.2). For this case the condition would be

$$\lim_{s\to\pm j\omega_p} W_{ii}(s) \to \infty\,, \qquad \forall \quad i = 1, 2, \cdots N\,,$$

where N is number of ports. Checking the denominators of diagonal elements for quadratic factors of form $s^2 + \omega_p^2$ is sufficient for determining the presence of the pole pair on imaginary axis.

The pole pair can be separated from the individual elements of the matrix . A rational function $W(s)$ having an imaginary pole pair at $\pm j\omega_p$, can be written as

$$W(s) = \frac{1}{s^2 + \omega_p^2}\frac{P}{Q}\,. \tag{3.11}$$

Let the residue of one of the poles be k, then after removing the pole pair from $W(s)$, the remaining matrix is:

$$\begin{aligned} W'(s) &= \frac{1}{s^2 + \omega_p^2}\frac{P}{Q} - \frac{2ks}{s^2 + \omega_p^2} \\ &= \frac{1}{s^2 + \omega_p^2}\left[\frac{P - 2kQs}{Q}\right]\,. \end{aligned} \tag{3.12}$$

The value of residue k is calculated such that the polynomial $P - 2kQs$ has the factor $s^2 + \omega_p^2$, thus canceling it out in the remainder function $W'(s)$. Long division can be applied with $P - 2kQs$ as dividend and $s^2 + \omega_p^2$ as divisor. The remainder would be a function of the residue k which can be found by equating the remainder to zero. The quotient P' would remain as the numerator of the remaining function $W'(s)$,

$$\begin{aligned} W'(s) &= \frac{1}{s^2+\omega_p^2}\left[\frac{P'(s^2+\omega_p^2)}{Q}\right] = \frac{P'}{Q} \\ &= W(s) - \frac{1}{\frac{s}{2k}+\frac{\omega_p^2}{2ks}} \qquad = W(s) - \frac{1}{D_1 s + \frac{1}{D_2 s}} . \end{aligned} \tag{3.13}$$

After extracting the pole pair from every element in which the pole is present, the general form of equation is:

$$\mathbf{W}(s) = \mathbf{C}_r^W \frac{2ks}{s^2+\omega_p^2} + \mathbf{W}^{'}(s) = \mathbf{C}_r^W \frac{1}{D_1 s + \frac{1}{D_2 s}} + \mathbf{W}^{'}(s) . \tag{3.14}$$

Extracting this pole pair does not effect the positive real character of the remaining matrix, according to condition 3 of positive real symmetric matrices (section 2.2.2). For impedance and admittance matrices the above equation can be written as

$$\mathbf{Z}(s) = \mathbf{C}_r^Z \frac{1}{Cs + \frac{1}{Ls}} + \mathbf{Z}^{'}(s) , \qquad \mathbf{Y}(s) = \mathbf{C}_r^Y \frac{1}{Ls + \frac{1}{Cs}} + \mathbf{Y}^{'}(s) . \tag{3.15}$$

Figs. 3.8 (a) and (b) show the sub-circuits extracted from impedance and admittance matrices in case 5, respectively.

3.2.7 Case 6

This case is dual to the case 5. Instead of having a pole pair, a zero pair at $\pm j\omega_p$ is present in given positive real symmetric matrix. The corresponding condition can be written as:

$$\lim_{s\to\pm j\omega} |\mathbf{W}(s)| = 0 .$$

These zeros would appear as poles in the inverse matrix from where they can be separated with same procedure discussed in case 5 above. The general equation for extracted and remaining matrix is

$$\mathbf{W}(s) = \left[\mathbf{C}_r^W \frac{2ks}{s^2+\omega_p^2} + \mathbf{W}^{'-1}(s)\right]^{-1} = \left[\mathbf{C}_r^W \frac{1}{D_1 s + \frac{1}{D_2 s}} + \mathbf{W}^{'-1}(s)\right]^{-1} , \tag{3.16}$$

which for impedance and admittance matrices is written as

$$\mathbf{Z}(s) = \left[\mathbf{C}_r^Y \frac{1}{Ls + \frac{1}{Cs}} + \mathbf{Z}^{'-1}(s)\right]^{-1} , \qquad \mathbf{Y}(s) = \left[\mathbf{C}_r^Z \frac{1}{Cs + \frac{1}{Ls}} + \mathbf{Y}^{'-1}(s)\right]^{-1} \tag{3.17}$$

Fig. 3.9 (a) and (b) show the sub-circuits corresponding to Eqs. 3.17 for impedance and admittance matrices, respectively.

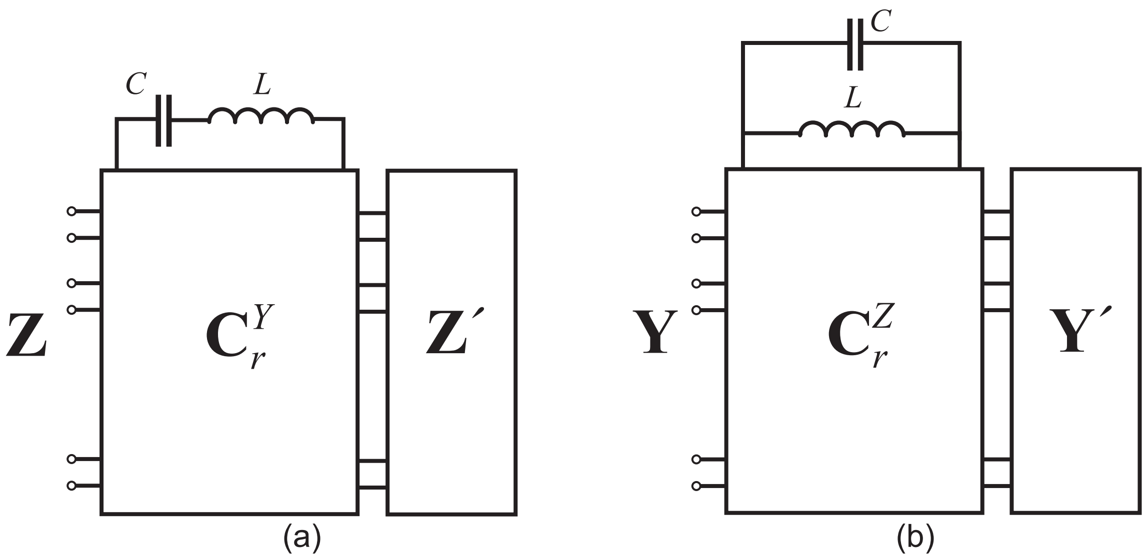


Figure 3.9: Sub-circuit realizations of case 6; (a) for impedance matrix and (b) for admittance matrix, according to Eqs. 3.17; the symbol used are explained in Figs. A.2 and A.4.

3.2.8 Case 7: Brune's Process

Cases 1 to 6 are related to poles or zeros of positive real symmetric matrices (PRSM) located on the imaginary axis of the complex frequency plane s, while case 7, which is also named as the Brune's process, deals with those PRSM whose poles and zeros are strictly in the left half plane of the complex frequency. Let $\mathbf{W}(s)$ be a PRSM with no pole or zero on the imaginary axis of s-plane, then

$$|\mathbf{W}(s)|_{s=j\omega} \neq 0 \quad \forall\, \omega \in \Re : \quad -\infty < \omega < \infty \,.$$

Brune's process strives to *produce* a zero, numerically, on the imaginary axis while preserving the positive real character of the remaining matrix. The location of the zero, $s = j\omega_{min}$, is determined during the process and, theoretically, there are many ways of determining it. Each different way yields a different value of ω_{min} and hence a different circuit realization. McMillan [28] has proposed the most general way, which forms an optimization problem [33] and requires additional information for incorporation in the algorithm. Due to its complexity the McMillan's approach is not taken in the algorithm. In contrast, Tellegen [27] has pointed out a specific yet a simple approach, which will be discussed in due coarse.

There are four steps involved in the Brune's process. The detailed discussion of each step is given below under specified heading, while a brief overview of each step in connection with flow diagram in Fig. 3.2 is as follows.

The first step of the process determines the frequency (i.e. $s = j\omega_{min}$) where the zero is to be *produced* and removes a real quantity from one of the diagonal elements of the given PRSM (which is realized in the equivalent circuit as a series resistance or parallel conductance). The value of ω_{min} forms three *situations* as shown in Fig. 3.2. If situation 1 ($\omega_{min} = \infty$) or situation 2 ($\omega_{min} = 0$) occurs, then the process ends prematurely and next three steps do not follow. But if ω_{min} has some finite value other than 0 (situation 3), then next three steps would be required to complete the process. The second step adds a pole either at infinity or at $s = 0$ in such a way that it preserves the positive real character of the matrix yet canceling the imaginary value of the determinant of the matrix at ω_{min}. This steps ends with a true zero-pair at $s = \pm j\omega_{min}$. Depending on the value of parameter α (discussed below), it yields two types of circuits. The third step is removal of the zero pair at $s = \pm j\omega_{min}$ which were formed in the previous step. Fourth and last step involves removal of a pole either at infinity or at $s = 0$, depending on the circuit type determined in second step.

3.2.8.1 Step 1: Determination of ω_{min} and removal of resistance or conductance 'A'

In this step the main idea is to remove a series resistor or parallel conductance, A, from one of the ports in such a manner that the real part of the given PRSM matrix $\mathbf{W}(s)$ (impedance or admittance respectively) becomes rank deficient at $s = j\omega_{min}$ while maintaining P.R. character. In [27] the extraction is done only for the first port, but it can be generalized to any port. The choice of the port is arbitrary and doesn't effect the algorithm itself. MATLAB implementation is done using the first port only.

Let the i^{th} port is chosen for the removal of the resistance or conductance A, and let

$$\mathbf{A}(\omega) = \Re[\mathbf{W}(s)]|_{s=j\omega} \,, \qquad \mathbf{I}(\omega) = \Im[\mathbf{W}(s)]|_{s=j\omega} \,,$$

then to decrease the rank of $\mathbf{A}(\omega)$, the value of A has to be chosen such that

$$\left|\mathbf{A}^{'}(\omega)\right| = \begin{vmatrix} A_{11}(\omega) & \cdots & A_{1i}(\omega) & \cdots & A_{1N}(\omega) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1i}(\omega) & \cdots & A_{ii}(\omega) - A & \cdots & A_{iN}(\omega) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1N}(\omega) & \cdots & A_{iN}(\omega) & \cdots & A_{NN}(\omega) \end{vmatrix} = 0 \,.$$

As determinant is a linear function of each row or column [45, §4] and [51, §5.1], one can expand the i^{th} column in the above determinant as

$$\left|\mathbf{A}^{'}(\omega)\right| = |\mathbf{A}(\omega)| - \begin{vmatrix} A_{11}(\omega) & \cdots & 0 & \cdots & A_{1N}(\omega) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1i}(\omega) & \cdots & A & \cdots & A_{iN}(\omega) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1N}(\omega) & \cdots & 0 & \cdots & A_{NN}(\omega) \end{vmatrix} = |\mathbf{A}(\omega)| - AM_{ii}(\omega) = 0 \,, \quad (3.18)$$

where $M_{ii}(\omega)$ is the minor of element $A_{ii}(\omega)$ in $\mathbf{A}(\omega)$. From Eq. 3.18 the value of A can be calculated for any given value of ω. The question is that which value of ω should be used? The answer lies in the fact that the matrix $\mathbf{A}^{'}(\omega)$ must remain positive semi-definite to preserve the P.R. character of the remaining matrix. For that the value of A should be equal to the value of the global minima and the frequency at that global minima would be the required frequency ω_{min}. Thus

$$A = min\left(\frac{|\mathbf{A}(\omega)|}{M_{ii}(\omega)}\right) = \frac{|\mathbf{A}(\omega_{min})|}{M_{ii}(\omega_{min})}\,. \tag{3.19}$$

Finding the global minima is a simple optimization problem. As the degrees of the polynomials in $\mathbf{W}(s)$ increases, it becomes more complex and less accurate to find the minima.

Summarizing the above discussion, one can proceed with following steps:

1. Choose the port i from which the resistance/ conductance is to be extracted.

2. Calculate the function

$$\frac{|\mathbf{A}(\omega)|}{M_{ii}(\omega)} = \frac{\left|\Re[\mathbf{W}(s)]|_{s=j\omega}\right|}{M_{ii}(\omega)}\,,$$

   where $M_{ii}(\omega)$ is the minor of $A_{ii}(\omega)$ in $\mathbf{A}(\omega)$

3. Find the global minima of above function, yielding the value of ω_{min} and A.

The constant A would represent a series resistance to be extracted from i^{th} port in case the given PRSM is impedance (Fig. 3.10 (Step 1)), otherwise it would be a parallel conductance at the same port for admittance matrix (Fig. 3.11 (Step 1)).

The given PRSM $\mathbf{W}(s)$ at imaginary axis of complex frequency plane s can be written as

$$\mathbf{W}(s=j\omega) = \Re[\mathbf{W}(s=j\omega)] + j\Im[\mathbf{W}(s=j\omega)] = \mathbf{A}(\omega) + j\mathbf{I}(\omega)\,. \tag{3.20}$$

After the subtraction of above calculated value of A from the corresponding port, the remaining matrix would be

$$\mathbf{W}^{'}(s=j\omega) = \mathbf{A}^{'}(\omega) + j\mathbf{I}(\omega)\,. \tag{3.21}$$

The matrix $\mathbf{I}(\omega)$ would remain unchanged (theorem B.4). At ω_{min} it would become

$$\mathbf{W}^{'}(s=j\omega_{min}) = \mathbf{A}^{'}(\omega_{min}) + j\mathbf{I}(\omega_{min})\,. \tag{3.22}$$

The matrix $\mathbf{A}^{'}(\omega_{min})$ is a real symmetric yet rank deficient matrix as discussed above. Depending on the value of ω_{min}, there will be three situations.

Situation 1: $\omega_{min} \to \infty$
As $\lim_{\omega \to \infty} \mathbf{I}(\omega) = \mathbf{0}$, the matrix $\mathbf{W}^{'}(s = j\omega_{min})$ is already rank deficient and a zero has already been created at $s = \infty$ by removing real value A. The Brune's process would terminate here and in next iteration of the algorithm case 2 would follow.

Situation 2: $\omega_{min} = 0$
In this situation, $\lim_{\omega \to 0} \mathbf{I}(\omega) = \mathbf{0}$, and a zero has already been created by removing real value A. The Brune's process would again terminate here and case 4 would follow in next iteration of algorithm.

Situation 3: ω_{min} = some finite value
Generally $\mathbf{I}(\omega_{min}) \neq \mathbf{0}$ and is not rank deficient matrix. This makes $\mathbf{W}^{'}(s = j\omega_{min})$ full rank matrix. To complete the Brune's process and to form true zero at $s = \pm j\omega_{min}$, further steps would be required.

3.2.8.2 Step 2: Determination of α and Addition of a Pole

This step makes the matrix $\mathbf{I}(\omega_{min})$ in Eq. 3.22 rank deficient, so that the whole matrix $\mathbf{W}^{'}(\omega_{min})$ become rank deficient. Let

$$\boldsymbol{\beta} = [\beta_1, \beta_2, \cdots, \beta_N]^T$$

be the null vector of $\mathbf{A}^{'}(\omega_{min})$:

$$\mathbf{A}^{'}(\omega_{min})\boldsymbol{\beta} = \mathbf{0}. \tag{3.23}$$

And let $\mathbf{H}$ has to be extracted to make matrix $\mathbf{I}(\omega_{min})$ rank deficient, then the matrix $\mathbf{H}$ should be chosen such that

$$(\mathbf{I}(\omega_{min}) - \mathbf{H})\,\boldsymbol{\beta} = \mathbf{0},$$

i.e. matrices $\mathbf{A}^{'}(\omega_{min})$ and $(\mathbf{I}(\omega_{min}) - \mathbf{H})$ share same null vector $\boldsymbol{\beta}$ (theorem B.1). As matrix $\mathbf{H}$ is rank one matrix, it can be written as [27]:

$$\mathbf{H} = \alpha \begin{pmatrix} h_1^2 & h_1h_2 & \cdots & h_1h_n \\ h_1h_2 & h_2^2 & \cdots & h_2h_n \\ \vdots & \vdots & \ddots & \vdots \\ h_1h_n & h_2h_n & \cdots & h_n^2 \end{pmatrix} = \alpha \boldsymbol{h}\boldsymbol{h}^T$$

where h_i for $i \in \{1, 2, 3, \cdots, N\}$ are real, $\alpha \in \{+1, -1\}$ and

$$\boldsymbol{h} = [h_1, h_2, \cdots, h_n]^T.$$

Determination of α: The value of α will be required to select the Brune's circuit type. Following calculation shows the determination of the value of α:

$$(\mathbf{I}(\omega_{min}) - \mathbf{H})\,\boldsymbol{\beta} = \mathbf{0}$$

$$\begin{aligned}
\mathbf{I}(\omega_{min})\boldsymbol{\beta} &= \mathbf{H}\boldsymbol{\beta} \\
\mathbf{I}(\omega_{min})\boldsymbol{\beta} &= \alpha\boldsymbol{h}\boldsymbol{h}^T\boldsymbol{\beta} \qquad (3.24)\\
\boldsymbol{\beta}^T\mathbf{I}(\omega_{min})\boldsymbol{\beta} &= \alpha\boldsymbol{\beta}^T\boldsymbol{h}\boldsymbol{h}^T\boldsymbol{\beta} \\
\boldsymbol{\beta}^T\mathbf{I}(\omega_{min})\boldsymbol{\beta} &= \alpha\left(\boldsymbol{\beta}.\boldsymbol{h}\right)^2 \\
&\Rightarrow \alpha = \operatorname{sgn}\left(\boldsymbol{\beta}^T\mathbf{I}(\omega_{min})\boldsymbol{\beta}\right) \qquad (3.25)
\end{aligned}$$

As $(\boldsymbol{\beta}.\boldsymbol{h})^2$ is positive, the value of α is the sign of $\boldsymbol{\beta}^T\mathbf{I}(\omega_{min})\boldsymbol{\beta}$.

Determination of $\boldsymbol{h}$: After the determination of α, the system of equations formed in Eq. 3.24 can solved simultaneously to determine the values of h_i's in the vector $\boldsymbol{h}$.

Determination of Brune Circuit Types: Considering two possible values of α and the given PRSM to be either impedance or admittance, there are four Brune circuit types. Table 3.3 shows the suitable Brune circuit type for the corresponding values of α and given PRSM.

Table 3.3: Suitable Brune's circuit types for corresponding values of α and type of PRSM

Type of given PRSM	$\alpha = -1$	$\alpha = +1$
Impedance	Type-I	Type-II
Admittance	Type-III	Type-IV

Type-I and Type-III corresponding to the value of $\alpha = -1$ have same mathematical procedure and the same goes for Type-II and Type-IV, thus they would be discussed in groups as follows:

Type I/III: $\alpha = -1$

When α is negative, the diagonal values of the matrix $\mathbf{H}$ are also negative. From this matrix, a circuit is intended to be calculated to fulfill two conditions. One one hand it should be equal to the matrix $\mathbf{H}$ at $s = j\omega_{min}$ and on other hand it should be subtracted from the matrix $\mathbf{W}'(s)$ so that the PRSM character is not disturbed. As the matrix $\mathbf{H}$ has to be subtracted from the imaginary part of $\mathbf{W}'(s)$, corresponding circuit would be reactive. To fulfill all requirements, let

$$\begin{aligned}
j\mathbf{H} = j\alpha\boldsymbol{h}\boldsymbol{h}^T &= j\omega_{min}D_1\boldsymbol{p}\boldsymbol{p}^T \\
&= \left[sD_1\boldsymbol{p}\boldsymbol{p}^T\right]\Big|_{s=j\omega_{min}} \; , \qquad (3.26)
\end{aligned}$$

where value of D_1 is negative to incorporate the negativity of α, thus the matrix $\boldsymbol{p}\boldsymbol{p}^T$ would have positive diagonal elements. The negativity of D_1 would be dealt with equivalent circuit representation.

Subtracting $sD_1\boldsymbol{p}\boldsymbol{p}^T$ from $\mathbf{W}'(s)$ gives

$$\mathbf{W}''(s) = \mathbf{W}'(s) - sD_1\boldsymbol{p}\boldsymbol{p}^T = \mathbf{W}'(s) + s(-D_1)\boldsymbol{p}\boldsymbol{p}^T \, . \qquad (3.27)$$

Because $-D_1$ is positive, thus *adding a pole at infinity with positive residue* to a positive real symmetric matrix would also result in positive real symmetric matrix. Note that in actual circuit, the value of D_1 would be used rather than $-D_1$.

For impedance PRSM, D_1 is an inductance L_1 (Fig. 3.10, Type-I, Step 2) and for admittance PRSM D_1 is a capacitance C_1 (Fig. 3.11, Type-III, Step 2):

$$\mathbf{Z}^{''}(s) = \mathbf{Z}^{'}(s) - sL_1\boldsymbol{pp}^T \qquad \text{(Type-I)} \tag{3.28a}$$

$$\mathbf{Y}^{''}(s) = \mathbf{Y}^{'}(s) - sC_1\boldsymbol{pp}^T \qquad \text{(Type-III)} \tag{3.28b}$$

Type II/IV: $\alpha = +1$

If α is positive, the diagonal elements of the matrix $\mathbf{H}$ are also positive. Although this matrix can be used to synthesize a reactive circuit as given above, but on subtraction from of $\mathbf{W}^{'}(s)$ it would not guarantee preservation of positive real character of remaining matrix. To overcome this difficulty let

$$\begin{aligned} j\mathbf{H} = j\alpha\boldsymbol{hh}^T &= \frac{-1}{j\omega_{min}D_1}\boldsymbol{pp}^T \\ &= \left[\frac{1}{s(-D_1)}\boldsymbol{pp}^T\right]\Bigg|_{s=j\omega_{min}}, \end{aligned} \tag{3.29}$$

where D_1 has positive value, and the value used in the circuit would be $-D_1$. This value can be avoided by using equivalent circuit as discussed later.

Subtracting $\frac{1}{s(-D_1)}\boldsymbol{pp}^T$ from $\mathbf{W}^{'}(s)$ gives

$$\begin{aligned} \mathbf{W}^{''}(s) &= \mathbf{W}^{'}(s) - \frac{1}{s(-D_1)}\boldsymbol{pp}^T \\ &= \mathbf{W}^{'}(s) + \frac{1}{sD_1}\boldsymbol{pp}^T . \end{aligned} \tag{3.30}$$

Because D_1 is positive, thus *adding a pole at* $s = 0$ *with positive residue* to a positive real symmetric matrix would also result in positive real symmetric matrix (PRSM).

For impedance PRSM, $-D_1$ would be a capacitance C_1 (Fig. 3.10, Type-II, Step 2) while for admittance PRSM, $-D_1$ would be an inductance L_1 (Fig. 3.11, Type-IV, Step 2).

$$\mathbf{Z}^{''}(s) = \mathbf{Z}^{'}(s) - \frac{1}{sC_1}\boldsymbol{pp}^T \qquad \text{(Type-II)} \tag{3.31a}$$

$$\mathbf{Y}^{''}(s) = \mathbf{Y}^{'}(s) - \frac{1}{sL_1}\boldsymbol{pp}^T \qquad \text{(Type-IV)} \tag{3.31b}$$

The addition of a pole to $\mathbf{W}^{'}(s)$ in all four types, forms a zero-pair in $\mathbf{W}^{''}(s)$ at $s = \pm j\omega_{min}$, which would be removed in next step.

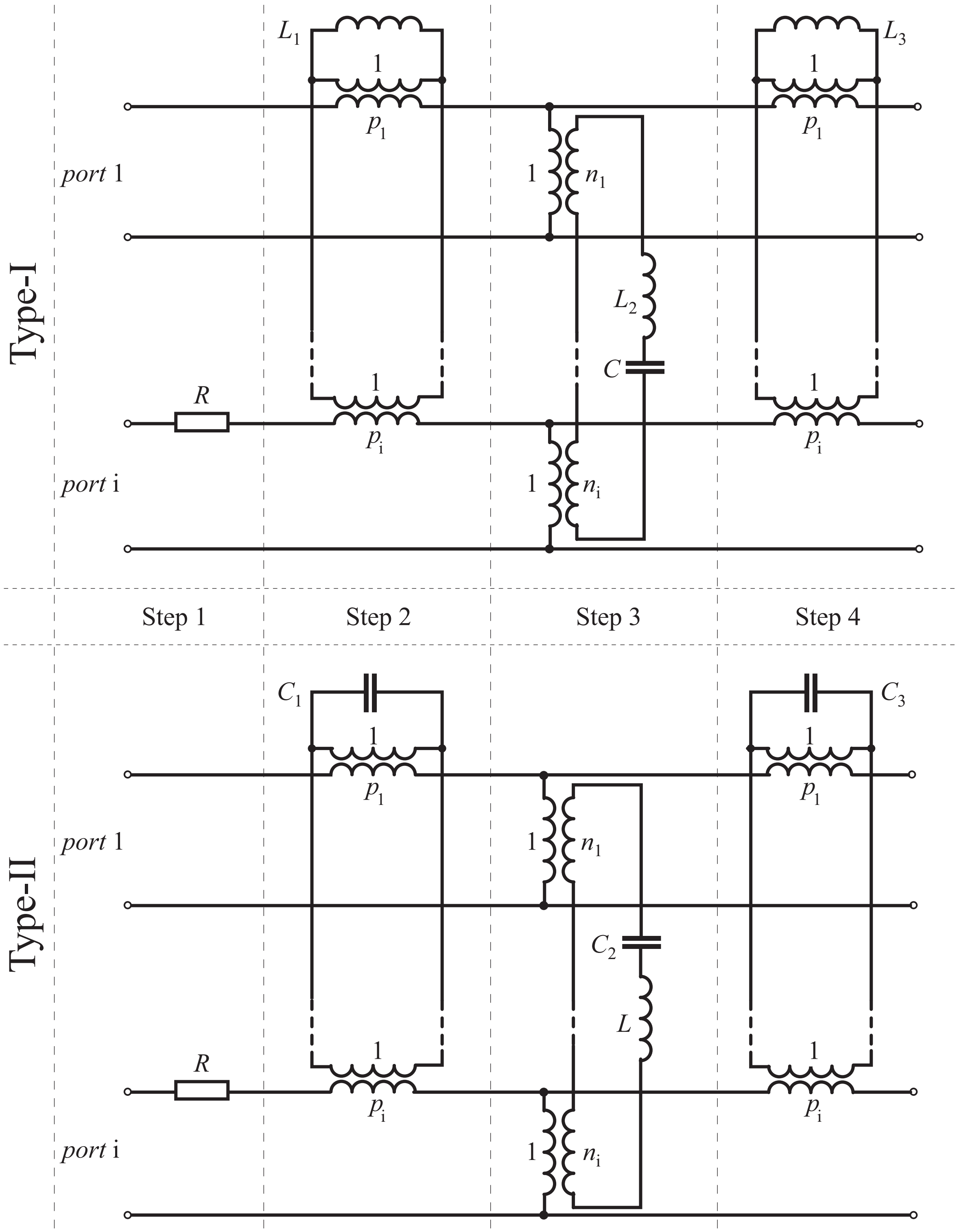


Figure 3.10: Case 7 networks; Extracted Brune's circuits of Type-I and Type-II

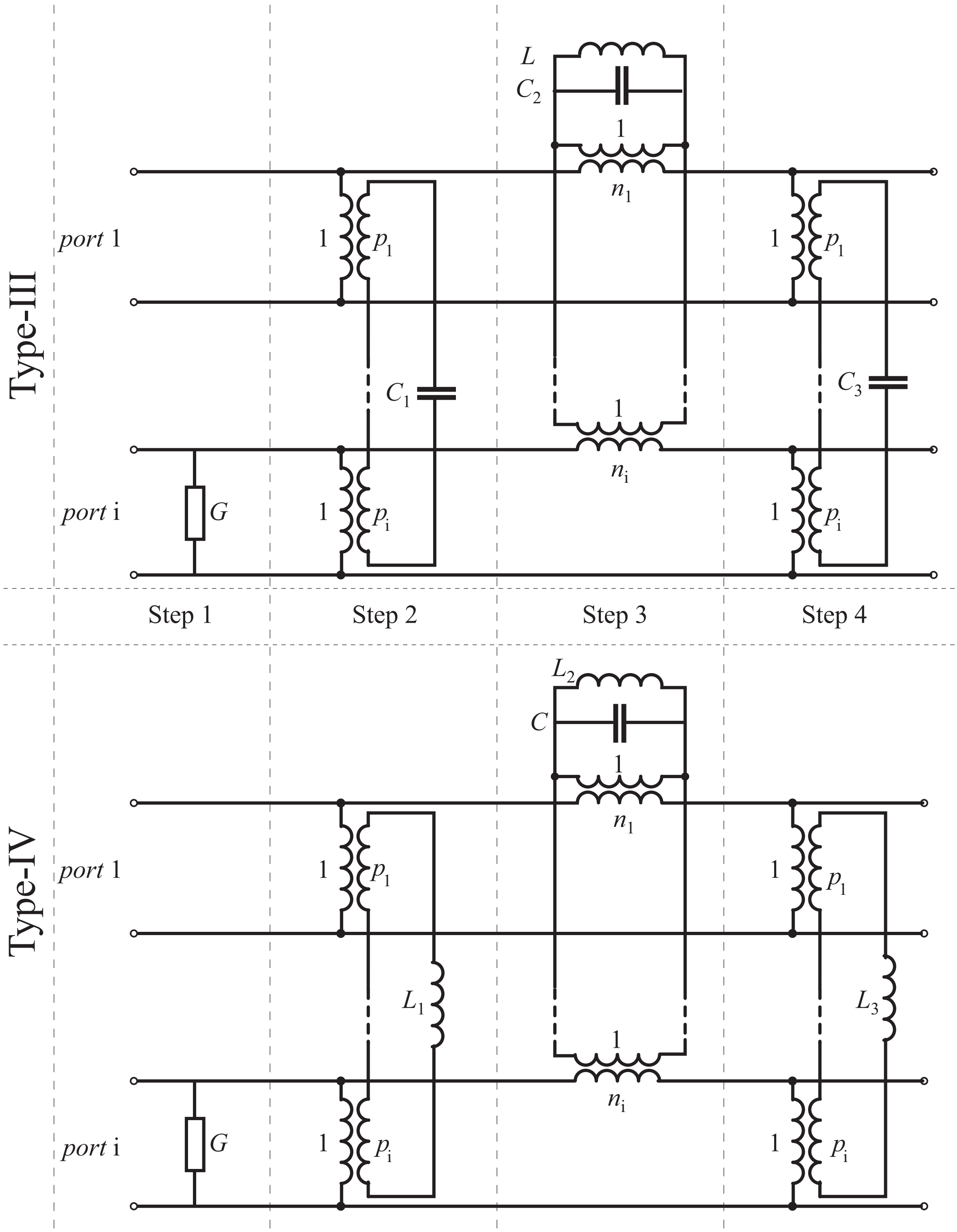


Figure 3.11: Case 7 networks; Extracted Brune's circuits of Type-III and Type-IV

3.2.8.3 Step 3: Removal of Zero Pair at $s = \pm j\omega_{min}$

This step is similar to Case 6 described above. The matrix $\mathbf{W}''(s)$ could be inverted and the pole pair can be removed from its inverse. Procedures describe in section 3.2.6 can be followed. In this step, Type-I and Type-II circuits would have similar circuits (Fig. 3.10, Step 3), and Type-III and Type-IV would have similar circuits (Fig. 3.11, Step 3). Removal of these zero pair from $\mathbf{W}''(s)$ can be written as:

$$\begin{aligned}\mathbf{W}'''(s) &= \left[[\mathbf{W}''(s)]^{-1} - \frac{2ks}{s^2+\omega_{min}^2}\boldsymbol{n}\boldsymbol{n}^T\right]^{-1} \\ &= \left[[\mathbf{W}''(s)]^{-1} - \frac{1}{\frac{s}{2k}+\frac{\omega_{min}^2}{2ks}}\boldsymbol{n}\boldsymbol{n}^T\right]^{-1}.\end{aligned} \tag{3.32}$$

The equations corresponding to each type of circuit are:

$$\mathbf{Z}'''(s) = \left[[\mathbf{Z}''(s)]^{-1} - \frac{1}{L_2 s+\frac{1}{Cs}}\boldsymbol{n}\boldsymbol{n}^T\right]^{-1} \quad \text{(Type-I)} \tag{3.33a}$$

$$\mathbf{Z}'''(s) = \left[[\mathbf{Z}''(s)]^{-1} - \frac{1}{Ls+\frac{1}{C_2 s}}\boldsymbol{n}\boldsymbol{n}^T\right]^{-1} \quad \text{(Type-II)} \tag{3.33b}$$

$$\mathbf{Y}'''(s) = \left[[\mathbf{Y}''(s)]^{-1} - \frac{1}{C_2 s+\frac{1}{Ls}}\boldsymbol{n}\boldsymbol{n}^T\right]^{-1} \quad \text{(Type-III)} \tag{3.33c}$$

$$\mathbf{Y}'''(s) = \left[[\mathbf{Y}''(s)]^{-1} - \frac{1}{Cs+\frac{1}{L_2 s}}\boldsymbol{n}\boldsymbol{n}^T\right]^{-1} \quad \text{(Type-IV)} \tag{3.33d}$$

The remaining matrix $\mathbf{W}'''(s)$ would still be positive real symmetric matrix and the extracted sub-circuit is realizable with positive values of inductance and capacitance. The reason being that the given matrix $\mathbf{W}''(s)$ is positive real symmetric matrix and containing zero-pair on imaginary axis. According to the third condition of positive real symmetric matrices the residue matrix ($2k\boldsymbol{n}\boldsymbol{n}^T$) of the corresponding poles in inverted matrix is positive definite (section 2.2.2).

3.2.8.4 Step 4: Removal of pole

After removal of a zero pair from the matrix $\mathbf{W}''(s)$, the remaining matrix $\mathbf{W}'''(s)$ would again have pole either at infinity (for Type-I and Type-III) or at $s = 0$ (for Type-II and Type-IV). The detailed proof of its existence is shown in section 4.1 below. This pole and its residue matrix would be highly dependent on the values extracted in previous steps (steps 2 and 3). The circuit topology and the elements would be same as those of Step 2. Only the value of the circuit element would be different; the turns ratios would be same as in step 2 (section 4.1).

After the removal of the pole, the equations can be written as:

$$\mathbf{Z}^{''''}(s) = \mathbf{Z}^{'''}(s) - sL_3\boldsymbol{p}\boldsymbol{p}^T \qquad \text{(Type-I)} \tag{3.34a}$$
$$\mathbf{Z}^{''''}(s) = \mathbf{Z}^{'''}(s) - \frac{1}{sC_3}\boldsymbol{p}\boldsymbol{p}^T \qquad \text{(Type-II)} \tag{3.34b}$$
$$\mathbf{Y}^{''''}(s) = \mathbf{Y}^{'''}(s) - sC_3\boldsymbol{p}\boldsymbol{p}^T \qquad \text{(Type-III)} \tag{3.34c}$$
$$\mathbf{Y}^{''''}(s) = \mathbf{Y}^{'''}(s) - \frac{1}{sL_3}\boldsymbol{p}\boldsymbol{p}^T . \qquad \text{(Type-IV)} \tag{3.34d}$$

The parameters L_3 and C_3 in each type of circuit are dependent on parameters L_1, L_2, C_1 and C_2, and transformer turns ratios vectors $\boldsymbol{p}$ and $\boldsymbol{n}$ (derived in steps 2 and 3, according to Eqs. 4.11, 4.21, 4.12 and 4.22 derived in next chapter). The corresponding sub-circuits are shown in Figs. 3.10 and 3.11.

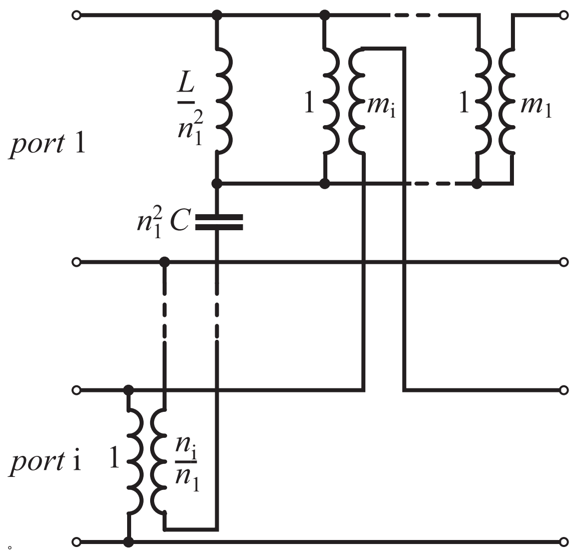
The Brune extraction procedure ends here. The remaining matrix $\mathbf{W}^{''''}(s)$ would be a positive real symmetric matrix as in every step of Brune's process the positive real character of the remaining matrix is preserved (section 4.3.1 and [28, §XVI]). The remaining problem is to remove the negative value elements, L_1 and C_1. Replacement of the part of Brune's extracted circuits with their equivalent ones, for this purpose, is described in next section.

3.2.9 Equivalent Brune's Circuit Types

Brune's equivalent circuits are required to avoid the negative elements formed during step 2 of the Brune's process (section 3.2.8.2). It also replaces four reactive elements (appearing in extracted circuit) by two elements. This is due to the fact that after every complete Brune's process the degree of the positive real symmetric matrix decreases by 2 [27]. The equivalent circuits make use of strong relation existing between the extracted parameters (section 4.1 in next chapter). The equivalent circuits and the corresponding relations are given below in brief. Detailed proof of equivalence is given in section 4.2 in next chapter. In the following, the extracted parameters $L_1, L_2, L_3, C_1, C_2, C_3$ and the transformer turns ratios vectors $\boldsymbol{p}$ and $\boldsymbol{n}$, are already known from the extraction procedures described above. Equivalent parameters, L and C, and new turns ratios $\boldsymbol{m}$ are calculated accordingly. Some parameters and turns ratios are same in both extracted and equivalent circuits.

The equivalent sub-circuit topology for extracted Brune sub-circuit of Type-I was given by Tellegen [27]. Other three topologies are new contribution in this work. Equivalent sub-circuit topology for Type-II is similar to Type-I with only replacement of elements, while sub-circuit topologies of Type-III and Type-IV are different.

3.2.9.1 Equivalent Brune's Circuit of Type-I



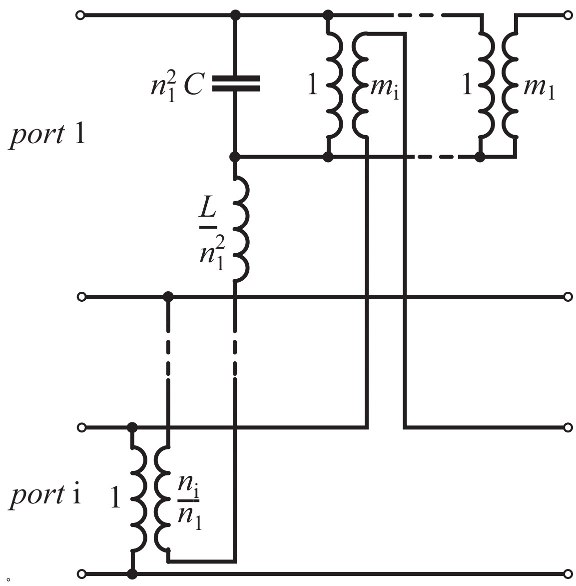
$$L = (\boldsymbol{p}.\boldsymbol{n})^2 L_1 + L_2 \tag{3.35a}$$
$$m_1 = \frac{A(\boldsymbol{p}.\boldsymbol{n}) L_1 + L_2}{L} \tag{3.35b}$$
$$m_k = -\frac{(\boldsymbol{p}.\boldsymbol{n}) n_1 L_1 p_k}{L} \tag{3.35c}$$
for $k = 2, 3, \cdots, N$

and it is valid if

$$L_3 = -\frac{L_1 L_2}{L} \tag{3.35d}$$
where $A = \sum_{k=2}^{N} p_k n_k$

Figure 3.12: Equivalent Brune's circuit of Type-I

3.2.9.2 Equivalent Brune's circuit of Type-II



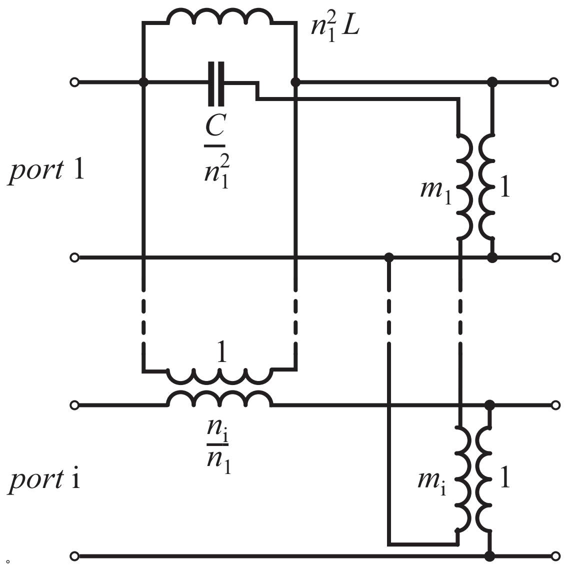
$$C = \frac{C_1 C_2}{(\boldsymbol{p}.\boldsymbol{n})^2 C_2 + C_1} \tag{3.36a}$$
$$m_1 = \frac{A(\boldsymbol{p}.\boldsymbol{n}) C_2 + C_1}{(\boldsymbol{p}.\boldsymbol{n})^2 C_2 + C_1} \tag{3.36b}$$
$$m_k = -\frac{(\boldsymbol{p}.\boldsymbol{n}) C_2 n_1 p_k}{(\boldsymbol{p}.\boldsymbol{n})^2 C_2 + C_1} \tag{3.36c}$$
for $k = 2, 3, \cdots, N$

with validity if

$$C_3 = -\frac{C_1 C_2}{C} \tag{3.36d}$$
where $A = \sum_{k=2}^{N} p_k n_k$

Figure 3.13: Equivalent Brune's circuit of Type-II

3.2.9.3 Equivalent Brune's circuit of Type-III



$$C = (\boldsymbol{p}.\boldsymbol{n})^2 C_1 + C_2 \tag{3.37a}$$

$$m_1 = \frac{A(\boldsymbol{p}.\boldsymbol{n}) C_1 + C_2}{(\boldsymbol{p}.\boldsymbol{n})^2 C_1 + C_2} \tag{3.37b}$$

$$m_k = -\frac{(\boldsymbol{p}.\boldsymbol{n}) n_1 C_1 p_k}{(\boldsymbol{p}.\boldsymbol{n})^2 C_1 + C_2} \tag{3.37c}$$

for $k = 2, 3, \cdots, N$

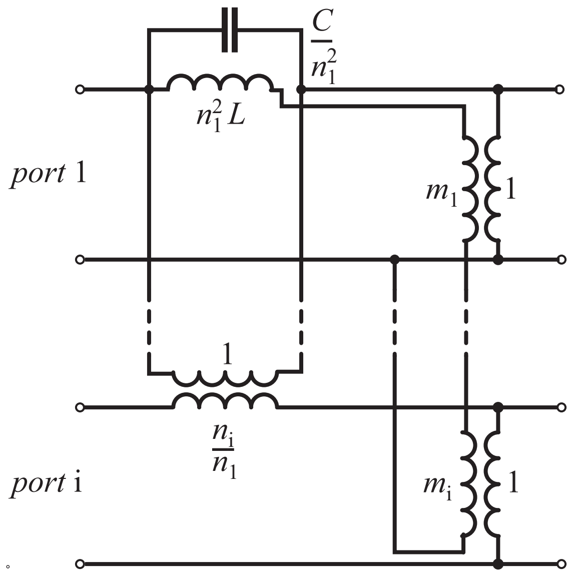
with validity if

$$C_3 = -\frac{C_1 C_2}{C} \tag{3.37d}$$

where $A = \sum_{k=2}^{N} p_k n_k$

Figure 3.14: Equivalent Brune's circuit of Type-III

3.2.9.4 Equivalent Brune's circuit of Type-IV



$$L = \frac{L_1 L_2}{(\boldsymbol{p}.\boldsymbol{n})^2 L_2 + L_1} \tag{3.38a}$$

$$m_1 = \frac{A(\boldsymbol{p}.\boldsymbol{n}) L_2 + L_1}{(\boldsymbol{p}.\boldsymbol{n})^2 L_2 + L_1} \tag{3.38b}$$

$$m_k = -\frac{(\boldsymbol{p}.\boldsymbol{n}) L_2 n_1 p_k}{(\boldsymbol{p}.\boldsymbol{n})^2 L_2 + L_1} \tag{3.38c}$$

for $k = 2, 3, \cdots, N$

with validity if

$$L_3 = -\frac{L_1 L_2}{L} \tag{3.38d}$$

where $A = \sum_{k=2}^{N} p_k n_k$

Figure 3.15: Equivalent Brune's circuit of Type-IV

3.3 Remarks on Stability and Time Complexity of Algorithm

The stability of the Brune multiport algorithm is can be defined as its ability to reduce the order of positive real symmetric matrix $\mathbf{W}(s)$ in each iteration. If at any iteration the order of the matrix $(\mathcal{O}[\mathbf{W}(s)])$ increases instead of decreasing then the algorithm becomes unstable. This situation arises when the numeric implementation fails to cancel common factors arising in the numerator and denominator of the determinant of $\mathbf{W}(s)$. Numeric errors, due to use of less number of digits in calculation and not discarding these errors actively, causes instability.

Time complexity increases with the increase of the degree of the given matrix. The main time consuming step is Step 1 of Brune's process(section 3.2.8.1) in which the Eq. 3.18 is a numeric optimization problem for finding the global minima of the function. For the matrix of order 40, first few iterations may take upto 45 minutes each (on machine with 2GB RAM and dual core processor of 2 GHz each). As the matrix reduces, the time per iteration also reduces.

3.4 Number of Elements of Circuit

There has been an interest to find the number of elements required to synthesize a circuit in any particular algorithm. Brune in [23] has shown that the minimum number of elements required in one cycle of Brune's process during the synthesis of one-port circuit are only four. This is true if the process doesn't terminate prematurely (situation 1 and situation 2), in which case it would require only two elements.

Similarly Tellegen [27] has shown that for N-port circuit of order $\mathcal{O}(\mathbf{W})$ can be synthesized in $\mathcal{O}(\mathbf{W})(N+1)+\frac{1}{2}N(N+1)$ elements. This expression is derived for the matrix which remains in case 7 and other cases(1 to 6) never occur.

The number of reactive elements in the circuit are equal to the order of the matrix given and the total number of elements required are equal to the number of independent parameters in the matrix [27].

4 Parameter Relationships and Equivalence of Circuits

In the Brune multiport algorithm, each of the seven cases and the stop criterion has a sub-circuit associated with it. The sub-circuit of a particular case is synthesized while the extraction procedure of that case is applied. The algorithm ensures that the resistors, capacitors and inductors of all sub-circuits have positive values, while the ideal transformers may assume negative turns ratios.

For the case 7, all four types of the extracted sub-circuit have an element with negative value. Moreover, there exists a special relationship between their parameters (i.e. values of reactive elements and ideal transformer turns ratios). This special relationship allows a part of the extracted sub-circuit to be replaced by an equivalent sub-circuit with all elements having positive values.

In this chapter, the relationship between the parameters of the extracted sub-circuits (of case 7) would be derived. Based on the relationships the equivalence between the extracted and the equivalent sub-circuits would be established. In the end the positive values of the elements of all sub-circuits of all cases would be discussed.

4.1 Parameters Relationships: Brune's Extracted Circuits

In the case 7 of Brune's multiport algorithm there are in general four steps (section 3.2.8). As described in section 3.2.8.4, the parameters extracted in step 4 are dependent on the parameters extracted in step 2 and step 3 (sections 3.2.8.2 and 3.2.8.3). The element extracted in step 1 (resistance or conductance) does not contribute in the relationship. The derivation of the relationship of these parameters will be carried out in detail. Because of the similarity, the derivations of the relationships for extracted Brune's circuit of Type-I and Type-III is done together, using the common symbols of table 3.2 . Similarly the derivations of the relationships for Brune's circuit of Type-II and Type-IV are grouped in one section.

Tellegen in [27] has derived the relationship for extracted Brune's circuit of Type-I. The derivation given here are more direct and in modern matrix notation. During the derivation, Sherman-Morrison Formula ([53, §2.7.1] and [54, §3.3.4]) for the calculation of inverses is used. Given an invertible square matrix $\mathbf{A}$ of order N and column vectors

$\boldsymbol{u}$ and $\boldsymbol{v}$ also of size N, and if $(1+\boldsymbol{v}^T\mathbf{A}^{-1}\boldsymbol{u}) \neq 0$, then

$$\left(\mathbf{A}+\boldsymbol{u}\boldsymbol{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\boldsymbol{u}\boldsymbol{v}^T\mathbf{A}^{-1}}{1+\boldsymbol{v}^T\mathbf{A}^{-1}\boldsymbol{u}} \tag{4.1}$$

Note that the denominator $(1+\boldsymbol{v}^T\mathbf{A}^{-1}\boldsymbol{u})$ is a scalar quantity.

4.1.1 Brune's Extracted Circuits: Type-I and Type-III

Extracted Brune's circuits of Type-I and Type-III are for impedance and admittance matrices respectively. But because the form of the equations is same, common symbols (as given in *the list of symbols*) would be used to do the derivation and in the end separate results would be written.

The impedance or admittance matrix is in case 7 when neither of its poles nor any of its zeros is located on the imaginary axis of complex frequency plane. In step 1 of the extraction procedure of case 7, a resistance or a conductance is subtracted form the given impedance or admittance matrix and the matrix $\mathbf{W}'(s)$ remains. The extraction of the resistance or conductance is such that the $\mathbf{W}'(s)$ is still positive real symmetric matrix. In step 2, a pole at $s=\infty$ is *added* and it can be written as

$$\mathbf{W}''(s) = \mathbf{W}'(s) - sD_1\boldsymbol{p}\boldsymbol{p}^T \qquad \text{(From Eq. 3.27)} \tag{4.2}$$

where the element D_1 is negative valued. In step 3, a zero-pair formed in step 2 (section 3.2.8.2) would be removed. The equation for that is

$$\mathbf{W}'''(s) = \left[\left[\mathbf{W}''(s)\right]^{-1} - \left(\frac{\boldsymbol{n}\boldsymbol{n}^T}{sD_2+\dfrac{1}{sD}}\right)\right]^{-1} \qquad \text{(From Eq. 3.32)} \tag{4.3}$$

As mentioned in section 3.2.8.4, a pole at ∞ would be formed after Step 3. Along with the formation of this pole, its matrix of residues would also be derived. This would give the dependency of parameter D_3 of Step 4 on the parameters D_2 and D_1 of Step 3 and Step 2, respectively ($\{L_3, L_2, L_1\}$ and $\{C_3, C_2, C_1\}$ for Type-I and Type-III respectively). The derivation would start with Eq. 4.3 of Step 3. For the sake of brevity symbol the variable dependency (s) is removed from the matrix symbols. Using Sherman-Morrison Formula of Eq. 4.1 on Eq. 4.3:

$$\mathbf{W}''' = \mathbf{W}'' + \frac{\mathbf{W}''\left(\dfrac{\boldsymbol{n}\boldsymbol{n}^T}{sD_2+\dfrac{1}{sD}}\right)\mathbf{W}''}{1-\left(\dfrac{\boldsymbol{n}^T\mathbf{W}''\boldsymbol{n}}{sD_2+\dfrac{1}{sD}}\right)} \tag{4.4}$$

Using Eq. 4.2 to replace $\mathbf{W}''$ and rearranging:

$$\mathbf{W}''' = \left[\mathbf{W}' - sD_1\boldsymbol{pp}^T\right] + \frac{\left[\mathbf{W}' - sD_1\boldsymbol{pp}^T\right]\boldsymbol{nn}^T\left[\mathbf{W}' - sD_1\boldsymbol{pp}^T\right]}{sD_2 + \frac{1}{sD} - \boldsymbol{n}^T\left[\mathbf{W}' - sD_1\boldsymbol{pp}^T\right]\boldsymbol{n}} \tag{4.5}$$

$$= \frac{\begin{array}{c} sD_2\mathbf{W}' + \frac{1}{sD}\mathbf{W}' - (\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n})\mathbf{W}' + sD_1(\boldsymbol{n}^T\boldsymbol{pp}^T\boldsymbol{n})\mathbf{W}' \\ -s^2D_1D_2\boldsymbol{pp}^T - \frac{D_1}{D}\boldsymbol{pp}^T + sD_1(\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n})\boldsymbol{pp}^T - \mathbf{W}'\boldsymbol{nn}^T\mathbf{W}' \\ -s^2D_1^2(\boldsymbol{n}^T\boldsymbol{pp}^T\boldsymbol{n})\boldsymbol{pp}^T - sD_1\boldsymbol{pp}^T\boldsymbol{nn}^T\mathbf{W}' \\ -sD_1\mathbf{W}'\boldsymbol{nn}^T\boldsymbol{pp}^T + s^2D_1^2\boldsymbol{pp}^T\boldsymbol{nn}^T\boldsymbol{pp}^T \end{array}}{sD_2 + \frac{1}{sD} - \boldsymbol{n}^T\mathbf{W}'\boldsymbol{n} + sD_1\boldsymbol{n}^T\boldsymbol{pp}^T\boldsymbol{n}} \tag{4.6}$$

Here one can easily verify that $\boldsymbol{n}^T\boldsymbol{pp}^T\boldsymbol{n} = \boldsymbol{p}^T\boldsymbol{nn}^T\boldsymbol{p} = (\boldsymbol{p}.\boldsymbol{n})^2$ thus canceling ninth and twelfth term in the numerator. Further dividing both numerator and denominator by s:

$$\mathbf{W}''' = \frac{\begin{array}{c} -s[D_1D_2\boldsymbol{pp}^T] + \frac{1}{s^2}\left[\frac{1}{D}\mathbf{W}'\right] \\ + \left[D_1(\boldsymbol{n}^T\boldsymbol{pp}^T\boldsymbol{n})\mathbf{W}' + D_2\mathbf{W}' + D_1(\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n})\boldsymbol{pp}^T \right. \\ \left. -D_1\mathbf{W}'\boldsymbol{nn}^T\boldsymbol{pp}^T - D_1\boldsymbol{pp}^T\boldsymbol{nn}^T\mathbf{W}'\right] \\ -\frac{1}{s}\left[\mathbf{W}'\boldsymbol{nn}^T\mathbf{W}' + (\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n})\mathbf{W}' + \frac{D_1}{D}\boldsymbol{pp}^T\right] \end{array}}{[D_2 + D_1\boldsymbol{n}^T\boldsymbol{pp}^T\boldsymbol{n}] - \frac{1}{s}[\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n}] + \frac{1}{s^2D}} \tag{4.7}$$

The matrix $\mathbf{W}'(s)$ was formed by removal of a real value from one of the diagonal elements of matrix $\mathbf{W}(s)$ (section 3.2.8.1). And by recalling the fact that the matrix $\mathbf{W}(s)$ is in case 7 of Brune's algorithm, which has no pole or zero on imaginary axis, it can easily be deduced that

$$\lim_{s\to\infty} \mathbf{W}'(s) = \mathbf{A} \tag{4.8}$$

where $\mathbf{A}$ is a real matrix.

From Eqs. 4.8 and 4.7 one can easily deduce that there exist a pole at ∞ in $\mathbf{W}'''(s)$ with residue matrix

$$\text{Res}_{s\to\infty}\mathbf{W}'''(s) = \frac{-D_1D_2}{D_2 + D_1\boldsymbol{n}^T\boldsymbol{pp}^T\boldsymbol{n}}\boldsymbol{pp}^T \tag{4.9}$$

$$= \frac{-D_1D_2}{D_2 + D_1(\boldsymbol{p}.\boldsymbol{n})^2}\boldsymbol{pp}^T \tag{4.10}$$

This pole is extracted in Step 4 of case 7 (section 3.2.8.4). Comparing them with Eqs. 3.34a and 3.34c, values of L_3 and C_3 are

$$L_3 = \frac{-L_1L_2}{L_2 + L_1(\boldsymbol{p}.\boldsymbol{n})^2} \qquad \text{(For Type-I)} \tag{4.11}$$

$$C_3 = \frac{-C_1C_2}{C_2 + C_1(\boldsymbol{p}.\boldsymbol{n})^2} \qquad \text{(For Type-III)} \tag{4.12}$$

4.1.2 Brune's Extracted Circuits: Type-II and Type-IV

For extracted Brune circuits of Type-II (for impedance) and of Type-IV (for admittance), step 2 *adds* a pole at $s = 0$:

$$\mathbf{W}'' = \mathbf{W}' + \frac{\boldsymbol{p}\boldsymbol{p}^T}{sD_1}. \qquad \text{(From Eq. 3.30)} \tag{4.13}$$

Step 3 removes a zero pair formed on the imaginary axis of complex frequency plane:

$$\mathbf{W}'''(s) = \left[\left[\mathbf{W}''(s)\right]^{-1} - \left(\frac{\boldsymbol{n}\boldsymbol{n}^T}{sD_2 + \frac{1}{sD}}\right)\right]^{-1} \qquad \text{(From Eq. 3.32)} \tag{4.14}$$

The formation of a pole at $s = 0$ in Step 3 of case 7 of Brune's algorithm (section 3.2.8.4) for Type-II and Type-IV Brune's circuits is derived in the following. The corresponding residue matrix would also be derived. Again for the sake of brevity symbol $\mathbf{W}$ would be used instead of $\mathbf{W}(s)$ which shows its dependency on s. It is to be noted that for the consistency in equations, the symbol D_1 is used as before and the corresponding circuit values C_1 and L_1 in actual circuit realizations would still have value of $-D_1$. Using Sherman-Morrison formula of Eq. 4.1 on Eq.4.14:

$$\mathbf{W}''' = \mathbf{W}'' + \frac{\mathbf{W}''\left(\frac{\boldsymbol{n}\boldsymbol{n}^T}{sD + \frac{1}{sD_2}}\right)\mathbf{W}''}{1 - \left(\frac{\boldsymbol{n}^T\mathbf{W}''\boldsymbol{n}}{sD + \frac{1}{sD_2}}\right)} \tag{4.15}$$

Using Eq. 4.13 to replace $\mathbf{W}''$ and rearranging

$$\mathbf{W}''' = \left[\mathbf{W}' + \frac{\boldsymbol{p}\boldsymbol{p}^T}{sD_1}\right] + \frac{\left[\mathbf{W}' + \frac{\boldsymbol{p}\boldsymbol{p}^T}{sD_1}\right]\boldsymbol{n}\boldsymbol{n}^T\left[\mathbf{W}' + \frac{\boldsymbol{p}\boldsymbol{p}^T}{sD_1}\right]}{sD + \frac{1}{sD_2} - \boldsymbol{n}^T\left[\mathbf{W}' + \frac{\boldsymbol{p}\boldsymbol{p}^T}{sD_1}\right]\boldsymbol{n}} \tag{4.16}$$

$$= \frac{\begin{array}{c} sD\mathbf{W}' + \frac{1}{sD_2}\mathbf{W}' - (\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n})\mathbf{W}' - \frac{1}{sD_1}(\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n})\mathbf{W}' \\ + \frac{D}{D_1}\boldsymbol{p}\boldsymbol{p}^T + \frac{\boldsymbol{p}\boldsymbol{p}^T}{s^2D_1D_2} - \frac{1}{sD_1}(\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n})\boldsymbol{p}\boldsymbol{p}^T - \frac{(\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n})\boldsymbol{p}\boldsymbol{p}^T}{s^2D_1^2} \\ + \mathbf{W}'\boldsymbol{n}\boldsymbol{n}^T\mathbf{W}' + \frac{1}{sD_1}\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n}\boldsymbol{n}^T\mathbf{W}' \\ + \frac{1}{sD_1}\mathbf{W}'\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T + \frac{\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T}{s^2D_1^2} \end{array}}{sD + \frac{1}{sD_2} - \boldsymbol{n}^T\mathbf{W}'\boldsymbol{n} - \frac{1}{sD_1}\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n}} \tag{4.17}$$

The eighth and twelfth terms in the numerator would cancel for the same reason that $\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n} = \boldsymbol{p}^T\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{p} = (\boldsymbol{p}.\boldsymbol{n})^2$. Further multiplying both numerator and denominator by s and rearranging:

$$\mathbf{W}''' = \frac{\begin{array}{r}\dfrac{\boldsymbol{p}\boldsymbol{p}^T}{sD_1D_2} + \dfrac{1}{D_2}\mathbf{W}' - \dfrac{1}{D_1}(\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n})\mathbf{W}' - \dfrac{1}{D_1}(\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n})\boldsymbol{p}\boldsymbol{p}^T \\ +\dfrac{1}{D_1}\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n}\boldsymbol{n}^T\mathbf{W}' + \dfrac{1}{D_1}\mathbf{W}'\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T \\ -s(\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n})\mathbf{W}' + s\mathbf{W}'\boldsymbol{n}\boldsymbol{n}^T\mathbf{W}' + s\dfrac{D}{D_1}\boldsymbol{p}\boldsymbol{p}^T \\ + s^2D\mathbf{W}'\end{array}}{\left[\dfrac{1}{D_2} - \dfrac{1}{D_1}\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n}\right] - s\boldsymbol{n}^T\mathbf{W}'\boldsymbol{n} + s^2D} \tag{4.18}$$

Following the argument given in the above section 4.1.1 the matrix $\mathbf{W}'(s)$ would be a constant at $s = 0$ and that there is a pole at $s = 0$ in $\mathbf{W}'''(s)$ having residue matrix

$$\begin{aligned}\mathrm{Res}_{s\to 0}\mathbf{W}'''(s) &= \lim_{s\to 0}(s\mathbf{W}'''(s)) \\ &= \frac{\boldsymbol{p}\boldsymbol{p}^T}{D_1 - D_2\boldsymbol{n}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{n}} \qquad (4.19)\\ &= \frac{\boldsymbol{p}\boldsymbol{p}^T}{D_1 - D_2(\boldsymbol{p}.\boldsymbol{n})^2} \qquad (4.20)\end{aligned}$$

This pole is extracted in Step 4 of case 7 (section 3.2.8.4). Comparing the residue calculated in Eq. 4.20 with residues of poles extracted in Eqs. 3.34b and 3.34d, values of C_3 and L_3 for Type-II and Type-IV, respectively, are

$$C_3 = -C_1 - C_2(\boldsymbol{p}.\boldsymbol{n})^2 \qquad \text{(For Type-II)} \tag{4.21}$$

$$L_3 = -L_1 - L_2(\boldsymbol{p}.\boldsymbol{n})^2 \qquad \text{(For Type-IV)} \tag{4.22}$$

4.2 Parameters Relationships: Brune's Extracted and Equivalent Circuits

Brune's equivalent circuits are required to avoid the negative elements appearing in Brune's extracted circuits (section 3.2.9). The equivalence between each extracted type (step 2 to step 4) of circuit with corresponding equivalent circuit would be shown in this section. Because of the similarities, only equivalence between Type-I circuits is shown step by step and a brief outline is given for other types.

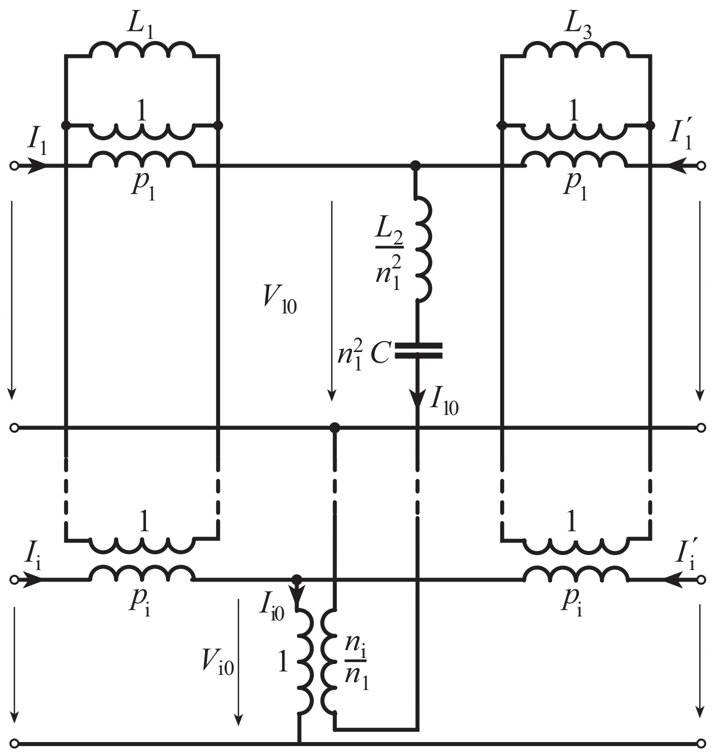


Figure 4.1: Extracted Brune's circuit of Type-I (step 2 to step 4)

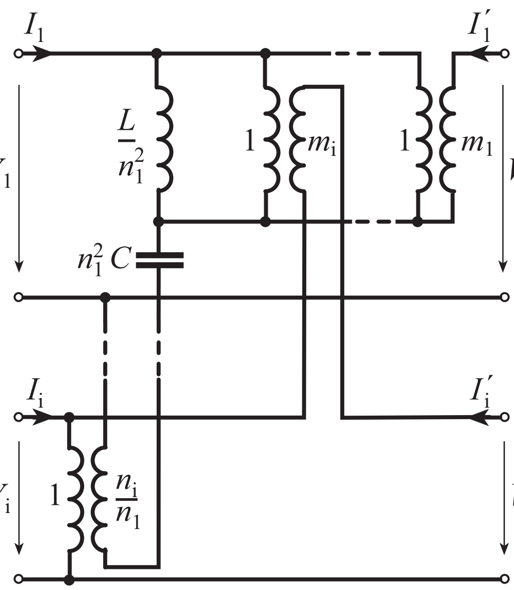


Figure 4.2: Equivalent Brune's circuit of Type-I

4.2.1 Brune Circuit Type-I

The extracted Brune's circuit of Type-I, shown in Fig. 4.1, is equivalent to the circuit shown in Fig. 4.2 if the relation in Eq. 4.11 is satisfied by parameters in extracted circuit. The relations between the parameters of these two circuits are derived here. Similar derivation is given in [27], taking $n_1 = -1$ and $p_1 = 1$. This derivation will generalize it by not specifying any value.

The total number of the ports of original circuit is N. As this circuit lies in the middle of the total synthesized circuit and we only want the equivalent of this circuit, the circuit is of $2N$ ports. In these figures the first port and any other ith port is shown. The values L_1, L_2, L_3, C, p's and n's in Fig. 4.1 are the parameters of the circuit available from the extraction procedure (section 3.2.8), while the parameters L and m's of Fig. 4.2 are dependent on extracted parameters. The value of the capacitor in both circuits would be same as will be obvious from the equations. In the derivation the values $I_1, I_1^{'}, I_i^{'}$ and V_i will be treated as independent variables and $V_1, V_1^{'}, V_i^{'}$ and I_i as dependent variables.

From Fig. 4.2

$$V_1 = s\frac{L}{n_1^2}\left(I_1 + m_1 I_1^{'} + \sum_{k=2}^{N} m_k I_k^{'}\right) + \frac{1}{sn_1^2 C}\left(I_1 + I_1^{'}\right) - \sum_{k=2}^{N}\frac{n_k}{n_1}V_k \tag{4.23a}$$

$$V_1^{'} = s\frac{L}{n_1^2}\left(I_1 + m_1 I_1^{'} + \sum_{k=2}^{N} m_k I_k^{'}\right) m_1 + \frac{1}{sn_1^2 C}\left(I_1 + I_1^{'}\right) - \sum_{k=2}^{N}\frac{n_k}{n_1}V_k \tag{4.23b}$$

$$V_i^{'} = s\frac{L}{n_1^2}\left(I_1 + m_1 I_1^{'} + \sum_{k=2}^{N} m_k I_k^{'}\right) m_i + V_i \tag{4.23c}$$

$$I_i = \frac{n_i}{n_1}\left(I_1 + I_1^{'}\right) - I_i^{'} \tag{4.23d}$$

From Fig. 4.1:

$$V_1 = sL_1\left(p_1 I_1 + \sum_{k=2}^{N} p_k I_k\right) p_1 + V_{10} \tag{4.24a}$$

$$V_1^{'} = sL_3\left(p_1 I_1^{'} + \sum_{k=2}^{N} p_k I_k^{'}\right) p_1 + V_{10} \tag{4.24b}$$

$$V_i^{'} = sL_3\left(p_1 I_1^{'} + \sum_{k=2}^{N} p_k I_k^{'}\right) p_i + V_{i0} \tag{4.24c}$$

$$V_i = sL_1\left(p_1 I_1 + \sum_{k=2}^{N} p_k I_k\right) p_i + V_{i0} \tag{4.24d}$$

$$I_i = I_{i0} - I_i^{'} = \frac{n_i}{n_1} I_{10} - I_i^{'} = \frac{n_i}{n_1}\left(I_1 + I_1^{'}\right) - I_i^{'} \tag{4.24e}$$

$$V_{10} = s\frac{L_2}{n_1^2}\left(I_1 + I_1^{'}\right) + \frac{1}{sn_1^2C}\left(I_1 + I_1^{'}\right) - \sum_{k=2}^{N}\frac{n_k}{n_1}V_{k0} \tag{4.24f}$$

The current I_i relations in two sets of equations are already same (compare Eqs. 4.24e and 4.23d). Putting the value of V_{10} from Eq. 4.24f into the Eqs. 4.24a and 4.24b and rearranging:

$$V_1 = s\left(p_1^2L_1 + \frac{L_2}{n_1^2}\right)I_1 + s\frac{L_2}{n_1^2}I_1^{'} + sp_1L_1\sum_{k=2}^{N}p_kI_k + \frac{(I_1 + I_1^{'})}{sn_1^2C} - \sum_{k=2}^{N}\frac{n_k}{n_1}V_{k0} \tag{4.25a}$$

$$V_1^{'} = s\frac{L_2}{n_1^2}I_1 + s\left(\frac{L_2}{n_1^2} + p_1^2L_3\right)I_1^{'} + sp_1L_3\sum_{k=2}^{N}p_kI_k^{'} + \frac{(I_1 + I_1^{'})}{sn_1^2C} - \sum_{k=2}^{N}\frac{n_k}{n_1}V_{k0} \tag{4.25b}$$

$$V_i^{'} = sp_1L_3p_iI_1^{'} + sL_3p_i\sum_{k=2}^{N}p_kI_k^{'} + V_{i0} \tag{4.25c}$$

$$V_{i0} = -sp_1L_1p_iI_1 - sL_1p_i\sum_{k=2}^{N}p_kI_k + V_i \tag{4.25d}$$

Now putting the value of I_i from Eq. 4.24e into Eqs. 4.25a and 4.25d we get:

$$V_1 = s\left(p_1^2L_1 + \frac{L_2}{n_1^2}\right)I_1 + s\frac{L_2}{n_1^2}I_1^{'} + sp_1L_1\sum_{k=2}^{N}p_k\left(\frac{n_k}{n_1}I_1 + \frac{n_k}{n_1}I_1^{'} - I_k^{'}\right)$$
$$+ \frac{1}{sn_1^2C}\left(I_1 + I_1^{'}\right) - \sum_{k=2}^{N}\frac{n_k}{n_1}V_{k0} \tag{4.26a}$$

$V_1^{'}$ (Same as above) (4.26b)

$V_i^{'}$ (Same as above) (4.26c)

$$V_{i0} = -sp_1L_1p_iI_1 - sL_1p_i\sum_{k=2}^{N}p_k\left(\frac{n_k}{n_1}I_1 + \frac{n_k}{n_1}I_1^{'} - I_k^{'}\right) + V_i \tag{4.26d}$$

Rearranging the terms in equations

$$V_1 = s\left(p_1^2L_1 + \frac{p_1}{n_1}L_1\sum_{k=2}^{N}p_kn_k + \frac{L_2}{n_1^2}\right)I_1 + s\left(\frac{p_1}{n_1}L_1\sum_{k=2}^{N}p_kn_k + \frac{L_2}{n_1^2}\right)I_1^{'}$$
$$- sp_1L_1\sum_{k=2}^{N}p_kI_k^{'} + \frac{1}{sn_1^2C}\left(I_1 + I_1^{'}\right) - \sum_{k=2}^{N}\frac{n_k}{n_1}V_{k0} \tag{4.27a}$$

$$V_{i0} = s\left(-p_1L_1p_i - \frac{L_1p_i}{n_1}\sum_{k=2}^{N}p_kn_k\right)I_1 - \frac{sL_1p_i}{n_1}\sum_{k=2}^{N}p_kn_kI_1^{'} + sL_1p_i\sum_{k=2}^{N}p_kI_k^{'}$$
$$+ V_i \tag{4.27b}$$

In Eq. 4.27b change the subscripts k to l, then i to k and then sum $\frac{n_k}{n_1}V_{k0}$ over k from 2 to N. Subtitute $\sum_{k=2}^{N} p_k n_k$ by A.

$$V_1 = s\left(p_1^2 L_1 + \frac{p_1}{n_1}L_1 A + \frac{L_2}{n_1^2}\right) I_1 + s\left(\frac{p_1}{n_1}L_1 A + \frac{L_2}{n_1^2}\right) I_1^{'} - sp_1 L_1 \sum_{k=2}^{N} p_k I_k^{'} + \frac{1}{sn_1^2 C}\left(I_1 + I_1^{'}\right) - \sum_{k=2}^{N} \frac{n_k}{n_1} V_{k0} \tag{4.28a}$$

$$\sum_{k=2}^{N} \frac{n_k}{n_1} V_{k0} = s\left(-\frac{p_1}{n_1}L_1 A - \frac{L_1}{n_1^2}A^2\right) I_1 - s\frac{L_1}{n_1^2}A^2 I_1^{'} + s\frac{L_1}{n_1}A\sum_{l=2}^{N} p_l I_l^{'} + \sum_{k=2}^{N} \frac{n_k}{n_1} V_k \tag{4.28b}$$

Subtitute $\sum_{k=2}^{N} n_k V_{k0}$ in Eqs. 4.28a and 4.25b by Eq. 4.28b and V_{i0} in Eq. 4.25c by Eq. 4.27b

$$V_1 = s\left(\left(\frac{A^2}{n_1^2} + 2\frac{A}{n_1}p_1 + p_1^2\right) L_1 + \frac{L_2}{n_1^2}\right) I_1 + s\left(\left(\frac{A^2}{n_1^2} + \frac{A}{n_1}p_1\right) L_1 + \frac{L_2}{n_1^2}\right) I_1^{'} - s\left(\frac{A}{n_1} + p_1\right) L_1 \sum_{k=2}^{N} p_k I_k^{'} + \frac{1}{sn_1^2 C}\left(I_1 + I_1^{'}\right) - \sum_{k=2}^{N} \frac{n_k}{n_1} V_k \tag{4.29}$$

$$V_1^{'} = s\left(\left(\frac{A^2}{n_1^2} + \frac{A}{n_1}p_1\right) L_1 + \frac{L_2}{n_1^2}\right) I_1 + s\left(\frac{A^2}{n_1^2}L_1 + \frac{L_2}{n_1^2} + p_1^2 L_3\right) I_1^{'} + s\left(-\frac{A}{n_1}L_1 + p_1 L_3\right) \sum_{k=2}^{N} p_k I_k^{'} + \frac{1}{sn_1^2 C}\left(I_1 + I_1^{'}\right) - \sum_{k=2}^{N} \frac{n_k}{n_1} V_k \tag{4.30}$$

$$V_i^{'} = s\left(-\frac{A}{n_1} - p_1\right) L_1 p_i I_1 + s\left(-\frac{A}{n_1}L_1 + p_1 L_3\right) p_i I_1^{'} + s\left(L_1 + L_3\right) p_i \sum_{k=2}^{N} p_k I_k^{'} + V_i \tag{4.31}$$

Comparing Eq. 4.29 with Eq. 4.23a, Eq. 4.30 with Eq. 4.23b and Eq. 4.31 with Eq. 4.23c, we get

$$L = \left(A + p_1 n_1\right)^2 L_1 + L_2 \quad = (\boldsymbol{p}.\boldsymbol{n})^2 L_1 + L_2 = -\frac{L_1 L_2}{L_3} \tag{4.32}$$

$$m_1 = \frac{\left(A^2 + A p_1 n_1\right) L_1 + L_2}{L} = \frac{A\left(\boldsymbol{p}.\boldsymbol{n}\right) L_1 + L_2}{\left(\boldsymbol{p}.\boldsymbol{n}\right)^2 L_1 + L_2} \tag{4.33}$$

$$m_k = -\frac{\left(A + p_1 n_1\right) n_1 L_1 p_k}{L} \quad = -\frac{\left(\boldsymbol{p}.\boldsymbol{n}\right) n_1 L_1 p_k}{\left(\boldsymbol{p}.\boldsymbol{n}\right)^2 L_1 + L_2} \qquad \text{for } k = 2, 3, \cdots, N \tag{4.34}$$

and it is valid if $L_3 = -\frac{L_1 L_2}{L} = -\frac{L_1 L_2}{(\boldsymbol{p}.\boldsymbol{n})^2 L_1 + L_2}$ where $A = \sum_{k=2}^{N} p_k n_k$, $\boldsymbol{p} = [p_1, p_2, \cdots, p_N]^T$ and $\boldsymbol{n} = [n_1, n_2, \cdots, n_N]^T$. This condition is already fulfilled by the Brune's extracted circuit of Type-I (Eq. 4.11)

4.2.2 Brune Circuits Type-II

The Brune circuit of type-II shown in Fig. 4.2.2 is equivalent to the circuit shown in Fig. 4.3. The relations between the parameters of these two circuits are derived here. As with previous circuit, the total number of the ports of original circuit is N, and in these figures the first port and any other ith port is shown. The values C_1, C_2, C_3, L, p's and n's in Fig. 4.2.2 are the parameters of the circuit available from the extraction procedure (section 3.2.8), while the parameters C and m's of Fig. 4.3 are dependent on the extracted parameters. T he value of the inductance in both circuits would be same as will be obvious from the equations. In the derivation the values I_1, I_1', I_i' and V_i will be treated as independent variables and V_1, V_1', V_i' and I_i as dependent variables.

The derivation equations for Type-II circuit are very similar to the equations for Type-I circuit in previous section. One can obtain all the equations by replacing sL_i with $1/sC_i$ and sC with $1/sL$. This simple change is due to the fact that the difference in the two types are only the change of the circuit elements with topology being the same.

The resulting relations connecting the two circuits are

$$C = \frac{C_1C_2}{(A+p_1n_1)^2 C_2 + C_1} = \frac{C_1C_2}{(\boldsymbol{p}.\boldsymbol{n})^2 C_2 + C_1} \tag{4.35}$$

$$m_1 = C\left(\frac{A(A+p_1n_1)C_2 + C_1}{C_1C_2}\right) = \frac{A(\boldsymbol{p}.\boldsymbol{n})C_2 + C_1}{(\boldsymbol{p}.\boldsymbol{n})^2 C_2 + C_1} \tag{4.36}$$

$$m_k = -\frac{(A+p_1n_1)Cn_1p_k}{C_1} = -\frac{(\boldsymbol{p}.\boldsymbol{n})C_2n_1p_k}{(\boldsymbol{p}.\boldsymbol{n})^2 C_2 + C_1} \quad \text{for } k = 2, 3, \cdots, N \tag{4.37}$$

with validity if

$$C_3 = -\frac{C_1C_2}{C} = -\left((\boldsymbol{p}.\boldsymbol{n})^2 C_2 + C_1\right) \tag{4.38}$$

where $A = \sum_{k=2}^{N} p_k n_k$, $\boldsymbol{p} = [p_1, p_2, \cdots, p_N]^T$ and $\boldsymbol{n} = [n_1, n_2, \cdots, n_N]^T$.

This condition is already fulfilled by the Brune's extracted circuit of Type-II (section 4.1.2, and Eq. 4.21)

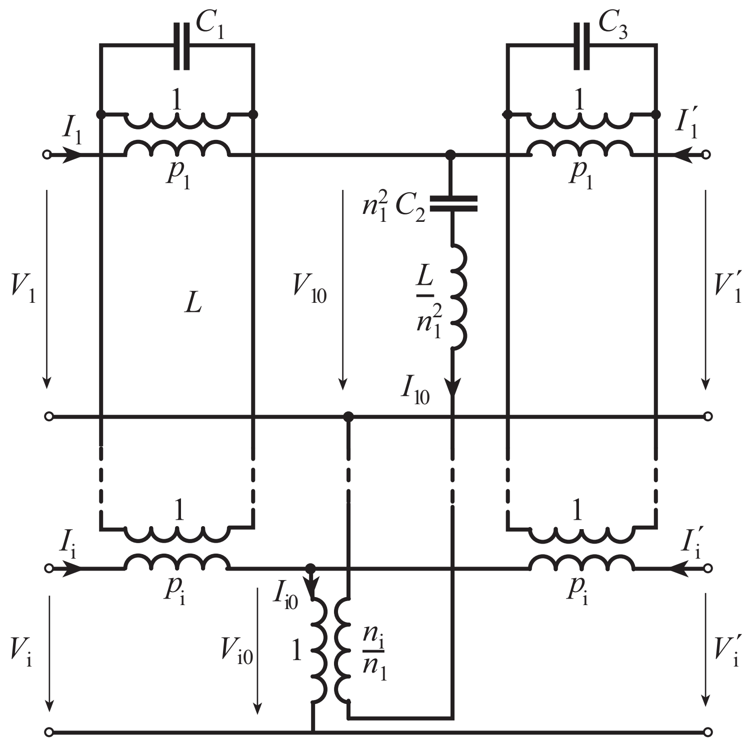


Figure 4.3: Extracted Brune's circuit of Type-II

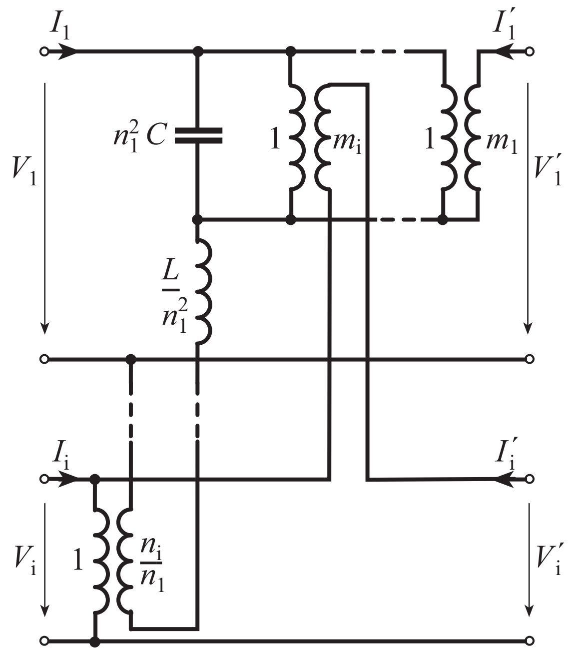


Figure 4.4: Equivalent Brune's circuit of Type-II

4.2.3 Brune Circuit Type-III

The extracted Brune circuit of type-III is shown in Fig. 4.5 and its equivalent circuit in Fig. 4.6. The relations between the parameters of these two circuits are derived here. As with previous circuit, the total number of the ports of original circuit is N, and in these figures the first port and any other ith port is shown. The values C_1, C_2, C_3, L, p's and n's in Fig. 4.5 are the parameters of the circuit available from the extraction procedure (section 3.2.8), while the parameters C and m's of Fig. 4.6 are dependent on the extracted parameters. The value of the inductance in both circuits would be same as will be obvious from the equations.

Due to duality, the derivation is same as in section 4.2.1. By replacing every voltage V_i with corresponding current I_i and vice versa, and replacing every capacitance C_i with corresponding inductance L_i and vice versa, one can easily find the resulting relations. In the derivation the values V_1, V_1', V_i' and I_i will be treated as independent variables and I_1, I_1', I_i' and V_i as dependent variables. It must be noted that the above replacement of voltages, currents, capacitance and inductance in equations of section 4.2.1 is valid if the sign conventions shown in Figs. 4.5 and 4.6 are followed. The resulting relations connecting the two circuits are

$$C = (A + p_1 n_1)^2 C_1 + C_2 \quad = (\boldsymbol{p}.\boldsymbol{n})^2 C_1 + C_2 \tag{4.39}$$

$$m_1 = \frac{(A^2 + A p_1 n_1) C_1 + C_2}{C} = \frac{A (\boldsymbol{p}.\boldsymbol{n}) C_1 + C_2}{(\boldsymbol{p}.\boldsymbol{n})^2 C_1 + C_2} \tag{4.40}$$

$$m_k = -\frac{(\boldsymbol{p}.\boldsymbol{n}) n_1 C_1 p_k}{C} \quad = -\frac{(\boldsymbol{p}.\boldsymbol{n}) n_1 C_1 p_k}{(\boldsymbol{p}.\boldsymbol{n})^2 C_1 + C_2} \tag{4.41}$$

$$\text{for } k = 2, 3, \cdots, N \tag{4.42}$$

and it is valid if

$$C_3 = -\frac{C_1 C_2}{C} = -\frac{C_1 C_2}{(\boldsymbol{p}.\boldsymbol{n})^2 C_1 + C_2} \tag{4.43}$$

where $A = \sum_{k=2}^{N} p_k n_k$, $\boldsymbol{p} = [p_1, p_2, \cdots, p_N]^T$ and $\boldsymbol{n} = [n_1, n_2, \cdots, n_N]^T$.

This condition is already fulfilled by the Brune's extracted circuit of Type-III (section 4.1.1, and Eq. 4.12)

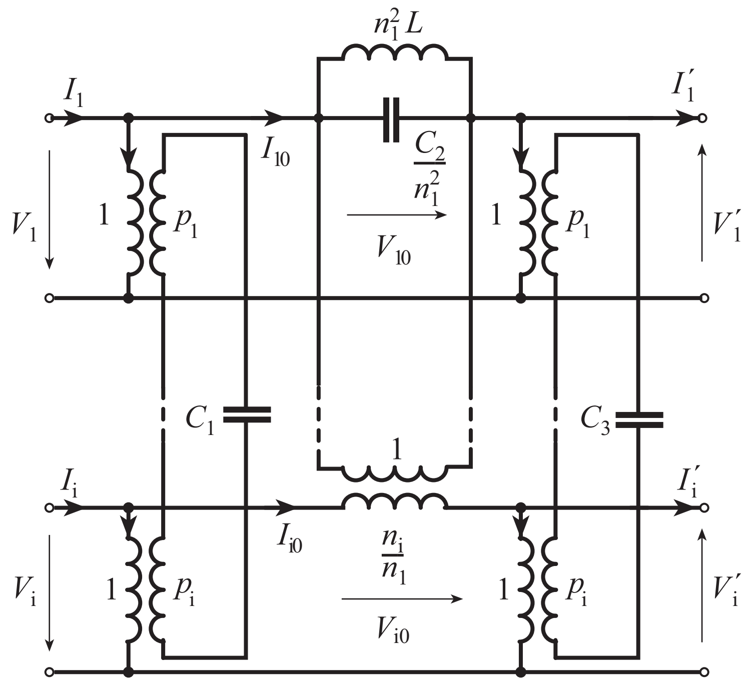


Figure 4.5: Extracted Brune's circuit of Type-III

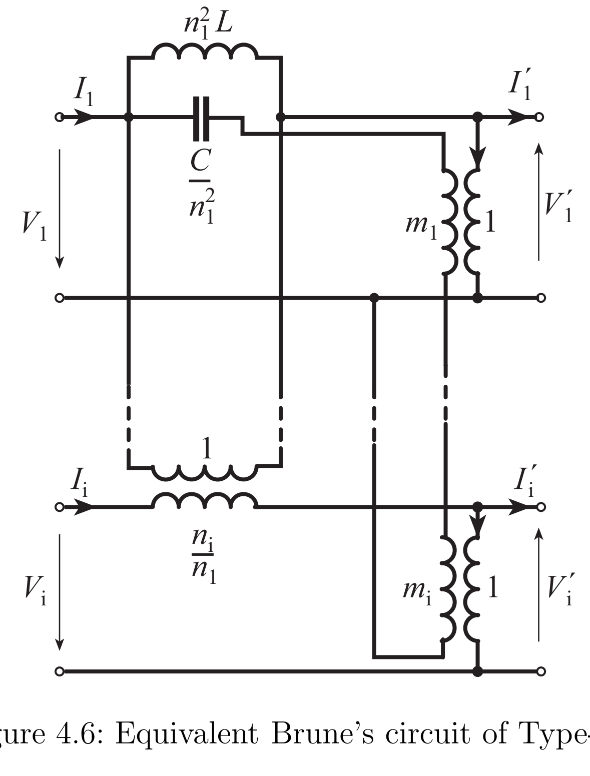


Figure 4.6: Equivalent Brune's circuit of Type-III

4.2.4 Brune Circuits Type-IV

Type-IV circuits are similar to Type-III circuits topologically but they are dual to Type-II circuits. The extracted and equivalent circuits are shown in Figs. 4.2.4 and 4.7, respectively.

These circuits are for given admittance matrices. Unlike Type-III circuit there are three inductance elements. The inductance L_1 is negative as derived in section 3.2.8. The equivalent circuit would replace L_1, L_2 and L_3 with one positive inductance L.

The derivation is very similar to the procedure given in section 4.2.1, with following changes:

1. Replace every voltage V_i with corresponding current I_i.
2. Replace every current I_i with corresponding voltage V_i.
3. Replace sL_i with $1/sL_i$.
4. Replace sC with $1/sC$.

The convention used would be as shown in Figs. 4.2.4 and 4.7, otherwise the above replacement is invalid.

The resulting relations are:

$$L = \frac{L_1 L_2}{(A + p_1 n_1)^2 L_2 + L_1} = \frac{L_1 L_2}{(\boldsymbol{p}.\boldsymbol{n})^2 L_2 + L_1} \tag{4.44}$$

$$m_1 = L\left(\frac{(A^2 + Ap_1 n_1) L_2 + L_1}{L_1 L_2}\right) = \frac{A(\boldsymbol{p}.\boldsymbol{n}) L_2 + L_1}{(\boldsymbol{p}.\boldsymbol{n})^2 L_2 + L_1} \tag{4.45}$$

$$m_k = -\frac{(A + p_1 n_1) L n_1 p_k}{L_1} = -\frac{(\boldsymbol{p}.\boldsymbol{n}) L_2 n_1 p_k}{(\boldsymbol{p}.\boldsymbol{n})^2 L_2 + L_1} \quad \text{for } k = 2, 3, \cdots, N \tag{4.46}$$

with validity if

$$L_3 = -\frac{L_1 L_2}{L} = -\left((\boldsymbol{p}.\boldsymbol{n})^2 L_2 + L_1\right) \tag{4.47}$$

where $A = \sum_{k=2}^{N} p_k n_k$, $\boldsymbol{p} = [p_1, p_2, \cdots, p_N]^T$ and $\boldsymbol{n} = [n_1, n_2, \cdots, n_N]^T$.

This condition is already fulfilled by the Brune's extracted circuit of Type-IV (section 4.1.2, and Eq. 4.22).

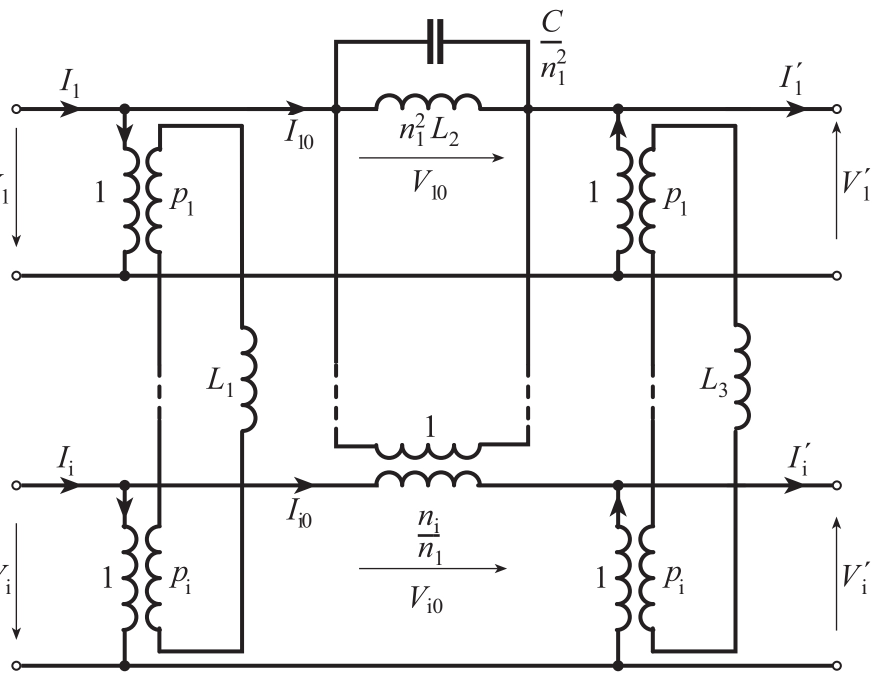


Figure 4.7: Extracted Brune's circuit of Type-IV

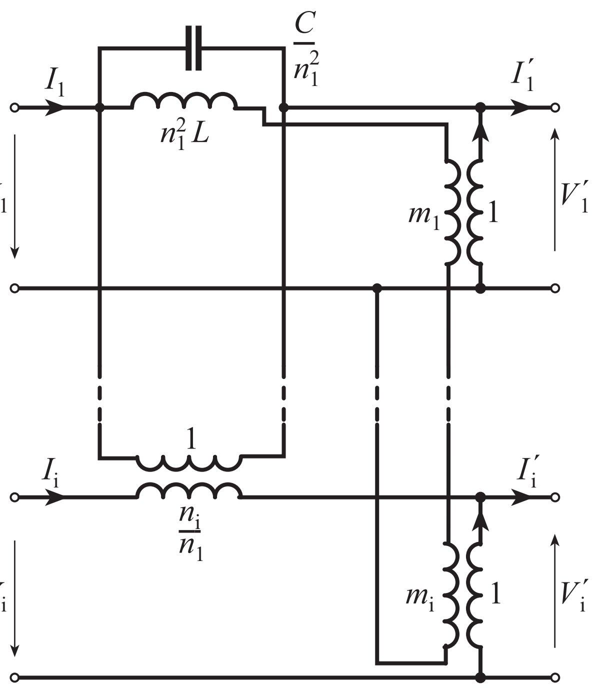


Figure 4.8: Equivalent Brune's circuit of Type-IV

4.3 Positivity of Elements

The values of the circuit elements in Brune's synthesis procedures are always positive. In cases 1 to 6, the elements are extracted with positive values. But in case 7 one of the extracted values is negative which can be avoided by using equivalent circuits.

4.3.1 Extracted Parameters

4.3.1.1 Cases 1 to 6

Given a PRSM in which there is a pole or zero on imaginary axis of complex frequency, the subtraction of this pole or zero would again result in PRSM [28, §XVI]. The pole (and zero which is a pole in inverse PRSM), would have residue matrix $\mathbf{K}^{\omega_p}$ which is positive definite 2.2.2. Thus every extracted element would have positive value. The transformer turns ratios can however be negative.

4.3.1.2 Case 7: Step 1

The element extracted in step 1 is either a resistance or a conductance. During the extraction process it is made sure that it remains positive. As the given matrix is positive real symmetric matrix, its real part on the imaginary axis is positive definite matrix for all frequencies. The positive real character is maintained while extracting the real value (synthesized as a resistance or a conductance).

4.3.1.3 Case 7: Step 2

The elements extracted in this step are made negative to preserve the positive real character of the remaining matrix. The subtraction of negative elements is actually addition of a pole with positive residue. The addition of two PRSM is again a PRSM [28, p. 544]. In Brune's circuits Type-I and Type-III, the elements are themselves negative and subtraction of these elements is actually addition of a pole with positive residue at ∞. However, in Type-II and Type-IV circuits the elements are made negative to make it addition of a pole at $s = 0$.

The negativity of the elements, formed in this step is dealt in the equivalent circuit.

4.3.1.4 Case 7: Step 3

The resultant matrix obtained from step 2, would be PRSM and would have zero pair on imaginary axis $s = j\omega$. The removal of this zero pair would again result in PRSM and a circuit (resulting from zero-pair) with positive values.

4.3.1.5 Case 7: Step 4

The resultant matrix obtained from step 3 is PRSM with a pole at ∞ or 0. Again, the poles is with positive definite residue matrix; thus the elements would be positive. After removal of the pole, the matrix is still PRSM [28, §XVI].

4.3.2 Equivalent Parameters

The equivalent circuits replace three circuit elements with one element. In Type-I and Type-IV, three inductance L_1, L_2 and L_3 are replaced by one inductance L, while in Type-II and Type-III circuits three capacitance C_1, C_2 and C_3 are replaced by one capacitance C. In all cases first element (L_1 or C_1) is negative and other two elements (L_2, L_3 or C_2, C_3) are positive, as discussed above. From Eqs. 4.32, 4.38, 4.43 and 4.47 it can be deduced that the equivalent elements (L or C) are positive.

5 Examples

The working of an algorithm is best understood through an example in which the algorithm is discussed step by step. The need of such demonstrative example becomes more important for an algorithm which is as complicated as Brune's algorithm. This need is first addressed in this chapter. An example demonstrating step by step working of Brune's algorithm is discussed.

Application of Brune's algorithm in modeling variety of microwave circuits are also presented. The purpose is to compare the accuracy of model response. As a raw tabular data from measurement or simulation is used, steps of data pre-processing and system identification are also involved. The second section is dedicated for discussion on the general procedure involved in developing these examples.

The step-by-step working of algorithm would not be shown in application examples. This is because the equivalent network models have very high orders and thus very complicated. A number of other examples can be found in [50, 52, 55–66]

5.1 Demonstration Example

This example is presented to elaborate the working of the Brune's algorithm. A positive real symmetric admittance matrix given in Eq. 5.1 is an input to the algorithm and the resulting output network is shown in Fig. 5.1.

Through inspection one can identify that the network contains only four basic circuit elements i.e. inductance, capacitance, conductance and ideal transformer. The success of the algorithm lies in the fact that except for the transformer turns ratios (which can be negative), all other elements have positive values. The number of reactive elements (inductance and capacitance) is 12 which is also the order of the equivalent circuit (section 2.3).

The exact coefficients in the terms of the polynomials are given. This format is adopted to avoid any numerical errors in this example as the purpose of this example is only to elaborate the working of the algorithm. The symbols used have extra index in the subscript which indicates the number of iteration of the algorithm.

An effort has been made to keep the network as small as possible yet containing all cases of the algorithm. Moreover to introduce un-symmetry in the network, the transformer turn-ratios are slightly different. The order of polynomials in Eq. 5.1 indicates that even for simple networks the use of numerical computational programs is un-avoidable. It is advisable for the reader to use *Mathematica* or MATLAB for solving the example to have deeper understanding of the algorithm.

The input to the algorithm is an admittance matrix given as

$$\mathbf{Y}_1(s) = \frac{1}{D_1}\begin{bmatrix} N_{11,1} & N_{12,1} \\ N_{21,1} & N_{22,1} \end{bmatrix} \tag{5.1}$$

where

$$D_1 = \begin{cases} s(s^2+4)(2396000s^8 + 35460405s^7 + 166883979s^6 \\ \quad +527995538s^5 + 805283399s^4 + 669903011s^3 \\ \quad\quad +307683728s^2 + 88471572s + 12038400) \end{cases} \tag{5.1a}$$

$$N_{11,1} = \begin{cases} 3450240s^{12} + 55317983.2s^{11} + 297238199.76s^{10} + 1217090834.62s^9 \\ \quad +3210391182.78s^8 + 6993939410.68s^7 + 10463825754.14s^6 \\ \quad\quad +10954997468.66s^5 + 8063875103.04s^4 + 4429859554.96s^3 \\ \quad\quad\quad +1620135968s^2 + 396683088s + 48153600 \end{cases} \tag{5.1b}$$

$$N_{12,1} = N_{21,1} = \begin{cases} 2875200s^{12} + 43982486s^{11} + 220929224.8s^{10} + 869845504.1s^9 \\ \quad +1992985209.1s^8 + 3762374473.8s^7 + 4279550537.1s^6 \\ \quad\quad +1202948201.5s^5 - 2648302205.2s^4 - 3204123096.8s^3 \\ \quad\quad\quad -1627284425.6s^2 - 482516174.4s - 62599680 \end{cases} \tag{5.1c}$$

$$N_{22,1} = \begin{cases} 2396000s^{12} + 36560405s^{11} + 193063719s^{10} + 840620052.45s^9 \\ \quad +2222788657.51s^8 + 5249151236.02s^7 + 8116866564.35s^6 \\ \quad\quad +9449883963.47s^5 + 8415735445.56s^4 + 5584236343.04s^3 \\ \quad\quad\quad +2298366097.28s^2 + 625024626.72s + 81379584 \end{cases} \tag{5.1d}$$

Preliminaries

First observation is that all four rational functions have common denominator, which generally may not be the case. In such situations one can make the denominator common by adding poles with zero residues to the rational functions in which they are not present.

Before actually starting the algorithm, the matrix has to be checked for its positive real character. Three conditions given in Section 2.2.2 are to be verified. First condition can be verified by factorizing the denominator D_1. As the numerators $N_{11,1}$, $N_{12,1}$, $N_{22,1}$ are a degree higher than the denominator, there is a pole at infinity. Other marginal poles are located at $s = 0$ and $s = \pm j2$ (as seen in D_1). For the verification of second condition one has to verify that $[N_{11,1}/D_1]|_{s=j\omega} \geq 0$ and $|\mathbf{Y}_1|_{s=j\omega} \geq 0$. The easiest way is to plot the functions in frequency band which is double of the highest frequency pole (in this case $0 \leq \omega < 10$), and verify by inspection. Other methods of verification are given in [39, Ch. 1]. The third condition for positive real character is related to the poles located on $j\omega$-axis. For each pole $s = 0$, $s \to \infty$ and $s = \pm j2$ one can extract the residues from each rational function and form matrix of residues $\mathbf{K}^0$, $\mathbf{K}^\infty$ and $\mathbf{K}^{\pm j2}$. All matrices of residues can seen to be positive definite, thus fulfilling the third condition.

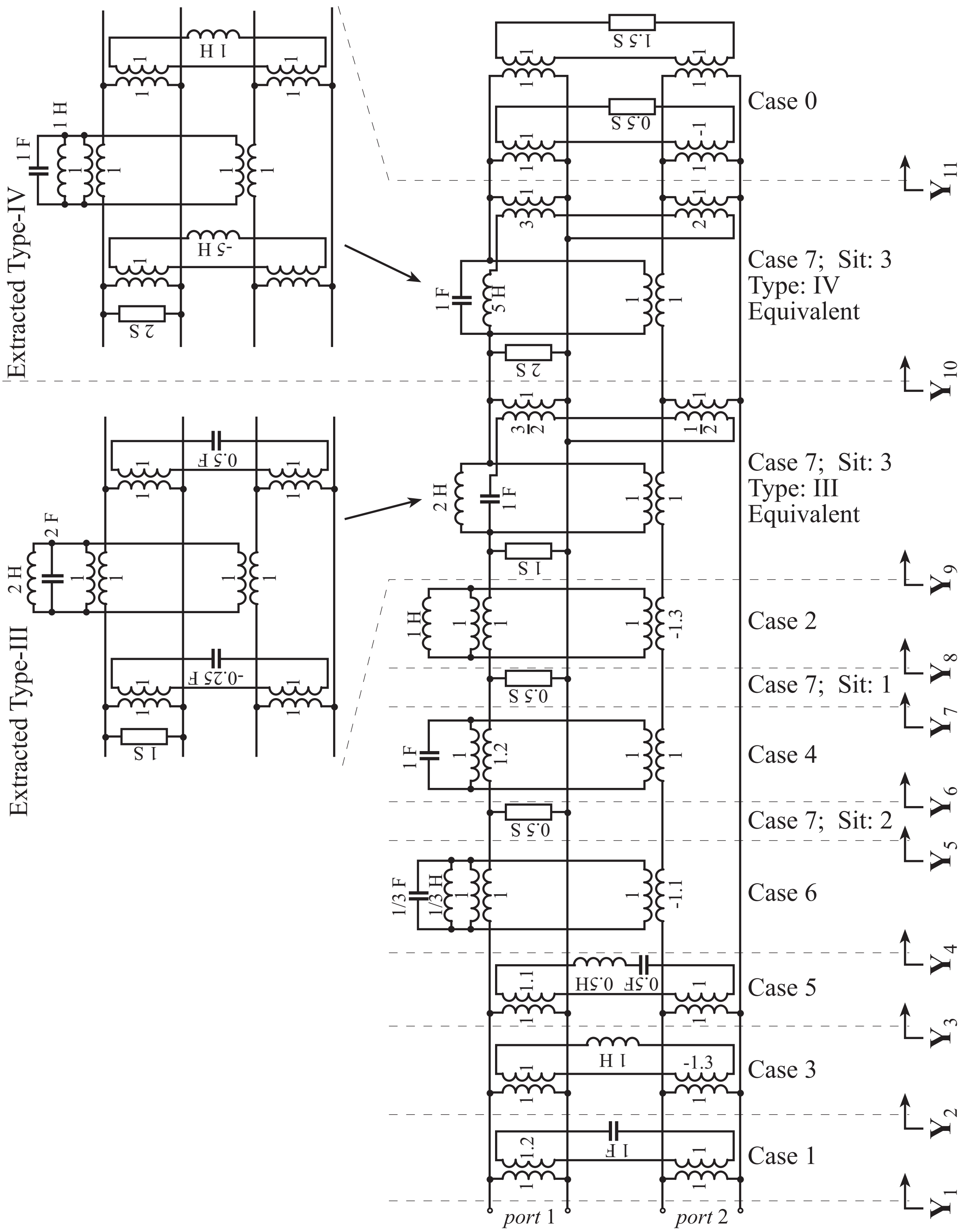


Figure 5.1: Brune's network diagram for admittance matrix given in Eq. 5.1

Iteration: 1

It is advisable to keep the flow-diagram given in Fig. 3.2 under consideration with the discussion of each iteration. At the start of each iteration the stop criterion has to be checked. If this criterion is not met then case 1 has to be checked and if that fails the next case is checked. This process is repeated until either some case is identified successfully or the default case 7 is reached.

Stop criterion (Case 0) occurs if the matrix in iteration is frequency independent real symmetric matrix. As $\mathbf{Y}_1$ has functions of s, stop criterion is not met. The algorithm will move on to checking for case 1. The given matrix lies in case 1 if any element of the matrix has a pole at infinity. A simple test is to see if the numerator polynomial is one degree higher than the denominator polynomial. As it is obvious from Eqs. 5.1a, 5.1b, 5.1c and 5.1d that all four numerator polynomials are one degree higher than the denominator polynomial, all four elements have a pole at infinity. Further cases would not be analyzed and after extraction of this pole the iteration would complete. These poles can be removed by long division (Section 3.2.2). One can write

$$\begin{aligned}\mathbf{Y}_1(s) &= s\begin{bmatrix} 1.44 & 1.2 \\ 1.2 & 1 \end{bmatrix} + \frac{1}{D_2}\begin{bmatrix} N_{11,2} & N_{12,2} \\ N_{21,2} & N_{22,2} \end{bmatrix} \\ &= s\begin{bmatrix} (1.2)^2 & (1.2)(1) \\ (1.2)(1) & (1)^2 \end{bmatrix} + \mathbf{Y}_2(s)\end{aligned} \tag{5.2}$$

where polynomials D_2, $N_{11,2}$, $N_{12,2}$, $N_{21,2}$ and $N_{22,2}$ are given in next iteration under Eqs. 5.3. The remaining reduced matrix $\mathbf{Y}_2$ would still be positive real symmetric matrix (Section 4.3.1). The extracted pole has associated residue matrix of rank one and would form a network in which a capacitance of 1F would be connect to port 1 and port 2 through transformers of turns ratios 1.2 and 1 respectively (Sections 3.2.2 and A.3). After extraction and network formation the remaining reduced matrix $\mathbf{Y}_2$ will continue for new iteration.

Iteration: 2

For this iteration the polynomials of $\mathbf{Y}_2$ are

$$D_2 = \begin{cases} s(s^2+4)(2396000s^8 + 35460405s^7 + 166883979s^6 + 527995538s^5 \\ \quad +805283399s^4 + 669903011s^3 + 307683728s^2 \\ \quad\quad +88471572s + 12038400) \end{cases} \tag{5.3a}$$

$$N_{11,2} = \begin{cases} 4255000s^{11} + 43124310s^{10} + 252525327.1s^9 \\ \quad +1089531369.18s^8 + 2988024775.96s^7 + 5382328807.58s^6 \\ \quad\quad +6968957061.62s^5 + 6274281533.76s^4 + 3920263300.24s^3 \\ \quad\quad\quad +1550794784s^2 + 396683088s + 48153600 \end{cases} \tag{5.3b}$$

$$N_{12,2} = N_{21,2} = \left\{ \begin{array}{l} 1430000s^{11} + 9167650s^{10} + 66040914.5s^{9} \\ \quad +225602031.1s^{8} + 424112278.2s^{7} + 44969748.3s^{6} \\ \quad\quad -2118752137.7s^{5} - 4139630179.6s^{4} - 3628786642.4s^{3} \\ \quad\quad\quad -1685068745.6s^{2} - 482516174.4s - 62599680 \end{array} \right. \tag{5.3c}$$

$$N_{22,2} = \left\{ \begin{array}{l} 1100000s^{11} + 16595740s^{10} + 170782894.45s^{9} + 749969342.51s^{8} \\ \quad +2467266073.02s^{7} + 4588049240.35s^{6} + 6681800347.47s^{5} \\ \quad\quad +7172962133.56s^{4} + 5230350055.04s^{3} + 2250212497.28s^{2} \\ \quad\quad\quad +625024626.72s + 81379584 \end{array} \right. \tag{5.3d}$$

New iteration would start with checking for stop criterion and case 1 (as described above); one after the other and it would turn out that $\mathbf{Y}_2$ neither lies in case 0 nor in case 1. For case 2 (zero at infinity) one has to take the determinant of $\mathbf{Y}_2$. A zero is defined through the determinant. Again it would turn out that the numerator and the denominator of the determinant $|\mathbf{Y}_2|$ are of same degree i.e. no zero at infinity.

After failure of case 2 check, case 3 (pole at $s = 0$) would be checked. The denominator D_2 has a factor s which indicates that there is a pole at $s = 0$. Removing the pole as described in Section 3.2.4 one can write

$$\begin{aligned} \mathbf{Y}_2(s) &= \frac{1}{s} \begin{bmatrix} 1 & -1.3 \\ -1.3 & 1.69 \end{bmatrix} + \frac{1}{D_3} \begin{bmatrix} N_{11,3} & N_{12,3} \\ N_{21,3} & N_{22,3} \end{bmatrix} \\ &= \frac{1}{s} \begin{bmatrix} (1)^2 & (1)(-1.3) \\ (1)(-1.3) & (-1.3)^2 \end{bmatrix} + \mathbf{Y}_3(s) \end{aligned} \tag{5.4}$$

Again $\mathbf{Y}_3$ is positive real symmetric matrix (Section 4.3.1) and its polynomials are given in Eqs. 5.5. The extracted pole can be synthesized by parallel network with an inductance of 1H and turns ratios 1 and -1.3 (Sections 3.2.4 and A.3). Fig. 5.1 shows the network. $\mathbf{Y}_3$ will continue to next iteration.

Iteration: 3

The polynomials for $\mathbf{Y}_3$ are as under:

$$D_3 = \left\{ \begin{array}{l} (s^2 + 4)(2396000s^{8} + 35460405s^{7} + 166883979s^{6} \\ \quad +527995538s^{5} + 805283399s^{4} + 669903011s^{3} \\ \quad\quad +307683728s^{2} + 88471572s + 12038400)\,, \end{array} \right. \tag{5.5a}$$

$$N_{11,3} = \left\{ \begin{array}{l} 4255000s^{10} + 40728310s^{9} + 217064922.1s^{8} + 913063390.18s^{7} \\ +2318187617.96s^{6} + 3909509492.58s^{5} + 4187071898.62s^{4} \\ +2745464209.76s^{3} + 1152179684.24s^{2} + 308021472s + 42796800\,, \end{array} \right. \tag{5.5b}$$

$$N_{12,3} = N_{21,3} = \left\{ \begin{array}{l} 1430000s^{10} + 12282450s^{9} + 112139441s^{8} \\ +455010403.8s^{7} + 1294900583.6s^{6} + 1959634857.8s^{5} \\ +1497698574.2s^{4} + 447832341.6s^{3} - 30277941.6s^{2} \\ -69463440s - 22464000\,, \end{array} \right. \tag{5.5c}$$

$$N_{22} = \begin{cases} 1100000s^{10} + 12546500s^9 + 110854810s^8 + 451738458s^7 \\ +1335241276s^6 + 2098984598s^5 + 1980414422s^4 \\ +1209260856s^3 + 552288744s^2 + 149925600s + 26956800\,. \end{cases} \tag{5.5d}$$

Again case 0 to case 4 would be checked and they would fail (description of case 4 would come later). Case 5 occurs when the matrix has a pole pair at some finite frequency on $j\omega$-axis. Noticing that there is a factor (s^2+4) in D_3, there is a pole pair at $s = \pm j2$.

The pole pair can be removed from $\mathbf{Y}_3$ through partial fractions or by applying the procedure described in Section 3.2.6 on each element of the matrix. One can again write

$$\begin{aligned} \mathbf{Y}_3(s) &= \frac{2s}{s^2+4}\begin{bmatrix} 1.21 & 1.1 \\ 1.1 & 1 \end{bmatrix} + \frac{1}{D_4}\begin{bmatrix} N_{11,4} & N_{12,4} \\ N_{21,4} & N_{22,4} \end{bmatrix} \\ &= \frac{1}{\frac{s}{2}+\frac{1}{s/2}}\begin{bmatrix} (1.1)^2 & (1.1)(1) \\ (1.1)(1) & (1)^2 \end{bmatrix} + \mathbf{Y}_4(s) \end{aligned} \tag{5.6}$$

The extracted parameters are synthesized as shown in Fig. 5.1. A series connection of inductance (0.5 H) and capacitance (0.5 F) connected to port 1 and port 2 through transformers with 1.1 and 1 turn-ratios respectively (Sections 3.2.6 and A.3).

Iteration: 4

The polynomials of $\mathbf{Y}_4$, remaining from previous iteration are

$$D_4 = \begin{cases} 2396000s^8 + 35460405s^7 + 166883979s^6 + 527995538s^5 + 805283399s^4 \\ +669903011s^3 + 307683728s^2 + 88471572s + 12038400\,, \end{cases} \tag{5.7a}$$

$$N_{11,4} = \begin{cases} 4255000s^8 + 34929990s^7 + 114230742s^6 + 369484201s^5 + 583515448s^4 \\ +482786863s^3 + 231844820s^2 + 69722136s + 10699200\,, \end{cases} \tag{5.7b}$$

$$N_{12,4} = N_{21,4} = \begin{cases} 1430000s^8 + 7011250s^7 + 28406550s^6 + 59820650s^5 \\ +19684200s^4 - 51271220s^3 - 54824850s^2 \\ -23986980s - 5616000\,, \end{cases} \tag{5.7c}$$

$$N_{22,4} = \begin{cases} 1100000s^8 + 7754500s^7 + 35534000s^6 + 86952500s^5 + 137114200s^4 \\ +140607800s^3 + 92151600s^2 + 31462200s + 6739200\,; \end{cases} \tag{5.7d}$$

and its determinant is

$$|\mathbf{Y}_4| = \frac{\begin{array}{c} 100(s^2+9)(11000s^6 + 51585s^5 + 113105s^4 \\ + 120890s^3 + 69541s^2 + 22055s + 3744) \end{array}}{\begin{array}{c} 2396000s^8 + 35460405s^7 + 166883979s^6 + 527995538s^5 + 805283399s^4 \\ +669903011s^3 + 307683728s^2 + 88471572s + 12038400 \end{array}} \tag{5.8}$$

In this iteration, after failing stop criterion and first five cases, the determinant of $\mathbf{Y}_4$ shows a factor (s^2+9) in its numerator. This indicates a zero pair at $s = \pm j3$. It can be extracted by inverting the matrix and extract the corresponding pole pair from each element (every zero of matrix is a pole in the inverse matrix); and then inverting back the remaining matrix. The extraction equation would be

$$\begin{aligned}[\mathbf{Y}_4]^{-1} &= \left(\frac{1}{D_5}\begin{bmatrix} N_{11,5} & N_{12,5} \\ N_{21,5} & N_{22,5}\end{bmatrix}\right)^{-1} + \frac{3s}{s^2+9}\begin{bmatrix} 1 & -1.1 \\ -1.1 & 1.21\end{bmatrix} \\ &= (\mathbf{Y}_5)^{-1} + \frac{1}{\frac{s}{3}+\frac{1}{\frac{1}{3}s}}\begin{bmatrix} (1)^2 & (1)(-1.1) \\ (1)(-1.1) & (-1.1)^2\end{bmatrix}, \end{aligned} \tag{5.9}$$

where the polynomials $N_{11,5}$, $N_{12,5}$, $N_{21,5}$, $N_{22,5}$ and D_5 are given in Eqs. 5.10. As the zero pair is extracted as a pole from the inverse matrix, it would be treated as an impedance matrix, in which an inductance (1/3H) parallel with a capacitance (1/3F) is connected to ports through transformers of turns ratios 1 and −1.1 (Sections 3.2.7 and A.2). The network is shown in Fig. 5.1 under 'Case 6'.

Iteration: 5

The remaining matrix from iteration 4, $\mathbf{Y}_5$, has following polynomials

$$D_5 = \begin{cases} 2396000s^6 + 28140405s^5 + 58655424s^4 + 56414477s^3 \\ \quad +28091690s^2 + 8674292s + 1337600 \end{cases}, \tag{5.10a}$$

$$N_{11,5} = \begin{cases} 4255000s^6 + 30936990s^5 + 57210387s^4 + 49994176s^3 \\ \quad +247388895s^2 + 7595896s + 1188800 \end{cases}, \tag{5.10b}$$

$$N_{12,5} = N_{21,5} = \begin{cases} 1430000s^6 + 3381250s^5 - 1486500s^4 \\ \quad -7935250s^3 - 6831000s^2 - 2802500s - 624000 \end{cases}, \tag{5.10c}$$

$$N_{22,5} = \begin{cases} 1100000s^6 + 4454500s^5 + 10158500s^4 + 12930500s^3 \\ \quad +9420700s^2 + 3371000s + 748800 \end{cases}; \tag{5.10d}$$

and its determinant is

$$|\mathbf{Y}_5| = \frac{\begin{matrix} 1100000s^6 + 5158500s^5 + 11310500s^4 + 12089000s^3 \\ +6954100s^2 + 2205500s + 374400 \end{matrix}}{\begin{matrix} 2396000s^6 + 28140405s^5 + 58655424s^4 + 56414477s^3 \\ +28091690s^2 + 8674292s + 1337600 \end{matrix}}. \tag{5.11}$$

Through the inspection of polynomials and the determinant, it is obvious that the stop criterion is not met and first six cases of the algorithm are not found. So the procedures of case 7 would be applied.

Step: 1 In this step a conductance would be extracted from any one-port in a manner that positive real character is not disturbed and a zero for real part of $\mathbf{Y}_5$ is formed on $j\omega$-axis. For this we choose port 1 (arbitrary choice). Calculating the real part of $\mathbf{Y}_5$ on $j\omega$-axis through the formula:

$$\mathbf{A}_5(\omega) = \frac{1}{2}\left[\mathbf{Y}_5(s) + \mathbf{Y}_5(-s)\right]\big|_{s=j\omega}; \tag{5.12}$$

and one can calculate the function

$$\frac{|\mathbf{A}_5(\omega)|}{M_{11,5}} = \frac{\begin{array}{c} 26356000000\omega^{12} + 347550459925\omega^{10} + 345931214400\omega^{8} \\ +125627522941\omega^{6} + 101699127006\omega^{4} \\ -20675737300\omega^{2} + 5007974400 \end{array}}{\begin{array}{c} 26356000000\omega^{12} + 364907016725\omega^{10} + 341561862550\omega^{8} \\ +217582632757\omega^{6} + 198164404012\omega^{4} \\ -43951474600\omega^{2} + 10015948800 \end{array}} . \tag{5.13}$$

The only minima of above function is at $\omega = 0$ with the value of 0.5 and the value of function at $\omega \to \infty$ is 1. Because the minimum value is given by $\omega = 0$, we shall take $\omega_{min} = 0$. Extracting 0.5S conductance from port 1:

$$\mathbf{Y}_6 = \mathbf{Y}_5 - \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} ; \tag{5.14}$$

where the polynomials of $\mathbf{Y}_6$ are given in Eqs. 5.15. Had we taken $\omega_{min} \to \infty$ and extracted 1S conductance from the port the real part would have become negative at $\omega_{min} = 0$. As $\omega_{min} = 0$, the imaginary part is already rank deficient at ω_{min} and a zero has already been formed at $s = 0$ for the matrix $\mathbf{Y}_6$. The Brune's process terminates prematurely (Case7, Situation: 2). This iteration ends here. The resulting circuit is shown in Fig. 3.3.

Iteration: 6

The remaining polynomials (pertaining to $\mathbf{Y}_6$) from previous iteration are:

$$D_6 = \left\{ \begin{array}{l} 2396000s^6 + 28140405s^5 + 58655424s^4 \\ \quad +56414477s^3 + 28091690s^2 + 8674292s + 1337600 \end{array} \right. , \tag{5.15a}$$

$$N_{11,6} = \left\{ \begin{array}{l} 3057000s^6 + 16866787.5s^5 + 27882675s^4 \\ +21786937.5s^3 + 10693050s^2 + 3258750s + 520000 \end{array} \right. , \tag{5.15b}$$

$$N_{12,6} = N_{21,6} = \left\{ \begin{array}{l} 1430000s^6 + 3381250s^5 - 1486500s^4 \\ -7935250s^3 - 6831000s^2 - 2802500s - 624000 \end{array} \right. , \tag{5.15c}$$

$$N_{22,6} = \left\{ \begin{array}{l} 1100000s^6 + 4454500s^5 + 10158500s^4 \\ +12930500s^3 + 9420700s^2 + 3371000s + 748800 \end{array} \right. ; \tag{5.15d}$$

The determinant $|\mathbf{Y}_6|$ shows a zero at $s = 0$ and it was expected from the outcome of previous iteration. The matrix $\mathbf{Y}_6$ is in case 4 (Section 3.2.5). The pole can be extracted by taking the inverse of the matrix and through partial fraction separating the pole from each element. The result is

$$[\mathbf{Y}_7]^{-1} = [\mathbf{Y}_6]^{-1} - \frac{1}{s} \begin{bmatrix} 1.44 & 1.2 \\ 1.2 & 1 \end{bmatrix} , = [\mathbf{Y}_6]^{-1} - \frac{1}{s} \begin{bmatrix} (1.2)^2 & (1.2)(1) \\ (1.2)(1) & (1)^2 \end{bmatrix} ; \tag{5.16}$$

where the polynomials of $\mathbf{Y}_7$ are given in Eq. 5.17. The extracted zero can be realized as shown in Fig. 5.1 (section labelled with 'Case 4').

Iteration: 7

The remaining polynomials (pertaining to $\mathbf{Y}_7$) from previous iteration are:

$$D_7 = 95840s^5 + 768253s^4 + 871910s^3 + 386901s^2 + 113304s + 10000\,, \quad (5.17a)$$
$$N_{11,7} = 122280s^5 + 652671.5s^4 + 998057s^3 + 622227.5s^2 + 202772s + 40600 \quad (5.17b)$$
$$N_{12,7} = N_{21,7} = 57200s^5 + 161650s^4 + 81240s^3 - 18310s^2 - 3300s - 4400\,, \quad (5.17c)$$
$$N_{22,7} = 44000s^5 + 146500s^4 + 237500s^3 + 158300s^2 + 52900s + 5600\,; \quad (5.17d)$$

and its determinant is

$$|\mathbf{Y}_7| = \frac{22000s^5 + 117250s^4 + 249250s^3 + 224950s^2 + 89750s + 20800}{95840s^5 + 768253s^4 + 871910s^3 + 386901s^2 + 113304s + 10000}\,. \quad (5.18)$$

Again through the inspection of polynomials and the determinant, it is obvious that the stop criterion is not met and first six cases of the algorithm are not found. So the procedures of case 7 would be applied.

Step: 1 In this step a conductance would be extracted from any one-port in a manner that positive real character is not disturbed and a zero for real part of $\mathbf{Y}_7$ is formed on $j\omega$-axis. For this we choose port 1 (arbitrary choice). Calculating the real part of $\mathbf{Y}_7$ on $j\omega$-axis through the formula:

$$\mathbf{A}_7(\omega) = \frac{1}{2}\left[\mathbf{Y}_7(s) + \mathbf{Y}_7(-s)\right]\big|_{s=j\omega}\,; \quad (5.19)$$

and calculation the function

$$\frac{|\mathbf{A}_7(\omega)|}{M_{11,7}} = \frac{\begin{array}{l}42169600\omega^{10} + 528310245\omega^8 + 560480906\omega^6 \\ +53875861\omega^4 + 8374160\omega^2 + 4160000\end{array}}{\begin{array}{l}84339200\omega^{10} + 1028460490\omega^8 + 776769812\omega^6 \\ -120401878\omega^4 + 44882720\omega^2 + 1120000\end{array}} \quad (5.20)$$

The only minima of above function is at $\omega = 0$ with the value of 3.7143 and the value of function at $\omega \to \infty$ is 0.5. Because the minimum value is given by $\omega \to \infty$, we shall take $\omega_{min} \to \infty$. Extracting 0.5S conductance from port 1:

$$\mathbf{Y}_8 = \mathbf{Y}_7 - \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}\,; \quad (5.21)$$

where the polynomials of $\mathbf{Y}_8$ are given in Eqs. 5.22. The imaginary part is already rank deficient at ω_{min} and a zero has already been formed at $s \to \infty$ in the matrix $\mathbf{Y}_8$. This iteration ends here (Case7, Situation: 1). The resulting circuit is shown in Fig. 3.3.

Iteration: 8

The remaining polynomials (pertaining to $\mathbf{Y}_8$) from previous iteration are:

$$D_8 = 95840s^5 + 768253s^4 + 871910s^3 + 386901s^2 + 113304s + 10000\,, \tag{5.22a}$$
$$N_{11,8} = 74360s^5 + 268545s^4 + 562102s^3 + 428777s^2 + 146120s + 35600\,, \tag{5.22b}$$
$$N_{12,8} = N_{21,8} = 57200s^5 + 161650s^4 + 81240s^3 - 18310s^2 - 3300s - 4400\,, \tag{5.22c}$$
$$N_{22,8} = 44000s^5 + 146500s^4 + 237500s^3 + 158300s^2 + 52900s + 5600\,; \tag{5.22d}$$

The determinant $|\mathbf{Y}_8|$ shows a zero at $s \to \infty$ and it was expected from the outcome of previous iteration. The matrix $\mathbf{Y}_8$ is in case 2 (Section 3.2.3). To extract the pole, take the inverse of the matrix and through long division separating the pole from each element. The result is

$$[\mathbf{Y}_9]^{-1} = [\mathbf{Y}_8]^{-1} - \frac{1}{s}\begin{bmatrix} 1 & -1.3 \\ -1.3 & 1.69 \end{bmatrix} = [\mathbf{Y}_8]^{-1} - \frac{1}{s}\begin{bmatrix} (1)^2 & (1)(-1.3) \\ (1)(-1.3) & (-1.3)^2 \end{bmatrix} \tag{5.23}$$

where the polynomials of $\mathbf{Y}_9$ are given in Eq. 5.24. The extracted zero can be realized as shown in Fig. 5.1 (section labelled with 'Case 2').

Iteration: 9

The remaining polynomials (pertaining to $\mathbf{Y}_9$) from previous iteration are:

$$D_9 = 160s^4 + 1280s^3 + 1428s^2 + 568s + 100\,, \tag{5.24a}$$
$$N_{11,9} = 480s^4 + 3157s^3 + 3218s^2 + 1157s + 356\,, \tag{5.24b}$$
$$N_{12,9} = N_{21,9} = -80s^4 - 1083s^3 - 1006s^2 - 267s - 44\,, \tag{5.24c}$$
$$N_{22,9} = 160s^4 + 917s^3 + 950s^2 + 349s + 56\,. \tag{5.24d}$$

Inspection through polynomials and the determinant $|\mathbf{Y}_9|$ shows that $\mathbf{Y}_9$ is in case 7. On the onset it is worth mentioning that this case have extracted circuit with an element having negative value. The extracted circuit can then be converted to equivalent circuit as shown in the Fig. 5.1.

Step: 1 This step involves extracting a conductance from either port so that positive real character is not disturbed and a zero is formed on $j\omega$-axis. For this we chose port 1 arbitrarily and calculate the real part of $\mathbf{Y}_9$ on $j\omega$-axis:

$$\mathbf{A}_9(\omega) = \frac{1}{2}\left[\mathbf{Y}_9(s) + \mathbf{Y}_9(-s)\right]\big|_{s=j\omega}\,,$$

and from the real part the function to calculate the minimum conductance and its corresponding frequency (Section 3.2.8.1):

$$\frac{|\mathbf{A}_9(\omega)|}{M_{11,9}} = \frac{4400\omega^8 + 53780\omega^6 + 35649\omega^4 - 4396\omega^2 + 1125}{1600\omega^8 + 49580\omega^6 + 25874\omega^4 + 1454\omega^2 + 350}\,; \tag{5.25}$$

For the above function the minima occurs at $\omega = 0$ and $\omega = \pm 0.5$ with values 3.2143 and 1 respectively. As the minimum value of 1 is given by $\omega = 0.5$, we take $\omega_{min} = 0.5$.

Subtracting the value from first element of $\mathbf{Y}_9$:

$$\mathbf{Y}_9^{'} = \mathbf{Y}_9 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.26}$$

It would form a conductance at port 1 (Fig. 5.1 under Extracted Type-III). This extraction makes the real part of $\mathbf{Y}_9^{'}$ rank-deficient at $s = \pm j\omega_{min} = \pm j0.5$. As ω_{min} is neither 0 nor ∞ (Situation: 3, Section 3.2.8.1), the Brune's process would continue to step 2.

Step: 2 First the value of $\mathbf{Y}_9^{'}$ would be found at $s = j\omega_{min} = j0.5$

$$\begin{aligned} \mathbf{Y}_9^{'}(s = j0.5) &= \begin{bmatrix} 0.6518 & -0.6518 \\ -0.6518 & 0.6518 \end{bmatrix} + j \begin{bmatrix} 0.0848 & -0.3348 \\ -0.3348 & 0.0848 \end{bmatrix} \\ &= \mathbf{A}_9^{'}(s = j0.5) + j\mathbf{I}_9(s = j0.5) \end{aligned} \tag{5.27}$$

The null vector of $\mathbf{A}_9^{'}(s = j0.5)$ is $\boldsymbol{\beta} = [1/\sqrt{2}, 1/\sqrt{2}]^T$. To make the whole matrix $\mathbf{Y}_9^{'}(s = j0.5)$ rank deficient, the matrix $\mathbf{I}_9(s = j0.5)$ should also have the same null vector $\boldsymbol{\beta}$ (Sections 3.2.8.2 and B.1). For this a matrix $\mathbf{H} = \alpha \boldsymbol{h}\boldsymbol{h}^T$, where $\boldsymbol{h} = [h_1, h_2]$, is to be subtracted from $\mathbf{I}_9(s = j0.5)$. The value of α can be calculated using Eq. 3.25:

$$\alpha = \operatorname{sgn}(\boldsymbol{\beta}^T \mathbf{I}_9(s = j0.5)\boldsymbol{\beta}) = -1\,. \tag{5.28}$$

Calculating the matrix $\mathbf{H}$ using Eq. 3.24: $\mathbf{I}_9(s = j0.5)\boldsymbol{\beta} = \alpha \boldsymbol{h}\boldsymbol{h}^T\boldsymbol{\beta}$, and solving simultaneous equations we have $h_1 = h_2 = 1/\sqrt{8}$.

As $\alpha = -1$ and it is an admittance matrix, Type-III Brune's circuit would be chosen. As the matrix $\mathbf{H}$ would be subtracted from the imaginary part $\mathbf{I}_9(s = j0.5)$, it should be translated to such reactive circuit which gives its values at appropriate frequency. For this let

$$j\mathbf{H} = j(-1)\frac{1}{8}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = j\frac{1}{2}\left(\frac{-1}{4}\right)\begin{bmatrix} (1)(1) & (1)(1) \\ (1)(1) & (1)(1) \end{bmatrix} = \left| sC_1 \begin{bmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{bmatrix} \right|_{s=j0.5} \tag{5.29}$$

With $C_1 = -1/4$F and $\boldsymbol{p} = [p_1, p_2]^T = [1, 1]^T$, subtracting $sC_1\boldsymbol{p}\boldsymbol{p}^T$ from $\mathbf{Y}_9^{'}$:

$$\mathbf{Y}_9^{''} = \mathbf{Y}_9^{'} - s\left(\frac{-1}{4}\right)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{5.30}$$

It should be noted that the degrees of the polynomials in the numerator has increased. Subtracting a pole with negative residue effectively is adding a pole with positive residue and it preserves the positive real character of $\mathbf{Y}_9^{''}$. Second part of extracted Type-III circuit in Fig. 5.1 shows addition of a pole. The whole process also made a zero pair at $s = \pm j0.5$, which would be removed in next step.

Step: 3 The determinant of $\mathbf{Y}_9''$ shows a zero pair at $s = \pm j0.5$:

$$\left|\mathbf{Y}_9''\right| = \frac{(4s^2+1)(10s^3+95s^2+96s+31)}{40s^4+320s^3+357s^2+142s+25}. \tag{5.31}$$

Removing the zero by inverting the matrix (as is done in case 6)

$$\begin{aligned}\mathbf{Y}_9''' &= \left(\left[\mathbf{Y}_9''\right]^{-1} - \frac{2s}{4s^2+1}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)^{-1} = \left(\left[\mathbf{Y}_9''\right]^{-1} - \frac{1}{2s+\frac{1}{2s}}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)^{-1}\\ &= \frac{\begin{bmatrix}5s^3+70s^2+183s+128 & 5s^3+40s^2+3s-22\\ 5s^3+40s^2+3s-22 & 5s^3+50s^2+63s+28\end{bmatrix}}{10s^2+60s+50}\end{aligned} \tag{5.32}$$

The removed zero pair would form the third part of the extracted Type-III network shown in Fig. 5.1.

Step: 4 Note that in remaining matrix $\mathbf{Y}_9'''$, the numerator polynomials have a degree higher than the denominator. This always occur after the third step of Brune's process (Section 4.1.1). It can be removed by long division.

$$\begin{aligned}\mathbf{Y}_{10} = \mathbf{Y}_9'''' &= \mathbf{Y}_9''' - s\left(\frac{1}{2}\right)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\\ &= \frac{\begin{bmatrix}20s^2+79s+64 & 5s^2-11s-11\\ 5s^2-11s-11 & 10s^2+19s+14\end{bmatrix}}{5s^2+30s+25}\end{aligned} \tag{5.33}$$

The last part would make fourth part of the extracted Type-III network shown in Fig.5.1.

Conversion to Equivalent Circuit: The extracted Type-III network has one element which is negative. It can be avoided by converting the whole network into an equivalent network. The relationship between the three capacitances is fulfilled as given in Eq. 4.43. Using the equations given in Section 4.2.3 one can find the parameters of equivalent circuit of Type-III shown in Fig. 5.1.

Iteration: 10

The remaining matrix $\mathbf{Y}_9$ from previous iteration is

$$\mathbf{Y}_{10} = \frac{\begin{bmatrix}20s^2+79s+64 & 5s^2-11s-11\\ 5s^2-11s-11 & 10s^2+19s+14\end{bmatrix}}{5s^2+30s+25}. \tag{5.34}$$

Inspection through polynomials and the determinant $|\mathbf{Y}_{10}|$ shows that $\mathbf{Y}_{10}$ is in case 7. On the onset it is worth mentioning that this case have extracted circuit with an element having negative value. The extracted circuit can then be converted to equivalent circuit as given in the Fig. 5.1.

Step: 1 This step involves extracting a conductance from either port so that positive real character is not disturbed and a zero is formed on $j\omega$-axis. For this we chose port 1 arbitrarily and calculate the real part of $\mathbf{Y}_{10}$ on $j\omega$-axis:

$$\mathbf{A}_{10}(\omega) = \frac{1}{2}\left[\mathbf{Y}_{10}(s) + \mathbf{Y}_{10}(-s)\right]\big|_{s=j\omega} \tag{5.35}$$

and from the real part the function to calculate the minimum conductance and its corresponding frequency (Section 3.2.8.1):

$$\frac{|\mathbf{A}_{10}(\omega)|}{M_{11,10}} = \frac{7\omega^4 + 14\omega^2 + 31}{2\omega^4 + 10\omega^2 + 14} \tag{5.36}$$

For the above function the minima occurs at $\omega = 0$ and $\omega = \pm 1$ with values 2.2143 and 2 respectively. As the minimum value of 2 is given by $\omega = 1$, we take $\omega_{min} = 1$.

Subtracting the value from first element of $\mathbf{Y}_{10}$:

$$\begin{aligned}\mathbf{Y}'_{10} &= \mathbf{Y}_{10} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \frac{1}{5s^2 + 30s + 25}\begin{bmatrix} 10s^2 + 19s + 14 & 5s^2 - 11s - 11 \\ 5s^2 - 11s - 11 & 10s^2 + 19s + 14 \end{bmatrix}\end{aligned} \tag{5.37}$$

It would form a conductance at port 1 (Fig. 5.1 under Extracted Type-IV). This extraction makes the real part of $\mathbf{Y}'_{10}$ rank-deficient at $s = \pm j\omega_{min} = \pm j1$. As ω_{min} is neither 0 nor ∞ (Situation: 3, Section 3.2.8.1), the Brune's process would continue to step 2.

Step: 2 First the value of $\mathbf{Y}'_{10}$ would be found at $s = j\omega_{min} = j1$

$$\mathbf{Y}'_{10}(s = j1) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + j\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \mathbf{A}'_{10}(s = j1) + j\mathbf{I}_{10}(s = j1) \tag{5.38}$$

The null vector of $\mathbf{A}'_{10}(s = j1)$ is $\boldsymbol{\beta} = [1/\sqrt{2}, 1/\sqrt{2}]^T$. To make the whole matrix $\mathbf{Y}'_{10}(s = j1)$ rank deficient, the matrix $\mathbf{I}_{10}(s = j1)$ should also have the same null vector $\boldsymbol{\beta}$ (Sections 3.2.8.2 and B.1). For this a matrix $\mathbf{H} = \alpha \boldsymbol{h}\boldsymbol{h}^T$, where $\boldsymbol{h} = [h_1, h_2]$, is to be subtracted from $\mathbf{I}_{10}(s = j1)$. The value of α as shown in Eq. 3.25 is

$$\alpha = \operatorname{sgn}(\boldsymbol{\beta}^T \mathbf{I}_{10}(s = j1)\boldsymbol{\beta}) = +1 \tag{5.39}$$

Calculating the matrix $\mathbf{H}$ using Eq. 3.24: $\mathbf{I}_{10}(s = j1)\boldsymbol{\beta} = \alpha \boldsymbol{h}\boldsymbol{h}^T\boldsymbol{\beta}$, and solving simultaneous equations we have $h_1 = h_2 = 1/\sqrt{5}$.

As $\alpha = +1$ and it is admittance matrix, Type-IV Brune's circuit would be chosen. As the matrix $\mathbf{H}$ would be subtracted from the imaginary part $\mathbf{I}_{10}(s = j1)$, it should be translated to such reactive circuit which gives its values at appropriate frequency. For this let

$$j\mathbf{H} = j\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{j(1)(-5)}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \left|\frac{1}{s(L_1)}\begin{bmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2 \end{bmatrix}\right|_{s=j1} \tag{5.40}$$

With $L_1 = -5\text{F}$ and $\boldsymbol{p} = [p_1, p_2]^T = [1, 1]^T$, subtracting $\frac{1}{sL_1}\boldsymbol{p}\boldsymbol{p}^T$ from $\mathbf{Y}'_{10}$:

$$\mathbf{Y}''_{10} = \mathbf{Y}'_{10} - \frac{1}{s(-5)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{pmatrix} \frac{2s^3+4s^2+4s+1}{s^3+6s^2+5s} & \frac{s^3-2s^2-s+1}{s^3+6s^2+5s} \\ \frac{s^3-2s^2-1s+1}{s^3+6s^2+5s} & \frac{2s^3+4s^2+4s+1}{s^3+6s^2+5s} \end{pmatrix} \tag{5.41}$$

It should be noted that the degrees of the polynomials in the numerator and denominator has increased. Subtracting a pole with negative residue effectively is adding a pole with positive residue and it preserves the positive real character of $\mathbf{Y}''_{10}$. Second part of extracted Type-IV circuit in Fig. 5.1 shows addition of a pole. The whole process also made a zero pair at $s = \pm j1$, which would be removed in next step.

Step: 3 The determinant of $\mathbf{Y}''_{10}$ shows a zero pair at $s = \pm j1$:

$$\left|\mathbf{Y}''_{10}\right| = \frac{(3s+2)(s^2+1)}{s(s+5)(s+1)}\,. \tag{5.42}$$

Removing the zero by inverting the matrix (as is done in case 6)

$$\begin{aligned}\mathbf{Y}'''_{10} &= \left(\left[\mathbf{Y}''_{10}\right]^{-1} - \frac{s}{s^2+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1} = \left(\left[\mathbf{Y}''_{10}\right]^{-1} - \frac{1}{s+\frac{1}{s}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} 2+1/s & 1+1/s \\ 1+1/s & 2+1/s \end{bmatrix}\end{aligned} \tag{5.43}$$

The removed zero pair would form the third part of the extracted Type-IV network shown in Fig. 5.1.

Step: 4 Note that in remaining matrix $\mathbf{Y}'''_{10}$, has a pole at $s = 0$. This always occur after the third step of Brune's process (section 4.1.2). which can be removed.

$$\mathbf{Y}_{11} = \mathbf{Y}''''_{10} = \mathbf{Y}'''_{10} - \frac{1}{s}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{5.44}$$

The last part would make fourth part of the extracted Type-IV network shown in Fig. 5.1.

Conversion to Equivalent Circuit The extracted Type-IV network has one element which is negative. It can be avoided by converting the whole network into an equivalent network. The relationship between the three inductances is as given in Eq. 4.47. Using the equations given in Section 4.2.4 one can find the parameters of equivalent circuit of Type-III shown in Fig. 5.1.

Iteration: 11

The remaining matrix, $\mathbf{Y}_{11}$, for this iteration is

$$\mathbf{Y}_{11} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{5.45}$$

As this matrix is complex frequency independent, the stop criterion is met and it would make the last stage of the whole network. The matrix is full rank matrix and to decompose it into one-rank matrices, spectral decomposition can be done. With eigen-values 1 and 3 with corresponding eigen-vectors $[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]^T$ and $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$ one can write (using diagonalization formula, section A.3 and A.2)

$$\begin{aligned}
\mathbf{Y}_{11} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} (1)^2 & (1)(-1) \\ (1)(-1) & (-1)^2 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} (1)^2 & (1)(1) \\ (1)(1) & (1)^2 \end{bmatrix}
\end{aligned} \tag{5.46}$$

Each of one-rank matrices on the right hand side of above equation represent one network similar to one in Fig. A.3. The first one has conductance of 0.5S connected with the ports through transformers of turn-ratios 1 and −1. Similarly the second one has conductance of 1.5S and turn-ratios 1 and 1. Last stage in Fig. 5.1 shows the circuits.

5.1.1 Demonstration Example as Impedance Matrix

The matrix given in Eq. 5.1 can be treated as an impedance matrix. The cases and the extraction procedures would be exactly same but the circuit topologies would be completely different. Fig. 5.2 shows the complete network of the matrix treated as an impedance matrix. Notice that with the exception of topology the values of the elements are same. Capacitances are converted to inductances and vice versa. Similarly, conductances have been changed to resistances. The network in Fig. 5.2 is dual to the network in Fig. 5.1.

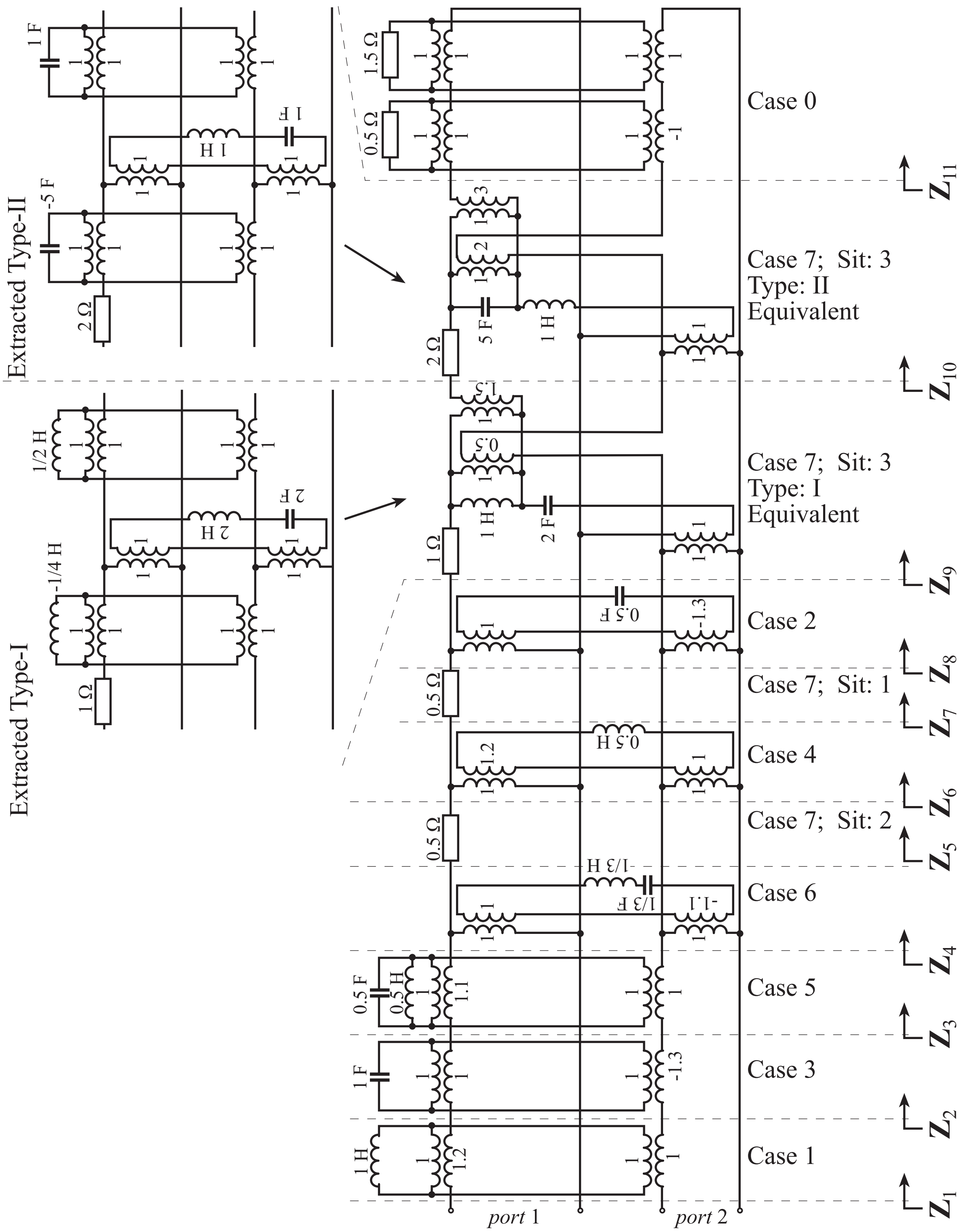


Figure 5.2: Brune's network diagram for impedance matrix given in Eq. 5.1

5.2 General Procedure

The general procedure of developing a circuit model from a microwave structure involves much more steps on the top of circuit synthesis. The steps can be divided into groups described below. Not all steps are required for each and every example. Some steps can be skipped and some times some special steps (like using Bartlett's bisection theorem for symmetrical structures) are also used. In the end the comparison of the responses is done. The flow of the steps are shown in Fig. 5.2.

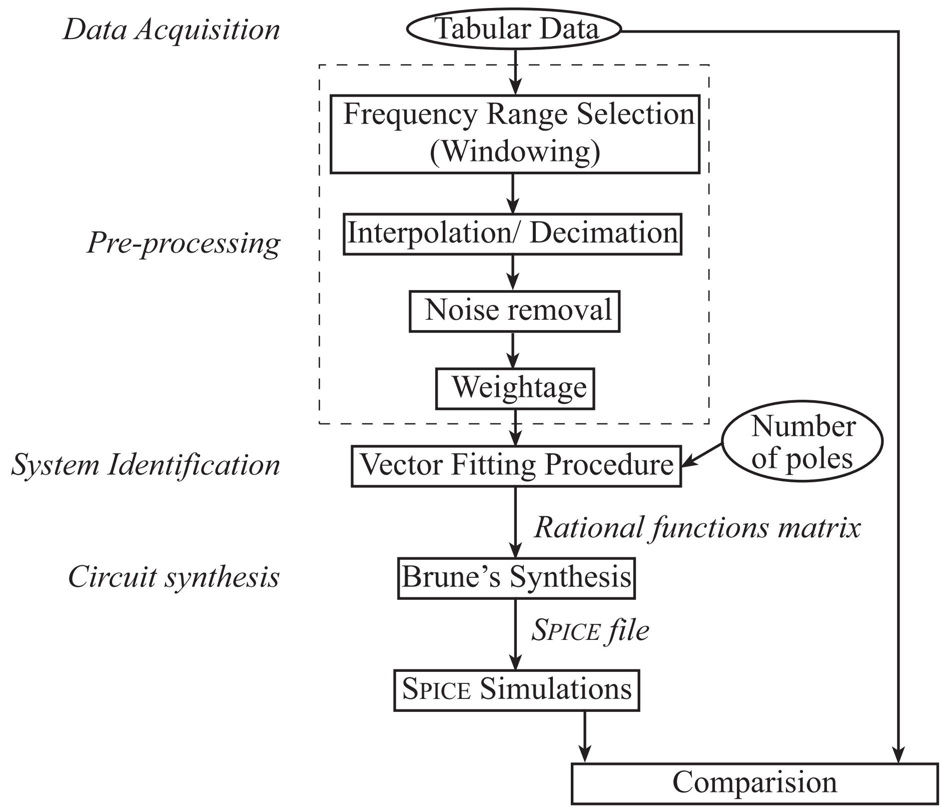


Figure 5.3: General steps followed in development of examples

5.2.1 Data Acquisition

Tabular data can be acquired from microwave structure either through measurement or through full wave simulation. There are some considerations and precautions required.

- In measurement, it is important to have both magnitude and phase of s-parameters by using vector analyzer. It is important for system identification procedures used in later stages of the whole procedure.

- Simulation results can be non-passive due to many reasons. Use of coarse mesh in the simulation or in case of time-domain simulations, simulating for not enough time can cause this problem. Experience has been that $\lambda/20$ mesh size, where *lambda* being the wavelength of highest frequency of simulation, does not cause any problem while $\lambda/10$ mesh size sometimes causes the data to be non-passive at many point throughout the frequency range.

- The tabular data must be checked for passivity at each frequency point. The real part of the given matrix must be positive definite at at each frequency point (section 2.2.2).

5.2.2 Data Pre-processing

Pre-processing of tabular data is usually required for system identification procedure. Selection of frequency range and interpolation/ decimation are minor steps but for the smoothness of curves, which is required for good fitting, special care must be taken. While smoothing, distortion of data may occur. Gaussian window proves to be helpful for smoothing the curves.

5.2.3 System Identification

Converting a tabular data into a matrix of rational functions can be termed both as *System Identification* [67] and as *Curve fitting*. Prony's method [11] and Vector Fitting (V.F.) procedure [8–10] are good examples. In developing of the examples, vector fitting procedure was used.

In vector fitting, a matrix of rational functions of the form

$$\mathbf{W}(s) = \mathbf{D}s + \mathbf{A} + \sum_{i=1}^{n} \frac{\mathbf{K}_i}{s - p_i}\,; \tag{5.47}$$

where $\mathbf{D}$ and $\mathbf{A}$ are real matrices and $\mathbf{K}_i$ is a residue matrix of pole p_i, is determined. The poles p_i appear in conjugate pairs and so does the corresponding residue matrices.

5.2.4 Circuit Synthesis, SPICE Simulations and Comparisons

Once a matrix of rational functions in complex frequency s is found, the Brune's algorithm can be applied to synthesize a circuit. Implementation of the algorithm has been done in Matlab. The code can also automatically write a SPICE file for the network. Once a SPICE file is obtained, SPICE simulator can be invoked to simulate it and generate the data for comparison.

5.3 Integrated Antenna Example

On-chip antenna are designed for inter-chip or intra-chip communication and to avoid the interconnects and buses. The communication link is characterized by near field communication and high coupling [56]. The network model is used for co-simulation with other circuitry behind the antennas.

Integrated antenna used as an example is shown in Fig. 5.4. The generalized Foster network model was developed in [56]. This model includes negative values for resistances which may cause instabilities. Brune's model was synthesized in [57] and results are presented here.

The S-parameters of the antenna link were measured for the frequency range upto 80 GHz. The data was converted to Z-parameters taking one frequency sample at a time. A few sample points violated the passivity condition (Section 2.2.2) and were removed. System identification is done through Vector-fitting procedure using 8 poles. A study on the determination of number of poles (complexity of model) is given in [68]. In the outcome of vector-fitting procedure the parameters (coefficients of polynomials) are mutually independent, eight poles in 2×2 matrix forms an order of 16.

The matrix of rational functions obtained is of form

$$\mathbf{Z}(s) = \mathbf{R} + \sum_{i=1}^{8} \frac{\mathbf{K}_i}{s - p_i}, \tag{5.48}$$

where $\mathbf{K}_i$ are residue matrix corresponding to pole p_i (occurring as conjugate pairs) and $\mathbf{R}$ is a real symmetric matrix.

Comparison of S-parameters from measurement data and from SPICE simulations is shown in Fig. 5.5

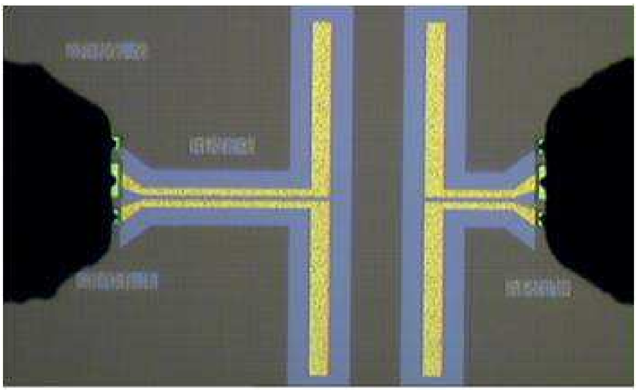


Figure 5.4: Layout of On-Chip antenna [56, 57]

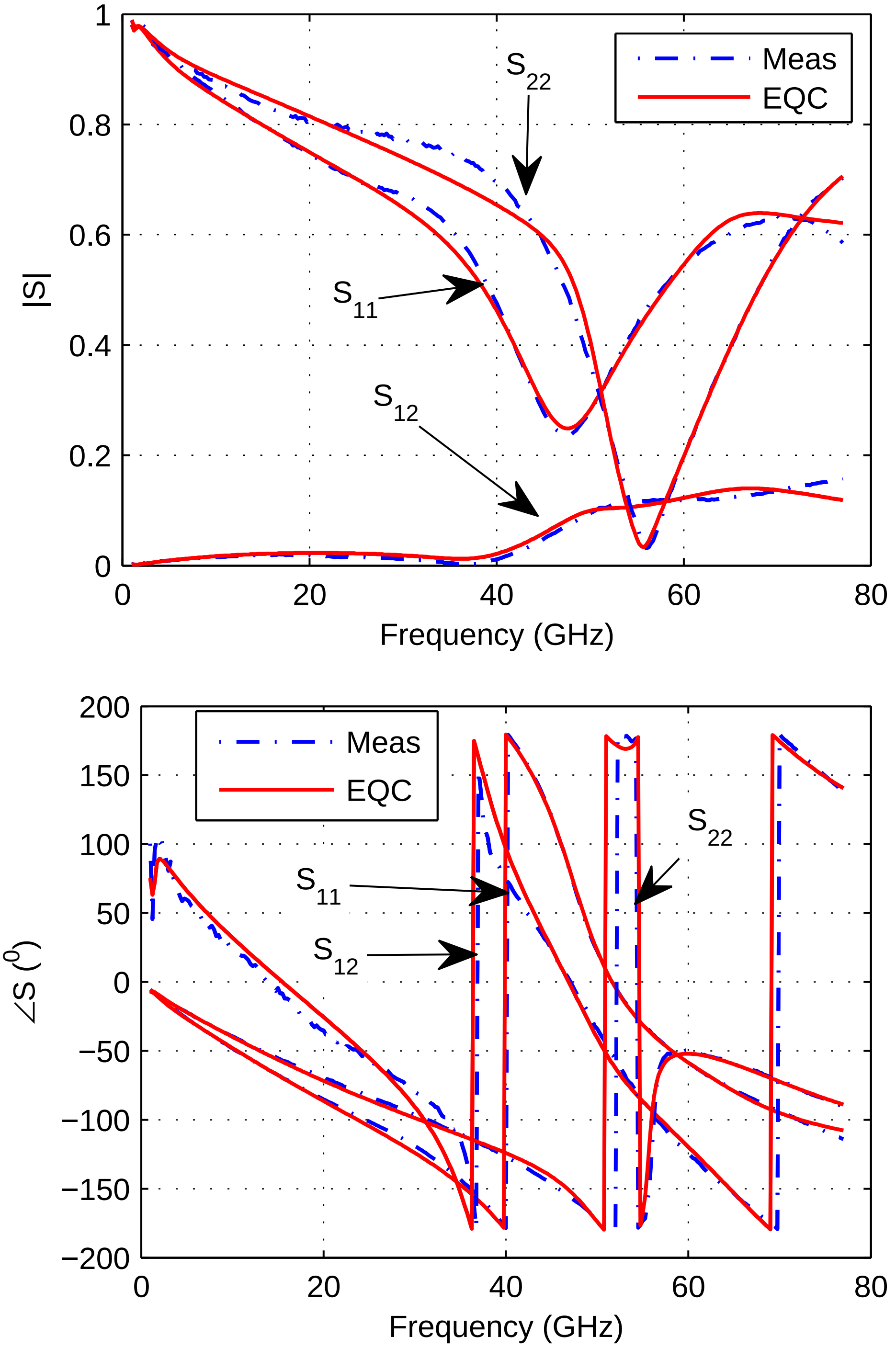


Figure 5.5: S-parameters comparison between measured data (Meas) and SPICE simulation of equivalent circuit (EQC) [57]

5.4 Filter Structure Example: Use of Symmetry

The symmetry of electromagnetic structure can be employed to decompose 2-port structure into equivalent network of one-port circuits, using Bartlett's bisection theorem [39, Ch. 6]. For a symmetric reciprocal two-port, two out of four parameters are independent i.e. $Z_{11} = Z_{22}$ and $Z_{12} = Z_{21}$. Bartlett's theorem defines two parameters Z_{oc} and Z_{sc} as

$$Z_{oc} = Z_{11} + Z_{12}$$
$$Z_{sc} = Z_{11} - Z_{12}$$

The equivalent circuit corresponding to Z_{oc} and Z_{sc} can be synthesized through Brune's algorithm and then connected in a lattice structure (shown in Fig. 5.6) to generate the response required.

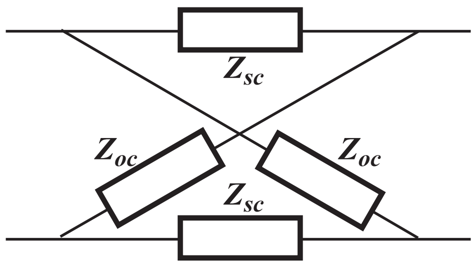


Figure 5.6: Lattice structure decomposition for symmetric two-port circuit [50]

The layout of 11^{th} order Chebyshev band-pass filter serves as an example to demonstrate the use of symmetry and is shown in Fig. 5.7. Z-parameter data was obtained from full wave simulation of the structure. Z_{oc} and Z_{sc} were obtained by applying the above equations to each data point. Vector-fitting procedure applied separately to each data set of Z_{oc} and Z_{sc} required 12 poles each. Brune's synthesis was done and SPICE simulations followed to give comparison results given in Figs. 5.8 and 5.9.

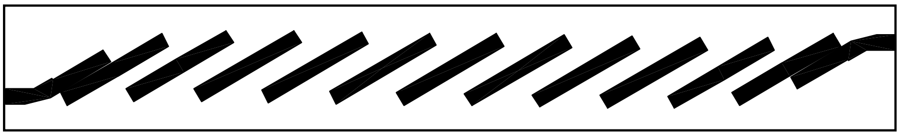


Figure 5.7: Layout of filter structure [50]

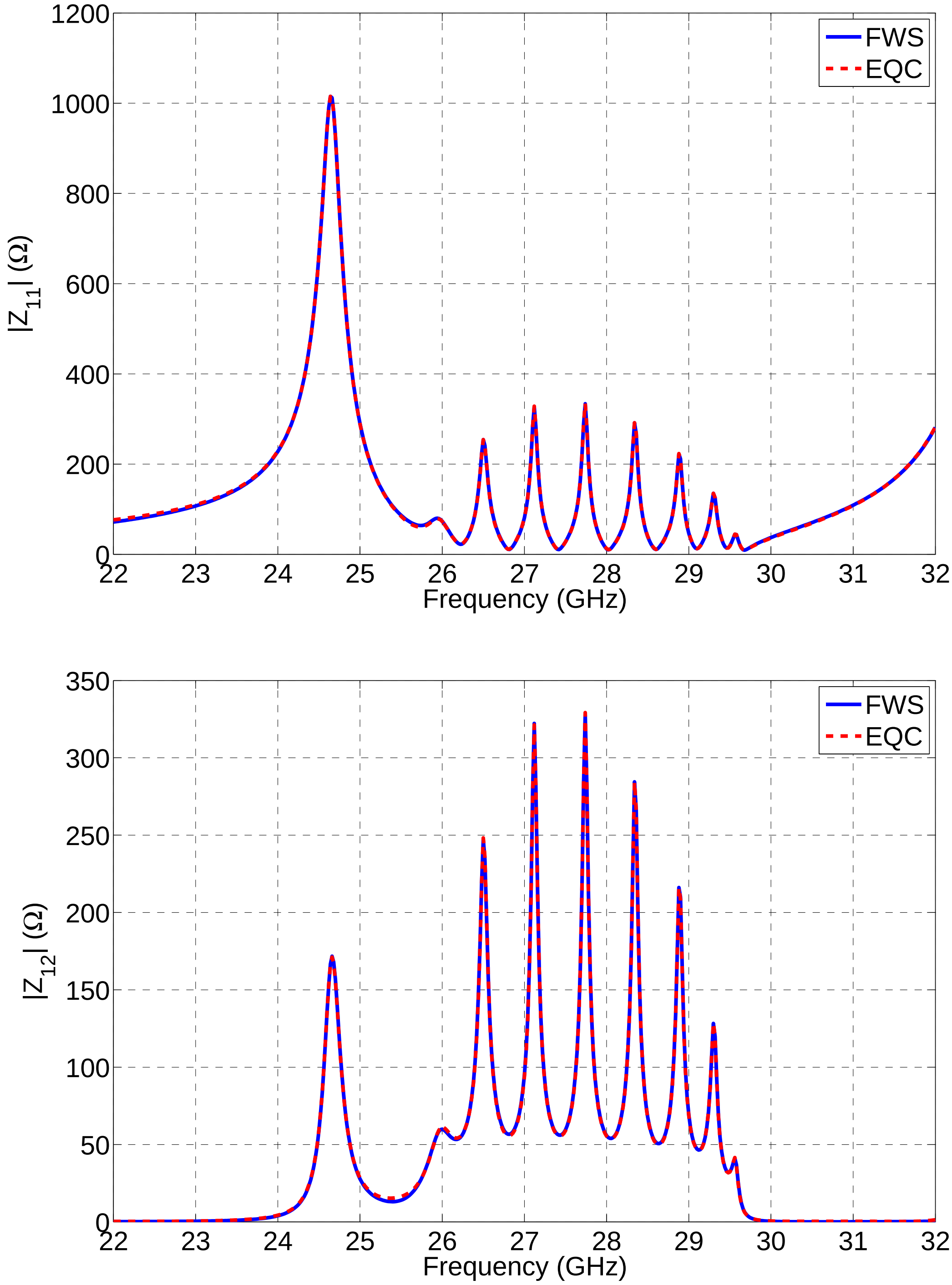


Figure 5.8: Comparison of the magnitude of Z_{11} and Z_{12} [50]

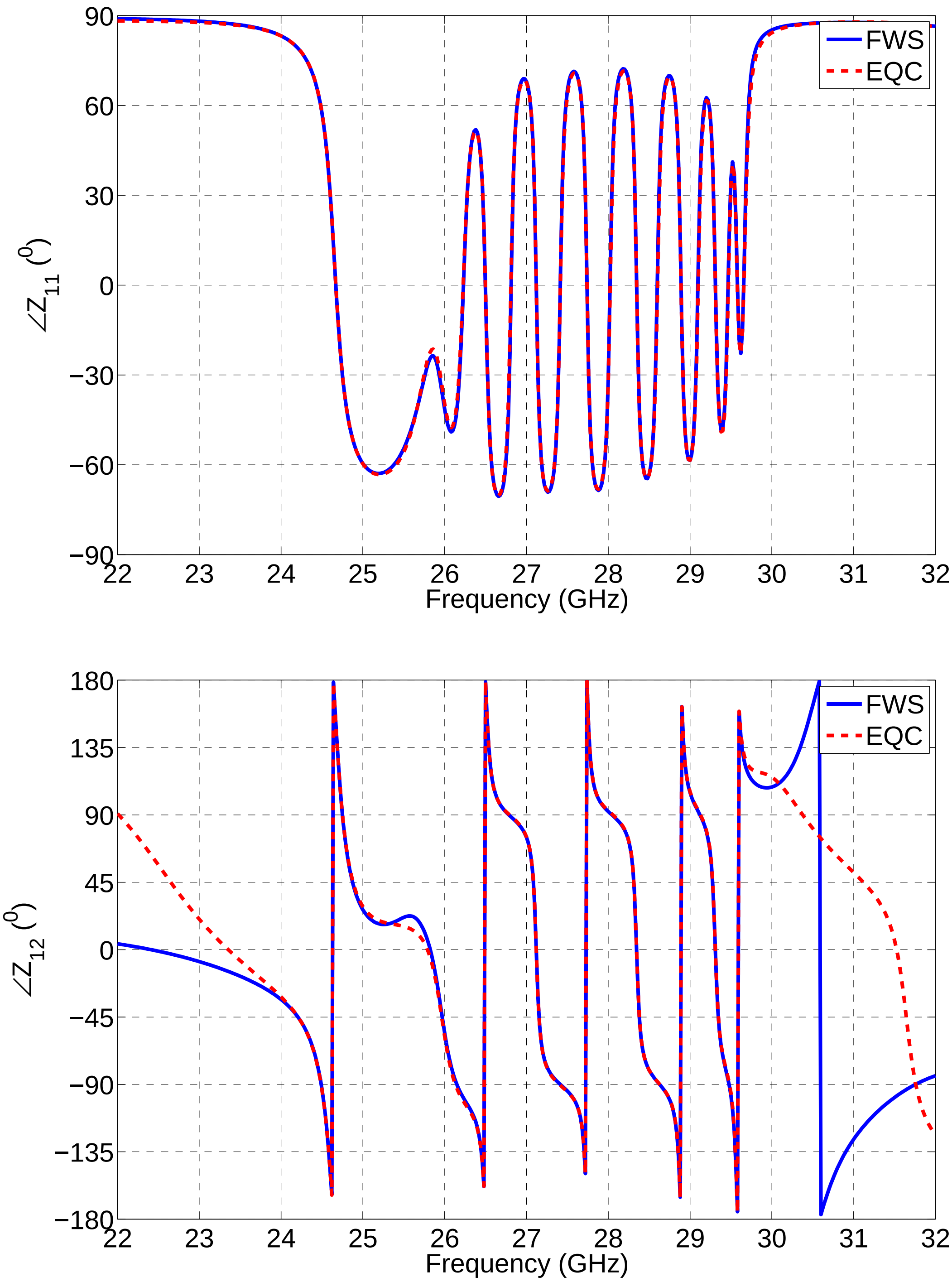


Figure 5.9: Comparison of the phase of Z_{11} and Z_{12} [50]

6 Conclusion and Outlook

The algorithm presented in this work synthesizes an equivalent and finite passive circuit from an impedance or an admittance represented as positive real symmetric matrix of rational functions. The resultant equivalent circuit is a cascaded circuit of fixed sub-circuit topologies. There are seven cases and a stop criterion each having a fixed extraction procedure and each having fix sub-circuit topology. In each iteration of the algorithm one of the cases is selected, depending on the location of poles and zeros; the corresponding extraction procedure is applied to extract a small part and parameter values of the corresponding sub-circuit topology are determined. The algorithm verifies the *realizability theorem* by McMillan [28] that there exists a finite passive network whose impedance or admittance is positive real symmetric matrix of rational functions.

An equivalent lumped element circuit model can be synthesized for the microwave or millimeter-wave circuits in a finite frequency band, if a proper rational function fitting forming a positive real symmetric matrix, from the tabular response data can be found. The equivalent circuit established in this manner would be a fixed model valid only in a frequency band. As all elements in the model have positive values it would remain stable at all frequencies.

The work can be extended in number of possible ways. One way is to develop the algorithm using most general approach used by McMillan for the Brune method (section 3.2.8, [28]). Contrary to Tellegen's approach in which only one resistor is extracted at one of the ports (Step 1 of Brune method), McMillan extracts a resistive multiport containing many resistors and ideal transformers. The approach forms an optimization problem and requires relatively more computational effort. Nevertheless it would lead to different forms of equivalent circuit models.

The algorithm may have applications in other physical systems for which a one to one analogy between different elements can be found. Analogous systems are frequently found in the area of control systems [69]. For example, in a translational mechanical system a spring mass system is analogous to capacitor-inductor system and force and displacement are analogous to current and voltage. Broadening the algorithm to include whole class of positive matrices (section 2.1) is also interesting. It would include non-reciprocal circuits as well. This, however, requires development of circuit simulators like SPICE which can handle such circuits.

A SPICE Implementations

A.1 Ideal Transformer & its implementation in SPICE3

The Fig. A.1 shows the standard symbol for an ideal transformer along with two implementations in SPICE3. Note that according to *dot-convention* [70, §2.4] if the primary current enters the dot-labeled terminal, the secondary current leaves the dot-labeled terminal on the other side and the polarity of voltage at the dot-labeled terminals is also same. This convention is not to be mixed with the standard convention of network theory in which all currents are taken to be entering the ports. The dot-convention for the transformers is followed through out this thesis. The *ideal transformer equation* is

$$\frac{I_1}{I_2} = \frac{V_2}{V_1} = \frac{n_2}{n_1} = n \tag{A.1}$$

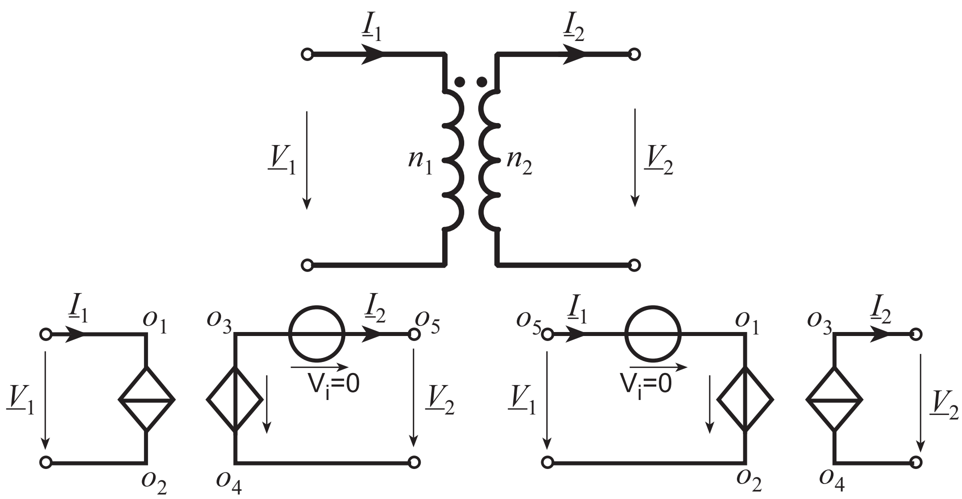


Figure A.1: Ideal Transformer with dot convention and two SPICE3 implementations; o's indicate the terminals used for SPICE lines

The SPICE3 model is adopted from [71]. It consists of two controlled sources i.e. voltage controlled voltage source ($VCVS$) and current controlled current source ($CCCS$) and an additional independent voltage source (V_i) in series with the VCVS. V_i is used for current sensing only and has zero voltage. One of the controlled sources is used on the primary and the other on the secondary side. This makes two possible implementations in SPICE3.

The symbols for VCVS, CCCS and independent voltage source in SPICE3 are E, F and V respectively [72]. The SPICE3 lines are as given as:

```
* n = n2/n1
* For First Type
F1   o1   o2   Vi   n
E2   o3   o4   o1   o2   n
Vi   o3   o5   DC   0    AC  0

* For Second Type
E1   o1   o2   o3   o4  1/n
F2   o4   o3   Vi   1/n
Vi   o5   o1   DC   0   AC  0
```

A.2 Realization of Impedance of form $\mathbf{Z}(s) = \mathbf{C}_r^Z z(s)$

Synthesis of an $N \times N$ impedance matrix, $\mathbf{Z}(s)$, which can be represented as a multiplication of frequency dependent *function*, $z(s)$, and a real matrix, $\mathbf{C}_r^Z$, is considered here. The real matrix $\mathbf{C}_r^Z$ is symmetric (due to *reciprocity* [14, Ch. 10] of circuit) and is of rank r where $1 \leq r \leq N$. The elements in the matrix would give the values of the turns ratios of the ideal transformers connecting network which connects impedance $z(s)$ with N-ports of the circuit.

Due to *spectral theorem*, any real symmetric matrix is diagonalizable by an orthogonal matrix [51, §5.5]:

$$\mathbf{C}_r^Z = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\mathbf{T} \tag{A.2}$$

where $\mathbf{Q}$ is an orthogonal matrix consisting of eigenvectors of $\mathbf{C}_r^Z$ and $\mathbf{\Lambda}$ is a diagonal matrix consisting of eigenvalues of $\mathbf{C}_r^Z$. We can write

$$\begin{aligned}\mathbf{\Lambda} &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r \end{pmatrix} \\ &= \sum_{i=1}^{r} \mathbf{\Lambda}_i \end{aligned} \tag{A.3}$$

where $\mathbf{\Lambda}_i$ is a diagonal matrix consisting of i^{th} non-zero eigenvalue λ_i. Combining Eqs. A.2 and A.3

$$\mathbf{Z}(s) = \mathbf{C}_r^Z z(s)$$

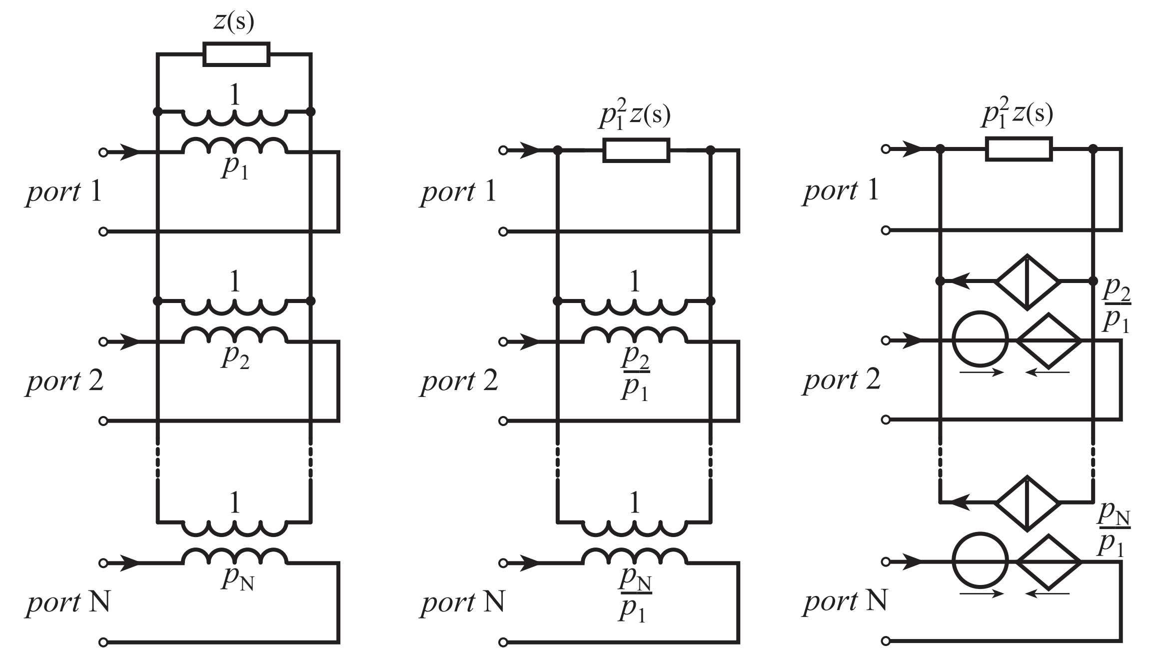


Figure A.2: Implementation of impedance matrix of form $\mathbf{C}_1^Z z(s)$: (a) & (b) shows two circuit realizations and (c) shows SPICE implementation

.

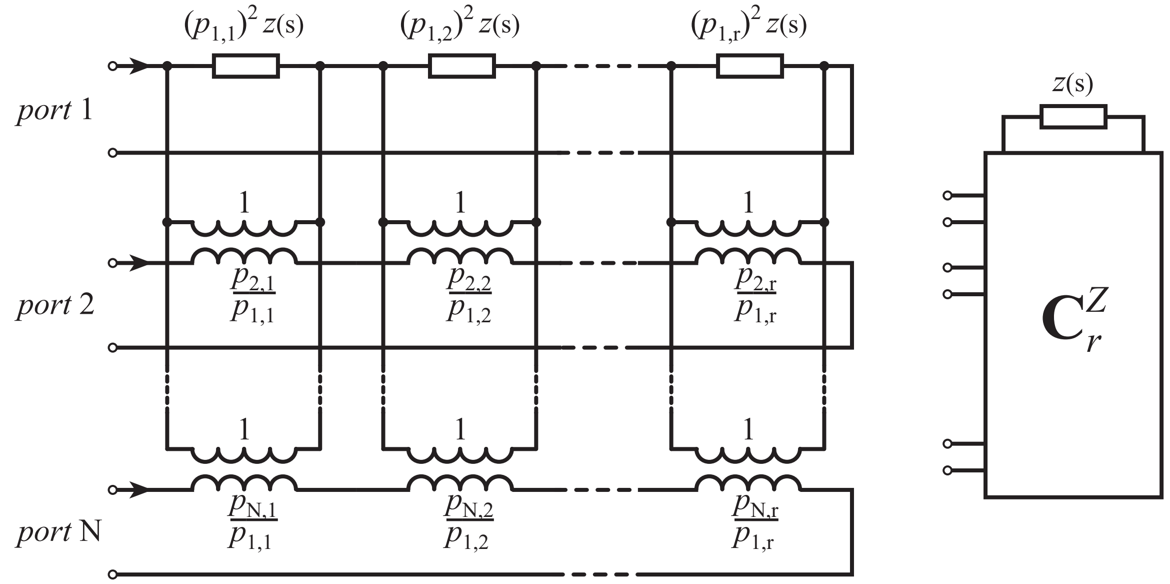


Figure A.3: Implementation of impedance matrix of form $\mathbf{C}_r^Z z(s)$, with its symbol. Note that connection matrix $\mathbf{C}_r^Z$ is of rank r.

$$
\begin{aligned}
&= \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathbf{T}} z(s) \; = \sum_{i=1}^{r} \mathbf{Q}\mathbf{\Lambda}_i \mathbf{Q}^{\mathbf{T}} z(s) \\
&= \sum_{i=1}^{r} \mathbf{C}^{Z}_{1,i} z(s) \qquad \text{(A.4)}
\end{aligned}
$$

The matrix $\mathbf{C}^Z_1$ is rank one matrix and can be written in following form [51, §2.4, p. 99]:

$$
\mathbf{C}^Z_1 = \mathbf{p}\mathbf{p}^T = \begin{pmatrix} p_1^2 & p_1p_2 & \cdots & p_1p_N \\ p_1p_2 & p_2^2 & \cdots & p_2p_N \\ \vdots & \vdots & \ddots & \vdots \\ p_1p_N & p_2p_N & \cdots & p_N^2 \end{pmatrix}, \qquad \text{(A.5)}
$$

where

$$
\boldsymbol{p} = [p_1, p_2, \cdots, p_N]^T. \qquad \text{(A.6)}
$$

It can be readily verify that the circuit for $\mathbf{C}^Z_1 z(s)$ is realized by the circuits given in Fig. A.2 and that $\boldsymbol{p}$ contains transformer turns ratios. Fig. A.2 also shows the SPICE3 realization. For complete matrix $\mathbf{C}^Z_r z(s)$, the realization is given in Fig. A.2.

A.3 Realization of Admittance of form $\mathbf{Y}(s) = \mathbf{C}^Y_r y(s)$

Analogous to n-port impedance as treated above, one can write for n-port admittance

$$
\begin{aligned}
\mathbf{Y}(s) &= \mathbf{C}^Y_r y(s) \\
&= \sum_{i=1}^{r} \mathbf{C}^Y_{1,i} y(s) \qquad \text{(A.7)}
\end{aligned}
$$

The circuit for $\mathbf{C}^Y_1 y(s)$ is Fig. A.3 and for $\mathbf{C}^Y_r y(s)$ it is Fig. A.4.

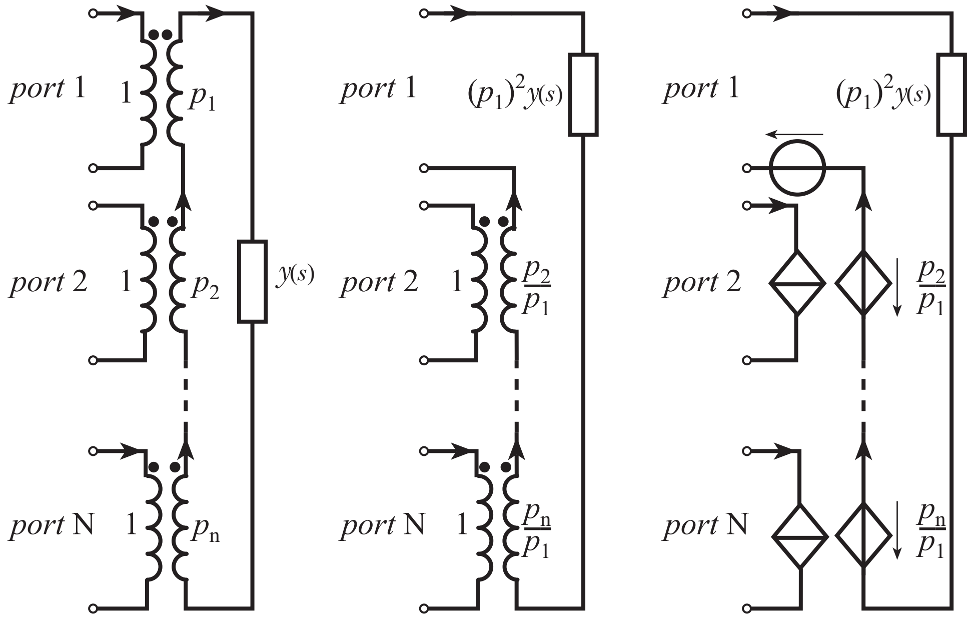


Figure A.4: Implementation of admittance matrix of form $\mathbf{C}_1^Y y(s)$: (a) & (b) shows two circuit realizations and (c) shows Spice implementation

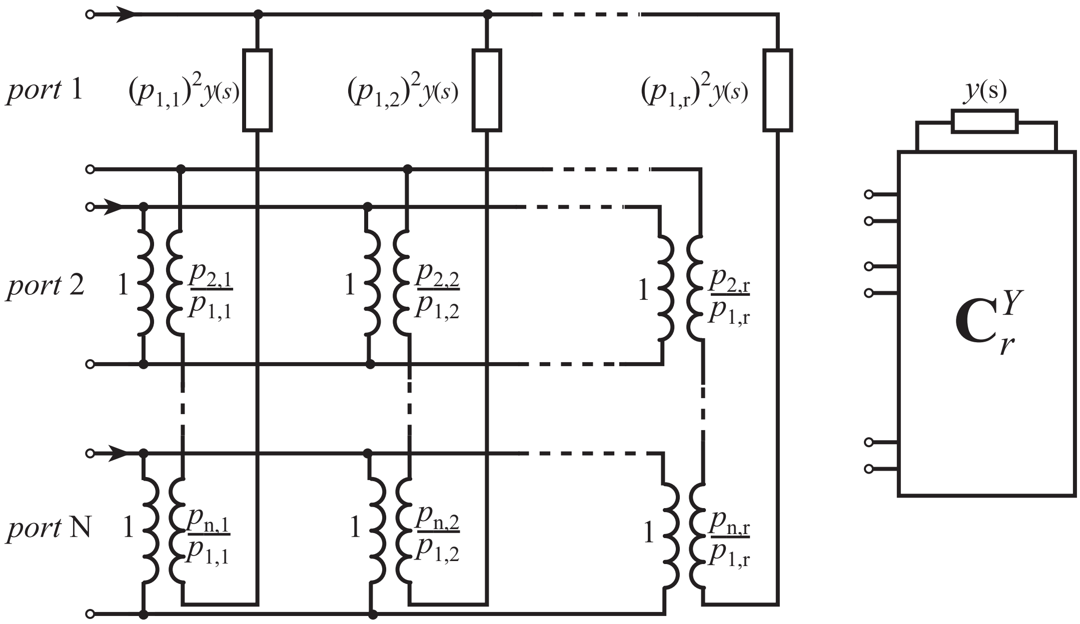


Figure A.5: Implementation of admittance matrix of form $\mathbf{C}_r^Y y(s)$, with its symbol. Note that connection matrix $\mathbf{C}_r^Y$ is of rank r.

B Proofs of Theorems

Theorem B.1. *Given a square matrix of complex numbers; if the nullity of its real and imaginary parts is one, then the complex matrix can only be rank deficient if and only if the null vector of the imaginary part is equal to the null vector of the real part or vice versa.*

Proof. Let $\mathbf{W}$ be a square matrix with complex number entries; an let $\mathbf{A}$ and $\mathbf{I}$ be its real and imaginary parts with nullity one and null vectors $\boldsymbol{\beta}_A$ and $\boldsymbol{\beta}_I$, respectively. Let

$$\boldsymbol{\beta}_A \neq \boldsymbol{\beta}_I \,,$$

then, because nullity is one for both matrices $\mathbf{A}$ and $\mathbf{I}$, $\boldsymbol{\beta}_I$ is not the null vector of $\mathbf{A}$ and $\boldsymbol{\beta}_A$ is not the null vector of $\mathbf{I}$ i.e.

$$\mathbf{I}\boldsymbol{\beta}_A \neq \mathbf{0} \,, \qquad \mathbf{A}\boldsymbol{\beta}_I \neq \mathbf{0} \,. \tag{B.1}$$

This leads to

$$\mathbf{W}\boldsymbol{\beta}_A = [\mathbf{A} + j\mathbf{I}]\boldsymbol{\beta}_A = j\mathbf{I}\boldsymbol{\beta}_A \neq \mathbf{0} \,, \tag{B.2}$$

$$\mathbf{W}\boldsymbol{\beta}_I = [\mathbf{A} + j\mathbf{I}]\boldsymbol{\beta}_I = \mathbf{A}\boldsymbol{\beta}_I \neq \mathbf{0} \,. \tag{B.3}$$

Neither $\boldsymbol{\beta}_A$ nor $\boldsymbol{\beta}_I$ is the null vector of $\mathbf{W}$ (i.e. $\mathbf{W}$ is full rank matrix). The only way they can be null vector of $\mathbf{W}$ is when

$$\boldsymbol{\beta}_A = \boldsymbol{\beta}_I = \boldsymbol{\beta} \,.$$

Conversely, a vector $\boldsymbol{\beta}$ has to make both matrices $\mathbf{A}$ and $\mathbf{I}$ equal to $\mathbf{0}$ to be the null vector of $\mathbf{W}$. □

Corollary B.2. *If the nullity of the real and imaginary parts is other than one and is not even equal, then the nullity of complex matrix is equal to the number of common null vectors between the real and imaginary parts.*

Corollary B.3. *The concept is extendable to matrices of quaternions and polynomials. Each part of the matrix must share the same null vector to make it the null vector of whole matrix.*

Theorem B.4. *Given a complex rational function of single complex variable s, with real coefficients; if a real number is subtracted from the function, then*

1. *on the real and imaginary axis of the complex variable, only the real part of the function would change and the imaginary part would remain unchanged.*

2. *both real and imaginary parts of the function would change on all the plane of the complex variable except at the real and imaginary axis of complex variable.*

Proof. Let the given function be $F(s)$ of a complex variable $s = \sigma + j\omega$, and let the subtracted real number be R then

$$F(s) - R = \frac{A}{B} - R = \frac{A - RB}{B} \tag{B.4}$$

where A and B are polynomials in s with real coefficients. We can separate the even and odd degree terms of numerator and denominator

$$F(s) - R = \frac{m_1 + n_1}{m_2 + n_2} = \frac{m_1 m_2 - n_1 n_2 + m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}\,, \tag{B.5}$$

where

$$\begin{aligned} m_1 &= A_e - RB_e\,, && \text{(even degree terms in numerator)} \\ n_1 &= A_o - RB_o\,, && \text{(odd degree terms in numerator)} \\ m_2 &= B_e\,, && \text{(even degree terms in denominator)} \\ n_2 &= B_o\,, && \text{(odd degree terms in denominator)} \end{aligned}$$

where A_e and B_e are even terms polynomials and A_o and B_o are odd terms polynomials, in polynomials A and B, respectively. From this we can calculate

$$m_1 m_2 - n_1 n_2 = A_e B_e - RB_e^2 + RB_o^2 - A_o B_o\,, \tag{B.6}$$

$$m_2 n_1 - m_1 n_2 = A_o B_e - A_e B_o\,. \tag{B.7}$$

Note that $(m_1 m_2 - n_1 n_2)$ contains only even degree terms and $(m_2 n_1 - m_1 n_2)$ contains only odd degree terms and is independent of R. The denominator $(m_2^2 - n_2^2)$ is contains only even degree terms and is independent of R.

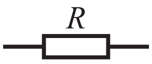
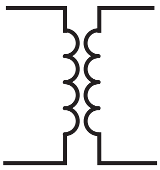
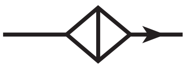
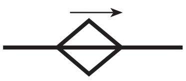
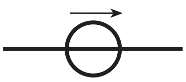
1. For $s = \sigma$ (real axis), the function $F(s)$ is real because of real coefficients, and it would change with value of R. There is no question of imaginary part. For $s = j\omega$ (imaginary axis), the polynomials $(m_1 m_2 - n_1 n_2)$ and $(m_2^2 - n_2^2)$ are real because of even degree terms and would form a the real part which is dependent on R; while the polynomial $(m_2 n_1 - m_1 n_2)$ would be pure imaginary due to odd degree terms and, with real denominator, would form imaginary part which would be independent of R.

2. For $s = \sigma + j\omega$ (whole plane except the axis), every term in $A - RB$ would be complex and both real and imaginary parts would be dependent on R. □

Corollary B.5. *Subtraction of a real matrix from the matrix of complex functions with real coefficients would change only real part of the matrix of functions on both axes of complex plane.*

List of Symbols and Abbreviations

$\mathbf{0}$	Matrix or vector having zero as all of its elements
s	Complex frequency
$\Re[.]$	Real part of
$\Im[.]$	Imaginary part of
$\sigma = \Re[s]$	Real part of complex frequency
$\omega = \Im[s]$	Imaginary part of complex frequency
P.R.	*P*ositive *R*eal
P.M.	*P*ositive *M*atrix
PRSM	*P*ositive *R*eal *S*ymmetric *M*atrix
PISM	*P*ositive *I*maginary *S*ymmetric *M*atrix
PRnSM	*P*ositive *R*eal *n*on-*S*ymmetric *M*atrix
PInSM	*P*ositive *P*maginary *n*on-*S*ymmetric *M*atrix
P.F.	*P*ositive *F*unction
$\mathbf{Z}$	Impedance matrix
$\mathbf{Y}$	Admittance matrix
$\mathbf{W}$, W	Symbol for positive matrix/ function; in case of PRSM/ P.R. function, represents both impedance $\mathbf{Z}$ and admittance $\mathbf{Y}$ matrices
$\mathbf{W}^T$, $\boldsymbol{\beta}^T$	Transpose of given matrix or vector
$\mathbf{W}^\dagger$, $\boldsymbol{\beta}^\dagger$	Hermitian (complex conjugate transpose)of given matrix or vector
$\mathbf{W}^{-1}$	Inverse of the given matrix
R	Resistance
$\mathbf{R}$	real symmetric matrix representing resistance network
G	Conductance
$\mathbf{G}$	real symmetric matrix representing conductance network
A	Common symbol for both resistance and conductance
$\mathbf{A}$,	real symmetric matrix representing resistance network, also used for real part of $\mathbf{W}$
$\mathbf{I}$,	Imaginary part of $\mathbf{W}$
$\forall$	for all
k, $\mathbf{K}^{\omega_p}$	residue or matrix of residues of the pole at ω_p
N	number of ports

$(O)(,)$	Degree or Order of
$\boldsymbol{\beta}$, $\boldsymbol{p}$, $\boldsymbol{n}$, $\boldsymbol{h}$	column vectors
$\boldsymbol{h}\boldsymbol{h}^T$	Matrix multiplication of matrices and/ or column vector
$\boldsymbol{p}.\boldsymbol{n}$	dot product of two vectors
$\mathbf{C}_r^W$, $\mathbf{C}_r^Z$, $\mathbf{C}_r^Y$	symmetric real matrix of rank r and type Z(impedance), Y(admittance) or W(common); it represents a transformer network connecting circuits to ports in particular configuration.
	Circuit symbol for Resistance
	Circuit symbol for Inductance
	Circuit symbol for Capacitance
	Circuit symbol for Transformer
VCVS	Voltage Controlled Voltage Source; the symbol used in Spice is E
CCCS	Current Controlled Current Source; the symbol used in Spice is F
	Circuit symbol for CCCS, arrow-head in the branch shows the direction of flow of current.
	Circuit symbol for VCVS, arrow-head is from positive to negative terminal.
	Circuit symbol for independent voltage source, arrow-head is from positive to negative terminal.

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