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Sequential R-Vine Copula Selection Using Different Weights.

Diplomarbeit

von

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Ich erkläre hiermit, dass ich die Diplomarbeit selbständig und nur mit den angegebenen Hilfsmitteln angefertigt habe.

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Contents

1 Introduction

The concept of R-Vine modeling has proven itself as very flexible way to model the pairwise dependence especially in high dimensions. There are many investigations that have been conducted such in (Brechmann and Czado 2011) and (Brechmann and Czado 2011). We base on the pair-copula constructions (PCC) from (Aas, Czado, Frigessi, and Bakken 2009) with different copula families and a very convenient graphical representation of an R-Vine model developed by (Bedford and Cooke 2002). An important issue however is that when selecting an R-Vine structure the number of possible R-Vines is growing rapidly with the dimension. For n variables there are $2 \cdot \binom{n-2}{2} \cdot \frac{n!}{2}$ possible R-Vines. Since, it is important to capture the most dependence in the early trees which can allow later truncation of the R-Vine (see (Brechmann 2010) for the concept of truncation and simplification) and significantly reduce computational effort.
So far, Kendall′s τ has been widely used as a measure of dependence for the construction of the first tree. This method has proven itself a good way to select high dependent variables among all possible pairs of variables. In this thesis we want to explore the possibility of constructing the trees using another measure of dependence. Now, we want to examine three alternative choices to Kendall′s τ namely tail cumulation, Hu dependence and exceedance dependence, proposed and analyzed by (Brechmann 2010). While Kendall′s τ models overall dependence, those alternative measures aim to model symmetrical or asymmetrical tail dependence. This can be a very desirable property in modeling financial data, since those show distinct joint behavior. After, the first tree is constructed by MST algorithm of Prim that maximizes the resulting sum of all weights. The construction of the R-Vine model is conducted sequentially, i.e. with estimating parameters and selection of an appropriate pair copula in each step (for more details see (Dissmann 2010)). For the selection of an appropriate pair-copula variable pair we chose smallest AIC, proposed by (Genest and Rémillard 2008) As a possible choice of a bivariate copula family we pick Gaussian, Student t, Gumbel copulas, as well as Gumbel rotated by $90°, 180°$ and $270°$ to model positive and negative asymmetrical dependence.
An extensive simulation study using different underlying scenarios was conducted to observe what properties of data would consider using a certain weight as a better fit.
Next we apply our methods on exchange rates ((Schepsmeier 2010)) to the US Dollar to evaluate the difference in the resulting models. Since this particular data set is rather of a small size (it contains 9 variables) we also investigate how close each of the five considered pairwise dependence measures is reflected in every of the resulting R-Vines. We evaluate each variable pair with respect to dependence measure coefficient and the tree of the R-Vine specification it occurs. Furthermore, we

analyze the dependence structure of the 16-dimensional data set of international financial indices ((Dissmann 2010)) with respect to different asset classes and a 30-dimensional data set of German DAX to observe the different tree structures resulting from different methods.

We investigated the resulting R-Vine models for each of those weights by comparing such criteria as the log likelihoods, Akaike and Bayesian Information Criteria ((Akaike 1973), (Schwarz 1978)). We also concentrate on the Vuong test comparison ((Vuong 1989),(Clarke 2003). This particular test helps us decide if one of the competing models can fit the data better than the other or if two models are considered to perform equally well.

This thesis is organized as follows. In Chapter 2 we summarize the mathematical background that will be needed, such as theory of copulas, where we mainly concentrate on the previously chosen copula families. We also replicate the basic graph theory that is necessary in later R-Vine construction in Chapter 3. Next we concentrate on the dependence measures especially we investigate the alternative dependence measures in more detail in Chapter 4. Chapter 5 and 6 are dedicated to sequential model selection principal and model comparison criteria. In following Chapter 7 the results of the simulation study are summarized. In Chapter 8 we finally apply methods on different data sets. We examine the data set of exchange rates and international financial indices. After, we have a look at 30-dimensional data set of German DAX ((Brechmann 2012) returns . In Section 8.4, we also investigate a slightly modified Hu dependence coefficient as a weight. Conclusions are provided in Chapter 9.

2 Mathematical Background

To be able to conduct the extensive simulation study and establish the best strategy for choosing the weights, we will provide the necessary mathematical background. We introduce the main concept of copulas and their properties and also look at the few certain families more closely. We also define measures of dependence with will be needed for the construction of the R-Vine tree and have a first impression of possible selection of weights. Finally, we see how bivariate copulas are used for constructing multivariate copula models and enable high-dimensional constructing.
First, we remind of a few facts about multivariate distributions, before going over to the copula definition. Since this subject is widely discussed in the respective literature (see for example (Embrechts, Lindskog, and McNeil.A 2003), (Nelsen 2006), (Sklar 1959), (Genest and Favre 2007) and others) we do not go into much detail. For all proofs we refer to the corresponding papers.

2.1 Multivariate Distributions

Definition 2.1. Multidimensional distribution

Let $X = (X_1, ..., X_n)$ be a random vector. The cumulative distribution function (cdf) of X is defined by

$$F_{X_1,...,X_n}(\boldsymbol{x}) = P(X \leq \boldsymbol{x}) = P(X_1 \leq x_1, ..., X_n \leq x_n).$$

For continuous X and F p-times differentiable, there exists a non-negative probability density function $f_{X_1,...,X_n}$, which results from using the joint distribution for two random variables :

$$f_{X_1,...,X_n}(x_1, ...x_n) = f_{X_1}(x_1) \cdot f_{X_2|X_1}(x_2 \mid x_1) \cdot ... \cdot f_{X_n|X_1,...X_{n-1}(x_n|x_1,...,x_{n-1})} \quad (2.1)$$

Hence, the joint cumulative distribution function is

$$F_{X_1,...,X_n}(\boldsymbol{x}) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f_{X_1,...,X_n}(u_1, ..., u_n) du_1 \cdots du_n,$$

where it holds $\int_{-\infty}^{\infty} f(u)du = 1$.
Accordingly, the expectation of a vector of random elements has to be understood component by component.

Definition 2.2. (Elliptical distribution)
An n-dimensional vector of random variables $X = (X_1, ..., X_n)$ is said to have an elliptical distribution with parameters μ, Σ and ϕ, where $\mu \in \mathbb{R}^n$, Σ a non-negative definite symmetric $n \times n$ matrix and some function $\phi : [0, \infty) \to \mathbb{R}$, when it holds:

$$\varphi_{X-\mu}(y) = \phi(y^T \Sigma y)$$

for the characteristic function $\varphi_{X-\mu}$ of $X - \mu$. The density function is given by

$$f_X(\boldsymbol{x}) = \lambda \mid \Sigma \mid^{-\frac{1}{2}} \phi\left((\boldsymbol{x}-\mu)^T \Sigma^{-\frac{1}{2}} (\boldsymbol{x}-\mu)\right) \tag{2.2}$$

with a constant λ and some suitable function ϕ, called generator function.

As it is known, if $X \sim Ep(\mu, \Sigma, \phi)$ and $E(X)$ exists, then $E(X) = \mu$. Moreover, if the second order moments exist Σ is up to a constant the covariance matrix of X, i.e., $Cov(X) = \Sigma$. In that case the constant λ equals to -2$\cdot\phi'(0) = -1$, where ϕ' stands for the derivative of ϕ. By setting $\lambda = (2\pi)^{-\frac{n}{2}}$, and the function $\phi(y) = exp(-\frac{1}{2}y)$ for some $y > 0$, (2.2) yields the multivariate normal density function. Since it is known that the characteristic function is given by $\varphi_X(y) = exp(iy^T\mu - \frac{1}{2}y^T\Sigma y)$, so that setting the generator $\phi(y) = exp(-\frac{1}{2}y)$ automatically gives $\phi'(0) = -1$.
Another famous example of elliptical distribution which we are also will be using in later simulations is the multivariate Student t distribution. Let ν denote the corresponding degrees of freedom. Using $\lambda = (\pi\nu)^{-\frac{n}{2}} \cdot \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$ and $\phi(y) = \left(1 + \frac{y}{\nu}\right)^{-(\nu+n)/2}$ for some positive y , the corresponding density function is defined as follows:

$$f_X(\mathbf{x}) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{(\pi\nu)^{n/2}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \mathbf{x}^T\mathbf{x}/\nu\right)^{-(\nu+n)/2}$$

Note that the components of X_i are uncorrelated, but not independent, and their marginal distribution is a univariate t distribution with parameter ν. The smaller ν is, the heavier are the tails of the distribution. The very heavy-tailed distribution with $\nu = 1$ is called the multivariate Cauchy distribution. The multivariate normal distribution is obtained as a limit case as $\nu \to 1$. Let $t_{\Sigma,\nu}$ denote the cdf of multivariate Student t distribution.

2.2 Copulas

A copula is a special distribution function, which characterizes the dependence between random variables. The crucial advantage of this particular function approach is that it enables one to investigate the dependence structure independent from marginal distributions, hence, it is a more convenient and elegant way to couple the margins of the variables to their joint distribution. This basically means the margins do not have to be selected from the same (parametrical) distribution family as the joint distribution. To generate a multivariate joint distribution we can select different marginal distributions for each margin and different copulas. We provide now details about this approach.

Let us consider a vector of random variables $X = (X_1, ...X_n)$ with $F_1(x_1), ..., F_n(x_n)$ the corresponding continuous marginal distribution functions. Let $f(x_1, ..., x_n)$ denote their joint density function and $F(x_1, ..., x_n)$ the cumulative distribution function. Next convert those to random variables having a uniform distribution:$(U_1, ..., U_n) = (F_1(X_1), ..., F_n(X_n))$

Definition 2.3. (Copula)
An n-dimensional copula $C : [0, 1]^n \to [0, 1]$ *is a multivariate cumulative distribution function of* $(U_1, ..., U_n)$ *:*

$$C(u_1, ..., u_n) = P(U_1 \leq u_1, ..., U_n \leq u_n) \tag{2.3}$$

and contains all information on the dependence structure between the components of $X = (X_1, ..., X_n)$*. The corresponding survival copula is defined as:*

$$\overline{C}(u_1, ..., u_n) = P(U_1 > u_1, ..., U_n > u_n) \tag{2.4}$$

There are two very important properties of copula function we will discuss. The first important property is so called $Sklar's\ theorem$(see (Nelsen 2006)), that reveals the suitability of the copula functions for dependency modeling and also provides the existence of the unique copula under relatively weak assumptions.

Theorem 2.1. *Sklar′s Theorem:*
Consider the n-dimensional random variables mentioned earlier with the margins $F_1, ..F_n$ *being continuous. there exist a unique copula*

$$C : [0, 1]^n \to [0, 1]$$

that satisfies:

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)) \tag{2.5}$$

and conversely

$$C(u_1, ...u_n) = F(F_1^{-1}(u_1), ..., F_n^{-1}(u_n)), \tag{2.6}$$

where $F_i^{-1}(u_i)$, for $i = 1, ...n$ are the quantile functions.
To obtain this result Sklar mainly uses the two following properties:

- *for $U \sim U(0,1)$ and G an univariate cumulative distribution function with G^{-1} its inverse cdf holds : $G^{-1}(U) \sim G$*

- *is G continuous and $X \sim G$, so is $G(X) \sim U(0,1)$, i.e. if $X \sim F$ with continuous joint cdf so is $(F_1(X_1), .., F_n(X_n)) \sim C$.In case $U \sim C$ holds $F^{-1}(U) \sim F$.*

In this way a copula describes a certain unique unit of stochastic (in-)dependence among a set of random variables.
Another property mentioned by (Nelsen 2006) is that a copula function can be estimated by the *Fréchet-Hoeffding upper* and *lower bounds*.

Theorem 2.2. *For $u_1, u_2 \in [0,1]$ define:*

$$C^-(u_1, u_2) := max\{u_1 + u_2 - 1, 0\} \text{ and } C^+(u_2, u_2) =: min\{u_1, u_2\}$$

For bivariate copulas we have $C^-(u_1, u_2) \leq C(u_1, u_2) \leq C^+(u_2, u_2)$.
In the n-variate case is $C^+(u_1, ..., u_n) = min\{u_1, ..., u_n\}$ but no longer a copula.

As the next step we are looking at the copula density, which can be obtained by a partially differentiation of the copula, when all derivatives exist.
Let $C(u_1, .., u_n)$ be a copula cdf, then copula density is defined as:

$$c_{1...n}(u_1, ..., u_n) = \frac{\partial^n C(u_1, ..., u_n)}{\partial u_1 \cdots \partial u_n} \tag{2.7}$$

Now if we know $F(x_1, ...x_n) = C(F_1(x_1), ..., F_n(x_n))$ has continuous univariate margins $F_1, ..., F_n$ with respective marginal densities f_i for $i = 1, ...n$. If C has all derivatives of order n, the density corresponding to F is given:

$$f(x_1, ...x_n) = c_{1...n}(F_1(x_1), ..., F_n(x_n)) \times \prod_{i=i}^{n} f_i(x_i) \tag{2.8}$$

For example, if we look at the independence copula $\Pi^n(u_1, ..., u_n) = u_1 \cdots u_n$. Its joint distributions computes:

$$F(x_1, ..., x_n) = \Pi^n(F_1(x_y), ..., F_n(x_n)) = F_1(x_1) \cdots F_n(x_n) \tag{2.9}$$

and the density :

$$\pi^n(F_1(x_1), ..., F_n(x_n)) = \frac{\partial^n \Pi^n(F_1(x_1), ..., F_n(x_n))}{\partial F_1(x_1) \cdots \partial F_n(x_n)} = 1$$

(see (Brechmann 2010)). For our further analysis we would need to introduce and discuss two important groups of copula families used for the modeling the most. That is the elliptical and Archimedean copulas.

2.3 Elliptical Copulas

The elliptical copula family includes all copulas that result from the multivariate elliptical distribution (see 2.1). Examples are the Normal or Student t distributions. The most famous example of this class of copula families is the Gauss-Copula. This copula corresponds to the multivariate Normal distribution.

Gaussian copula

For the Gaussian copula with the distribution function of the multivariate normal distribution Φ_Σ with zero means, unit variances and positive definite and symmetric correlation matrix Σ following holds.

Derived from the Sklar′s theorem the multivariate Gaussian copula is given as:

$$C_\Sigma^G(u_1, ..., u_n) = \Phi_\Sigma(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n))$$

With $x - i = \Phi^{-1}(u_i)$ for $i = 1, ..., n$. Using the definition of elliptical copula it leads to another more implied definition:

$$C_\Sigma^G = \frac{1}{(2\pi)^{\frac{n}{2}}} | \Sigma |^{\frac{1}{2}} \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} exp\left(-\frac{1}{2}\boldsymbol{x}^T \Sigma^{-1} \boldsymbol{x}\right) \mathrm{d}x_1 ... \mathrm{d}x_n$$

which leads to the corresponding density

$$c(u_1, ..., u_n) = | \Sigma |^{-\frac{1}{2}} \; exp\left(1/2 \cdot \boldsymbol{x}'(\boldsymbol{I}_n - \Sigma^{-1})\boldsymbol{x}\right)$$

Student t copula

Another elliptical copula is derived from the multivariate Student t distribution. Analogously to the normal distributed copula this copula type is derived from univariate t-distributed margins and under Sklar$'$s theorem. Let us denote the correlation matrix Σ as before and the ν stand for the number of degrees of freedom. With $t_{\Sigma,\nu}$ the cdf of multivariate Student t distribution and t_ν being the univariate standard Student t distribution$'$ cdf, define:

$$C^t(u_1, ..., u_n) = t_{\Sigma,\nu}(t_\nu^{-1}(u_1), ..., t_\nu^{-1}(u_n))$$

Analog to the Gauss normal distributed copula, the t-copula is derived as:

$$C^t_{\Sigma,\nu} = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{(\pi\nu)^{\frac{n}{2}} \cdot \Gamma\left(\frac{\nu}{2}\right) \mid \Sigma \mid^{\frac{1}{2}}} \cdot \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_n)} \left(1 + \frac{1}{\nu}\boldsymbol{x}^T\Sigma^{-1}\boldsymbol{x}\right)^{-(\frac{\nu+n}{2})} \mathrm{d}x_1...\mathrm{d}x_n$$

Note that for $\nu \to \infty$ the defined t-copula converges to the Gaussian copula.
The corresponding density((Joe 1997)) is given by:

$$c_{\Sigma,\nu,}(\boldsymbol{x}) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{(\pi\nu)^{n/2}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \boldsymbol{x}^T\Sigma^{-1}\boldsymbol{x}/\nu\right)^{-(\nu+n)/2} \mid \Sigma \mid^{-1/2} .$$

However, the elliptical copulas class is mainly described by the property to model the symmetrical tail dependence (see in the section Dependence measures). Due to the famous asymmetries in upper in lower dependence in the financial data it is very useful to be able to capture the extreme events while working with real financial data.
While the two copulas of the elliptical family we have discuss are only able to measure the symmetric dependence (Normal copula independence and t-copula the symmetric dependence) of the joint distribution, the next copula class, the Archimedean copulas, serve the purpose to model the asymmetric behavior in the tails.

2.4 Archimedean Copulas:

To introduce Archimedean copulas we refer to (Joe 1997) and (Nelsen 2006).
Let $\phi : [0, 1] \to [0, \infty]$ define a strict decreasing *generator* function with the following properties:

- ϕ convex
- $\phi(0) = \infty$ and $\phi(1) = 0$
- ϕ^{-1} completely monotonic inverse of the generator ϕ

Archimedean copula in dimension n is defined:

$$C_\phi(u_1, ..., u_n) = \phi^{-1}\left(\sum_{i=1}^{n} \phi(u_i)\right)$$

To derive the corresponding density of Archimedean copulas first we define the Laplace transform(see (Joe 1997)) of Laplace-Stieltjes transform for a non-negative random variable. Is Y a non-negative random variable, then

$$\psi_Y(t) = E[e^{-tY}] = \int_0^\infty e^{-ty} dF_Y(y) \text{ , for some } t \geq 0$$

is the Laplace transform of Y.
Using this tool the definition of the Archimedean copula can be rewritten as:

$$C(u_1, ..., u_n) = \phi^{-1}\left(\sum_{i=1}^{n} \phi(u_1)\right) = \psi\left(\sum_{i=1}^{n} \psi^{-1}(u_i)\right)$$

To be able to derive a density of an Archimedean copula there is one more condition that is needed to be fulfilled. In order for all the mixed derivatives till order n to be non-negative the derivatives of ψ need to change sign till order n as $(-1)^j \psi^{(j)} \geq 0$, for $j = 1, ..., n$ and $(\psi^{-1})' \leq 0$. Now finally the density is :

$$c_\psi(u_1, ..., u_n) = \frac{\partial^n C_\psi(u_1, ..., u_n)}{\partial u_1 \cdots \partial u_n} = \psi^n\left(\sum_{i=1}^{n} \psi^{-1}(u_1)\right) \cdot \prod_{i=1}^{n} (\psi^{-1})'(u_i)$$

The best examples for this copula class are the Gumbel-, Clayton- and the Frank-copulas. Gumbel copula function is especially convenient due to the ability of the bivariate copulas of that family to be nested in each other for modeling in higher dimensions. Table 2.1 offers an overview with the respective generator and copula functions of these examples.

Note that Gumbel copula operates only in the upper tail dependence aria (positive dependence, when large (or small for lower) values occur together) for the parameter $\theta \in [1, \infty)$. For the $\theta = 1$ the Gumbel copula models the stochastic independence of the univariate distributions. Clayton copula is defined for all $\theta > 0$,

	Generator Function	Copula
Gumbel	$\phi(t) = (-\log t)^{\theta}$	$C_{\theta}^{Gu}(u_1, ..., u_n) = \left(exp\left(\sum_{i=1}^{n} -(ln)^{\theta}\right)^{\frac{1}{\theta}}\right)$
Clayton	$\phi(t) = 1/\theta \cdot (t^{-\theta} - 1)$	$C_{\theta}^{Cl}(u_1, ..., u_n) = (u_1^{-\theta} + \cdots u_n^{-\theta} - n + 1)^{-\frac{1}{\theta}}$
Frank	$\phi(t) = -log\left(\frac{e^{-t\theta}-1}{e^{-\theta}-1}\right)$	$C_{\theta}^{F}(u_1, ..., u_n) = -\frac{1}{\theta} ln\left(1 + \frac{\prod_{i=1}^{n}(exp(-\theta u_i)-1)}{(exp(-\theta)-1)^{n-1}}\right)$

Table 2.1: Generator function and copula function of Gumbel, Clayton and Frank copula families.

for $\theta \to 0$ however one gets $C = \Pi$. For the Frank copula it holds $\theta \neq 0$ and $n \geq 2$ For our further simulation studies we however are going to concentrate on three copula families, namely Gaussian, Student t and Gumbel.

2.5 Dependence Measures

In this section we want to investigate the bivariate dependence of random variables which would allow us to explain and measure the dependencies between large number of random variables. It is important to measure these dependence adequately for the future purpose of the model construction (extensively discussed in (Dissmann 2010)). We first discuss the most common measure of dependence that is *Spearmann′* ρ and *Pearsons correlation*. We are going to have a little closer look at the *Kendall′s* τ as it is related to the simulation study. In this Chapter we mainly follow (Kurowicka and Cooke 2006) and (Nelsen 2006).

Pearson correlation

Pearson correlation also *product moment correlation* is the most common measure of the linear correlation of two random variables. For two variables X, Y we consider $E(X), E(Y)$ expectations and $Var(X), Var(Y)$ variances to be finite. The Pearson correlation coefficient is given by

$$\rho(X,Y) = \frac{E(XY) - E(X) \cdot E(Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

This measure is able to fully specify the dependence between the normal distributed random variables, whenever the requirement of finite means and variances is met. (Kurowicka and Cooke 2006) prove that the Pearson correlation is

not invariant under non-linear strictly increasing transformations. The possible values also depend on marginal distributions of X and Y. Another measures of dependence, such as Kendall′s τ and estimated Spearman ′s ρ, avoid the above disadvantages, so-called measures of association, depend on ranks.
To derive the appropriate estimate of pearson correlation we consider a sample of given pairs of observations $(x_i, y_i), i = 1, ..., N$. Then the estimated ρ is given by:

$$\hat{\rho}(X,Y) = \frac{\sum_{i=1}^{N}(x_i - \overline{X})(y_i - \overline{Y})}{\sqrt{\sum_{i=1}^{N}(x_i - \overline{X})^2}\sqrt{\sum_{i=1}^{N}(y_i - \overline{Y})^2}},$$

where $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$. The values of Pearson correlation lie in $[-1, 1]$. In case X and Y are independent, $\rho(X,Y) = 0$. The reversal however does not hold.

Spearman′s ρ

Spearman′s ρ or the *rank correlation* for two random variables X and Y with cdf ′s F_X and F_Y is defined as:

$$\rho^r(X,Y) = \rho(F_X(X), F_Y(X)).$$

Based on that notification (Kurowicka and Cooke 2006) define the population version of rank correlation as:

$$\rho^r(X,Y) = 3(P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)),$$

for two independent vectors (X_1, Y_2) and (X_2, Y_2). Let the distribution function of (X_1, Y_1) be denoted by F_{XY} with the marginal distributions F_X , F_Y . X_2, Y_2 are independent with distributions F_X and F_Y. The Spearman rank correlation can also be expressed in terms of copula. Because the X_2, Y_2 are independent their joint distribution function equals $F_X \cdot F_Y$ and their copula is the independence copula Π. Let $u = F_X$ and $v = F_Y$ be observations from the uniform random variables $U = F(X)$ and $V = F(Y)$. If the copula of (X_1, Y_1) is denoted by C then the Spearman′s ρ modifies to

$$\rho^r(X,Y) = 12\int\int_{[0,1]^2} C(u,v)dudv - 3$$

As U and V both in fact have mean 1/2 and variance 1/12 the previous formula yields:

$$\rho^r(X,Y) = 12E(UV) - 3 = \frac{E(UV) - E(U)E(V)}{\sqrt{Var(U)}\sqrt{Var(Y)}}$$

and the Spearman′s ρ is the identical to Pearson correlation of ranks, i.e. is a correlation of random variables transformed into uniform random variables with values in $[-1, 1]$ (see (Nelsen 2006)).

Let (x_i, y_i) be the N given pairs of observations of the vector (X, Y). Let R_i^x denote the rank of the corresponding x_i and R_i^y the rank of the corresponding y_i for each $i = 1, ..., N$. The empirical version of Spearman′s ρ is given by:

$$\hat{\rho}^r(X,Y) = \frac{\sum_{i=1}^{N}(R_i^x - \overline{R^x})((R_i^y - \overline{R^y})}{\sqrt{\sum_{i=1}^{N}(R_i^x - \overline{R^x})^2}\sqrt{\sum_{i=1}^{N}(R_i^y - \overline{R^y})^2}},$$

where $\overline{R^x} = \frac{1}{N}\sum_{i=1}^{N} R_i^x$ and $\overline{R^y} = \frac{1}{N}\sum_{i=1}^{N} R_i^y$.

Kendall′s τ

The most common measure of association is the *Kendall′s* τ (Kendall 1938)). Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed copies of (X, Y). Then *Kendall′s* τ is defined as:

$$\tau(X,Y) = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$$

In case X, Y are independent, $\tau(X, Y) = 0$. In terms of copulas the Kendall′s τ is expressed as follows:

$$\tau(X,Y) = 4\int\int_{[0,1]^2} C(u,v)dC(u,v) - 1$$

for continuous X, Y with $u = F_X$ and $v = F_Y$. This measure is always well defined, invariant under continuous increasing transformations. It is also independent of the margins which makes it a good choice for later modeling. Figures 2.1 to 2.4 present pair and contour plots of the mainly discussed copula families for different values of τ.
For the empirical version of Kendall′s τ (Nelsen 2006) defines *concordant* and *discordant* pair of variables. Two continuous random variables X and Y are said to form a concordant pair if large values of one are associated with large values of the other one. By the discordance are the large values of one random variable associated with small values of the other respectively. For (X, Y) and their pairs of observation (x_i, y_i) and (x_j, y_j) this means

- $x_i < x_j$ and $y_i < y_j$ for concordant pairs

- $x_i > x_j$ and $y_i < y_j$ for discordant pairs

or alternatively

- $(x_i - x_j)(y_i - y_j) > 0$ for concordant

- $(x_i - x_j)(y_i - y_j) < 0$ for discordant

Suppose, we have a sample of N pairs of observations. Let CP denote the number of concordant pairs, DP the number of discordant pairs and $T_x(T_y)$ the number of tied pairs among all pairs, respectively. Kendall′s Tau can be estimated from an underlying data set by:

$$\hat{\tau}(X,Y) = \frac{CP - DP}{\sqrt{CP + DP + T_x}\sqrt{CP + DP + T_y}}$$

For continuous random variables (X, Y) the estimated Kendall′s $\hat{\tau}(X, Y)$ can be expressed as:

$$\hat{\tau}(X,Y) = \frac{CP - DP}{\binom{n}{2}}$$

Note that asymmetric behavior in the tails will not be detected through this measure of dependence.

Tail Dependence

Like the dependence measures discussed earlier the measure of *Tail Dependence* reflects the dependence between the random variables. The difference lies in the fact that measures of association operate on the whole space $[0, 1]^2$. Tail dependence measures the dependencies in the upper-right or/and the lower-upper quadrant of that space, so that large values of one variable leads to large values of the other, and small values occur with small values of the two variables respectively. To define *upper and lower tail dependence* of two random variables X and Y we denote F_X, F_Y to be marginal distribution functions of the variables. The concept of tail dependence is used for the purpose of measuring dependence that arises from random variables in the presence of extreme events, i.e. how likely it is for one risk variable to take an extreme value, given that another risk variable takes an extreme value. The tail dependence measures in this way co-movements in the lower and the upper tail dependence of bivariate distribution respectively. The most common definition for upper tail dependence coefficient is presented by (Joe 1997).

Definition 2.4. (Tail dependence) *The upper tail dependence parameter* λ_U *of two random variables* X *and* Y *with marginal distributions* F_X *and* F_Y *respectively is defined as*

$$\lambda_U = \lim_{u \nearrow 1} P\left(Y > F_Y^{-1}(u) \mid X > F_X^{-1}(u)\right)$$

presumed the limit exists.
In the similar way the lower tail dependence can be defined as

$$\lambda_L = \lim_{u \searrow 0} P\left(Y \leq F_Y^{-1}(u) \mid X \leq F_X^{-1}(u)\right)$$

with the same assumption that the limit exists.
Expressed in terms of copula of X, Y these measures are given as

$$\lambda_U = \lim_{u \nearrow 1} \frac{1 - 2u - C(u,u)}{1-u}$$

for upper and

$$\lambda_L = \lim_{u \searrow 0} \frac{C(u,u)}{u} \tag{2.10}$$

for lower tail dependence.(Proof can be found in (Nelsen 2006)).
Note that the parameter λ_U is only defined in $(0,1]$ and copula C is said to have no upper tail dependence if $\lambda_U = 0$. Similar expressions hold for lower tail dependence parameter λ_L. In following there are a two examples to represent tail dependence of different copula families: Student t and *rotated Gumbel*. For the the first example of elliptical copula family of t copulas with ν degrees of freedom and the correlation matrix Σ, proposed by (Demarta and McNeil 2005) the λ_L is calculated as follows. For the λ_L defined as in (2.10) applying the l'Hospital rule and using the property of copula function yields

$$\lambda_L = \lim_{u \searrow 0} \frac{dC(u,u)}{du} = \lim_{u \searrow 0} P(U_2 \leq u \mid U_1 = u) + \lim_{u \searrow 0} P(U_1 \leq u \mid U_2 = u)$$

with random variables (U_1, U_2) and C being their distribution function. For a pair of continuous variables X, Y such as $X = t_\nu^{-1}(U_1)$ and $Y = t_\nu^{-1}(U_2)$ with $(X, Y) \sim t_2(\nu, 0, \Sigma)$. Next we benefit from the property of exchangeability of the two variables X, Y (for properties see (Nelsen 2006)) the equation sums up to

$$\lambda_L^t = 2 \lim_{t \searrow -\infty} P(Y \leq t \mid X = t)$$

The conditional probability leads to

$$\left(\frac{\nu + t^2}{\nu + 1}\right)^{-1/2} \left(\frac{Y - \rho t}{\sqrt{1 - \rho^2}}\right) \sim t_1(\nu + 1, 0, 1)$$

with ρ being a non-diagonal element of the P matrix. Going to the limit it finally yields out the tail dependence coefficient of the t copula

$$\lambda_L^t = 2t_{\nu+1}(-\frac{\sqrt{1+\nu}\sqrt{1-\rho}}{\sqrt{1+\rho}})$$

The lower tail dependence of rotated Gumbel copula is fairly easy to calculate.

$$\lambda_L = \lim_{u \searrow 0} \frac{C(u, u)}{u} = \lim_{u \searrow 0} \frac{u - C_{-\theta}^{Gu}(u, 1-u)}{u} =$$

$$= 1 - \lim_{u \searrow 0} \frac{exp\left(-((-\log u)^{-\theta} + (-\log(1-u))^{-\theta})^{-1/\theta}\right)}{u} =$$

$$= 1 - \lim_{u \searrow 0} exp\left(-((-\log u)^{-\theta} + (-\log(1-u)^{\theta})^{-1/\theta} - \log u\right)$$

Since, the limit

$$exp\left(-((-\log u)^{-\theta} + (-\log(1-u)^{\theta})^{-1/\theta} - \log u\right) \xrightarrow[u \searrow 0]{} 0$$

it follows $\lambda_L = 0$. See also (Brechmann 2010) for the example of λ_U.

Table 2.2 offers an overview for these two measures of dependence in terms of the defining parameter for the bivariate copula families discussed in the previous section and a few others.

Copula family	Parameter	Kendall$'$ s Tau	upper tail dependence	lower tail dependence
Gaussian	$-1 < \rho < 1$	$\frac{2}{\pi}\arcsin(\rho)$	0	0
t	$-1 < \rho < 1$, df $\nu \geq 1$	$\frac{2}{\pi}\arcsin(\rho)$	$2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}})$	
Clayton	$\theta > 0$	$\frac{\theta}{1+\theta}$	0	$2^{-\frac{1}{\theta}}$
Gumbel	$\theta \geq 1$	$1-\frac{1}{\theta}$	$2-2^{-\frac{1}{\theta}}$	0
Frank	$\theta < 0, \theta > 0$	$1-\frac{4}{\theta}+4\frac{D_1(\theta)}{\theta}$	0	0
Joe-Clayton	$\theta \geq 1$ $\delta > 0$	$1-\frac{2}{\delta(2-\theta)}+\frac{4}{\theta^2\delta})\cdot$ $\cdot B(\frac{2-2\theta}{\theta}+1, \delta+2)$	$2-2^{1/\theta}$	$2^{-1/\delta}$

Table 2.2: Kendall$'$s τ's, upper and lower tail dependence parameters of bivariate copula families. D_1 denotes the Debey function $D_1(\theta) = \int_0^\theta \frac{x/\theta}{exp(x)-1}dx$ and $B(x, y) = \int_0^1 t^{1+x}(t-1)^{y-1}dt$ the Beta function.

The classical estimator of upper/lower tail dependence coefficient is the one proposed by (Huang 1992):

Suppose we have i.i.d. observations $(x_i, y_i), i = 1, ..., n$ *for two random variables* X *and* Y *with copula* C*. The corresponding marginal order of statistics we denote by* $min\{x_1, ..., x_n\} =: x_{(1)} \leq ... \leq x_{(n)} = max\{x_1, ..., x_n\}$ *and* $min\{y_1, ..., y_n\} =: y_{(1)} \leq ... \leq y_{(n)} = max\{y_1, ..., y_n\}$ *respectively.*
We substitute the theoretical copula C *by its empirical equivalent* $\hat{C}_n$

$$\hat{C}_n(\frac{i}{n}, \frac{j}{n}) = \frac{1}{n}\sum_{t=1}^{n} \mathbf{I}(x_t \leq x_{(i)}), y_t \leq y_{(j)}) \tag{2.11}$$

For $i = 0$ and $j = 0$ we set:

$$\hat{C}_n(\frac{i}{n}, \frac{j}{n}) = 0$$

Next we determine a sequence k_n *of natural numbers such as*

$$k_n \xrightarrow{n\to\infty} \infty \textit{ and } k/n \xrightarrow{n\to\infty} 0.$$

Inserting (5) into (3) and (4) the estimators of the upper/lower tail dependence parameter are defined as:

$$\hat{\lambda}_U^{n,k_n} = \frac{\hat{C}_n(1-\frac{k_n}{n}, 1-\frac{k_n}{n})}{1-(1-\frac{k_n}{n})} = \frac{1}{k_n}\sum_{t=1}^{n} \mathbf{I}(x_t > x_{(n-k_n)}, y_t > y_{(n-k_n)})$$

$$\hat{\lambda}_L^{n,k_n} = C_n(\frac{k_n}{n}, \frac{k_n}{n})/\frac{k_n}{n} = \frac{1}{k_n}\sum_{t=1}^{n} \mathbf{I}(x_t < x_{(k_n)}, y_t < y_{(k_n)}),$$

where $\mathbf{I}$ *stands for the indicator function.*

The choice of k_n is however not random. The empirical results of later studies have led to so called ‘ square root of n rule ‘ : $k_n \approx \sqrt{n}$. If k_n is asymptotically equal to $\sqrt{n}$ and assuming that partial derivatives of C exist and are continuous, then λ_U^{n,k_n} is weakly consistent and asymptotically unbiased estimator of λ_U. For the purpose of precision k is often set equal to $\lfloor\sqrt{n}\rfloor$. The disadvantage of this rule is that it makes the estimation of tail dependence quite difficult, as it cuts down the number of observations being used.
(Brechmann 2010) suggests further investigation of upper/lower tail dependence. In conclusion the estimators we defined are found to be well suitable for our further applications. A generalization of the notion of lower tail dependence was proposed

by (Schmid and Schmidt 2007) and offers a tail dependence measure for the multivariate case.

2.6 Bivariate copula families

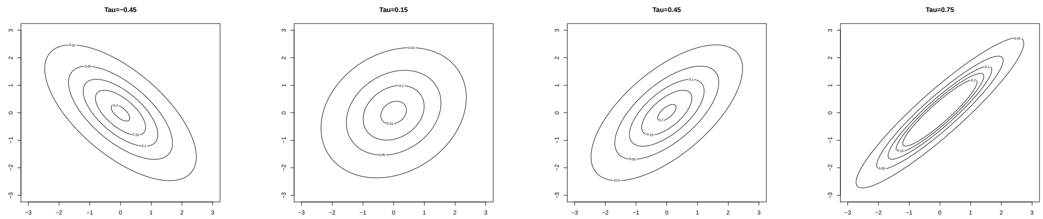
Referring to the last Chapter we will here investigate the copula families on that we later base our simulation studies, namely the Gauss- , Gumbel- and the t copulas. It is important to know the properties of the bivariate copulas used for later construction of the pair copulas. Those will be extracted to the high-dimensional copulas by pair copula construction. In this section the copulas used will be briefly discussed but first we refer to work of (Joe 1997) and (Bedford and Cooke 2001), to see the extension in the n dimensions.
Now let us go back to the bivariate copula families and have a quick overview for three copula families we said we will consider relevant for our investigations and modeling. We concentrate this time on the bivariate case. It has been pointed out earlier that the Gumbel copula can only measure the positive dependence between random variables. We however need to capture both, positive as well as negative dependence for proper modeling. For that purpose there is a rotated version of the Gumbel copula. This idea is applicable on asymmetric copulas (Gumbel or Clayton) which is the case in our simulation studies, and provides certain flexibility. Recall the definition of the survival copula (2.3) where instead of u_1 and u_2 we consider $1 - u_1$ and $1 - u_2$ respectively. Similarly, $90°$ we set the rotation of the $(1 - u_1, u_2)$ and the third possible rotation, the Gumbel copula rotated by $270°$ with $(u_1, 1 - u_2)$. In this way we will be able to cover both possible asymmetrical dependencies, positive and negative, while using the Gumbel copula families.

All the convergence patterns of multivariate copulas also hold for the bivariate ones, i.e. depending on the parameter ρ Gauss copula as well as Student t copula exhibit complete positive (when $\rho \to 1$) or complete negative ($\rho \to -1$) dependence.

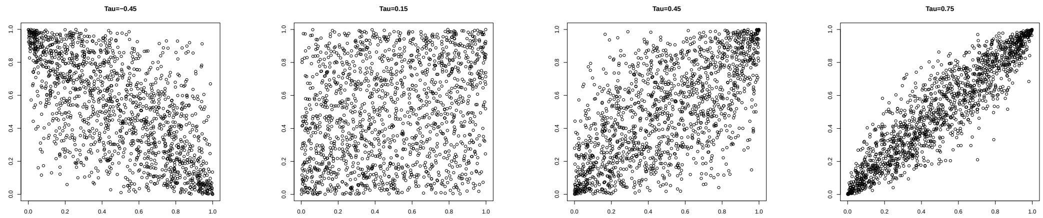
For Gumbel copula it holds : for $\theta \to 1$ the Gumbel copula stands for independence, when $\theta \to \infty$ it exhibits complete positive dependence.
The behavior of main copula families is illustrated in Figures 2.1 - 2.4.

Copula	Density function	Parameter
Gauss	$c_\rho^G(u_1,u_2) = \frac{1}{\sqrt{1-\rho^2}} \cdot exp\left(-\frac{\rho^2(x_1^2+x_2^2)-2\rho x_1 x_2}{2(1-\rho^2)}\right)$	$\rho \in (-1,1)$
t	$c_t(u_1,u_2) = \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\nu\pi\sqrt{1-\rho^2 \cdot dt_\nu(x_1)dt_\nu(x_2)}} \left(1+\frac{x_1^2+x_2^2-2\rho}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}$	$\rho \in (-1,1)$
Gumbel	$c_\theta^{Gu}(u_1,u_2) = \frac{(\log u_1 \cdot \log u_2)^{\theta-1}}{u_1 u_2} \cdot exp\Big((-\log u_1)\theta + (-\log u_2)^\theta\Big)^{(2-1/\theta)} \cdot$ $\cdot \frac{\left((-\log u_1)^\theta+(-\log u_2)^\theta\right)^{1/\theta}+\theta-1}{\left((-\log u_1)\theta+(-\log u_2)^\theta\right)^{1/\theta}}$	$\theta \in [1,\infty)$
rotated Gumbel	$180^\circ : c_{180}(u_1,u_2) = c_\theta^{Gu}(1-u_1, 1-u_2)$ $90^\circ : c_{90}(u_1,u_2) = c_{-\theta}^{Gu}(1-u_1, u_2)$ $270^\circ : c_{270}(u_1,u_2) = c_{-\theta}^{Gu}(u_1, 1-u_2)$	
Clayton	$c^{Cl}(u_1,u_2) = (1+\theta)(u_1 \cdot u_2)^{-1-\theta}(u_1^{-\theta} + u_2^{-theta} - 1)^{-2-1/\theta}$	$\theta \in (0,\infty)$
Frank	$c^F(u_1,u_2) = \frac{\theta(e^{-\theta}-1)\cdot e^{-\theta(u_1+u_2)}}{(e^{-\theta}-1+(e^{-\theta u_1}-1)(e^{\theta u_2}-1))^2}$	$\theta \in \mathbb{R}/\{0\}$

Table 2.3: Copula density functions of different copula families and corresponding parameter range.



(a) $\tau = -0.45$ (b) $\tau = 0.15$ (c) $\tau = 0.45$ (d) $\tau = 0.75$



(e) $\tau = -0.45$ (f) $\tau = 0.15$ (g) $\tau = 0.45$ (h) $\tau = 0.75$

Figure 2.1: Pair- and contour-plots of Gaussian copula for different values of Kendall′s τ

2.7 Pair Copulas Construction

The idea of constructing multivariate distributions using two-dimensional copulas was originally proposed by (Bedford and Cooke 2002) and explicitly discussed in (Aas, Czado, Frigessi, and Bakken 2009). Specifying bivariate copulas of selected pairs of random variables and extending to conditional dependence for all marginal distribution functions. Decomposition is constructed as follows. Due to Sklar′s theorem, every multivariate distribution F with the marginals $F_1(x_1), ..., F_n(x_n)$ can be rewritten as

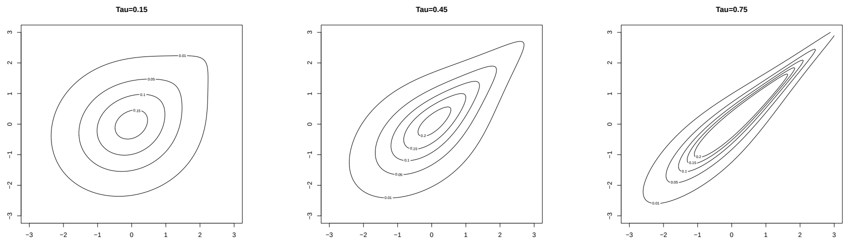
$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)$$

for some appropriate n-dimensional copula C. To derive the density function f, we use the chain rule and get

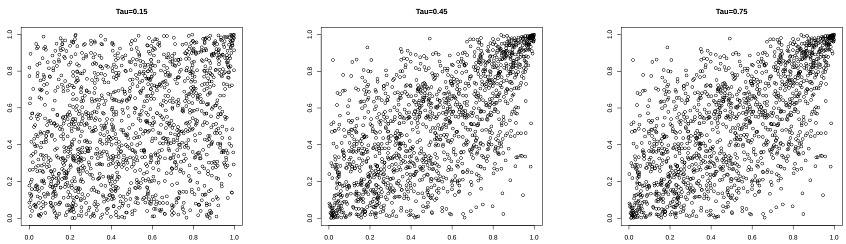
$$f(x_1, ..., x_n) = c_{1,...,n}(F_1(x_1), ..., F_n(x_n)) \cdot f_1(x_1) \cdot ... \cdot f_n(x_n),$$

where $c_{1...n}(\cdot)$is a n-variate copula density .
Now, the pair-copula decomposition we get for bivariate case is:

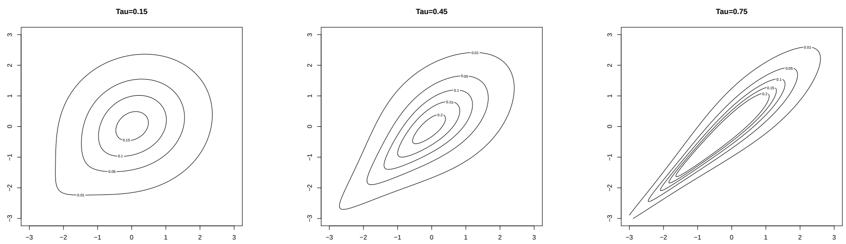
$$f_{12}(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2),$$



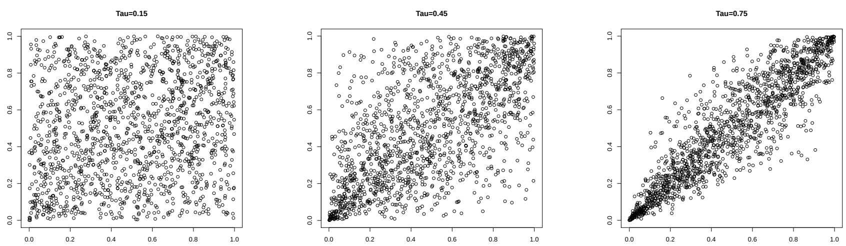
(a) $\tau = 0.15$ (b) $\tau = 0.45$ (c) $\tau = 0.75$



(d) $\tau = 0.15$ (e) $\tau = 0.45$ (f) $\tau = 0.75$

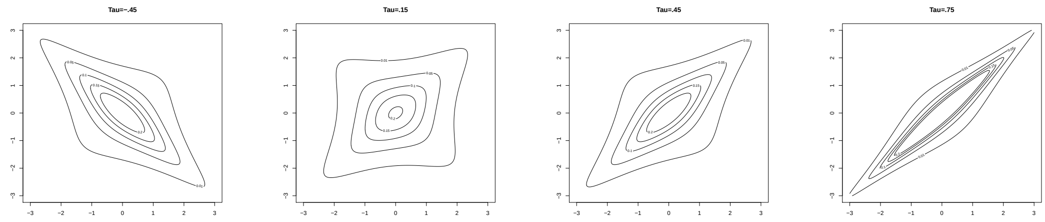


(g) $\tau = 0.15$ (h) $\tau = 0.45$ (i) $\tau = 0.75$

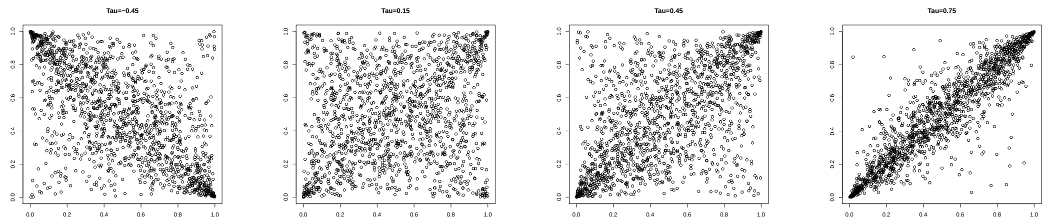


(j) $\tau = 0.15$ (k) $\tau = 0.45$ (l) $\tau = 0.75$

Figure 2.2: Pair- and contour-plots of Gumbel (a to f) und Gumbel survival (g to l) copulas for different values of Kendall′s τ



(a) $\tau = -0.45$ (b) $\tau = 0.15$ (c) $\tau = 0.45$ (d) $\tau = 0.75$



(e) $\tau = -0.45$ (f) $\tau = 0.15$ (g) $\tau = 0.45$ (h) $\tau = 0.75$

Figure 2.3: Pair- and contour-plots of Student t copula with degree of freedom 2 for different values of Kendall's τ.

with $c_{12}(X_1, X_2)$ being an appropriate pair-copula density of X_1 and X_2. Using the last equation we derive the conditional densities of this factorization under definition of conditional density. It is given as:

$$f_{2|1}(x_2 \mid x_1) = \frac{f_{12}(x_1, x_2)}{f_1(x_1)} = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2)$$

Since the $f(x_2, x_1)/f(x_1)$ defines the conditional density we can extend the 3-dimensional case to

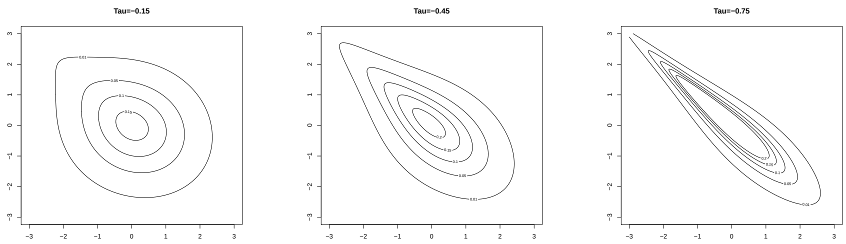
$$\begin{aligned} f_{3|12}(x_3 \mid x_1, x_2) &= \frac{f_{2,3|1}(x_2, x_3 \mid x_1)}{f_{2|1}(x_2 \mid x_1)} \\ &= c_{13|2}(F_{1|2}(x_1 \mid x_2), F_{3|2}(x_3 \mid x_2)) \cdot c_{32}(F_3(x_3), F_2(x_2)) \cdot f_2(x_2) \end{aligned} \tag{2.12}$$

for an appropriate pair-copula $32|1$ and transformed variables $F_{1|2}(x_1|x_2)$ and $F_{3|2}(x_3|x_2)$. Now recall the definition of the joint density function:

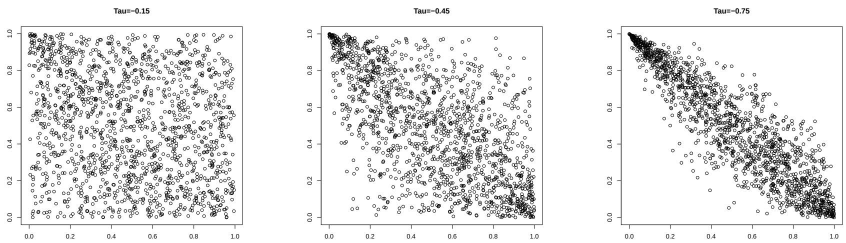
$$f(x_1, ..., x_n) = f(x_1)f(x_2 \mid x_1)f(x_3 \mid x_1, x_2) \cdots f(x_n \mid x_1, ..., x_{-1}). \tag{2.13}$$

Using the decomposition (2.8), the joint density of X_1, X_2 and X_3 is given by

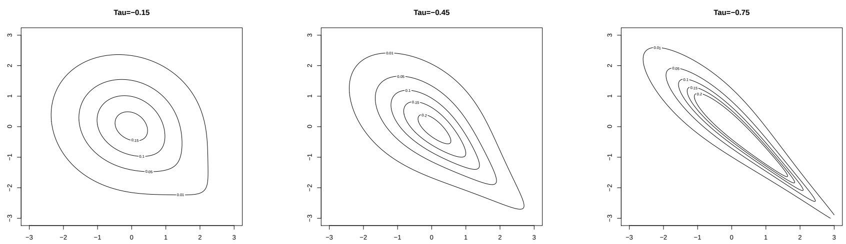
$$f_{123}(x_1, x_2, x_3) = f_{2|13}(x_2 \mid x_1, x_3)f_{3|1}(x_3 \mid x_1)f_1(x_1) =$$



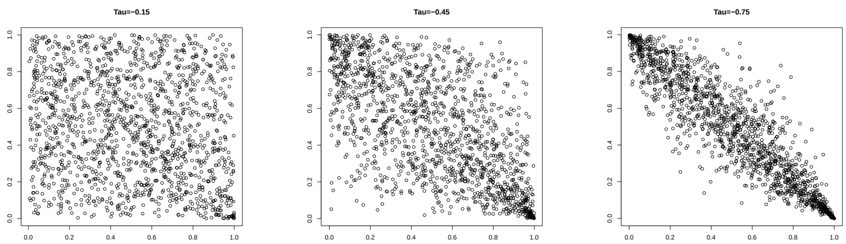
(a) $\tau = -0.15$ (b) $\tau = -0.45$ (c) $\tau = -0.75$



(d) $\tau = -0.15$ (e) $\tau = -0.45$ (f) $\tau = -0.75$



(g) $\tau = -0.15$ (h) $\tau = -0.45$ (i) $\tau = -0.75$



(j) $\tau = -0.15$ (k) $\tau = -0.45$ (l) $\tau = -0.75$

Figure 2.4: Pair- and contour-plots of rotated Gumbel by 90°(a to f) und by 270°(g to l) for different values of Kendall′s τ.

$$= c_{23|1}(F_{2|1}(x_2 \mid x_1), F_{3|1}(x_3 \mid x_1)) \cdot c_{12}(F_1(x_1), F_2(x_2)) f_2(x_2) \cdot$$
$$\cdot\, c_{13}(F_1(x_1), F_3(x_3)) f_3(x_3) \cdot f_2(x_2) \cdot f_1(x_1) \qquad (2.14)$$

Note, this formula derived is only defined trough bivariate copulas.
To generalize this results to the n-dimensional random vector $X = (X_1, ..., X_n)$ and using the join density we need to define to sets of arbitrary components γ_i, and a matching vector γ, containing the γ_j as components. Let us denote $\gamma \backslash \gamma_j$ as γ_{-j}. The factorization from above is then for $i, j = 1, ..., n$:

$$f(x_i \mid \gamma) = c_{ij|\gamma_{-j}}(F(x_i \mid \gamma_{-j}), F(x_j \mid \gamma_{-j})) f(x_i \mid \gamma_{-j})$$

Going one step back we apply the pair copulas with the bivariate copula distribution function $C_{ij|\gamma_{-j}}$. Applying the marginal conditional distribution functions $F(x_i)$ and marginal densities (Joe 1996) we obtain

$$F(x_i \mid \boldsymbol{\gamma}) = \frac{\partial C_{ij|\boldsymbol{\gamma_{-j}}}(F(x_i \mid \boldsymbol{\gamma_{-j}}), F(x_j \mid \boldsymbol{\gamma_{-j}}))}{\partial F(x_j \mid \boldsymbol{\gamma_{-j}})} \qquad (2.15)$$

where $C_{ij} \mid \boldsymbol{\gamma_{-j}}$ denotes a bivariate copula distribution function and $\boldsymbol{\gamma_{-j}}$ is a vector with out the $j-$th component. Recall that the decomposition was however not unique. This also holds for the above formula. To define an n-dimensional vine copula there are $\binom{n}{2}$ bivariate copula needed. More over all of them can be specified completely independent of each other. This leaves us with a very large amount of possible construction especially for large dimensions. This makes the question of optimal modeling a priority. For this purpose we will introduce the concept of *regular vines* or *R-Vines* in next chapter.

3 Regular Vines

In this section we introduce the theoretical background of regular vines or short R-Vines. Based on the results of (Bedford and Cooke 2001), (Bedford and Cooke 2002) and (Dissmann 2010) we will see how the R-Vine is stored in terms of a matrix, which makes the further investigation more tractable, and the section also contains an overview of the graphical representation of an R-Vine model to mirror the pair copula construction discussed in the previous section.

3.1 Graph theory

Constructing an R-Vine model we first need to introduce definitions from basic graph theory that we refer to later. In this section we only list the "tools"explicitly used for model construction. For further references see for example graph theory/optimization literature, such as (Diestel 2006), (Harris, Hirst, and Mossinghoff 2008).

Definition 3.1. A pair $G = (N, E)$, where N is an arbitrary finite set and $E \in \{\{n_i, n_j\} : n_i, n_j \in N\}$ is called a graph. Elements of N are called nodes, and elements of E edges of the graph. A graph $G' = (N', E')$ with $N' \subset N$ and $E' \subset E$ is called subgraph. If there exist a function $\omega : E \rightarrow \mathbb{R}^+$ the graph G is called weighted.

Two nodes are *connected* if and only if there is an edge linked to both of the nodes. The degree $d(n_i)$ of node n_i is the number of edges attached to it. Since the order of n_i, n_j is arbitrary, the graph G is called *undirected.* Next definition is important and helps to define R-Vine trees later.

Definition 3.2. A path in the G graph is a sequence of nodes such that for every one of them there is an edge connecting to the next node, i.e.

$$\text{for } N = \{n_1, ..., n_k\}, k \geq 2, \text{ it holds } \{n_i, n_{i+1}\} \in E, \text{ for } i = 1, ..., k-1.$$

If the start node n_1 and the end node n_k are linked by an edge, the path is called a cycle.

Is the graph G undirected and acyclic, it is called a *tree* $T = (N, E)$. For a tree T it is equivalent:

- T is a tree
- T is connected by (N-1) edges

- any path connecting two nodes in T is unique

Algorithm of Prim and maximum spanning tree

Algorithm of Prim can be found in various literature(see for example (Cormen, Leiserson, and Stein 2001),(Grama, Gupta, and Karypus 2003)) and is a sequential method used to find the minimum spanning tree. This algorithm delivers a solution if the graph is connected, since there will always be a path to every node (see (Dissmann 2010)). However, it also can be applied to find a maximum spanning tree in analog way. It operates on weighted graphs and searches for a subgraph that connects all nodes of the graph and maximizes the resulting sum of weights.

Algorithm 1. *Prim's algorithm*

Input *A non-empty connected weighted graph* $G = (N, E)$
Output *Maximum spanning tree* $T_m = (N_m, E_m)$
1. Initialize$N_m = \{n_m\}$ *with node* $n_m \in E$ *as a starting point and* $E_m = \{\}$
*2.**while*** $N_m \neq N$ ***do***

- *select an edge* $e\prime$ *in* E *such that it connects a node in* N_m *with a new node* $n\prime$ *in* N *with* $e\prime \notin E_m$*,i.e. the new node is not already connected with selected edges, and the weight* $\mid \omega \mid$ *is maximal*
- *add the new node to* N_m *and the new edge* $e\prime$ *to* E_m *respectively*

*3.**end while***

This algorithm repeatedly adds the shortest edge incident to N_m and delivers a tree connecting all nodes with the maximum possible value on weights ω_{ij} (positive or negative) for edges e_{ij} connecting the two nodes i and j(for proof see (Dissmann 2010)).In case of C-Vine one needs to look for a spanning star instead of the tree, and- in case of D-Vine -a Hamiltonian path (see (Dissmann 2010), (Brechmann 2010) for details).

3.2 Regular Vines

The R-Vine is a graphical model widely discussed in (Bedford and Cooke 2001, Bedford and Cooke 2002) and by (Kurowicka 2009, Kurowicka and Cooke 2006) to organize different pair copula constructions. They define a regular vine or R-Vine as a nested set of $n-1$ trees such that nodes of a tree are built of the edges of the previous tree and the edges are joint only if they share a common node in the previous tree. More formal definition is:

Definition 3.3. A sequence of linked trees $T_1, ..., T_{n-1}$ is a vine on n elements if

(i) $T_1 = (N_1, E_1)$ with $N_1 = \{1, ..., n\}$

(ii) for $i = 2, .., n-1$ T_i is a connected tree with nodes $N_i = E_{i-1}$
It is called **regular vine** if additionally holds the

(iii) **proximity condition**: for $i = 2, ..., n-1$ and $\{e_l, e_r\} \in E_i$ with $e_l = \{e_{l_1}, e_{l_2}\}, e_r = \{e_{r_1}, e_{r_2}\}$ then exactly one of the e_{l_i} equals one of the e_{r_i}.
This condition guarantees that there is only edge $\{e_l, e_r\}$ in the tree T_i when e_l and e_r share a common node in the tree T_{i-1}.

The vine is called a **D-Vine**, if additionally each node in the first tree T_1 has a degree not higher then 2.
If each tree T_i has an unique node of degree $n-i$ it is called **canonical or C-Vine** and the node with the maximal degree in T_1 is called the **root node** .
To be able to define the pair copula constructions based on R-Vines we follow (Kurowicka and Cooke 2006) and define constraint, conditioned and conditioning sets.

Definition 3.4. For an edge $e \in E_i$

(i) For two nodes n_i, n_j if n_i is an element of n_j it is said to be **m-child** of n_j. Node n_i is called **m-descendent** of n_j if through the relation $n_i \in n_{i+1} \in ... \in n_j$ one can reach the node i from the node j. The **complete union** U_e^c of an $e \in E_i$ is the subset of all nodes in N_1 consisting of m-descendants of e.

(ii) The **constraint set** associated with an edge $e \in E_i, i \leq n-1$ is the complete union U_e^c,i.e. all the variables that can be reached from the edge e.

(iii) For $e \in E_i, i = 1, ..., n-1$ and $e_l, e_r \in E_{i-1}$, if $e = \{e_l, e_r\}$ then $D_e = U_{e_l}^c \cap U_{e_r}^c$ is the **conditioning set** associated with e,i.e. the intersection of the complete unions of e_l and e_r. For $e \in E_1$ this set is empty.

(iv) First, the symmetric difference of two sets of elements A and B is commonly defined by $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Then, the **conditioned set** associated with e is given by the symmetric difference of the complete unions of e_l and e_r:

$$\{C_{e,e_l}, C_{e,e_r}\} = U_{e_l} \triangle U_{e_r} = \{U_{e_l}^c \setminus D_e, U_{e_r}^c \setminus D_e\}.$$

Finally we note a few properties of an R-Vine (see (Kurowicka and Cooke 2006, Dissmann 2010) for further references). For a regular vine with the set of trees $(T_1, ..., T_{n-1})$it holds:

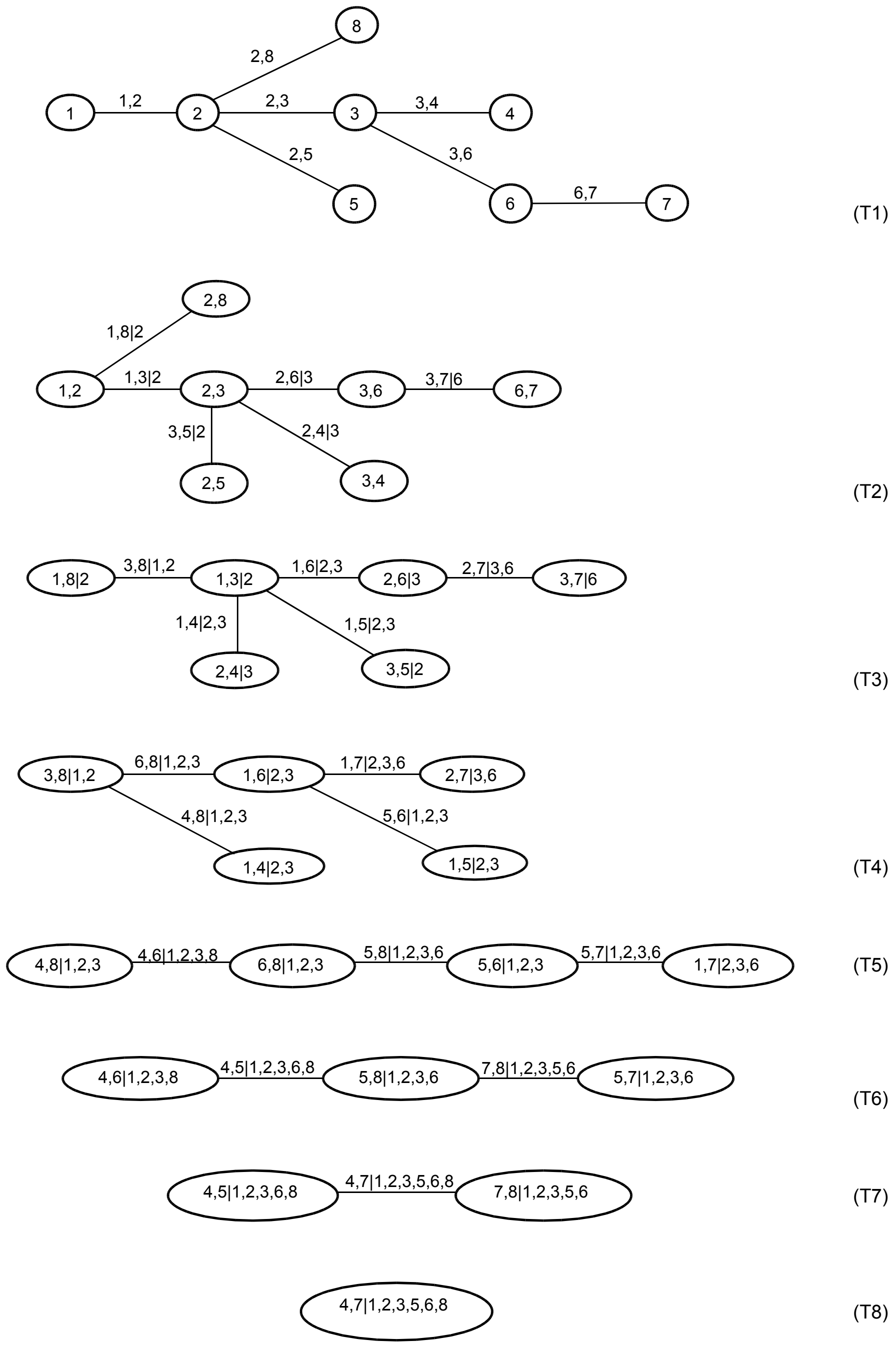
- the number of edges is $n(n-1)/2$
- each conditioned set is a doubleton, each pair of variables occurs exactly once as a conditioned set
- it two edges have the same conditioning set, it is same edge.

3.3 R-Vine copula

As mentioned earlier the R-Vine model is very useful for the specification of the bivariate copulas in a pair copula construction. In the first tree there are $n-1$ bivariate copulas to be selected, which represent the dependencies between variables. They are given by the conditioned sets of the edges. For the next tree we have conditional copulas between these variables given by conditioned sets and also conditional on the values of the variables that are given by the conditioning sets of the first tree edges. We are however mostly interested how to define the vine copula density to express the distribution of an R-Vine. First recall the construction of the bivariate copulas from the previous section.

$$F(x_i \mid \boldsymbol{\gamma}) = \frac{\partial C_{ij|\boldsymbol{\gamma}_{-j}}(F(x_i \mid \boldsymbol{\gamma}_{-j}), F(x_j \mid \boldsymbol{\gamma}_{-j}))}{\partial F(x_j \mid \boldsymbol{\gamma}_{-j})} \tag{3.1}$$

Figure 3.1: An example of 8-dimensional R-Vine



where $C_{ij} \mid \boldsymbol{\gamma_{-j}}$ denotes a bivariate copula distribution function and $\boldsymbol{\gamma_{-j}}$ is a vector without the j-th component. Now let $\mathcal{V} = (\mathcal{T}_\infty, ...\mathcal{T}_{\backslash-\infty})$ be an R-Vine on n elements with margins $F_1, ..., F_n$. (Kurowicka and Cooke 2006) define a multivariate distribution based on an R-Vine as follows:

Definition 3.5. For an R-Vine V on n elements:

- Let $\mathbf{F} = (F_1, .., F_n)$ be a vector of invertible continuous marginal distribution functions and $\mathcal{V}$ an R-Vine on n elements. Further for set E_i of edges of the tree T_i let $\mathcal{B}= \{B_e \mid i = 1, ..., n-1; e \in E_i\}$ denote a copula set with pair copula B_e. Then $(\mathbf{F}, \mathcal{V}, \mathcal{B})$ is called a **copula vine specification**.

- The joint distribution $\mathbf{F}$ of a vector $(X_1, ..X_n)$ of random variables with margins F_i is said to realize the R-Vine copula specification, when for each edge $e \in E_i$ with $e = \{e_l, e_r\}$ B_e is the bivariate copula of conditional distributions (X_l, X_r) conditioned on X_{D_e}, where $X_{D_e} := \{x_i, i \in D_e\}$. In this case it is called an **R-Vine distribution**.

For an edge $e \in E$ with conditioned elements e_l, e_r and the conditioning set by D_e denote the conditional copula by $C_{e_l,e_r|D_e}$ and its density by $c_{e_l,e_r|D_e}$. For $i = 1, ..., n$ let the density functions f_i correspond to margins F_i , respectively. For an R-Vine copula specification on n elements the uniquely specified vine-dependent distribution that realizes this copula specification has the density:

$$f_{1...n}(x_1, ..., x_n) = \prod_{i=1}^{n} f_i(x_i) \prod_{i=1}^{n-1} \prod_{e \in E_i} c_{e_l,e_r|X_{D_e}}(F(x_{e_l} \mid X_{D_e}), F(x_{e_r} \mid X_{D_e})) \quad (3.2)$$

This formula can be simplified for a D-Vine or C-Vine. We however only present the general case of the R-Vine in this thesis. (see (Bedford and Cooke 2001) and (Kurowicka and Cooke 2006), (Aas, Czado, Frigessi, and Bakken 2009) for proof and further references).

3.4 Matrix representation

This advantageous way of presenting and storing an R-Vine matrix was used by (Kurowicka 2009) and explored in detail by (Dissmann 2010). Due to this method it is possible to store all the information from the nested trees in one n-dimensional matrix. As mentioned earlier we identify an R-Vine by the constraint set. Hence, we first define a constraint set of a lower triangular matrix.

Definition 3.6. The i-th constraint set of a lower triangular matrix $M = (m_{i,j})_{i,j=1,...,n}$ is given by

$$\mathcal{C}_M(i) = \{(\{m_{i,i}, m_{k,i}\}, D) \mid k = i+1, ..., n; D = \{m_{k+1,i}, ..., n_i\} \tag{3.3}$$

for $i = 1, ..., n-1$.

Note that conditioning set D is empty for $k = n$.
Then the constraint set of M is simply defined as the union of the constraint sets of the elements, i.e.

$$\mathcal{C}_M = \mathcal{C}_M(1) \cup ... \cup \mathcal{C}_M(n-1)$$

To give an impression on how this specification allows to read its entries we give an 5-dimensional example.

$$M = \begin{pmatrix} m_{1,1} & & & & \\ m_{2,1} & m_{2,2} & & & \\ m_{3,1} & m_{3,2} & m_{3,3} & & \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} \end{pmatrix}$$

All the diagonal entries ($m_{1,1}, m_{2,2}$ and so on) stand for elements of the constraint set. We pick an element in the same column such as $m_{3,1}$ for example and all the elements after in the same column ($m_{4,1}, m_{5,1}$) generate an element of $\mathcal{C}_M(1)$ namely $(\{m_{1,1}, m_{3,1}\}, \{m_{4,1}, m_{5,1}\})$ that corresponds to an edge of T_3 in the R-Vine. Using this notation we follow (Dissmann, Czado, Brechmann, and Kurowicka 2012) to define the R-Vine matrix.

> **Definition 3.7.** A lower triangular matrix $M = (m_{i,j})_{i,j=1,...,n}$ is called R-Vine matrix if it holds that

(i) $\{m_{i,i}, ..., m_{n,i}\} \subset \{m_{j,j}, ..., m_{n,j}\}$ for $1 \leq j < i \leq n$,

(ii) $m_{i,i} \notin \{m_{i+1,i+1}, ..., m_{n,i+1}\}$ for $i = 1, ..., n-1$,

(iii) for $i = 1, ..., n-1$ and for all $k = i+1, ..., n-1$:
$(m_{k,i}, \{m_{k+1,i}, ..., m_{n,i}\}) \in B_M(i+1) \cup ... \cup B_M(n-1) \cup \tilde{B}_M(i+1) \cup ... \cup \tilde{B}_M(n-1)$,
where $B_M(i) := \{(m_{i,i}, D) : k = i+1, ..., n; D = \{m_{k,i}, ..., m_{n,i}\}\}$
and $\tilde{B}_M(i) := \{(m_{k,i}, D) : k = i+1, ..., n; D = \{m_{i,i}\} \cup \{m_{k+1,i}, ..., m_{n,i}\}\}$

Parts (i) and (ii) explore how the combination of, the process whereby the entries of a column are reflected in the column to the left and the fact that there is a new entry on the diagonal in every column, results in the sequential addition of new variables to the R-Vine matrix from right to left, as previously described by both (Kurowicka 2009) and (Dissmann 2010). The third definition part serves as a better understanding of the way the matrix works and this is necessary for definition. However it agrees to the proximity condition of Definition 3.
The way we have defined the R-Vine matrix allows deriving a few useful properties. First, it can be shown that all elements of a column are in fact different and for a given n-dimensional R-Vine matrix deleting the first row and the first column leaves an $(n-1)$-dimensional R-Vine matrix. This n property can be expended. (Dissmann 2010) has a very detailed discussion of the construction and storing all the information of an R-Vine model by the matrix system. Based on those considerations we will illustrate an R-Vine matrix for a 5-dimensional example. Recall the matrix of the previous example

$$M = \begin{pmatrix} m_{1,1} & & & & \\ m_{2,1} & m_{2,2} & & & \\ m_{3,1} & m_{3,2} & m_{3,3} & & \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} \end{pmatrix}$$

or in more convenient representation let it have the entries:

$$M = \begin{pmatrix} 3 & & & & \\ 4 & 4 & & & \\ 5 & 5 & 1 & & \\ 1 & 2 & 5 & 2 & \\ 2 & 1 & 2 & 5 & 5 \end{pmatrix}$$

Given the conditions 1. and 2. of Definition 3.7 are satisfied and hence, we have a R-Vine matrix. It is defined by 4 trees $T_1, ...T_4$ each of them specified by $5-i+1$ nodes and $4-i+1$ edges,$i = 1, ..., 4$. The nodes of the first tree T_1 are $1, 2, 3, 4$ and 5 ; and the edges of T_1 and nodes of T_2 are given by $\{\{m_{i,i}, m_{n,i}\} : i = 1, ..., n-1$: here $\{\{m_{1,1}, m_{5,1}\}, \{\{m_{2,2}, m_{5,2}\}, \{\{m_{3,3}, m_{5,3}\}, \{\{m_{4,4}, m_{5,4}\}$.
In this example this corresponds to $\{1, 4\}, \{1, 2\}, \{2, 3\}, \{3, 5\}$. In similar way the edges of T_2 and nodes of T_3 are given by $\{\{m_{i,i}, m_{n-1,i} \mid m_{n,i}\} : i = 1, ..., n-2\}$,which are
$\{\{m_{1,1}, m_{4,1} \mid m_{5,1}\}, \{m_{2,2}, m_{4,2} \mid m_{5,2}\}, \{m_{3,3}, m_{4,3} \mid m_{5,3}\}\}$ or $\{1, 5 \mid 2\}, \{1, 3 \mid 2\}, \{2, 4 \mid 3\}$, respectively.
In trees T_3 and T_4 we have: $\{\{m_{1,1}, m_{3,1} \mid m_{4,1}, m_{5,1}\}, \{m_{2,2}, m_{3,2} \mid m_{4,2}, m_{5,2}\}\}$ specified as $\{3, 5 \mid 1, 2\}, \{4, 5 \mid 1, 2\}$. The last tree T_5 is given by $\{m_{1,1}, m_{2,1} \mid m_{3,1}, m_{4,1}, m_{5,1}\}\}$, $i.e.\{3, 4 \mid 1, 2, 5\}$.
In general for every tree T_j the edges are specified by diagonal element combined

with the element in row $n-j+1$ and conditioned on last elements of columns $i = 1, ..., n-1$,i.e.

$$\{m_{i,i}, m_{n-j+1,i} \mid m_{n-j+2,i}, ..., m_{n,i}\}.$$

It is important to notice that the R-Vine matrix representation is not unique and there are 2^{n-1} different possible matrices for a given R-Vine.
Besides the matrix M (Dissmann 2010) shows a way to store the chosen bivariate copula families as well as corresponding parameters of an R-Vine in two additional matrices. We denote them $F = (f_{i,j})$ and $P = (p_{i,j})$ for $i, j = 1, ..., n$, respectively. As previously seen, the entry m_i, j of M describes the copula of the variables $\{m_{i,i}, m_{j,i} \mid m_{j+1,i}, ..., m_{n,i}\}$ for $i < j$, the corresponding entry of matrix F , $f_{i,j}$ describes the type and the corresponding entry of P, $p_{i,j}$ describes the parameter of this copula. For the copula families that require two parameters a third matrix $P2 = (p2_{i,j})$ is defined in a similar manner. For example for the 5-dimensional matrix M defined earlier the entries of F and P given by

$$F = \begin{pmatrix} f_{1,1} & & & & \\ f_{2,1} & f_{2,2} & & & \\ f_{3,1} & f_{3,2} & f_{3,3} & & \\ f_{4,1} & f_{4,2} & f_{4,3} & f_{4,4} & \\ f_{5,1} & f_{5,2} & f_{5,3} & f_{5,4} & f_{5,5} \end{pmatrix} \qquad P = \begin{pmatrix} p_{1,1} & & & & \\ p_{2,1} & p_{2,2} & & & \\ p_{3,1} & p_{3,2} & p_{3,3} & & \\ p_{4,1} & p_{4,2} & p_{4,3} & p_{4,4} & \\ p_{5,1} & p_{5,2} & p_{5,3} & p_{5,4} & p_{5,5} \end{pmatrix}$$

where $f_{3,1}$ stands for the copula type of $\{m_{1,1}, m_{3,1} \mid m_{4,1}, m_{5,1}\}$,i.e. $c_{3,5|1,2}$ and $p_{3,1}$ is the parameter of this copula.

4 Dependence weights and their estimates

In this section we investigate different choices of weights used for the selection of R-Vine. First, we want to discover and examine the best choices of building the first tree in the R-Vine models, as we know the selection of the first tree T_1 plays most important role in the following selection of the best copulas and fitting of copula parameters. Due to the very large number of R-Vines in high dimensions this approach enables us to obtain the good fitting model without calculation of all possible R-Vines. This saves a significant amount of time and computational effort. Once determined, the first tree as one that allows us to capture the most significant dependencies between the variables, often in the financial data leads to later trees that contain weakly or even independent pairs. We use heuristic methods to sequentially specify the next tree and then continue with this process until tree T_{n-1} is constructed. In each step once we have to select the structure of the tree and, for each pair the bivariate copula family has to be chosen. Finally for each chosen copula family the corresponding parameters need to be estimated.

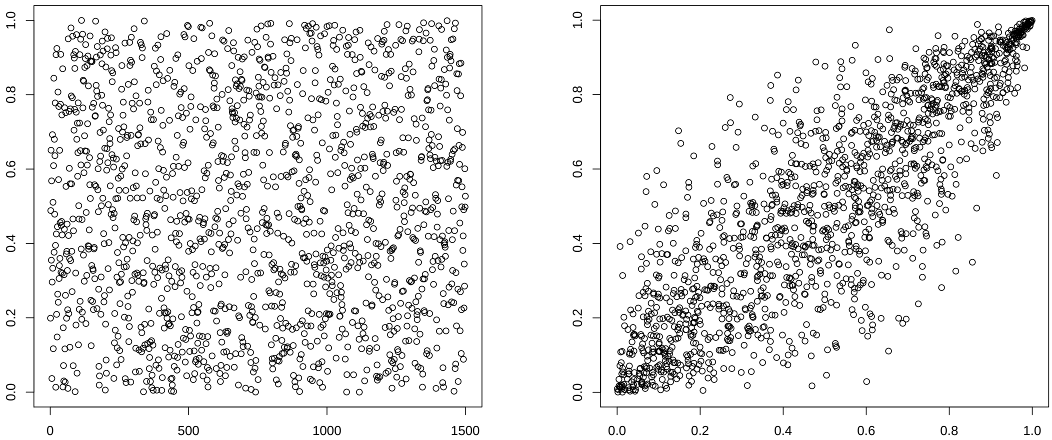
Our first choice of the weight would be Kendall's τ, since it has proven itself as adequate dependence measure and is widely used for sequential R-Vine selection. To remind the empirical Kendall's τ is given by

$$\hat{\tau}(X, Y) = \frac{C - D}{\sqrt{C + D + T_x}\sqrt{C + D + T_y}} \tag{4.1}$$

(see section 2.5). Starting with the selection of R-Vine structure using alternative weights we mainly follow (Brechmann 2010) and choose weights proposed in his thesis. We concentrate on those who allow us to capture the most tail dependence which is a useful property for financial data.
For calculation define a pair of random variables (X, Y) with marginal distributions $F_X(x)$ and $F_Y(y)$. The joint distribution function is given as $F_{X,Y}(x, y)$. So do (U, V) denote the transformed versions of (X, Y) as $U := F_X(X)$, $V := F_Y(Y)$.

4.1 Tail Cumulation

An alternative to Kendall's τ that we are going to use for the choice of weights is the tail cumulation. This measure of dependence is based on a graphical observation. We want to investigate those pairs of observations that show strong



(a) (b)

Figure 4.1: An example of distribution without tail dependence (a) while the plot on the right shows very strong tail dependence (b). Note, that upper tail dependence is larger than the lower tail dependence.

dependence in the tails. For that purpose we look at the upper-right and lower-left quadrants of $[0,1]^2$.
The idea behind this is simply to compare our dependent data observations to those of independent data. To be more specific, the boundaries that are calculated from independent observations are applied to dependent observations. For example, for two random variables U_1 and U_2 that are independent and uniformly distributed on $[0,1]$ the upper boundary u^{upper} is derived from:

$$P(U_1 > u^{upper}, U_2 > u^{upper}) = \sigma,$$

given that $\sigma \in [0,1]$. If we say for example $\sigma = 15\%$, this would mean we marked 15% of all observations that are placed in the upper right corner of the scatter plot. Figure 4.1 shows an example of independent (left) observations and those who show certain dependence in the tails (right).
Similarly, the lower boundary is defined as:

$$P(U_1 \leq u^{lower}, U_2 \leq u^{lower}) = \sigma$$

Calculation of the boundaries is also fairly easy. It holds:

$$\sigma = P(U_1 > u^{upper}, U_2 > u^{upper})$$

Due to independence of the variables U_1 and U_2 we have:

$$P(U_1 > u^{upper}, U_2 > u^{upper}) = P(U_1 > u^{upper})P(U_2 > u^{upper}).$$

Since $P(U_1 > u^{upper})P(U_2 > u^{upper}) = (u^{upper})^2$ the result is $u^{upper} = \sqrt{\sigma}$ for upper boundaries and in a similar way $u^{lower} = 1 - \sqrt{\sigma}$ for lower boundaries.

In the case that variables X and Y are dependent one can observe that the boundaries of the scatter plot in upper right and lower left corner contain more than 15% of all observations each.

Now let $(U_1^i, U_2^i), i = 1, ..., n$ denote independent copies of (U_1, U_2). Define

$$N^{upper} := \sum_{i=1}^{n} \boldsymbol{I}_{\{U_1^i > u^{upper}, U_2^i > u^{upper}\}} \tag{4.2}$$

for the upper boundary quadrant, i.e. for $[u^{upper}, 1]^2$ and similarly

$$N^{lower} := \sum_{i=1}^{n} \boldsymbol{I}_{\{U_1^i \le u^{upper}, U_2^i \le u^{upper}\}}$$

for the lower quadrant corner, i.e. $[0, u^{lower}]^2$. In the next step we can define the upper and lower tail cumulation as:

$$\gamma^{upper} := \tfrac{N^{upper}}{n} - \sigma \text{ and } \gamma^{lower} := \tfrac{N^{lower}}{n} - \sigma$$

Using data (u_1^i, u_2^i), γ^{upper} is estimated by $\hat{\gamma}_{upper} = \frac{n^{upper}}{n} - \sigma$, where $n^{upper} = \sum_{i=1}^{n} \boldsymbol{I}_{\{u_1^i > u_1, u_2^i > u_2\}}$. Similarly $\hat{\gamma} lower$ is derived. For the case of observations $(x_i, y_i) \in \mathbb{R}^2$ we transform first to $[0,1]^2$ by using the empirical cdf's $\hat{F}_X(x) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}_{\{x_i \le x\}}$ for X and $\hat{F}_Y(y) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}_{\{y_i \le y\}}$ for Y respectively. Then we can define the corresponding estimated upper and lower tail cumulation as:

$$\hat{\gamma}^{upper} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(\hat{F}_X(x_i) > u^{upper}, \hat{F}_Y(y_i) > u^{upper}) - \sigma \tag{4.3}$$

$$\hat{\gamma}^{lower} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(\hat{F}_X(x_i) \le u^{lower}, \hat{F}_Y(y_i) \le u^{lower}) - \sigma.$$

If we look at this definition more closely we obtain that positive values of upper and lower tail dependence show movement in the tails, as it means there are more dependent observations than set by independent σ boundaries. In our simulation study we are going to use the boundaries $u^{upper} = u^{lower} = 0.1$ as this is the best choice according to a Monte Carlo simulation on study contained in (Brechmann 2010).

4.2 Hu Dependence

The idea of this dependence measure is proposed by Hu (2006). In order to investigate both degree and structure of dependence between random variables we are going to use a copula family that consist of three different copulas, Gaussian copula, Gumbel- as well as Gumbel survival copula. According to the investigations by (Hu 2006) and (Brechmann 2010) this copula mixture will allow us to capture different dependence structures, that often occur while using financial data. We refer to notation of (Brechmann 2010) since it is a little more convenient. First recall that the bivariate Gaussian copula with parameter $\rho \in [-1, 1]$ given by

$$C_\rho{}^{Ga}(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$$

has symmetrical dependence structure. Further

$$C_{\theta_1}{}^{Gu}(u, v) = exp(-((-log(u))^{\frac{1}{\theta_1}} + (-log(v))^{\frac{1}{\theta_1}})^{\theta_1})$$

is a Gumbel copula with association parameter $\theta_1 \in (0, 1]$. Here, dependence becomes weaker with increasing θ_1. This copula is asymmetric about $(0.5, 0.5)$ and has positive right tail dependence, while $\lambda_l = 0$.(see also Table 2.1.) The third copula function that has been applied in the work of (Hu 2006) is the Gumbel survival copula, i.e. Gumbel copula rotated by 180°. According to section 2.6 it is defined as:

$$C_{\theta_2}{}^{GS}(u, v) := u + v - 1 + C_{\theta_2}{}^{Gu}(1 - u, 1 - v),$$

With parameter $\theta_2 \in (0, 1]$. This copula captures the lower tail dependence. We are particularly interested in this property since recent events in the financial markets have shown, that large losses for different risks tend to occur more often at the same time compared to large gains.

Let $\omega_1, \omega_2 \in [0, 1]$ be weights such that $\omega_1+\omega_2 \leq 1$ and consider the mixture copula

$$C^{mixture}(u, v) = (1 - \omega_1 - \omega_2)C_\rho{}^{Ga}(u, v) + \omega_1 C_{\theta_1}{}^{Gu}(u, v) + \omega_2 C_{\theta_2}{}^{SG}(u, v) \quad (4.4)$$

Now suppose we have a series of observations from $C^{mixture}(u, v)$ and we are interested in the calculation of weights. Basically we want to know 'how much' our mixed model that is unknown to us, should contain of each copula it consist of, i.e. Gaussian, Gumbel and Gumbel survival. In order to determine the weights a solution of the following optimization problem :

$$max_{\omega_1,\omega_2}\sum_{i=1}^{n} log((1-\omega_1-\omega_2)c_{\hat{\rho}}{}^{Ga}(x_i,y_i)+\omega_1 c_{\hat{\theta}_1}{}^{Gu}(x_i,y_i)+\omega_2 c_{\hat{\theta}_2}{}^{SG}(x_i,y_i)) \quad (4.5)$$

subject to ω_1, ω_2 non-negative with $\omega_1 + \omega_2 \leq 1$, needs to be provided.
We follow the approach suggested in (Brechmann 2010).
Empirical Kendall′ s τ as well as inversion formulas (see Table 1) are used to estimate the parameters of dependence ρ, θ_1 and θ_2, keeping in mind that due to same inversion formulas of Gumbel and Gumbel survival copulas it holds that $\hat{\theta}_1 = \hat{\theta}_2$. To maximize the log likelihood of $C^{mixture}$ copula density with respect to ω_1 and ω_2 from (3.4.) (Brechmann 2010) suggests the adaptive barrier method of (Lange 1999). This yields the estimates $\hat{\omega}_1$ and $\hat{\omega}_2$. We call $\hat{\omega}_1$ the upper Hu dependence coefficient and $\hat{\omega}_2$ the lower Hu dependence coefficient. To the value $(1 - \hat{\omega}_1 - \hat{\omega}_2)$ (Brechmann 2010) refers as "normality "of the data. He also shows that including this measure in the mixture model is important as it allows certain flexibility in case that tail dependence of the data does not exhibit strong asymmetry. According to Monte Carlo simulation study provided in (Brechmann 2010) for different copula families different choice of Kendall′s τ based on $R = 1000$ repetitions and $n = 1000$ observations, Hu dependence performs quite well for our selection of copula families. For the Gaussian and Gumbel copulas it is well estimated with increasing accuracy. Due to the properties of Student t copula to display lower as well as upper tail dependence it is best estimated by mixture of Gumbel and Gumbel survival copula each to 50%.

4.3 Exceedance Dependence

The further tool to investigate the asymmetric dependence is the so-called exceedance correlation. The definition of this dependence measure proposed by (Longin and Solnic 2001) and (Ang and Chen 2002) is:

$$Excorr(X,Y) = \begin{cases} Corr(X,Y|X \leq \delta_1, Y \leq \delta_2), & \text{for } \delta_1 \leq 0, \delta_2 \leq 0 \\ Corr(X,Y|X > \delta_1, Y > \delta_2), & \text{for } \delta_1 > 0, \delta_2 > 0. \end{cases}$$

As one can see this measure is based on Pearson′s product-moment correlation coefficient.
The discussion from (Ang and Chen 2002) has shown that there exists asymmetry in the exceedance correlation: large positive returns are much less correlated than large negative returns. It is also rather difficult to calculate due to it being affected by marginal characteristics. Since we want a measure that is independent

of marginal distributions we consider the idea of (Brechmann 2010) to use the $Kendall's\ \tau$

instead, as this measure does not change under strictly increasing transformations and is independent of the marginal distributions of the variables X and Y. So in order to measure the joint tail behavior of two random variables we define the *upper* and *lower exceedance* $Kendall'$s τ as:

$$\tau^{upper}(X,Y) := \tau(X,Y|X \leq \delta_1, Y \leq \delta_2)$$
$$\tau^{lower}(X,Y) := \tau(X,Y|X > 1-\delta_1, Y > 1-\delta_2)$$

According to (Nelsen 2006) these can be expressed in terms of the copula function as:

$$\tau^{upper}(X,Y) = \frac{4}{(1-u-v+C(1-\delta_1,1-\delta_2))^2}\times$$

$$\times \int_{1-\delta_2}^{1}\int_{1-\delta_2}^{1}(1-u-v+C(u,v))dC(u,v)-1$$

and

$$\tau^{lower}(X,Y) = \frac{4}{C(\delta_1,\delta_2)^2}\int_0^{\delta_2}\int_0^{\delta_1}C(u,v)dC(u,v)-1,$$

respectively.
Out of all discussed copula families only the lower exceedance τ of the Clayton copula is relatively easy to calculate and equals $\frac{2}{2+\theta}$, others are rather difficult due to the computation of the integrals (see also (Brechmann 2010)).
The empirical versions $\hat{\tau}^{upper}$ and $\hat{\tau}^{lower}$ of these measure are derived using the corresponding empirical Kendall$'$s $\hat{\tau}$ under given conditions. During the estimations it is important to choose the thresholds δ_1 and δ_2 correctly as they determine the number of observations used in the estimation and play a crucial role when measuring exceedance dependence. Empirical studies show that the theoretical value of exceedance τ are consistent with the empirical values when using $\delta_1^u = \delta_2^u = 0.8$ for upper and $\delta_1^l = \delta_2^l = 0.2$ for lower measures.

4.4 Rotated Measures

In this section we will briefly discuss the rotated measures discussed in (Brechmann 2010). These measures are needed for capturing the negative dependence between

Dependency measure	Estimates of rotated dependence measures
tail dependence	$\hat{\lambda}_r^{upper} = \frac{1}{k}\sum_{t=1}^{n} \mathbf{I}(x_t > x_{(n-k)}, y_t \leq y_{(k)})$ $\hat{\lambda}_r^{lower} = \frac{1}{k}\sum_{t=1}^{n} \mathbf{I}(x_t \leq x_{(k)}, y_t > y_{(n-k)})$
tail cumulation	$\hat{\gamma}_r^{upper} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(\hat{F}(x_i) > u^{upper}, \hat{F}(y_i) \leq u^{upper}) - \sigma$ $\hat{\gamma}_r^{lower} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(\hat{F}(x_i) \leq u^{lower}, \hat{F}(y_i) > u^{lower}) - \sigma$
exceedance dependence	$\hat{\tau}_r^{upper}(X,Y) = \hat{\tau}(X,Y\|X \leq \delta_1, Y > \delta_2)$ $\hat{\tau}_r^{lower}(X,Y) = \hat{\tau}(X,Y\|X > 1-\delta_1, Y \leq 1-\delta_2)$
Hu dependence	use rotated Gumbel copula and rotated Gumbel copula (270°)

Table 4.1: Overview of the weight measures corresponding to a 90° rotation of (Brechmann 2010).

observations, since through the measures discussed in the previous sections one can only investigate the positive tail behavior. Hence, to describe the dependence of variables with underlying rotated copulas, such as rotated Gumbel for example, the dependency measures given in previous section are slightly modified. Table 4.1 gives an overview for dependency measures rotated by 90° . Note that due to rotations not being unique (clockwise or counterclockwise) rotated measures cannot be simply added to non-rotated(see (Brechmann 2010)).

4.5 Discussion of dependence weights

In this thesis we are searching for the most appropriate weight when selecting an R-Vine tree structure sequentially. The appropriate weight measure will be determined by an extensive simulation study. To capture the behavior in the tails, especially asymmetric tail behavior we use measures of tail dependence instead of Kendall′s τ and Spearman′s ρ. Due to their symmetry, elliptical copulas obtain the same values of upper and lower tail dependence, i.e. $\lambda_U = \lambda_L$. The asymptotic

tail dependence parameter is zero for the Gaussian copula and positive for the t copula. This means if we suspect tail dependence in the data we should discard the Gaussian copula as a model. The symmetric tail dependence of the observations generated by the Student t copula is however well estimated. The Gumbel copula allows for good upper tail dependence while the lower one is zero. In case of exceedance dependence, i.e. exceedance Kendall′s τ, the variable pairs display strong correlation in lower left and upper right quadrant. Note, that theoretical and empirical values of estimates of exceedance dependence are consistent. The measure of tail cumulation used under same settings delivers much poorer results (see (Brechmann 2010)). It fails to discriminate the pairs with strong dependence from those without, with the exception of tail asymmetry of Gumbel with certain parameter values. The results of estimates of the Student t copula are indistinguishable from those of Gaussian copula.
For the Hu dependence (Brechmann 2010) obtained good estimates for Gumbel,t and also Gaussian copulas. It also provides more accurate results if the dependence in the pair of variables increases.
For our studies and simulations we are going to use absolute values of Kendall′s τ and Spearman′s ρ as absolute values as we require positive weight. The thresholds for exceedance dependence are set as $\delta_1 = \delta_2 = 0.2$ due to the best estimation results (see (Brechmann 2010)). Using the tail cumulation we set the boundaries by 0.1. Later on we concentrate on the *max* measure which is the maximum of the two weights to capture the maximal asymmetric tail dependence.
Perhaps, one word about the computation of the discussed weight measures. Except for the Hu dependence every weight we considered took approximately the same computational effort. The calculation of the Hu dependence however was significantly slower due to the underlying optimization problem being more complex than the calculation for the other weight measures.

5 Sequential R-Vine Model Selection

Based on the fact that the main idea of R-Vine is to capture the most significant dependencies in the first tree, we now have a look at the selecting the appropriate model. It is an important step as we know there is a large number of possible R-Vine representations and it grows when dimension becomes larger. The approach of (Dissmann, Czado, Brechmann, and Kurowicka 2012) centers on choosing the R-Vine tree structure sequentially starting from the first tree T_1 until the last tree T_{n-1}. To keep the model parsimonious we like to capture the strongest dependencies as measured by the weights introduced in the last section in the first tree. For this we compute the positive weight λ_{ij} for all possible pair of variables. Out of all possible edges (ij) we select the first tree as the tree which maximizes the sum of weights contained in the tree, i.e. we maximize:

$$\max_{\substack{T\ tree \\ on\ n\ nodes}} \sum_{\substack{e=(i,j) \\ edge\ in\ T}} \lambda_{i,j}, \tag{5.1}$$

This maximization is facilitated using the maximum spanning tree (MST) algorithm of Prim. For all pairs of T_1 select an appropriate copula and fit the corresponding parameters. For the construction of next tree we use this copulas and corresponding parameters to transform the observations using (3.1). They will be used as input values to compute the empirical weight of all pairs of variables, considered to built the next tree, i.e. all pairs that satisfy the proximity condition of Definition 3.3.

This procedure is repeated until all trees are specified. Algorithm 5.1 summarizes this procedure for our selection of the edge weights. It can also be found in (Dissmann, Czado, Brechmann, and Kurowicka 2012). There the Kendall's τ is used as the edge weight, but it can be extended to any other of the weights presented in Section 4.

Algorithm 2. *Sequential method to select an R-vine model*

Input: *Data* $(x_{l1}, ...x_{ln})$, $l = 1, ..., N$ *(realizations of i.i.d. random vectors).*
Output: *R-vine copula specification, i.e.,* $\mathcal{V}, \mathcal{B}$.

1. *Calculate the weight* $\lambda_{i,j}$ *corresponding to the chosen method for all possible variable pairs* $\{i, j\}$, *where* $1 \leq i < j \leq n$.

2. *Select the spanning tree that maximizes the sum of absolute edge weights, as in Algorithm (5.1)*

3. *For each edge* (i, j) *in the selected spanning tree, select a copula and estimate the corresponding parameter(s). Then transform* $\hat{F}_{i|j}(x_{li} \mid x_{lj})$ *and* $\hat{F}_{j|i}(x_{lj} \mid x_{li})$, $l = 1, ..., N$, *using the fitted copula* $\hat{C}_{ij}$

4. *Iteration over the trees:*
   for $k = 2, ..., n-1$ ***do***

5. *Calculate the corresponding weight* $\lambda_{i,j|D}$ *for all conditional variable pairs* $(i, j \mid D)$ *that can be part of tree* T_i, *i.e., all edges fulfilling the proximity condition*

6. *Among these edges, select the spanning tree that maximizes the sum of absolute selected edge weights, i.e.,*

$$\max \sum_{\substack{e=(i,j|D) \\ edge\ in\ T}} \lambda_{i,j|D},$$

7. *For each edge $(i, j \mid D)$ in the selected spanning tree, select a conditional copula and estimate the corresponding parameter(s). Then transform $\hat{F}_{i|j\cup D}(x_{li} \mid x_{lj}, \boldsymbol{x}_{lD})$ and $\hat{F}_{j|i\cup D}(x_{lj} \mid x_{li}, \boldsymbol{x}_{lD})$, $l = 1, ..., N$, using the fitted copula $\hat{C}_{ij|D}$*

8. ***end for***

5.1 Fitting the copula and its parameters

Once the first tree is defined we then select pair copulas from the families discussed previously (Gaussian, Student t, Gumbel and Gumbel survival), that are believed to indicate well performing properties (see (Brechmann, Czado, and Aas 2012)) to model positive and negative dependence as well as asymmetrical behavior and investigate whether the variables are independent. Suppose we have a parametric copula family

$$\boldsymbol{C} = \{C_\theta \mid \theta \in \Theta, \Theta \in \mathbb{R}^t\}, t \geq 1 \tag{5.2}$$

which is assumed to model the dependance between two random variables X and Y and C denotes unknown distribution function of (X, Y), i.e. the null hypothesis is given as

$$H_0 : C \in \boldsymbol{C}. \tag{5.3}$$

This requires estimation of the parameters θ. Let (x_i, y_i) denote a given random sample from (X, Y) for $i = 1, ..., n$. To estimate the parameters of these observations a certain maximum likelihood method is used ((Genest, Choudi, and Riverst 1995) and (Genest and Favre 2007, Pedreira Collazo and Canela 2007). The empirical versions $\hat{F}_X(x) = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{I}_{\{x_i \leq x\}}$ and $F_Y(y) = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{I}_{\{y_i \leq y\}}$ are used to replace the unknown margins F_X, F_Y. Then the observations can be transformed into a set of points (u_i, v_i) such as

$$u_i = \hat{F}_Y(x_i)\ and\ v_i = \hat{F}_Y(y_i), \qquad i = 1, ..., n$$

and the likelihood is defined as:

$$l^{ML}(\boldsymbol{\theta}) = \sum_{j=1}^{N} log(c_{\boldsymbol{\theta}}(u_i, v_i)).$$

Maximizing with respect to $\boldsymbol{\theta}$ delivers the desired estimate $\hat{\boldsymbol{\theta}}$. The approach for copula selection we use is based on the AIC$= -2log(max.likelihood) + 2k$. (see section 6.2.) and has been discussed in (Manner 2007) and investigated in (Brechmann 2010) and is considered to be a reasonable selection criterion. This method compares two competing models according to their AIC value. The AIC for each possible family is computed and we choose the copula with smallest AIC.

5.2 Choosing the independence copula

We want to perform the test of independence in our simulations since it would enable us to evaluate the pairs of variables that do not exhibit dependence. Also an independence test conducted in advance may significantly simplify the model construction. For this we test for the null hypothesis

$$H_0 : C = \Pi \tag{5.4}$$

(Genest and Favre 2007) investigate the bivariate independence test based on Kendall′s τ, since we know from Section 2.5 it holds $\tau(X, Y) = 0$ in case variables X and Y are independent. To test the hypothesis the empirical version of Kendall′s $\hat{\tau}$ from (4.1) is used. In fact, under H_0 the statistic $\hat{\tau}$ is asymptotically normal with mean zero and variance equal to $\frac{2(2N+5)}{9N(N-1)}$. If we set α as an appropriate level of significance, the H_0 should be rejected when

$$\sqrt{\frac{9N(N-1)}{2(2N+5)}} \mid \hat{\tau}(X, Y) \mid > \Phi^{-1}(1 - \alpha/2),$$

where $\Phi^{-1}(1 - \alpha/2)$ is the $(1 - \alpha/2)$ quantile of the standard normal distribution.

6 Model comparison of R-Vines

Once R-Vine models have been selected, we would need to compare them to each other in order to choose the "better"one. To perform this task we need a procedure of model selection, a test that would be appropriate for non-nested model selection. We choose the Vuong test. This test is a relative discrimination test, which means it would not reject both of the competing models. It also specifies the significance level to which the decision has been made. Before we investigate the Vuong test we will look at the underlying approaches: Kullback-Leibler information criterion and Akaike and Bayesian information criteria. For this section we refer to the papers of (Clarke 2003, Clarke 2007) and (Vuong 1989).

6.1 Kullback-Leibler information criterion (KLIC)

This criteria proposed by (Kullback and Leibler 1951) is a measure of closeness.

It is widely used in developing model discrimination tests. Consider a statistical model class $F = \{f(X \mid; \theta), \theta \in \Theta\}$. KLIC measures the distance between the unknown true density h_0 and the approximate model based on estimate $\hat{\theta}$ of the pseudo-true value of θ, defined as:

$$KLIC = E_0(\log h_0(X)) - E_0(\log f(X \mid \hat{\theta})) \tag{6.1}$$

Where E_0 stands for the expectation with respect to the true model.
To find the model contained in F that is nearest to the true one, KLIC is minimized. However, due to the fact that the true model specification is unknown, the KLIC can not be directly estimated, then the closest model must be the one which maximizes $E_0(\log f(\mathbf{X} \mid \hat{\theta}))$. In other words we are looking for a model whose expected log-likelihood is larger in comparison to a rival model of F. The expectation is estimated by the average over the log likelihood contribution of each observation.

6.2 Akaike information criterion (AIC)

The AIC , introduced by (Akaike 1973) combines the principal of measuring the distance between two models using KLIC and a measure for model complexity.

Definition 6.1. Let $\underset{\sim}{\hat{\theta}}$ denote the maximum likelihood estimate of $\hat{\theta}$ for a given parameter vector $\theta_{\boldsymbol{X}} = (\theta_1, ..., \theta_k)$. The Akaike information criterion is defined by

$$
\begin{aligned}
AIC &= -2 \sum_{i=1}^{n} \log f(x_i \mid \underset{\sim}{\hat{\theta}}) + 2k \qquad (6.2) \\
&= -2 \log L(\underset{\sim}{\hat{\theta}}) + 2k
\end{aligned}
$$

for i.i.d. observations $\mathbf{X}$: $x_i, i = 1, ..., n$ of X.

The first term of the equation measures the inaccuracy of the model while the second penalizes the log-likelihood when there are additional free parameters included in the model. So if there are several competing models, the parameters will be estimated by the maximum likelihood method and then the values of AIC need to be computed and compared in order to find the one with the minimum value of AIC - the chosen best model.
Note that if the sample size gets significantly large the first term of AIC increases with it, but the second penalty term does not since it is fixed. Hence, the term $2k$ has only a small effect on the AIC for large n, which displays the need for a stronger penalty term. An alternative method that satisfies this requirement is the Bayesian information criterion.

6.3 Bayesian information criterion (BIC)

The (Schwarz 1978) Bayesian information criterion is the most popular extension of the AIC to compare nested models. It is defined as

$$
\begin{aligned}
BIC &= -2 \sum_{i=1}^{n} \log f(x_i \mid \underset{\sim}{\hat{\theta}}) + k \log n \qquad (6.3) \\
&= -2 \log L(\underset{\sim}{\hat{\theta}}) + k \log n
\end{aligned}
$$

with k the number of free parameters to be estimated and $L(\underset{\sim}{\hat{\theta}})$ denotes the maximized log likelihood of the model considered. Given two different models to be compared the one with the lower BIC value is the one to be preferred.

6.4 The Vuong test

In this section the discrimination test of Vuong ((Vuong 1989)) is presented. This test offers an improvement to the presented model selection criteria such as AIC or BIC. It operates on relative and not absolute terms. This means if the test passes a certain pre-specified level it will choose a model that is closer to the true model specification even if both model are in the far distance from that true specification.

The Vuong test is based on the Kullback-Leibler criterion. Consider two given models with corresponding densities

$$f(\cdot \mid \hat{\theta}_f) \text{ and } g(\cdot \mid \hat{\theta}_g)$$

Then the corresponding maximum likelihood estimates $\hat{\theta}_f, \hat{\theta}_g$ need to be compared. Using KLIC (Vuong 1989) defines the null hypothesis of the test as

$$H_0 : E_0 \left(\log \frac{f(\mathbf{X} \mid \hat{\theta}_f)}{g(\mathbf{X} \mid \hat{\theta}_g)} \right) = 0 \tag{6.4}$$

or equivalently

$$E_0(\log f(\mathbf{X} \mid \hat{\theta}_f)) = E_0(\log g(\mathbf{X} \mid \hat{\theta}_g)) \tag{6.5}$$

that the two models are considered to be equally close to the true model. Is one of the expected values larger than the other, so is the corresponding model to be preferred. The expected value of the null hypothesis is however unknown. To consistently estimate the expected value and to be able to directly compare the two models in terms of closeness to the true specification, Vuong demonstrates that under fair general assumptions it holds

$$\frac{1}{n} LR_n(\hat{\theta}_{f,n}, \hat{\theta}_{g,n}) \xrightarrow{a.s.} E_0 \left(\log \frac{f(\mathbf{X} \mid \hat{\theta}_f)}{g(\mathbf{X} \mid \hat{\theta}_g)} \right), where \tag{6.6}$$

$$LR_n(\hat{\theta}_{f,n}, \hat{\theta}_{g,n}) = L_{f,n}(\hat{\theta}_{f,n}) - L_{g,n}(\hat{\theta}_{g,n}) \tag{6.7}$$

and

$$\hat{\omega}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\log \frac{f(x_i \mid \hat{\theta}_{f,n})}{g(x_i \mid \hat{\theta}_{g,n})} \right)^2 - \left(\frac{1}{n} \sum_{i=1}^{n} \log \frac{f(x_i \mid \hat{\theta}_{f,g})}{g(x_i \mid \hat{\theta}_{g,n})} \right)^2 \tag{6.8}$$

denote the estimated standard deviation. According to (Vuong 1989) under the null hypothesis H_0:

$$\frac{LR_n(\hat{\theta}_{f,n}, \hat{\theta}_{g,n})}{\hat{\omega}_n \cdot \sqrt{n}} \xrightarrow{D} N(0,1) \; as \; n \to \infty \tag{6.9}$$

i.e. the likelihood ratio statistic is asymptotically normally distributed. This means that the null hypothesis H_0 will be rejected at level α when the resulting likelihood

$$\left| \frac{LR_n(\hat{\theta}_{f,n}, \hat{\theta}_{g,n})}{\hat{\omega}_n \cdot \sqrt{n}} \right| > \Phi^{-1}(1 - \alpha/2) \tag{6.10}$$

and model F_θ is preferred over model G_θ if the value of likelihood is larger $\Phi^{-1}(1 - \alpha/2)$ and model G_θ over model F_θ if this value is smaller $\Phi^{-1}(1 - \alpha/2)$.
In a further investigation (Vuong 1989) shows that the above test is sensitive to the possibility that the models may have a different number of parameters. He proposes a **corrected** or **adjusted Vuong test** which takes the dimensions of the models into account. Vuong suggests to use either AIC or BIC for the correction of the test. The adjusted statistic is then defined as

$$\tilde{L}R_n(\hat{\theta}_{f,n}, \hat{\theta}_{g,n}) := LR_n(\hat{\theta}_{f,n}, \hat{\theta}_{g,n}) - P_f - P_g \tag{6.11}$$

where P_f, P_g denote the number of estimated parameters of the corresponding models. In similar way the Schwarz correction of the Vuong test is defined as

$$\tilde{L}R_n(\hat{\theta}_{f,n}, \hat{\theta}_{g,n}) = LR_n(\hat{\theta}_{f,n}, \hat{\theta}_{g,n}) - (P_f/2)\log n - (P_g/2)\log n \tag{6.12}$$

7 Simulation Study

In this Chapter we apply the strategies we discussed in the previous Chapters on different scenarios. We simulate 1000 times from a given R-Vine considered as the “true “model for each of twenty different scenarios with different set of pair copula families and parameters. For each of these scenarios an R-Vine is sequentially specified using one of considered strategies, i.e. different weights in selection of trees. For this purpose we notify the strategies by numbers according to the weight used:

- ***Strategy* 1 :** Kendall′s τ
- ***Strategy* 2 :** Tail Cumulation. We chose maximum of lower and upper coefficient using boundaries=0.1, i.e.

$$\max\{\hat{\gamma}^{upper}, \hat{\gamma}^{lower} : u^{upper} = u^{lower} = 0.1\} \text{ (see (4.3))}$$

- ***Strategy* 3 :** Hu Dependence. Maximum of lower and upper coefficient to capture max asymmetrical tail dependence, i.e.

$$\max\{\omega_1, \omega_2\} \text{ (see (4.4))}.$$

- ***Strategy* 4 :** Exceedance Dependence. Maximum of lower coefficient with thresholds=0.2 and upper coefficient with thresholds=0.8, i.e.

$$\max\{\hat{\tau}^{upper}, \hat{\tau}^{lower} : \delta_1^u = \delta_2^u = 0.8, \delta_1^l = \delta_2^l = 0.2\} \text{ (see Section 4.3)}.$$

We repeat this 100 times and then compare the resulting R-Vine models by Vuong test . We hope to specify a “better “Strategy or perhaps an alternative to the widely used method based on Kendall′s τ. Pairwise independence tests are made to reduce number of parameters.

7.1 Scenarios and Methods

We consider the 8-dimensional R-Vine model given in Chapter 3 Figure 3.1. to be our “true “model, so that the original tree structure, copula families and parameters are known to us for each scenario we choose. In terms of matrix representation as described earlier the R-Vine matrix is given by:

$$M = \begin{pmatrix} 4 & & & & & & & \\ 7 & 8 & & & & & & \\ 5 & 7 & 5 & & & & & \\ 6 & 5 & 7 & 7 & & & & \\ 8 & 6 & 6 & 1 & 6 & & & \\ 1 & 3 & 1 & 2 & 1 & 1 & & \\ 2 & 1 & 3 & 3 & 2 & 3 & 2 & \\ 3 & 2 & 2 & 6 & 3 & 2 & 3 & 3 \end{pmatrix}$$

We summarize for each scenario:

- 8-dimensional R-Vine
- 1000 sample size
- 100 data sets

We apply the strategies mentioned earlier. Our goal is to see which of this models will enable us to get the “closest fit “to the original R-Vine model.
As mentioned earlier we will use 20 different scenarios. Figure 7.1 provides an overview. Clearly, they are based on different copula types and strength of dependence. To summarize:

- Copula type:
  (i) all mixed pair copulas (non-elliptical)
  (ii) T1-T3 Student t , T4-T7 Gauss (elliptical)
- Dependence structure
  (i) all trees allow for dependent pair copulas (dependent)
  (ii) T4-T7 contain only independence copulas (independent)
  (iii) T4-T7 contain only Gaussian copulas (simplified)
- Dependence strength:
  (i) monotonically increasing
  (ii) monotonically decreasing
  (iii) constant strong
  (iv) constant weak

From the Figure 7.1 we can see that the Scenarios 21-24 correspond to the Scenarios 17-20 with respect to the higher order trees and that leaves us with 20 scenarios in total. The precise specifications with corresponding family and parameter matrices can be found in Appendix A.
At last we recall the copula families we choose for modeling in this simulation study. Those are:

- Gauss copula with no tail dependence
- Student t copula, exhibits tail dependence
- Gumbel copula, to capture upper tail dependence

- rotated Gumbel copula

  (i) 90° : no tail dependence

  (ii) 180° : lower tail dependence

  (iii) 270° : no tail dependence

Note also, t and Gaussian copulas belong to symmetrical copulas while Gumbel and rotated Gumbel are asymmetric. From the previously conducted studies it is know that this list of copulas is able to model positive as well as negative dependence.
We will use maximum likelihood method to estimate the parameters and the AIC method for the pair-copula selection. We decide in favor of the model with smallest AIC. In addition, we want to see which of the "changed "factors in a R-Vine model specification would have a greater influence on the resulting fitted model. For this purpose we simulated from the "true "model under same conditions two further models. In the first one we estimated the parameters leaving the tree structure and the pair-copulas identical to the "true "ones. In the second we estimated parameters as well as the pair-copulas. Later in this Chapter we will discuss the results of the log likelihood comparison for all models.

7.2 Results of the non-nested model comparison using the Vuong Test.

In this section we summarize the results of the model comparison based on the Vuong test, discussed in Chapter 6. We compare our strategies pairwise. Let the H_0 of the test be that our two models to be compared perform equally well, i.e. we can not choose the Strategy providing a "better fitted "model. The decision is made at level $\alpha = 0.05$. Figure 7.2 displays the results of the test. Every "$\approx$"sign means that the null hypothesis H_0 could not be rejected the test could not prefer one model over the other. We add the number of times out of 100 when we could not reject H_0 and in case it is larger then 50 will say the models can not be distinguished in terms of Vuong test. If it is not the case, we count how often one of two compared models performed better. However, we need to keep in mind that while giving additional information on the statistical significance (see Chapter 6), the Vuong test does not deliver any information on the general goodness-of-fit. To check the reliability of the performed test we have a look at the normal QQ plots for each Scenario. It will give us an impression on whether the normality assumption is met and the statistics are normally distributed. These are found in Appendix A. However, due to the simulation size of 100 data sets being rather

Figure 7.1: Overview of scenarios for the simulation study. Precise specifications can be found in Appendix A.1.

	Copula families	Dependence structure	Correlation strength
Scenario1	mixed	T4-T7 indep	mon. decreasing
Scenario2	mixed	T4-T7 indep	const. weak
Scenario3	mixed	T4-T7 indep	const. strong
Scenario4	mixed	T4-T7 indep	mon. increasing
Scenario5	mixed	T4-T7 Gauss	mon. decreasing
Scenario6	mixed	T4-T7 Gauss	const. weak
Scenario7	mixed	T4-T7 Gauss	const. strong
Scenario8	mixed	T4-T7 Gauss	mon. increasing
Scenario9	mixed	all dependent	mon. decreasing
Scenario10	mixed	all dependent	const. weak
Scenario11	mixed	all dependent	const. strong
Scenario12	mixed	all dependent	mon. increasing
Scenario13	T1-3: t; T4-7: Gauss	T4-T7 indep	mon. decreasing
Scenario14	T1-3: t; T4-7: Gauss	T4-T7 indep	const. weak
Scenario15	T1-3: t; T4-7: Gauss	T4-T7 indep	const. strong
Scenario16	T1-3: t; T4-7: Gauss	T4-T7 indep	mon. increasing
Scenario17	T1-3: t; T4-7: Gauss	T4-T7 Gauss	mon. decreasing
Scenario18	T1-3: t; T4-7: Gauss	T4-T7 Gauss	const. weak
Scenario19	T1-3: t; T4-7: Gauss	T4-T7 Gauss	const. strong
Scenario20	T1-3: t; T4-7: Gauss	T4-T7 Gauss	mon. increasing
Scenario21	T1-3: t; T4-7: Gauss	all dependent	mon. decreasing
Scenario22	T1-3: t; T4-7: Gauss	all dependent	const. weak
Scenario23	T1-3: t; T4-7: Gauss	all dependent	const. strong
Scenario24	T1-3: t; T4-7: Gauss	all dependent	mon. increasing

small, we expect skewness in the QQ plots. Hence, in the most cases we would relax the assumption in order to validate the test.

7.3 Ranking of strategies Performance

Once the test results are summarized, we provide an overview in Table 7.1 using ranking for validation. This would give us a better impression on the results of the Vuong test.

Scenarios with mixed copulas	Kendall	TailCum	HuDep	ExceedDep
Scenario1	2	1	0	1
Scenario2	3	0	0	2
Scenario3	2	0	2	1
Scenario4	2	0	3	1
Scenario5	2	0	3	1
Scenario6	0	1	3	2
Scenario7	2	0	3	1
Scenario8	0	1	3	2
Scenario9	2	0	3	1
Scenario10	1	0	2	3
Scenario11	3	0	2	1
Scenario12	1	0	3	2
Scenarios with elliptical copulas	**Kendall**	**TailCum**	**HuDep**	**ExceedDep**
Scenario13	2	0	0	2
Scenario14	2	0	0	2
Scenario15	1	0	3	1
Scenario16	2	0	3	1
Scenario17	2	0	1	2
Scenario18	2	0	3	1
Scenario19	2	0	3	1
Scenario20	2	0	3	1

Table 7.1: Ranking on the Vuong test results. Each number represents the “score“of the corresponding Strategy.

In Table 7.1 each number denotes the points each Strategy receives for performing better than the others. For example, in Scenario 11 the Strategy 1 performed better than other three strategies, i.e. gained 3 points. Analogously, 2 points are given when a Strategy performs better than two others. Whenever the test could not decide between two strategies, we will give them equal number of points. For convinience, we denote each Strategy with appropriate abbreviation.

Figure 7.2: Results of the Vuong test. Here M stands for "Model chosen by Strategy "and the number for the number of times certain Strategy was preferred over other, respectively.

	M1>M2	M1 $\approx$ M2	M1<M2	M1>M3	M1 $\approx$ M3	M1<M3	M1>M4	M1 $\approx$ M4	M1<M4	
Scenario1	50	46	4	100	0	0	35	64	1	ind
Scenario2	75	24	1	78	21	1	55	40	5	ind
Scenario3	99	1	0	31	38	31	64	13	23	ind
Scenario4	76	19	5	4	4	92	60	28	12	ind
Scenario5	85	7	8	16	21	63	52	31	17	sim
Scenario6	9	38	53	0	4	96	9	31	60	sim
Scenario7	91	6	3	18	16	66	85	7	8	sim
Scenario8	16	32	52	1	2	97	20	27	53	sim
Scenario9	97	3	0	24	35	41	79	19	2	dep
Scenario10	53	42	5	31	36	33	35	24	41	dep
Scenario11	92	7	1	44	23	33	71	14	15	dep
Scenario12	63	35	2	0	15	85	26	34	40	dep
Scenario13	65	33	2	65	34	1	32	55	13	ind
Scenario14	93	7	0	84	16	0	35	60	5	ind
Scenario15	95	5	0	25	30	45	25	54	21	ind
Scenario16	61	24	15	11	12	77	47	28	25	ind
Scenario17	83	5	12	11	5	84	84	10	6	sim
Scenario18	81	17	2	59	38	3	28	52	20	sim
Scenario19	99	1	0	21	15	64	41	24	35	sim
Scenario20	93	6	1	34	30	36	43	32	25	sim
TOTAL	1476	358	166	657	395	948	926	647	427	

	M 2>M3	M 2 $\approx$ M3	M 2<M3	M 2>M4	M 2 $\approx$ M4	M 2<M4	M4>M3	M4 $\approx$ M3	M4<M3	
Scenario1	96	4	0	11	58	31	100	0	0	ind
Scenario2	23	57	20	26	36	38	34	49	17	ind
Scenario3	1	7	92	16	11	73	24	11	65	ind
Scenario4	1	5	94	25	32	43	4	3	93	ind
Scenario5	3	8	89	13	21	66	8	18	74	sim
Scenario6	1	9	90	18	38	44	11	20	69	sim
Scenario7	1	2	97	22	24	54	2	3	95	sim
Scenario8	1	3	96	31	33	36	1	6	93	sim
Scenario9	3	9	88	10	12	78	9	17	74	dep
Scenario10	24	23	53	22	31	47	42	32	26	dep
Scenario11	4	10	86	16	18	66	20	15	65	dep
Scenario12	0	5	95	18	23	59	7	18	75	dep
Scenario13	30	55	15	12	43	45	58	39	3	ind
Scenario14	20	62	18	4	34	62	65	32	3	ind
Scenario15	8	11	81	2	11	87	26	25	49	ind
Scenario16	12	7	81	36	17	47	10	11	79	ind
Scenario17	4	4	92	30	18	52	5	5	90	sim
Scenario18	28	39	33	6	27	67	57	30	13	sim
Scenario19	3	4	93	4	5	91	23	14	63	sim
Scenario20	6	10	84	8	11	81	35	22	43	sim
TOTAL	269	334	1397	330	503	1167	541	370	1089	

Figure 7.3: Results on the normality assumption for every scenario. If the assumption is not fulfilled it is denoted with "⊝ ".Note, that mostly it is the case when Strategy 3 is preferred over one of the other strategies.

Scenario 1	$M1 > M2$ ✓	$M1 > M3$ ✓	$M1 > M4$ ✓	$M2 > M3$ ✓	$M2 \approx M3$ ✓	$M4 > M3$ ✓
Scenario 2	$M1 > M2$ ✓	$M1 > M3$ ✓	$M1 > M4$ ✓	$M2 \approx M3$ ✓	$M2 < M4$ ✓	$M4 > M3$ ✓
Scenario 3	$M1 > M2$ ✓	$M1 = M3$ ✓	$M1 > M4$ ✓	$M2 < M3$ ⊝	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 4	$M1 > M2$ ✓	$M1 < M3$ ⊝	$M1 > M4$ ✓	$M2 < M3$ ⊝	$M2 < M4$ ✓	$M4 < M3$ ⊝
Scenario 5	$M1 > M2$ ⊝	$M1 < M3$ ✓	$M1 > M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 6	$M1 < M2$ ✓	$M1 < M3$ ✓	$M1 < M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 7	$M1 > M2$ ✓	$M1 < M3$ ✓	$M1 > M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 8	$M1 < M2$ ✓	$M1 < M3$ ✓	$M1 < M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 9	$M1 > M2$ ⊝	$M1 < M3$ ✓	$M1 > M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 10	$M1 > M2$ ✓	$M1 < M3$ ✓	$M1 < M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 > M3$ ✓
Scenario 11	$M1 > M2$ ✓	$M1 > M3$ ✓	$M1 > M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 12	$M1 > M2$ ✓	$M1 < M3$ ✓	$M1 < M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 13	$M1 > M2$ ✓	$M1 > M3$ ✓	$M1 \approx M4$ ✓	$M2 \approx M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 14	$M1 > M2$ ✓	$M1 > M3$ ✓	$M1 \approx M4$ ✓	$M2 \approx M3$ ✓	$M2 < M4$ ✓	$M4 > M3$ ✓
Scenario 15	$M1 > M2$ ✓	$M1 < M3$ ✓	$M1 \approx M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 16	$M1 > M2$ ✓	$M1 < M3$ ✓	$M1 > M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 17	$M1 > M2$ ⊝	$M1 < M3$ ⊝	$M1 > M4$ ⊝	$M2 < M3$ ⊝	$M2 < M4$ ✓	$M4 < M3$ ⊝
Scenario 18	$M1 > M2$ ✓	$M1 > M3$ ✓	$M1 \approx M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 > M3$ ✓
Scenario 19	$M1 > M2$ ✓	$M1 < M3$ ✓	$M1 > M4$ ✓	$M2 < M3$ ⊝	$M2 < M4$ ✓	$M4 < M3$ ✓
Scenario 20	$M1 > M2$ ⊝	$M1 < M3$ ✓	$M1 > M4$ ✓	$M2 < M3$ ✓	$M2 < M4$ ✓	$M4 < M3$ ✓

To have a better impression on the ranking of strategies we analyze them by counting their overall performance. This means we see how often each of them "earned"a certain place. We also analyze the performance of each individual Strategy according to the properties of the scenarios. From Figure 7.4 we can see that Strategy 2 preforms very poorly for both of the copula family choices. Strategy 3 tends to compete for the first place. Strategy 1 performs well for both of the choices and seems to be rather consistent with its ranking. Strategy 4 tends to perform well and take 2nd and 3rd places. Note also that it is never "outperformed"by any other Strategy. However, it is easier to rank the strategies when applied on scenarios with elliptical copula families. Here, first place is reserved for Strategy 3, followed by the Strategy 1. Strategy 4 takes third place. Finally, the least preferable is the Strategy 2.

For an overview with a different perspective Figure 7.5 shows the total number of each Strategy fitting best, i.e. taking first place, when we separate scenarios according to their dependence and correlation features. With color yellow we mark the Strategy outcome that scores best and with green we mark the second place winner. Since this Figure only reflects the number of times each Strategy performed best, it does not contain all outcome numbers. Those are summarized in Appendix A.

We can see that for Scenarios with independent trees we would choose the Strategy 1 while 3 takes place 2 followed by Strategy 4. For dependent and simplified scenarios,i.e. Scenarios using Gaussian copulas in trees $T_4 - T_7$, Strategy 3 takes first place while 1 and 4 would be equal next choice. Strategy 2 would be least preferable. This pattern is partly repeated if we have a look at the ranking according to the correlation properties. Strategy 3 is winner when the correlation monotonically decreases and offers second best choice for other types of correlation. It is also preferred in case of constant strong correlation as well as monotonically non-decreasing. Strategy 4 would be the preferred in case of constant weak correlation. All in one we could say that the Strategy 3 seems to offer the best fit tightly followed by Strategy 1 and Strategy 4 as second choice. The model fitted using the Strategy 2 approach seems to be rather to decline.

mixed copulas	Kendall	TailCum	HuDep	ExceedDep
Rank 1	4	0	8	1
Rank 2	4	1	2	5
Rank 3	2	2	0	6
Rank 4	2	9	2	0

t/Gauss copulas	Kendall	TailCum	HuDep	ExceedDep
Rank 1	3	0	5	3
Rank 2	4	0	1	0
Rank 3	1	0	0	5
Rank 4	0	8	2	0

Figure 7.4: Ranking of the individual strategies denoted corresponding to the weight used according to their place.

T4-T7	Kendall	TailCum	HuDep	ExceedDep
independent	5	0	4	2
dependent	1	0	2	1
simplified	1	0	7	1

dependence structure	Kendall	TailCum	HuDep	ExceedDep
mon.decreasing	2	0	3	1
const.weak	3	0	1	3
const.strong	2	0	4	0
mon.increasing	0	0	5	0

Figure 7.5: Ranking of the individual strategies according to the number of times they took first place.

7.4 Discussion of the Log Likelihood Values

In following we introduce and analyze the log likelihood values of the performed simulation. We denote the introduced model as follows:

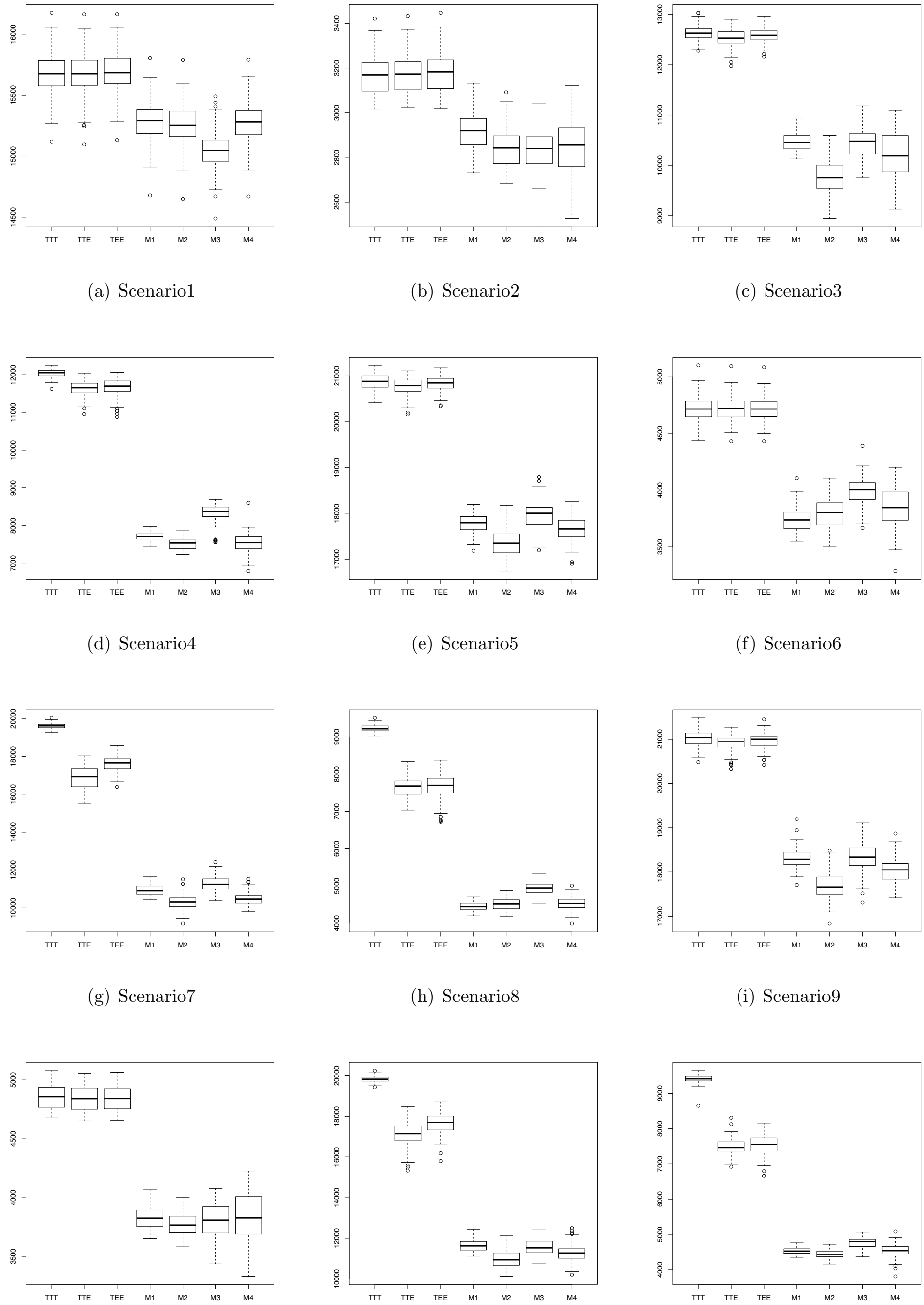
- TTT : "true "model
- TTE : parameters estimated
- TEE : parameters and pair-copulas estimated
- M1 to M4: model choosing weights by Strategy 1 to 4, respectively.

For each Strategy and each scenario the corresponding 100 log likelihood points are summarized in a boxplot. For a better comparison the Figures 7.8 to 7.9 show the log likelihoods relative to the "true "model,i.e. in percentage to the value of the TTT as median.
At the first look one can immediately observe the crucial role that the choice of the tree structure plays. In compare to the TTE and TEE model, where the tree structure was maintained, all of the four strategies we use show a large difference in the likelihood values. In some of them one can see that while we "lose points"through estimation of the parameters and/or pair-copulas, the impact is still relatively small in compare to the jumps when the tree structure is fitted. Since we use the same selection procedure for each of our strategies (maximum likelihood for parameter estimation and smallest AIC for selection of the copula families),we can analyze the results by corresponding groups and not individually. The selected tree structure becomes the most important factor for the evaluation of the strategies performance. In following we investigate the performance of the strategies according to the underlying scenario.

Scenarios with mixed copulas vs. scenarios with elliptical copulas

To compare the differences in the performance of scenarios using mixed copula families (Scenarios 1-12) to those who use t-copula in trees 1 to 3 and Gaussian copulas in trees 4 to 7 (Scenarios 13-20) we have a look at Figures 7.6 and 7.7 and observe the differences. At first, we want to see how the TTE and TEE models behave in compare to the TTT model in each of this Figures. Both of them show rather small differences in the likelihood values and seem to have a similar pattern. The main difference lies in the likelihood values of the model chosen by our strategies. In scenarios with mixed copulas M1-M4 show much larger "loss"till up to 30-35% while in those with t/Gauss copulas this difference is visual but not that major. By elliptical scenarios we can also observe that the "loss"range is rather undeviating and held around 10%, while the scenarios with mixed copulas exhibit

Figure 7.6: Log likelihoods values corresponding to 12 non-elliptical scenarios.

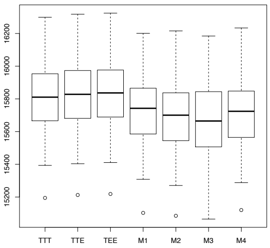


(a) Scenario1 (b) Scenario2 (c) Scenario3

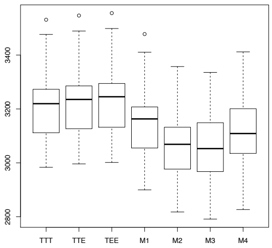
(d) Scenario4 (e) Scenario5 (f) Scenario6

(g) Scenario7 (h) Scenario8 (i) Scenario9

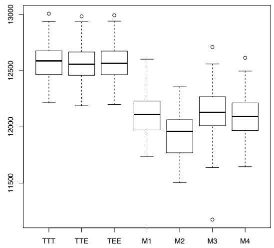
Figure 7.7: Log likelihoods values corresponding to 8 elliptical scenarios.



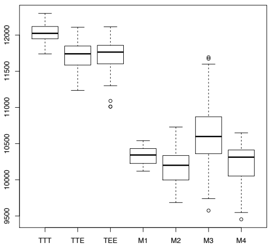
(m) Scenario13



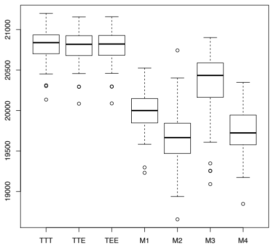
(n) Scenario14



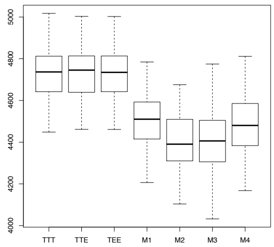
(o) Scenario15



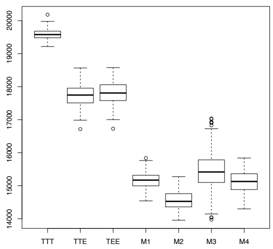
(p) Scenario16



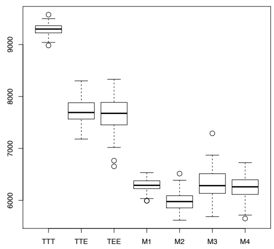
(q) Scenario17



(r) Scenario18

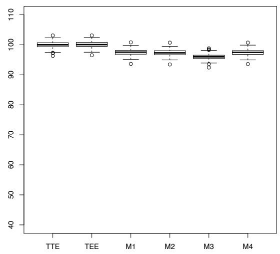


(s) Scenario19

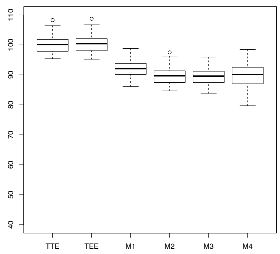


(t) Scenario20

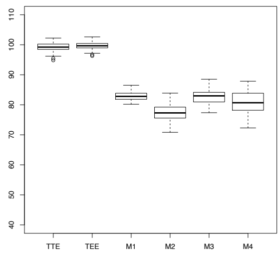
Figure 7.8: Log likelihoods values in % of the models corresponding to 12 non-elliptical scenarios .



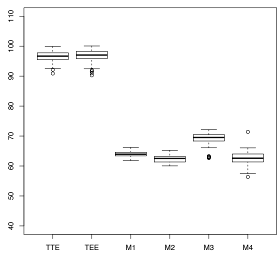
(a) Scenario1



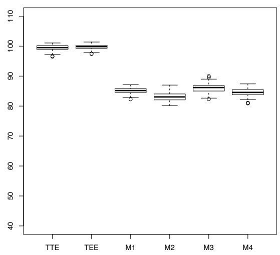
(b) Scenario2



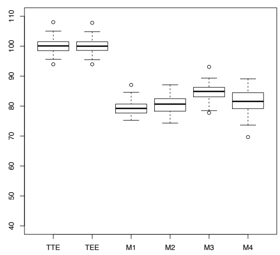
(c) Scenario3



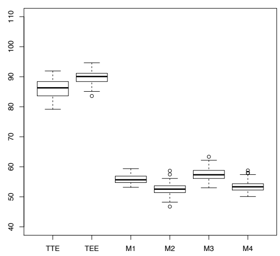
(d) Scenario4



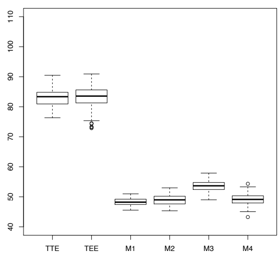
(e) Scenario5



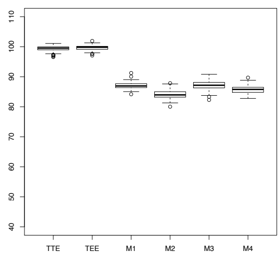
(f) Scenario6



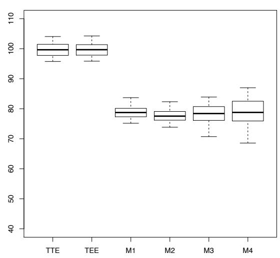
(g) Scenario7

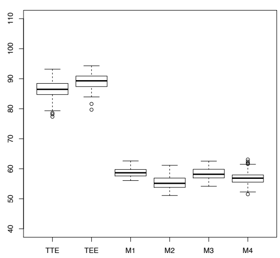


(h) Scenario8



(i) Scenario9





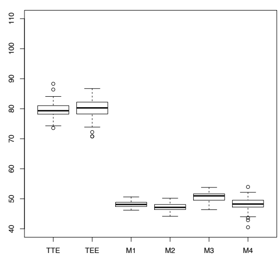
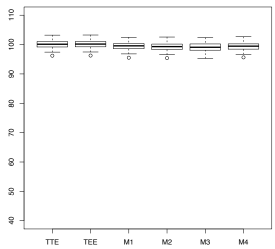
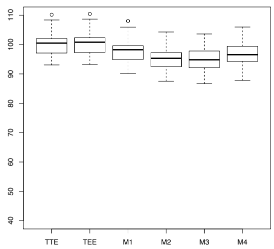


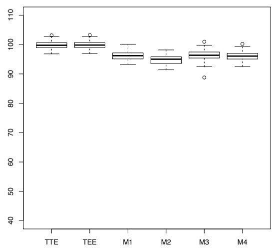
Figure 7.9: Log likelihoods values in % of the models corresponding to 8 elliptical scenarios.



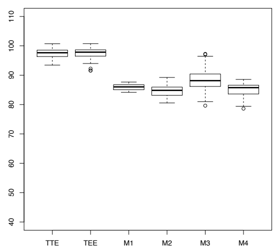
(m) Scenario13



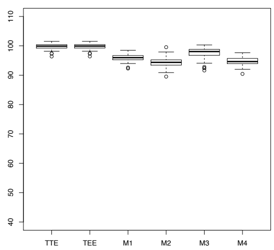
(n) Scenario14



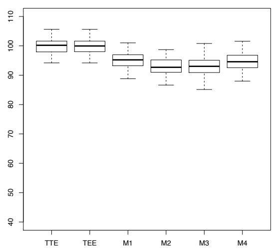
(o) Scenario15



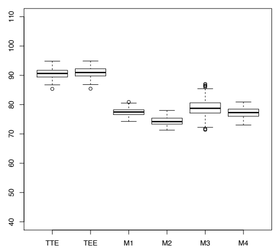
(p) Scenario16



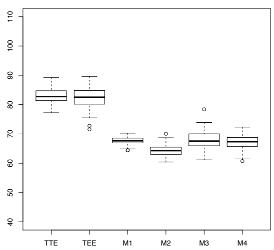
(q) Scenario17



(r) Scenario18



(s) Scenario19



(t) Scenario20

various numbers from relatively small (Scenario 1,2) to rather large (4,7,8,11,12). Now, if we have a look at the log likelihood values of the M1 to M4, we can observe that similar to Vuong test results, Strategy 4 seems to perform medium well, while Strategy 2 delivers rather poor results due to its inflexibility. The most interesting strategies appear to be 1 and 3. The Strategy 3 seems to be either best or worst choice, due to the extreme behavior of the underlying weight. Nevertheless, it seems to offer a good solution in both of the scenario types. Note also that Strategy 1 and Strategy 3 often deliver almost equally good results. This was also the case in the Vuong test performance. At last , we note that all four strategies seem to perform in very similar patterns for mixed and elliptical scenarios according to their correlation properties and the dependence properties. The exception are Scenarios 6 and 18 with constant weak correlation and simplified higher trees.

(i) Scenarios with monotonically increasing correlation

When looking at the performance of the monotonically increasing correlation scenarios we notice the drop of the likelihood values between TTT, TTE and TEE and models and those chosen by the Strategies 1 to 4. While by Scenarios 4 and 16 with independent higher order trees TTE and TEE model seems to still perform quite well, Scenarios(4,12,20) with dependent trees exhibit “loss“within this comparisons. The overall likelihood values also drop rapidly for all strategies. This means that the dependence in higher trees is rather problematic. In the case of increasing dependence the Strategy 3 is clearly preferred.

(ii) Scenarios with monotonically decreasing correlation

When comparing scenarios with monotonically decreasing correlation to other correlation types scenarios the first notice is that the likelihood value of these scenarios are the highest. Hence, those correlation features are more realistic. Scenarios with assumed independence in higher order trees (1 and 13) show relatively small drop in likelihood value between strategies but out rule the Strategy 3 in their choice. However, the likelihood value itself is smaller than those of the Scenarios 5,9 and 17. This indicates that the independence in higher trees is rather difficult to capture. On the other hand if the we are given dependent or even simplified trees in the scenarios, the likelihood of the models increases and delivers ordered Strategy choices.

(iii) Scenarios with constant strong correlation

If we look at the Scenarios 3,7,11,15 and 19 which assume constant strong correlation we observe that the overall likelihood values are rather large, yet slightly smaller than those of the scenarios with monotonically decreasing correlation. Again, the dependence in higher trees provides a little larger values of likelihood. Next thing we notice is that the Scenarios 7,11, and 19 lose value in the estimation TTE and TEE already which can be considered to occur due to the dependence in the higher trees property, while scenarios 3 and 15 with independent trees keep the values of TTT, TTE and TEE almost equal. However, all strategies show similar pattern in their performance.

(iv) Scenarios with constant weak correlation

In the Scenarios with constant weak correlation (2,6,10,14 and 18) the likelihood values are very small in compare to any other correlation type. Hence, it is easier to underestimate the difference in the performance of strategies and make a wrong choice. We also notice that analog to scenarios with monotonically decreasing correlation and independent higher trees the loss in likelihood by strategies in compare to TTT, TTE and TEE models is very small and the Strategy 1 is the one with highest value of likelihood in compare to others. Due to the low correlation value we expect the weights be rather centered and this is the case in Strategy 1. Strategies 2,3 and 4 looking for the most tail dependence are clearly lower.

7.5 Conclusions

We want to summarize the results from the Vuong test comparison as well as the results we were able to derive from the likelihoods.
First of all, the major role in goodness-of- fit is played by the tree structure,i.e. it has the biggest impact on the likelihood value. The estimation of parameters and pair-copulas are rather unimportant. This however is different for monotonically increasing and constant strong correlations and dependent higher order trees. If we compare mixed copulas scenarios to the one using only elliptical copulas, all t/Gauss copula scenarios have a smaller "loss"in the likelihood values. In mixed scenarios the best choice is Strategy 3 followed by Strategy 1. Strategy 1 and 3 also perform almost equally well if we have all dependent trees. In case of independent higher trees we need to look at the correlation. If the correlation is increasing or constant strong, we favor the Strategy 3. When the correlation is constant weak or decreasing Strategy 1 is clearly best choice. The weak correlation causes very low likelihood values and rather bad goodness-of-fit. Strategy 4 seems to perform consistently well in every environment, but is hardly the best choice. Strategy 2 loses to all others Strategies in mixed as well as elliptical copula scenario types, except just a few.

8 Applications

In this section we will apply the strategies discussed in previous chapters on three different data sets. The first data set we look at is the exchange rates from (Schepsmeier 2010) and (Dissmann 2010). Due to the relatively small size of nine variables we will be able to have a better chance to analyze the outcome. We compare the different choices of first tree made by each strategy using the maximum spanning tree algorithm and also to see which of them shows the biggest change in the likelihood values. We also investigate the performance of our strategies applied on the (Dissmann 2010) financial indices data set and the German DAX data set which due to the large size we have summarized the results in a brief overview. In our applications we will use the sequential method from Chapter 5 to select the appropriate R-Vine. Individual copula parameters are estimated via maximum likelihood. For the selection of copula family we use the smallest AIC approach. For each of variable pair we apply independence test with level 0.05.We use the same copula families selection as in the simulation study (Gaussian,Student t, Gumbel and rotated Gumbel). Also, for t-copulas with degree of freedom larger than 30 instead the Gaussian copula will be used.

8.1 Exchange Rates Data Set

The exchange rates data set contains 9 variables, each with respect to the US-Dollar from 7/22/2005 to 7/17/2009. The notation is given below in Table 8.1.

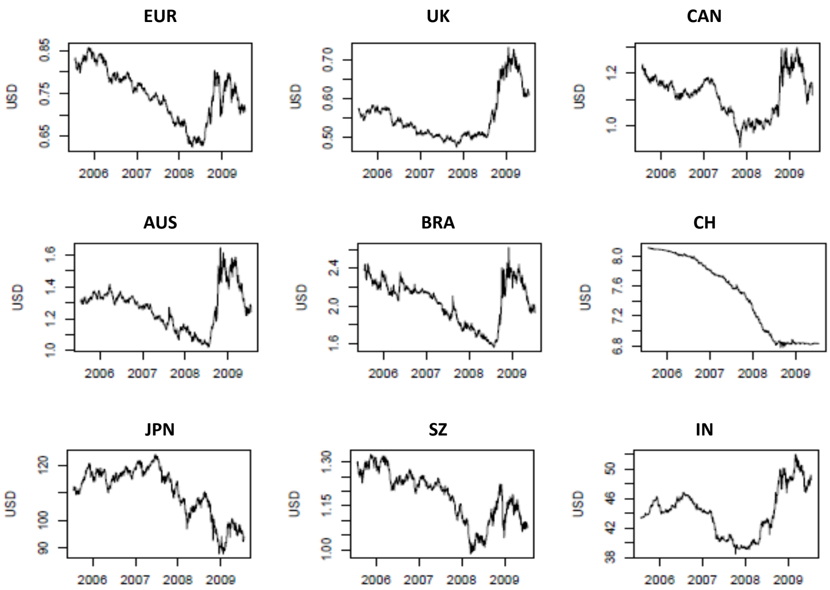
notation	currency
EUR	Euro
UK	British Pound
CAN	Canadian Dollar
Kendall AUS	Australian Dollar
BRA	Brazilian Real
CH	Chinese Yuan
JPN	Japanese Yen
SZ	Swiss Frank
IN	Indian Rupee

Table 8.1: Short names and corresponding currency of exchange rates.

Figure 8.1. shows ARMA(1, 1) and GARCH(1, 1) models fitted to individual time series. For detailed information on parameters see (Dissmann 2010). Residuals were transformed using their empirical distribution function. According to

dependence structure based on Kendall′ tau (pairs-plots are given in Figure B.1, we can observe those pairs of variables that display stronger dependencies in comparison to others. For example, pairs using EUR, UK or SZ. We apply our four strategies and compare the different R-Vines that every strategy chooses as appropriate.

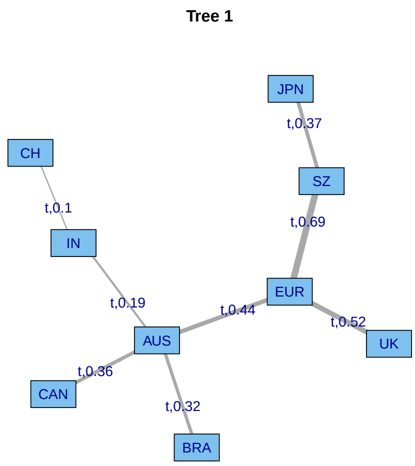
Figure 8.1: Time series development of exchange rates with respect to the US-Dollar from 7/22/2005 to 7/17/2009.



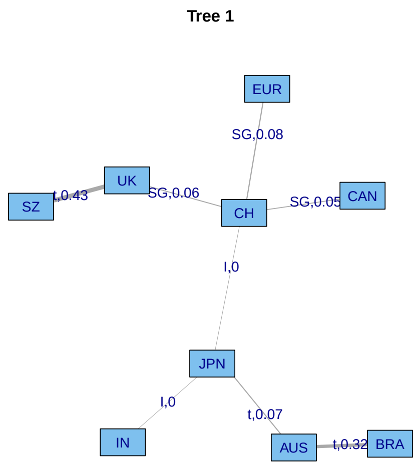
Comparison of the resulting first tree

Since we are using different weight measures in our strategies we are interested in whether and how significantly the selected R-Vine models differ from each other. For this purpose we first look at the first tree T_1 of each of the models. Since the weights were different we are mainly interested if our strategies select different pairs of variables among all possible pairs to maximize the resulting sum. Figure 8.2 shows the T_1 for three of our strategies, without the Strategy 2 which uses tail cumulation as weight. The reason is that the first tree of Strategy 1 and Strategy 2 coincide, i.e. Strategy 2 chooses same pairs of variables as most important. This can also be obtained in the overview of R-Vine matrices and family matrices in Appendix B.1. The width of the edge in the trees is denoted with the copula family that was selected as appropriate and the corresponding value of theoretical

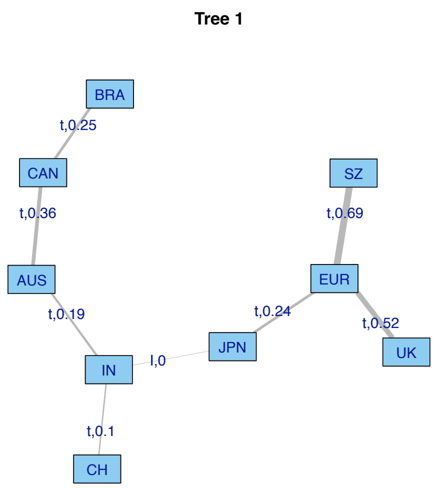
Figure 8.2: First trees of the full R-vine copula model based on Strategies 1,3 and 4 for the exchange rates data set . The edge labels indicate empirical Kendall's τ and the bivariate copula families between the respective variables.



(a) T_1 of R-Vine using Kendall's τ or tail cumulation as weight



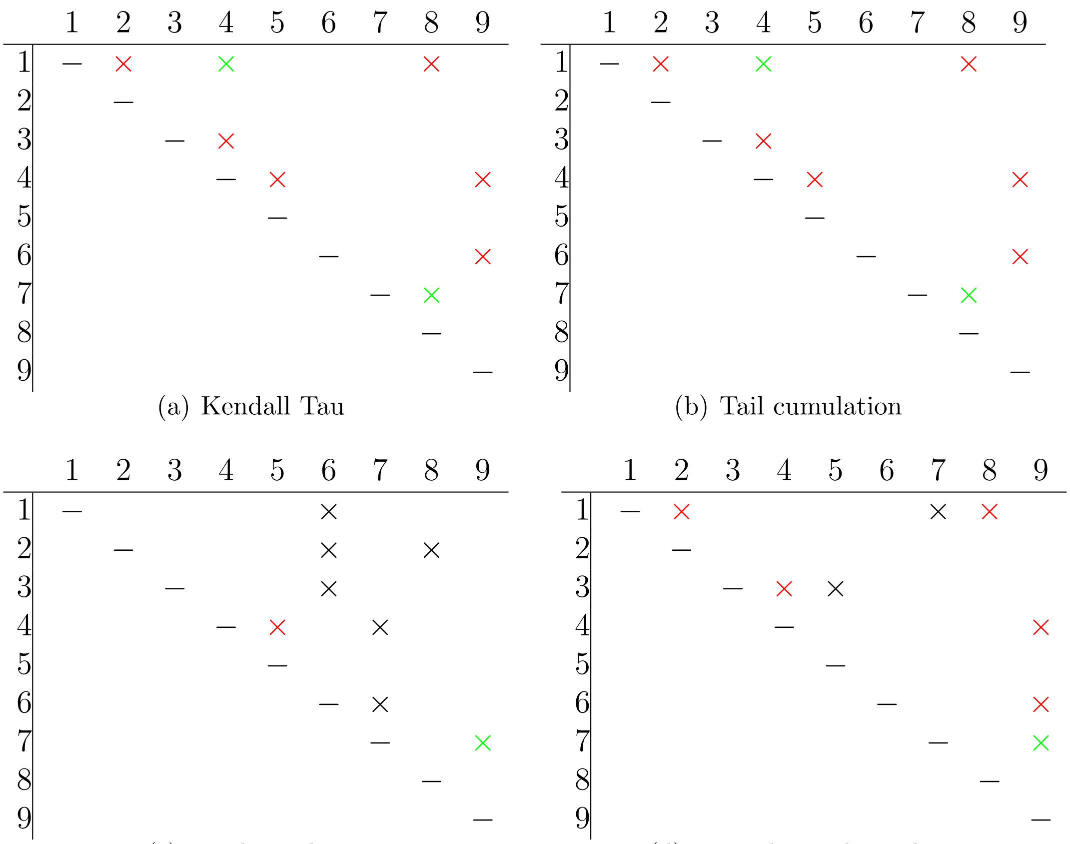
(b) T_1 of R-Vine using Hu dependence as weight



(c) T_1 of R-Vine using exceedance dependence as weight

Kendall's τ. The larger this value the thicker the edge line is .
At first we can immediately observe that selected pairs of variables differ corresponding to the applied strategy. Since, Strategy 1(S1) and Strategy 2(S2) select same pairs of variables out of all passible pairs, we compare Strategies 3(S3) and 4(S4) to the first tree chosen by the Strategy 1. We can see that there is a strong connection between EUR,UK,SZ and AUS, CAN as well few others which were selected by the Strategy 4, while IN and JPN are connected by independence copula. Strategy 4 prefers mostly different pairs in the first tree to maximize the sum of absolute values of underlying weight. However, it chooses AUS,BRA like in the Strategy 1 and IN,JPN as in Strategy 4. Remarkable is also the choice if appropriate copula families. While Strategies 1,2,4 are almost exclusively occupied by t copula to model symmetrical tail dependence, Strategy 3 chooses many Gumbel survival to mirror the lower tail dependence.
Figure 8.3 gives an overview of all selected pairs of the exchange rates data set according to the applied strategy. It allows us to compare whether and which variable pairs were selected multiple time. Since we know that S1 and S2 choice of the first tree coincide (see Figure 8.2 (a) and (b)) we want to have a look at the other two models in comparison to S1. The Strategy 4 using exceedance dependence as weight show 5 variable pairs that coincide with the S1 and so it is more that 50% of the T_1 respectively. While Strategy 3 selects only 1 pair namely BRA,AUS similar to S1, hence displays a totally different perspective on the dependence features of the underlying data set. We summarize the selection of the variable pairs in the matrix below. Here, each number mirrors the number of times the corresponding pair was selected applying different strategies. Note, that all the pairs chosen one time correspond to the Strategy 3 choice with the exception of pairs EUR,JPN and CAN,BRA, witch were found appropriate by Strategy 4.In total, 47% of all selected pairs were chosen only once.

	EUR	UK	CAN	AUS	BRA	CH	JPN	SZ	IN
EUR	–	3		2		1	1	3	
UK		–				1		1	
CAN			–	3	1	1			
AUS				–	3		1		3
BRA					–				
CH						–	1		3
JPN							–	2	2
SZ								–	
IN									–



(a) Kendall Tau (b) Tail cumulation

(c) Hu dependence (d) Exceedance dependence

Figure 8.3: Individual selection of pairs for each strategy. Black mark indicates the pairs selected once, green for pairs selected twice and red for three time selected pairs. Numbers represent the exchange rates data set variables according to their order in Table 8.1

Maximum Log Likelihood Estimation

Next, we want to estimate the log likelihood for each individual strategy, since we want to find out whether any of our strategies are likely to fit the data better than Strategy 1. Table 8.2 gives and overview of the likelihood values for each selected R-Vine corresponding to each strategy and the respective AIC values. Additionally, it gives number of parameters and BIC values. We can observe that the biggest difference is the Hu dependence strategy S3. Opposite to the other strategies it does not truncate till the last tree, since the Hu dependence weight shows quite extreme behavior. Respectively, it needs much larger number of parameters to specify the dependence. Strategy 4 truncates later than the 1st and 2nd and accordingly has larger number of parameters and a slightly better

AIC and log likelihood value. Table 8.2(bottom) offers an overview of values for truncated R-Vines with respect to the strategy. Those values corresponding to tree wise analysis for all strategies selection can be found in Tables 8.3 and 8.4.

Weight	Number of parameters	Log Likelihood	AIC	BIC
Kendall′s τ	33	2219.28	-4372.56	-4210.38
Tail cumulation	33	2212.52	-4359.49	-4196.86
Hu dependence	47	2202.59	-4309.18	-4073.27
Exceedance dependence	37	2204.18	-4334.36	-4152.51

Weight	Truncation	Number of parameters	Log Likelihood	AIC	BIC
Kendall′s τ	T2	25	2158.61	-4267.21	-4144.35
Tail cumulation	T3	26	2156.96	-4261.93	-4134.14
Hu dependence	none	47	2202.59	-4309.18	-4073.27
Exceedance dependence	T4	34	2192.41	-4316.81	-4155.31

Table 8.2: Log Likelihoods, AIC, BIC and number of parameters for non- truncated (top) and truncated R-Vines (bottom) according to the strategy used for exchange rates. Truncation based on Vuong test. Second column in the bottom Table gives the tree number after which the R-Vine was truncated.

However, Strategy 1 seems to be the best fit according to the values of AIC and likelihoods if we compare the overall differences. The number of parameters varies the most. We see that using different weights increases the number of parameters needed for specification. The differences in values of AIC and BIC as well as likelihoods appear to be rather small of less then 2 %. In this way the choice of different weights instead of Kendall′s τ does not improve the results significantly.

Comparison via Vuong test

Next, we want to compare the R-Vine model chosen according to different Strategies by the Vuong test (see Chapter 6 for more detail on Vuong test), similarly to the simulation study. In order to do so we use the R-Vine based on Kendall τ as our “true“model and simulate from this R-Vine the same way we did in the simulation study with 1000 data points and 100 repetitions.

All four R-Vines are sequentially specified by using four weights according to the four strategies. We compare the results of the Vuong test to see if any of

Table 8.3: Tree wise log likelihoods using Kendall's τ (top) and tail cumulation (bottom) as weight for exchange rates data set

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	16	1973.42	-3914.84	-3836.21
T2	9	185.19	-352.37	-308.14
truncation	$\sum = 25$	$\sum =2158.61$	$\sum = - 4267.21$	$\sum = - 4144.38$
T3	3	16.10	-26.19	-11.45
T4	2	33.82	-63.64	-53.81
T5	2	8.93	-13.85	-4.03
T6	1	1.83	-1.66	3.26
TOTAL	33	2219.28	-4372.56	-4210.38

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	16	1973.42	-3914.84	-3836.21
T2	8	150.59	-285.18	-245.86
T3	2	32.95	-61.90	-52.07
truncation	$\sum = 26$	$\sum =2156.96$	$\sum = - 4261.93$	$\sum = - 4134.14$
T4	2	12.10	-20.194	-10.365
T5	2	32.65	-61.30	-51.47
T6	2	8.83	-13.67	-3.84
T7	1	1.98	-1.96	2.95
TOTAL	33	2212.52	-4359.49	-4196.86

Table 8.4: Tree wise log loglikelihoodslikelihoods using Hu dependence (top) and exceedance dependence (bottom) as weight for exchange rates data set

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	9	451.51	-885.02	-840.78
T2	12	318.18	-612.36	-553.38
T3	10	563.95	-1107.90	-1058.75
T4	5	676.79	-1343.58	-1319.01
T5	5	50.32	-90.65	-66.08
T6	3	73.98	-141.96	-127.21
T7	2	57.28	-108.56	-93.81
T8	1	10.57	-19.15	-14.24
TOTAL	47	2202.59	-4309.18	-4073.27

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	14	1561.71	-3095.41	-3026.60
T2	9	290.61	-563.22	-518.99
T3	6	254.05	-496.09	-466.60
T4	5	86.00	-162.09	-137.51
truncation	$\sum = 34$	$\sum = 2192.41$	$\sum =$- 4316.81	$\sum =$ - 4155.31
T5	2	2.62	-1.25	8.58
T6	-	-	-	-
T7	1	9.15	-16.29	-11.38
TOTAL	37	2204.18	- 4334.36	- 4152.51

the strategies are considered better fit or fit the data equally good according to the test. Results of the comparison are summarized in the Table 8.5. First of all, Strategy 3 is clearly outperformed by the three other strategies. According to Vuong test this particular choice is not a better fit than Strategy 1 and 2, since they perform so similarly. While comparing the Strategy 3 and 4 we can see the test considers those two models equally a few more times that others but it still prefers S4 over S3. Between Strategy 1 and Strategy 4 as well as between the 2nd and 4th the test could not decide. Clearly, the specific measures of tail dependence especially Hu dependence measuring the asymmetrical tail dependence are not the best choice according to the Vuong test. We could expect this result if we have a look at the pair-plots of the transformed copula data of exchange rates of (Dissmann 2010) . The respective tail dependence is mostly not present or not very strong. The exception is the pair EUR,SZ. An overview is to find in Figure B.1. Let us summarize. Based on likelihoods and on Vuong test the model could be ordered in terms of better performance as follows:

Comparison	Exchange Rates		Comparison	Exchange Rates
$S1 > S2$	3		$S2 > S3$	90
$S1 \approx S2$	97		$S2 \approx S3$	10
$S1 < S2$	0		$S2 < S3$	0
$S1 > S3$	93		$S2 > S4$	34
$S1 \approx S3$	7		$S2 \approx S4$	66
$S1 < S3$	0		$S2 < S4$	0
$S1 > S4$	46		$S3 > S4$	1
$S1 \approx S4$	54		$S3 \approx S4$	32
$S1 < S4$	0		$S3 < S4$	67

Table 8.5: Results of the Vuong test comparison for R-Vines fitted according to four strategies using the selected model by Kendall′s Tau as underlying “true“model. Notation: S1: Kendall′s τ; S2: Tail cumulation; S3: Hu dependence; S4: Exceedance dependence.

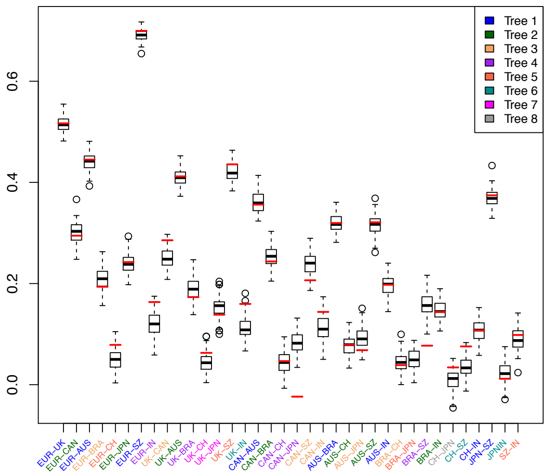
$$\text{Strategy 1} \approx \text{Strategy 2} \approx \text{Strategy 4} > \text{Strategy 3}$$

Comparison of different pairs of variables of simulated R-Vines

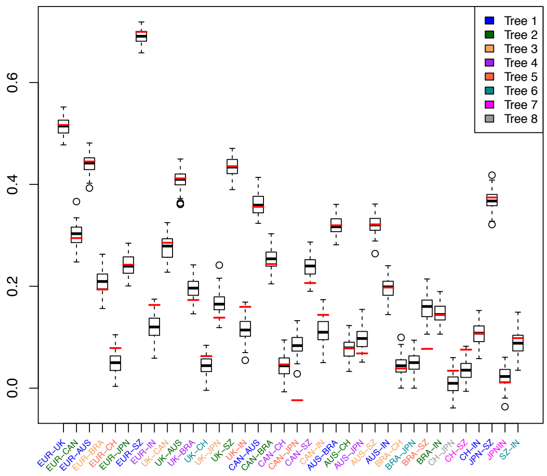
The exchange rates data set we investigate contains 9 variables and therefore 36 possible pairs of variables. We want to investigate how good the dependence of those pairs is measured with each of our chosen dependence measures, i.e.

Kendall′s τ, upper and lower tail cumulation, upper and lower Hu dependence as well as upper and lower exceedance dependence. First we estimate each of these dependence measures for all pairs of variables from copula data. Next, for each of our four fitted R-Vines we summarize the 100 values for every dependence estimate in a boxplot. We want to compare the values of “true“estimate and the median value of each boxplot. The nearer those two lines are to each other, the smaller the difference in the estimates and simulated values. For better impression the results are listed in the Figures 8.4 to 8.10. The red line in every boxplot stands for the “true“estimate value of every pair according to the coefficient chosen (Kendall′s τ, upper/lower tail cumulation etc.). Especially, since we cannot estimate all pairs chosen for the first tree of each R-Vine model to maximize the sum of edge weights, it is visibly reflected in the boxplots figures also given the additional information on the tree in which this particular pair occurs. Advantageously, each table contains results on all four strategies for each of the seven measures, which provides easier comparison of their fit.
At first if we look at the boxplots of Kendall′s τ for each of the fitted R-Vine we see that they deliver very similar results. Almost all of the variable pairs red lines stay in the corresponding boxplots. Since strategies do not always choose same pair of variables in the same order trees the estimation varies and shows slightly different results. An example is the CAN-JPN dependence. It is clearly better measured in the R-Vine models fitted using Hu dependence and exceedance dependence. While Kendall Tau and Tail Cumulation methods′ “true“estimate lies far outside the corresponding boxplots. On the other hand we observe that this pair of variables was selected in T_3 in better estimating models while the model based on tail cumulation chooses this pair in T_7 and Kendall′s τ model in T_5, which clearly leads to underestimation, but also would not be as important as using wrong measurements in the first trees. We can also see this in boxplots of upper and lower tail cumulation with the variable pairs CAN-JPN and BRA-JPN. In the estimates of upper tail cumulation the pair BRA-SZ displays deviation in higher order trees when using Kendall′s τ and tail cumulation as weights. The models using Hu dependence and exceedance dependence as weights are more accurate according to the boxplot display of empirical values of τ and upper and lower tail cumulation. For the estimation of upper and lower exceedance dependence we indicate more pairs that display deviance. The change in distance of empirical values of simulated data and the one from the underlying data among the different R-Vine models becomes not so obvious. If we compare upper and lower Hu dependence to other measures the first thing that occurs is that estimator takes the whole span from 0 to 1 for each selected model, since the Hu dependence coefficient tends to display rather extreme values. Note that it also has more outliers then other strategies and is only partially able to capture the estimated values of the “true“model.

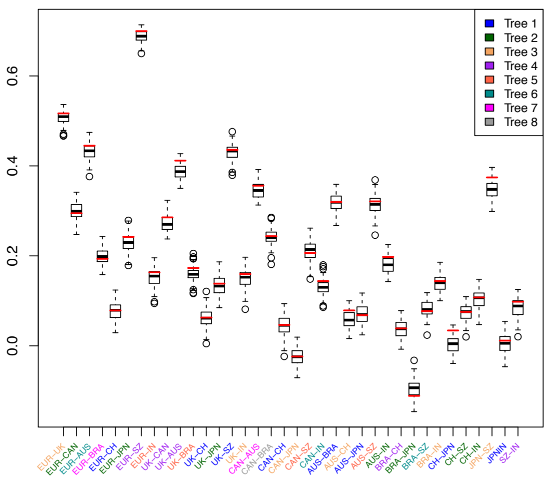
Figure 8.4: Boxplots of pairwise empirical Kendall's τ based on 100 simulations using the selected model by Kendall's τ(top left), tail cumulation(top right), Hu dependence(bottom left) and exceedance dependence(bottom right).



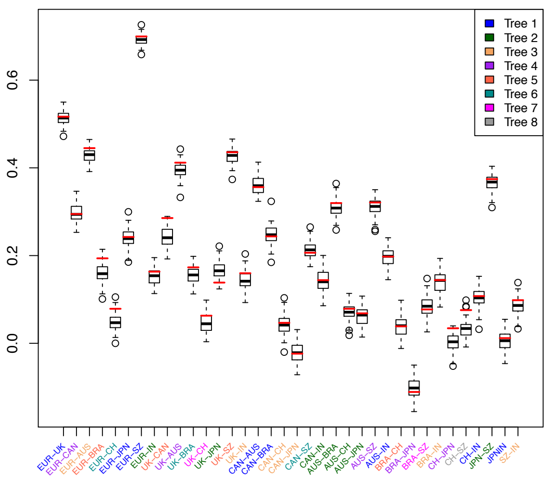
i Method Kendalls Tau



ii Method Tail Cumulation

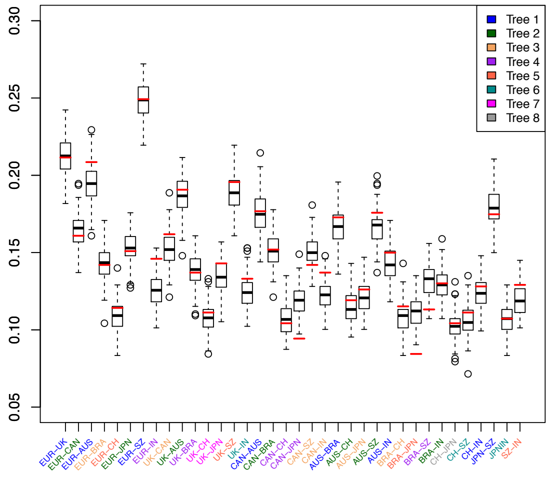


iii Method Hu Dependence

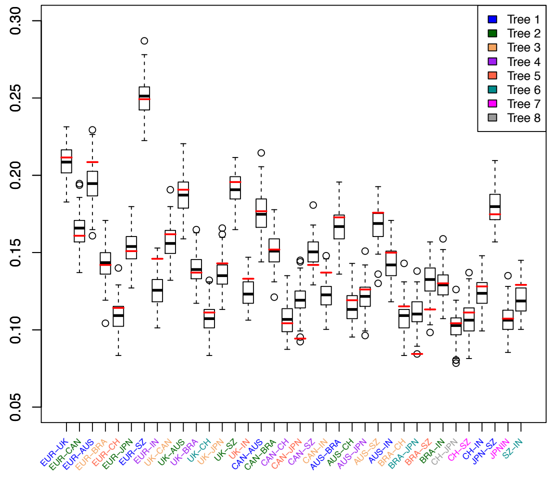


iv Method Exceedance Dependence

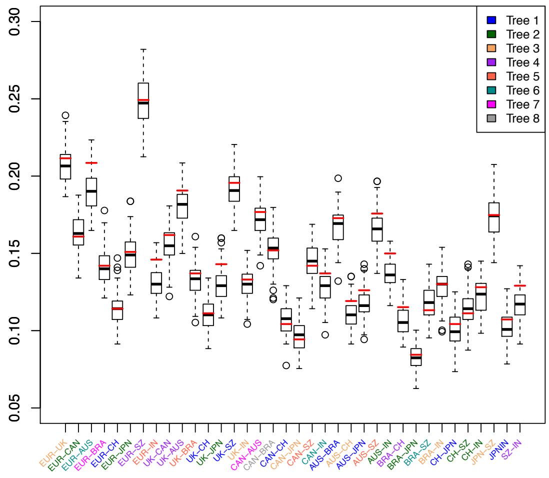
Figure 8.5: Boxplots of pairwise empirical upper tail cumulation based on 100 simulations using the selected model by Kendall′s τ(top left), tail cumulation(top right), Hu dependence(bottom left) and exceedance dependence(bottom right).



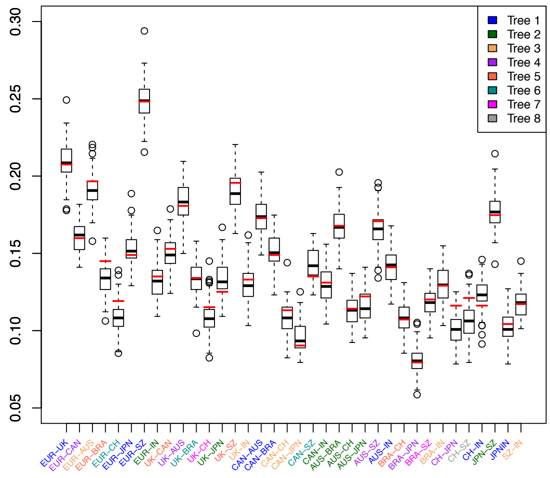
v Method Kendalls Tau



vi Method Tail Cumulation

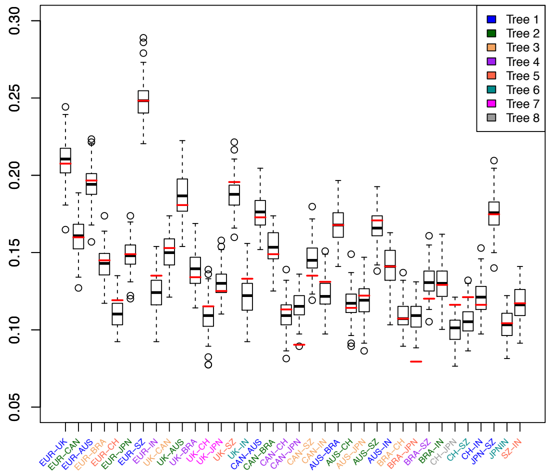


vii Method Hu Dependence

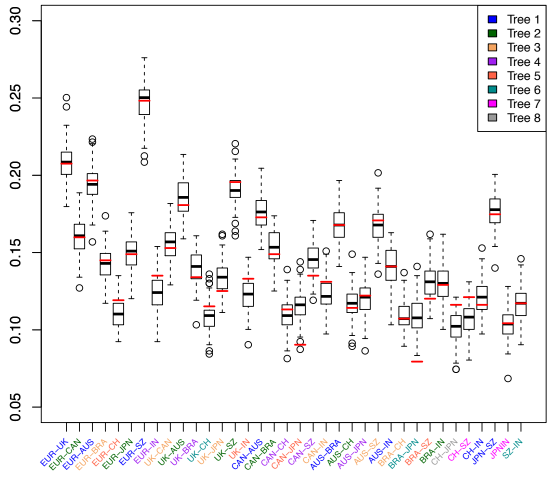


viii Method Exceedance Dependence

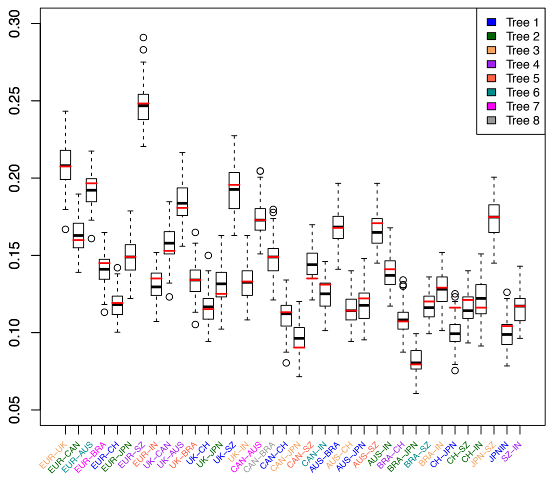
Figure 8.6: Boxplots of pairwise empirical lower tail cumulation based on 100 simulations using the selected model by Kendall's τ(top left), tail cumulation(top right), Hu dependence(bottom left) and exceedance dependence(bottom right).



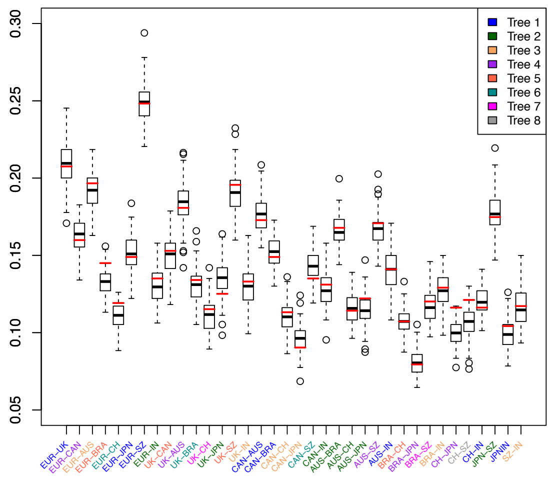
ix Method Kendalls Tau



x Method Tail Cumulation

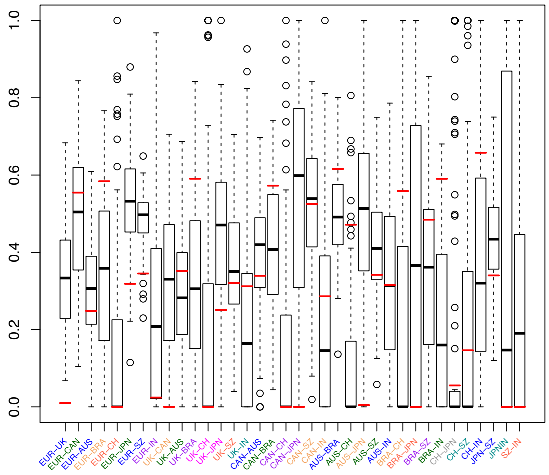


xi Method Hu Dependence

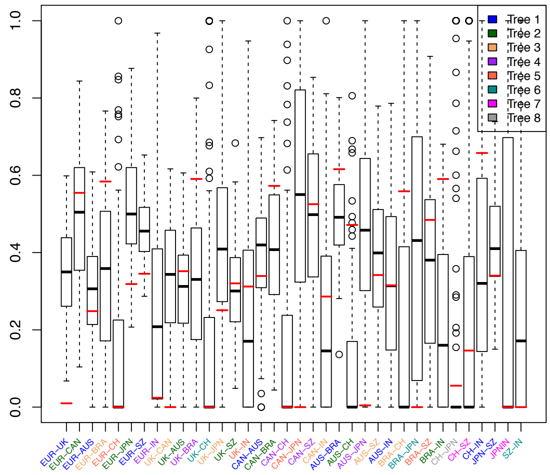


xii Method Exceedance Dependence

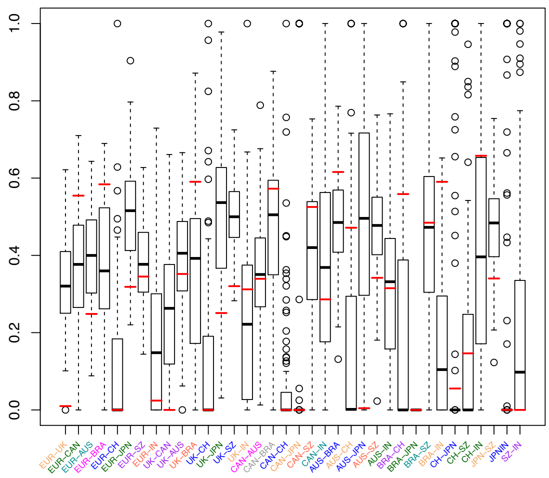
Figure 8.7: Boxplots of pairwise empirical upper Hu dependence based on 100 simulations using the selected model by Kendall's τ(top left), tail cumulation(top right), Hu dependence(bottom left) and exceedance dependence(bottom right).



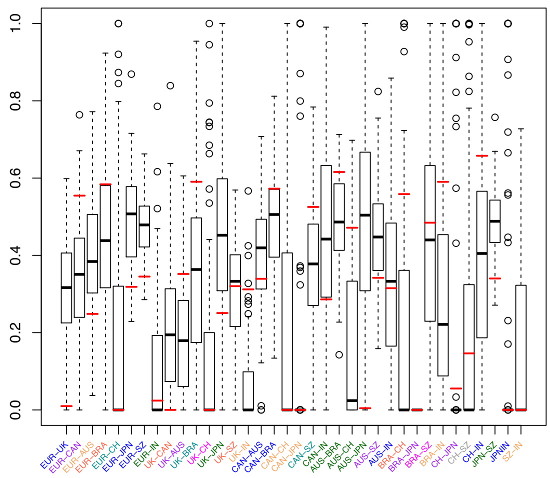
xiii Method Kendalls Tau



xiv Method Tail Cumulation

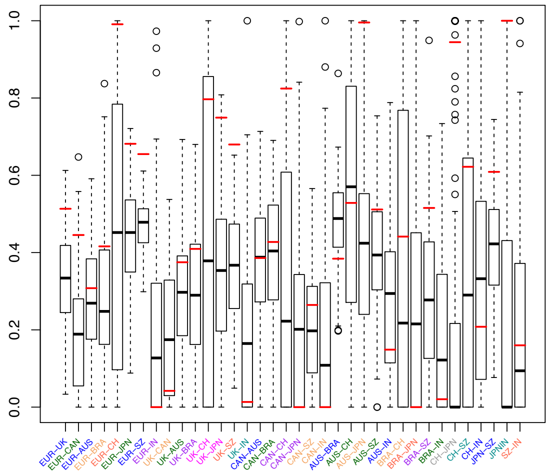


xv Method Hu Dependence

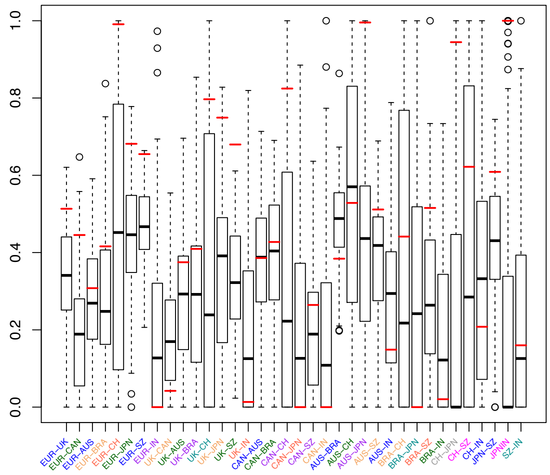


xvi Method Exceedance Dependence

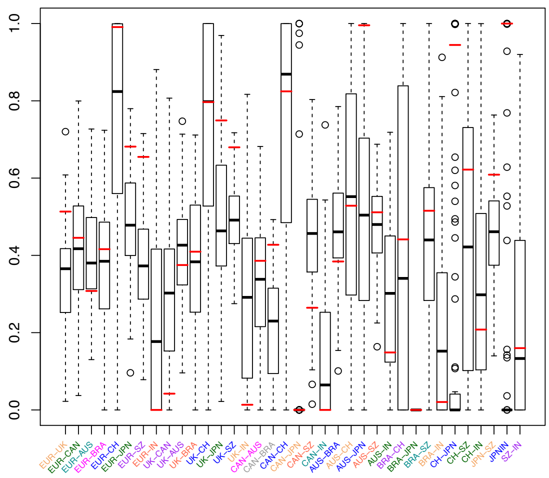
Figure 8.8: Boxplots of pairwise empirical lower Hu dependence based on 100 simulations using the selected model by Kendall′s τ(top left), tail cumulation(top right), Hu dependence(bottom left) and exceedance dependence(bottom right)



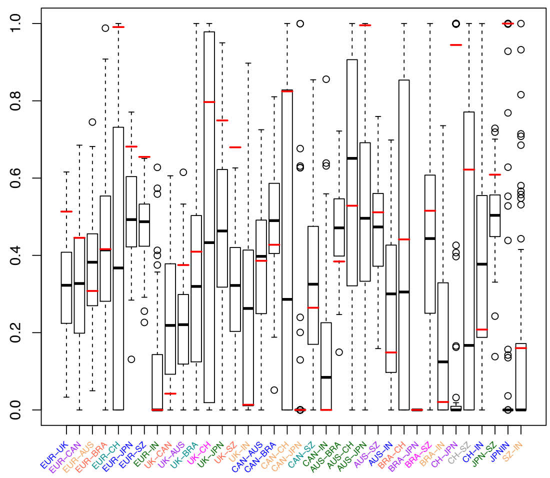
xvii Method Kendalls Tau



xviii Method Tail Cumulation

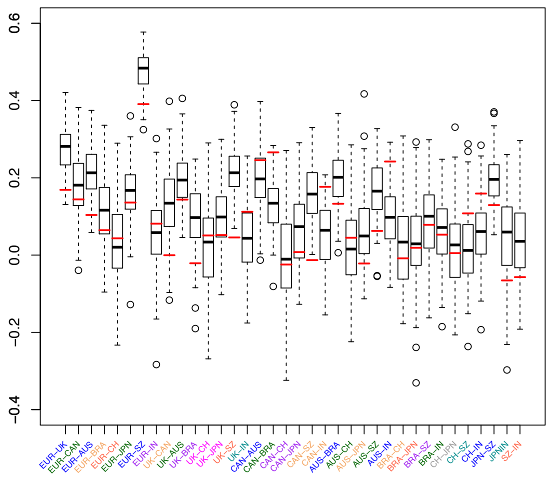


xix Method Hu Dependence

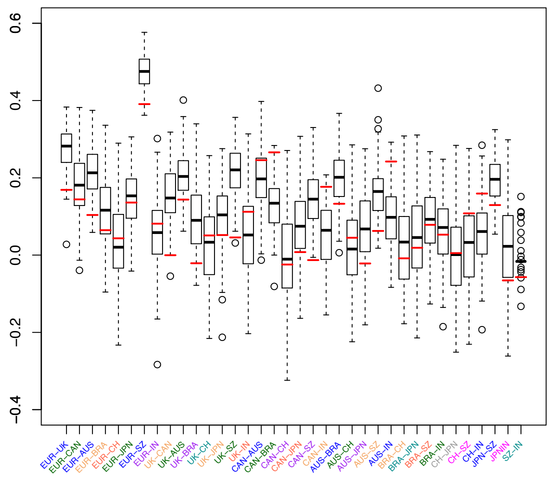


xx Method Exceedance Dependence

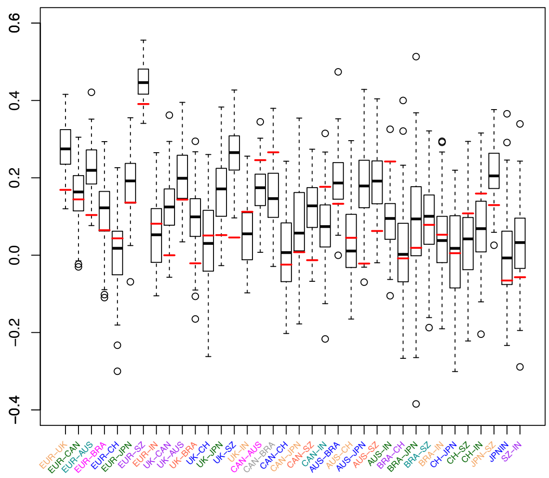
Figure 8.9: Boxplots of pairwise empirical upper exceedance dependence based on 100 simulations using the selected model by Kendall's τ(top left), tail cumulation(top right), Hu dependence(bottom left) and exceedance dependence(bottom right).



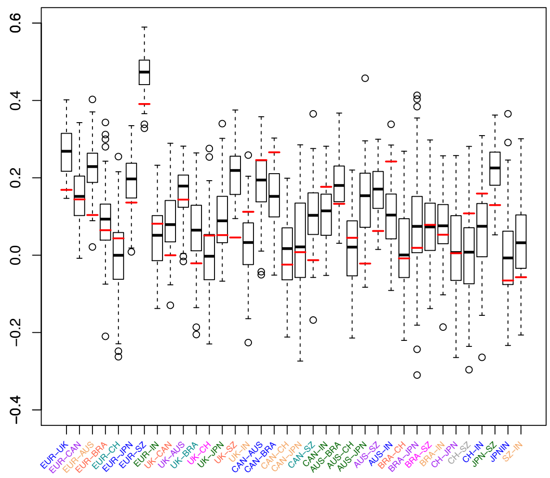
xxi Method Kendalls Tau



xxii Method Tail Cumulation

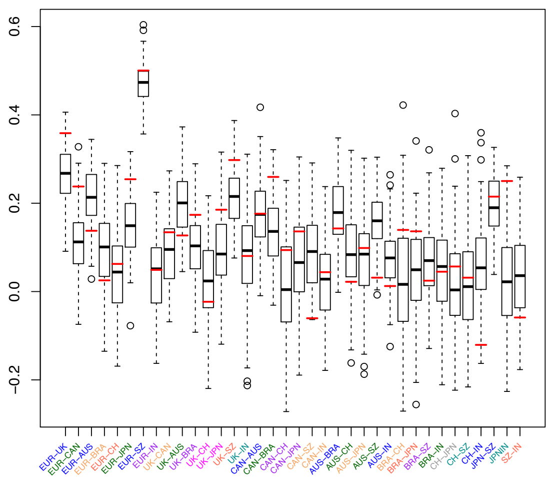


xxiii Method Hu Dependence

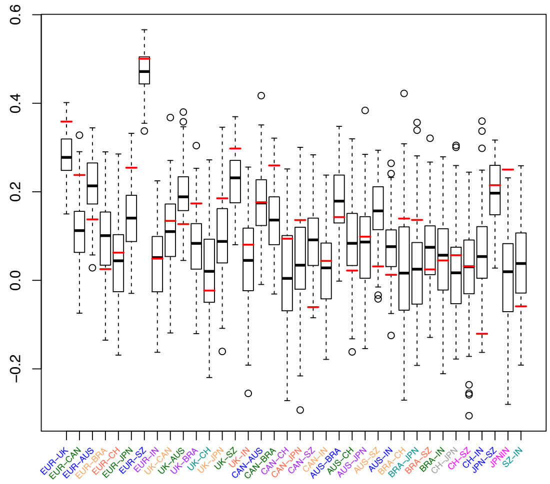


xxiv Method Exceedance Dependence

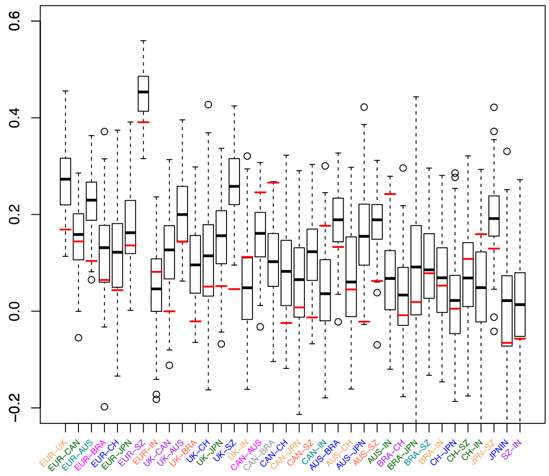
Figure 8.10: Boxplots of pairwise empirical lower exceedance dependence based on 100 simulations using the selected model by Kendall's τ(top left), tail cumulation(top right), Hu dependence(bottom left) and exceedance dependence(bottom right).



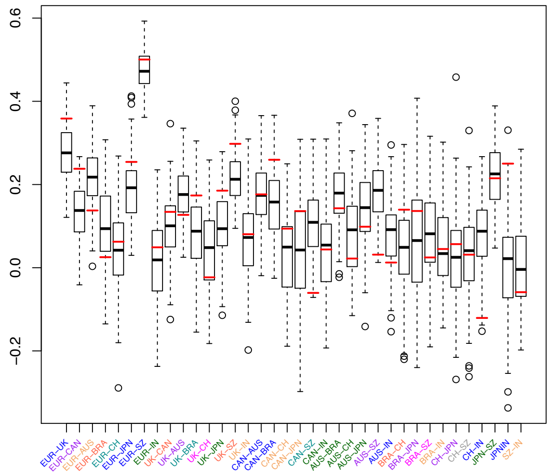
xxv Method Kendalls Tau



xxvi Method Tail Cumulation



xxvii Method Hu Dependence



xxviii Method Exceedance Dependence

Strategy	τ	upper tail cumulation	lower tail cumulation	upper hu dependence	lower hu dependence	upper exceedance dependence	lower exceedance dependence
Kendall′s τ	0.056	0.003	0.018	2.088	3.193	0.269	0.275
Tail cumulation	0.057	0.003	0.018	1.987	3.201	0.256	0.289
Hu dependence	0.005	0.002	0.015	1.387	2.395	0.308	0.292
Exceedance dependence	0.010	0.001	0.016	1.391	2.880	0.237	0.276

Strategy	τ	upper tail cumulation	lower tail cumulation	upper hu dependence	lower hu dependence	upper exceedance dependence	lower exceedance dependence
Kendall′s τ	0.047	0.005	0.017	3.107	6.809	0.490	0.576
Tail cumulation	0.049	0.005	0.018	2.594	7.053	0.483	0.496
Hu dependence	0.011	0.004	0.019	2.379	4.574	0.459	0.697
Exceedance dependence	0.018	0.003	0.018	4.795	6.000	0.265	0.587

Table 8.6: Top Table: Exchange rates, sum over all pairs of variables $\sum_{i,j}(\omega^e_{ij} - \overline{\hat{\omega}_{ij}})^2$, where ω^e stands for the $\{(i,j), i < j\}$ parameter estimated of underlying data and $\overline{\hat{\omega}_{ij}}$ represents the mean of the simulated values. Bottom Table : Exchange rates, over all pairs of variables and all 100 simulated observations $\sum_{i,j}\sum_{k=1}^{100}\left(\omega^e_{ij} - \overline{\hat{\omega}_{ij}}^k\right)^2$,where ω^e_{ij} stands for the observed parameter, and $\overline{\hat{\omega}_{ij}}^k$ represents the kth simulated value.

Figures 8.4 to 8.10 reflect the individual differences quite well. They also give the tree number in which the corresponding pair of variables occurs. It can be observed in the legend and corresponds to the given color. Now, we want to compute the overall difference for each dependence measure and each selected model and display in one value. First, for each measure we compute squared difference between the median value and the value computed from underlying data for each pair of variables. We sum over all pairs of variables. The resulting values are easier to compare with each other and are summarized in Table 8.6 (top). According to this values upper and lower tail dependence as well as Kendall′s τ are best captured with the corresponding R-Vine using Hu dependence as weight. It also has best results when measuring lower tail cumulation, while for upper tail cumulation the best model is the one based on exceedance dependence. However, we can also observe that both measures upper as well as lower tail cumulation have almost the same value which can be due to the definition of this measure as less accurate. The upper exceedance dependence is clearly better captured with the exceedance dependence R-Vine when lower exceedance dependence has almost same value for models using Kendall′s τ and exceedance dependence as weights. We want to have a more exact value and take every simulated estimate into account. For that we change the measure a little. This time we sum over squared differences between estimated values and each simulated observation for all 100 simulations over all 36 pairs. However, with most of our earlier observations verified, we can determine that the model using Hu dependence is the best measure for both Hu dependencies. Kendall′s τ is also best captured with Hu dependence tightly followed by exceedance dependence. Upper and lower tail cumulation display very similar values for all models and upper exceedance dependence is best captured with exceedance dependence R-Vine. While in the Table 8.6 (top) the lower exceedance dependence prefers R-Vine fitted using Kendall′s τ but also only by little difference in all values, the corresponding value in Table 8.6 (bottom) clearly displays the value of tail cumulation model as smallest due to this measure being similar in the definition but with slightly different choices of the tail aria considered.

8.2 International Financial Indices

So far, we have investigated the exchange rates data set with 9 variables, which is relatively small. The data set we want to investigate next has 16 variables and therefore larger size which is more applicable in real financial data. Daily returns of financial indices data set was introduced in (Dissmann 2010). It is based on three different asset classes and different regions. The 2336 data points result from a series of log returns and time series from 01/01/2001 until 12/14/2009 and 12/29/2000 till 12/14/2009 respectively. We summarize according to asset class:

- **Equity**

Short Name	Long Name	Region
Dax	DAX30 PERFORMANCE	Germany
STOXX50	DJ STOXX 50	Europe
S&P500	S&P 500 COMPOSITE	USA
MSCI-WORLD	MSCI WORLD U$	Global
MSCI-EE	MSCI EM EASTERN EUROPE U$	Eastern Europe

- **Fixed Income**

Short Name	Long Name	Region
IBOXX-G-3-5	IBOXX EURO SOV.GERMANY 3-5 YRS	Germany
IBOXX-G-7-10	IBOXX EURO SOV.GERMANY 7-10 YRS	Germany
IBOXX-E-1-3	IBOXX EURO SOV.EZONE 1-3 YRS	Eurozone
IBOXX-E-5-7	IBOXX EURO SOV.EZONE 5-7 YRS	Eurozone
IBOXX-E-10+	IBOXX EURO SOV.EZONE 10+ YRS	Eurozone
BOXX-E-A	IBOXX EURO CORP.A RATED ALL MATS.	Eurozone
BOXX-E-AA	IBOXX EURO CORP.AA RATED ALL MATS.	Eurozone
BOXX-E-AAA	IBOXX EURO CORP.AAA RATED ALL MATS.	Eurozone
BOXX-E-BBB	IBOXX EURO CORP.BBB RATED ALL MATS.	Eurozone

- **Commodity**

Short Name	Long Name	Region
Comm	DJ UBS-Spot Commodity Index	Global
Gold	MLCX Gold Total Return	Global

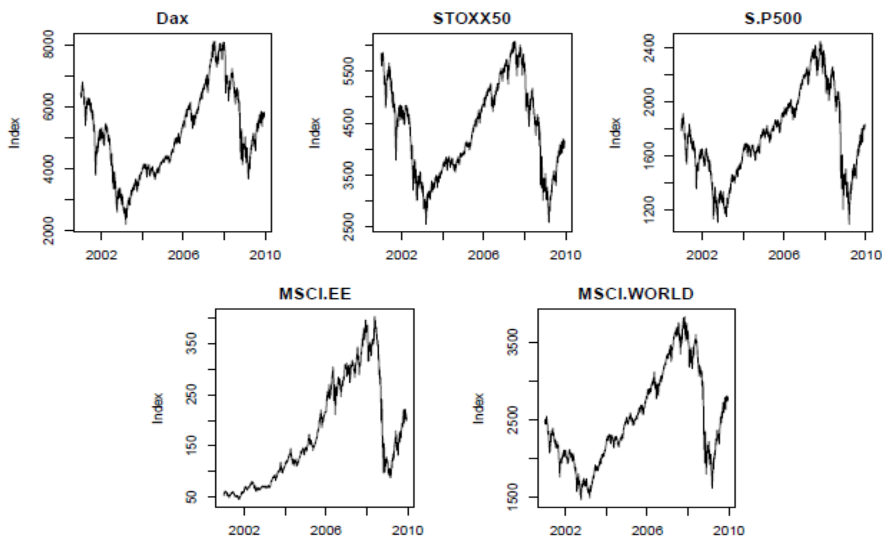
All of the indices are stated in the respective home currency and global indices in USD. All named indices also divide in term of dividends into following groups:

- total return, to catch the effect of dividend reinvestment: for all fixed income indices plus Gold and S&P500;

- net total return, to capture this effect after deduction of withholding tax: STOXX50 and MSCI-WORLD;
- price return group for the purpose to capture the change in the prices of the index components: Dax, MSCI-EE and Comm.

Additionally, the maturity of the selected German and Euro government bonds are disjointed. For each time series (Dissmann 2010) uses ARMA(1,1) and GARCH(1,1) models and standardizes transformed residuals to obtain pseudo-observations. For information on data set and time series analysis see (Dissmann 2010). Appendix B.2 Figure B.2 shows the within group dependence of the financial indices variables.

Figure 8.11: Time series developing of international equity indices from 12/29/2000 to 12/14/2009.



Comparison of the selected models

Since the financial indices data set is based on qualitative relation of 16 variables and 120 possible pairs, we want to compare the choice of R-Vine that each strategy makes. In particular, we have a look at the first tree since it is not dependent on the choice of copula and only sums the selected edge values. Using different weights clearly effects the selected R-Vine model, since we can observe

Figure 8.12: Time series developing of international equity indices from 12/29/2000 to 12/14/2009.

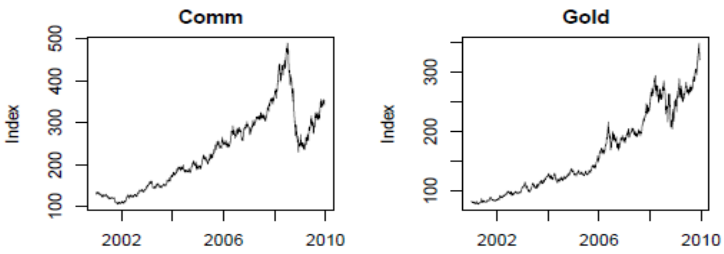
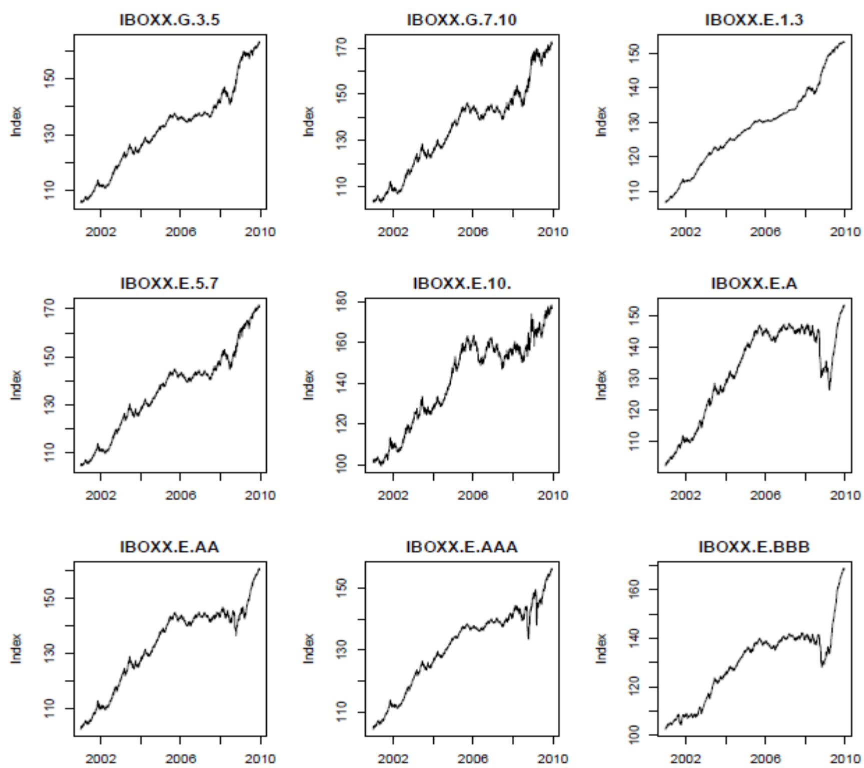


Figure 8.13: Time series developing of international bond indices from 12/29/2000 to 12/14/2009.



partly different variable pairs. The connecting pair of variable varies according to the strategy applied. However, the strongest dependence between the government and corporate bond are chosen in most of possible pairs. Every model chooses the pair Gold-Comm, hence it has strong economical value. The strong dependence among equity variables is reflected in every model except $S3$ using Hu dependence. The most asymmetric dependence is captured between those variables and commodities. Hence, if we need to model explicitly tail dependence especially asymmetrical, the better choice would be exceedance or even Hu dependence, despite the fact that it requires more resources. Information on choice of variable pair, copula families and corresponding parameters are to find in Appendix. B.2. Remarkably, for all pairs the dependence is given with t copula indicating symmetric tail dependence.
From Figure 8.14 we can see that Strategy 3 again selected pairs differ most from the rest of the strategies. Here already 60 % of pairs were selected only once. S1 and S2 again deliver similar choice although do differ in a few selected pairs such as 2-6 or 1-5. Strategy 4 choses 4 pairs of total 15 witch do not repeat in any other strategy but keeps the rest of the selection similar to S1 and S2. Below a summarized overview.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	–	3		2	1									1		
2		–		1	2	1										
3			–	3											1	
4				–	1								1		1	1
5					–										1	1
6						–		3	3				1	1		
7							–		3	3				1		
8								–			1			1		2
9									–			3		1		
10										–				1		
11											–	3		3	1	
12												–	2			1
13													–			1
14														–		
15															–	4
16																–

For convinience here we use numbers from 1 to 16 to represent every variable. The numbers correspond to previous notation in Section 8.2 starting with Dax=1, STOXX50=2 and so on till Gold=16.

Log likelihoods comparison for four selected R-Vines

Analogous to the exchange rates data set we want to analyze the log likelihoods for every selected R-Vine in order to see if there is a significant difference in the values to indicate a better fit. Tables 8.9-8.12 give tree wise log likelihoods, AIC and BIC values, which we summarize and reflect in the Table 8.7.
Hu dependence again needs the largest number of parameters for specification since it uses 65 t copulas (in comparison exceedance dependence only needs 121 parameters with 46 t copulas in its specification). The values of log likelihoods and AIC/BIC do not display large differences not larger than 2 %, but do indicate the order of the better data fit. Strategy 1 is again the better choice. Followed by Strategy 4 which seems to perform better then in exchange rates comparison of likelihoods. However, it is tightly followed by S2 and the difference is rather insignificant. Strategy 3 is last in this comparison. Now we want to see the behavior of strategies when truncating R-Vines. In the Table 8.7 below we have an overview of likelihoods, AIC/BIC values for selected R-Vines truncated according to the Vuong test. Additionally, the tree number before truncation is given in column 2.

Weight	Number of parameters	Log Likelihood	AIC	BIC
Kendall's τ	113	36334.43	-72442.85	-71792.35
Tail cumulation	148	35973.99	-71691.97	-70955.13
Hu dependence	157	35580.86	-70859.71	-69960.46
Exceedance dependence	121	36082.32	-71920.63	-71218.33

Weight	Truncation	Number of parameters	Log Likelihood	AIC	BIC
Kendall's τ	T9	111	36329.62	-72437.23	-71792.35
Tail cumulation	T6	91	34858.43	-69530.86	-68995.50
Hu dependence	T11	154	35570.69	-70845.37	-69963.46
Exceedance dependence	T9	111	35938.89	-71657.77	-71024.55

Table 8.7: Log Likelihoods, AIC, BIC and number of parameters for non-truncated (top) and truncated (bottom) R-Vines selected according to different weights for international financial indices.

Figure 8.14: Variable pair selected in T_1 for every Strategy. Numbering of the variables according to the list of Section 8.2. Pairs marked by $\color{blue}{\times}$ - selected by all strategies, $\color{red}{\times}$ - three strategies, $\color{green}{\times}$ - twice, $\times$ - only once.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	–	$\color{red}{\times}$		$\color{green}{\times}$												
2		–			$\color{green}{\times}$	$\times$										
3			–	$\color{red}{\times}$												
4				–												
5					–										$\times$	
6						–		$\color{red}{\times}$	$\color{red}{\times}$							
7							–		$\color{red}{\times}$	$\color{red}{\times}$						
8								–								
9									–			$\color{red}{\times}$				
10										–						
11											–	$\color{red}{\times}$		$\color{red}{\times}$		
12												–	$\color{green}{\times}$			
13													–			
14														–		
15															–	$\color{blue}{\times}$
16																–

(a) Kendall′s Tau

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	–	$\color{red}{\times}$		$\color{green}{\times}$	$\times$											
2		–														
3			–	$\color{red}{\times}$												
4				–											$\times$	
5					–											
6						–		$\color{red}{\times}$	$\color{red}{\times}$							
7							–		$\color{red}{\times}$	$\color{red}{\times}$						
8								–								
9									–			$\color{red}{\times}$				
10										–						
11											–	$\color{red}{\times}$		$\color{red}{\times}$	$\times$	
12												–	$\color{green}{\times}$			
13													–			
14														–		
15															–	$\color{blue}{\times}$
16																–

(b) Tail cumulation

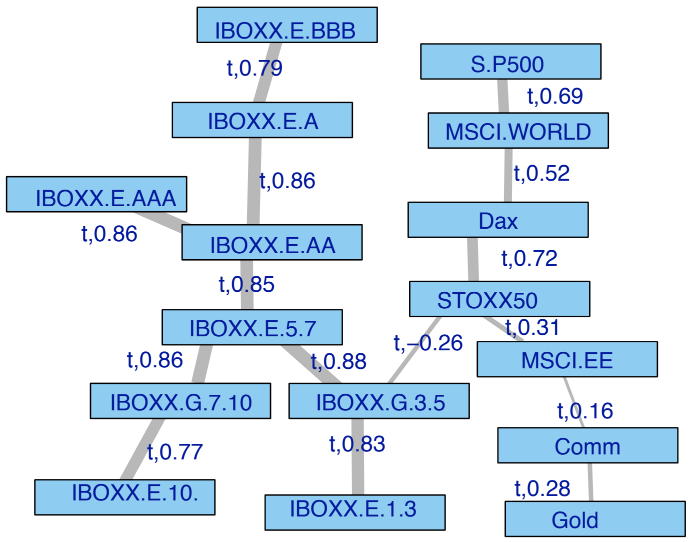
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	–														$\times$	
2		–			$\color{green}{\times}$											
3			–												$\times$	
4				–												$\times$
5					–											$\times$
6						–								$\times$		
7							–							$\times$		
8								–			$\times$			$\times$		$\color{green}{\times}$
9									–					$\times$		
10										–				$\times$		
11											–					
12												–				$\times$
13													–			$\times$
14														–		
15															–	$\color{blue}{\times}$
16																–

(c) Hu dependence

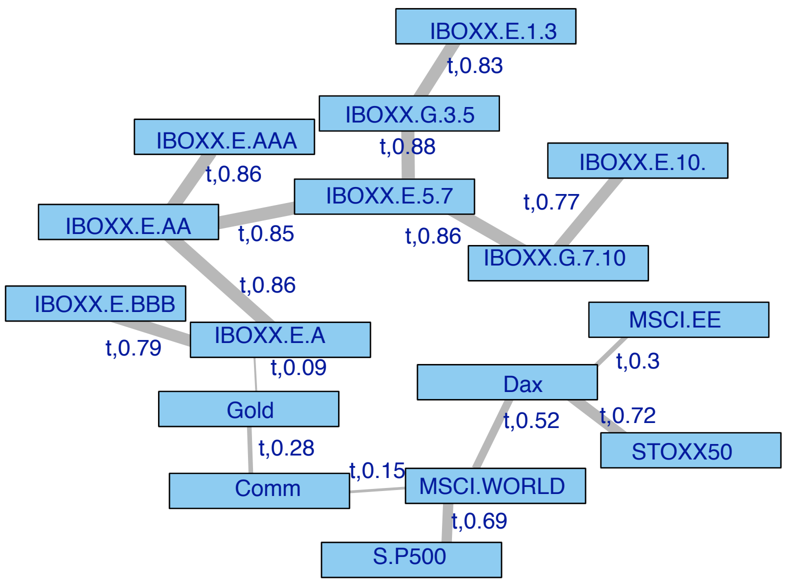
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	–	$\color{red}{\times}$														
2		–		$\times$												
3			–	$\color{red}{\times}$												
4				–	$\times$								$\times$			
5					–											
6						–		$\color{red}{\times}$	$\color{red}{\times}$				$\times$			
7							–		$\color{red}{\times}$	$\color{red}{\times}$						
8								–								$\color{green}{\times}$
9									–			$\color{red}{\times}$				
10										–						
11											–	$\color{red}{\times}$		$\color{red}{\times}$		
12												–				
13													–			
14														–		
15															–	$\color{blue}{\times}$
16																–

(d) Exceedance dependence

Figure 8.15: First tree of R-Vine model fitted using Strategies 1 and 2 for the international financial indices data set.

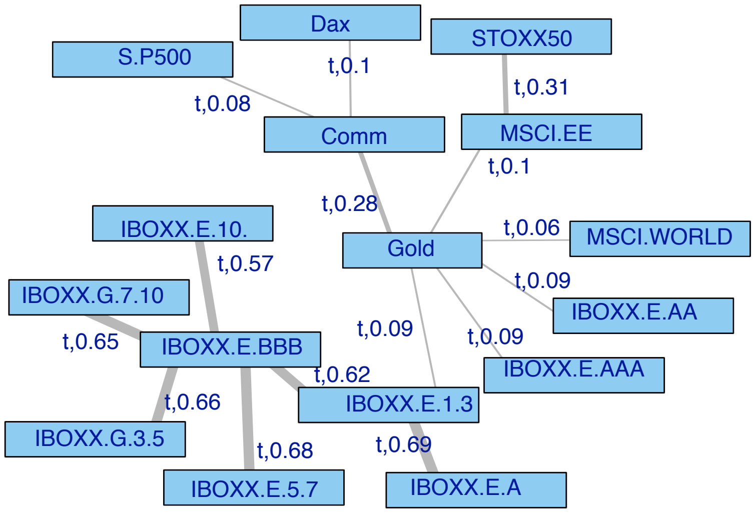


(a) T_1 of R-Vine using Kendalls τ as weight

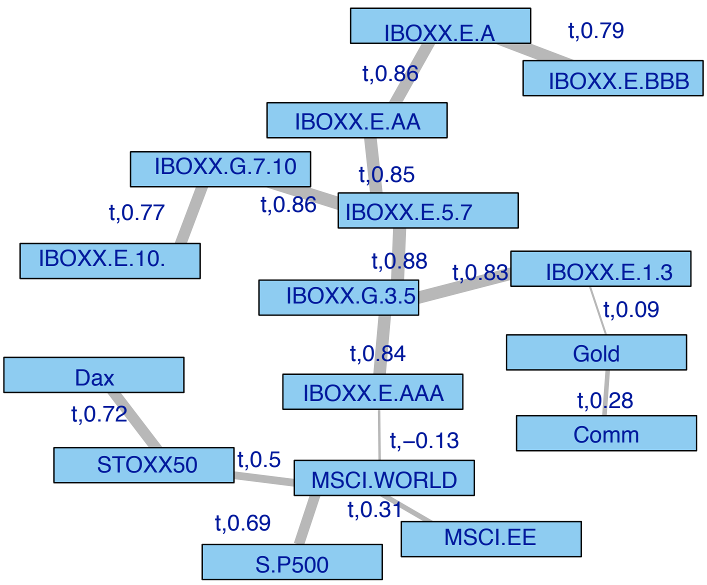


(b) T_1 of R-Vine using tail cumulation as weight

Figure 8.16: First tree of R-Vine model fitted using Strategies 3 and 4 for the international financial indices data set (continued).



(a) T_1 of R-Vine using Hu dependence as weight



(b) T_1 of R-Vine using exceedance dependence as weight

We observe the that R-Vine selected based on Strategy 2 using tail cumulation as weight truncates earlier in compare to other strategies and so looses the additional information. As result the likelihood as well as AIC and BIC values are the smallest of all four. Hu dependence Strategy 3 truncates very late as expected and need more parameters. Kendall′s τ based model seems to be the better fit again. However, if we take in the consideration the size of the data set we might as well say that all strategies are reasonably close to the optimal values of Strategy 1.

Result of the Vuong test comparison

So far we conclude that according to the log likelihoods and the AIC/BIC values all of our strategies offer quite a good fit. Similarly to the exchange rates data set, also here Strategy 1 shows best results and Strategy 3 performs as least preferable. Now we want to see if those conclusions can be supported and perform a Vuong test. The results of the test are given in Table 8.8. We observe that Strategy 3 clearly "loses"to all other strategies in 100 % of cases. Combined with the results of likelihoods its an indicator against the Hu dependence weight in this data set. According to the test, Strategies 2 and 4 fit the data equally well and the test could not prefer one over another when comparing Strategies 1 and 4. When comparing S1 and S2 the test shows that H_0 could be rejected 60 times in favor of S1.

Comparison	Financial Indices		Comparison	Financial Indices
$S1 > S2$	60		$S2 > S3$	100
$S1 \approx S2$	40		$S2 \approx S3$	0
$S1 < S2$	0		$S2 < S3$	0
$S1 > S3$	100		$S2 > S4$	6
$S1 \approx S3$	0		$S2 \approx S4$	87
$S1 < S3$	0		$S2 < S4$	7
$S1 > S4$	48		$S3 > S4$	0
$S1 \approx S4$	52		$S3 \approx S4$	0
$S1 < S4$	0		$S3 < S4$	100

Table 8.8: Results of Vuong test comparison for R-Vines fitted according to four strategies using the selected model by Kendall′s Tau as underlying "true"model. Notation: S1: Kendall′s τ; S2: Tail cumulation; S3: Hu dependence; S4: Exceedance dependence.

So, to summarize the results:

$$\text{Strategy 1} > \text{Strategy 2} \approx \text{Strategy 4} > \text{Strategy 3}.$$

Table 8.9: Financial Indices, tree wise log likelihoods and AIC/BIC values of R-Vine model selected using Kendall's τ as weight

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	30	32358.57	-64657.13	-64484.43
T2	27	2291.50	-4592.00	-4373.57
T3	17	877.77	-1721.55	-1623.69
T4	15	.416.28	-802.56	-716.21
T5	13	156.82	-287.65	-212.21
T6	5	154.20	-298.41	-269.62
T7	2	20.29	-36.86	-25.07
T8	1	27.05	-52.10	-46.34
T9	1	27.13	-52.25	-46.50
truncation	$\sum = 111$	$\sum =36329.62$	$\sum = -\ 72437.23$	$\sum = -\ 71798.25$
T10	1	1.92	-1.84	3.91
T11	-	-	-	-
T12	1	2.89	-3.77	1.98
TOTAL	113	36334.43	-72442.85	-71792.35

Table 8.10: Financial Indices, tree wise log likelihoods and AIC/BIC values of R-Vine model selected using tail cumulation as weight

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	30	32139.62	-64219.25	-64046.55
T2	16	1546.38	-3060.77	-2968.66
T3	16	652.36	-1268.71	-1165.09
T4	14	300.05	-572.10	-491.51
T5	8	89.70	-163.393	-117.34
T6	7	130.32	-246.63	-206.34
truncation	$\sum$ =91	$\sum$ = 34858.43	$\sum$ = -69530.86	$\sum$ = -68995.50
T7	7	44.14	-74.28	-33.99
T8	5	18.66	-29.32	-6.29
T9	7	133.92	-253.84	-213.54
T10	4	34.63	-61.25	-38.23
T11	2	25.43	-46.86	-35.35
T12	3	13.23	-22.46	-10.95
T13	3	404.35	-802.69	-785.42
T14	4	421.89	-835.79	-812.76
T15	2	19.30	-34.61	-23.10
TOTAL	148	35973.99	-71691.97	-70955.13

Table 8.11: Financial Indices, tree wise log likelihoods and AIC/BIC values of R-Vine model selected using Hu dependence as weight

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	30	9936.70	-19813.40	-19640.70
T2	27	9195.45	-18338.90	-18187.23
T3	22	11215.12	-22386.24	-22259.6 0
T4	19	2293.37	-4550.75	-4447.13
T5	17	870.74	-1713.75	-1632.89
T6	11	572.48	-1122.95	-1059.63
T7	10	229.04	-438.08	-380.52
T8	6	126.91	-241.82	-207.28
T9	6	73.10	-136.20	-107.41
T10	4	119.20	-230.40	-207.37
T11	2	938.57	-1873.13	-1861.62
truncation	$\sum = 154$	$\sum = 35570.69$	$\sum =$ -70845.37	$\sum =$ -69963.39
T12	1	5.38	-8.77	-3.01
T13	2	4.79	-5.57	5.94
TOTAL	157	35580.86	-70859.71	-69960.46

Table 8.12: Financial Indices, tree wise log likelihoods and AIC/BIC values of R-Vine model selected using exceedance dependence“as weight

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	30	31710.78	-63361.57	-63188.87
T2	20	2169.15	-4298.30	-4183.16
T3	16	1126.40	-2220.79	-2128.69
T4	14	455.60	-883.21	-802.62
T5	12	207.25	-392.51	-329.19
T6	4	68.29	-122.57	-82.28
T7	8	16.86	-25.73	-2.70
T8	5	32.76	-59.525	-42.25
T9	2	151.78	-293.57	-264.78
truncation	$\sum$ =111	$\sum$ = 35938.89	$\sum$ = - 71657.77	$\sum$ = -71024.55
T10	1	36.53	-63.06	-34.28
T11	2	68.77	-133.55	-122.03
T12	4	13.31	-22.63	-11.11
T13	2	16.77	-29.54	-18.02
T14	1	8.04	-14.09	-8.33
TOTAL	121	36082.32	-71920.63	-71218.33

8.3 German DAX Indices

ID.DE	Company Name		ID.DE	Company Name
ADS	Adidas		HEI	HeidelbergCement
ALV	Allianz		HEN3	Henkel
BAS	BASF		IFX	Infineon Technologies
BAYN	Bayer		LHA	Lufthansa
BEI	Beiersdorf		LIN	Linde
BMW	BMW		MAN	MAN
CBK	Commerzbank		MEO	Metro
DAI	Daimler		MRK	Merk
DB1	Deutsche Börse		MUV2	Munich Re
DBK	Deutsche Bank		RWE	RWE
DPW	Deutsche Post		SAP	SAP
DTE	Deutsche Telekom		SDF	K+S
EOAN	E·ON		SIE	Siemens
FME	Fresenius Medical Care		TKA	ThyssenKrupp
FRE	Fresenius SE		VOW3	Volkswagen

Table 8.13: Variables of German DAX data set.

So far we have investigated how good four strategies we chose can fit the data and observed that the strategy using Kendall′s τ as its weight can not be outperformed by other choice of weight selection. However, the exchange rates data set with 9 variables is rather small. The data set of daily returned of financial indices has more variables, which is also preferred for modeling since it is rather the case for the real financial world. Now we want to apply the strategies to a bigger size of data in order to see if that might influence the decision to chose one strategy over another. The German DAX data set of (Brechmann 2012) contains 30 variables which correspond to the 30 most important German stocks. The log returns and time series observations from 01/2005 till 07/2011 are summarized in 1158 data points.
Similarly to financial indices data set, (Brechmann 2012) organizes German DAX in 10 groups according to their "field". Those are:

financials: ALV, CBK, DBK, DB1, MUV2	industrials : MAN, SIE, TKA
chemicals : BASF, BAYN, K+S, LIN	healthcare : FME, FRE, MRK
consumer goods : ADS, BEI, MEO, HEN3	auto industry: BMW, DAI, VOW3
IT and communication : DTE, IFX, SAP	utilities : EOAN, RWE
logistic and transportation : DPW, LHA	building materials : HEI

The big size of that particular data set makes it difficult to simulate and perform the Vuong test for comparison. But we are still able to see if different choices the strategies make when selecting first tree and the following estimation lead to a significant changes in likelihood values and the performance of AIC and BIC tests. Obviously, in this particular case based on results in Table 8.24 we would prefer S2 since it shows largest values and reasonable number of parameters. However, the overall differences in values of S1 and S2 are very small it indicates that both could be considered as optimal fit.
In Figures 8.18-19 the first tree selection for each model are given. The tree structures look almost "C-Vine like"with the DBK.DE in the role of root node. This is self explicable considering the economic value of the Deutsche Bank on the German stock market with hight dependence on others. The model resulting from Strategy 3 again shows different perspective according to the preferable asymmetrical distinctions, by selecting different pairs of variables as well as different pair-copulas.

Figure 8.17 summarizes the variable pairs selected in every strategy. Overview of individually selected pairs are given in Appendix B.3.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	–								1	3												1								
2		–		1				2		3						1			1				1	2		1				
3			–	2									1	1						2				1				2	1	
4				–										1												1		1		
5					–					2																1				1
6						–	1	2								1	1						1							
7							–			3			1												1		1			1
8								–	1	2												1								2
9									–	2				1				1			1									
10										–	2	2	1	1		1	1	2	1		2	1	1	1		1		3		
11											–	1		1									1							
12												–		2								1								
13													–								1				3					
14														–	2					1	1					1	1	1	1	
15															–		1		1											
16																–				1										
17																	–			1								1		1
18																		–												2
19																			–				1					1		
20																				–										
21																					–						1		1	
22																						–								1
23																							–							1
24																								–	1					1
25																									–					
26																										–				
27																											–		2	
28																												–	2	
29																													–	
30																														–

Figure 8.17: An overview matrix of selected variable pairs applying S1-S4 for German DAX data set. The number represent each variable of Table 8.13 alphabetically, i.e. starting with ADS=1, ALV=2 and so on till VOW3=30.

Figure 8.18: First tree T_1 of R-Vine model fitted using weights Kendall's τ (top) and Tail cumulation (bottom) for German DAX.

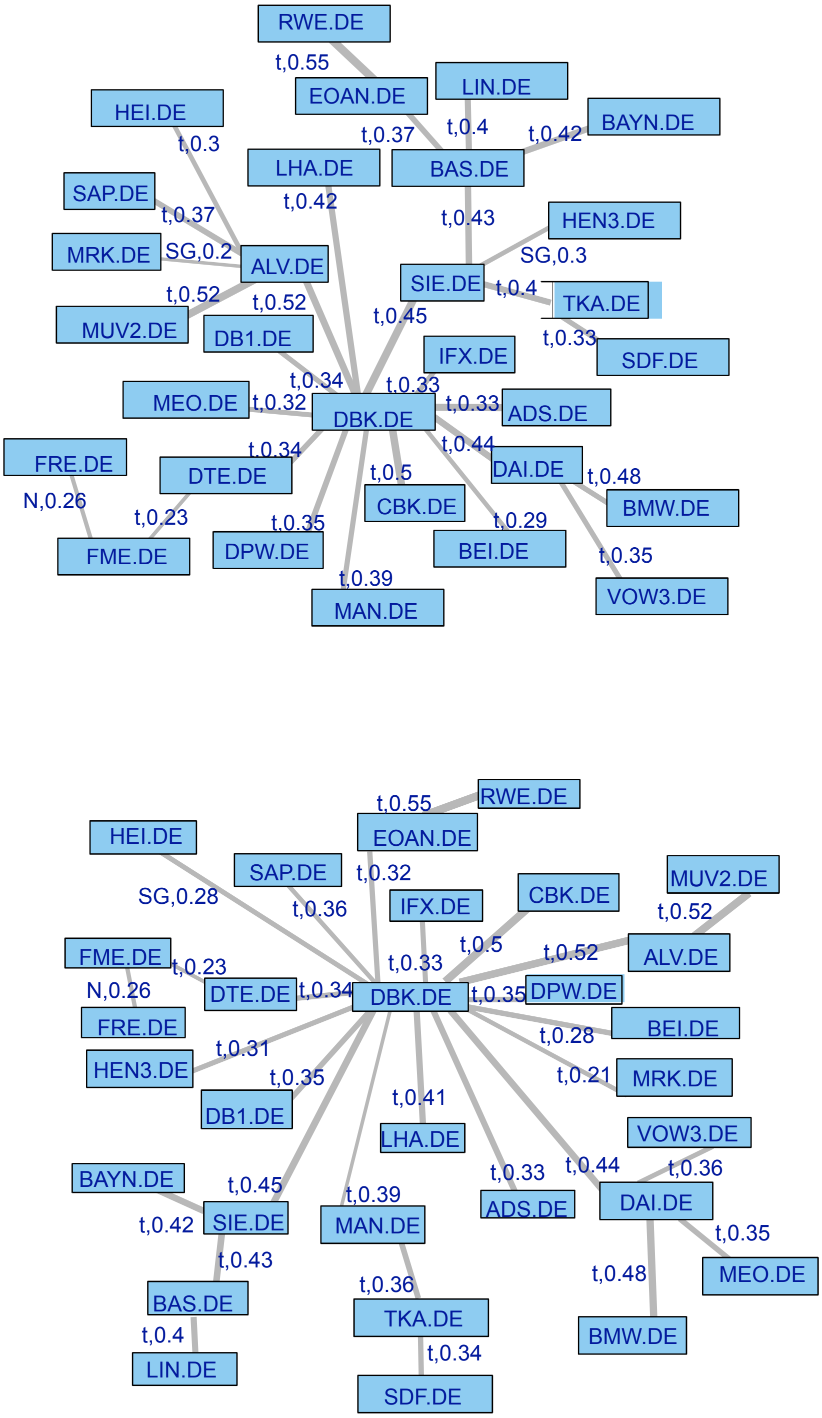
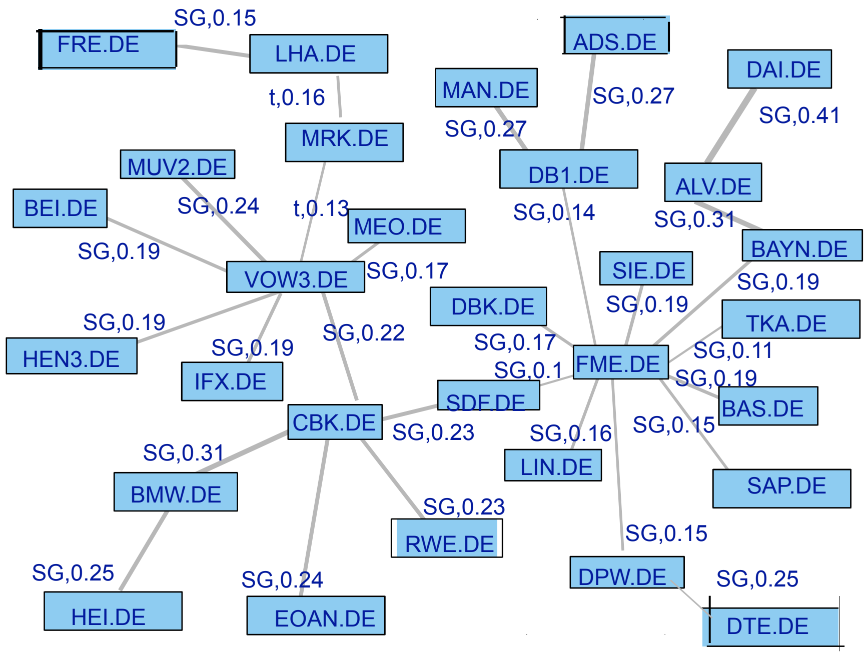
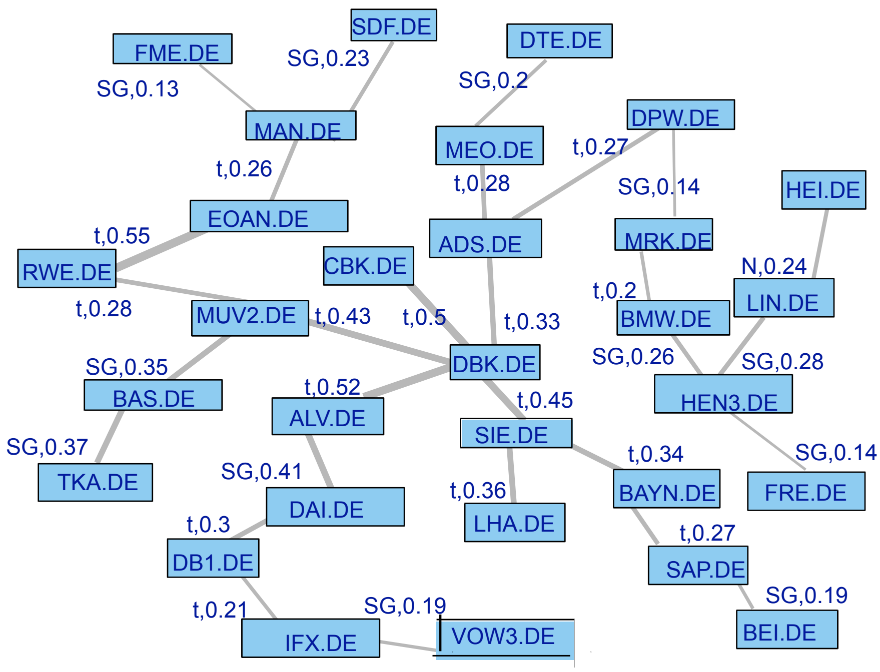


Figure 8.19: First tree of R-Vine model fitted using weights Hu dependence (top) and Exceedance dependence (bottom) for German DAX (continued).





Weight	Number of parameters	Log Likelihood	AIC	BIC
Kendall′s τ	336	9041.55	-17411.10	-15712.81
Tail Cumulation	331	9043.56	-17425.13	-15752.10
Hu Dependence	393	8952.17	-17118.34	-15131.94
Exceedance Dependence	370	8905.32	-17070.64	-15200.49

Weight	Truncation	Number of parameters	Log Likelihood	AIC	BIC
Kendall′s τ	T5	214	8553.40	-16678.80	-15597.14
Tail Cumulation	T5	202	8445.05	-16486.09	-15465.10
Hu Dependence	T12	334	8246.13	-16264.03	-14575.84
Exceedance Dependence	T10	277	8271.12	-15988.25	-14588.17

Table 8.14: Log Likelihoods, AIC, BIC and number of parameters for non-truncated (top) and truncated (bottom) R-Vines selected according to different weights for DAX data set.

8.4 Expanding the Hu Dependence Weight

So far, we have observed that the Strategy 3 using maximum of the upper and lower hu dependence coefficient has not performed very well against other strategies. First, it is of cause due to the specifics of the underlying data. For example, when using exchange rates data set which did not show heavy tails, also did not prefer this Strategy over the other in the comparison of log likelihoods or using the Vuong test. Recall the definition of the mixture copula according to (Brechmann 2010):

$$C^{mixture}(u,v) = (1-\omega_1-\omega_2){C_\rho}^{Ga}(u,v) + \omega_1 {C_{\theta_1}}^{Gu}(u,v) + \omega_2 {C_{\theta_2}}^{SG}(u,v) \quad (8.1)$$

where $\omega_1, \omega_2 \in [0,1]$ are called Hu dependence coefficients such that $\omega_1 + \omega_2 \leq 1$. This coefficients show the corresponding “closeness“of the underlying data set to Gumbel, Gumbel survival copulas. The coefficient $(1-\omega_1-\omega_2)$ indicates the closeness to the Gaussian copula, i.e. the normality of the data.
Before when we chose the weight of the edge in the MST for $S3$ we took the

Table 8.15: Tree wise log likelihoods and AIC/BIC values of R-Vine selected using Kendall′s τ as weight for German DAX.

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	55	6734.67	-13359.34	-13081.35
T2	51	1049.49	-1996.98	-1739.21
T3	41	393.66	-705.33	-498.10
T4	35	216.71	-363.54	-186.64
T5	32	158.80	-253.59	-91.85
truncation	$\sum = 214$	$\sum = 8553.40$	$\sum = -16678.80$	$\sum =-15597.14$
T6	17	63.78	-93.56	-7.63
T7	15	59.97	-89.95	-14.13
T8	13	44.62	-63.23	2.48
T9	8	42.51	-69.03	-28.59
T10	6	20.18	-28.36	1.97
T11	8	26.59	-37.18	3.25
T12	12	40.75	-57.51	3.15
T13	6	28.67	-45.51	-15.00
T14	6	28.10	-44.19	-13.87
T15	6	23.07	-34.15	-0.38
T16	8	45.68	-75.36	-34.92
T17	2	4.53	-5.06	5.05
T18	3	14.10	-22.21	-7.05
T19	3	6.43	-6.86	8.30
T20	3	15.96	-25.92	-1.76
T21	3	11.41	-16.82	-1.65
T22	1	8.16	-14.32	-9.27
T23	-	-	-	-
T24	1	0.18	1.64	6.95
T25	-	-	-	-
T26	1	3.45	-4.90	0.15
TOTAL	336	9041.55	-17411.10	-15712.81

Table 8.16: Tree wise log likelihoods and AIC/BIC values of R-Vine selected using tail cumulation as weight for German DAX

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	56	6591.49	-13070.98	-12787.93
T2	49	1111.95	-2125.21	-1878.24
T3	45	448.37	-806.75	-579.30
T4	33	166.59	-267.19	-100.39
T5	19	126.64	-215.27	-119.24
truncation	$\sum$ =202	$\sum$ =8445.05	$\sum$ = -16486.09	$\sum$ = -15465.10
T6	15	70.15	-110.31	-34.49
T7	17	80.50	-127.01	-41.08
T8	16	75.64	-119.29	-38.41
T9	10	70.64	-120.41	-69.87
T10	10	72.06	-124.13	-73.57
T11	15	72.19	-114.37	-38.55
T12	6	17.47	-22.94	7.38
T13	9	26.43	-34.86	10.63
T14	3	6.55	-7.10	8.06
T15	6	19.48	-26.97	3.36
T16	3	17.70	-29.40	-14.24
T17	5	15.61	-21.22	4.05
T18	5	14.51	-19.01	6.26
T19	2	4.55	-5.09	5.02
T20	2	10.49	-16.98	-6.87
T21	-	-	-	-
T22	1	5.19	-8.37	-3.32
T23	1	4.54	-7.09	-2.03
T24	1	3.87	-5.75	-0.69
T25	2	11.36	-18.73	-8.62
TOTAL	331	9043.56	-17425.13	-15752.10

Table 8.17: Tree wise log likelihoods and AIC/BIC values of R-Vine selected using Hu dependence as weight for German DAX

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	31	2207.71	-4353.41	-4196.73
T2	49	2335.71	-4573.41	-4325.74
T3	42	1182.69	-2281.39	-2069.10
T4	42	853.52	-1623.39	-1410.75
T5	37	426.74	-779.48	-592.47
T6	33	423.52	-781.04	-614.24
T7	23	334.17	-622.34	-506.09
T8	18	137.56	-239.12	-148.14
T9	23	281.13	-516.27	-400.01
T10	10	63.38	-106.76	-56.21
T11	17	158.02	-282.05	-196.12
T12	9	61.86	-105.72	-60.23
truncation	$\sum = 334$	$\sum =8246.13$	$\sum = -16264.03$	$\sum = -14575.84$
T13	5	18.21	-26.42	-1.14
T14	9	56.15	-94.30	-48.81
T15	6	16.20	-94.30	9.92
T16	4	17.41	-26.81	-6.59
T17	5	109.70	-209.41	-184.13
T18	7	26.62	-39.25	-3.87
T19	4	46.02	-84.04	-63.83
T20	3	10.81	-15.62	-0.46
T21	5	135.51	-261.03	-235.75
T22	7	22.11	-30.21	5.17
T23	1	5.64	-9.27	-4.22
T24	-	-	-	-
T25	1	2.44	-2.89	2.16
T26	-	-	-	-
T27	2	19.33	-34.66	-24.55
TOTAL	393	8952.17	-17118.34	-15131.94

Table 8.18: Tree wise log likelihoods and AIC/BIC results of R-Vine selected using exceedance dependence as weight for German DAX

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	45	4700.39	-9310.78	-9083.33
T2	47	1438.30	-2782.60	-2545.04
T3	42	741.59	-1399.19	-1186.90
T4	33	510.65	-955.31	-788.51
T5	21	169.23	-296.55	-190.31
T6	23	269.45	-492.90	-376.65
T7	19	136.98	-235.97	-139.93
T8	18	116.24	-196.48	-105.50
T9	17	95.99	-157.99	-72.07
T10	12	92.28	-160.57	-99.91
truncation	$\sum = 277$	$\sum =8271.12$	$\sum =$ - 15988.25	$\sum =$ -14588.17
T11	8	47.23	-78.46	-38.02
T12	14	83.17	-138.33	-67.57
T13	9	91.81	-165.63	-120.14
T14	9	42.59	-67.18	-21.69
T15	9	126.57	-235.13	-189.64
T16	6	35.12	-58.24	-27.92
T17	7	43.32	-72.65	-37.27
T18	3	14.27	-22.55	-7.38
T19	2	7.99	-11.98	-1.87
T20	5	43.65	-77.29	-52.02
T21	5	29.44	-48.97	-23.60
T22	1	3.14	-4.28	0.77
T23	6	21.13	-30.26	0.07
T24	2	6.49	-8.98	1.13
T25	4	26.05	-44.11	-23.89
T26	-	-	-	-
T27	1	6.35	-10.69	-5.64
T28	-	-	-	-
T29	2	5.87	-7.75	2.36
TOTAL	370	8905.32	-17070.64	-15200.05

Table 8.19: Exchange rates, tree wise log likelihoods and AIC/BIC results of R-Vine model selected using adjusted Hu dependence as weight

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	11	348.35	-674.87	-620.80
T2	11	463.67	-905.35	-851.28
T3	9	661.890	-1305.78	-1261.55
truncation	$\sum$= 31	$\sum$=1437.99	$\sum$= - 2885.99	$\sum$= - 2733.64
T4	7	36.44	-20.194	-24.47
T5	7	680.71	-61.30	-1313.01
T6	3	30.62	-13.67	-40.49
T7	2	2.76	-1.96	8.31
T8	1	0.46	-1.96	5.99
TOTAL	33	2224.98	-4347.96	-4097.31

$max\{\omega_1, \omega_2\}$, i.e. the maximum of the two coefficient indicating either strong upper or lower tail dependence to model the maximum asymmetrical dependence. Now we want to expend this measure by adding the third coefficient namely the coefficient of the closeness to Gaussian copula with no tail dependence. This means we define the new weight: *Hu dependence adjusted* $= max\{\omega_1, \omega_2, 1 - \omega_1 - \omega_2\}$. Since this new weight theoretically does not show such extreme behavior it is likely to display different results in comparison of the fitted models. Although it is rather not predictable, since it might simplify the model when using Gaussian copulas only. Despite this, we want to investigate the R-Vine model fitted using this weight and whether it would be able to offer a better fit. We proceed the same way as before so that we can compare the outcome of the models by the same criteria. Due to the size of the data we did not perform this comparison for German DAX data set.

Log Likelihood Values

After the R-Vine was fully specified using the Hu dependence adjusted for the

Table 8.20: Financial Indices, tree wise log likelihoods and AIC/BIC results of R-Vine model selected using “Adjusted Hu dependence“as weight

Tree	Number of parameters	Log likelihood	AIC	BIC
T1	30	9936.70	-19813.40	-19640.70
T2	28	9200.09	-18344.18	-18183.00
T3	25	11330.15	-22610.30	-22466.39
T4	19	23224.95	-4611.91	-4502.53
T5	18	886.91	-1737.83	-1634.21
T6	14	587.47	-1146.95	-1066.35
T7	13	221.16	-416.33	-341.49
T8	10	132.48	-244.95	-187.39
T9	8	89.66	-163.32	-117.27
T10	7	122.10	-230.18	-189.89
T11	6	955.35	-1899.70	-1864.16
truncation	$\sum$ =178	$\sum$ = 35787.03	$\sum$ = -71218.06	$\sum$ = -70154.10
T12	4	10.54	-9.54	-3.01
T13	3	5.86	-6.09	5.94
T14	2	1.32	-8.77	-3.01
T15	1	1.41	-5.57	5.94
TOTAL	198	35806.17	-71236.35	-70154.10

Weight	Number of parameters	Log Likelihood	AIC	BIC
Kendall's τ	33	2219.28	-4372.56	-4210.38
Tail cumulation	33	2212.52	-4359.49	-4196.86
Hu dependence	47	2202.59	-4309.18	-4073.27
Exceedance dependence	37	2204.18	-4334.36	-4152.51
Hu Dep adjusted	51	2224.98	-4347.96	-4097.31

Weight	Truncation	Number of parameters	Log Likelihood	AIC	BIC
Kendall's τ	T2	25	2158.61	-4267.21	-4144.35
Tail cumulation	T3	26	2156.96	-4261.93	-4134.14
Hu dependence	none	47	2202.59	-4309.18	-4073.27
Exceedance dependence	T4	34	2192.41	-4316.81	-4155.31
Hu Dep adjusted	T3	31	1473.98	-2885.99	-2733.64

Table 8.21: Log Likelihoods, AIC, BIC and number of parameters for non-truncated (top) and truncated (bottom) R-Vines according to four previous strategies plus adjusted Hu dependence weight for Exchange rates data set.

specification of the trees, we compute the tree wise likelihoods for the new model. Like the rest of the R-Vine model fitted using strategies proposed earlier we mainly want to compare the possible improvement in the log likelihoods , which would be an indicator to one model fit the data better than others. Below Tables give the likelihood values as well as AIC/BIC and the number of parameters as we had in the previous sections for S1 to S4. Additionally, the values of the adjusted model are displayed.

Note, that the values vary strongly in compare to the original Hu dependence weight, since the adjusted weight is a combination of strong asymmetrical dependence versus no tail dependence quality of the variable pair. If we have a look at the results for exchange rates, the new model truncates very soon and loses in the respective values due to less explicit specification. For non- truncated models we see large improvement in the log likelihood value. However if we take into account the number of parameters needed, the improvement is not significant in comparison to the respective values of our previous models. The same result indicated also the values of the AIC/BIC.(see Table 8.21)

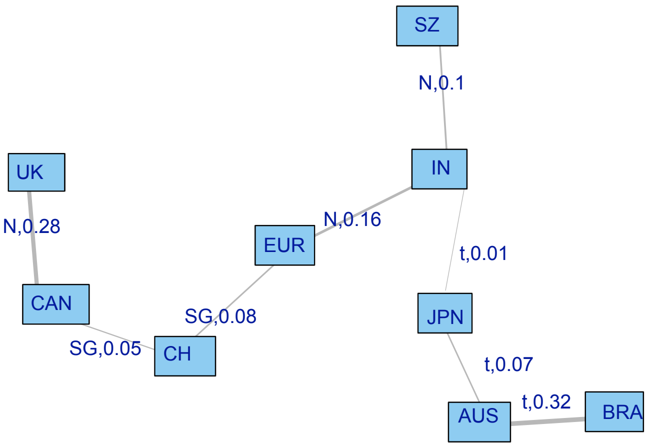
Weight	Number of parameters	Log Likelihood	AIC	BIC
Kendall's τ	113	36334.43	-72442.85	-71792.35
Tail Cumulation	148	35973.99	-71691.97	-70955.13
Hu dependence	157	35580.86	-70859.71	-69960.46
Exceedance Dependence	121	36082.32	-71920.63	-71218.33
Hu Dep adjusted	198	35806.17	-71236.35	-70154.10

Weight	Truncation	Number of parameters	Log Likelihood	AIC	BIC
Kendall's τ	T9	111	36329.62	-72437.23	-71792.35
Tail Cumulation	T6	91	34858.43	-69530.86	-68995.50
Hu dependence	T11	154	35570.69	-70845.37	-69963.46
Exceedance Dependence	T9	111	35938.89	-71657.77	-71024.55
Hu Dep adjusted	T11	178	35787.03	-71218.06	-70193.38

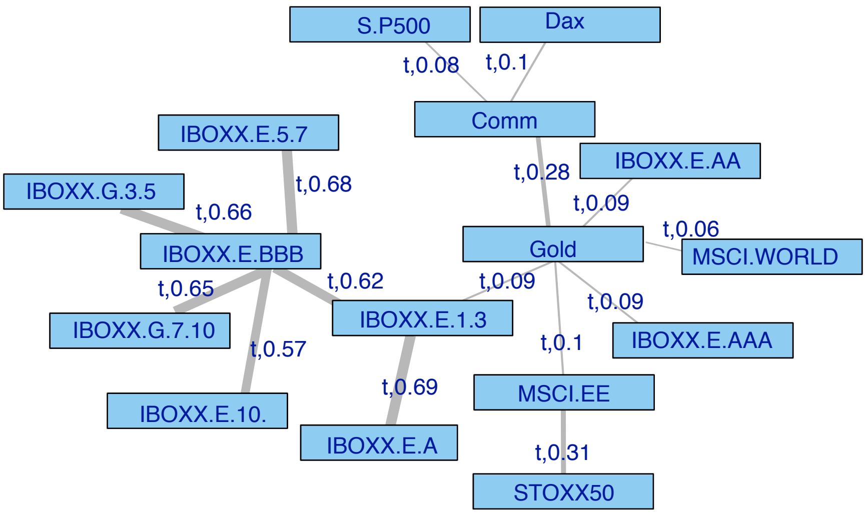
Table 8.22: Log Likelihoods, AIC, BIC and number of parameters for non-truncated (top) and truncated (bottom) R-Vines according to four strategies plus adjusted Hu dependence weight for International financial indices data set.

Since the international financial indices show stronger dependence between variables, especially among variables in the same asset group, the new adjusted measure seems to not be suitable for this data set. When looking at the values in Table 8.22 we observe that the adjusted model delivers the smallest likelihood as well as smallest AIC/BIC values while the number of parameters used is enormous. Although the displayed values are slightly higher than the corresponding values for not adjusted Hu dependence weight, it still smaller than the other strategies based on different weights. However, since the values are so large the differences can be taken as not very significant. Note also that while this method applied on exchange rates delivers a different tree structure than using non-adjusted Hu dependence, the first tree of International financial indices with adjusted and non-adjusted weights coincide. The choice of the appropriate copula families in later selection differ. Corresponding matrix representations of the R-Vine specifications based on adjusted weight for both models can be found in Appendix B.3.

Figure 8.20: T_1 of R-Vine using Hu dependence adjusted as weight applied of:



(a) exchange rates data set



(b) international financial indices data set

Comparison	Exchange Data	Financial Indices
$S1 > S3^{adj}$	98	100
$S1 \approx S3^{adj}$	2	0
$S1 < S3^{adj}$	0	0
$S2 > S3^{adj}$	96	100
$S2 \approx S3^{adj}$	4	0
$S2 < S3^{adj}$	0	0
$S4 > S3^{adj}$	76	100
$S4 \approx S3^{adj}$	24	0
$S4 < S3^{adj}$	0	0

Table 8.23: The results of Vuong test comparison for Strategies S1, S2, S4 against $S3^{adj}$ using the corresponding weights.

Vuong Test Statistics

We want to have a look at the Vuong test comparison in order to decide if the adjusted model could be a better fit to the data set. For the data sets of exchange rates and financial indices the adjusted Hu dependence model was simulated from the model based on Kendalls tau as before. For those 100 fitted R-Vines we perform Vuong test to compare if it is closer to the original model than fitted R-Vines using Strategies 1 to 4, i.e. tail cumulation. Kendalls tau or exceedance dependence. The results are found in Table 8.23. Clearly, the Strategy $S3^{adj}$ does not offer a better fit according to the Vuong test statistics. Hence, based on the Vuong test comparison and the values of AIC/BIC plus log likelihoods we do not consider the adjusted weight and corresponding resulting R-Vine model to be efficient.

9 Conclusions

In this thesis, we analyzed four different approaches to build the first tree of an R-Vine model. We wanted to capture the most dependence in the first tree since it has the most impact in an R-Vine and might allow further simplifications or even truncation. It is also known that the more dependence is captured in early trees the more likely it is for further trees to use independence copulas and reduce the estimation errors. We based our research on the sequential model selection. However, alongside $Kendall's$ τ as the edge weight we explored three more weights, namely tail cumulation and exceedance dependence based on Kendalls tau to model the joint tail dependence as well as the Hu dependence coefficient which allows us to model the asymmetrical tail dependence. Since we concentrated on alternative weight to measure tail dependence, we use the 5 copula families that seem best suiTable for that purpose. The MST algorithm of Prim is applied for the selection of the first tree for each of those approaches. An extensive simulation study was conducted in order to see how each Strategy using corresponding weight is able to fit the data. Mainly we wanted to see if there is a reasonable alternative to the commonly used Kendalls tau weight for the construction of the maximum valued first tree as we have seen that it has the biggest influence on the likelihood values of an R-Vine specification. In the simulation study we used 20 various scenarios each of them with certain properties, such as different dependence structures, strength of correlation and copula family types. The resulting R-Vines were compared by the Vuong test. We could observe that all strategies performance results differ conditionally on the scenario. Hence, if the properties of the underlying data are know it can be advantageous to use an alternative weight instead of Kendalls tau to offer a different perspective. This could be especially useful for large data sets. However, if the underlying original model specification is not fully known, it is better to stick to the Kendalls tau approach, since this has shown good results and was less computationally expensive as for example the approach of Hu dependence.

To evaluate the strategies we apply our four methods on three different data sets. Different weights resulted in different tree structure as well as the copula types selection. We compared the models by analyzing the log likelihood estimation and AIC/BIC tests. We also used the Vuong test to see if any model is considered as a better fit.

The first data set used was exchange rates to the US Dollar of nine variables. We analyzed the log likelihoods and AIC/BIC values for the resulted R-Vine specifications. The differences were not extreme except of the Hu dependence model did need greater parameter estimation. This result was confirmed by the Vuong test performance. The next data set, the international financial indices with 16 variables showed that independent from the Strategy applied, the strongest de-

pendencies were still kept. The results of the comparison approach support the conclusions we have made on the exchange rates application. Hence, for an unknown underlying specification the Hu dependence as weight is not a good choice. Finally, for the 30-dimensional data set of German DAX development with the same observations we now can tell that Kendall's τ can not be completely outperformed. This might be a starting point for further analysis with more asymmetrical dependence in application data.
In order to improve the Hu dependence weight to make it more adjustable to data that might not exhibit strong asymmetry in the tail we slightly changed the weight and repeated the investigation on data sets of exchange rates and financial indices. The resulting specifications differed from the R-Vines based on the original Hu dependence Strategy either in the choice of the variable pairs as in exchange rates or in choice of pair-copula specification as in the financial indices data set. However, it could not show better results in the comparison methods.
In conclusion, the three alternative approaches we investigated, i.e. tail cumulation, Hu dependence and exceedance dependence offer a quite good alternative to the commonly used Kendall's τ. Using one of those weight could be especially useful if the underlying data is known to exhibit certain properties that coincide with the ones we have seen in our simulation study. For unknown data it might be better to stick to the approach based on Kendalls tau as it performs well in high dimensions and also requires less resources compare to the alternatives, especially the Hu dependence weight.

Appendix

A Additional Information on Simulation Study

The R-Vine Matrix of the underlying “true “model is given by:

$$\text{M}=\begin{pmatrix} 4 & & & & & & & \\ 7 & 8 & & & & & & \\ 5 & 7 & 5 & & & & & \\ 6 & 5 & 7 & 7 & & & & \\ 8 & 6 & 6 & 1 & 6 & & & \\ 1 & 3 & 1 & 2 & 1 & 1 & & \\ 2 & 1 & 3 & 3 & 2 & 3 & 2 & \\ 3 & 2 & 2 & 6 & 3 & 2 & 3 & 3 \end{pmatrix}$$

the copula family matrices for the non-elliptical and for the elliptical scenarios are given by:

$$F^{mixed} = \begin{pmatrix} 1 & & & & & & \\ 24 & 14 & & & & & \\ 4 & 1 & 2 & & & & \\ 34 & 1 & 14 & 34 & & & \\ 1 & 24 & 24 & 2 & 1 & & \\ 2 & 14 & 2 & 34 & 14 & 1 & \\ 24 & 2 & 4 & 1 & 34 & 4 & 2 \end{pmatrix} F^{elliptical} = \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & & & \\ 2 & 2 & 2 & 2 & 2 & & \\ 2 & 2 & 2 & 2 & 2 & 2 & \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

A detailed overview given in Figures A.1-A.8. The list of used copula families:

- 0 = independence copula
- 1 = Gaussian copula
- 2 = Student t copula (t-copula)
- 4 = Gumbel copula
- 14 = rotated Gumbel copula (180 degrees; i.e. survival Gumbel)
- 24 = rotated Gumbel copula (90 degrees)
- 34 = rotated Gumbel copula (270 degrees)

Further, we denote the parameter matrix with P and with P2 matrix which contains information on degreed of freedom.

Figure A.1: The structure of the underlying R-Vine

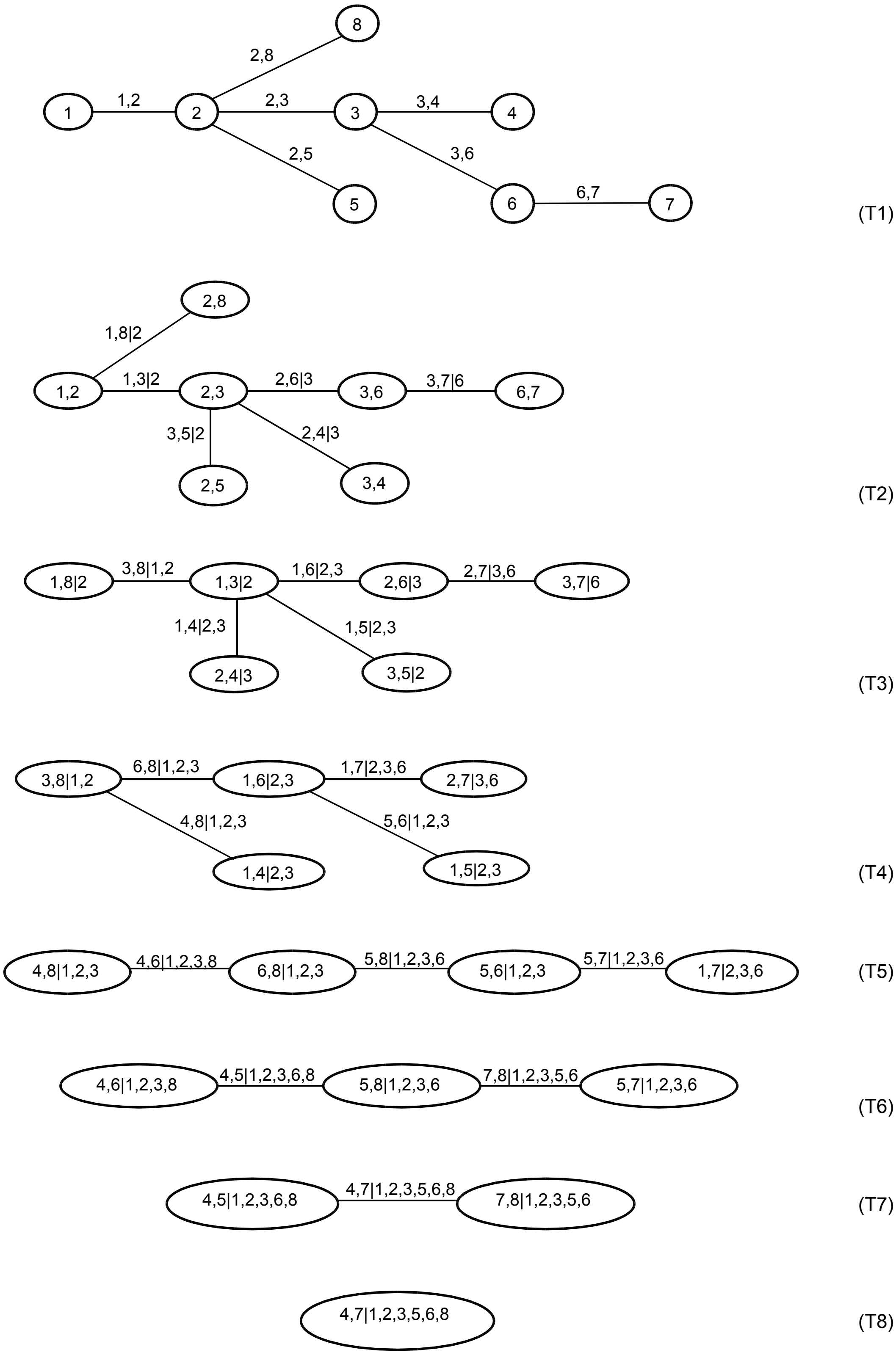


Figure A.2: The family matrices F, Kendall's τ and degrees of freedom values (P2) corresponding to Scenarios 1 to 5.

1 = Gaussian copula	24 = rotated Gumbel copula (90 degrees)	ParamaTer <0
2 = Student t copula (t-copula)	34 = rotated Gumbel copula (270 degrees)	ParameTer <0
4 = Gumbel copula		

Scenario1: mixed T4-T7 independent mon. decreasing

F1

0						
0	0					
0	0	0				
0	0	0	0			
1	24	24	2	1		
2	14	2	34	14	1	
24	2	4	1	34	4	2

Kendalls

0						
0	0					
0	0	0				
0	0	0	0			
0.19	-0.27	-0.23	0.23	0.21		
0.54	0.5	-0.57	-0.59	0.53	0.54	
-0.9	0.85	0.92	0.87	-0.9	0.88	-0.91

P2

			8			
2		4				
	3					2

Scenario2: mixed T4-T7 independent const. weak

F2

0						
0	0					
0	0	0				
0	0	0	0			
1	24	24	2	1		
2	14	2	34	14	1	
24	2	4	1	34	4	2

Kendalls

0						
0	0					
0	0	0				
0	0	0	0			
0.35	-0.29	-0.4	0.32	0.34		
0.29	0.33	-0.36	-0.35	0.37	0.33	
-0.3	0.36	0.35	0.35	-0.29	0.41	-0.33

P2

			8			
2		4				
	3					2

Scenario3: mixed T4-T7 independent const.sTring

F3

0						
0	0					
0	0	0				
0	0	0	0			
1	24	24	2	1		
2	14	2	34	14	1	
24	2	4	1	34	4	2

Kendalls

0						
0	0					
0	0	0				
0	0	0	0			
0.65	-0.6	-0.7	0.63	0.66		
0.59	0.69	-0.65	-0.66	0.67	0.68	
-0.71	0.65	0.68	0.64	-0.64	0.7	-0.63

P2

			8			
2		4				
	3					2

Scenario4: mixed T4-T7 independent mon. increasing

F4

0						
0	0					
0	0	0				
0	0	0	0			
1	24	24	2	1		
2	14	2	34	14	1	
24	2	4	1	34	4	2

Kendalls

0						
0	0					
0	0	0				
0	0	0	0			
0.9	-0.92	-0.85	0.87	0.91		
0.49	0.5	-0.52	-0.6	0.43	0.5	
-0.19	0.23	0.29	0.22	-0.2	0.21	-0.25

P2

			8			
2		4				
	3					2

Scenario5: mixed T4-T7 Gauss mon. decreasing

F5

1						
1	1					
1	1	1				
1	1	1	1			
1	24	24	2	1		
2	14	2	34	14	1	
24	2	4	1	34	4	2

Kendalls

0.2						
0.25	0.26					
0.32	0.31	0.3				
0.43	0.39	0.45	0.41			
0.53	-0.5	-0.54	0.49	0.48		
0.73	0.69	-0.64	-0.66	0.71	0.7	
-0.9	0.85	0.92	0.87	-0.9	0.88	-0.91

P2

			8			
2		4				
	3					2

Figure A.3: The family matrices F, Kendall's τ and degrees of freedom values (P2) corresponding to Scenarios 6 to 10.

Scenario6: mixed T4-T7 Gauss const. weak

F6							
	1						
	1	1					
	1	1	1				
	1	1	1	1			
	1	24	24	2	1		
	2	14	2	34	14	1	
	24	2	4	1	34	4	2

Kendalls							
	0.35						
	0.4	0.31					
	0.28	0.37	0.36				
	0.31	0.29	0.39	0.35			
	0.35	-0.29	-0.4	0.32	0.34		
	0.29	0.33	-0.36	-0.35	0.37	0.33	
	-0.3	0.36	0.35	0.35	-0.29	0.41	-0.33

P2							
				8			
	2		4				
		3					2

Scenario7: mixed T4-T7 Gauss const. strong

F7							
	1						
	1	1					
	1	1	1				
	1	1	1	1			
	1	24	24	2	1		
	2	14	2	34	14	1	
	24	2	4	1	34	4	2

Kendalls							
	0.61						
	0.65	0.68					
	0.63	0.67	0.71				
	0.7	0.59	0.73	0.69			
	0.65	-0.6	-0.7	0.63	0.66		
	0.59	0.69	-0.65	-0.66	0.67	0.68	
	-0.71	0.65	0.68	0.64	-0.64	0.7	-0.63

P2							
				8			
	2		4				
		3					2

Scenario8: mixed T4-T7 Gauss mon. increasing

F8							
	1						
	1	1					
	1	1	1				
	1	1	1	1			
	1	24	24	2	1		
	2	14	2	34	14	1	
	24	2	4	1	34	4	2

Kendalls							
	0.9						
	0.76	0.65					
	0.61	0.58	0.63				
	0.52	0.49	0.53	0.5			
	0.4	-0.42	-0.39	0.42	0.38		
	0.28	0.31	-0.33	-0.29	0.26	0.32	
	-0.19	0.23	0.29	0.22	-0.2	0.21	-0.25

P2							
				8			
	2		4				
		3					2

Scenario9: mixed all dependent mon. decreasing

F9							
	1						
	24	14					
	4	1	2				
	34	1	14	34			
	1	24	24	2	1		
	2	14	2	34	14	1	
	24	2	4	1	34	4	2

Kendalls							
	0.2						
	-0.25	0.26					
	0.32	0.31	0.3				
	-0.43	0.39	0.45	-0.41			
	0.53	-0.5	-0.54	0.49	0.48		
	0.73	0.69	-0.64	-0.66	0.71	0.7	
	-0.9	0.85	0.92	0.87	-0.9	0.88	-0.91

P2							
			6				
				8			
	2		4				
		3					2

Scenario10: mixed all dependent const. weak

F10							
	1						
	24	14					
	4	1	2				
	34	1	14	34			
	1	24	24	2	1		
	2	14	2	34	14	1	
	24	2	4	1	34	4	2

Kendalls							
	0.35						
	-0.4	0.31					
	0.28	0.37	0.36				
	-0.31	0.29	0.39	-0.35			
	0.35	-0.29	-0.4	0.32	0.34		
	0.29	0.33	-0.36	-0.35	0.37	0.33	
	-0.3	0.36	0.35	0.35	-0.29	0.41	-0.33

P2							
			6				
				8			
	2		4				
		3					2

Figure A.4: The family matrices F, Kendall's τ and degrees of freedom values (P2) corresponding to Scenarios 11 to 16.

Scenario11: mixed all dependent const. strong

F11						
1						
24	14					
4	1	2				
34	1	14	34			
1	24	24	2	1		
2	14	2	34	14	1	
24	2	4	1	34	4	2

Kendalls						
0.61						
-0.65	0.68					
0.63	0.67	0.71				
-0.7	0.59	0.73	-0.69			
0.65	-0.6	-0.7	0.63	0.66		
0.59	0.69	-0.65	-0.66	0.67	0.68	
-0.71	0.65	0.68	0.64	-0.64	0.7	-0.63

P2						
		6				
			8			
2		4				
	3					2

Scenario12: mixed all dependent mon. increasing

F12						
1						
24	14					
4	1	2				
34	1	14	34			
1	24	24	2	1		
2	14	2	34	14	1	
24	2	4	1	34	4	2

Kendalls						
0.9						
-0.76	0.65					
0.61	0.58	0.63				
-0.52	0.49	0.53	-0.5			
0.4	-0.42	-0.39	0.42	0.38		
0.28	0.31	-0.33	-0.29	0.26	0.32	
-0.19	0.23	0.29	0.22	-0.2	0.21	-0.25

P2						
		6				
			8			
2		4				
	3					2

Scenario13: T1-3: t; T4-T7 independent mon.decreasing

F13						
0						
0	0					
0	0	0				
0	0	0	0			
2	2	2	2	2		
2	2	2	2	2	2	
2	2	2	2	2	2	2

Kendalls						
0.19	0.27	0.23	0.23	0.21		
0.54	0.5	0.57	-0.59	0.53	0.54	
0.9	-0.85	0.92	0.87	0.9	0.88	0.91

P2						
10	6	7	8	4		
9	7	8	9	6	7	
2	5	4	2	3	5	2

Scenario14: T1-3: t; T4-T7 independent const. weak

F14						
0						
0	0					
0	0	0				
0	0	0	0			
2	2	2	2	2		
2	2	2	2	2	2	
2	2	2	2	2	2	2

Kendalls						
0.35	0.29	0.4	0.32	0.34		
0.29	0.33	0.36	-0.35	0.37	0.33	
0.3	-0.36	0.35	0.35	0.29	0.41	0.33

P2						
10	6	7	8	4		
9	7	8	9	6	7	
2	5	4	2	3	5	2

Scenario15: T1-3: t; T4-T7 independent const. strong

F15						
0						
0	0					
0	0	0				
0	0	0	0			
2	2	2	2	2		
2	2	2	2	2	2	
2	2	2	2	2	2	2

Kendalls						
0.65	0.6	0.7	0.63	0.66		
0.59	0.69	0.65	-0.66	0.67	0.68	
0.71	-0.65	0.68	0.64	0.64	0.7	0.63

P2						
10	6	7	8	4		
9	7	8	9	6	7	
2	5	4	2	3	5	2

Figure A.5: The family matrices F, Kendall′s τ and degrees of freedom values (P2) corresponding to Scenarios 17 to 20.

Scenario16: T1-3: t; T4-T7 independent mon. increasing

F16

0						
0	0					
0	0	0				
0	0	0	0			
2	2	2	2	2		
2	2	2	2	2	2	
2	2	2	2	2	2	2

Kendalls

0.9	0.92	0.85	0.87	0.91		
0.49	0.5	0.52	-0.6	0.43	0.5	
0.19	-0.23	0.29	0.22	0.2	0.21	0.25

P2

10	6	7	8	4		
9	7	8	9	6	7	
2	5	4	2	3	5	2

Scenario17: T1-3: t; T4-T7 Gauss mon. decreasing

F17

1						
1	1					
1	1	1				
1	1	1	1			
2	2	2	2	2		
2	2	2	2	2	2	
2	2	2	2	2	2	2

Kendalls

0.2						
0.25	0.26					
0.32	0.31	0.3				
0.43	0.39	0.45	0.41			
0.53	0.5	0.54	0.49	0.48		
0.73	0.69	0.64	-0.66	0.71	0.7	
0.9	-0.85	0.92	0.87	0.9	0.88	0.91

P2

10	6	7	8	4		
9	7	8	9	6	7	
2	5	4	2	3	5	2

Scenario18: T1-3: t; T4-T7 Gauss const. weak

F18

1						
1	1					
1	1	1				
1	1	1	1			
2	2	2	2	2		
2	2	2	2	2	2	
2	2	2	2	2	2	2

Kendalls

0.35						
0.4	0.31					
0.28	0.37	0.36				
0.31	0.29	0.39	0.35			
0.35	0.29	0.4	0.32	0.34		
0.29	0.33	0.36	-0.35	0.37	0.33	
0.3	-0.36	0.35	0.35	0.29	0.41	0.33

P2

10	6	7	8	4		
9	7	8	9	6	7	
2	5	4	2	3	5	2

Scenario19: T1-3: t; T4-T7 Gauss const. strong

F19

1						
1	1					
1	1	1				
1	1	1	1			
2	2	2	2	2		
2	2	2	2	2	2	
2	2	2	2	2	2	2

Kendalls

0.61						
0.65	0.68					
0.63	0.67	0.71				
0.7	0.59	0.73	0.69			
0.65	0.6	0.7	0.63	0.66		
0.59	0.69	0.65	-0.66	0.67	0.68	
0.71	-0.65	0.68	0.64	0.64	0.7	0.63

P2

10	6	7	8	4		
9	7	8	9	6	7	
2	5	4	2	3	5	2

Scenario20: T1-3: t; T4-T7 Gauss mon. increasing

F20

1						
1	1					
1	1	1				
1	1	1	1			
2	2	2	2	2		
2	2	2	2	2	2	
2	2	2	2	2	2	2

Kendalls

0.9						
0.76	0.65					
0.61	0.58	0.63				
0.52	0.49	0.53	0.5			
0.4	0.42	0.39	0.42	0.38		
0.28	0.31	0.33	-0.29	0.26	0.32	
0.19	-0.23	0.29	0.22	0.2	0.21	0.25

P2

10	6	7	8	4		
9	7	8	9	6	7	
2	5	4	2	3	5	2

Figure A.6: The normal QQ-plots and statistics of Scenario 1.

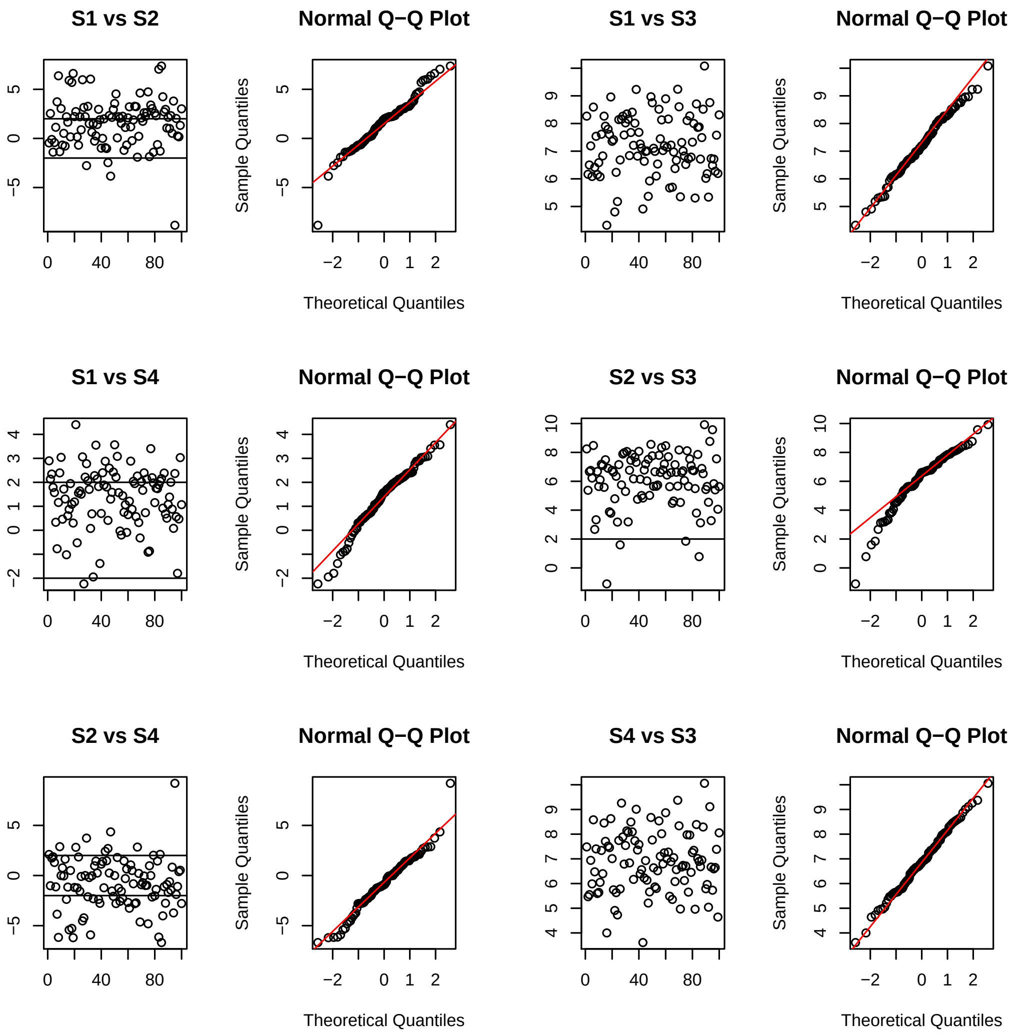


Figure A.7: The normal QQ-plots and statistics of Scenario 2.

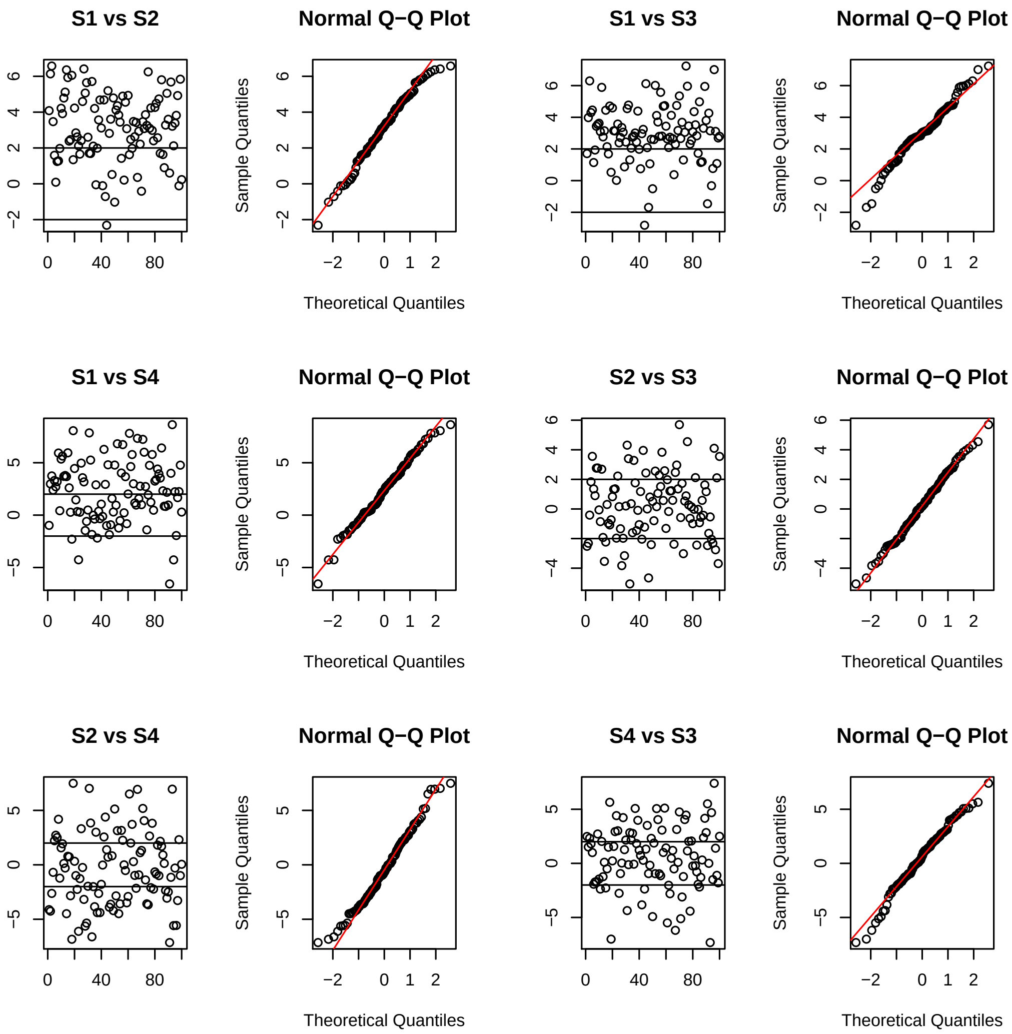


Figure A.8: The normal QQ-plots and statistics of Scenario 3.

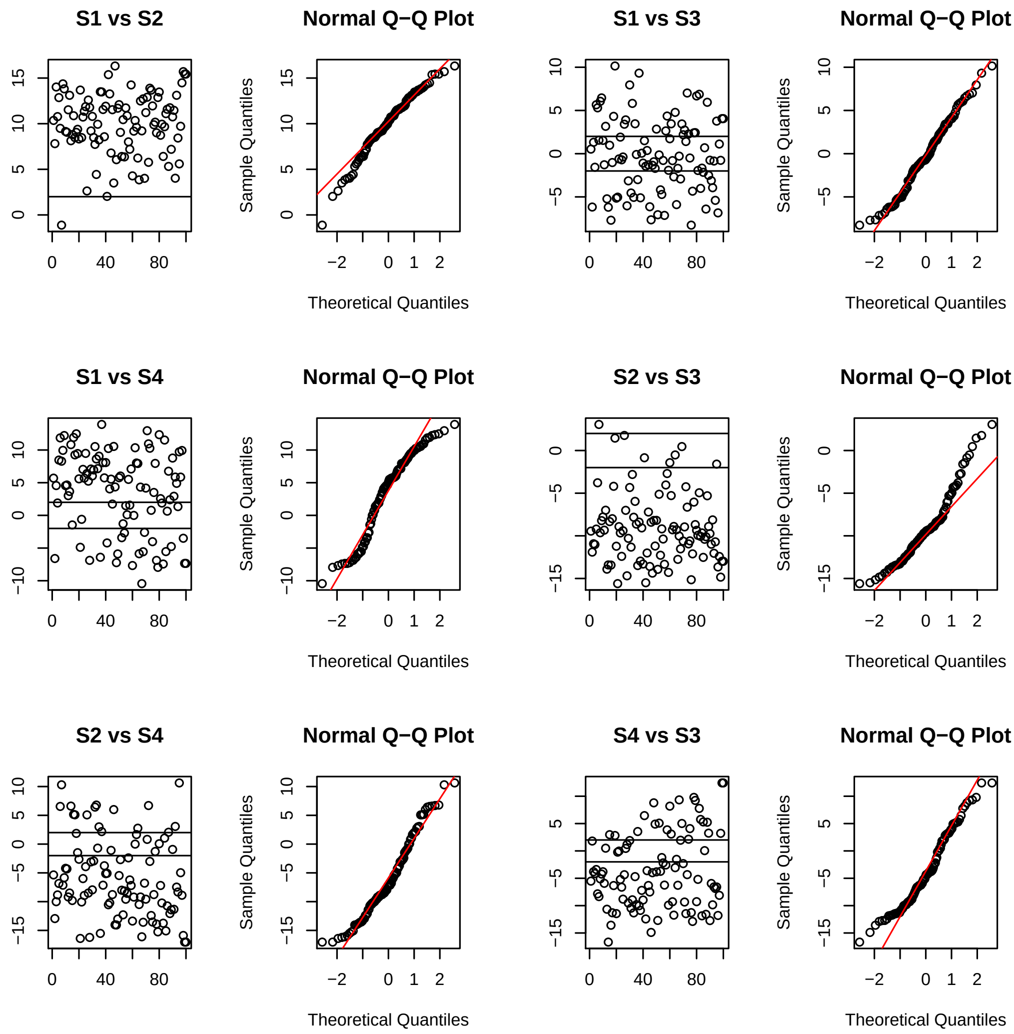


Figure A.9: The normal QQ-plots and statistics of Scenario 4.

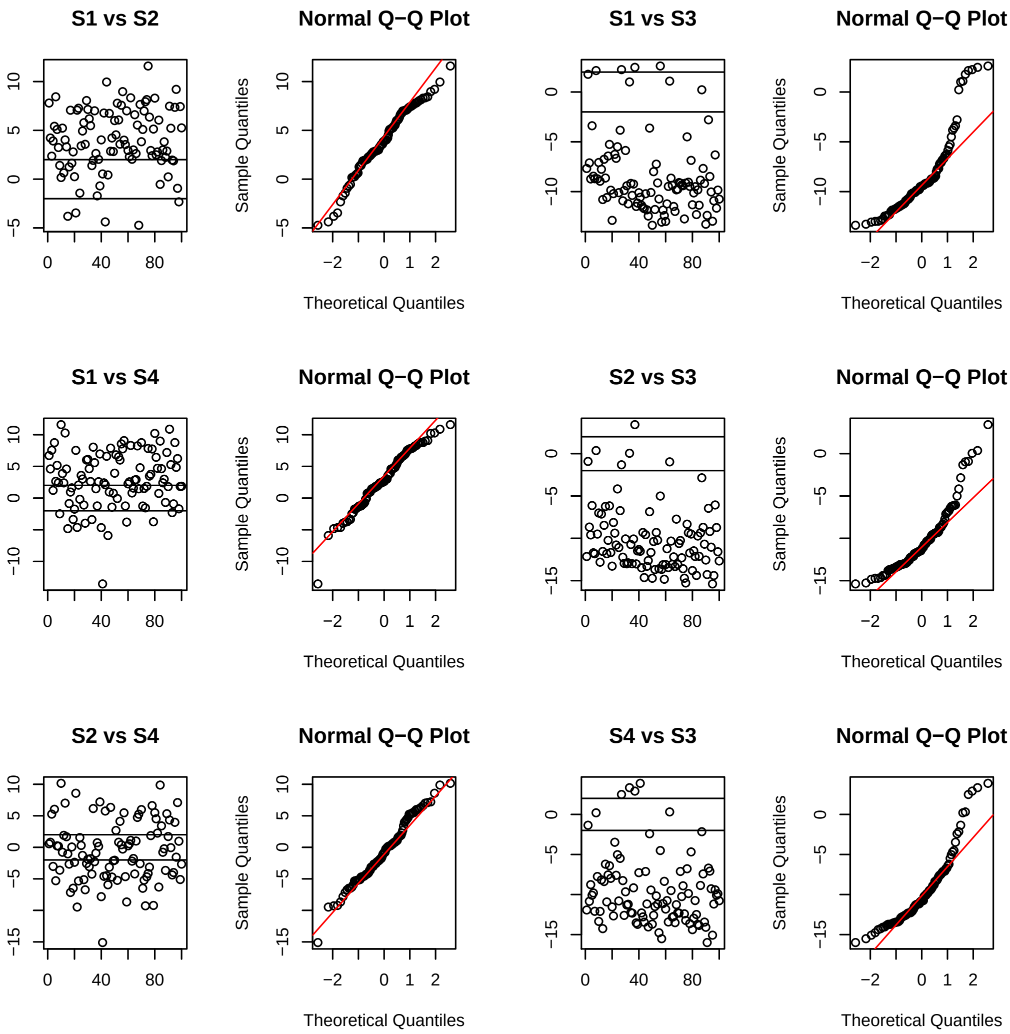


Figure A.10: The normal QQ-plots and statistics of Scenario 5.

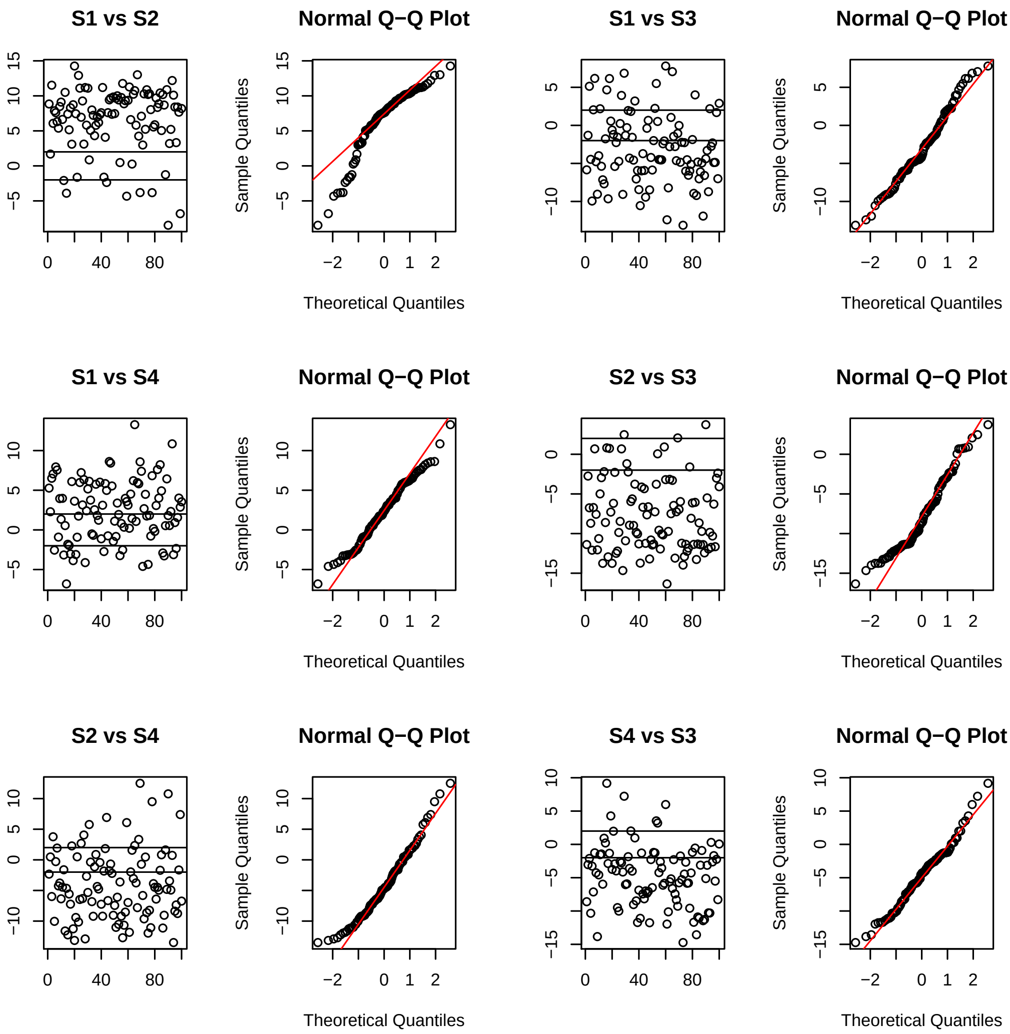


Figure A.11: The normal QQ-plots and statistics of Scenario 6.

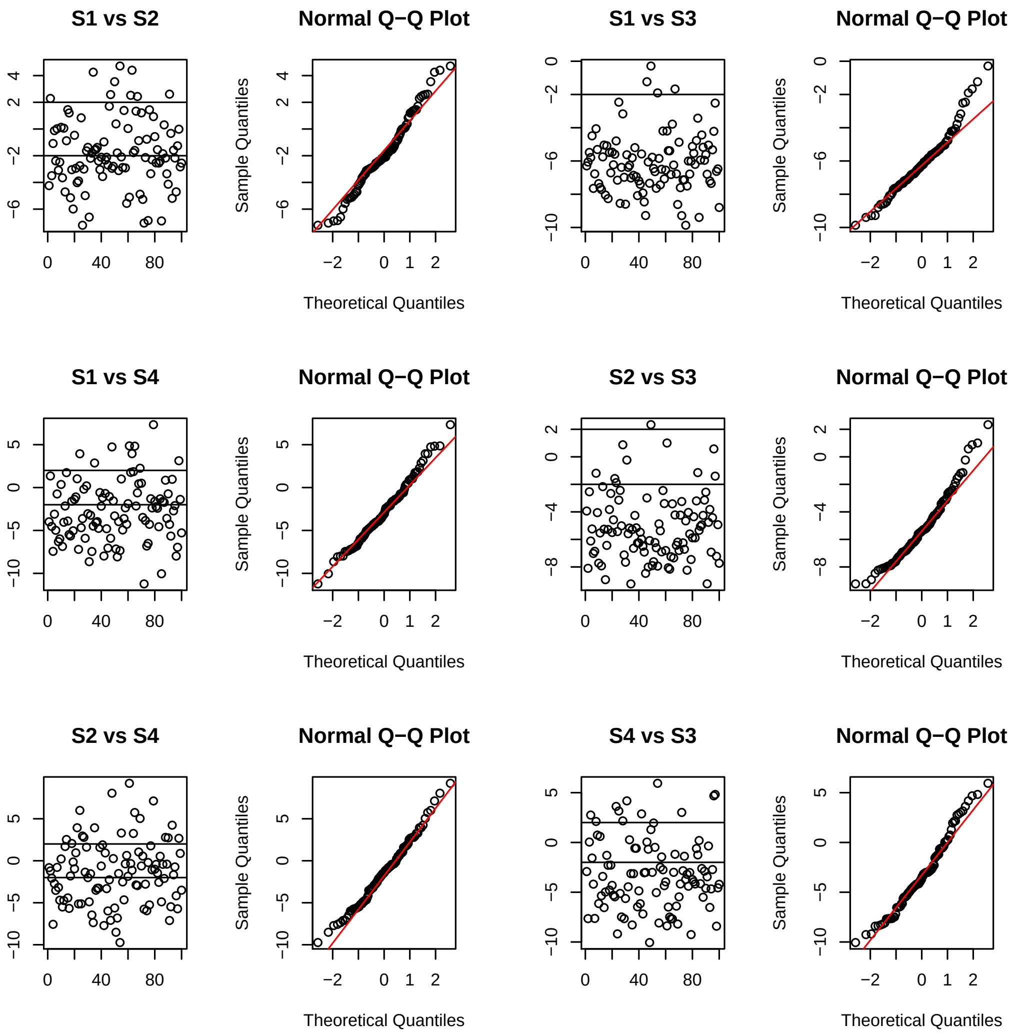


Figure A.12: The normal QQ-plots and statistics of Scenario 7.

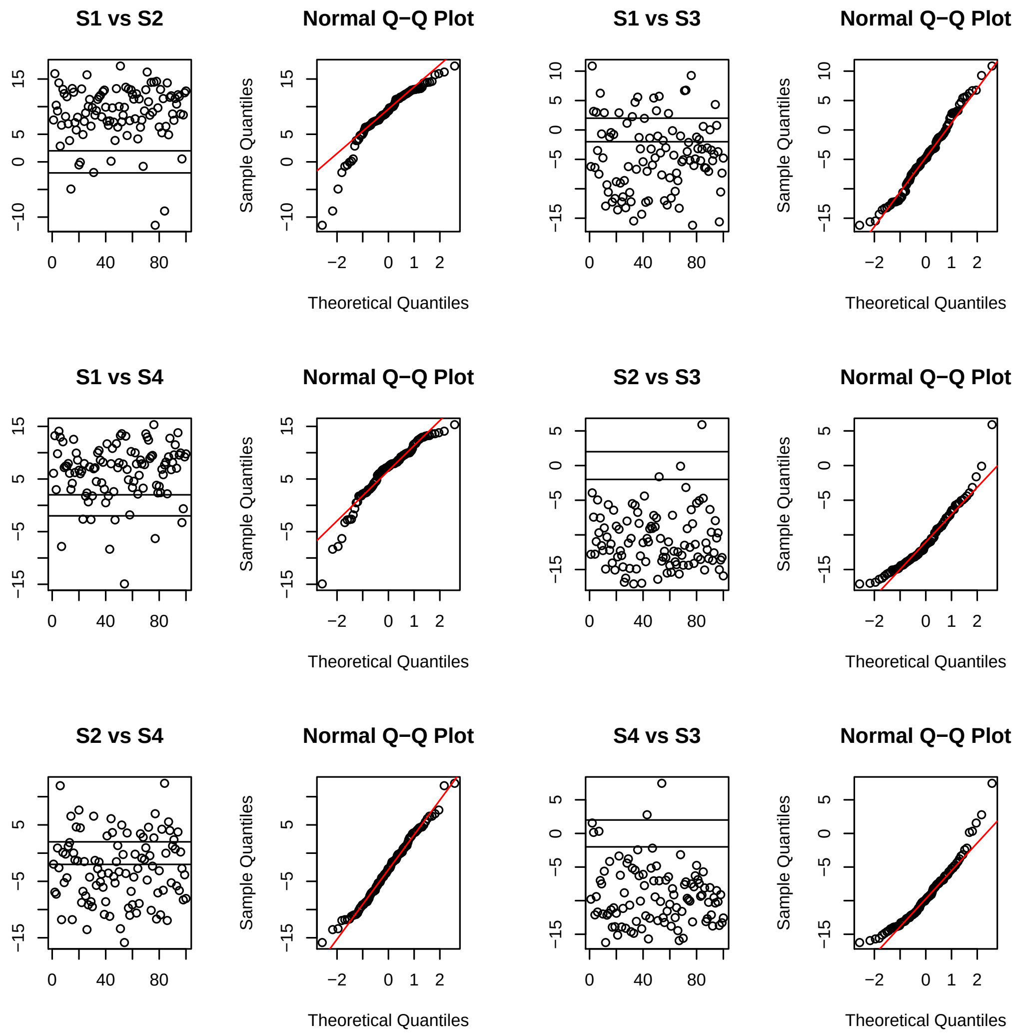


Figure A.13: The normal QQ-plots and statistics of Scenario 8.

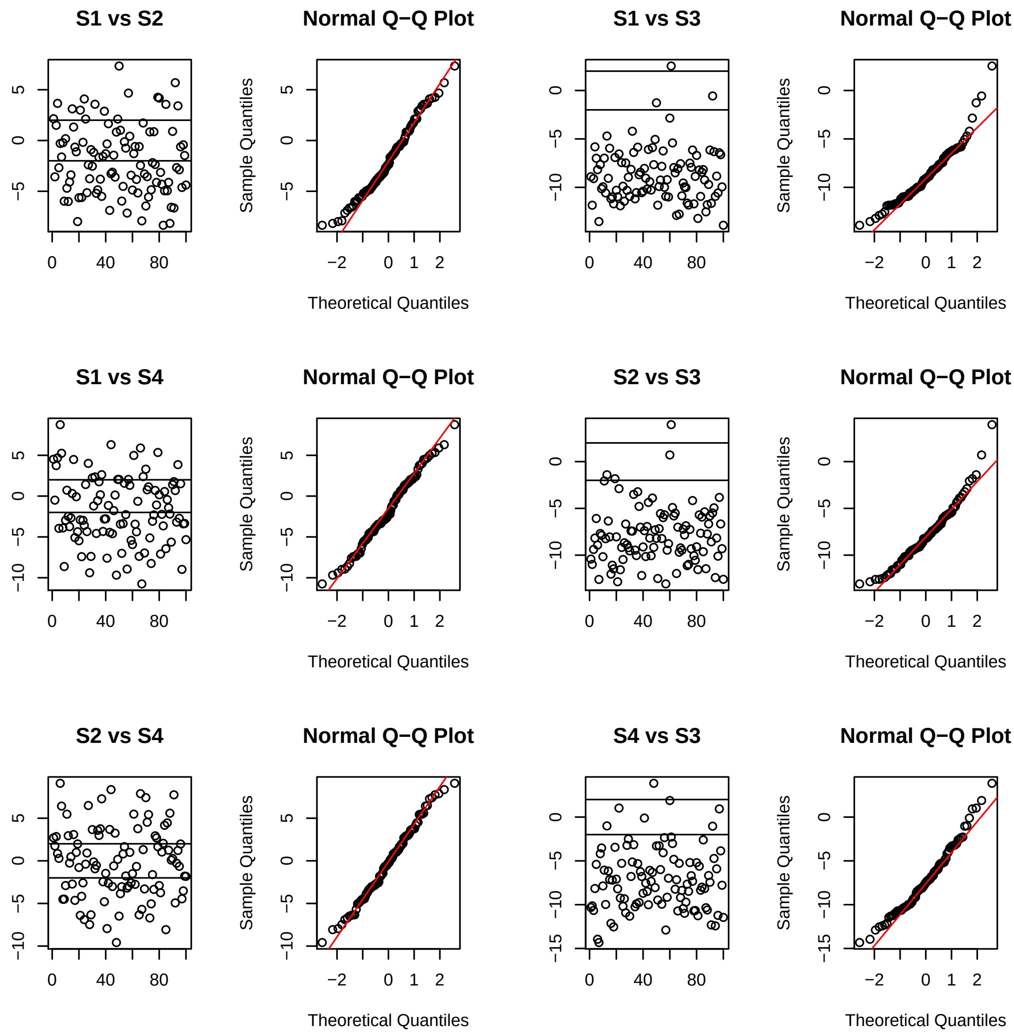


Figure A.14: The normal QQ-plots and statistics of Scenario 9.

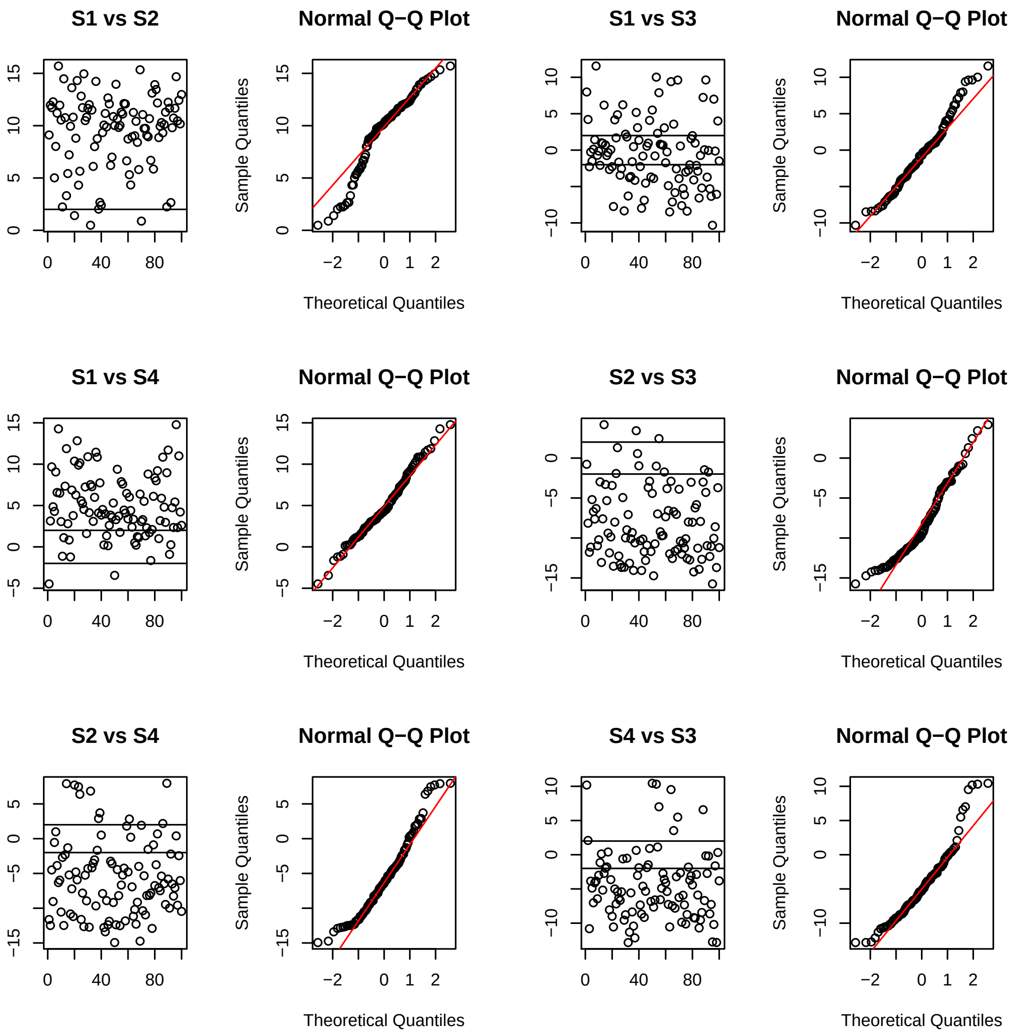


Figure A.15: The normal QQ-plots and statistics of Scenario 10.

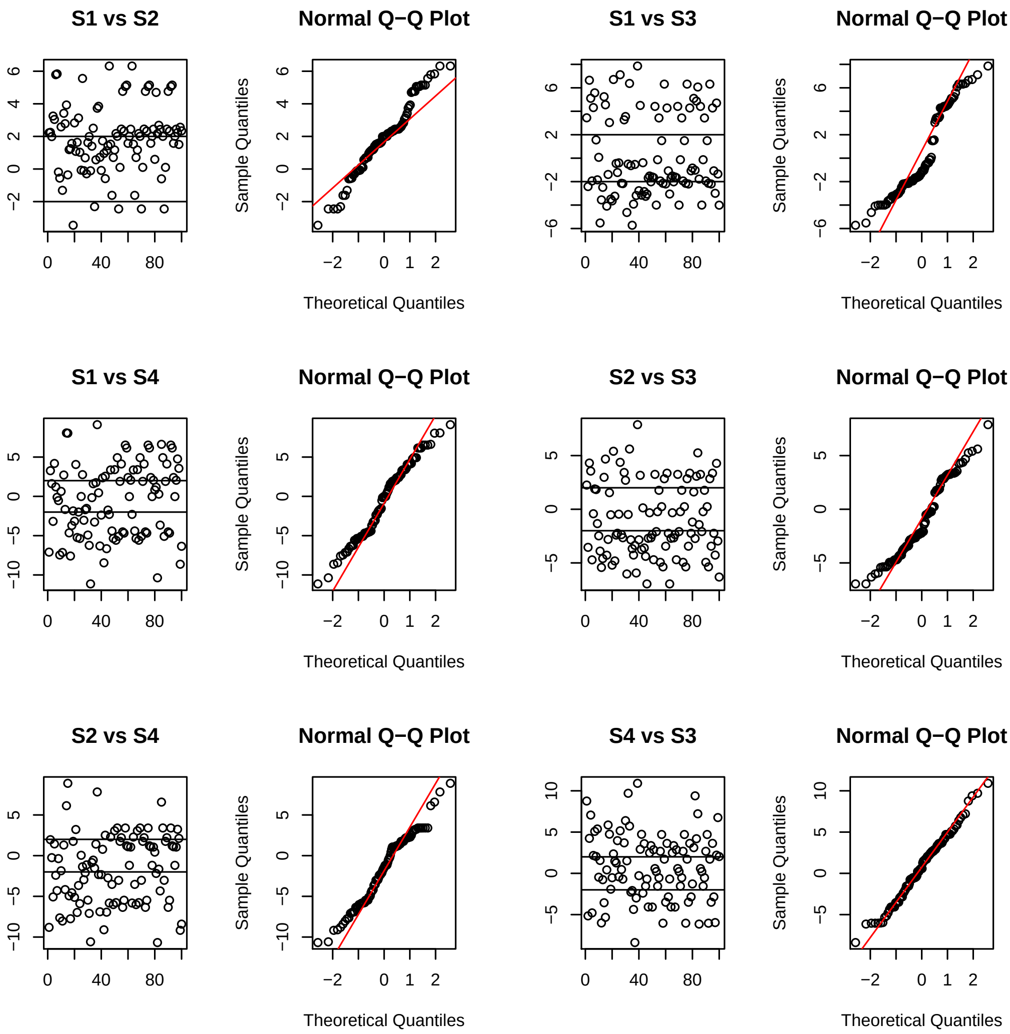


Figure A.16: The normal QQ-plots and statistics of Scenario 11.

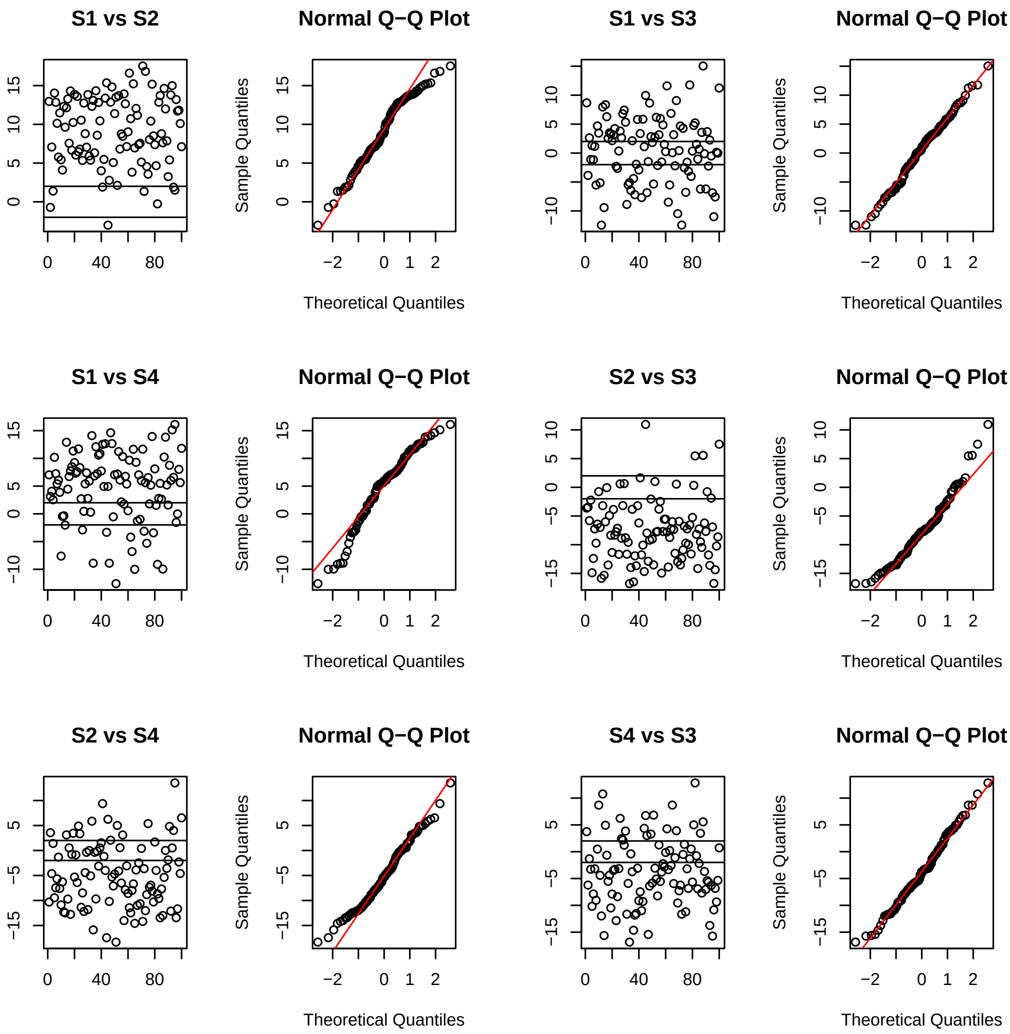


Figure A.17: The normal QQ-plots and statistics of Scenario 12.

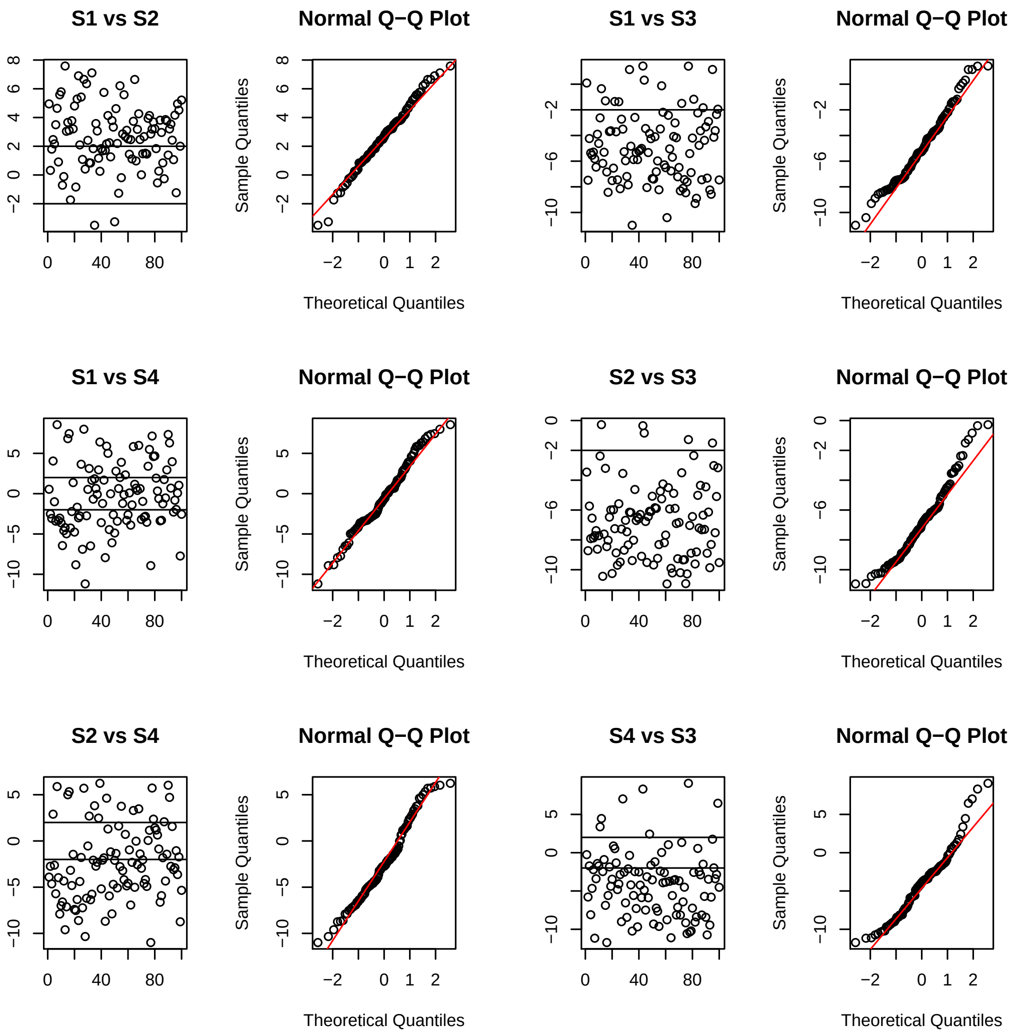


Figure A.18: The normal QQ-plots and statistics of Scenario 13.

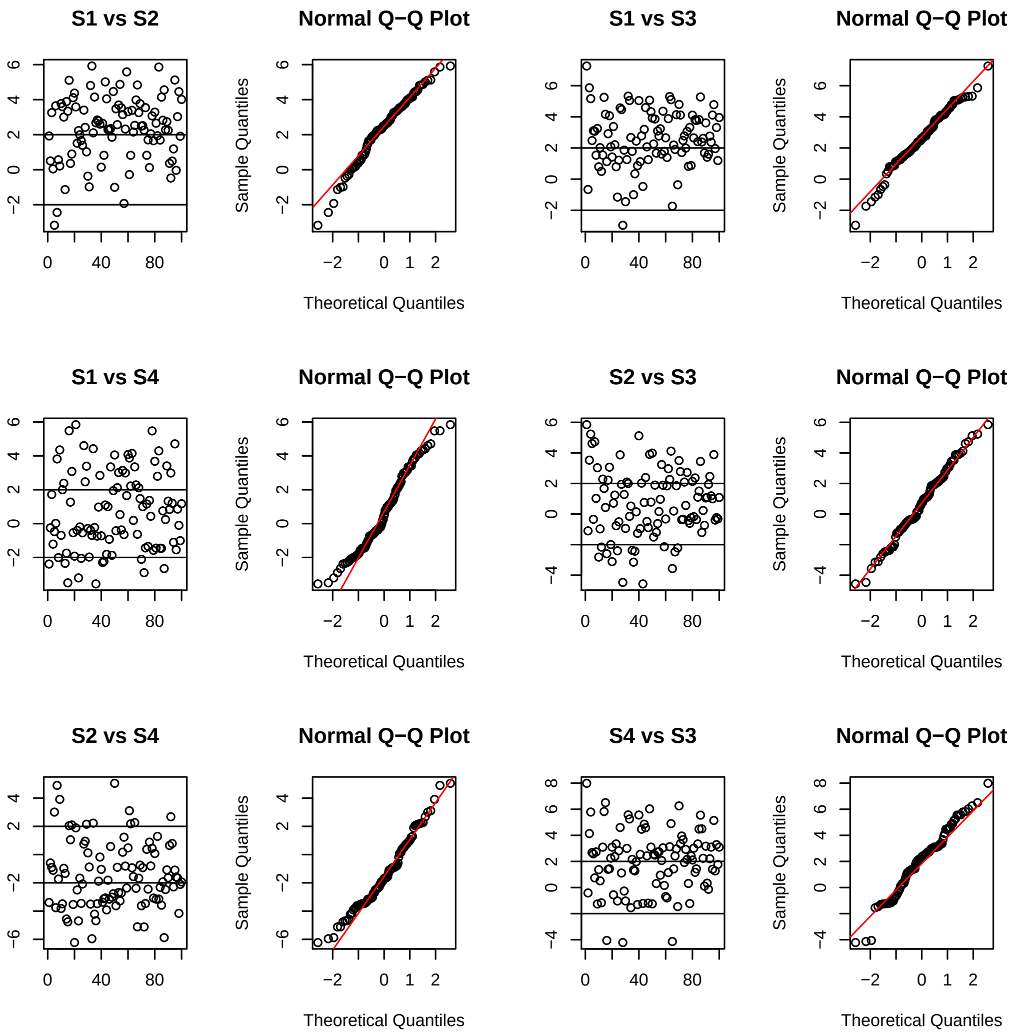


Figure A.19: The normal QQ-plots and statistics of Scenario 14.

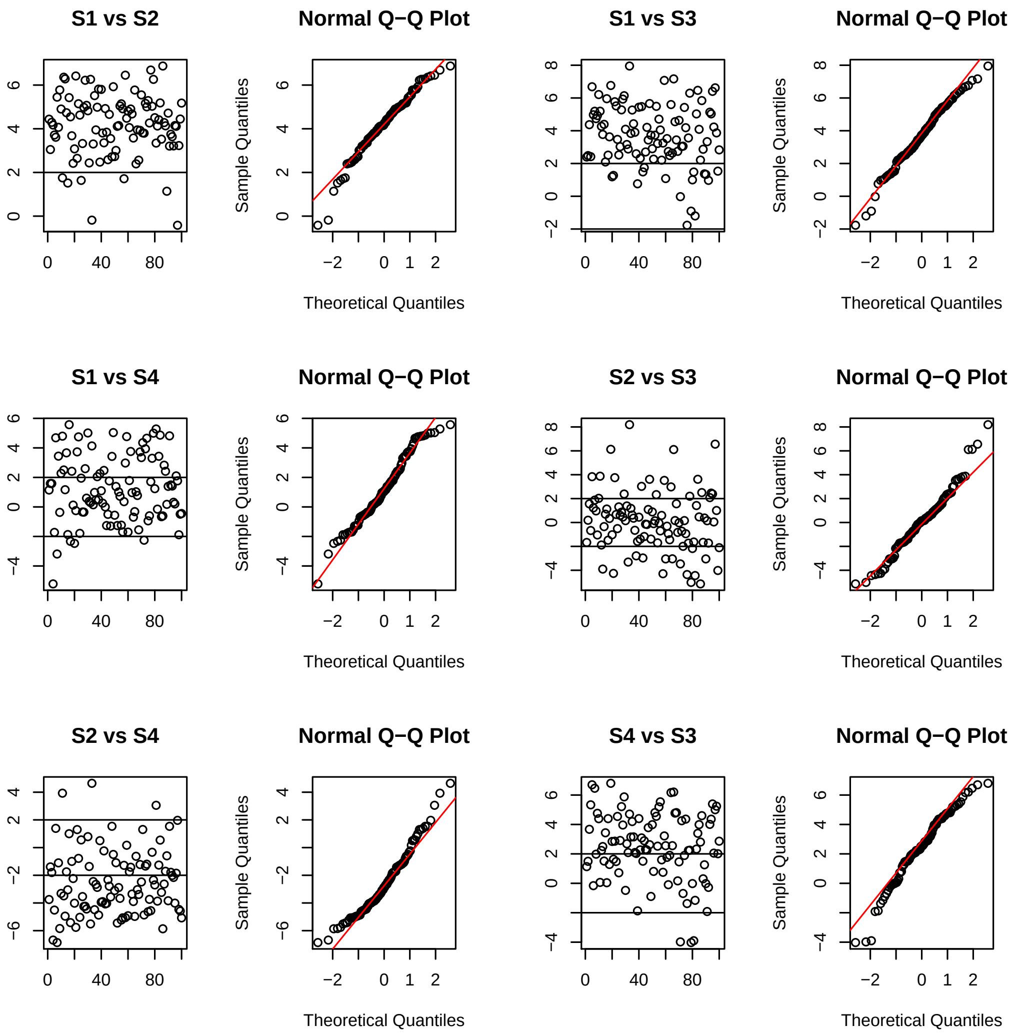


Figure A.20: The normal QQ-plots and statistics of Scenario 15.

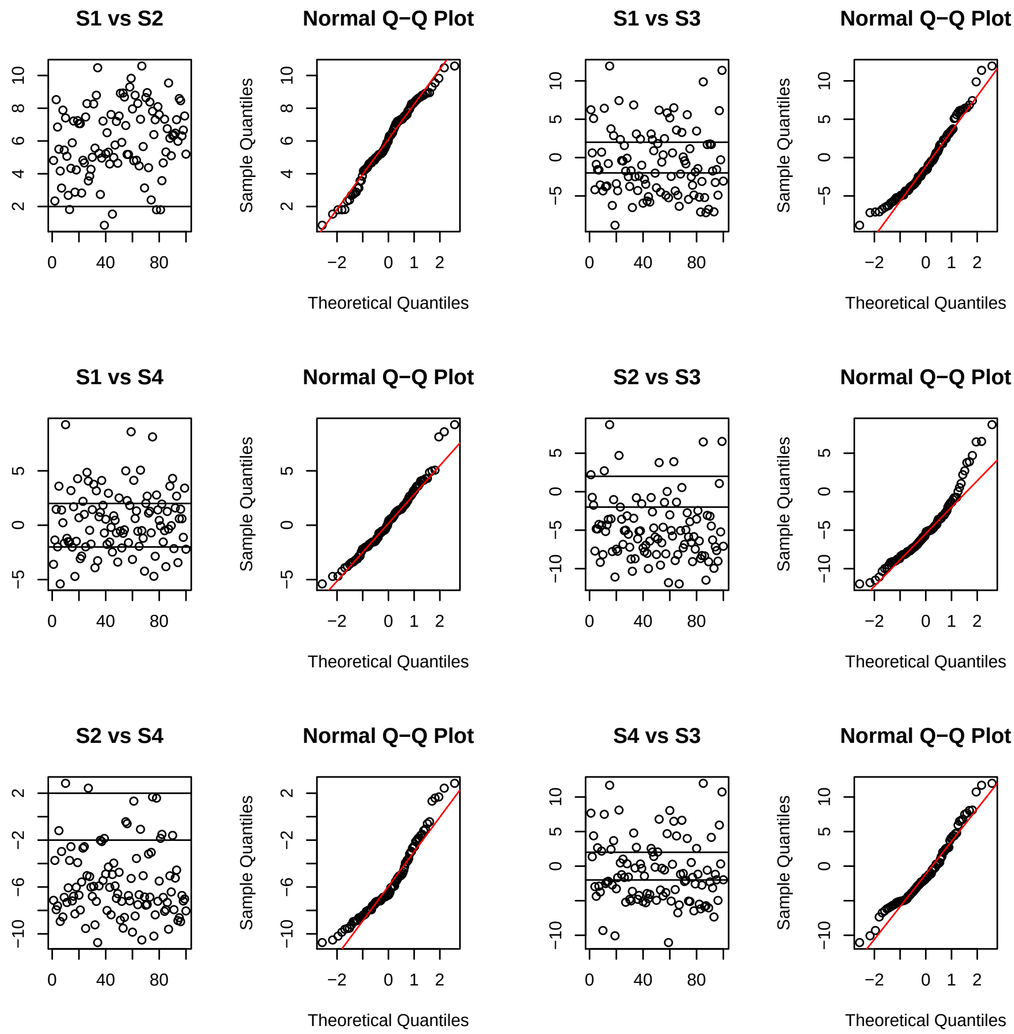


Figure A.21: The normal QQ-plots and statistics of Scenario 16.

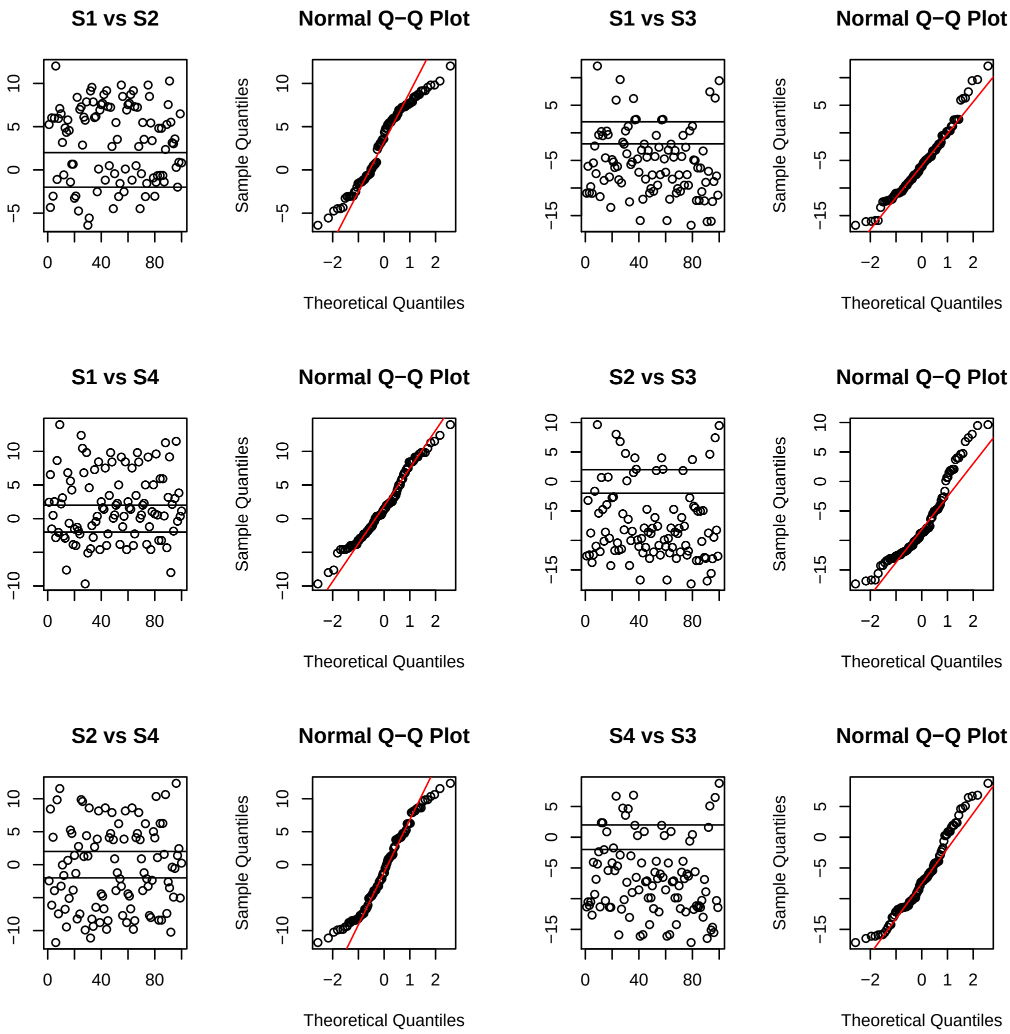


Figure A.22: The normal QQ-plots and statistics of Scenario 17.

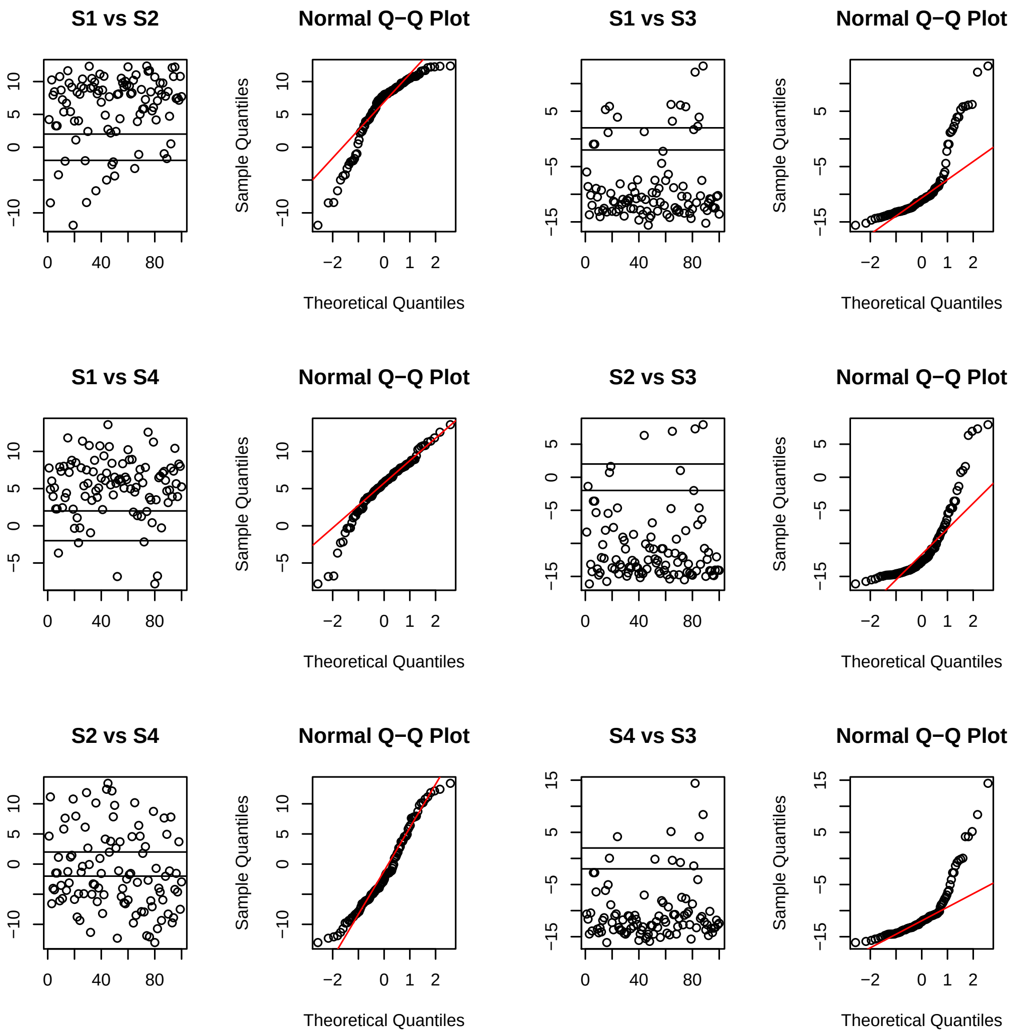


Figure A.23: The normal QQ-plots and statistics of Scenario 18.

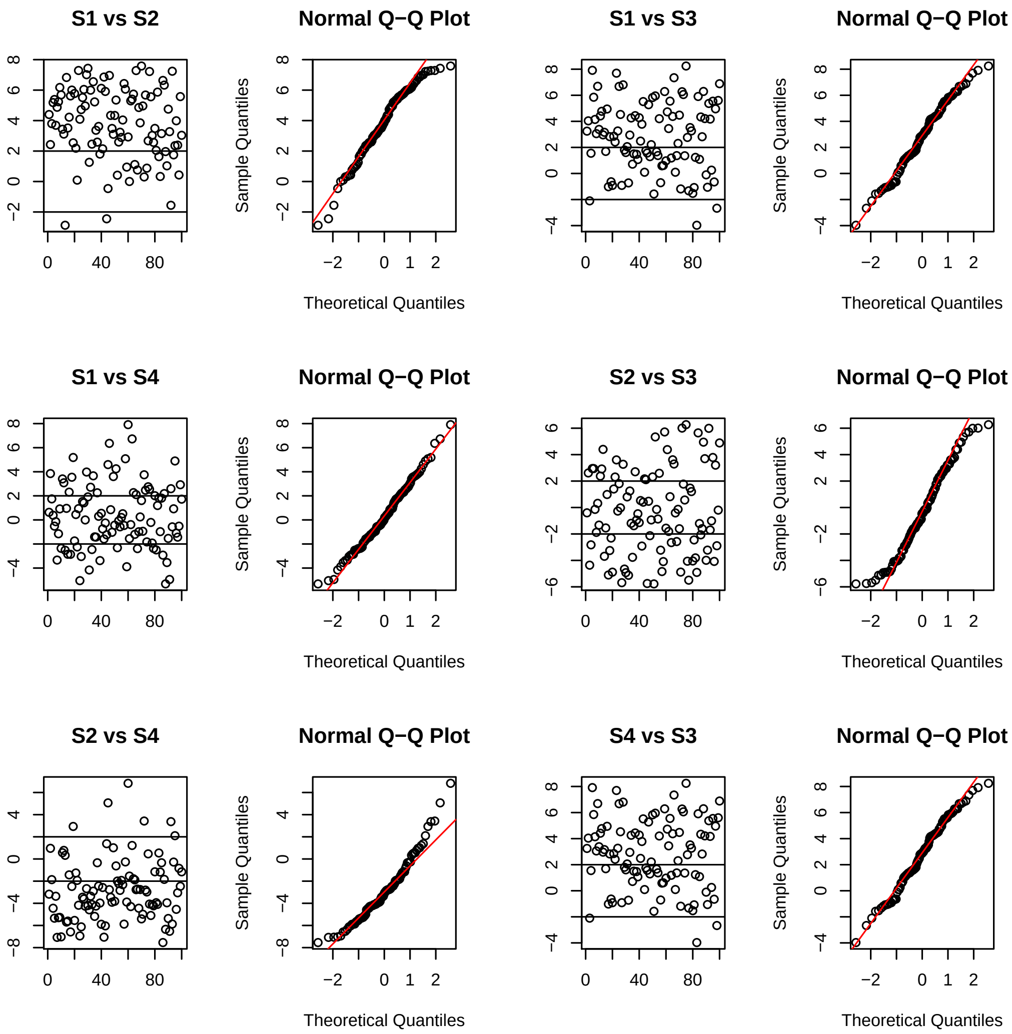


Figure A.24: The normal QQ-plots and statistics of Scenario 19.

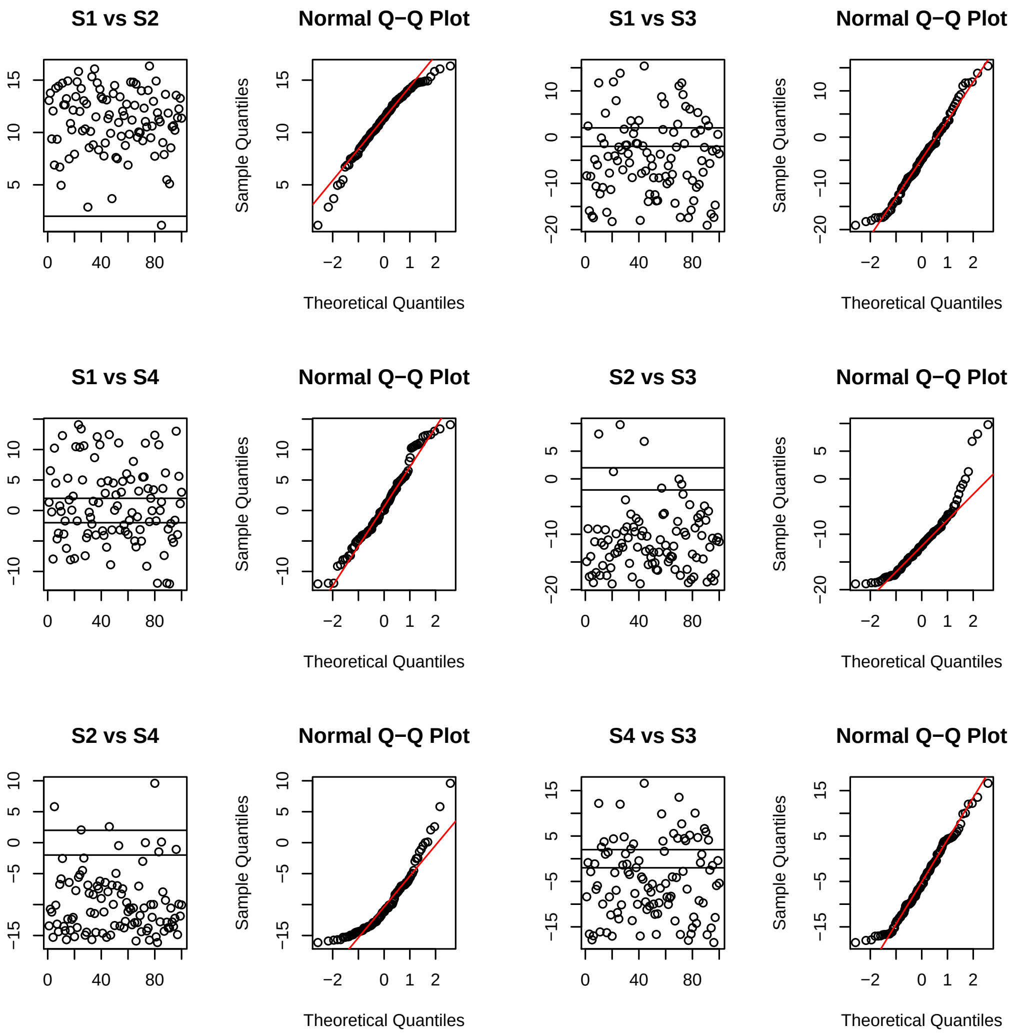


Figure A.25: The normal QQ-plots and statistics of Scenario 20.

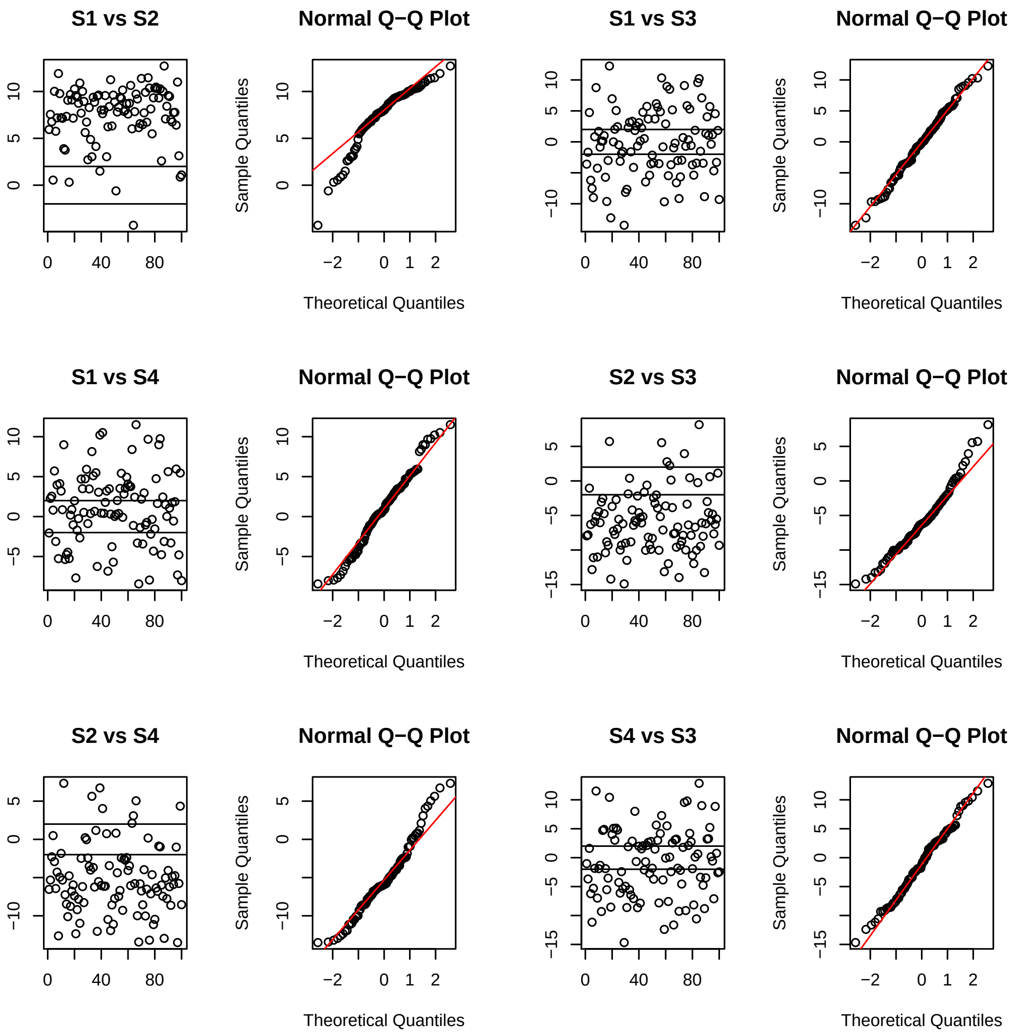


Figure A.26: Ranking for R-Vines corresponding to four strategies applied on Scenarios with respective features:

independent	Kendall	TailCum	HuDep	ExceedDep
Rank 1	5	0	4	2
Rank 2	2	1	0	3
Rank 3	1	0	0	3
Rank 4	0	7	4	0

simplified	Kendall	TailCum	HuDep	ExceedDep
Rank 1	1	0	7	1
Rank 2	5	0	1	2
Rank 3	0	2	0	5
Rank 4	2	6	0	0

dependent	Kendall	TailCum	HuDep	ExceedDep
Rank 1	1	0	2	1
Rank 2	1	0	2	1
Rank 3	2	0	0	2
Rank 4	0	4	0	0

(a) Scenarios with dependence structure in trees $T4 - T7$. Best ranking of the model is marked yellow.

mon.decreasing	Kendall	TailCum	HuDep	ExceedDep
Rank 1	2	0	3	1
Rank 2	3	1	0	1
Rank 3	0	0	0	3
Rank 4	0	4	2	0

mon.increasing	Kendall	TailCum	HuDep	ExceedDep
Rank 1	0	0	5	0
Rank 2	3	0	0	2
Rank 3	1	1	0	3
Rank 4	1	4	0	0

const.weak	Kendall	TailCum	HuDep	ExceedDep
Rank 1	3	0	1	3
Rank 2	0	0	2	2
Rank 3	1	1	0	0
Rank 4	1	4	2	0

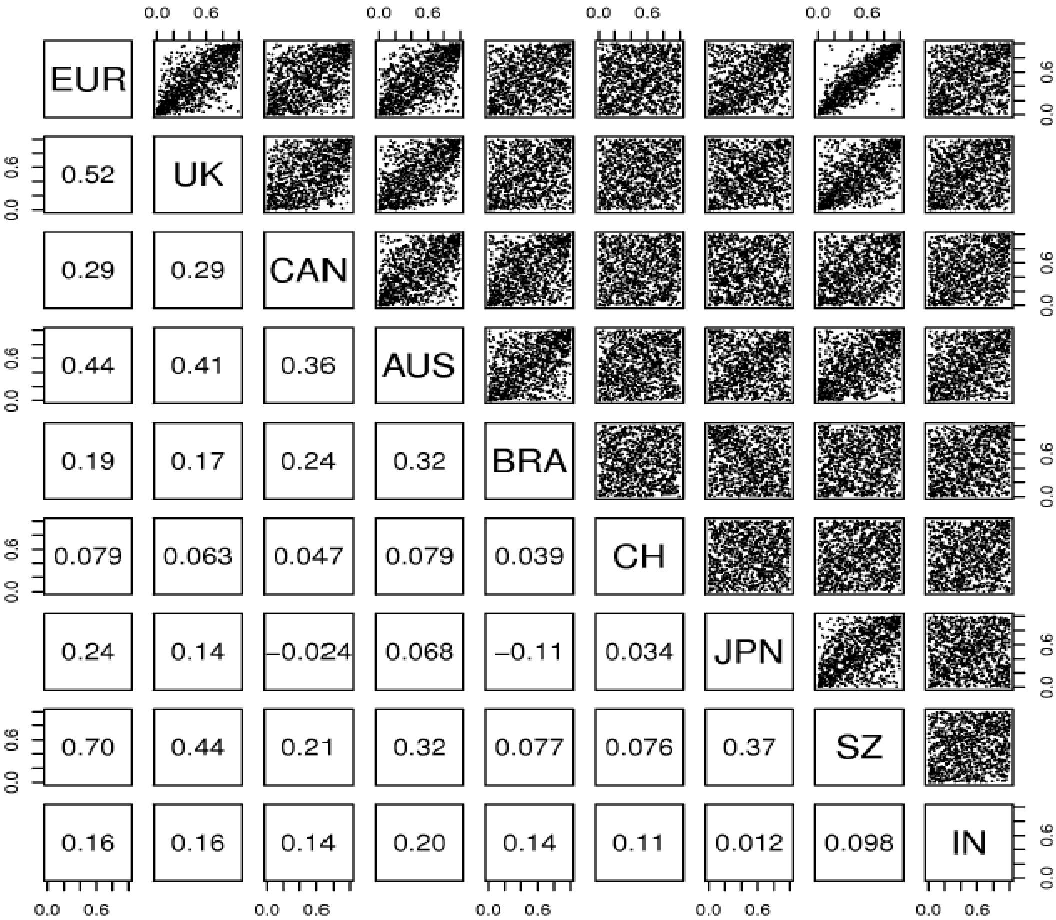
const.strong	Kendall	TailCum	HuDep	ExceedDep
Rank 1	2	0	4	0
Rank 2	2	0	1	1
Rank 3	1	0	0	4
Rank 4	0	5	0	0

(b) Ranking for R-Vines corresponding to four strategies applied on Scenarios with respective correlation strength. Best ranking of the model is marked yellow.

B Additional Information on Applications

B.1 Exchange Rates

Figure B.1: Pairs-plots for the transformed copula data and the empirical values of Kendall′s τ of Dissmann (2010) for the Exchange rates data set.



Matrices corresponding to R-Vine selected using Kendall's τ as weight:

$$
\mathrm{M}=\begin{pmatrix}
7 & & & & & & & & \\
6 & 2 & & & & & & & \\
2 & 6 & 8 & & & & & & \\
9 & 9 & 6 & 1 & & & & & \\
5 & 5 & 9 & 6 & 3 & & & & \\
3 & 8 & 5 & 9 & 6 & 5 & & & \\
4 & 3 & 3 & 5 & 9 & 6 & 4 & & \\
1 & 4 & 4 & 3 & 5 & 9 & 6 & 9 & \\
8 & 1 & 1 & 4 & 4 & 4 & 9 & 6 & 6
\end{pmatrix}
\quad
\mathrm{F}=\begin{pmatrix}
0 & & & & & & & & \\
0 & 0 & & & & & & & \\
0 & 0 & 0 & & & & & & \\
0 & 0 & 1 & 0 & & & & & \\
34 & 0 & 1 & 0 & 0 & & & & \\
1 & 0 & 1 & 0 & 0 & 0 & & & \\
34 & 1 & 1 & 0 & 0 & 0 & 0 & & \\
34 & 2 & 34 & 4 & 2 & 1 & 14 & 0 & \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}
$$

$$
\mathrm{P}=\begin{pmatrix}
0 & & & & & & & & \\
0 & 0 & & & & & & & \\
0 & 0 & 0 & & & & & & \\
0 & 0 & 0.07 & 0 & & & & & \\
-1.06 & 0 & -0.08 & 0 & 0 & & & & \\
-0.15 & 0 & -0.24 & 0 & 0 & 0 & & & \\
-1.06 & 0.07 & -0.12 & 0 & 0 & 0 & 0 & & \\
-1.23 & 0.26 & -1.17 & 1.13 & 0.18 & 0.10 & 1.04 & 0 & \\
0.55 & 0.72 & 0.88 & 0.64 & 0.53 & 0.48 & 0.30 & 0.16 & 0
\end{pmatrix}
$$

$$
\mathrm{P2}=\begin{pmatrix}
0 & & & & & & & & \\
0 & 0 & & & & & & & \\
0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 11.80 & 0 & 0 & 15.38 & 0 & 0 & 0 & \\
7.54 & 8.55 & 4.00 & 10.28 & 8.45 & 6.06 & 14.77 & 14.68 & 0
\end{pmatrix}
$$

$$
\tau=\begin{pmatrix}
0 & & & & & & & & \\
0 & 0 & & & & & & & \\
0 & 0 & 0 & & & & & & \\
0 & 0 & 0.05 & 0 & & & & & \\
-0.06 & 0 & -0.05 & 0 & 0 & 0 & & & \\
-0.09 & 0 & -0.16 & 0 & 0 & 0 & 0 & & \\
-0.06 & 0.04 & -0.08 & 0 & 0 & 0 & 0 & & \\
-0.19 & 0.17 & -0.14 & 0.11 & 0.12 & 0.07 & 0.04 & 0 & \\
0.37 & 0.51 & 0.69 & 0.44 & 0.36 & 0.32 & 0.19 & 0.10 & 0
\end{pmatrix}
$$

Matrices corresponding to R-Vine selected using Tail cumulation as weight:

$$M=\begin{pmatrix}7\\6&8\\9&6&2\\5&9&6&1\\3&5&9&6&3\\4&3&5&9&6&5\\2&4&3&5&9&6&4\\1&2&4&3&5&9&6&9\\8&1&1&4&4&4&9&6&6\end{pmatrix}\quad F=\begin{pmatrix}0\\0&0\\0&1&0\\34&1&0&0\\1&1&1&0&0\\34&1&0&0&0&0\\0&34&1&0&0&0&0\\34&0&2&4&2&1&14&0\\2&2&2&2&2&2&2&2&0\end{pmatrix}$$

$$P=\begin{pmatrix}0\\0&0\\0&0.08&0\\-1.06&-0.08&0&0\\-0.14&-0.24&0&0&0\\-1.06&-0.13&0&0&0&0\\0&-1.16&0.07&0&0&0&0\\-1.23&0&0.26&1.13&0.18&0.10&1.05&0\\0.55&0.88&0.72&0.64&0.53&0.48&0.30&0.16&0\end{pmatrix}$$

$$P2=\begin{pmatrix}0\\0&0\\0&0&0\\0&0&0&0\\0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&11.80&0&15.38&0&0&0\\7.55&4.02&8.55&10.28&8.45&6.06&14.78&14.68\end{pmatrix}$$

$$\tau=\begin{pmatrix}0\\0&0\\0&0.05&0\\-0.06&-0.05&0&0\\-0.09&-0.15&0&0&0\\-0.06&-0.08&0&0&0&0\\0&-0.14&0.04&0&0&0&0\\-0.19&0&0.17&0.11&0.12&0.07&0.04&0\\0.37&0.69&0.51&0.44&0.36&0.32&0.19&0.10&0\end{pmatrix}$$

Matrices corresponding to R-Vine selected using Hu dependence as weight:

$$\text{M=}\begin{pmatrix}5 &&&&&&&& \\ 3 & 3 &&&&&&& \\ 1 & 4 & 1 &&&&&& \\ 8 & 9 & 4 & 8 &&&&& \\ 2 & 8 & 9 & 4 & 2 &&&& \\ 6 & 2 & 8 & 9 & 4 & 4 &&& \\ 9 & 7 & 2 & 7 & 9 & 6 & 6 && \\ 7 & 1 & 7 & 6 & 7 & 9 & 9 & 9 & \\ 4 & 6 & 6 & 2 & 6 & 7 & 7 & 7 & 7\end{pmatrix} \quad \text{F=}\begin{pmatrix}0 &&&&&&&& \\ 4 & 0 &&&&&&& \\ 1 & 2 & 0 &&&&&& \\ 0 & 4 & 2 & 0 &&&&& \\ 0 & 24 & 2 & 2 & 0 &&&& \\ 0 & 1 & 2 & 0 & 2 & 0 &&& \\ 1 & 2 & 2 & 2 & 2 & 14 & 0 && \\ 1 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & \\ 2 & 14 & 14 & 2 & 14 & 2 & 0 & 0 & 0\end{pmatrix}$$

$$\text{P=}\begin{pmatrix}0 &&&&&&&& \\ 1.09 & 0 &&&&&&& \\ 0.21 & 0.28 & 0 &&&&&& \\ 0 & 1.05 & 0.35 & 0 &&&&& \\ 0 & -1.04 & 0.19 & 0.22 & 0 &&&& \\ 0 & 0.14 & 0.79 & 0 & 0.53 & 0 &&& \\ 0.11 & -0.23 & 0.70 & 0.52 & 0.23 & 1.03 & 0 && \\ -0.23 & 0.45 & 0.36 & 073 & 0.20 & 0.28 & 0.16 & 0 & \\ 0.48 & 1.05 & 1.09 & 0.62 & 1.06 & 0.11 & 0 & 0 & 0\end{pmatrix}$$

$$\text{P2=}\begin{pmatrix}0 &&&&&&&& \\ 0 & 0 &&&&&&& \\ 0 & 16.57 & 0 &&&&&& \\ 0 & 0 & 15.10 & 0 &&&&& \\ 0 & 0 & 17.07 & 26.43 & 0 &&&& \\ 0 & 0 & 8.44 & 0 & 18.76 & 0 &&& \\ 0 & 13.10 & 16.16 & 13.58 & 20.19 & 0 & 0 && \\ 0 & 9.66 & 4.83 & 0 & 4.03 & 14.41 & 14.68 & 0 & \\ 6.06 & 0 & 0 & 4.53 & 0 & 3.38 & 0 & 0 & 0\end{pmatrix}$$

$$\tau=\begin{pmatrix}0 &&&&&&&& \\ 0.08 & 0 &&&&&&& \\ 0.13 & 0.18 & 0 &&&&&& \\ 0 & 0.05 & 0.23 & 0 &&&&& \\ 0 & -0.04 & 0.12 & 0.14 & 0 &&&& \\ 0 & 0.09 & 0.58 & 0 & 0.36 & 0 &&& \\ 0.07 & -0.15 & 0.49 & 0.35 & 0.15 & 0.03 & 0 && \\ -0.15 & 0.30 & 0.23 & 0.05 & 0.13 & 0.18 & 0.10 & 0 & \\ 0.32 & 0.05 & 0.08 & 0.43 & 0.06 & 0.07 & 0 & 0 & 0\end{pmatrix}$$

Matrices corresponding to R-Vine selected using Exceedance dependence as weight:

$$M=\begin{pmatrix} 8 \\ 6 & 2 \\ 5 & 6 & 6 \\ 3 & 5 & 1 & 5 \\ 2 & 3 & 5 & 1 & 3 \\ 4 & 4 & 7 & 7 & 1 & 1 \\ 9 & 9 & 3 & 9 & 7 & 4 & 4 \\ 7 & 7 & 4 & 4 & 9 & 9 & 7 & 9 \\ 1 & 1 & 9 & 3 & 7 & 7 & 9 & 7 & 7 \end{pmatrix} \quad F=\begin{pmatrix} 0 \\ 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 34 & 1 & 0 & 1 & 2 & 0 \\ 1 & 14 & 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 14 & 2 & 4 & 1 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 \end{pmatrix}$$

$$P=\begin{pmatrix} 0 \\ 0 & 0 \\ -0.16 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.64 & 0 \\ -1.09 & 0.24 & 0 & -0.22 & 0.23 & 0 \\ -0.13 & 1.06 & 0 & 0.09 & -0.10 & 0.61 & 0 \\ 0.51 & 0 & 1.05 & 0.34 & 1.05 & 0.25 & 0.09 & 0 \\ 0.88 & 0.72 & 0.16 & 0.38 & 0.53 & 0.36 & 0.30 & 0 & 0 \end{pmatrix}$$

$$P2=\begin{pmatrix} 0 \\ 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27.74 & 0 \\ 0 & 0 & 0 & 0 & 18.34 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15.26 & 0 \\ 10.02 & 0 & 0 & 6.47 & 0 & 0 & 4.15 & 0 \\ 4.00 & 8.55 & 14.68 & 5.81 & 8.45 & 4.38 & 14.78 & 0 & 0 \end{pmatrix}$$

$$\tau=\begin{pmatrix} 0 \\ 0 & 0 \\ -0.10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.04 & 0 \\ -0.08 & 0.16 & 0 & -0.14 & 0.15 & 0 \\ -0.08 & 0.05 & 0 & 0.06 & -0.06 & 0.41 & 0 \\ 0.34 & 0 & 0.04 & 0.22 & 0.05 & 0.16 & 0.06 & 0 \\ 0.69 & 0.51 & 0.10 & 0.25 & 0.36 & 0.24 & 0.19 & 0 & 0 \end{pmatrix}$$

Matrices corresponding to R-Vine selected using adjusted Hu dependence as weight:

$$M^{adj}=\begin{pmatrix}2 &&&&&&&& \\ 8 & 8 &&&&&&& \\ 5 & 3 & 5 &&&&&& \\ 4 & 6 & 3 & 3 &&&&& \\ 7 & 1 & 6 & 4 & 6 &&&& \\ 9 & 5 & 1 & 7 & 4 & 4 &&& \\ 1 & 4 & 9 & 9 & 7 & 1 & 1 && \\ 6 & 7 & 7 & 1 & 9 & 9 & 7 & 9 & \\ 3 & 9 & 4 & 6 & 1 & 7 & 9 & 7 & 7\end{pmatrix} \quad F^{adj}=\begin{pmatrix}0 &&&&&&&& \\ 14 & 0 &&&&&&& \\ 1 & 34 & 0 &&&&&& \\ 1 & 1 & 4 & 0 &&&&& \\ 2 & 2 & 34 & 2 & 0 &&&& \\ 14 & 14 & 2 & 2 & 4 & 0 &&& \\ 2 & 2 & 1 & 4 & 1 & 2 & 0 && \\ 14 & 2 & 1 & 2 & 4 & 2 & 2 & 0 & \\ 1 & 1 & 2 & 14 & 14 & 2 & 1 & 2 & 0\end{pmatrix}$$

$$\text{P}=\begin{pmatrix}0 &&&&&&&& \\ 1.02 & 0 &&&&&&& \\ -0.06 & -1.01 & 0 &&&&&& \\ 0.20 & 0.05 & 1.09 & 0 &&&&& \\ -0.39 & 0.84 & -1.01 & 0.31 & 0 &&&& \\ 1.04 & 1.07 & 0.10 & -0.22 & 1.02 & 0 &&& \\ 0.66 & 0.48 & 0.11 & 1.07 & 0.03 & 0.61 & 0 && \\ 1.05 & 0.56 & -0.23 & 0.45 & 1.09 & 0.28 & 0.37 & 0 & \\ 0.43 & 0.15 & 0.48 & 1.05 & 1.09 & 0.11 & 0.25 & 0.01 & 0\end{pmatrix}$$

$$\text{P2}=\begin{pmatrix}0 &&&&&&&& \\ 0 & 0 &&&&&&& \\ 0 & 0 & 0 &&&&&& \\ 0 & 0 & 0 & 0 &&&&& \\ 13.17 & 7.60 & 0 & 14.49 & 0 &&&& \\ 0 & 0 & 27.79 & 14.69 & 0 & 0 &&& \\ 10.05 & 20.23 & 0 & 0 & 0 & 17.07 & 0 && \\ 0 & 7.80 & 0 & 9.66 & 0 & 13.83 & 5.42 & 0 & \\ 0 & 0 & 6.06 & 0 & 0 & 3.38 & 0 & 9.15 & 0\end{pmatrix}$$

$$\tau=\begin{pmatrix}0 &&&&&&&& \\ 0 & 0 &&&&&&& \\ -0.04 & -0.01 & 0 &&&&&& \\ 0.13 & 0.04 & 0.08 & 0 &&&&& \\ -0.02 & 0.64 & -0.01 & 0.20 & 0 &&&& \\ 0.04 & 0.01 & 0.06 & -0.14 & 0.02 & 0 &&& \\ 0.45 & 0.32 & 0.07 & 0.07 & 0.02 & 0.42 & 0 && \\ 0.05 & 0.37 & -0.15 & 0.30 & 0.08 & 0.18 & 0.24 & 0 & \\ 0.28 & 0.10 & 0.32 & 0.05 & 0.08 & 0.07 & 0.16 & 0.01 & 0\end{pmatrix}$$

Figure B.2: $T_1 - T_4$ corresponding to R-Vine selected using Kendall's τ as weight.

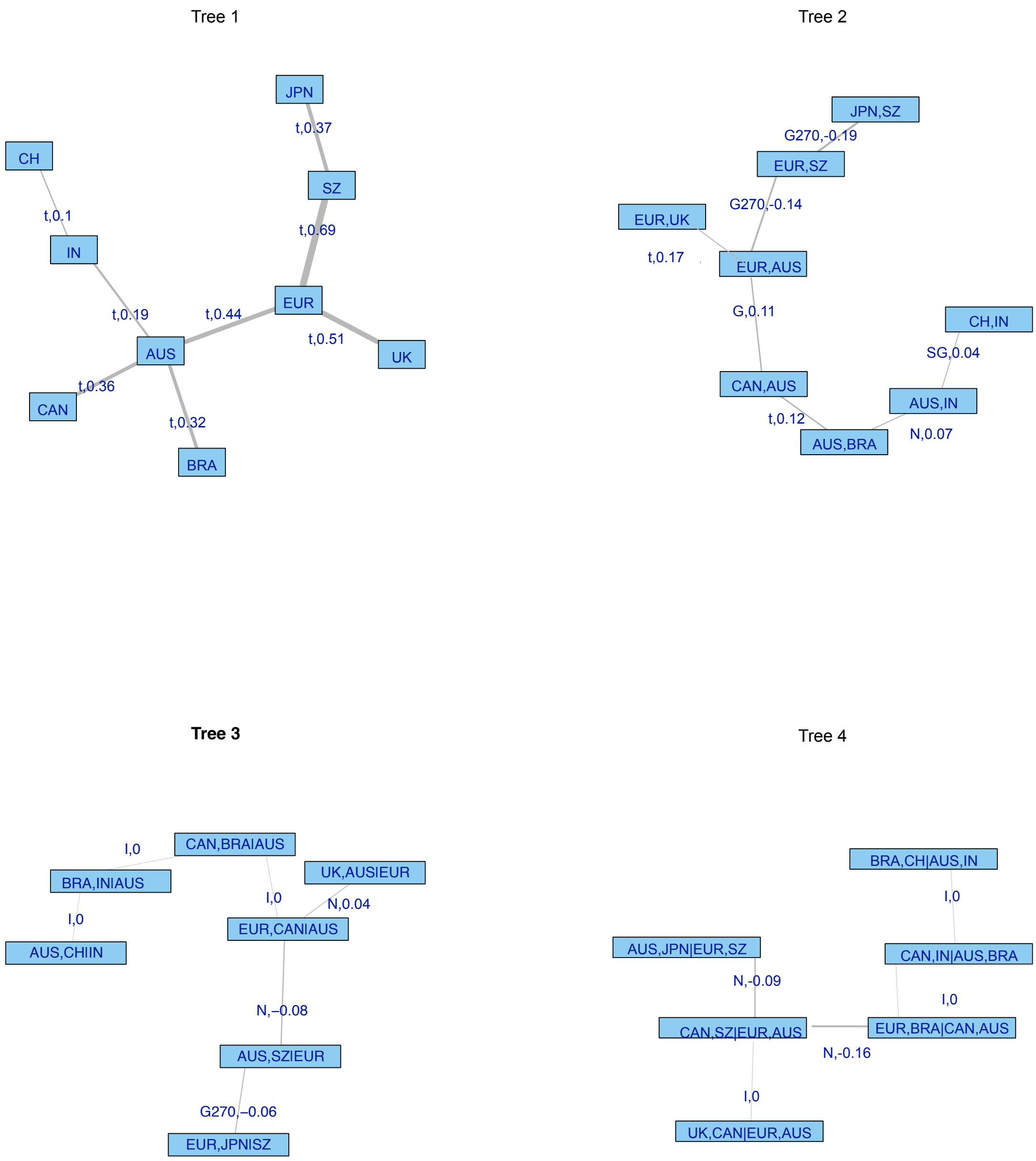


Figure B.3: $T_5 - T_9$ corresponding to R-Vine selected using Kendall′s τ as weight.

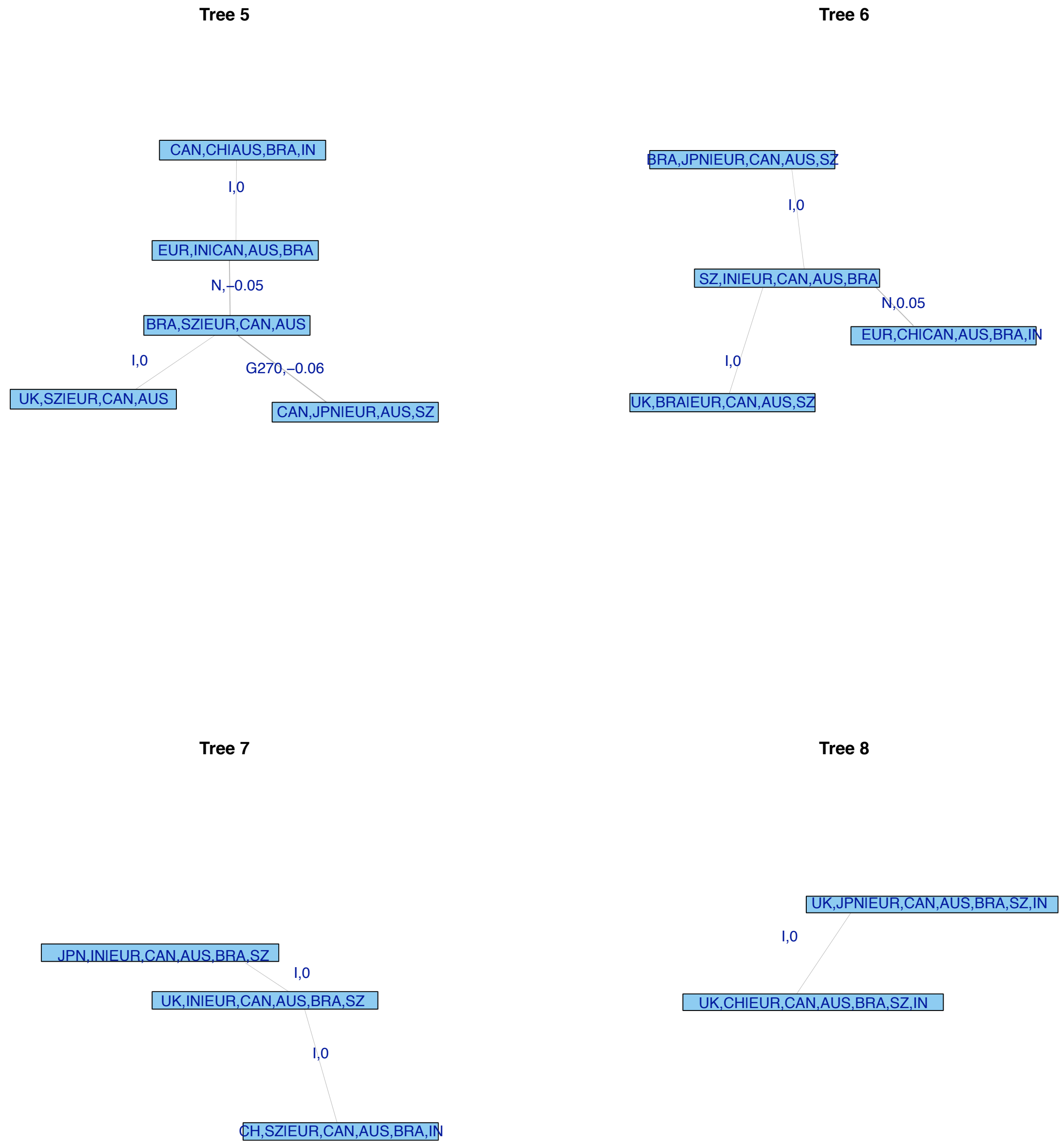


Figure B.4: $T_1 - T_4$ corresponding to R-Vine selected using Kendall's τ as weight.

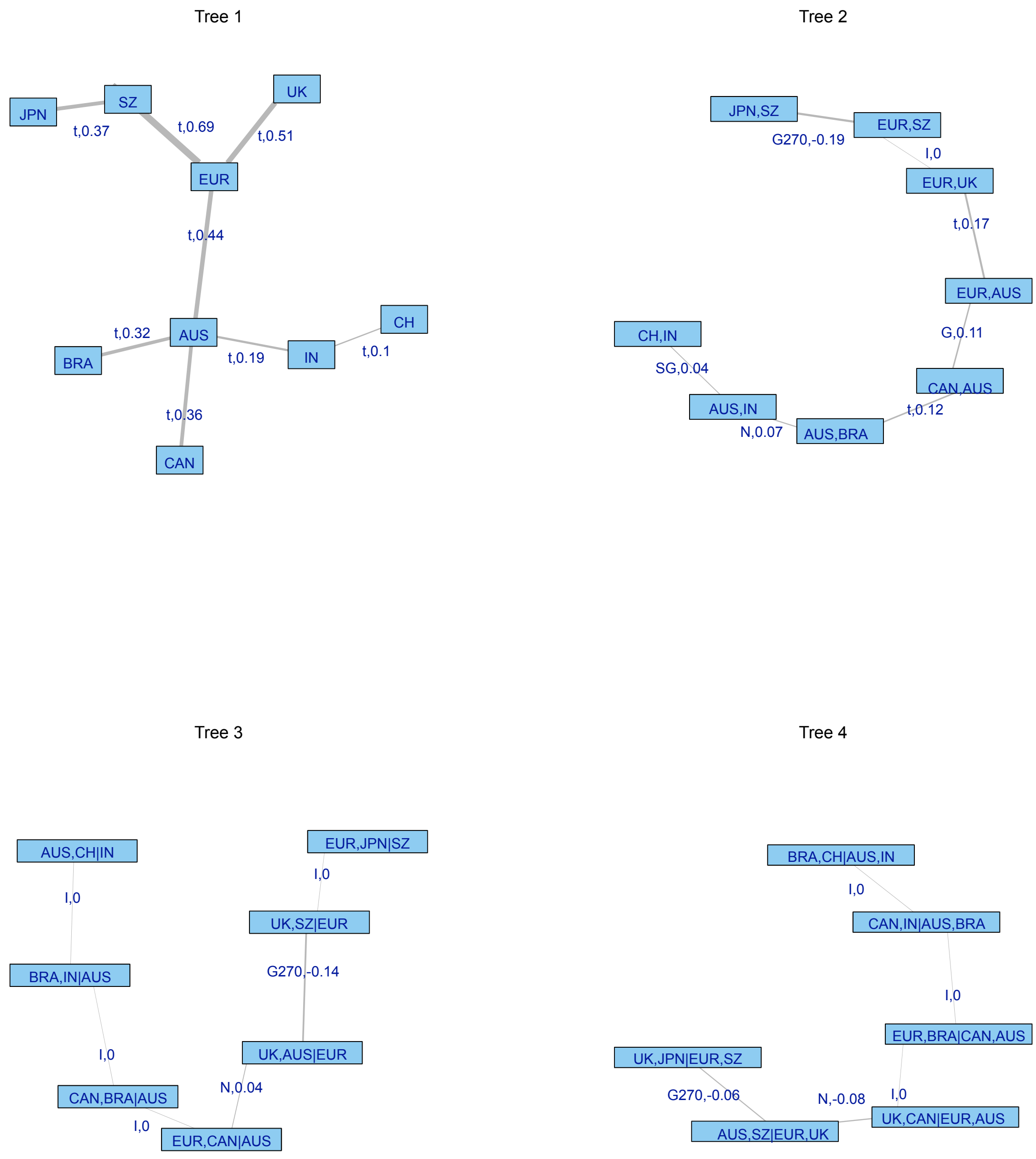


Figure B.5: $T_5 - T_9$ corresponding to R-Vine selected using Tail cumulation as weight.

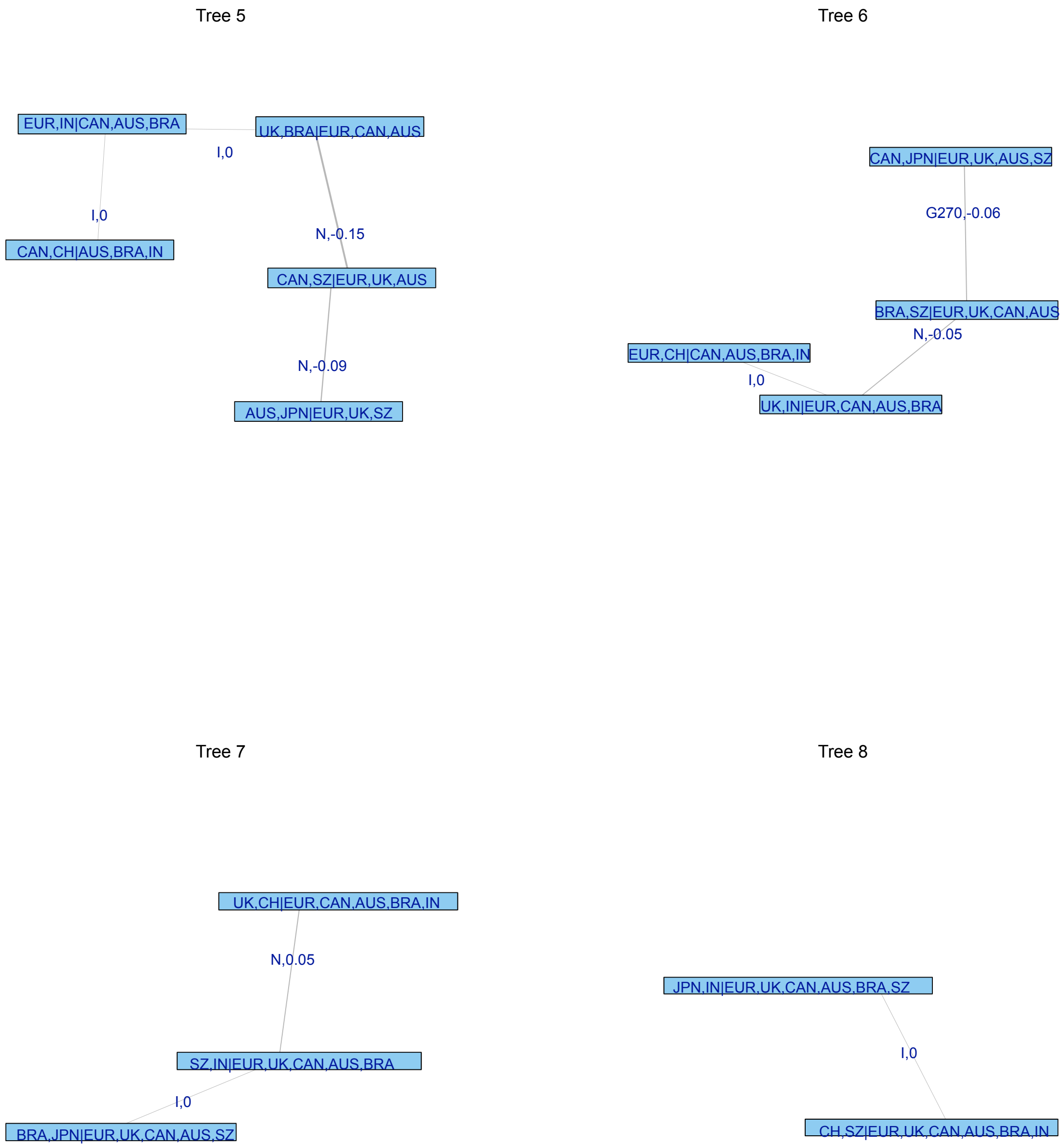


Figure B.6: $T_1 - T_4$ corresponding to R-Vine selected using Hu dependence as weight.

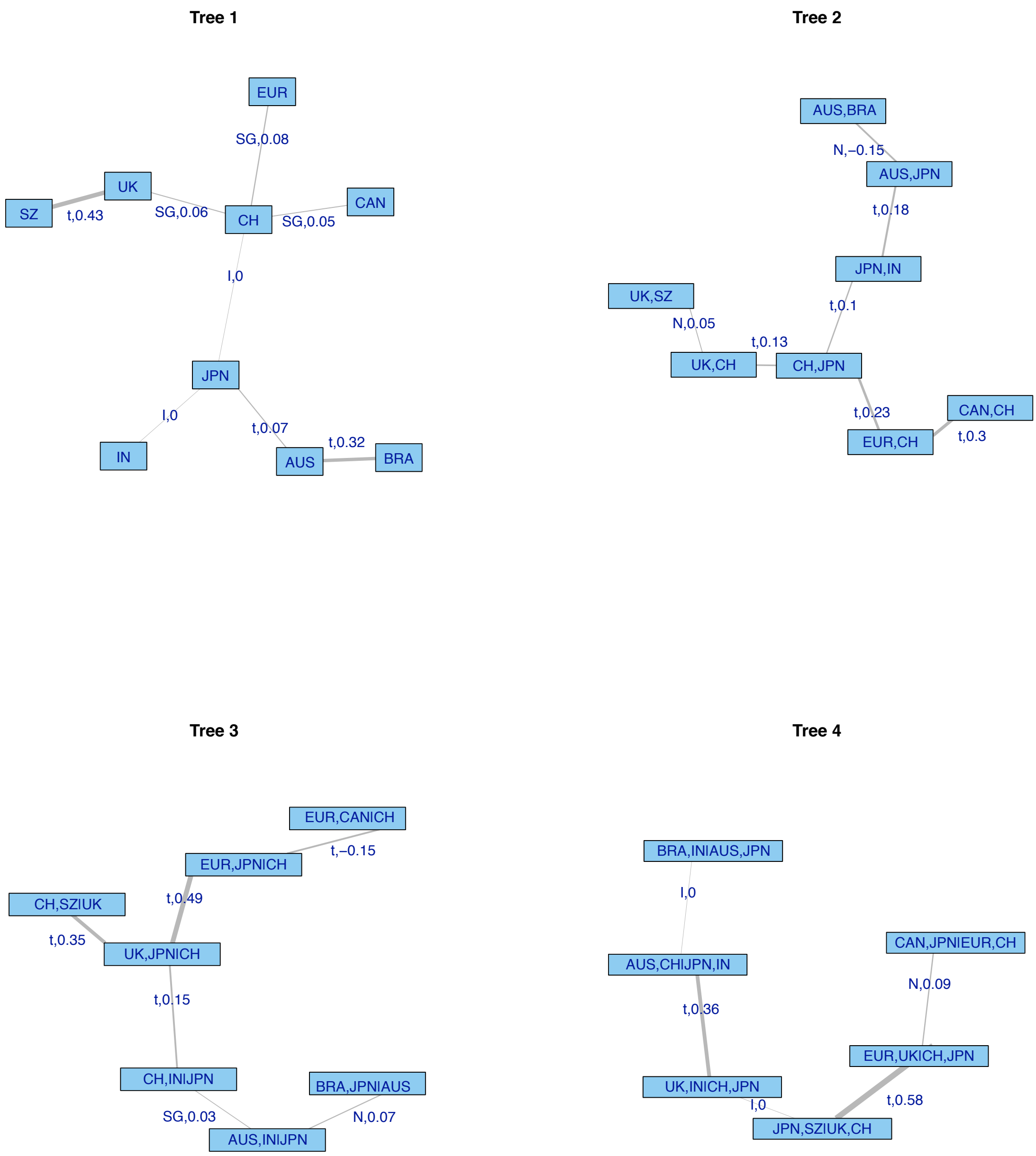


Figure B.7: $T_5 - T_9$ corresponding to R-Vine selected using Hu dependence as weight.

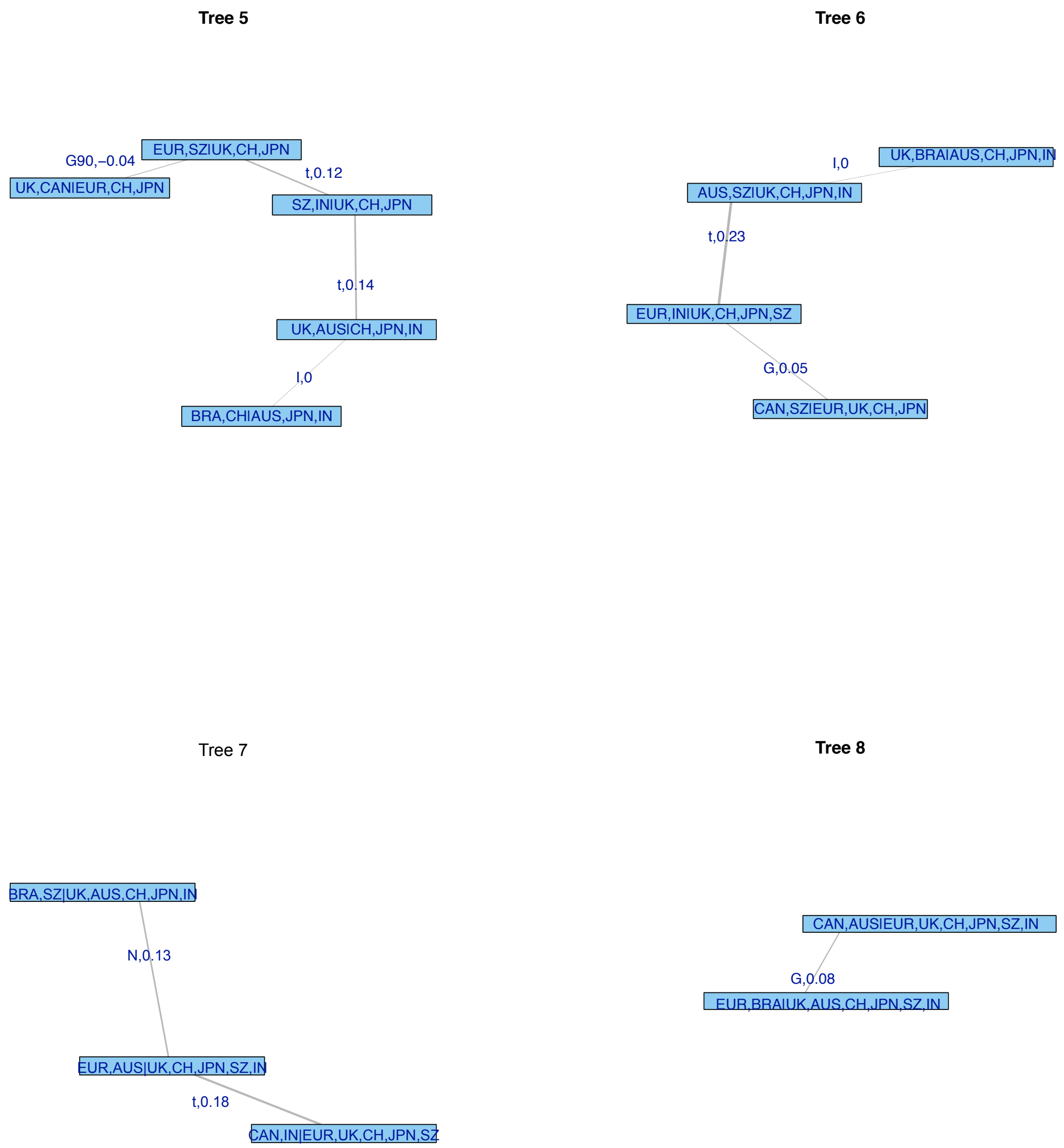


Figure B.8: $T_1 - T_4$ corresponding to R-Vine selected using Exceedance dependence as weight.

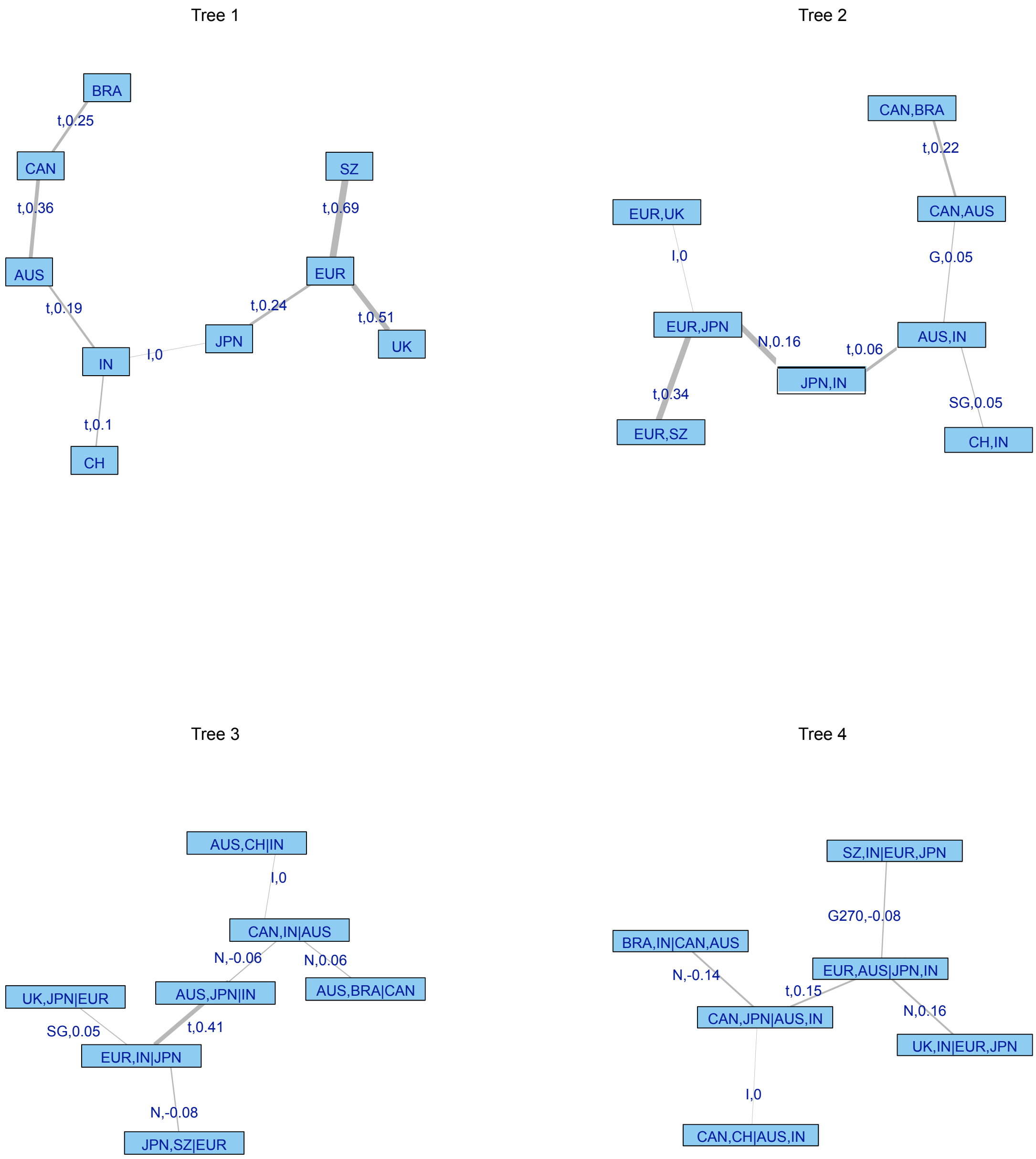


Figure B.9: $T_5 - T_9$ corresponding to R-Vine selected using Exceedance dependence as weight.

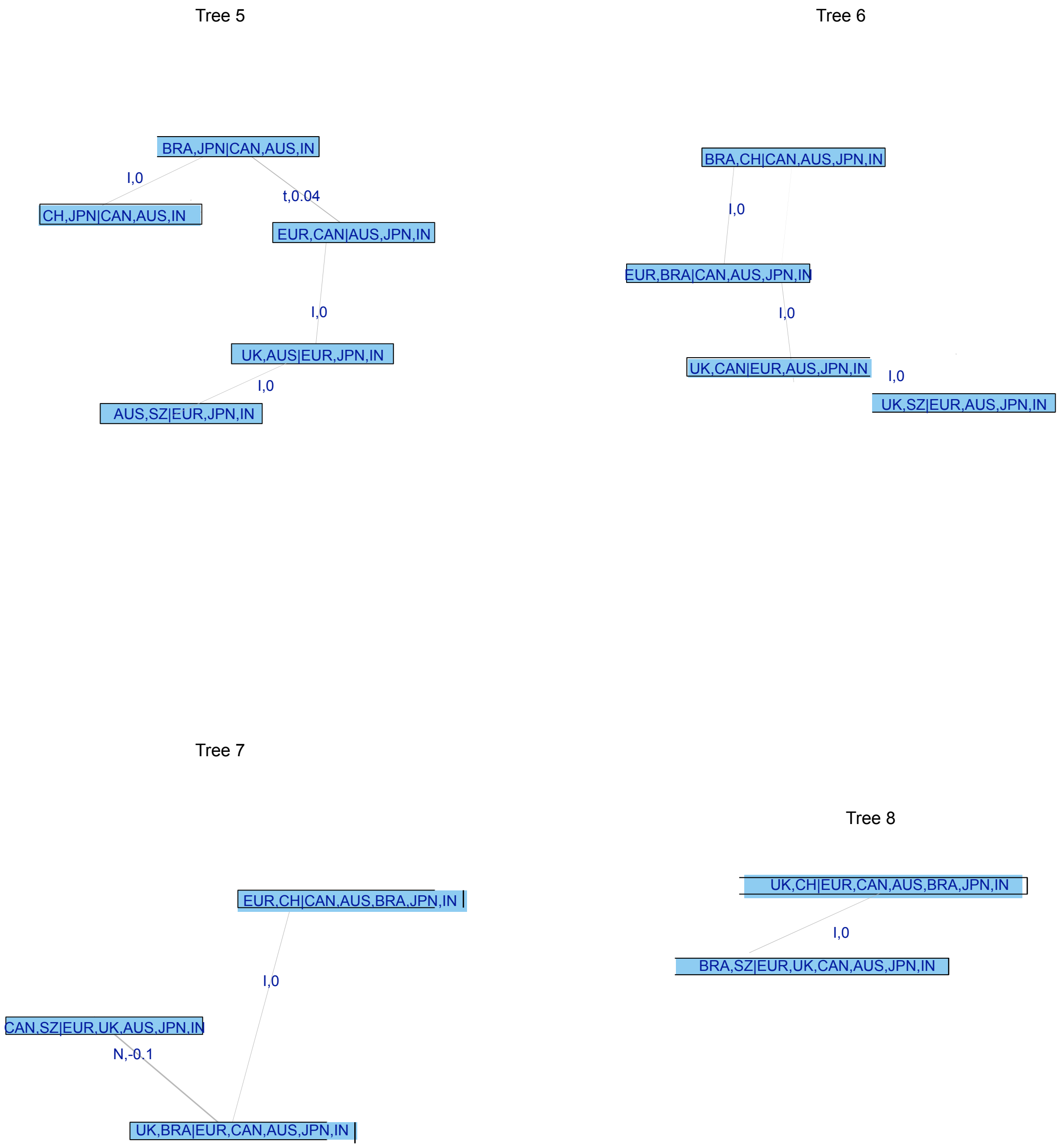


Figure B.10: $T_1 - T_4$ corresponding to R-Vine selected using adjusted Hu dependence as weight.

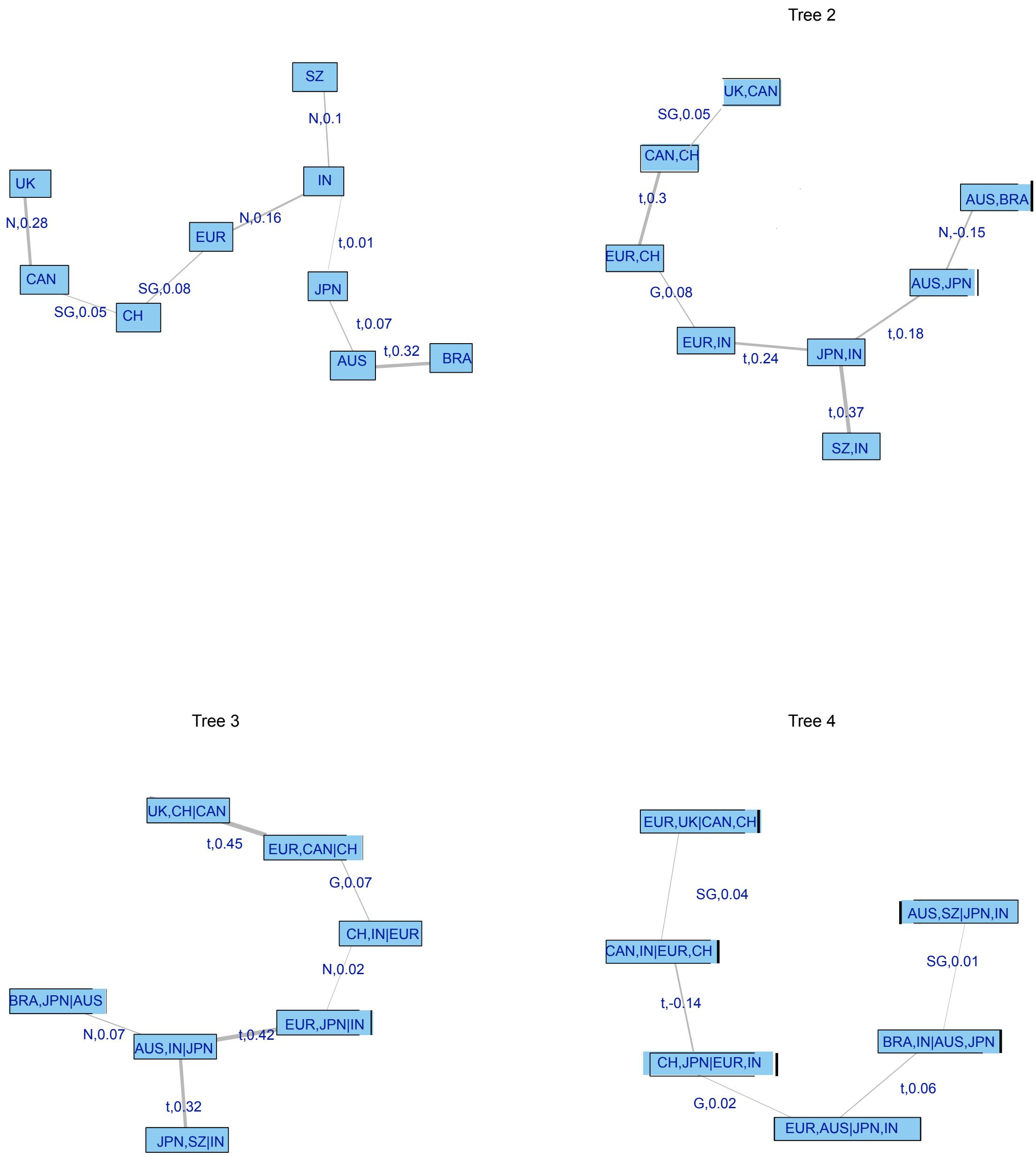
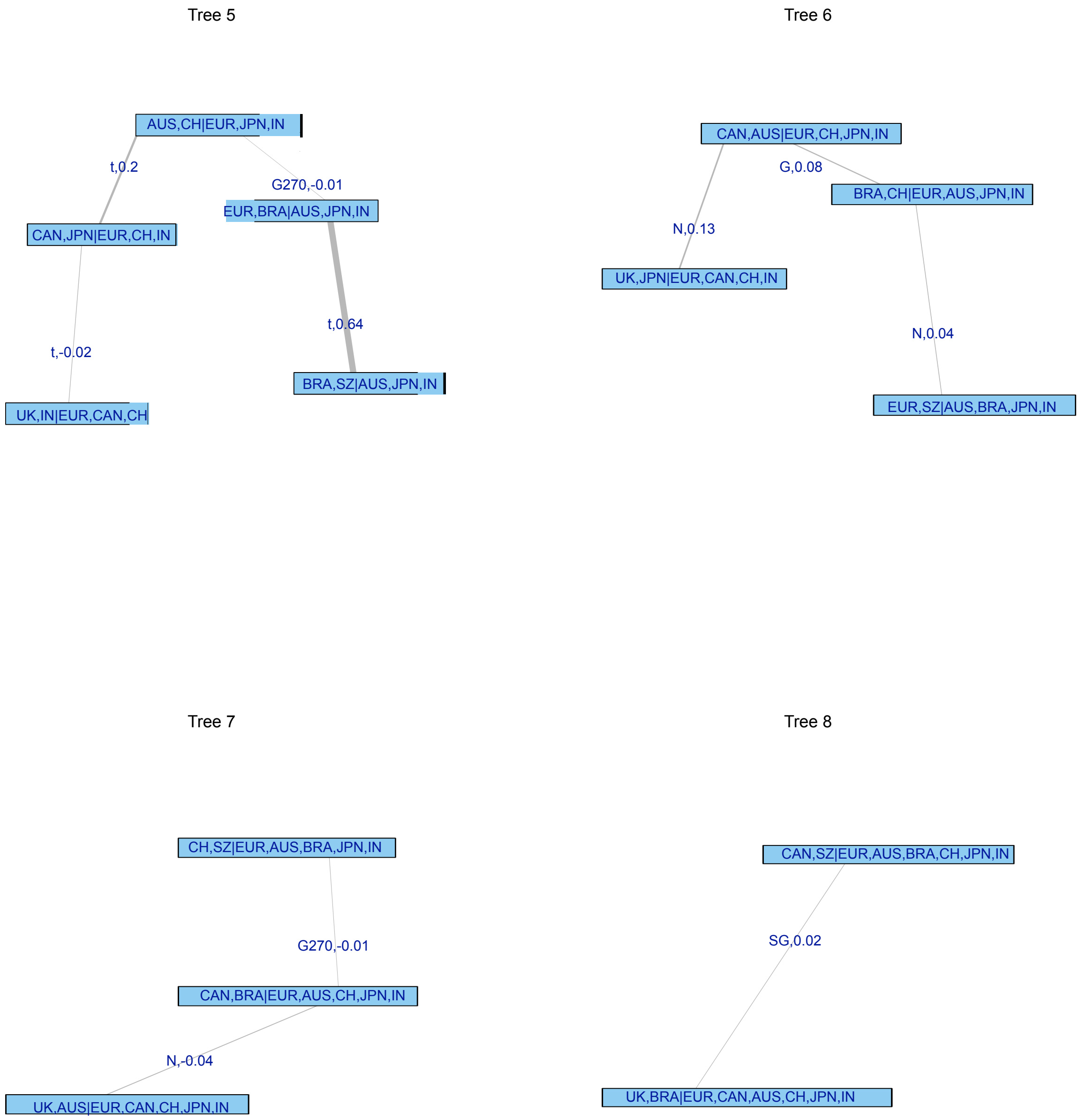


Figure B.11: $T_5 - T_9$ corresponding to R-Vine selected using adjusted Hu dependence as weight.

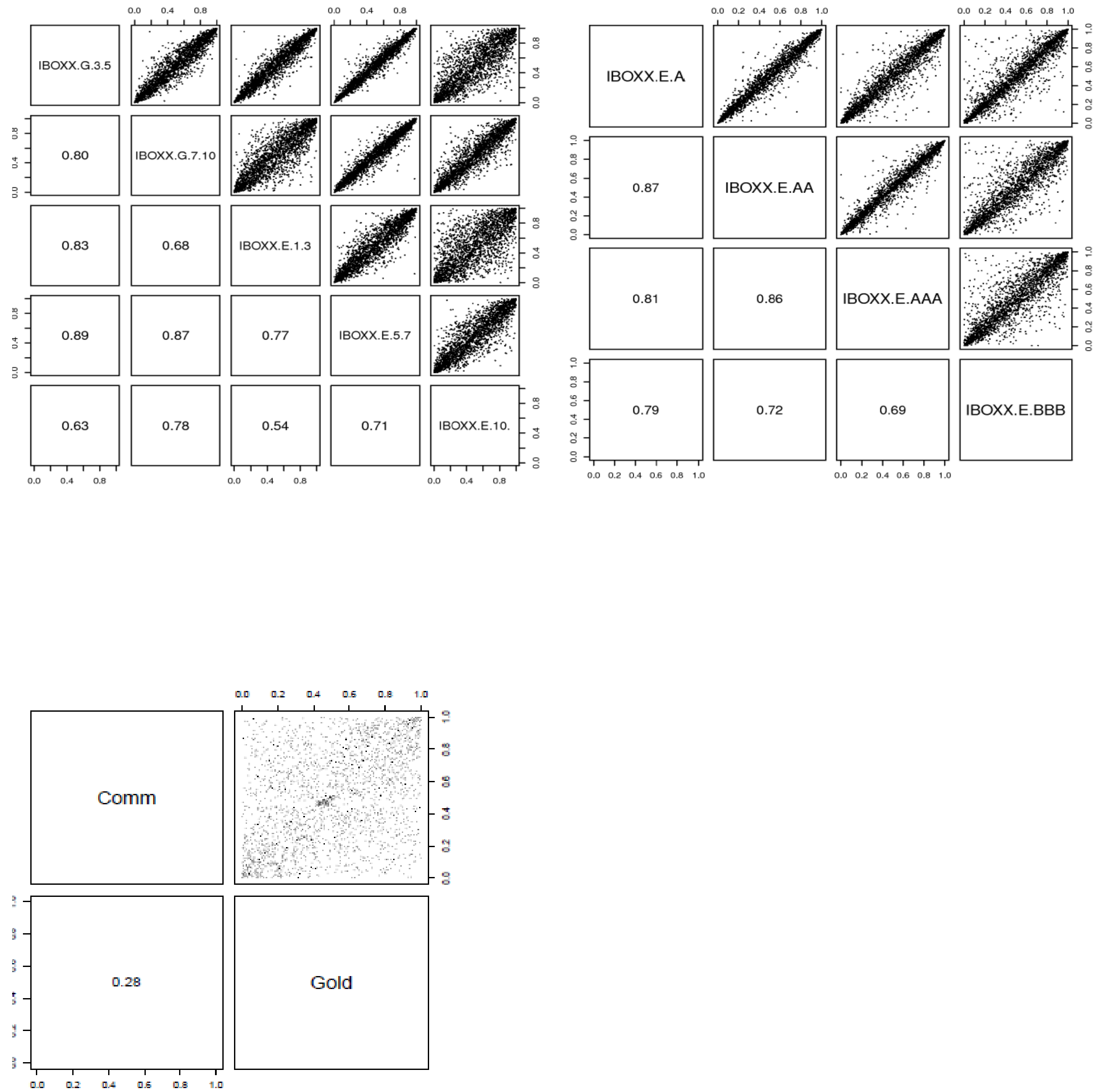


B.2 International Financial Indices

Below an overview of international financial indices with corresponding number from 1 to 16.

	Short Name	Long Name	Region
1	Dax	DAX30 PERFORMANCE	Germany
2	STOXX50	DJ STOXX 50	Europe
3	S&P500	S&P 500 COMPOSITE	USA
4	MSCI-WORLD	MSCI WORLD U$	Global
5	MSCI-EE	MSCI EM EASTERN EUROPE U$	Eastern Europe
6	IBOXX-G-3-5	IBOXX EURO SOV.GERMANY 3-5 YRS	Germany
7	IBOXX-G-7-10	IBOXX EURO SOV.GERMANY 7-10 YRS	Germany
8	IBOXX-E-1-3	IBOXX EURO SOV.EZONE 1-3 YRS	Eurozone
9	IBOXX-E-5-7	IBOXX EURO SOV.EZONE 5-7 YRS	Eurozone
10	IBOXX-E-10+	IBOXX EURO SOV.EZONE 10+ YRS	Eurozone
11	BOXX-E-A	IBOXX EURO CORP.A RATED ALL MATS.	Eurozone
12	BOXX-E-AA	IBOXX EURO CORP.AA RATED ALL MATS.	Eurozone
13	BOXX-E-AAA	IBOXX EURO CORP.AAA RATED ALL MATS.	Eurozone
14	BOXX-E-BBB	IBOXX EURO CORP.BBB RATED ALL MATS.	Eurozone
15	Comm	DJ UBS-Spot Commodity Index	Global
16	Gold	MLCX Gold Total Return	Global

Figure B.12: Pairs-plots for the transformed copula data and the empirical values of Kendall′s τ of Schepsmeier (2010) for the Equity financial indices (top left), Fixed Income (top right) and Commodities (bottom).



Method Kendalls Tau

$$
M=\begin{pmatrix}
16 \\
14 & 15 \\
11 & 14 & 5 \\
13 & 11 & 14 & 3 \\
12 & 13 & 11 & 14 & 4 \\
10 & 12 & 13 & 11 & 14 & 1 \\
7 & 10 & 12 & 13 & 11 & 14 & 2 \\
9 & 7 & 10 & 12 & 13 & 11 & 14 & 8 \\
8 & 9 & 7 & 10 & 12 & 13 & 11 & 14 & 10 \\
6 & 8 & 9 & 7 & 10 & 12 & 13 & 11 & 14 & 6 \\
3 & 6 & 8 & 9 & 7 & 10 & 12 & 13 & 11 & 14 & 7 \\
4 & 3 & 6 & 8 & 9 & 7 & 10 & 12 & 13 & 11 & 14 & 9 \\
1 & 4 & 3 & 6 & 8 & 9 & 7 & 10 & 12 & 13 & 11 & 14 & 13 \\
2 & 1 & 4 & 2 & 6 & 8 & 9 & 7 & 6 & 12 & 13 & 11 & 14 & 12 \\
5 & 2 & 1 & 1 & 2 & 6 & 8 & 9 & 9 & 7 & 12 & 13 & 11 & 14 & 14 \\
15 & 5 & 2 & 4 & 1 & 2 & 6 & 6 & 7 & 9 & 9 & 12 & 12 & 11 & 11 & 11
\end{pmatrix}
$$

$$
F=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\
2 & 24 & 2 & 0 & 0 & 0 & 14 & 2 & 2 & 2 & 1 & 0 \\
0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 0 & 4 & 1 & 2 & 0 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 34 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}
$$

$$
P=\begin{pmatrix}
0.00 \\
0.00 & 0.00 \\
0.00 & 0.00 & 0.000 \\
0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.06 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & -0.05 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.15 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.14 & 0.00 & 0.00 \\
0.12 & 0.00 & 0.06 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
-0.28 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.11 & 0.19 & 0.00 & 0.00 & 0.00 \\
0.13 & -1.10 & -0.048 & 0.00 & 0.00 & 0.00 & 1.05 & 0.13 & 0.09 & -0.09 & -0.11 & 0.00 \\
0.00 & 0.13 & -0.23 & -0.10 & 0.00 & 0.00 & -0.06 & 0.18 & 0.18 & 0.34 & -0.13 & -0.09 & 0.00 \\
-0.16 & 0.00 & 1.13 & -0.17 & 0.06 & 0.00 & 0.07 & -0.26 & -0.44 & 0.13 & -0.17 & 0.00 & 0.04 & 0.00 \\
0.05 & 0.06 & 0.08 & -0.21 & 0.17 & -1.02 & 0.06 & -0.53 & -0.40 & -0.57 & 0.32 & 0.42 & 0.08 & -0.24 & 0.00 \\
0.43 & 0.25 & 0.47 & 0.88 & 0.73 & 0.90 & -0.40 & 0.96 & 0.93 & 0.98 & 0.97 & 0.97 & 0.971 & 0.98 & 0.94 & 0
\end{pmatrix}
$$

$$
P2=\begin{pmatrix}
0.00 \\
0.00 & 0.00 \\
0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
25.93 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 24.14 & 0.00 & 0.00 & 0.00 \\
11.22 & 0.00 & 22.21 & 0.00 & 0.00 & 0.00 & 0.00 & 26.37 & 11.77 & 13.18 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 15.86 & 0.00 & 0.00 & 0.00 & 23.24 & 8.64 & 6.50 & 10.15 & 11.49 & 0.00 \\
15.12 & 0.00 & 0.00 & 0.00 & 22.54 & 0.00 & 0.00 & 5.24 & 8.05 & 4.42 & 3.41 & 0.00 & 8.88 & 0.00 \\
12.84 & 11.88 & 21.13 & 9.09 & 11.53 & 0.00 & 16.92 & 2.84 & 2.81 & 1.75 & 3.29 & 3.03 & 4.26 & 4.23 & 0.00 \\
6.25 & 20.27 & 7.79 & 6.93 & 9.92 & 6.16 & 6.59 & 3.79 & 3.63 & 2.67 & 3.00 & 1.46 & 2.45 & 3.14 & 1.54 & 0
\end{pmatrix}
$$

$$
\tau=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0.04 & 0 & 0 \\
0 & 0 & 0 & 00 & 0 & 0 \\
0 & 0 & 0 & -0.03 & 0 & 0 & 0 \\
0 & 0 & 0 & 00 & 0 & 0 & 0.10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.09 & 0 & 0 \\
0.07 & 0 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.18 & 0 & 0 & 0 & 0 & 0 & 0.07 & 0.12 & 0 & 0 & 0 \\
0.08 & -0.09 & -0.03 & 0 & 0 & 0 & 0.04 & 0.08 & 0.06 & -0.06 & -0.07 & 0 \\
0 & 0.09 & -0.15 & -0.07 & 0 & 0 & -0.04 & 0.12 & 0.12 & 0.22 & -0.09 & -0.06 & 0 \\
-0.10 & 0 & 0.11 & -0.11 & 0.04 & 0 & 0.04 & -0.17 & -0.29 & 0.09 & -0.11 & 0 & 0.03 & 0 \\
0.03 & 0.04 & 0.05 & -0.14 & 0.11 & -0.02 & 0.04 & -0.35 & -0.26 & -0.39 & 0.21 & 0.27 & 0.05 & -0.15 & 0 \\
0.28 & 0.16 & 0.31 & 0.69 & 0.52 & 0.72 & -0.26 & 0.83 & 0.77 & 0.88 & 0.86 & 0.85 & 0.86 & 0.86 & 0.78 & 0
\end{pmatrix}
$$

Method Tail Cumulation

$$
M=\begin{pmatrix}
8 \\
10 & 6 \\
7 & 10 & 14 \\
14 & 7 & 10 & 3 \\
3 & 14 & 7 & 10 & 2 \\
2 & 3 & 9 & 7 & 10 & 5 \\
5 & 2 & 13 & 9 & 7 & 10 & 1 \\
1 & 5 & 12 & 13 & 9 & 7 & 10 & 10 \\
4 & 1 & 3 & 12 & 13 & 9 & 7 & 4 & 7 \\
15 & 4 & 2 & 11 & 12 & 13 & 9 & 15 & 4 & 9 \\
16 & 15 & 5 & 16 & 11 & 12 & 13 & 16 & 15 & 4 & 13 \\
11 & 16 & 1 & 2 & 16 & 11 & 12 & 11 & 16 & 15 & 4 & 12 \\
13 & 11 & 4 & 5 & 15 & 16 & 11 & 13 & 11 & 16 & 15 & 4 & 4 \\
12 & 13 & 15 & 1 & 4 & 15 & 16 & 12 & 13 & 11 & 16 & 15 & 11 & 11 \\
9 & 12 & 16 & 15 & 5 & 4 & 15 & 9 & 12 & 13 & 11 & 16 & 16 & 15 & 16 \\
6 & 9 & 11 & 4 & 1 & 1 & 4 & 7 & 9 & 12 & 12 & 11 & 15 & 16 & 15 & 15
\end{pmatrix}
$$

$$
F=\begin{pmatrix}
0 \\
2 & 0 \\
2 & 2 & 0 \\
4 & 2 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 14 & 0 \\
1 & 1 & 2 & 1 & 24 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 2 & 24 & 1 & 0 & 1 & 0 & 24 & 0 \\
0 & 0 & 1 & 2 & 24 & 1 & 34 & 0 & 0 & 1 & 0 \\
0 & 0 & 14 & 1 & 2 & 0 & 1 & 14 & 0 & 24 & 24 & 0 \\
2 & 0 & 2 & 1 & 0 & 14 & 2 & 2 & 2 & 0 & 0 & 2 & 0 \\
2 & 2 & 0 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 0 & 24 & 2 & 0 \\
2 & 0 & 0 & 2 & 2 & 4 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}
$$

$$
P=\begin{pmatrix}
0 \\
0.11 & 0 \\
-0.31 & -0.43 & 0 \\
1.03 & -0.54 & 0 & 0 \\
0 & 1.02 & -0.10 & 0 & 0 \\
0 & 0 & -0.07 & 0 & 0 & 0 \\
0 & 0 & 0.06 & 0 & -0.05 & 1.02 & 0 \\
0.05 & -0.06 & -0.20 & -0.05 & -1.01 & 0 & 0.07 & 0 \\
0.05 & -0.06 & 0 & 0 & -0.05 & 0 & 0 & 0.09 & 0 \\
0 & -0.06 & 0.09 & -1.04 & -0.08 & 0 & -0.06 & 0 & -1.03 & 0 \\
0 & 0 & 0.05 & -0.27 & -1.05 & -0.07 & -1.04 & 0 & 0 & -0.10 & 0 \\
0 & 0 & 1.04 & -0.15 & -0.07 & 0 & -0.16 & 1.03 & 0 & -1.02 & -1.03 & 0 \\
0.19 & 0 & 0.13 & -0.23 & 0 & 1.05 & -0.21 & -0.06 & -0.13 & 0 & 0 & -0.15 & 0 \\
0.04 & 0.35 & 0 & -0.22 & 0.14 & 0.17 & -0.18 & 0.11 & -0.17 & 0 & 0 & -1.05 & -0.17 & 0 \\
-0.53 & 0 & 0 & -0.19 & 0.15 & 1.15 & 0 & -0.40 & 0.32 & 0.42 & 0.08 & 0 & 0 & -0.09 & 0 \\
0.96 & 0.98 & 0.94 & 0.88 & 0.90 & 0.45 & 0.73 & 0.93 & 0.98 & 0.97 & 0.98 & 0.98 & 0.24 & 0.15 & 0.43 & 0
\end{pmatrix}
$$

$$
P2=\begin{pmatrix}
0 \\
20.35 & 0 \\
7.77 & 19.48 & 0 \\
0 & 2.94 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10.97 & 0 & 0 & 0 \\
0 & 0 & 8.81 & 0 & 0 & 0 & 0 \\
0 & 0 & 4.40 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 17.83 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 25.18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 25.50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11.26 & 0 & 16.39 & 0 & 0 & 0 & 14.74 & 8.26 & 10.15 & 0 & 0 & 28.99 & 0 \\
6.92 & 4.11 & 0 & 9.86 & 11.66 & 0 & 13.36 & 5.17 & 3.42 & 0 & 0 & 0 & 12.39 & 0 \\
2.84 & 0 & 0. & 14.82 & 15.05 & 0 & 0 & 2.81 & 3.30 & 3.03 & 4.26 & 0 & 0 & 0 & 0 \\
3.79 & 2.68 & 1.54 & 6.93 & 6.16 & 8.72 & 9.92 & 3.63 & 3.00 & 1.47 & 2.45 & 3.14 & 10.11 & 9.19 & 6.26 & 0
\end{pmatrix}
$$

$$
\tau=\begin{pmatrix}
0 \\
0.07 & 0 \\
-0.20 & -0.29 & 0 \\
0.03 & -0.36 & 0 & 0 \\
0 & 0.02 & -0.07 & 0 & 0 \\
0 & 0 & -0.04 & 0 & 0 & 0 \\
0 & 0 & 0.04 & 0 & -0.03 & 0.02 & 0 \\
0.03 & -0.04 & -0.13 & -0.03 & -0.01 & 0 & 0.05 & 0 \\
0.03 & -0.04 & 0 & 0 & -0.03 & 0 & 0 & 0.05 & 0 \\
0 & -0.04 & 0.06 & -0.04 & -0.05 & 0 & -0.04 & 0 & -0.03 & 0 \\
0 & 0 & 0.03 & -0.18 & -0.05 & -0.05 & -0.03 & 0 & 0 & -0.06 & 0 \\
0 & 0 & 0.04 & -0.09 & -0.04 & 0 & -0.10 & 0.03 & 0 & -0.02 & -0.03 & 0 \\
0.12 & 0 & 0.08 & -0.14 & 0 & 0.05 & -0.13 & -0.03 & -0.08 & 0 & 0 & -0.10 & 0 \\
0.02 & 0.23 & 0 & -0.14 & 0.09 & 0.11 & -0.12 & 0.07 & -0.11 & 0 & 0 & -0.05 & -0.11 & 0 \\
-0.35 & 0 & 0 & -0.12 & 0.10 & 0.13 & 0 & -0.26 & 0.21 & 0.27 & 0 & 0 & 0 & -0.06 & 0 \\
0.83 & 0.88 & 0.78 & 0.69 & 0.72 & 0.30 & 0.52 & 0.77 & 0.86 & 0.85 & 0.86 & 0.86 & 0.15 & 0.09 & 0.28 & 0
\end{pmatrix}
$$

Method Hu Dependence

$$
\mathrm{M}=\begin{pmatrix}
2 \\
7 & 7 \\
9 & 1 & 9 \\
6 & 3 & 1 & 6 \\
10 & 15 & 3 & 1 & 10 \\
1 & 4 & 15 & 3 & 1 & 14 \\
3 & 5 & 4 & 15 & 3 & 1 & 11 \\
14 & 12 & 5 & 4 & 15 & 3 & 1 & 1 \\
11 & 13 & 12 & 5 & 4 & 15 & 3 & 8 & 3 \\
8 & 11 & 13 & 12 & 5 & 4 & 15 & 13 & 8 & 15 \\
13 & 16 & 11 & 13 & 12 & 5 & 4 & 12 & 13 & 8 & 4 \\
12 & 8 & 16 & 11 & 13 & 12 & 5 & 5 & 12 & 13 & 8 & 5 \\
15 & 6 & 8 & 16 & 11 & 13 & 12 & 4 & 5 & 12 & 13 & 8 & 8 \\
4 & 9 & 6 & 816 & 11 & 13 & 16 & 4 & 5 & 12 & 13 & 12 & 12 & 12 \\
16 & 10 & 10 & 108 & 16 & 16 & 3 & 16 & 4 & 5 & 12 & 13 & 13 & 16 & 16 \\
5 & 14 & 14 & 14 & 14 & 8 & 8 & 15 & 15 & 16 & 16 & 16 & 16 & 16 & 13 & 13
\end{pmatrix}
$$

$$
\mathrm{F}=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 14 & 0 \\
1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
34 & 2 & 2 & 2 & 0 & 2 & 0 & 1 & 0 & 0 \\
34 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 0 & 0 & 0 \\
2 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 0 & 0 & 34 & 0 \\
14 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 2 & 0 \\
2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}
$$

$$
\mathrm{P}=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
-0.05 & 0.04 & 0 & 0 \\
0 & 0 & 0 & -0.06 & 0 \\
0.74 & 0 & 0 & 0 & 0 & 0 \\
-0.29 & -0.06 & 0 & 0 & 0 & 1.01 & 0 \\
0.10 & 0.16 & 0 & -0.06 & 0 & 0 & 0.10 & 0 \\
0.12 & 0.14 & 0.16 & -0.11 & 0 & 0 & 0 & 0 & 0 \\
-1.02 & 0.09 & 0.12 & 0.32 & 0 & 0.10 & 0 & -0.04 & 0 & 0 \\
-1.03 & 0 & 0.16 & 0.40 & 0.36 & 0.12 & 0.08 & -0.23 & 0 & 0 & 0 \\
-0.25 & -0.31 & 0 & 0.37 & 0.45 & -0.21 & 0.13 & 0.15 & 0 & 0 & -1.02 & 0 \\
1.03 & -0.21 & -0.35 & 1.02 & 0.56 & -0.13 & 0.66 & 0.60 & -0.25 & -0.06 & 0 & -0.07 & 0 \\
0.63 & 0.86 & 0.94 & 0.89 & 0 & 0.82 & 0.80 & -0.09 & 0.89 & 0.13 & -0.16 & -0.04 & 0 & 0 \\
-0.13 & 0.81 & 0.67 & 0.44 & 0.23 & 0 & 0.05 & 0.56 & -0.10 & 0.20 & 0.45 & -0.18 & 0.92 & 0.97 & 0 \\
0.47 & 0.85 & 0.87 & 0.86 & 0.78 & 0.82 & 0.88 & 0.15 & 0.12 & 0.430 & 0.10 & 0.15 & 0.14 & 0.14 & 0.13 & 0
\end{pmatrix}
$$

$$
\mathrm{P2}=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
7.24 & 0 & 0 & 0 & 0 & 0 \\
19.22 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 11.73 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 9.57 & 10.99 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12.19 & 11.55 & 9.44 & 0 & 20.24 & 0 & 0 & 0 & 0 \\
0 & 0 & 10.70 & 5.87 & 6.47 & 0 & 0 & 23.18 & 0 & 0 & 0 \\
14.27 & 11.69 & 0. & 5.77 & 5.48 & 9.12 & 0 & 28.20 & 0 & 0 & 0 & 0 \\
0 & 2.49 & 3.58 & 0 & 4.64 & 13.43 & 6.30 & 17.93 & 0 & 0 & 0 & 24.65 & 0 \\
8.27 & 3.11 & 2.46 & 3.95 & 0 & 3.27 & 3.81 & 14.74 & 8.80 & 29.80 & 22.28 & 17.08 & 0 & 0. \\
12.48 & 3.92 & 5.24 & 4.30 & 4.56 & 0 & 17.08 & 6.82 & 11.18 & 9.10 & 9.64 & 13.96 & 3.40 & 2.51 & 0 \\
7.80 & 2.54 & 2.37 & 2.55 & 3.40 & 3.67 & 4.52 & 8.17 & 12.40 & 6.26 & 7.98 & 10.23 & 9.97 & 9.47 & 7.95 & 0
\end{pmatrix}
$$

$$
\tau=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
-0.03 & 0.03 & 0 & 0 \\
0 & 0 & 0 & -0.04 & 0 \\
0.53 & 0 & 0 & 0 & 0 & 0 \\
-0.19 & -0.04 & 0 & 0 & 0 & 0.01 & 0 \\
0.06 & 0.10 & 0 & -0.04 & 0 & 0 & 0.07 & 0 \\
0.08 & 0.09 & 0.10 & -0.07 & 0 & 0 & 0 & 0 & 0 \\
-0.02 & 0.06 & 0.07 & 0.21 & 0 & 0.06 & 0 & -0.03 & 0 & 0 \\
-0.03 & 0 & 0.10 & 0.26 & 0.23 & 0.07 & 0.05 & -0.15 & 0 & 0 & 0 \\
-0.16 & -0.20 & 0 & 0.24 & 0.30 & -0.14 & 0.08 & 0.10 & 0 & 0 & -0.02 & 0 \\
0.03 & -0.13 & -0.22 & 0.02 & 0.38 & -0.08 & 0.46 & 0.41 & -0.16 & -0.04 & 0 & -0.04 & 0 \\
0.44 & 0.66 & 0.79 & 0.70 & 0 & 0.61 & 0.59 & -0.05 & 0.70 & 0.08 & -0.10 & -0.02 & 0 & 0 \\
-0.08 & 0.61 & 0.47 & 0.29 & 0.15 & 0 & 0.03 & 0.38 & -0.06 & 0.13 & 0.30 & -0.11 & 0.74 & 0.86 & 0 \\
0.31 & 0.64 & 0.67 & 0.66 & 0.57 & 0.62 & 0.69 & 0.10 & 0.08 & 0.28 & 0.06 & 0.10 & 0.09 & 0.09 & 0.08 & 0
\end{pmatrix}
$$

Method Exceedance Dependence

$$
M=\begin{pmatrix}
1 \\
15 & 2 \\
14 & 15 & 5 \\
11 & 14 & 15 & 3 \\
16 & 11 & 14 & 15 & 15 \\
8 & 16 & 11 & 14 & 10 & 10 \\
12 & 8 & 16 & 11 & 14 & 14 & 4 \\
10 & 12 & 8 & 16 & 11 & 11 & 14 & 16 \\
7 & 10 & 12 & 8 & 4 & 16 & 11 & 14 & 8 \\
9 & 7 & 10 & 12 & 7 & 8 & 16 & 11 & 14 & 7 \\
6 & 9 & 7 & 10 & 13 & 12 & 8 & 7 & 11 & 14 & 13 \\
13 & 6 & 9 & 7 & 12 & 4 & 12 & 13 & 7 & 11 & 14 & 6 \\
3 & 13 & 6 & 9 & 9 & 13 & 7 & 12 & 13 & 12 & 11 & 14 & 9 \\
5 & 3 & 13 & 6 & 6 & 6 & 9 & 9 & 12 & 13 & 12 & 11 & 14 & 12 \\
4 & 5 & 3 & 13 & 8 & 9 & 6 & 6 & 9 & 6 & 9 & 12 & 11 & 14 & 14 \\
2 & 4 & 4 & 4 & 16 & 7 & 13 & 8 & 6 & 9 & 6 & 9 & 12 & 11 & 11 & 11
\end{pmatrix}
$$

$$
F=\begin{pmatrix}
0 \\
2 & 0 \\
2 & 2 & 0 \\
4 & 2 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 14 & 0 \\
1 & 1 & 2 & 1 & 24 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 2 & 24 & 1 & 0 & 1 & 0 & 24 & 0 \\
0 & 0 & 1 & 2 & 24 & 1 & 34 & 0 & 0 & 1 & 0 \\
0 & 0 & 14 & 1 & 2 & 0 & 1 & 14 & 0 & 24 & 24 & 0 \\
2 & 0 & 2 & 1 & 0 & 14 & 2 & 2 & 2 & 0 & 0 & 2 & 0 \\
2 & 2 & 0 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 0 & 24 & 2 & 0 \\
2 & 0 & 0 & 2 & 2 & 4 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}
$$

$$
P=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0.08 & 0.10 & 0 \\
-0.09 & 0.11 & 1.02 & -0.07 & 0 \\
0 & -0.22 & 0.08 & 0 & 0 & 0 \\
0 & 0 & 1.03 & 0 & 0 & 0 & 0 \\
0 & 0.05 & 0 & -0.29 & 0 & 0 & 0.12 & 0 \\
0 & 0.05 & 0 & 0 & 0.17 & 0 & 0.10 & 0 & 0 \\
0 & -0.08 & 0 & 0 & 0 & 0.14 & 0.14 & 0.05 & 0.04 & 0 \\
0 & 0 & -0.05 & 0 & 0 & 0.08 & 0 & 0 & 0 & -0.10 & 0 \\
-0.07 & -0.11 & 0.05 & 0.06 & 0 & 0.06 & 0.05 & 0 & -0.27 & -0.14 & 0.07 & 0 \\
0 & -0.29 & -0.06 & 0 & 0 & 0.18 & -0.06 & 0 & 0.19 & 0.26 & 0.12 & 0.06 & 0 \\
0 & -0.23 & -0.08 & 1.03 & -1.02 & -0.44 & 0 & 0 & 0.04 & 0.22 & 0.55 & 0 & -0.05 & 0 \\
0.29 & 0.23 & -0.28 & 0 & -0.12 & -0.40 & -1.08 & 1.03 & -0.53 & -0.57 & 0.47 & 0 & 0 & -0.24 & 0 \\
0.90 & 0.71 & 0.46 & 0.88 & 0.43 & 0.93 & -0.21 & 0.14 & 0.96 & 0.98 & 0.97 & 0.98 & 0.97 & 0.98 & 0.94 & 0
\end{pmatrix}
$$

$$
P2=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 19.50 & 0 & 0 & 21.96 & 0 \\
0 & 0 & 0 & 0 & 14.12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 20.18 & 13.28 & 0 & 27.96 & 0 \\
0 & 0 & 0 & 0 & 0 & 10.09 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.98 & 13.93 & 11.34 & 0 \\
0 & 11.17 & 0 & 0 & 0 & 9.59 & 0 & 0 & 10.87 & 4.28 & 7.72 & 21.33 & 0 \\
0 & 9.06 & 28.34 & 0 & 0 & 8.06 & 0 & 0 & 6.92 & 5.43 & 3.78 & 0 & 9.29 & 0 \\
16.68 & 15.28 & 21.86 & 0 & 13.81 & 2.81 & 0 & 0 & 2.84 & 1.75 & 4.41 & 0 & 0 & 4.23 & 0 \\
6.16 & 7.20 & 9.13 & 6.93 & 6.26 & 3.63 & 10.09 & 9.97 & 3.79 & 3.00 & 1.79 & 2.68 & 1.47 & 3.14 & 1.54 & 0
\end{pmatrix}
$$

$$
\tau=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0.05 & 0.06 & 0 \\
-0.06 & 0.07 & 0.02 & -0.04 & 0 \\
0 & -0.14 & 0.05 & 0 & 0 & 0 \\
0 & 0 & 0.03 & 0 & 0 & 0 & 0 \\
0 & 0.03 & 0 & -0.18 & 0 & 0 & 0.08 & 0 \\
0 & 0.03 & 0 & 0 & 0.11 & 0 & 0.07 & 0 & 0 \\
0 & -0.05 & 0 & 0 & 0 & 0.09 & 0.09 & 0.03 & 0.03 & 0 \\
0 & 0 & -0.03 & 0 & 0 & 0.05 & 0 & 0 & 0 & -0.07 & 0 \\
-0.04 & -0.07 & 0.03 & 0.04 & 0 & 0.04 & 0.03 & 0 & -0.18 & -0.09 & 0.04 & 0 \\
0 & -0.19 & -0.04 & 0 & 0 & 0.12 & -0.04 & 0 & 0.12 & 0.16 & 0.07 & 0.03 & 0 \\
0 & -0.15 & -0.05 & 0.03 & -0.02 & -0.29 & 0 & 0 & 0.02 & 0.14 & 0.37 & 0 & -0.03 & 0 \\
0.19 & 0.15 & -0.18 & 0 & -0.08 & -0.26 & -0.08 & 0.03 & -0.35 & -0.39 & 0.31 & 0 & 0 & -0.15 & 0 \\
0.72 & 0.50 & 0.31 & 0.69 & 0.28 & 0.77 & -0.13 & 0.09 & 0.83 & 0.86 & 0.84 & 0.88 & 0.85 & 0.86 & 0.78 & 0
\end{pmatrix}
$$

Method Adjusted Hu Dependence

$$M^{adj}=\begin{pmatrix}
2 \\
7 & 7 \\
9 & 1 & 9 \\
6 & 3 & 1 & 6 \\
10 & 15 & 3 & 1 & 10 \\
1 & 4 & 15 & 3 & 1 & 14 \\
3 & 5 & 4 & 15 & 3 & 1 & 11 \\
14 & 12 & 5 & 4 & 15 & 3 & 1 & 1 \\
11 & 13 & 12 & 5 & 4 & 15 & 3 & 8 & 3 \\
8 & 11 & 13 & 12 & 5 & 4 & 15 & 13 & 8 & 15 \\
13 & 16 & 11 & 13 & 12 & 5 & 4 & 12 & 13 & 8 & 4 \\
12 & 8 & 16 & 11 & 13 & 12 & 5 & 5 & 12 & 13 & 8 & 5 \\
15 & 6 & 8 & 16 & 11 & 13 & 12 & 4 & 5 & 12 & 13 & 8 & 8 \\
4 & 9 & 6 & 8 & 16 & 11 & 13 & 16 & 4 & 5 & 12 & 13 & 12 & 12 \\
16 & 10 & 10 & 10 & 8 & 16 & 16 & 3 & 16 & 4 & 5 & 12 & 13 & 13 & 16 \\
5 & 14 & 14 & 14 & 14 & 8 & 8 & 15 & 15 & 16 & 16 & 16 & 16 & 16 & 13 & 13
\end{pmatrix}$$

$$F^{adj}=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
2 & 1 & 24 & 1 & 1 & 0 \\
2 & 1 & 1 & 34 & 1 & 14 & 0 \\
1 & 2 & 1 & 1 & 24 & 1 & 1 & 0 \\
1 & 2 & 2 & 1 & 24 & 34 & 1 & 1 & 0 \\
34 & 2 & 2 & 2 & 14 & 2 & 1 & 1 & 1 & 0 \\
34 & 24 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 34 & 0 \\
2 & 2 & 24 & 2 & 2 & 2 & 1 & 2 & 2 & 14 & 34 & 0 \\
14 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 1 & 1 & 34 & 2 & 0 \\
2 & 2 & 2 & 2 & 24 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}$$

$$P^{adj}=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0.74 & -0.03 & -1.00 & 0.04 & 0.02 & 0 \\
-0.29 & -0.06 & -0.03 & -1.01 & 0.03 & 1.02 & 0 \\
0.10 & 0.19 & 0.01 & -0.07 & -1.02 & -0.03 & 0.10 & 0 \\
0.12 & 0.14 & 0.17 & -0.12 & -1.01 & -1.00 & -0.02 & -0.03 & 0 \\
-1.02 & 0.09 & 0.12 & 0.31 & 1.01 & 0.09 & 0.03 & -0.04 & 0 & 0 \\
-1.03 & -1.00 & 0.16 & 0.41 & 0.37 & 0.11 & 0.07 & -0.23 & -0.02 & -1.03 & 0 \\
-0.25 & -0.31 & -1.00 & 0.37 & 0.45 & -0.22 & 0.12 & 0.15 & 0 & 1.00 & -1.02 & 0 \\
1.03 & -0.21 & -0.35 & 1.02 & 0.57 & -0.13 & 0.67 & 0.60 & -0.25 & -0.06 & -1.01 & -0.06 & 0 \\
0.63 & 0.86 & 0.94 & 0.89 & -1.01 & 0.82 & 0.80 & -0.09 & 0.89 & 0.13 & -0.16 & -0.04 & -0.02 & 0 \\
-0.13 & 0.81 & 0.67 & 0.44 & 0.23 & 0.03 & 0.05 & 0.56 & -0.10 & 0.20 & 0.45 & -0.18 & 0.92 & 0.97 & 0 \\
0.47 & 0.85 & 0.87 & 0.86 & 0.78 & 0.82 & 0.88 & 0.15 & 0.12 & 0.43 & 0.10 & 0.15 & 0.14 & 0.14 & 0.13 & 0
\end{pmatrix}$$

$$P2^{adj}=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
7.05 & 0 & 0 & 0 & 0 & 0 \\
19.53 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 9.71 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 9.67 & 9.91 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 11.63 & 11.48 & 11.53 & 0 & 21.45 & 0 & 0 & 0 & 0 \\
0 & 0 & 10.71 & 5.89 & 5.96 & 0 & 0 & 22.59 & 0 & 0 & 0 \\
14.27 & 11.69 & 0 & 5.81 & 5.41 & 7.76 & 0 & 28.20 & 16.16 & 0 & 0 & 0 \\
0 & 2.49 & 3.58 & 0 & 4.63 & 13.66 & 5.38 & 17.93 & 0 & 0 & 0 & 24.35 & 0 \\
8.27 & 3.11 & 2.46 & 3.95 & 0 & 3.16 & 3.81 & 14.74 & 8.80 & 29.80 & 22.28 & 17.08 & 3.59 & 0 \\
12.48 & 3.92 & 5.24 & 4.29 & 4.56 & 20.01 & 17.08 & 6.82 & 11.18 & 9.10 & 9.64 & 13.96 & 3.40 & 2.51 & 0 \\
7.80 & 2.54 & 2.37 & 2.55 & 3.40 & 3.67 & 4.52 & 8.17 & 12.40 & 6.26 & 7.98 & 10.23 & 9.97 & 9.47 & 7.95 & 0
\end{pmatrix}$$

$$\tau=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0.53 & -0.02 & 0 & 0.03 & 0.01 & 0 \\
-0.19 & -0.04 & -0.02 & -0.01 & 0.02 & 0.02 & 0 \\
0.06 & 0.12 & 0.01 & -0.04 & -0.02 & -0.02 & 0.06 & 0 \\
0.08 & 0.09 & 0.11 & -0.07 & -0.01 & -0.01 & -0.01 & -0.02 & 0 \\
-0.02 & 0.06 & 0.07 & 0.20 & 0.01 & 0.06 & 0.02 & -0.03 & 0 & 0 \\
-0.03 & 0 & 0.10 & 0.27 & 0.24 & 0.07 & 0.05 & -0.15 & -0.01 & -0.03 & 0 \\
-0.16 & -0.20 & 0 & 0.24 & 0.30 & -0.14 & 0.08 & 0.10 & 0 & 0.01 & -0.02 & 0 \\
0.03 & -0.13 & -0.22 & 0.02 & 0.38 & -0.08 & 0.47 & 0.41 & -0.16 & -0.04 & -0.01 & -0.04 & 0 \\
0.44 & 0.66 & 0.79 & 0.70 & 0 & 0.61 & 0.59 & -0.05 & 0.70 & 0.08 & -0.10 & -0.02 & -0.01 & 0 \\
-0.08 & 0.61 & 0.47 & 0.29 & 0.15 & 0.02 & 0.03 & 0.38 & -0.06 & 0.13 & 0.30 & -0.11 & 0.74 & 0.86 & 0 \\
0.31 & 0.64 & 0.67 & 0.66 & 0.57 & 0.62 & 0.69 & 0.10 & 0.07 & 0.28 & 0.06 & 0.10 & 0.09 & 0.09 & 0.08 & 0
\end{pmatrix}$$

B.3 German DAX

Below we summarize the variables of the German DAX data set. Each corresponding to a respective number.

ID.DE	Company Name		ID.DE	Company Name	
1	ADS	Adidas	16	HEI	HeidelbergCement
2	ALV	Allianz	17	HEN3	Henkel
3	BAS	BASF	18	IFX	Infineon Technologies
4	BAYN	Bayer	19	LHA	Lufthansa
5	BEI	Beiersdorf	20	LIN	Linde
6	BMW	BWM	21	MAN	MAN
7	CBK	Commerzbank	22	MEO	Metro
8	DAI	Daimler	23	MRK	Merk
9	DB1	Deutsche Börse	24	MUV2	Munich Re
10	DBK	Deutsche Bank	25	RWE	RWE
11	DPW	Deutsche Post	26	SAP	SAP
12	DTE	Deutsche Telekom	27	SDF	K+S
13	EOAN	E·ON	28	SIE	Siemens
14	FME	Fresenius Medical Care	29	TKA	ThyssenKrupp
15	FRE	Fresenius SE	30	VOW3	Volkswagen

Table B.1: Variables of German DAX data set.

Figures B.13 and B.14 offer an overview of variable pair selected according to strategy. Red color represent a pair that has been selected by three strategies out of four, green for two strategies and black for one.

Figure B.13: Overview of selected variable pairs for German DAX data set for each strategy according to the used weight

(a) Kendall's τ

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	–									×																				
2		–								×						×			×				×	×		×				
3			–	×									×							×								×		
4				–																										
5					–					×																				
6						–		×																						
7							–			×																				
8								–		×																				×
9									–	×																				
10										–	×	×						×			×	×						×		
11											–	×		×																
12												–		×																
13													–												×					
14														–	×															
15															–															
16																–														
17																	–											×		
18																		–												
19																			–											
20																				–										
21																					–									
22																						–								
23																							–							
24																								–						
25																									–					
26																										–				
27																											–		×	
28																												–		×
29																													–	
30																														–

(b) Tail cumulation

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	–									×																				
2		–								×														×						
3			–	×																×								×		
4				–																										
5					–					×																				
6						–		×																						
7							–			×																				
8								–		×												×								×
9									–	×																				
10										–	×	×	×			×		×	×		×		×			×		×		
11											–																			
12												–		×																
13													–												×					
14														–	×															
15															–	×														
16																–														
17																	–													
18																		–												
19																			–											
20																				–										
21																					–								×	
22																						–								
23																							–							
24																								–						
25																									–					
26																										–				
27																											–		×	
28																												–		
29																													–	
30																														–

Figure B.14: Overview of selected variable pairs for German DAX data set for each strategy according to the used weight (continued)

(c) Hu dependence

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	–								×																					
2		–		×						×																				
3			–											×																
4				–											×															
5					–																									×
6						–	×									×														
7							–						×												×		×			×
8								–																						
9									–					×							×									
10										–				×																
11											–	×		×																
12												–																		
13													–																	
14														–						×						×	×	×	×	
15															–				×											
16																–														
17																	–													×
18																		–											×	
19																			–				×							
20																				–										
21																					–									
22																						–								×
23																							–							×
24																								–						×
25																									–					
26																										–				
27																											–			
28																												–		
29																													–	
30																														–

(d) Exceedance dependence

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	–									×												×								
2		–						×		×																				
3			–																					×					×	
4				–																						×		×		
5					–																					×				
6						–											×						×							
7							–			×																				
8								–	×																					
9									–									×												
10										–														×				×		
11											–												×							
12												–										×								
13													–								×				×					
14														–							×									
15															–		×													
16																–				×										
17																	–			×										
18																		–											×	
19																			–									×		
20																				–										
21																					–						×			
22																						–								
23																							–							
24																								–	×					
25																									–					
26																										–				
27																											–			
28																												–		
29																													–	
30																														–

Method Kendalls Tau

$$
M=\begin{pmatrix}
21 \\
30 & 9 \\
25 & 30 & 16 \\
15 & 25 & 30 & 27 \\
6 & 15 & 25 & 30 & 4 \\
14 & 6 & 15 & 25 & 30 & 30 \\
23 & 14 & 6 & 15 & 25 & 20 & 25 \\
27 & 23 & 14 & 6 & 15 & 24 & 20 & 15 \\
4 & 27 & 23 & 14 & 6 & 23 & 24 & 20 & 6 \\
20 & 4 & 27 & 23 & 14 & 26 & 23 & 24 & 20 & 14 \\
24 & 20 & 4 & 26 & 23 & 7 & 26 & 23 & 24 & 20 & 23 \\
13 & 24 & 20 & 7 & 26 & 25 & 7 & 26 & 23 & 24 & 20 & 26 \\
12 & 13 & 24 & 18 & 7 & 15 & 18 & 7 & 26 & 23 & 24 & 20 & 7 \\
17 & 12 & 13 & 13 & 18 & 18 & 29 & 18 & 7 & 26 & 13 & 24 & 20 & 18 \\
29 & 17 & 12 & 12 & 13 & 29 & 19 & 29 & 18 & 7 & 12 & 13 & 24 & 20 & 20 \\
3 & 29 & 17 & 17 & 12 & 19 & 5 & 19 & 29 & 18 & 17 & 12 & 13 & 24 & 13 & 24 \\
26 & 3 & 29 & 19 & 17 & 5 & 22 & 5 & 19 & 29 & 29 & 17 & 12 & 13 & 12 & 13 & 29 \\
7 & 26 & 3 & 5 & 19 & 22 & 1 & 22 & 5 & 19 & 3 & 29 & 17 & 12 & 17 & 12 & 13 & 19 \\
18 & 7 & 26 & 22 & 5 & 1 & 6 & 1 & 22 & 5 & 5 & 3 & 29 & 17 & 19 & 17 & 12 & 13 & 5 \\
19 & 18 & 7 & 24 & 22 & 14 & 11 & 6 & 1 & 22 & 22 & 5 & 3 & 29 & 5 & 19 & 17 & 12 & 13 & 22 \\
5 & 19 & 18 & 1 & 24 & 13 & 8 & 11 & 14 & 1 & 1 & 22 & 5 & 3 & 22 & 5 & 19 & 17 & 12 & 13 & 1 \\
16 & 5 & 19 & 11 & 1 & 12 & 2 & 8 & 13 & 11 & 11 & 1 & 22 & 5 & 24 & 22 & 5 & 3 & 17 & 12 & 13 & 11 \\
2 & 16 & 5 & 8 & 11 & 11 & 15 & 2 & 12 & 8 & 7 & 11 & 1 & 22 & 1 & 29 & 22 & 5 & 3 & 17 & 12 & 13 & 8 \\
9 & 2 & 22 & 2 & 8 & 17 & 14 & 13 & 11 & 2 & 8 & 7 & 11 & 1 & 11 & 1 & 1 & 22 & 2 & 3 & 17 & 12 & 13 & 2 \\
22 & 11 & 1 & 10 & 2 & 3 & 12 & 3 & 17 & 13 & 19 & 8 & 18 & 11 & 8 & 11 & 11 & 1 & 11 & 2 & 3 & 17 & 12 & 13 & 12 \\
1 & 8 & 11 & 4 & 10 & 2 & 17 & 17 & 3 & 3 & 18 & 19 & 8 & 8 & 2 & 3 & 8 & 11 & 8 & 11 & 2 & 3 & 17 & 12 & 13 & 13 \\
11 & 22 & 8 & 20 & 29 & 28 & 10 & 28 & 2 & 17 & 28 & 18 & 28 & 19 & 10 & 8 & 2 & 8 & 28 & 8 & 11 & 2 & 3 & 17 & 3 & 17 & 3 \\
8 & 1 & 28 & 3 & 28 & 10 & 28 & 10 & 28 & 28 & 26 & 28 & 19 & 2 & 29 & 28 & 10 & 28 & 1 & 28 & 8 & 28 & 2 & 3 & 17 & 10 & 17 & 10 \\
28 & 28 & 10 & 28 & 20 & 6 & 3 & 12 & 10 & 10 & 10 & 10 & 2 & 28 & 28 & 10 & 3 & 10 & 22 & 1 & 28 & 8 & 28 & 28 & 28 & 28 & 10 & 17 & 28 \\
10 & 10 & 2 & 29 & 3 & 8 & 13 & 14 & 8 & 12 & 2 & 2 & 10 & 10 & 3 & 2 & 28 & 2 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 3 & 28 & 28 & 17 & 17
\end{pmatrix}
$$

$$
F=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 14 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\
24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 14 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 4 & 0 & 0 & 14 & 1 & 0 & 0 & 0 & 4 & 4 & 1 & 0 & 0 & 0 \\
14 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 4 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 14 & 1 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 14 & 2 & 2 & 1 & 0 & 0 & 0 & 34 & 1 & 0 \\
14 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 14 & 4 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 4 & 4 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 14 & 0 & 2 & 0 & 0 & 0 & 0 & 14 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 1 & 0 \\
14 & 2 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 14 & 0 & 2 & 0 & 14 & 14 & 2 & 2 & 14 & 0 & 0 \\
0 & 1 & 4 & 0 & 0 & 0 & 1 & 0 & 2 & 14 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 4 & 4 & 14 & 1 & 2 & 4 & 0 & 0 \\
1 & 1 & 2 & 4 & 1 & 0 & 0 & 4 & 2 & 2 & 1 & 2 & 4 & 2 & 2 & 2 & 4 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 0 \\
4 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 14 & 2 & 14 & 2 & 1 & 1 & 14 & 14 & 4 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 14 & 0 \\
14 & 14 & 14 & 1 & 2 & 2 & 4 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 14 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 1 & 4 & 2 & 14 & 14 & 2 & 2 & 2 & 2 & 14 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 14 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 14 & 0
\end{pmatrix}
$$

Method Kendalls Tau

P=

$$
\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.07 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.01 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.14 & 0 & 0.07 & 0 & 0 & 0 & 0.06 & 0 \\
0 & 0 & 0 & 0 & 1.05 & 0 & 1.00 & 0.10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0.08 & 0 \\
-1.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.11 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.03 & 0 & 0 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.15 & 1.04 & 0 & 0 & 1.07 & 0.08 & 0 & 0 & 0 & 1.05 & 1.05 & 0.07 & 0 & 0 & 0 \\
1.06 & 0.07 & 0 & 0 & 0.10 & 0 & 0 & 0 & 0 & 0 & 0.08 & 1.04 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.15 & 0 & 1.05 & 0.08 & 0 & 0.14 & 0 & -0.07 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.05 & 0 & 0 & 0 & 0 & 0 & 0 & 1.03 & 0 & 0 & 1.05 & 0.14 & 0.10 & 0 & 0 \\
0.07 & 0.11 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 1.05 & 0.08 & 0.07 & 0.07 & 0 & 0 & 0 & -1.04 & -0.07 & 0 \\
1.05 & 0 & 0 & -0.06 & 0.09 & 0 & 0 & 0 & 0 & 0 & 0.10 & 0 & 0 & 0 & 1.03 & 1.03 & 0 & 0.10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.13 & 0 & 0.08 & 0 & 0.06 & 0 & 0 & 0 & 0.06 & 1.04 & 1.05 & 0 & 0 \\
0.07 & 0 & 0 & 0 & 0.09 & 0 & 0 & 0 & -0.07 & 0 & 0 & 1.07 & 0 & 0.05 & 0 & 0 & 0 & 0 & 1.09 & 0 & 0.07 & 0 \\
0 & 0 & 0.09 & 0 & 0 & 0 & 0 & 0.08 & 0 & 0 & 0.06 & 0 & 0 & 0 & 0.07 & 0 & 0.06 & 0.08 & 0 & 0.15 & 0.06 & 0.06 & 0 \\
1.03 & 0.08 & 0 & 1.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.11 & 0 & 1.06 & 0 & 0.09 & 0 & 1.03 & 1.06 & 0.08 & 0.11 & 1.06 & 0 & 0 \\
0 & 0.06 & 1.03 & 0 & 0 & 0 & 0.11 & 0 & 0.06 & 1.04 & 0 & 0.10 & 0 & 0.09 & 0.07 & 0 & 0.13 & 1.03 & 1.00 & 1.06 & 0.11 & 0.09 & 1.02 & 0 & 0 \\
0.12 & 0.09 & 0.09 & 1.03 & 0.12 & 0 & 0 & 1.02 & 0.08 & 0.08 & 0.08 & 0.05 & 1.03 & 0.07 & 0.13 & 0.09 & 1.07 & 0.12 & 0.09 & 0.12 & 0 & 0.10 & 0.13 & 0.11 & 0.21 & 0 \\
1.07 & 0.13 & 0.17 & 0.08 & 0 & 0.05 & 0.09 & 0.10 & 1.06 & 0.12 & 1.04 & 0.21 & 0.08 & 0.17 & 1.06 & 1.06 & 1.04 & 0.16 & 0.09 & 0.11 & 0.11 & 0.11 & 0.14 & 0.12 & 0.07 & 1.03 & 0 \\
1.15 & 1.11 & 1.08 & 0.19 & 0.15 & 0.09 & 1.05 & 0.19 & 0.15 & 0.09 & 0.10 & 0.24 & 0.10 & 0.16 & 0.17 & 1.06 & 0.16 & 0.14 & 0.14 & 0.18 & 0.18 & 0.14 & 0.21 & 0.18 & 0.12 & 0.14 & 0.15 & 0 \\
0.31 & 0.25 & 0.20 & 0.19 & 0.25 & 0.20 & 1.12 & 0.09 & 1.22 & 1.08 & 0.13 & 0.24 & 0.25 & 0.22 & 1.23 & 0.20 & 0.31 & 0.29 & 0.21 & 0.24 & 0.25 & 0.25 & 0.34 & 0.32 & 0.20 & 0.23 & 0.33 & 0.22 & 0 \\
0.58 & 0.51 & 0.45 & 0.50 & 0.61 & 0.53 & 0.76 & 0.40 & 0.68 & 0.35 & 1.25 & 0.54 & 0.71 & 0.49 & 0.59 & 0.73 & 0.58 & 0.61 & 0.43 & 0.49 & 0.50 & 0.52 & 0.64 & 0.73 & 0.50 & 0.55 & 0.63 & 0.62 & 1.43 & 0
\end{pmatrix}
$$

P2=

$$
\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 23.07 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 26.40 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 16.54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27.45 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.72 & 0 & 16.43 & 29.48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.89 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 14.78 & 0 & 0 & 0 & 0 & 25.64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.19 & 0 & 23.35 & 0 & 0 & 0 \\
0 & 12.53 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 19.70 & 0 & 0 & 0 & 23.18 & 14.74 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 23.96 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21.73 & 0 & 0 & 0 \\
0 & 0 & 28.59 & 0 & 0 & 0 & 0 & 0 & 27.00 & 26.29 & 0 & 20.72 & 0 & 24.82 & 21.43 & 13.88 & 0 & 0 & 0 & 29.19 & 0 & 0 & 0 & 13.90 & 11.83 & 0 \\
0 & 0 & 0 & 0 & 0 & 13.77 & 22.65 & 14.54 & 0 & 27.27 & 0 & 17.24 & 0 & 0 & 0 & 0 & 0 & 0 & 12.62 & 26.09 & 23.86 & 0 & 13.66 & 0 & 15.44 & 0 & 0 \\
0 & 0 & 0 & 0 & 12.64 & 17.02 & 0 & 22.12 & 0 & 18.90 & 0 & 19.79 & 24.08 & 0 & 10.74 & 0 & 0 & 13.42 & 19.19 & 0 & 0 & 17.97 & 0 & 11.52 & 27.30 & 16.19 & 18.03 & 0 \\
10.22 & 14.84 & 20.88 & 22.03 & 10.68 & 0 & 0 & 17.06 & 0 & 0 & 17.79 & 24.17 & 9.47 & 16.95 & 0 & 17.00 & 7.96 & 15.62 & 21.71 & 12.61 & 13.16 & 18.19 & 8.16 & 8.33 & 19.51 & 9.38 & 11.87 & 12.19 & 0 \\
9.34 & 9.87 & 17.20 & 6.59 & 4.63 & 11.34 & 4.63 & 0 & 8.77 & 13.43 & 0 & 6.92 & 5.34 & 13.74 & 11.52 & 4.22 & 6.45 & 6.36 & 10.18 & 8.14 & 12.76 & 5.59 & 5.64 & 4.52 & 7.52 & 5.45 & 4.11 & 5.36 & 0 & 0
\end{pmatrix}
$$

$$M=\begin{pmatrix}
25\\
7 & 7\\
24 & 30 & 24\\
19 & 27 & 30 & 19\\
18 & 29 & 27 & 30 & 18\\
30 & 22 & 29 & 27 & 30 & 15\\
27 & 6 & 22 & 29 & 27 & 30 & 14\\
29 & 5 & 6 & 22 & 29 & 27 & 30 & 12\\
22 & 11 & 5 & 6 & 22 & 29 & 27 & 30 & 4\\
6 & 16 & 11 & 5 & 6 & 22 & 29 & 27 & 30 & 22\\
5 & 15 & 16 & 11 & 5 & 6 & 22 & 29 & 27 & 30 & 30\\
11 & 14 & 15 & 16 & 11 & 5 & 6 & 22 & 29 & 27 & 27 & 6\\
16 & 12 & 14 & 15 & 16 & 11 & 5 & 6 & 22 & 29 & 29 & 27 & 5\\
8 & 4 & 12 & 14 & 15 & 16 & 11 & 5 & 6 & 20 & 20 & 29 & 27 & 11\\
1 & 20 & 4 & 12 & 14 & 8 & 16 & 11 & 5 & 13 & 13 & 20 & 29 & 27 & 16\\
17 & 13 & 20 & 4 & 12 & 1 & 8 & 16 & 11 & 3 & 3 & 13 & 20 & 29 & 27 & 8\\
21 & 8 & 13 & 20 & 4 & 17 & 1 & 8 & 16 & 26 & 26 & 3 & 13 & 20 & 29 & 27 & 1\\
9 & 1 & 8 & 13 & 20 & 21 & 17 & 1 & 8 & 23 & 23 & 26 & 3 & 13 & 20 & 29 & 27 & 20\\
4 & 17 & 1 & 8 & 13 & 9 & 21 & 17 & 1 & 16 & 16 & 23 & 26 & 3 & 13 & 20 & 29 & 27 & 13\\
20 & 21 & 17 & 1 & 8 & 4 & 9 & 21 & 17 & 2 & 2 & 16 & 23 & 26 & 3 & 13 & 20 & 29 & 27 & 3\\
23 & 9 & 21 & 17 & 1 & 20 & 4 & 9 & 21 & 9 & 9 & 2 & 16 & 23 & 26 & 3 & 13 & 17 & 29 & 27 & 26\\
15 & 23 & 9 & 21 & 17 & 23 & 20 & 4 & 9 & 1 & 1 & 9 & 2 & 16 & 23 & 26 & 3 & 21 & 17 & 29 & 27 & 23\\
26 & 3 & 23 & 9 & 21 & 26 & 23 & 20 & 23 & 5 & 5 & 1 & 9 & 2 & 2 & 23 & 26 & 9 & 21 & 17 & 29 & 27 & 17\\
2 & 18 & 3 & 23 & 9 & 2 & 26 & 23 & 26 & 11 & 11 & 5 & 1 & 9 & 9 & 2 & 23 & 23 & 9 & 21 & 17 & 29 & 27 & 2\\
3 & 26 & 18 & 3 & 23 & 3 & 2 & 26 & 13 & 17 & 17 & 11 & 8 & 1 & 1 & 9 & 2 & 26 & 23 & 9 & 21 & 17 & 29 & 27 & 9\\
14 & 24 & 26 & 18 & 3 & 28 & 3 & 2 & 2 & 28 & 28 & 17 & 21 & 8 & 17 & 1 & 9 & 13 & 26 & 23 & 9 & 21 & 2 & 29 & 27 & 28\\
12 & 28 & 28 & 26 & 2 & 13 & 28 & 3 & 10 & 21 & 21 & 28 & 28 & 21 & 28 & 17 & 21 & 2 & 2 & 26 & 23 & 9 & 9 & 21 & 29 & 27 & 10\\
28 & 19 & 19 & 28 & 26 & 10 & 13 & 28 & 28 & 6 & 6 & 21 & 11 & 28 & 21 & 28 & 17 & 10 & 3 & 2 & 2 & 28 & 21 & 9 & 21 & 29 & 27 & 21\\
10 & 2 & 10 & 2 & 28 & 12 & 10 & 13 & 20 & 10 & 10 & 10 & 17 & 17 & 8 & 21 & 28 & 28 & 28 & 10 & 28 & 2 & 28 & 28 & 28 & 21 & 29 & 27 & 29\\
13 & 10 & 2 & 10 & 10 & 14 & 12 & 10 & 3 & 8 & 8 & 8 & 10 & 10 & 10 & 10 & 10 & 3 & 10 & 28 & 10 & 10 & 10 & 10 & 10 & 10 & 21 & 29 & 27 & 27
\end{pmatrix}$$

$$F=\begin{pmatrix}
0\\
0 & 0\\
0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 14 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 4 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 14 & 0 & 0 & 24 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 14 & 2 & 14 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 2 & 0 & 1 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 14 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 14 & 0 & 2 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 24 & 0 & 0 & 1 & 1 & 14 & 1 & 0 & 14 & 2 & 0 & 0 & 4 & 1 & 0 & 1\\
0 & 0 & 0 & 14 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 4 & 0 & 14 & 0 & 0 & 4 & 1 & 0 & 1 & 0\\
2 & 0 & 1 & 0 & 4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 & 14 & 0 & 14 & 0\\
1 & 2 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 4 & 1 & 0 & 4 & 1 & 1 & 0 & 14 & 0 & 2 & 0 & 1 & 0\\
1 & 0 & 1 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 14 & 14 & 1 & 1 & 14 & 1 & 1 & 14 & 0 & 1 & 0\\
2 & 1 & 0 & 0 & 4 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 4 & 2 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
14 & 0 & 2 & 1 & 2 & 0 & 0 & 4 & 0 & 14 & 0 & 0 & 14 & 0 & 1 & 4 & 1 & 14 & 4 & 1 & 14 & 0 & 1 & 4 & 1 & 0\\
1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 4 & 14 & 1 & 1 & 2 & 2 & 14 & 4 & 1 & 1 & 2 & 2 & 0\\
2 & 4 & 2 & 14 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 14 & 2 & 2 & 2 & 2 & 2 & 2 & 14 & 2 & 14 & 2 & 2 & 2 & 14 & 0\\
2 & 2 & 2 & 14 & 2 & 2 & 2 & 14 & 2 & 14 & 2 & 2 & 14 & 2 & 2 & 1 & 14 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 14 & 2 & 0\\
2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 14 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}$$

Method Tail Cumulation

P=

$$
\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.13 & 0.12 & 0 & 0 \\
0 & 0 & 0 & 1.04 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.08 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.04 & 0 & 0 & 0 & 1.04 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 1.04 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.10 & 0 & -0.07 & 0 & 1.03 & 0.08 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.12 & 0 & 0 & 0 & -0.05 & 0 & 0 & 0 & 0.06 & 0 \\
0 & 0 & 0 & 0 & 0.10 & 0 & 0 & 0 & 0 & 1.07 & 0 & 0 & -1.04 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.07 & 0 & 1.03 & 0 & 0 & 0 & 0 & 1.04 & 0.06 & 1.05 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.07 & 0 & 0 & 0 & 0 & 0 & 0 & 0.07 & 0 & 0 \\
0.06 & 0 & 0.08 & 0.09 & 0 & 0.07 & 0.09 & 1.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0.06 & 0 & 0 & 0 \\
0 & 1.01 & 0 & 0 & 0.10 & 0 & 0 & 1.03 & 0 & 0 & 0 & 0 & 1.04 & 0 & 0.10 & 0 & 0 & 0 & 0 \\
0.07 & 0 & 0 & 0 & -1.04 & 0 & 0 & 0.07 & 0.11 & 1.04 & 0.09 & 0 & 1.05 & 0.08 & 0 & 0.14 & 0.07 & 1.03 & 0.12 & 0 \\
0 & 0 & 0 & 1.04 & 0 & 0.13 & 0 & 0.12 & 0 & 0.15 & 0 & 1.04 & 0 & 1.06 & 0 & 0 & 1.06 & 0.08 & 0 & 0.14 & 0 \\
0.12 & 0 & 0.12 & 0 & 1.04 & 0.08 & 0 & 0.13 & 0 & 0 & 0 & 0 & 0 & 0 & 1.02 & 0 & 0.07 & 0 & 1.13 & 0 & 1.05 & 0 \\
0.14 & 0.05 & 0 & 0.17 & 0 & 0 & 0 & 0.05 & 0.10 & 0 & 1.07 & 0.07 & 0 & 1.05 & 0.06 & 0.08 & 0 & 1.07 & 0 & 0.10 & 0 & 0.09 & 0 \\
0.14 & 0 & 0.09 & 0 & 1.06 & 0.06 & 0.10 & 0 & 0 & 0 & 1.02 & 0 & 0 & 1.06 & 1.03 & 0.07 & 0.18 & 1.03 & 0.07 & 0.10 & 1.08 & 0 & 0.10 & 0 \\
0.19 & 0.10 & 0 & 0 & 1.04 & -0.09 & 0 & 0 & 0.08 & 0.13 & 0.16 & 0 & 0 & 1.03 & 0.09 & 0 & 0.08 & 0 & 0.07 & 0.10 & 0.06 & 0 & 0 & 0 & 0 \\
1.09 & 0 & 0.09 & 0.08 & 0.11 & 0 & 0 & 1.07 & 0 & 1.08 & 0 & 0 & 1.05 & 0 & 0.19 & 1.04 & 0.14 & 1.09 & 1.05 & 0.12 & 1.05 & 0 & 0.09 & 1.03 & 0.07 & 0 \\
0.12 & 0.12 & 0 & 0.07 & 0 & 0.08 & 0.12 & 0.09 & 0.13 & 0.11 & 0.13 & 0.09 & 0.11 & 0.09 & 1.09 & 1.06 & 0.14 & 0.13 & 0.13 & 0.13 & 1.05 & 1.05 & 0.10 & 0.15 & 0.05 & 0.14 & 0 \\
0.11 & 1.05 & 0.20 & 1.07 & 0.17 & 0.11 & 0.09 & 0.15 & 0.23 & 0.11 & 0.15 & 0.17 & 0.13 & 1.04 & 0.17 & 0.10 & 0.27 & 0.16 & 0.29 & 0.14 & 1.07 & 0.12 & 1.06 & 0.17 & 0.29 & 0.18 & 1.08 & 0 \\
0.29 & 0.17 & 0.09 & 1.08 & 0.28 & 0.25 & 0.20 & 1.24 & 0.22 & 1.09 & 0.25 & 0.14 & 1.22 & 0.24 & 0.19 & 0.23 & 1.22 & 0.26 & 0.33 & 0.25 & 0.31 & 0.25 & 0.13 & 0.30 & 0.27 & 0.32 & 1.23 & 0.33 & 0 \\
0.45 & 0.76 & 0.40 & 0.35 & 0.50 & 0.71 & 0.73 & 0.60 & 0.49 & 0.50 & 0.61 & 0.53 & 0.68 & 0.43 & 0.53 & 1.40 & 0.64 & 0.50 & 0.54 & 0.47 & 0.58 & 0.51 & 0.33 & 0.54 & 0.49 & 0.73 & 0.59 & 0.63 & 0.65 & 0
\end{pmatrix}
$$

P2=

$$
\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25.95 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 21.20 & 0 & 0 & 0 & 26.17 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 22.02 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 22.52 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 27.63 & 0 & 0 & 18.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 29.01 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16.46 & 0 & 0 & 17.70 & 0 & 12.01 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 13.03 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
29.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 25.30 & 0 & 0 & 0 & 0 & 0 & 22.30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25.30 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 29.67 & 12.99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 28.74 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 23.78 & 0 & 16.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 19.37 & 0 & 0 & 0 & 20.11 & 0 & 20.86 & 0 & 0 & 25.01 & 12.57 & 0 & 0 & 0 & 0 & 19.79 & 18.86 & 0 & 0 & 0 & 0 & 22.03 & 19.24 & 0 \\
15.57 & 0 & 22.12 & 0 & 0 & 24.64 & 13.76 & 13.40 & 14.70 & 14.41 & 12.64 & 0 & 0 & 0 & 21.00 & 21.49 & 9.71 & 12.53 & 15.13 & 17.93 & 12.91 & 0 & 24.96 & 15.51 & 11.52 & 0 & 0 & 0 \\
11.44 & 11.64 & 17.06 & 0 & 8.78 & 9.47 & 17.00 & 0 & 16.95 & 0 & 10.68 & 15.16 & 0 & 14.54 & 9.27 & 0 & 0 & 13.16 & 12.35 & 22.77 & 10.22 & 14.84 & 18.32 & 10.87 & 8.96 & 8.33 & 0 & 11.87 & 0 \\
9.70 & 4.63 & 0 & 13.43 & 7.52 & 5.34 & 4.22 & 7.26 & 13.74 & 6.59 & 4.63 & 11.34 & 8.77 & 10.18 & 5.59 & 0 & 5.64 & 12.76 & 7.74 & 7.21 & 9.34 & 9.87 & 10.38 & 7.34 & 5.01 & 4.52 & 11.52 & 4.11 & 5.36 & 0
\end{pmatrix}
$$

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M=\begin{pmatrix}
8 \\
18 & 20 \\
17 & 18 & 26 \\
22 & 17 & 18 & 21 \\
24 & 22 & 17 & 18 & 1 \\
15 & 24 & 22 & 17 & 18 & 2 \\
19 & 15 & 24 & 22 & 17 & 18 & 18 \\
23 & 19 & 15 & 24 & 22 & 17 & 9 & 17 \\
5 & 23 & 19 & 15 & 24 & 22 & 10 & 9 & 22 \\
30 & 5 & 23 & 19 & 15 & 24 & 3 & 10 & 9 & 24 \\
25 & 30 & 5 & 23 & 19 & 15 & 29 & 3 & 10 & 9 & 15 \\
13 & 25 & 30 & 5 & 23 & 19 & 28 & 29 & 3 & 10 & 9 & 19 \\
16 & 13 & 25 & 30 & 5 & 23 & 4 & 28 & 29 & 3 & 10 & 9 & 23 \\
6 & 16 & 13 & 25 & 30 & 5 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 5 \\
7 & 6 & 16 & 13 & 25 & 30 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 30 \\
27 & 7 & 6 & 16 & 13 & 25 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 25 \\
21 & 27 & 7 & 6 & 16 & 13 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 13 \\
1 & 21 & 27 & 7 & 6 & 16 & 15 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 16 \\
20 & 1 & 21 & 27 & 7 & 6 & 19 & 15 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 6 \\
26 & 2 & 1 & 12 & 27 & 7 & 23 & 19 & 15 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 7 \\
12 & 9 & 2 & 11 & 12 & 27 & 16 & 23 & 19 & 15 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 27 \\
11 & 26 & 9 & 10 & 11 & 12 & 6 & 16 & 23 & 19 & 16 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 12 \\
10 & 12 & 10 & 2 & 10 & 11 & 13 & 6 & 16 & 23 & 6 & 16 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 9 & 9 \\
3 & 10 & 3 & 3 & 2 & 10 & 25 & 13 & 6 & 16 & 13 & 6 & 16 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 11 & 10 \\
29 & 3 & 29 & 29 & 3 & 3 & 24 & 25 & 13 & 6 & 25 & 13 & 6 & 16 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 10 & 11 & 3 \\
28 & 11 & 12 & 28 & 29 & 29 & 5 & 24 & 25 & 13 & 7 & 25 & 13 & 6 & 16 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 3 & 28 & 11 & 11 \\
9 & 28 & 11 & 4 & 28 & 28 & 7 & 5 & 24 & 25 & 5 & 7 & 25 & 13 & 6 & 16 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 4 & 28 & 29 & 28 \\
14 & 4 & 4 & 14 & 4 & 9 & 22 & 7 & 5 & 7 & 30 & 5 & 7 & 25 & 13 & 6 & 16 & 27 & 14 & 11 & 12 & 4 & 28 & 29 & 4 & 28 & 29 & 4 \\
4 & 29 & 28 & 1 & 14 & 14 & 17 & 22 & 7 & 5 & 23 & 30 & 5 & 7 & 25 & 13 & 6 & 7 & 27 & 14 & 11 & 14 & 4 & 3 & 29 & 4 & 4 & 29 & 29 \\
2 & 14 & 14 & 9 & 9 & 4 & 30 & 30 & 30 & 30 & 19 & 23 & 30 & 30 & 7 & 7 & 7 & 6 & 7 & 27 & 14 & 11 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14
\end{pmatrix}
$$

$$
F=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 0 \\
1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 2 & 0 & 0 \\
1 & 0 & 14 & 1 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 14 & 4 & 14 & 0 & 0 & 1 & 0 & 0 \\
0 & 14 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 4 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 14 & 1 & 14 & 0 & 0 \\
1 & 2 & 14 & 24 & 0 & 2 & 1 & 4 & 0 & 0 & 1 & 1 & 4 & 0 & 2 & 0 & 2 & 1 & 0 & 0 \\
0 & 1 & 1 & 4 & 0 & 14 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 14 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 4 & 2 & 2 & 2 & 1 & 1 & 2 & 0 & 0 & 2 & 4 & 0 & 1 & 2 & 1 & 14 & 2 & 14 & 0 \\
1 & 0 & 1 & 14 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 4 & 0 & 14 & 1 & 1 & 1 & 14 & 1 & 0 & 0 & 2 & 0 \\
2 & 14 & 1 & 14 & 1 & 2 & 4 & 14 & 2 & 0 & 0 & 2 & 0 & 1 & 0 & 14 & 2 & 0 & 1 & 1 & 1 & 4 & 1 & 0 \\
4 & 1 & 1 & 2 & 1 & 2 & 2 & 4 & 4 & 1 & 2 & 0 & 1 & 2 & 1 & 14 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 14 & 1 & 1 & 2 & 1 & 2 & 0 & 14 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 1 & 0 \\
1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 0 \\
4 & 2 & 2 & 14 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 14 & 2 & 2 & 4 & 2 & 2 & 1 & 2 & 2 & 2 & 0 \\
4 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 1 & 14 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 2 & 2 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 0
\end{pmatrix}
$$

Method Hu Dependence

$$
P=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0.09 & 0.17 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.09 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.10 & 0 & 0 & 0 \\
0 & 0 & 0.06 & 0.07 & 0.10 & 0 & 0.10 & 0.07 & 0 \\
0.22 & 0 & 0 & 0.09 & 0 & 0.39 & 0 & 0 & 0.08 & 0 \\
0 & 0 & 0 & 0 & 1.03 & 0 & 0 & 0 & 0 & 0.07 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.20 & 1.06 & 0 & 0 & 0.16 & 0 & 0 \\
0.08 & 0 & 1.01 & 0.08 & 0.09 & 0 & 0 & 0.09 & 0 & 0.14 & 0 & 0 & 0 \\
0.37 & 0 & 0 & 0 & 0 & 0 & 1.05 & 1.05 & 1.04 & 0 & 0 & 0.17 & 0 & 0 \\
0 & 1.03 & 0 & 0 & 0 & 0 & 0.10 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.06 & 0 & 0.10 & 0.06 & 0.08 & 0 & 0 & 0 & 0 \\
0.16 & 0 & -0.07 & 0.12 & 0 & 0 & 0 & 0.09 & 0.16 & 1.09 & 0 & 0.12 & 0.07 & 0 & 0 & 0 & 0 \\
0.10 & 0 & 0 & 0 & 0 & 0.07 & 0.08 & 0 & 0 & 0.09 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.11 & 0.19 & 0 & 0 & 0 & 0 & 0.15 & 1.07 & 0 & 1.04 & 0.09 & 1.02 & 0 & 0 \\
0.09 & 0.10 & 1.06 & -1.03 & 0 & 0.19 & 0.06 & 1.06 & 0 & 0 & 0.29 & 0.17 & 1.05 & 0 & 0.07 & 0 & 0.19 & 0.09 & 0 & 0 \\
0 & 0.09 & 0.14 & 1.06 & 0 & 1.04 & 0 & 0.07 & 0.11 & 0 & 0 & 0 & 0 & 1.08 & 0 & 0 & 1.03 & 0 & 0.21 & 0.12 & 0 \\
0.15 & 0 & 0 & 1.09 & 0.09 & 0.09 & 0.09 & 0.07 & 0.10 & 0.20 & 0 & 0 & 0.13 & 1.01 & 0 & 0.10 & 0.15 & 0.14 & 1.08 & 0.47 & 1.08 & 0 \\
0.20 & 0 & 0.20 & 1.07 & 0.12 & 0.10 & 0 & 0.12 & 0.11 & 0.07 & 0 & 1.05 & 0 & 1.07 & 0.08 & 0.13 & 0.25 & 1.04 & 0.08 & 0 & 0 & 0.05 & 0 \\
0.11 & 1.03 & 0.16 & 1.06 & 0.11 & 0.43 & 1.02 & 1.03 & 0.17 & 0 & 0 & 0.21 & 0 & 0.11 & 0 & 1.03 & 0.27 & 0 & 0.27 & 0.16 & 0.15 & 1.11 & 0.09 & 0 \\
1.10 & 0.23 & 0.12 & 0.24 & 0.11 & 0.21 & 0.15 & 1.08 & 1.04 & 0.20 & 0.10 & 0 & 0.14 & 0.11 & 0.11 & 1.02 & 0.19 & 0 & 0.17 & 0.28 & 0.31 & 0.06 & 0.20 & 0.22 & 0 \\
0.26 & 0.10 & 0.15 & 0.33 & 0.11 & 0.18 & 0.10 & 0.13 & 0.14 & 1.07 & 0.10 & 0.14 & 0.07 & 0.14 & 0.14 & 0 & 1.11 & 0.12 & 0.16 & 0.19 & 0.24 & 0 & 0.07 & 0.34 & 0.15 & 0 \\
0.21 & 0.25 & 0.10 & 0.21 & 0.27 & 0.44 & 0.25 & 0.20 & 0.18 & 0.24 & 0.19 & 0.37 & 0.12 & 0.09 & 0.35 & 0 & 0.20 & 0 & 0.25 & 0.18 & 0.25 & 0.25 & 0.18 & 0.18 & 0.32 & 0.22 & 0 \\
1.03 & 0.35 & 0.17 & 1.04 & 0.27 & 0.36 & 0.23 & 0.22 & 0.23 & 0.41 & 0.07 & 0.26 & 0.21 & 0.19 & 0.10 & 0.09 & 0.13 & 1.15 & 0.19 & 0.34 & 1.07 & 0.24 & 0.36 & 0.30 & 0.47 & 0.33 & 0.46 & 0 \\
1.15 & 0.47 & 0.51 & 0.32 & 1.12 & 0.15 & 0.24 & 0.37 & 0.36 & 0.27 & 0.23 & 1.29 & 0.22 & 0.31 & 0.13 & 0.71 & 0.21 & 0.28 & 0.19 & 0.23 & 0.26 & 0.30 & 0.33 & 0.56 & 0.53 & 0.31 & 0.46 & 0.38 & 0 \\
1.69 & 1.19 & 1.17 & 1.36 & 1.37 & 1.44 & 1.24 & 1.23 & 1.21 & 1.32 & 1.17 & 0.25 & 0.20 & 1.24 & 1.28 & 1.30 & 1.31 & 1.33 & 1.44 & 1.29 & 1.12 & 1.33 & 1.16 & 1.21 & 1.23 & 1.18 & 1.23 & 1.23 & 1.13 & 0
\end{pmatrix}
$$

$$
P2=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 25.17 & 0 & 0 & 15.04 & 0 \\
0 & 0 & 0 & 0 & 0 & 11.92 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 22.48 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 22.46 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27.47 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 25.67 & 0 & 28.17 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.48 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25.29 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 16.88 & 0 & 0 & 0 & 0 & 0 & 19.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 21.87 & 0 & 0 & 0 & 15.13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.93 & 0 & 24.98 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 28.81 & 19.86 & 16.73 & 0 & 0 & 14.22 & 0 & 0 & 21.03 & 0 & 0 & 0 & 19.81 & 0 & 0 & 11.93 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21.50 & 0 \\
26.15 & 0 & 0 & 0 & 0 & 11.72 & 0 & 0 & 15.05 & 0 & 0 & 19.31 & 0 & 0 & 0 & 0 & 16.23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 15.10 & 0 & 11.52 & 24.08 & 0 & 0 & 0 & 11.48 & 0 & 0 & 15.85 & 0 & 0 & 0 & 0 & 19.25 & 23.00 & 13.41 & 19.00 & 19.96 & 13.39 & 0 \\
13.38 & 0 & 20.14 & 12.36 & 0 & 22.92 & 28.59 & 19.09 & 0 & 0 & 0 & 0 & 18.42 & 0 & 26.18 & 0 & 0 & 22.13 & 13.82 & 26.74 & 10.88 & 0 & 20.66 & 13.45 & 0 & 0 \\
0 & 18.92 & 0 & 28.63 & 15.66 & 16.61 & 29.49 & 15.01 & 0 & 11.72 & 21.65 & 10.41 & 0 & 17.01 & 25.81 & 0 & 21.21 & 0 & 16.79 & 24.41 & 24.45 & 0 & 15.26 & 0 & 9.59 & 25.41 & 0 \\
0 & 10.37 & 13.57 & 0 & 20.68 & 7.29 & 0 & 0 & 27.29 & 12.30 & 0 & 12.06 & 0 & 9.30 & 0 & 0 & 0 & 0 & 25.39 & 14.10 & 0 & 12.12 & 10.99 & 0 & 7.10 & 12.09 & 8.59 & 0 \\
0 & 6.24 & 6.23 & 18.29 & 0 & 22.62 & 25.71 & 7.44 & 10.72 & 9.15 & 0 & 0 & 10.74 & 14.49 & 15.63 & 5.31 & 13.24 & 0 & 11.79 & 0 & 8.66 & 0 & 11.48 & 6.11 & 5.55 & 9.02 & 4.79 & 7.59 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7.66 & 10.81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
M=\begin{pmatrix}
5 \\
16 & 7 \\
20 & 16 & 26 \\
15 & 20 & 16 & 4 \\
7 & 15 & 20 & 16 & 30 \\
17 & 17 & 15 & 20 & 16 & 29 \\
6 & 6 & 17 & 15 & 20 & 16 & 18 \\
30 & 26 & 6 & 17 & 15 & 20 & 16 & 9 \\
12 & 4 & 30 & 6 & 17 & 15 & 20 & 16 & 3 \\
22 & 12 & 12 & 30 & 6 & 17 & 15 & 20 & 16 & 19 \\
23 & 22 & 22 & 12 & 12 & 6 & 17 & 15 & 20 & 16 & 8 \\
11 & 23 & 23 & 22 & 22 & 12 & 6 & 17 & 15 & 20 & 16 & 16 \\
18 & 11 & 11 & 23 & 23 & 22 & 12 & 6 & 17 & 15 & 20 & 28 & 20 \\
9 & 29 & 18 & 11 & 11 & 23 & 22 & 12 & 6 & 17 & 15 & 2 & 28 & 15 \\
1 & 1 & 9 & 18 & 29 & 11 & 23 & 22 & 12 & 6 & 17 & 27 & 2 & 28 & 17 \\
27 & 27 & 1 & 9 & 1 & 18 & 11 & 23 & 22 & 12 & 6 & 14 & 27 & 2 & 28 & 6 \\
14 & 14 & 27 & 1 & 27 & 9 & 1 & 11 & 23 & 22 & 12 & 21 & 14 & 27 & 2 & 28 & 12 \\
8 & 3 & 14 & 27 & 14 & 1 & 27 & 1 & 11 & 23 & 22 & 12 & 21 & 14 & 27 & 2 & 28 & 22 \\
21 & 19 & 8 & 14 & 3 & 27 & 14 & 27 & 1 & 11 & 23 & 22 & 12 & 21 & 14 & 27 & 2 & 28 & 23 \\
13 & 28 & 21 & 8 & 19 & 14 & 3 & 14 & 27 & 1 & 11 & 13 & 22 & 12 & 21 & 14 & 27 & 2 & 28 & 11 \\
25 & 21 & 13 & 21 & 28 & 8 & 19 & 3 & 14 & 27 & 1 & 25 & 13 & 22 & 12 & 21 & 14 & 27 & 2 & 28 & 1 \\
29 & 13 & 25 & 13 & 21 & 21 & 28 & 19 & 8 & 14 & 27 & 24 & 25 & 13 & 22 & 12 & 21 & 14 & 27 & 2 & 28 & 28 \\
3 & 25 & 29 & 25 & 13 & 13 & 21 & 28 & 21 & 8 & 14 & 10 & 24 & 25 & 13 & 22 & 13 & 21 & 14 & 27 & 2 & 27 & 2 \\
2 & 30 & 3 & 29 & 25 & 25 & 13 & 21 & 13 & 21 & 28 & 1 & 10 & 24 & 25 & 13 & 25 & 13 & 21 & 14 & 27 & 14 & 27 & 27 \\
24 & 18 & 2 & 3 & 24 & 19 & 25 & 13 & 25 & 13 & 21 & 11 & 1 & 10 & 24 & 25 & 23 & 25 & 13 & 21 & 14 & 21 & 14 & 10 & 14 \\
19 & 24 & 24 & 2 & 10 & 28 & 24 & 25 & 19 & 25 & 13 & 23 & 11 & 1 & 10 & 24 & 11 & 23 & 25 & 13 & 21 & 13 & 21 & 24 & 10 & 21 \\
10 & 9 & 19 & 24 & 2 & 2 & 10 & 24 & 28 & 2 & 25 & 6 & 23 & 11 & 1 & 10 & 24 & 11 & 24 & 25 & 13 & 25 & 13 & 25 & 24 & 10 & 10 \\
28 & 8 & 10 & 19 & 8 & 10 & 2 & 10 & 2 & 24 & 24 & 15 & 6 & 23 & 11 & 1 & 10 & 24 & 10 & 24 & 25 & 2 & 25 & 13 & 25 & 24 & 13 & 13 \\
4 & 2 & 28 & 10 & 9 & 24 & 8 & 2 & 10 & 10 & 10 & 17 & 15 & 6 & 23 & 11 & 1 & 10 & 1 & 10 & 24 & 24 & 24 & 14 & 13 & 25 & 25 & 24 & 25 \\
26 & 10 & 4 & 28 & 18 & 3 & 9 & 8 & 24 & 28 & 2 & 20 & 17 & 17 & 6 & 23 & 22 & 1 & 11 & 1 & 10 & 10 & 10 & 21 & 21 & 13 & 24 & 25 & 24 & 24
\end{pmatrix}
$$

$$
F=\begin{pmatrix}
0 \\
2 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 4 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 4 & 0 & 2 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 0 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 14 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 14 & 1 & 0 & 0 \\
2 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 1 & 0 & 1 & 0 & 1 & 0 \\
14 & 4 & 0 & 1 & 1 & 14 & 14 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 14 & 0 \\
4 & 0 & 0 & 4 & 0 & 0 & 0 & 14 & 0 & 0 & 1 & 0 & 14 & 1 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 1 & 0 & 2 & 0 & 14 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 14 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 4 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 4 & 0 \\
0 & 0 & 14 & 1 & 1 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 24 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 14 & 1 & 0 & 2 & 1 & 2 & 0 & 2 & 4 & 1 & 2 & 0 & 0 & 2 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 14 & 1 & 14 & 2 & 1 & 1 & 1 & 4 & 0 & 4 & 0 & 1 & 1 & 1 & 2 & 1 & 0 \\
0 & 2 & 4 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 4 & 0 & 4 & 1 & 0 & 1 & 4 & 2 & 4 & 2 & 0 \\
2 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 4 & 2 & 1 & 1 & 1 & 14 & 1 & 14 & 4 & 14 & 1 & 14 & 0 \\
2 & 1 & 1 & 2 & 1 & 4 & 1 & 1 & 2 & 1 & 1 & 14 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 0 & 1 & 14 & 2 & 0 \\
2 & 2 & 1 & 4 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 14 & 1 & 2 & 2 & 0 \\
1 & 2 & 2 & 2 & 1 & 4 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 14 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 14 & 14 & 4 & 2 & 2 & 0 \\
14 & 2 & 2 & 2 & 14 & 14 & 2 & 2 & 14 & 2 & 14 & 1 & 14 & 14 & 14 & 2 & 14 & 2 & 14 & 2 & 2 & 2 & 2 & 14 & 14 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}
$$

Method Exceedance Dependence

P=

$$
\begin{pmatrix}
0 \\
0.05 & 0 \\
0 & 0 & 0 \\
0.13 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0.13 & 0 & 0 & 0.13 & 1.03 & 0 \\
0 & 0 & 0.06 & 0 & 0 & 0.10 & 0 \\
0.09 & -0.10 & 0 & 1.04 & 0 & 0.09 & -0.07 & 0 \\
0.08 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.12 & -0.06 & 1.04 & 0 & 0.14 & 0 & 0 & 0 & 0 & 0 \\
0.08 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0.23 & 0 & 0 \\
0 & 0 & 0 & 0.08 & 0 & 0 & 0 & 0 & 0 & 0 & 0.09 & 0 \\
1.03 & 0 & 0 & 0.13 & 0 & 0 & 1.04 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.17 & 0 & 0 & 0 & 0 & 0 & 0.07 & 0.09 & 0 & 0.09 & 1.08 & 0 \\
0.10 & 0 & 0 & 0 & 0 & 0.08 & 0.06 & 0.08 & 0 & 0 & 0 & 1.11 & 0.13 & 0 & 0 \\
0.08 & 0 & 1.06 & 0 & 0 & 0 & 0 & 0 & 0.06 & 1.07 & 0.38 & 0 & 0.17 & 0 & 0.09 & 0 \\
1.06 & 1.04 & 0 & 0.08 & 0.08 & 1.04 & 1.04 & 0 & 0 & 0.07 & 0 & 0.08 & 0 & 0 & 0 & 1.11 & 0 \\
1.03 & 0 & 0 & 1.03 & 0 & 0 & 0 & 1.07 & 0 & 0 & 0.07 & 0 & 1.07 & 0.30 & 0.07 & 0.13 & 0.11 & 0 \\
0 & 0.07 & 0.08 & 0.10 & 0 & 0.25 & 0 & 1.08 & 0.07 & 0.10 & 0 & 0.08 & 0 & 0.09 & 0.08 & 0 & 0 & 1.05 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.06 & 0 & 0 & 0.14 & 1.04 & 0.14 & 0 & 0 & 0 & 0.09 & 0.11 & 0 & 0.11 & 0 & 0 \\
0.08 & 0.06 & 0 & 0 & 0 & 1.07 & 0.13 & 0 & 0.07 & 0 & 0.13 & 0 & 0.08 & 0 & 0.09 & 0.19 & 0.16 & 0 & 0 & 1.04 & 0 \\
0 & 0 & 1.02 & 0.10 & 0.09 & 0.19 & 0.11 & 0 & 0.09 & 0 & 0 & 0 & 0.14 & 0 & 0.16 & 0.04 & -1.04 & 0 & 0 & 0.08 & 0.13 & 0 \\
0 & 0 & 0 & 0.16 & 0 & 0 & 0.12 & 0.15 & 1.07 & 0.11 & 0 & 0.15 & 0.15 & 0.09 & 0 & 0.12 & 1.05 & 0.09 & 0.15 & 0 & 0 & 0.11 & 0 \\
0.08 & 0 & 0.11 & 0 & 0 & 0 & 0 & 1.06 & 0.19 & 1.09 & 0.19 & 0.12 & 0.16 & 0.06 & 1.07 & 0 & 1.12 & 0 & 0.09 & 0.06 & 0.10 & 0.06 & 0.19 & 0 \\
0 & 0.05 & 1.07 & 0.35 & 0 & 0.11 & 0 & 0 & 0.27 & 0 & 0.25 & 0.14 & 0.14 & 0.09 & 1.03 & 0 & 1.07 & 0.08 & 0 & 0.18 & 1.05 & 0.24 & 1.03 & 0.11 & 0 \\
0.12 & 0 & 0.08 & 0 & 0.08 & 0.23 & 0 & 0.10 & 0 & 0 & 0 & 0 & 0.16 & 1.05 & 0.15 & 0.14 & 0.14 & 0.12 & 1.04 & 0.07 & 1.12 & 1.08 & 1.07 & 0.12 & 1.04 & 0 \\
0.20 & 0.12 & 0.13 & 0.13 & 0.12 & 1.06 & 0.17 & 0.07 & 0.31 & 0.20 & 0.14 & 1.15 & 0.14 & 0 & 0.20 & 0.39 & 0.18 & 0.15 & 0.11 & 0.14 & 0.10 & 0.19 & 0 & 0.06 & 1.07 & 0.38 & 0 \\
0.19 & 0.08 & 0.29 & 1.02 & 0.43 & 0.25 & 0.27 & 0.20 & 0.19 & 0.17 & 0.13 & 0.12 & 0.33 & 0.19 & 0.22 & 0.23 & 0.34 & 0.13 & 0.21 & 0.13 & 0.12 & 0.23 & 0.09 & 1.14 & 0.10 & 0.29 & 0.16 & 0 \\
0.25 & 0.25 & 0.42 & 0.24 & 0.23 & 1.14 & 0.34 & 0.29 & 0.40 & 0.39 & 0.34 & 0.22 & 0.15 & 1.12 & 0.18 & 0.41 & 0.24 & 0.35 & 0.20 & 0.41 & 0.19 & 0.25 & 0.52 & 1.06 & 1.16 & 1.06 & 0.29 & 0.18 & 0 \\
1.24 & 0.71 & 0.42 & 0.51 & 1.24 & 1.58 & 0.33 & 0.46 & 1.55 & 0.53 & 1.69 & 0.37 & 1.38 & 1.17 & 1.36 & 0.31 & 1.26 & 0.42 & 1.16 & 0.41 & 0.50 & 0.65 & 0.73 & 1.30 & 1.14 & 0.40 & 0.62 & 0.76 & 0.43 & 0
\end{pmatrix}
$$

P2=

$$
\begin{pmatrix}
0 \\
14.17 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
21.79 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 13.76 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 21.32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15.98 & 0 & 0 & 17.76 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 25.32 & 0 & 0 & 0 & 0 & 0 & 0 & 14.50 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
26.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 29.48 & 0 & 0 & 0 & 17.66 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 28.02 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 23.57 & 0 & 0 & 0 & 0 & 0 \\
17.30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 20.68 & 0 & 0 & 24.45 & 0 & 0 & 0 & 0 & 0 & 20.41 & 23.94 & 0 & 0 & 0 & 0 & 20.66 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13.55 & 0 & 23.53 & 0 & 13.58 & 0 & 0 & 26.81 & 0 & 0 & 23.26 & 0 \\
21.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 17.73 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20.31 & 0 & 0 \\
0 & 19.78 & 0 & 11.58 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.03 & 21.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13.41 & 0 & 27 & 0 \\
12.84 & 0 & 0 & 0 & 0 & 18.59 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.58 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14.50 & 0 & 0 & 14.78 & 0 & 0 & 0 & 0 & 9.26 & 0 & 0 & 0 & 0 & 0 & 10.85 & 16.67 & 16.13 & 28.54 & 0 & 0 & 0 & 15.31 & 0 & 0 & 0 & 14.71 & 0 \\
10.64 & 20.57 & 0 & 0 & 0 & 0 & 0 & 0 & 9.19 & 10.77 & 21.57 & 0 & 15.58 & 0 & 9.73 & 20.31 & 10.81 & 0 & 0 & 16.55 & 22.11 & 20.21 & 8.89 & 0 & 0 & 26.84 & 15.76 & 0 \\
0 & 9.47 & 9.24 & 17.93 & 0 & 0 & 10.71 & 8.94 & 8.96 & 12.33 & 14.89 & 0 & 17.87 & 0 & 0 & 9.21 & 14.21 & 13.74 & 11.52 & 8.33 & 13.77 & 9.72 & 5.90 & 0 & 0 & 0 & 8.75 & 12.09 & 0 \\
0 & 5.34 & 5.72 & 4.02 & 0 & 0 & 15.00 & 11.53 & 0 & 5.55 & 0 & 0 & 0 & 0 & 0 & 9.81 & 0 & 7.66 & 0 & 7.66 & 12.76 & 5.36 & 4.52 & 0 & 0 & 8.35 & 6.51 & 4.63 & 6.39 & 0
\end{pmatrix}
$$

Method Kendalls Tau

$$
\tau=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.04 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.01 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.09 & 0 & 0.04 & 0 & 0 & 0 & 0.04 & 0 \\
0 & 0 & 0 & 0 & 0.05 & 0 & 0.01 & 0.07 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03 & 0 & 0.05 & 0 \\
-0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.07 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.03 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.09 & 0.04 & 0 & 0 & 0.06 & 0.05 & 0 & 0 & 0 & 0.05 & 0.05 & 0.05 & 0 & 0 & 0 \\
0.06 & 0.05 & 0 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.04 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.09 & 0 & 0.04 & 0.05 & 0 & 0.09 & 0 & -0.04 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03 & 0 & 0 & 0.04 & 0.09 & 0.07 & 0 & 0 \\
0.04 & 0.07 & 0 & 0 & 0 & 0 & 0 & 0 & 0.03 & 0.05 & 0.05 & 0.05 & 0.05 & 0 & 0 & 0 & -0.04 & -0.05 & 0 \\
0.04 & 0 & 0 & -0.04 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0.06 & 0 & 0 & 0 & 0.03 & 0.03 & 0 & 0.06 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0 & 0.05 & 0 & 0.04 & 0 & 0 & 0 & 0.04 & 0.04 & 0.05 & 0 & 0 \\
0.05 & 0 & 0 & 0 & 0.06 & 0 & 0 & 0 & -0.04 & 0 & 0 & 0.06 & 0 & 0.03 & 0 & 0 & 0 & 0 & 0.08 & 0 & 0.05 & 0 \\
0 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0.04 & 0 & 0 & 0 & 0.05 & 0 & 0.04 & 0.05 & 0 & 0.10 & 0.04 & 0.04 & 0 \\
0.03 & 0.05 & 0 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.07 & 0 & 0.06 & 0 & 0.06 & 0 & 0.03 & 0.06 & 0.05 & 0.07 & 0.06 & 0 & 0 \\
0 & 0.04 & 0.03 & 0 & 0 & 0 & 0.07 & 0 & 0.04 & 0.04 & 0 & 0.06 & 0 & 0.06 & 0.04 & 0 & 0.09 & 0.03 & 0.01 & 0.06 & 0.07 & 0.06 & 0.02 & 0 & 0 \\
0.07 & 0.06 & 0.06 & 0.03 & 0.08 & 0 & 0 & 0.02 & 0.05 & 0.05 & 0.05 & 0.03 & 0.03 & 0.04 & 0.09 & 0.06 & 0.07 & 0.08 & 0.06 & 0.08 & 0 & 0.07 & 0.08 & 0.07 & 0.13 & 0 \\
0.07 & 0.08 & 0.11 & 0.05 & 0 & 0.03 & 0.06 & 0.07 & 0.06 & 0.08 & 0.04 & 0.13 & 0.05 & 0.11 & 0.06 & 0.06 & 0.04 & 0.10 & 0.05 & 0.07 & 0.07 & 0.07 & 0.09 & 0.08 & 0.05 & 0 & 0 \\
0.13 & 0.10 & 0.08 & 0.12 & 0.10 & 0.06 & 0.05 & 0.13 & 0.10 & 0.06 & 0.06 & 0.15 & 0.07 & 0.10 & 0.11 & 0.06 & 0.11 & 0.09 & 0.09 & 0.12 & 0.11 & 0.09 & 0.14 & 0.11 & 0.08 & 0.09 & 0.10 & 0 \\
0.20 & 0.16 & 0.13 & 0.12 & 0.16 & 0.13 & 0.11 & 0.06 & 0.18 & 0.08 & 0.09 & 0.16 & 0.16 & 0.14 & 0.19 & 0.13 & 0.20 & 0.18 & 0.14 & 0.15 & 0.16 & 0.16 & 0.22 & 0.21 & 0.13 & 0.15 & 0.22 & 0.14 & 0 \\
0.39 & 0.34 & 0.30 & 0.34 & 0.42 & 0.35 & 0.55 & 0.26 & 0.48 & 0.23 & 0.20 & 0.37 & 0.50 & 0.33 & 0.40 & 0.52 & 0.40 & 0.42 & 0.28 & 0.32 & 0.33 & 0.35 & 0.44 & 0.52 & 0.33 & 0.37 & 0.43 & 0.45 & 0.3 & 0
\end{pmatrix}
$$

Method Tail Cumulation

$$
\tau=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.08 & 0.07 & 0 & 0 \\
0 & 0 & 0 & 0.04 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.08 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.04 & 0 & 0 & 0 & 0.04 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.04 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.06 & 0 & -0.04 & 0 & 0.03 & 0.05 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0.08 & 0 & 0 & 0 & -0.03 & 0 & 0 & 0 & 0.04 & 0 \\
0 & 0 & 0 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0.06 & 0 & 0 & -0.04 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.04 & 0 & 0.03 & 0 & 0 & 0 & 0 & 0.04 & 0.04 & 0.05 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0.04 & 0 & 0 & 0 & 0.04 & 0 & 0 & 0 \\
0.04 & 0 & 0.05 & 0.06 & 0 & 0.04 & 0.06 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0.04 & 0 & 0 & 0 & 0 \\
0 & 0.01 & 0 & 0 & 0.06 & 0 & 0 & 0.03 & 0 & 0 & 0 & 0 & 0.04 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0 \\
0.05 & 0 & 0 & 0 & -0.04 & 0 & 0 & 0.04 & 0.07 & 0.04 & 0.06 & 0 & 0.05 & 0.05 & 0 & 0.09 & 0.05 & 0.03 & 0.08 & 0 & 0 \\
0 & 0 & 0 & 0.04 & 0 & 0.08 & 0 & 0.07 & 0 & 0.09 & 0 & 0.04 & 0 & 0.05 & 0 & 0 & 0.05 & 0.05 & 0 & 0.09 & 0 & 0 \\
0.08 & 0 & 0.08 & 0 & 0.04 & 0.05 & 0 & 0.08 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 & 0 & 0.05 & 0 & 0.12 & 0 & 0.05 & 0 & 0 & 0 \\
0.09 & 0.03 & 0 & 0.11 & 0 & 0 & 0 & 0.03 & 0.06 & 0 & 0.06 & 0.05 & 0 & 0.05 & 0.04 & 0.05 & 0 & 0.07 & 0 & 0.06 & 0 & 0.06 & 0 & 0 & 0 & 0 \\
0.09 & 0 & 0.06 & 0 & 0.06 & 0.04 & 0.06 & 0 & 0 & 0 & 0.02 & 0 & 0 & 0.06 & 0.03 & 0.04 & 0.12 & 0.03 & 0.04 & 0.07 & 0.08 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0 \\
0.12 & 0.06 & 0 & 0 & 0.04 & -0.06 & 0 & 0 & 0.05 & 0.08 & 0.10 & 0 & 0 & 0.03 & 0.05 & 0 & 0.05 & 0 & 0.04 & 0.07 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0.08 & 0 & 0.06 & 0.05 & 0.07 & 0 & 0 & 0.07 & 0 & 0.07 & 0 & 0 & 0.05 & 0 & 0.12 & 0.04 & 0.09 & 0.08 & 0.04 & 0.08 & 0.05 & 0 & 0.06 & 0.03 & 0.05 & 0 & 0 & 0 & 0 & 0 \\
0.08 & 0.08 & 0 & 0.05 & 0 & 0.05 & 0.08 & 0.06 & 0.08 & 0.07 & 0.08 & 0.06 & 0.07 & 0.06 & 0.08 & 0.05 & 0.09 & 0.09 & 0.09 & 0.09 & 0.05 & 0.04 & 0.07 & 0.09 & 0.03 & 0.09 & 0 & 0 & 0 & 0 \\
0.07 & 0.04 & 0.13 & 0.07 & 0.11 & 0.07 & 0.06 & 0.09 & 0.15 & 0.07 & 0.10 & 0.11 & 0.08 & 0.04 & 0.11 & 0.06 & 0.17 & 0.11 & 0.19 & 0.09 & 0.07 & 0.08 & 0.05 & 0.11 & 0.19 & 0.11 & 0.07 & 0 & 0 & 0 \\
0.19 & 0.11 & 0.06 & 0.08 & 0.18 & 0.16 & 0.13 & 0.20 & 0.14 & 0.08 & 0.16 & 0.09 & 0.18 & 0.16 & 0.12 & 0.15 & 0.18 & 0.16 & 0.21 & 0.16 & 0.20 & 0.16 & 0.08 & 0.19 & 0.18 & 0.21 & 0.19 & 0.22 & 0 & 0 \\
0.30 & 0.55 & 0.26 & 0.23 & 0.33 & 0.50 & 0.52 & 0.41 & 0.33 & 0.34 & 0.42 & 0.35 & 0.48 & 0.28 & 0.35 & 0.28 & 0.44 & 0.33 & 0.36 & 0.31 & 0.39 & 0.34 & 0.21 & 0.36 & 0.32 & 0.52 & 0.40 & 0.43 & 0.45 & 0
\end{pmatrix}
$$

Method Hu Dependence

$$\tau=\begin{pmatrix}
0 \\
0 & 0 \\
0 & 0 & 0 \\
0 & 0.06 & 0.11 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.06 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.07 & 0 & 0 & 0 \\
0 & 0 & 0.04 & 0.04 & 0.06 & 0 & 0.06 & 0.05 & 0 \\
0.14 & 0 & 0 & 0.06 & 0 & 0.26 & 0 & 0 & 0.05 & 0 \\
0 & 0 & 0 & 0 & 0.03 & 0 & 0 & 0 & 0 & 0.05 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.13 & 0.06 & 0 & 0 & 0.10 & 0 & 0 \\
0.05 & 0 & 0.01 & 0.05 & 0.06 & 0 & 0 & 0.06 & 0 & 0.09 & 0 & 0 & 0 \\
0.24 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.05 & 0.04 & 0 & 0 & 0.11 & 0 & 0 \\
0 & 0.03 & 0 & 0 & 0 & 0 & 0.06 & 0 & 0.06 & 0.10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.04 & 0 & 0.07 & 0.04 & 0.05 & 0 & 0 & 0 & 0 \\
0.10 & 0 & -0.05 & 0.08 & 0 & 0 & 0 & 0.06 & 0.10 & 0.08 & 0 & 0.07 & 0.05 & 0 & 0 & 0 & 0 \\
0.07 & 0 & 0 & 0 & 0 & 0.05 & 0.05 & 0 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.07 & 0.12 & 0 & 0 & 0 & 0 & 0.10 & 0.07 & 0 & 0.04 & 0.05 & 0.02 & 0 & 0 \\
0.06 & 0.07 & 0.05 & -0.03 & 0 & 0.12 & 0.04 & 0.06 & 0 & 0 & 0.19 & 0.11 & 0.04 & 0 & 0.04 & 0 & 0.12 & 0.06 & 0 & 0 \\
0 & 0.05 & 0.09 & 0.05 & 0 & 0.04 & 0 & 0.04 & 0.07 & 0 & 0 & 0 & 0 & 0.07 & 0 & 0 & 0.03 & 0 & 0.13 & 0.08 & 0 \\
0.10 & 0 & 0 & 0.08 & 0.06 & 0.06 & 0.06 & 0.05 & 0.06 & 0.13 & 0 & 0 & 0.08 & 0.01 & 0 & 0.07 & 0.10 & 0.09 & 0.08 & 0.31 & 0.07 & 0 \\
0.13 & 0 & 0.13 & 0.07 & 0.07 & 0.06 & 0 & 0.08 & 0.07 & 0.05 & 0 & 0.04 & 0 & 0.07 & 0.05 & 0.09 & 0.16 & 0.04 & 0.05 & 0 & 0 & 0.03 & 0 \\
0.07 & 0.03 & 0.10 & 0.06 & 0.07 & 0.28 & 0.02 & 0.03 & 0.11 & 0 & 0 & 0.13 & 0 & 0.07 & 0 & 0.03 & 0.17 & 0 & 0.18 & 0.10 & 0.10 & 0.10 & 0.06 & 0 \\
0.09 & 0.15 & 0.08 & 0.16 & 0.07 & 0.13 & 0.09 & 0.07 & 0.04 & 0.13 & 0.06 & 0 & 0.09 & 0.07 & 0.07 & 0.02 & 0.12 & 0 & 0.11 & 0.18 & 0.20 & 0.04 & 0.13 & 0.14 & 0 \\
0.17 & 0.06 & 0.10 & 0.21 & 0.07 & 0.11 & 0.06 & 0.08 & 0.09 & 0.06 & 0.06 & 0.09 & 0.04 & 0.09 & 0.09 & 0 & 0.10 & 0.08 & 0.10 & 0.12 & 0.15 & 0 & 0.04 & 0.22 & 0.09 & 0 \\
0.13 & 0.16 & 0.06 & 0.14 & 0.17 & 0.29 & 0.16 & 0.13 & 0.12 & 0.15 & 0.12 & 0.24 & 0.08 & 0.06 & 0.23 & 0 & 0.13 & 0 & 0.16 & 0.11 & 0.16 & 0.16 & 0.12 & 0.12 & 0.20 & 0.14 & 0 \\
0.03 & 0.23 & 0.11 & 0.04 & 0.17 & 0.23 & 0.15 & 0.14 & 0.15 & 0.27 & 0.04 & 0.17 & 0.14 & 0.12 & 0.07 & 0.06 & 0.08 & 0.13 & 0.12 & 0.22 & 0.06 & 0.16 & 0.24 & 0.20 & 0.31 & 0.22 & 0.30 & 0 \\
0.13 & 0.31 & 0.34 & 0.20 & 0.10 & 0.10 & 0.16 & 0.24 & 0.23 & 0.18 & 0.15 & 0.22 & 0.14 & 0.20 & 0.09 & 0.51 & 0.14 & 0.18 & 0.12 & 0.15 & 0.17 & 0.19 & 0.21 & 0.38 & 0.36 & 0.20 & 0.30 & 0.25 & 0 \\
0.41 & 0.16 & 0.15 & 0.27 & 0.27 & 0.31 & 0.19 & 0.19 & 0.17 & 0.24 & 0.15 & 0.16 & 0.13 & 0.19 & 0.22 & 0.23 & 0.24 & 0.25 & 0.31 & 0.23 & 0.10 & 0.25 & 0.14 & 0.17 & 0.19 & 0.15 & 0.19 & 0.19 & 0.11 & 0
\end{pmatrix}$$

Method Exceedance Dependence

$$\tau=\begin{pmatrix}
0 \\
0.03 & 0 \\
0 & 0 & 0 \\
0.08 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0.08 & 0 & 0 & 0.08 & 0.03 & 0 \\
0 & 0 & 0.04 & 0 & 0 & 0.07 & 0 \\
0.06 & -0.06 & 0 & 0.04 & 0 & 0.06 & -0.04 & 0 \\
0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.08 & -0.04 & 0.03 & 0 & 0.09 & 0 & 0 & 0 & 0 & 0 \\
0.05 & 0 & 0 & 0.03 & 0 & 0 & 0 & 0 & 0.15 & 0 & 0 \\
0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0.06 & 0 \\
0.03 & 0 & 0 & 0.08 & 0 & 0 & 0.04 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.11 & 0 & 0 & 0 & 0 & 0 & 0.04 & 0.06 & 0 & 0.06 & 0.07 & 0 \\
0.06 & 0 & 0 & 0 & 0 & 0.05 & 0.04 & 0.05 & 0 & 0 & 0 & 0.10 & 0.08 & 0 & 0 \\
0.05 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0.04 & 0.06 & 0.25 & 0 & 0.11 & 0 & 0.06 & 0 \\
0.05 & 0.04 & 0 & 0.05 & 0.05 & 0.04 & 0.04 & 0 & 0 & 0.05 & 0 & 0.05 & 0 & 0 & 0 & 0.10 & 0 \\
0.03 & 0 & 0 & 0.03 & 0 & 0 & 0 & 0.06 & 0 & 0 & 0.05 & 0 & 0.06 & 0.20 & 0.04 & 0.09 & 0.07 & 0 \\
0 & 0.04 & 0.05 & 0.06 & 0 & 0.16 & 0 & 0.07 & 0.04 & 0.06 & 0 & 0.05 & 0 & 0.06 & 0.05 & 0 & 0 & 0.05 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.04 & 0 & 0 & 0.09 & 0.03 & 0.09 & 0 & 0 & 0 & 0.06 & 0.07 & 0 & 0.07 & 0 & 0 \\
0.05 & 0.04 & 0 & 0 & 0 & 0.06 & 0.09 & 0 & 0.05 & 0 & 0.08 & 0 & 0.05 & 0 & 0.06 & 0.12 & 0.10 & 0 & 0 & 0.04 & 0 \\
0 & 0 & 0.02 & 0.07 & 0.06 & 0.12 & 0.07 & 0 & 0.06 & 0 & 0 & 0 & 0.09 & 0 & 0.10 & 0.03 & -0.04 & 0 & 0 & 0.05 & 0.09 & 0 \\
0 & 0 & 0 & 0.10 & 0 & 0 & 0.08 & 0.10 & 0.06 & 0.07 & 0 & 0.10 & 0.10 & 0.05 & 0 & 0.07 & 0.05 & 0.06 & 0.10 & 0 & 0 & 0.07 & 0 \\
0.05 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0.05 & 0.12 & 0.08 & 0.12 & 0.08 & 0.10 & 0.04 & 0.07 & 0 & 0.11 & 0 & 0.06 & 0.04 & 0.07 & 0.04 & 0.12 & 0 \\
0 & 0.03 & 0.06 & 0.23 & 0 & 0.07 & 0 & 0 & 0.17 & 0 & 0.16 & 0.09 & 0.09 & 0.06 & 0.03 & 0 & 0.06 & 0.05 & 0 & 0.12 & 0.05 & 0.16 & 0.03 & 0.07 & 0 \\
0.08 & 0 & 0.05 & 0 & 0.05 & 0.15 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0.10 & 0.04 & 0.10 & 0.09 & 0.09 & 0.07 & 0.04 & 0.05 & 0.11 & 0.07 & 0.06 & 0.07 & 0.04 & 0 \\
0.13 & 0.08 & 0.08 & 0.09 & 0.08 & 0.06 & 0.11 & 0.05 & 0.20 & 0.13 & 0.09 & 0.13 & 0.09 & 0 & 0.13 & 0.25 & 0.12 & 0.10 & 0.07 & 0.09 & 0.07 & 0.12 & 0 & 0.04 & 0.07 & 0.25 & 0 \\
0.12 & 0.05 & 0.19 & 0.02 & 0.28 & 0.16 & 0.18 & 0.13 & 0.12 & 0.11 & 0.08 & 0.08 & 0.21 & 0.12 & 0.14 & 0.15 & 0.22 & 0.08 & 0.13 & 0.08 & 0.08 & 0.15 & 0.06 & 0.12 & 0.07 & 0.19 & 0.10 & 0 \\
0.16 & 0.16 & 0.27 & 0.16 & 0.15 & 0.12 & 0.22 & 0.19 & 0.26 & 0.26 & 0.22 & 0.14 & 0.09 & 0.11 & 0.11 & 0.27 & 0.16 & 0.23 & 0.13 & 0.27 & 0.12 & 0.16 & 0.35 & 0.06 & 0.14 & 0.05 & 0.19 & 0.12 & 0 \\
0.19 & 0.50 & 0.27 & 0.34 & 0.19 & 0.37 & 0.21 & 0.30 & 0.35 & 0.36 & 0.41 & 0.24 & 0.28 & 0.14 & 0.26 & 0.20 & 0.20 & 0.28 & 0.14 & 0.27 & 0.33 & 0.45 & 0.52 & 0.23 & 0.13 & 0.26 & 0.43 & 0.55 & 0.28 & 0
\end{pmatrix}$$

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