Suboptimal event-triggered control for networked control systems

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The advent of networked control systems urges the digital control design to incorporate communication constraints efficiently. In order to accommodate this requirement, this article studies the joint design of controller and event-trigger for linear stochastic systems in the presence of a resource-limited communication channel which exhibits packet dropouts and time-delay. The event-trigger situated at the sensor decides at every sampling instance, whether to send information over the communication channel to the controller. The design approach is formulated as a stochastic average-cost optimization problem, where the communication constraints are reflected as an additional cost penalty of the average transmission rate. Different conditions on the communication model are given where the joint optimal design can be split into a separate control and event-trigger design. Based on these results, two suboptimal design approaches are developed. By using drift criteria, stability guarantees of the closed-loop system for both approaches are derived in terms of bound moment stability. Numerical simulations illustrate the efficacy of the event-triggered approach compared with optimal time-triggered controllers.

1 Introduction

With the emergence of networked control systems, where sensors, controllers and actuators can be viewed as self-contained entities exchanging information over a common communication network, various benefits for the control system can be envisioned. These benefits comprising modularity, flexibility and ease of maintenance come at the price of various challenges imposed by the digital communication system that have to be tackled in the digital control design. Besides facing time-delays and packet dropouts introduced by packet-based transmission of data, the control design must take into account that the communication medium is a sparse resource that needs to be used efficiently. Common digital control design assumes that measurements are obtained periodically to update control inputs. However, various results in the literature show that substantial improvements are achieved, when replacing periodic sampling with event-triggered sampling schemes [2, 6, 17, 20]. The work in [2] shows that event-triggered control can reduce the state variance by a factor of 3 compared to a time-triggered periodic minimum variance controller with equal average transmission rate. This work is extended in [6,17] to multiple control loops sharing a common communication network, where it is shown that event-triggered scheduling outperforms periodic scheduling schemes with respect to minimizing an LQ cost of each control system. The work in [20] shows that an event-triggered task execution for control tasks can enlarge the maximum allowable transfer interval that guarantees global asymptotic stability compared to periodic task execution.

Despite of the fact that time-delays and packet-dropout are unavoidable in digital communication networks, these effects have merely been addressed by some works for the design of event-triggered control systems [5, 18, 22, 24]. On the other hand, the study of time-delay and packet-dropouts mainly focuses on periodic transmission schemes [8, 19] or on finding upper bounds on the sampling period [10]. In [18] the design of optimal level-triggered impulse control under packet dropouts with multiple loops sharing a common network is considered. The work in [5] derives the optimal event-based controller in the presence of packet dropouts and under a constraint on the number of transmissions. In [24], optimal event-based estimators are designed by imposing a penalty on sending updates over a communication network with a fixed transmission delay. In the framework of model predictive control, the work in [22] designs an event-triggered controller over a UDP-like communication channel with time-delay and packet dropouts. Rather than specifying the event-triggering rule for transmitting information to the controller, the results specify an upper bound on consecutive transmissions.

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The contribution of this article is to develop numerically efficient algorithms for the optimization-based design of event-triggered controllers for linear stochastic processes with time-delay and stochastic packet-dropouts in the feedback loop. The event-trigger situated at the sensor decides upon current observations, whether a state update shall be transmitted over a communication channel to the controller. The controller is situated at the actuator and adjusts the control inputs to stabilize the plant. In contrast to other work, we assume that the control input may not be constant but is allowed to vary between transmission times. Inspired by work in [24], the common design objective for the event-trigger and the controller is to minimize a long-term average-cost criterion comprised of a quadratic control cost and a communication cost penalizing transmissions over the digital communication system. The communication penalty reflects the awareness of the design approach with respect to the communication constraint.

The occurrence of both time-delay and packet-dropouts in the feedback loop requires an innovative design approach for the optimal design of event-triggered controllers, as the problem is generally very hard to solve. Because of the distributed information pattern evoked by the time-delay and the packet droputs in the feedback-loop, it can not be separated into tractable subproblems in contrast to the case of ideal communication systems as shown in [15]. This motivates us to identify different conditions for the communication model, where the separate design of event-trigger and controller leads to the optimal solution. For that reason a TCP-like communication system is assumed, i.e. the communication system is equipped with an acknowledgement channel that informs the event-trigger, whether a transmission has been successful. The acknowledgement channel is error-free, but may delay information.

It turns out that the separate design is possible, if either (i) the acknowledgement channel is delay-free or (ii) the feedback link is error-free or (iii) intervals between subsequent transmission times are restricted to be equal or greater than the round-trip time. Inspired by these conditions, we develop two suboptimal design approaches. In this article, the notion of *suboptimality* refers to optimal event-triggered controllers under certain assumptions in the design. The first approach called *waiting strategy* assumes that the event-trigger is idle for the duration of a round-trip time after transmitting information. This restriction enables an efficient solution algorithm separating control and event-trigger design into two tractable problems. The controller is given by a certainty equivalence controller and the event-trigger can be calculated by dynamic programming. The second approach is called *dropout estimation strategy* and assumes the controller to be a certainty equivalence controller. In contrast to the waiting strategy, there are no restrictions on the duration between subsequent transmissions. In such case, we show that the optimal event-trigger needs only to have finite memory, where the number of variables to be taken into account scales linearly with the round-trip time.

The notion of drift criteria that have been introduced in [13] to analyze asymptotic properties of Markov chains, turns out to offer appropriate mathematical tools to analyze closed-loop stability of the event-triggered system under time-delays and packet dropouts. Under these criteria, we are able to derive sufficient conditions to guarantee bounded moment stability for both design approaches.

Finally, numerical results indicate that the proposed suboptimal algorithms outperform standard time-triggered controller and approach a lower bound of the cost closely.

A preliminary version of this work dealing with finite horizon problems first appeared in the conference paper [14].

The remainder of this article is organized into four sections. Section 2 introduces the problem statement. Conditions for separating the underlying optimization problem into subproblems and the suboptimal design approaches are derived in section 3. Stability analysis of the closed-loop behavior is conducted in section 4. Section 5 validates numerically the efficacy of the suboptimal algorithm.

Notation. In this article, the operators $\mathrm{tr}[\cdot]$ and $(\cdot)^{\mathrm{T}}$ denote the trace and the transpose operator of a square matrix, respectively. The symbol '$\wedge$' denotes the logical AND-operation. The operator $\mathbb{1}_{(\cdot)}$ denotes the indicator function. The expectation operator is denoted by $\mathsf{E}[\cdot]$ and the conditional expectation is denoted by $\mathsf{E}[\cdot|\cdot]$. Sans-serif variables, e.g. x_k, indicate realizations of random variables. A sequence of a random process $\{x_k\}_k$ is denoted by $\boldsymbol{X}^k = [x_0, \ldots, x_k]$ for its complete history and $\boldsymbol{X}_l^k = [x_l, \ldots, x_k]$ for a specific time interval $\{l, \ldots, k\}$. If $l > k$, then $\boldsymbol{X}_l^k$ is an empty sequence.

2 Problem Statement

The system under consideration is illustrated in Fig. 1 and can be viewed as a control system with a resource-constrained feedback link. The resource-constrained communication channel $\mathcal{N}$ delays and drops information. The control system consists of a process $\mathcal{P}$, an event-trigger $\mathcal{E}$ and a controller $\mathcal{C}$. The stochastic discrete-time process $\mathcal{P}$ to be controlled is described by the following time-invariant difference equation

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times d}$. The variables, x_k and u_k denote the state and the control input. They are taking values in $\mathbb{R}^n$ and $\mathbb{R}^d$, respectively. The system noise w_k takes values in $\mathbb{R}^n$ and is an i.i.d. (independent identically distributed) zero-mean Gaussian distributed sequence with covariance matrix Σ_w. The initial state, x_0 is Gaussian with mean $\bar{x}_0$ and covariance Σ_{x_0}.

The event-trigger output $\delta_k \in \{0, 1\}$ is defined as follows.

$$\delta_k = \begin{cases} 1 & \text{update is sent} \\ 0 & \text{nothing transmitted} \end{cases}$$

The system model for the communication system is given by an erasure channel in the forward link. When $\delta_k = 1$, packet dropouts are modeled as a Bernoulli process $\{q_k\}_k$ defined as

$$q_k = \begin{cases} 1 & \text{update successfully transmitted} \\ 0 & \text{packet dropout occurred} \end{cases}$$

with packet dropout probability $\beta = \mathsf{P}[q_k = 0|\delta_k = 1]$ and q_k takes a value of 0, if $\delta_k = 0$. We assume a TCP-like communication protocol as introduced in [19] for networked control systems. The main feature of TCP-like communication protocols is that a binary acknowledgement is sent over the reverse link to the event-trigger, whenever a packet has been transmitted successfully. It is assumed that the reverse link is error-free. Most point-to-point protocols for wired connections fulfill this assumption. For example, the CAN-Bus protocol achieves such behavior by letting each transmitting node compare its priority with the other nodes that want to access the bus. Forward and reverse link delay packets by the duration of T_1 and T_2, respectively. Both, T_1 and T_2, are positive and non-negative integer values and are known apriori. If only upper bounds on these delays are known, a buffering approach can be used to obtain constant time-delays equal to the bounds and the subsequent analysis can still be carried out.

Let $(\Omega, \mathcal{A}, \mathsf{P})$ denote the probability space generated by the random variables x_0, $\{w_k\}_k$ and $\{q_k\}_k$. These variables are also called primitive random variables. System parameters and statistics are known to the event-trigger and controller. The event-trigger $\mathcal{E}$ situated at the sensor side has access to the complete observation history and decides, whether the controller $\mathcal{C}$ should receive an update.

If the event-trigger decides to update the controller, it transmits the current state over the erasure channel with delay T_1 to the controller. As we assume that the state measurements are noise-free, the received signal at the controller can be defined as

$$y_{k+T_1} = \begin{cases} x_k, & \delta_k = 1 \wedge q_k = 1 \\ \emptyset, & \text{otherwise} \end{cases} \tag{2}$$

with $y_0 = \cdots = y_{T_1-1} = \emptyset$. The admissible policies for the event-trigger and the controller at time k are defined as Borel-measurable functions of their past available data, i.e.,

$$\delta_k = \mathrm{f}_k(\mathcal{I}_k^{\mathcal{E}}), \qquad u_k = \gamma_k(\mathcal{I}_k^{\mathcal{C}}).$$

The observation history $\mathcal{I}_k^{\mathcal{E}}$ and $\mathcal{I}_k^{\mathcal{C}}$ of the event-trigger and controller, respectively, are defined as

$$\mathcal{I}_k^{\mathcal{E}} = \{\boldsymbol{X}^k, \boldsymbol{Q}^{k-T_1-T_2}\}, \quad \mathcal{I}_k^{\mathcal{C}} = \boldsymbol{Y}^k.$$

Let $\mathcal{U}$ be the set of all admissible policy pairs (f, γ), where the event-triggering policies is given by $\mathrm{f} = \{\mathrm{f}_1, \mathrm{f}_2, \ldots\}$ and control policies is given by $\gamma = \{\gamma_1, \gamma_2, \ldots\}$. The cost function is defined as

$$J(\mathrm{f}, \gamma) = \limsup_{N\to\infty} \frac{1}{N} \mathsf{E}\left[\sum_{k=0}^{N-1} x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k + \lambda \delta_k\right], \tag{3}$$

where the weighting matrix $Q \in \mathbb{R}^{n\times n}$ is positive semi-definite and $R \in \mathbb{R}^{d\times d}$ is positive definite. The positive factor λ can be regarded as the weight of penalizing information exchange between sensor and controller.

Then, the design objective is to find the pair $(\mathrm{f}, \gamma) \in \mathcal{U}$ that minimizes the long-term average-cost criterion J

$$\min_{(\mathrm{f},\gamma)\in\mathcal{U}} J(\mathrm{f}, \gamma), \tag{4}$$

where J is defined in (3). In the following, we assume that the pair (A, B) is controllable and the pair $(A, Q^{\frac{1}{2}})$ is observable with $Q = (Q^{\frac{1}{2}})^{\mathrm{T}} Q^{\frac{1}{2}}$.

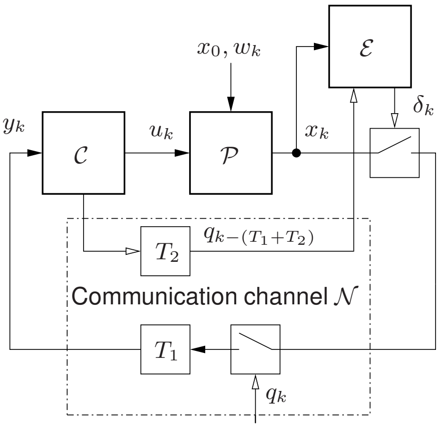

Fig. 1 System model of the networked control system with plant $\mathcal{P}$, event-trigger $\mathcal{E}$, controller $\mathcal{C}$ and a resource-constrained communication system $\mathcal{N}$. The forward link is an erasure channel and the reverse link carries the acknowledgement. Both links contain transmission delays T_1, T_2.

3 Design Approach

This section is consists of three main parts. First, conditions of the communication systems are derived that enable structural characterization of the optimal solution. These structural properties allow an efficient calculation of the optimal event-trigger. As the conditions for the communication system are generally not satisfied, we develop two different suboptimal design procedures, a waiting strategy and a dropout estimation strategy, which are discussed in the two remaining subsections.

3.1 Structural Properties

Finding the optimal solution by minimizing J given by (3) over all admissible pairs (f, γ) is a hard problem. This is mainly due to the fact that the event-trigger and the controller have different information available, which prevents the direct application of dynamic programming. It is well known that such optimization problems with so-called distributed information patterns generally lack of systematic solution algorithms [23]. However, in case of the absence of packet dropouts and time-delays, it has been shown in [15] for finite horizon problems that the optimal solution exhibits structural properties that allow an efficient design by separating the minimization into a set of subproblems.

This motivates us to ask whether there exist conditions for the communication system, where similar structural results hold as for ideal communication. In order to give the precise statement on such structural result, we introduce the following definition

Definition 3.1 (Dominating policies) A set of policy pairs $\mathcal{U}' \subset \mathcal{U}$ is called a *dominating class of policies* for the optimization defined by (4), if for any feasible $(\mathrm{f}, \gamma) \in \mathcal{U}$, there exists a feasible $(\mathrm{f}', \gamma') \in \mathcal{U}'$, such that

$$J(\mathrm{f}', \gamma') \leq J(\mathrm{f}, \gamma),$$

where J is the cost function defined by (3).

Once a dominating class of policies is obtained, Definition 3.1 implies that we can restrict the solutions of the optimization problem (4) to such policies. We will see that the set of policy pairs, where the controller is a certainty equivalence controller is a dominating class of policies, if the information pattern is nested. A certainty equivalence controller is given by the solution of the related deterministic control problem, where all primitive random variables are set to their means, and replacing the state variable by its least squares estimate within the deterministic solution. Therefore, the optimal solution splits into two problems, i.e. calculating the certainty equivalence controller and determining the optimal event-trigger. As already indicated, the key feature that enables such separation is given by the fact that the information pattern is nested. For the present problem, the nestedness property means that the information $\mathcal{I}_k^{\mathcal{C}}$ at the controller at time k is available at the event-trigger at time $k+1$. In other words, the σ-algebra $\sigma(\mathcal{I}_k^{\mathcal{C}})$ generated by $\mathcal{I}_k^{\mathcal{C}}$ is embedded in the σ-algebra $\sigma(\mathcal{I}_{k+1}^{\mathcal{E}})$ generated by $\mathcal{I}_{k+1}^{\mathcal{E}}$. This observation is formalized in the following lemma.

Lemma 3.2 *If the information pattern is nested, i.e.*

$$\sigma(\mathcal{I}_{k-1}^{\mathcal{C}}) \subset \sigma(\mathcal{I}_k^{\mathcal{E}}),$$

then the set $\mathcal{U}^{CE} = \{(\mathrm{f}, \gamma^{CE}) | \, \mathrm{f}$ *is an admissible event-triggering policy*$\}$ *is a dominating class of policies, where* γ^{CE} *is given by*

$$u_k = \gamma^{CE}(\mathcal{I}_k^{\mathcal{C}}) = -L \, \mathsf{E}[x_k | \mathcal{I}_k^{\mathcal{C}}]$$

with

$$
\begin{aligned}
&L=(B^{\mathrm T}PB+R)^{-1}B^{\mathrm T}PA,\\
&P=A^{\mathrm T}(P-PB(B^{\mathrm T}PB+R)^{-1}B^{\mathrm T}P)A+Q.
\end{aligned}
\tag{5}
$$

Proof. Assume we are given a pair $(\mathrm{f},\gamma)\in\mathcal{U}$ with finite costs J. We define a reparameterization (g,γ), where $\mathrm{g}=\{\mathrm{g}_1,\mathrm{g}_2,\ldots\}$ is the new event-trigger with g_k defined as a function of these primitive random variables $\{x_0,\mathbf{W}^{k-1},\mathbf{Q}^k-T_1-1\}$ such that

$$
\mathrm{g}_k(x_0,\mathbf{W}^{k-1},\mathbf{Q}^{k-T_1-1})=\mathrm{f}_k(\mathcal{I}_k^{\mathcal{E}}),\quad k\geq 0,\ \mathsf{P}\text{-a.s.}
\tag{6}
$$

when both systems are using a control law γ. As we assume that $\sigma(\mathcal{I}_{k-1}^{\mathcal{C}})\subset\sigma(\mathcal{I}_k^{\mathcal{E}})$ and γ is known, the control inputs $\mathbf{U}^{k-1}$ and dropouts $\mathbf{Q}^{k-T_1-1}$ can be fully recovered at the event-trigger. This implies also that $x_0,\mathbf{W}^{k-1}$ can be fully recovered from $\mathbf{X}^k$ and vice versa. Therefore, the event-triggering law f can always be replaced by g and vice versa assuming $\sigma(\mathcal{I}_{k-1}^{\mathcal{C}})\subset\sigma(\mathcal{I}_k^{\mathcal{E}})$. Notice that the pair (f,γ) and (g,γ) produce identical random variables u_k and δ_k almost surely by (6) and therefore yield the same cost J.

In the following, we fix the event-trigger g and find the optimal law γ^* minimizing J. Using an identity represented by lemma 6.1 of chapter 8 in [1], the cost can be rewritten as

$$
\begin{aligned}
J=\limsup_{N\to\infty}\frac{1}{N}\Bigg(&\mathsf{E}\left[\sum_{k=0}^{N-1}(u_k+Lx_k)^{\mathrm T}(B^{\mathrm T}PB+R)(u_k+Lx_k)\right]+\\
&+\mathsf{E}\left[x_0^{\mathrm T}Px_0+\sum_{k=0}^{N-1}w_k^{\mathrm T}P(Ax_k+Bu_k)+(Ax_k+Bu_k)^{\mathrm T}Pw_k+w_k^{\mathrm T}Pw_k+\lambda\delta_k\right]\Bigg)
\end{aligned}
$$

where L and P are given by (5). As w_k is zero-mean and statistically independent of x_k and u_k and as g is a function of primitive random variables, the second expectation is a constant independent of the control law chosen for any N and can therefore be omitted from the optimization. Let Δ_k be the estimation error defined by $\Delta_k=x_k-\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]$ and the matrix Γ be defined as $\Gamma=B^{\mathrm T}PB+R$. By replacing x_k by $\Delta_k+\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]$, we obtain

$$
\begin{aligned}
&\mathsf{E}\left[(u_k+Lx_k)^{\mathrm T}\Gamma_k(u_k+Lx_k)\right]=\\
&=\mathsf{E}\left[(u_k+L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}])^{\mathrm T}\Gamma(u_k+L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}])\right]+2\,\mathsf{E}\left[(u_k+L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}])^{\mathrm T}\Gamma L\Delta_k\right]+\mathsf{E}\left[\Delta_k^{\mathrm T}L^{\mathrm T}\Gamma L\Delta_k\right]
\end{aligned}
$$

Because u_k and $\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]$ are measurable functions with respect to $\mathcal{I}_k^{\mathcal{C}}$, the cross term vanishes as shown in the following.

$$
\begin{aligned}
\mathsf{E}\left[\mathsf{E}[(u_k+L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}])^{\mathrm T}\Gamma L\Delta_k|\mathcal{I}_k^{\mathcal{C}}]\right]&=\mathsf{E}\left[(u_k+L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}])^{\mathrm T}\Gamma L\,\mathsf{E}[\Delta_k|\mathcal{I}_k^{\mathcal{C}}]\right]\\
&=\mathsf{E}\left[(u_k+L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}])^{\mathrm T}\Gamma L\,\mathsf{E}[\Delta_k|\mathcal{I}_k^{\mathcal{C}}]\right]\\
&=\mathsf{E}\left[(u_k+L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}])^{\mathrm T}\Gamma L(\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]-\mathsf{E}[\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]|\mathcal{I}_k^{\mathcal{C}}]])\right]\\
&=0.
\end{aligned}
$$

Similarly to the proof of lemma 5.2.1 in [4] and [15], it can be shown that the estimation error Δ_k is a function of primitive random variables that is independent of the control law γ. Hence, the term $\mathsf{E}\left[\Delta_k^{\mathrm T}L^{\mathrm T}\Gamma L\Delta_k\right]$ is a constant that is independent of the control law γ. Therefore, we attain the minimum by choosing γ to be γ^{CE}.

Summarizing these results, we obtain

$$
J(\mathrm{f},\gamma)=J(\mathrm{g},\gamma)\geq\min_{\gamma}J(\mathrm{g},\gamma)=J(\mathrm{g},\gamma^{\mathrm{CE}})=J(\mathrm{f}',\gamma^{\mathrm{CE}}),
\tag{7}
$$

where f' satisfies (6). The statement in (7) states that for any given pair $(\mathrm{f},\gamma)\in\mathcal{U}$, we find another pair $(\mathrm{f}',\gamma^{\mathrm{CE}})$ where $J(\mathrm{f},\gamma)\geq J(\mathrm{f}',\gamma^{\mathrm{CE}})$. Therefore, we have shown that the set of solutions given by $(\mathrm{f}',\gamma^{\mathrm{CE}})$ is a dominating class of policies. This concludes the proof. □

Based on the above Lemma, we are able to identify conditions for the communication system, where the set of pairs $\mathcal{U}^{\mathrm{CE}}$ is a dominating class of policies. These conditions are given by the following three propositions.

Proposition 3.3 *Let $T_1\geq 0$ and $T_2\geq 1$. If the packet dropout probability β is 0, then $\mathcal{U}^{\mathrm{CE}}$ is a dominating class of policies.*

Proof. In case of $\beta = 0$, we observe that $\{q_k\}_k$ is deterministic and does not carry any information. As the data $\mathbf{Y}^{k-T_2-1}$ can be reconstructed by $\mathcal{I}_k^{\mathcal{E}}$ and equation (2), the information pattern is nested. By applying Lemma 3.2, the proof is complete. □

Proposition 3.4 *Let $\beta \in [0,1]$ and $T_1 \geq 0$. If the delay T_2 is equal to 1, then $\mathcal{U}^{CE}$ is a dominating class of policies.*

Proof. In case of $T_2 = 1$, we are able to reconstruct $\mathbf{Y}^{k-T_2-1}$ by equation (2), as the event-trigger has the data $\delta^k - 1$, $\mathbf{Q}^{k-T_2-1}$ and $\mathbf{X}^k$ are available at time k. Applying Lemma 3.2 concludes the proof. □

In order to formulate the next proposition, we define the number of unacknowledged packets in the communication system denoted by M_k as

$$M_k = \sum_{l=k-T_1+T_2}^{k-1} \delta_l$$

In TCP-like networks it is common to bound M_k by a so-called TCP window size [21]. The next proposition shows that setting the TCP window size to 1 enables separation.

Proposition 3.5 *Let $\beta \in [0,1]$, $T_1 \geq 0$ and $T_2 \geq 1$. If the communication system only allows one unacknowledged packet, i.e. $M_k \leq 1$ for all $k \geq 1$, then $\mathcal{U}^{\mathrm{CE}}$ is a dominating class of policies.*

Proof. Suppose the event-trigger decides to transmit information at time k_1, i.e. $\delta_{k_1} = 1$. The subsequent sequence of δ_k is predefined as $[\delta_{k_1}, \ldots, \delta_{k_1+T_1+T_2-1}] = [0, \ldots, 0]$. Therefore, no decision are taken at the event-trigger during this period. At time step $k_1 + T_1 + T_2$, the event-trigger may again decide to transmit information. But as $\delta_k = 0$ for $k \in \{k_1, \ldots, k_1+T_1+T_2-1\}$, the event-trigger knows the history of $\mathsf{Q}^{k_1+T_1+T_2-1}$ and is able to reconstruct $\mathsf{Y}^{k_1+T_1+T_2-1}$ by the information available at the event-trigger. In case no transmissions occurred prior to time k the same arguments hold, as $\delta_l = 0$ for $l \in \{0, \ldots, k-1\}$ By using Lemma 3.2, we find that $\mathcal{U}^{\mathrm{CE}}$ is a dominating class of policies. □

The results in Proposition 3.5 motivate us to propose a special class of event-triggered controllers, which is studied in the following section.

3.2 Waiting Strategy

The main idea of the waiting strategy is to wait for the acknowledgement before sending the next message to the controller. Setting the TCP window size to 1 enforces the event-trigger to wait for the length of a round trip time $T_1 + T_2$, before transmitting again information. The benefits of such approach are two-fold. Waiting for the acknowledgment before sending the next packet enhances predictability of the event-trigger for communication and diminishes the possibility of congestion in the communication network. Second, such restriction facilitates the solution of the optimization problem by reducing it to numerically tractable subproblems. Besides the structural property due to Proposition 3.5, it turns out that the optimal event-trigger is described by a decision function in $\mathbb{R}^n$ that can be found by value iteration.

Based on Proposition 3.5, we restrict our attention to find the optimal solution in $\mathcal{U}^{\mathrm{CE}}$ that satisfies $M_k \leq 1$ for all $k \geq 0$. Therefore, the remaining problem reduces to finding

$$\min_{\mathrm{f}} J^{\mathcal{E}}, \quad \text{s.t. } M_k \leq 1, \tag{8}$$

where

$$J^{\mathcal{E}} = \limsup_{N\to\infty} \frac{1}{N} \mathsf{E}\left[\sum_{k=0}^{N-1} (x_k - \hat{x}_k^{\mathcal{C}})^{\mathrm{T}} \Gamma (x_k - \hat{x}_k^{\mathcal{C}}) + \lambda \delta_k\right] \tag{9}$$

with $\hat{x}_k^{\mathcal{C}} = \mathsf{E}[x_k | \mathcal{I}_k^{\mathcal{C}}]$. In order to embed the constraint $M_k \leq 1$ into the system evolution, we define the following variable s_k representing the state of the communication network

$$s_{k+1} = \begin{cases} T_1 + T_2 - 1 & \delta_k = 1 \wedge s_k = 0 \\ s_k - 1 & \delta_k = 0 \wedge s_k > 0 \\ 0 & \delta_k = 0 \wedge s_k = 0 \end{cases} \tag{10}$$

with $s_0 = 0$ and the following modified interconnection relation which differs from Eq. (2)

$$y_{k+T_1} = \begin{cases} x_k, & \delta_k = 1 \wedge q_k = 1 \wedge s_k = 0, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{11}$$

Equation (11) implies that even if $\delta_k = 1$, the update will be blocked, when $s_k > 1$. This reflects exactly the behavior of the waiting strategy, as choosing $\delta_k = 1$ when $s_k > 0$, will have no effect on the system evolution, although the communication penalty λ is paid. Therefore, the optimal event-triggering law always selects $\delta_k = 0$ for $s_k > 0$.

Since the least-squares estimate $\hat{x}_k^{\mathcal{C}} = \mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]$ may depend explicitly on the choice of f, the optimization problem (8) can not be formulated in the framework of dynamic programming directly and remains hard to solve. The work in [11] and [9] show that for first-order systems and for random walks in arbitrary dimensions, respectively, the optimal event-trigger exhibits certain symmetry properties that imply that the least-squares estimator is independent of f and given by a linear predictor [11]. By assuming that the cost function is radially symmetric and that the the transition kernel is radially symmetric and radially non-increasing, the proof techniques in [9] can be used to show that the linear predictor is optimal also for higher-order systems. However, a rigorous proof in the general case for higher-order systems does not exist. Nevertheless, we assume in the following that this symmetry property is also present for any arbitrary higher-order system. Then, the least-squares estimate $\hat{x}_k^{\mathcal{C}} = \mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]$ is given by

$$\hat{x}_{k+T_1}^{\mathcal{C}} = \begin{cases} A^{T_1} x_k - \sum_{m=0}^{T_1-1} A^{T_1-m-1} BL\hat{x}_{k+m}^{\mathcal{C}} & \text{for } \delta_k = 1 \wedge\ q_k = 1 \wedge s_k = 0 \\ (A - BL)\hat{x}_{k+T_1-1}^{\mathcal{C}} & \text{otherwise} \end{cases} \tag{12}$$

with initial condition

$$\hat{x}_k^{\mathcal{C}} = (A - BL) \cdots (A - BL)\bar{x}_0,\ k = 0, \ldots, T_1 - 1. \tag{13}$$

Similar to [24], the optimization problem (8) can then be written as

$$\begin{aligned} &\min_{\mathrm{f}} \limsup_{N\to\infty} \frac{1}{N} \mathsf{E}\left[\sum_{k=0}^{N-1} (1 - \mathbb{1}_{\{s_k=0\}} q_k \delta_k) e_k^{\mathrm{T}} (A^{T_1})^{\mathrm{T}} \Gamma A^{T_1} e_k + \lambda\delta_k\right] \\ &e_{k+1} = \big(1 - \mathbb{1}_{\{s_k=0\}} q_k \delta_k\big) A e_k + w_k, \quad e_0 = x_0 - \bar{x}_0, \end{aligned} \tag{14}$$

where the evolution of s_k is given by (10). The variable e_k can be considered as the estimation of a one-step ahead prediction at the controller assuming a time-delay $T_1 = 0$. It is interesting to see that this variable is crucial for the event-trigger for arbitrary time-delay T_1. Another important property that is attributed to the waiting strategy is that the signal e_k can be calculated at the event-trigger, whenever $s_k = 0$. For $s_k \neq 0$, it is easy to see that $\delta_k = 0$. Therefore, the optimization problem can be viewed as an average cost problem with full state information $[e_k, s_k]$. Such problem can be solved numerically in the framework of dynamic programming by applying value iteration.

In summary, we have developed a numerically tractable algorithm for determining the optimal event-triggered controller. By restricting the optimal policies to a waiting strategy, the initial optimization problem with distributed information pattern reduces to the calculation of control gain L given by (5), a least-squares estimator defined by (12) and the solution of a dynamic program stated by (14).

3.3 Dropout Estimation Strategy

The waiting strategy discussed in the last section is certainly suboptimal, as there are circumstances, where another update should be sent, although the outstanding acknowledgement has not been received yet. For example, this would be the case, if a significant state disturbance is observed, while the event-trigger has to wait. This fact motivates us to relax the waiting strategy and allow update transmissions before an outstanding acknowledgement has been received.

In the following, we restrict our attention to policies in $\mathcal{U}^{\mathrm{CE}}$. In other words, the control law is assumed to be the certainty equivalence controller γ^{CE} given by Lemma 3.2. We assume a linear predictor as the suboptimal state estimator which is similar to (12) that does not take into account the event-triggering law, i.e.,

$$\hat{x}_{k+T_1}^{\mathcal{C}} = \begin{cases} A^{T_1} x_k - \sum_{m=0}^{T_1-1} A^{T_1-m-1} BL\hat{x}_{k+m}^{\mathcal{C}} & \text{for } \delta_k = 1 \wedge\ q_k = 1, \\ (A - BL)\hat{x}_{k+T_1-1}^{\mathcal{C}} & \text{otherwise} \end{cases} \tag{15}$$

with initial conditions given by Eq. (13). The above state estimator differs from (12) merely by its independence of s_k.

Based on the least squares estimator (15), finding the optimal event-trigger is the solution of the following optimization problem

$$\begin{aligned}&\min_{\mathrm{f}} \limsup_{N\to\infty} \frac{1}{N}\,\mathsf{E}\left[\sum_{k=0}^{N-1}(1-q_k\delta_k)e_k^{\mathrm{T}}(A^{T_1})^{\mathrm{T}}\Gamma A^{T_1}e_k+\lambda\delta_k\right]\\ &e_{k+1}=\big(1-q_k\delta_k\big)Ae_k+w_k,\quad e_0=x_0-\bar{x}_0.\end{aligned}\tag{16}$$

Although, the above optimization problem differs only slightly from the dynamic program for solving the optimal waiting strategy given by (14), there is a major burden in solving problem (16), as the variable e_k is generally not perfectly known at the event-trigger.

Remark 3.6 It should be noted that the cases of no packet dropouts, i.e. $\beta = 0$, or one-step delayed acknowledgement channel, i.e. $T_2 = 1$, constitute special situations, where e_k can be fully recovered at the event-trigger. Due to Proposition 3.3 and 3.4, respectively, the optimal event-triggered controller $(\mathrm{f}^*, \gamma^*)$is given by $\gamma^* = \gamma^{\mathrm{CE}}$ with a state estimator given by (15) and f^* is the solution of the dynamic program stated in (16).

Problems with partial state information can be restated as problems with perfect state information by considering the information available as the current state as described in chapter 5 in [4]. As the information state $\mathcal{I}_k^{\mathcal{E}}$ increases with time, such approach is not suitable for the above infinite horizon problem. Due to this fact, we need to reduce the data in $\mathcal{I}_k^{\mathcal{E}}$ to its essential quantities, which are known as sufficient statistics. The main feature of a sufficient statistic $\mathcal{I}_k^{\mathrm{red}}$ is that the optimal policy f^* can be rewritten as

$$\mathrm{f}_k^*(\mathcal{I}_k^{\mathcal{E}}) = \tilde{\mathrm{f}}_k(\mathcal{I}_k^{\mathrm{red}}).$$

A well known sufficient statistic is given by the conditional distribution $\mathsf{P}_{e_k|\mathcal{I}_k^{\mathcal{E}}}$ for the above problem with partial state information, see also section 5.4 in [4].

The subsequent paragraph is concerned with the calculation of $\mathsf{P}_{e_k|\mathcal{I}_k^{\mathcal{E}}}$. Such problem can be posed in the framework of optimal filtering of discrete-time Markov jump linear systems, which is studied in [7], with unknown discrete mode q_k and observation $\mathsf{P}_{e_k|\mathcal{I}_k^{\mathcal{E}}}$. In this case it is well known that the optimal nonlinear filter can be described by a bank of Kalman filters, which requires exponentially increasing memory and computation with time [3]. However, the event-trigger receives a delayed version of the discrete mode q_k through the acknowledgement channel. Based on results by [12], such additional information leads to nonlinear filters that require merely finite memory and their computation has a polynomial complexity that does not increase over time. This fact is stated formally in the subsequent lemma.

Lemma 3.7 *A sufficient statistics for the optimal event-trigger* f^* *solving the optimization problem (16) is given by the information*

$$\mathcal{I}_k^{red} = \{e_{k-T_1-T_2+1}, \mathbf{X}_{k-T_1-T_2+1}^k, \delta_{k-T_1-T_2+1}^k\},$$

where $\mathcal{I}_k^{red} = \{e_0, \mathbf{X}^k\}$ *for* $k < T_1 + T_2$*. The* σ*-algebra of* $\mathcal{I}_k^{red}$ *is a subset of* $\sigma(\mathcal{I}_k^{\mathcal{E}})$*.*

Proof. The initial value of e_k is given by $x_0 - \bar{x}_0$ and is therefore known by the event-trigger, i.e. $\sigma(\mathcal{I}_k^{\mathrm{red}}) \subset \sigma(\mathcal{I}_k^{\mathcal{E}})$ for $k \le T_1 + T_2 - 1$. On the other hand, $\mathcal{I}_k^{\mathrm{red}} = \{e_0, \mathbf{X}^k\}$ is the complete information available at the event-trigger at time $k \le T_1 + T_2 - 1$. This implies $\sigma(\mathcal{I}_k^{\mathrm{red}}) = \sigma(\mathcal{I}_k^{\mathcal{E}})$ for $k \le T_1 + T_2 - 1$.
For $k \ge T_1 + T_2$, the event-trigger obtains additional information $\mathbf{Q}^{k-T_1+T_2}$ that enables us to determine $e_{k-T_1-T_2+1}$ from the difference equation of e_k defined in the optimization problem (16), where the noise sequence $\mathbf{W}^{k-T_1-T_2}$ can be recovered from knowledge of $\mathbf{X}^{k-T_1-T_2+1}$ and the control inputs $\mathbf{U}^{k-T_1-T_2}$.
As the sequence δ^{k-1} is known at the event-trigger, the state e_k conditioned on δ^{k-1} has the Markov property , i.e. given the current state e_k and the sequence δ^{k-1}, the future evolution of e_k is statistically independent of past observations. Therefore, we obtain

$$\mathsf{P}_{e_k|\mathcal{I}_k^{\mathrm{red}}} = \mathsf{P}_{e_k|\mathcal{I}_k^{\mathcal{E}}}\tag{17}$$

As $\mathsf{P}_{e_k|\mathcal{I}_k^{\mathcal{E}}}$ is in general a sufficient statistics for problems with partial state information, equation (17) implies that $\mathcal{I}_k^{\mathrm{red}}$ is also a sufficient statistics. This completes the proof. □

The resulting event-trigger is called dropout estimation strategy as it internally estimates the unknown discrete modes that have not been acknowledged at the event-trigger in order to determine the conditional distribution of e_k. This suggests that $\mathsf{P}_{e_k|\mathcal{I}_k^{\mathcal{E}}}$ is given by a finite set of point masses increasing with T_2. For the calculation of the conditional distribution, we define the discrete mode i_k at time step k as

$$i_k = q_k \delta_k = \begin{cases} 1, & \text{successful state update} \\ 0, & \text{no update} \end{cases}$$

We also define the probability matrix of i_k conditioned on δ_k as

$$\mathcal{T} = \left[\begin{array}{cc} 1 & 0 \\ 1-\beta & \beta \end{array} \right]$$

which satisfies

$$\mathsf{P}[i_k = j | \delta_k = l] = \mathcal{T}_{l+1,j+1}, \quad j,l \in \{0,1\}.$$

In the following, we assume that $T_1 = 1$ for illustrative purposes, but the main principles for computing $\mathsf{P}_{e_k|\mathcal{I}_k^{\mathrm{red}}}$ also carry over to arbitrary forward delays T_1. We further define $\hat{x}_k^{\mathcal{C},i_{k-T_2}^{k-1}}$ as the state estimate at time k given the sequence $i_{k-T_2}^{k-1}$ with initial condition $\hat{x}_{k-T_2}^{\mathcal{C}}$. The term $\hat{x}_k^{\mathcal{C},i_{k-T_2}^{k-1}}$ can be calculated recursively by Equation (15). With the estimator

$$\hat{x}_{k+1}^{\mathcal{C}} = g(\hat{x}_k^{\mathcal{C}}, x_k, i_k) = \begin{cases} Ax_k - BL\hat{x}_k^{\mathcal{C}} & \text{for } i_k = 1 \\ (A - BL)\hat{x}_k^{\mathcal{C}} & \text{for } i_k = 0 \end{cases}$$

this yields for $T_1 = 1$

$$\hat{x}_k^{\mathcal{C},i_{k-T_2}^{k-1}} = g(\cdot, x_{k-1}, i_{k-1}) \circ \cdots \circ g(\hat{x}_{k-T_2}^{\mathcal{C}}, x_{k-T_2}, i_{k-T_2})$$

According to [12], we then have

$$\mathsf{P}_{i_{k-T_2}^{k-1}|\mathcal{I}_k^{\mathrm{red}}} = \frac{\alpha(i_{k-T_2}^{k-1}, \mathcal{I}_k^{\mathrm{red}})}{\sum_{i_{k-T_2}^{k-1}} \alpha(i_{k-T_2}^{k-1}, \mathcal{I}_k^{\mathrm{red}})}$$

where $\sum_{i_{k-T_2}^{k-1}}$ denotes the sum over all possible permutations of $i_{k-T_2}^{k-1}$, i.e. $\sum_{i_{k-T_2}^{k-1}} = \sum_{i_{k-T_2}=0}^{1} \cdots \sum_{i_{k-1}=0}^{1}$ and

$$\alpha(i_{k-T_2}^{k-1}, \mathcal{I}_k^{\mathrm{red}}) = \prod_{t=0}^{T_2-1} \mathcal{T}_{\delta_{k-t-1}+1, i_{k-t-1}+1} p_{k-t}(x_{k-t})$$

where $p_{k-t}(\cdot)$ is probability density function of the conditional probability distribution of x_{k-t} given $\{x_{k-t-1}, e_{k-T_2}, i_{k-T_2}^{k-t-1}\}$, which is described by the multivariate normal distribution $\mathcal{N}(Ax_{k-t-1} - BL\hat{x}_{k-t-1}^{\mathcal{C},i_{k-T_2}^{k-t-2}}, \Sigma_w)$. Finally, we obtain $\mathsf{P}_{e_k|\mathcal{I}_k^{\mathrm{red}}}$ by computing the points

$$e_k = x_k - \hat{x}_k^{\mathcal{C},i_{k-T_2}^{k-1}}$$

which have a probability $\mathsf{P}_{i_{k-T_2}^{k-1}|\mathcal{I}_k^{\mathrm{red}}}$. The number of point masses can be reduced by taking into account that $\alpha(i_{k-T_2}^{k-1}, \mathcal{I}_k^{\mathrm{red}})$ is zero whenever $i_{k-t-1} = 1$ and $\delta_{k-t-1} = 0$, which corresponds to $\mathcal{T}_{1,2} = 0$. Hence, the number of point masses depends on the number of transmission attempts during a round trip time. In case no transmissions occurred during this period, we recover the waiting strategy as there is no ambiguity for e_k with probability one.

Remark 3.8 In contrast to the initial problem stated by (16), the information state $\mathcal{I}_k^{\mathrm{red}}$ does not increase in time, but is bounded by the round-trip time $T_1 + T_2$. Therefore, finding the optimal event-trigger f is feasible for practical implementation in the case of infinite horizon problems with a moderate round-trip time.

4 Stability Analysis

This section is concerned with the asymptotic behavior of the resulting closed-loop system. The notion of stability is defined in terms of bounded moment stability. The dynamical system described by a time-homogeneous Markov chain is said to be bounded moment stable, if it is ergodic with a stationary distribution whose second-order moment is bounded.

In the following, we assume that the event-trigger being of the type of a waiting strategy is stationary, i.e. $\mathrm{f}_k = \mathrm{f}^{\mathrm{S}}$, and satisfies

$$\mathrm{f}^{\mathrm{S}}(e_k, s_k) = 1 \text{ for } \|e_k\|_2 > \bar{c}, s_k = 0. \tag{18}$$

This assumption does not put severe restrictions on the design, as $\bar{c}$ may be chosen arbitrarily large. When Γ is not positive definite, but only non-negative definite, optimal solutions of (14) generally violate assumption (18) for any $\bar{c}$ as is shown in [16]. But as $\bar{c}$ can be arbitrarily large, there always exists an ϵ-optimal event-triggering law taking assumption (18) into account. The following lemma gives a sufficient condition for bounded moment stability of the process e_k, when using the waiting strategy. This result will also enable implications for the asymptotic behavior of the state x_k for both the waiting strategy and the dropout estimation strategy.

Lemma 4.1 *Let the event-trigger be of the type of a waiting strategy satisfying constraint (18). If the packet dropout probability β and the round-trip time $T_1 + T_2$ satisfy*

$$\beta < \frac{1}{\|A\|_2^{2(T_1+T_2)}}, \tag{19}$$

then the process e_k evolving by the difference equation defined in (14) is bounded moment stable.

P r o o f. Based on results in chapter 14 in [13], we use Lyapunov-like drift criteria to show bounded moment stability of the process e_k. The drift operator is defined as

$$\Delta h(e_k, s_k) = \mathsf{E}[h(e_{k+1}, s_{k+1})|e_k, s_k] - h(e_k, s_k), \quad e_k \in \mathbb{R}^n, s_k \in \{0, 1, \ldots, T_1 + T_2 - 1\},$$

where h is a real-valued function of the state (e_k, s_k). Let us fix an arbitrary stationary event-trigger $\mathrm{f} = [\mathrm{f}^{\mathrm{S}}, \mathrm{f}^{\mathrm{S}}, \ldots]$ which satisfies assumption (18). Then, it is straight forward to prove that the drift of quadratic functions of $\{e_k, s_k\}$ inside a compact set $\mathcal{O}$. is bounded. Thus, it suffices to consider the set of states $\{e_k, s_k\}$ outside of the compact set $\mathcal{O}$ [13]. As the compact set can be chosen arbitrarily large, we assume that

$$\{\{e_k, s_k\} | \|e_k\|_2 \leq M, s_k \in \{0, 1, \ldots, T_1 + T_2 - 1\}\} \subset \mathcal{O}.$$

By taking assumption (18) into account, the evolution of e_k for $\|e_k\|_2 > \bar{c}$ is given by

$$e_{k+T_1+T_2} = (1 - q_k) A^{T_1+T_2} e_k + \sum_{l=k}^{k+T_1+T_2-1} A^{k+T_1+T_2-1-l} w_l, \quad k \in \{m + l(T_1 + T_2) | l = 0, 1, \ldots\}. \tag{20}$$

where m is the first exit-time of $\mathcal{O}$. It is clear that the underlying sampled Markov chain evolving by (20) is ψ-irreducible and aperiodic. As s_k does not appear in (20) anymore, it can be omitted in the following. According to theorem 14.01 in [13], in order to have bounded moment stability, we need to ensure that

$$\Delta h(e_k) \leq -\epsilon \|e_k\|_2^2, \quad e_k \in \mathbb{R}^n \backslash \bar{\mathcal{O}}, \tag{21}$$

where $\epsilon > 0$ and $\bar{\mathcal{O}} \supset \{e_k | \|e_k\|_2 \leq \bar{c}\}$ is compact. Let us take $h(e_k) = \|e_k\|_2^2$. The drift $\Delta h(e_k)$ can be written as

$$\Delta h(e_k) = \mathsf{E}[\|(1 - q_k) A e_k + w_k\|_2^2 | e_k] - \|e_k\|_2^2$$

Due to mutual statistical independence of w_k, q_k and e_k and the fact that w_k is i.i.d. and zero-mean with covariance matrix Σ_w, the drift term can be written as

$$\Delta h(e_k) = \mathsf{E}[1 - q_k] \|A^{T_1+T_2} e_k\|_2^2 + \mathrm{tr}[\sum_{l=k}^{k+T_1+T_2-1} (A^{k+T_1+T_2-1-l})^{\mathrm{T}} A^{k+T_1+T_2-1-l} \Sigma_w] - \|e_k\|_2^2. \tag{22}$$

The term $\mathsf{E}[1-q_k]$ is the average packet drop probability given by β. On the other hand, we have $\|A^{T_1+T_2}e_k\|_2 \leq \|A\|_2^{T_1+T_2}\|e_k\|$. Therefore, the drift is bounded by

$$\Delta h(e_k) \leq (\beta\|A\|_2^{2(T_1+T_2)} - 1)\|e_k\|_2^2 + c_1,$$

where c_1 is a constant summarizing the trace-term in (22). Condition (19) ensures that we can find appropriate ϵ and $\bar{\mathcal{O}}$, such that the drift criteria given by (21) is satisfied. This completes our proof. $\square$

Bounded moment stability of e_k implies that the cost J defined by 3 is finite, as we assumed that (A, B) to be controllable and the pair $(A, Q^{\frac{1}{2}})$ to be observable. This in turn means that the process state x_k is bounded moment stable, when the condition (19) in Lemma 4.1 is satisfied.

Remark 4.2 As the class of event-triggers described by the waiting strategy is a subset of the collection of event-triggers described by the dropout estimation strategy, the solution of the optimization problem (16) will not be worse than the solution of (14) for identical system parameters. Therefore, we can conclude that whenever condition (19) is satisfied, stability is guaranteed for the closed-loop system resulting from the optimal solution in (16).

5 Numerical Simulations

Suppose a scalar process $\mathcal{P}$ defined by (1) with $A = 1$, $B = 1$ and variances $\Sigma_w = 1$, $\Sigma_{x_0} = 1$ and mean $\bar{x}_0 = 0$. The cost function J is defined by (3), where Q and R are selected apriori to $Q = 1$ and $R = 10$. Subsequently, we analyze the performance with respect to diverse settings of the communication network. We consider three different packet dropout probabilities $\beta \in \{0,\ 0.25,\ 0.5\}$, a forward delay $T_1 = 1$ and two different reverse delays $T_2 \in \{1, 2\}$. The proposed suboptimal algorithms are compared with the optimal time-triggered controller. The time-triggered strategy does not need an acknowledgement channel and is therefore independent of T_2. The optimal transmission timings of the time-triggered controller are calculated by the deterministic dynamic programming algorithm. In addition, a lower bound on the cost is determined by assuming that packet dropouts are instantaneously known at the event-trigger after transmission. Besides implying that $\mathcal{U}^{\mathrm{CE}}$ is a dominating class of policies due to Lemma 3.2, the variable e_k defined in (16) is known at the event-trigger at time k. Therefore, the solution of optimization problem (16) can be solved directly by dynamic programming, which will then yield a lower bound. It should be noticed that this bound is not tight for non-zero packet dropout probability β and $T_1 + T_2 \geq 2$. It can not be attained by any event-triggered controller, as it imposes that the event-trigger guesses q_k at any time step $k+1$ correctly with probability 1.

For all event-triggered controllers it turns out that γ is a certainty equivalence controller given in Lemma 3.2 with a linear control gain $L = 0.27$. The weighting Γ in (9) is given by $\Gamma = 1$. The least-squares estimator is given by (12) for the waiting strategy and by (15) for all other approaches. According to Lemma 4.1, we attain bounded moment stability for any packet dropout probability β smaller than 1.

The difference in cost between the approaches is reflected in $J^{\mathcal{E}}$ defined by (9). The objective function $J^{\mathcal{E}}$ consists of a weighted mean squared error $\mathrm{MSE}(\Gamma)$ given by

$$\mathrm{MSE}(\Gamma) = \limsup_{N\to\infty} \frac{1}{N}\,\mathsf{E}\left[\sum_{k=0}^{N-1}(x_k - \hat{x}_k^{\mathcal{C}})^{\mathrm{T}}\Gamma(x_k - \hat{x}_k^{\mathcal{C}})\right]$$

and the weighted average transmission rate λr, where r is defined as

$$r = \limsup_{N\to\infty} \frac{1}{N}\,\mathsf{E}\left[\sum_{k=0}^{N-1}\delta_k\right].$$

A comparative study is illustrated in Fig. 2 for different qualities of the communication system, where we are interested in the trade-off between the weighted mean squared error and the average transmission rate. Fig. 2 draws the achievable cost pairs $[\mathrm{MSE}(\Gamma), r]$ for varying communication penalty λ, where cost pairs with r close to 0 corresponds to large λ and pairs with a transmission rate r close to 1 corresponds to a vanishing λ.

In all depicted scenarios in Fig. 2, the dropout estimation strategy outperforms the optimal time-triggered scheme and approaches the lower bound very closely. For the case of no packet dropouts, i.e. $\beta = 0$, the dropout estimation strategy is equal to the lower bound, as both optimal solutions coincide, because q_k is deterministic in this case. The waiting strategy also outperforms the optimal time-triggered scheme and deviates only slightly from the lower bound for low transmission rates. Fig. 2 also reveals the natural upper bound on the transmission rate r given by $\frac{1}{T_1+T_2}$. Evidently, the dropout estimation strategy shows better performance than the waiting strategy at the cost of additional computations due to the filtering procedure.

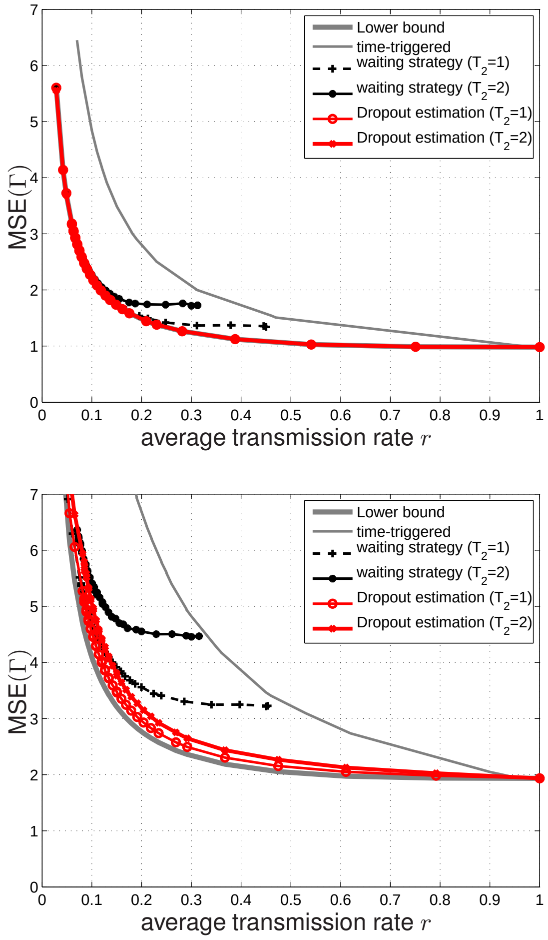

Fig. 2 Performance comparison: without packet dropouts **(top, left)**; for packet dropout probability $\beta = 0.25$ **(top, right)**; for packet dropout probability $\beta = 0.5$ **(bottom, left)**.

6 Conclusion

Due to its distributed information structure, the optimal design of event-triggered control is a challenging problem in the field of networked control systems. This article addresses this problem for controlling linear stochastic systems in the presence of time-delays and packet-dropouts within the feedback loop. By using an acknowledgement channel, conditions for the communication system can be identified that enable a structural characterization of the optimal event-triggered controller. The structural properties allow an efficient optimal design of the event-triggered controller. These conditions do not hold for a general communication system, but restrictions on the communication protocol may recover these conditions.

Despite of these restrictions, which facilitate the design procedure significantly, numerical simulations indicate that the suboptimal procedures outperform time-triggered control systems, while marginally deviate from a lower bound on the system performance.

For future investigations, it is of interest to analyze the proposed algorithms in a setting with multiple control loops sharing a common communication network, where time-delays are varying and packet-dropouts have more complicated statistical models.

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