# Energy Efficiency Optimization in MIMO Broadcast Channels with Fairness Constraints

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***Abstract*—We consider the minimization of the energy per bit (or, equivalently, the maximization of the energy efficiency) in multiple-input multiple-output (MIMO) broadcast channels with rate balancing constraints for fairness between users, and we propose a framework to find the globally optimal energy-efficient solution, which relies on dirty paper coding (DPC). The main idea is to introduce a continuous, differentiable, and concave optimal rate function, which transforms the problem into a convex-concave fractional program in one scalar variable. This not only allows an efficient solution, but also an intuitive interpretation of the method as well as the possibility to extend the method to related problems. In numerical simulations, we compare the energy efficiency with and without fairness constraints.**

***Index Terms*—broadcast channels, energy efficiency, fairness, multiple-input multiple-output (MIMO), rate balancing.**

## I. Energy-Efficient Communications

Aiming at reducing energy costs and designing environmentally friendly communication systems, energy efficiency has become an important design criterion (e.g., [1], [2]). Potential to increase the energy efficiency has been identified on all abstraction layers of communication systems (e.g., [1]).

In this work, we want to focus on the particularities of multiuser multiple-input multiple-output (MIMO) communication systems instead of performing a detailed study of the energy efficiency on the circuit level. However, relying only on a Shannon-rate perspective (dashed curve in Fig. 1) is not a sufficiently accurate model for energy efficiency considerations. Due to the sublinear growth of the rate as a function of the transmit power, the most energy-efficient strategy from such an abstract point of view would always be to transmit with vanishing transmit power and data rate [1], which is unrealistic. Not only would it lead to infinitely long delays, but it would also require the devices to be switched on during an infinite amount of time so that the circuit power consumption, which is present in addition to the transmit power in a more accurate model, would lead to a prohibitive amount of consumed energy.

Therefore, a term modeling the circuit power has to be incorporated into the optimization in order to penalize slow transmission (e.g., [1], [3] and the solid curve in Fig. 1).

Many works on energy efficiency have studied point-to-point transmission (e.g., [4]–[7]). In this case, the energy efficiency optimization is computationally tractable since the energy per bit is pseudoconvex in the transmit power (cf. [6] and Fig. 1) so that that every local minimum is a global one.

In multiuser systems with multiple antennas, additional aspects have to be considered [2]. First, additional optimization variables (transmit covariance matrices or beamforming vectors) come into play, and inter-user interference has to be taken into account. Second, fairness might become an additional objective of the optimization. While the latter aspect has been mostly disregarded in the existing literature, several approaches have been presented for the former.

One method is to assume orthogonal frequency division multiple access (OFDMA), i.e., exclusive assignment of sub-carriers to users (e.g., [8]–[10]). With this restriction, only a subclass of all possible transmit strategies is considered, which introduces a suboptimality. The same is true for the restriction to zero-forcing beamforming (e.g., [11], [12]). Other proposed suboptimal solutions are a random beamforming scheme for multiple-input single-output (MISO) broadcast channels [13], and a heuristic selection criterion for MIMO broadcast channels based on uniform power allocation [14].

The framework proposed in this paper can compute the globally optimal energy-efficient transmit strategy in, amongst others, MIMO broadcast channels with and without fairness constraints. Since other solution methods for the case without fairness constraints have been published recently (for the special case of single-antenna systems with two users in [15] and for the general MIMO case with arbitrary numbers of users in [16], [17]), we focus on the case where fairness is considered (in the form of rate balancing constraints, cf., e.g., [18]–[20]—the study of energy efficiency in combination with other fairness metrics is left open for future research). To the best of our knowledge, the combination of energy efficiency and fairness in MIMO broadcast channels has only been studied for the special case of single-antenna receivers with a restriction to linear transceivers [12], [21].

Instead of in a specialized algorithm for one particular problem (as in [16], [17] based on the iterative waterfilling method [22]), we were rather interested in deriving a framework that can be applied to solve a wider class of energy efficiency optimizations in multiuser systems. The idea is to reuse existing algorithms from the multiuser MIMO literature, which have not been developed under an energy efficiency point of view (e.g., [19], [23]) as building block for an energy efficiency optimization. The second ingredient is the fractional programming [24], which has been previously applied to optimize the energy efficiency of point-to-point transmission (e.g., [3]–[6]). As linking element, we need a technique called sensitivity analysis [25, Section 5.6]. As the optimization problem is split into one part specific to energy efficiency

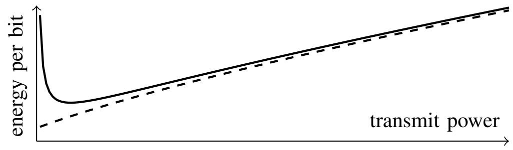


Fig. 1. Energy per bit as a pseudoconvex function of the transmit power considering (solid curve) and neglecting (dashed curve) the circuit power.

optimization and one part specific to the underlying system model, various energy efficiency problems can be treated by replacing the building block that represents the system model. The method is a nontrivial generalization of frameworks that exist for the point-to-point case (e.g., [3]). Thus, some well-studied simpler energy efficiency problems can be solved as special cases (cf. Section V).

*Notation:* $\mathbf{1} = [1, \dots, 1]^{\mathrm{T}}$ and $(\bullet_k)_{\forall k} = (\bullet_1, \dots, \bullet_K)$.

## II. Problem Formulation and Decomposition

The energy per bit is the ratio of the total energy consumed and the total amount of data transmitted. In a time-invariant environment, these quantities are obtained by multiplying the total transmission time $T$ by the total power $P_{\text{total}}$ and the total rate $R$, respectively, i.e.,

$$E_{\text{b}} = \frac{P_{\text{total}} T}{RT} = \frac{P_{\text{total}}}{R} = \frac{\alpha P + P_{\text{c}}}{R}. \tag{1}$$

The last equality is due to [3], where it was shown that many power models found in the literature are equivalent to this form. The value of $\alpha$ depends, amongst others, on the efficiency of the power amplifier, and $P_{\text{c}}$ models power consumption of the other circuit components. In a multiuser system, $P$ is the sum of the powers invested for various users, and $R = \sum_{k=1}^{K} r_k$ is the sum of per-user rates $r_k$.

We consider the minimization of the energy per bit (equivalent to maximizing the energy efficiency $\frac{1}{E_{\text{b}}}$ [3]) over all possible transmit strategies $\mathcal{S}$, where the sum transmit power $P$ and the per-user rates $r_k$ depend on the chosen strategy:

$$\min_{\mathcal{S} \in \mathbb{S}} \quad \alpha \frac{P(\mathcal{S}) + c}{\sum_{k=1}^{K} r_k(\mathcal{S})} \tag{2}$$

where $c = \alpha^{-1} P_{\text{c}}$ is called *circuit power constant*. The set of feasible strategies $\mathbb{S}$ is specific to the considered system and will be defined later.

We drop the constant factor $\alpha$ and introduce the rate region

$$\mathcal{R}(P) = \{\boldsymbol{r} \in \mathbb{R}_{0,+}^{K} \mid \exists \mathcal{S} \in \mathbb{S} : (P, \boldsymbol{r}) = (P(\mathcal{S}), \boldsymbol{r}(\mathcal{S}))\} \tag{3}$$

i.e., the set of all rate vectors $\boldsymbol{r} = [r_1, \dots, r_K]^{\mathrm{T}}$ achievable with a given transmit power $P$. The optimization then reads

$$\min_{P \geq 0, \boldsymbol{r}} \ \frac{P + c}{\sum_{k=1}^{K} r_k} \quad \text{s.t.} \quad \boldsymbol{r} \in \mathcal{R}(P) \tag{4}$$

where the objective function reflects the fact that an energy efficiency optimization is carried out while the properties of the particular communication system are captured by the constraint set. Note that $P$ and, thus, $\mathcal{R}(P)$ are no constants, which is in contrast to the classical sum rate maximization in MIMO broadcast channels [23].

Since a strategy can only be optimal for (4) if it maximizes the denominator for a given value of the numerator, we have

$$\min_{P \geq 0} \frac{P + c}{R(P)} \qquad \text{with} \qquad R(P) = \max_{\boldsymbol{r} \in \mathcal{R}(P)} \mathbf{1}^{\mathrm{T}} \boldsymbol{r} \tag{5}$$

where the sum rate is considered as a function of $P$ that captures the specifics of the underlying system model.[1] The outer minimization has the typical objective function of an energy efficiency problem and can be intuitively understood as the extension from the single-stream case (e.g., [1]) to the multiuser case. To find out whether this objective function still has the familiar pseudoconvex shape shown in Fig. 1, the properties of $R(P)$ have to be studied. As these properties depend on the particularities of the considered system, we have to introduce the system model at this point.

We consider the downlink of a communication system, where the base station is equipped with $M$ antennas and serves $K$ users, each equipped with an individual number of antennas $N_k$. The time-invariant channel $\boldsymbol{H}_k^{\mathrm{H}} \in \mathbb{C}^{N_k \times M}$ between the base station and user $k$ is assumed to be frequency flat and perfectly known. The data transmission can be described by

$$\boldsymbol{y}_k = \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{s}_k + \sum_{k' \in \mathcal{I}_k} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{s}_{k'} + \boldsymbol{w}_k, \tag{6}$$

where $\boldsymbol{w}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_k})$ is additive circularly symmetric complex Gaussian noise, which is independent across users and independent of the transmitted Gaussian data symbols $\boldsymbol{s}_{k'}$. Using dirty paper coding (DPC) [23], interference caused by users encoded before user $k$ can be suppressed so that the set of interferers $\mathcal{I}_k$ contains the users encoded after user $k$.

A transmit strategy $\mathcal{S} \in \mathbb{S}_{\text{BC}}$ for the MIMO broadcast channel can be described by the transmit covariance matrices $(\boldsymbol{Q}_k)_{\forall k}$ in the dual uplink [23] and by the DPC encoding order. Moreover, we can apply time-sharing, i.e., $\mathcal{S} \in \mathbb{S}_{\text{BC}}$ can consist of several different strategies $\mathcal{S}_i$, which are applied one after another during fractions $t_i$ of the total time. The resulting rate region (3) with $\mathbb{S} = \mathbb{S}_{\text{BC}}$ is the convex capacity region $\mathcal{R}_{\text{BC}}(P)$ of the MIMO broadcast channel [26].

To avoid that some users are served with a much lower rate than others, we introduce rate balancing constraints (e.g., [18]–[20]), which are a generalization of max-min-fairness [18]. The idea is to restrict the set $\mathbb{S}_{\text{fair}} \subset \mathbb{S}_{BC}$ to strategies $\mathcal{S}$ for which the ratios of per-user rates equal predefined values:

$$r_k(\mathcal{S}) / r_1(\mathcal{S}) = \rho_k / \rho_1 \quad \forall k, \ \forall \mathcal{S} \in \mathbb{S}_{\text{fair}} \tag{7}$$

for given relative rate targets $\rho_k$ $\forall k$. In the special case that all $\rho_k$ are equal, all users are served with the same rate. We use $\mathcal{R}_{\text{fair}}(P)$ to denote the rate region (3) with $\mathbb{S} = \mathbb{S}_{\text{fair}}$.

## III. Sensitivity Analysis for the Inner Problem

We study the sensitivity of the sum rate to a change in sum power, i.e., the behavior of the optimal rate function $R(P)$.

For the case of rate balancing, where $\mathcal{R}(P) = \mathcal{R}_{\text{fair}}(P)$, the function can be evaluated by solving (e.g., [19])

$$\max_{\boldsymbol{r} \in \mathcal{R}_{\text{BC}}(P), R_0} R_0 \quad \text{s.t.} \quad \boldsymbol{r} \geq R_0 \boldsymbol{\rho} \tag{8}$$

[1] Note that an optimal rate function was also used in [6], [11], but only for the less general special case of parallel orthogonal channels.

with $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_K]^{\mathrm{T}}$. Without loss of generality, we scale $\boldsymbol{\rho}$ such that $\mathbf{1}^{\mathrm{T}}\boldsymbol{\rho} = 1$ and, consequently, $R_0 = \mathbf{1}^{\mathrm{T}}\boldsymbol{r}$. Due to convexity of $\mathcal{R}_{\mathrm{BC}}$ [26], (8) is a convex problem, and we can switch to the dual problem (cf., e.g., [25, Section 5.2.3])

$$\min_{\boldsymbol{\mu} \geq \mathbf{0}} \left( \max_{R_0} \ R_0(1 - \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho}) + \max_{\boldsymbol{r} \in \mathcal{R}_{\mathrm{BC}}(P)} \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r} \right) \tag{9}$$

with the vector of Lagrangian multipliers $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_K]^{\mathrm{T}}$, which obviously has to fulfill $\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho} = 1$ in the optimal solution. Thus, the rate function for the fairness-constrained case reads

$$R(P) = R_{\mathrm{fair}}(P) = \min_{\boldsymbol{\mu} \in \mathbb{R}^K_{0,+}: \ \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho} = 1} \ \max_{\boldsymbol{r} \in \mathcal{R}_{\mathrm{BC}}(P)} \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r} \tag{10}$$

where the inner problem is a weighted sum rate maximization for given weights $\boldsymbol{\mu}$ and given sum transmit power $P$. For algorithmic aspects, the reader is referred to [19]. We are instead interested in the following properties of the solution.

*Proposition 1:* $R_{\mathrm{fair}}(P)$ in (10) is increasing, concave, and differentiable for $P \in ]0, \infty[$. The derivative is given by the optimal Lagrangian multiplier $\lambda$ associated with the power constraint of the weighted sum rate maximization solved for optimal Lagrangian multipliers $\boldsymbol{\mu}$.

*Proof:* The proof is based on a sensitivity analysis (cf. [25, Section 5.6]) and a study of the uniqueness of the optimal solution and can be found in Appendix A. ■

If no fairness constraints are considered, we have

$$R(P) = R_{\mathrm{BC}}(P) = \max_{\boldsymbol{r} \in \mathcal{R}_{\mathrm{BC}}(P)} \mathbf{1}^{\mathrm{T}}\boldsymbol{r} \tag{11}$$

which can be efficiently evaluated by standard solvers or by the sum power iterative waterfilling algorithm proposed in [22], and the following proposition holds.[2]

*Proposition 2:* $R_{\mathrm{BC}}(P)$ in (11) is increasing, concave, and differentiable for $P \in ]0, \infty[$. The derivative with respect to $P$ is given by the optimal Lagrangian multiplier $\lambda$ associated with the power constraint of the sum rate maximization.

*Proof:* As (11) is a special case of (14) for $\boldsymbol{\mu} = \mathbf{1}$, the same reasoning as in the proof of Proposition 1 applies. ■

The properties stated in Proposition 1 and 2 are visualized in Fig. 2. $R(P)$ has the same logarithmic shape as the Shannon rate of a single data stream, and the objective function $\frac{P+c}{R(P)}$ of the energy-per-bit minimization (5) has the same shape as the single-stream version in [1].

## IV. Solution to the Outer Problem

Since $R(P)$ is concave and differentiable as shown above, (5) is a convex-concave fractional program with a pseudoconvex objective function due to [24]. Thus, any stationary point is a global optimizer. Convex-concave fractional programming has been previously applied to optimize the energy efficiency of point-to-point transmission (e.g., [3]–[6]). Above sensitivity analysis extends the applicability to broadcast scenarios.

[2]For the special case where only scalar per-stream powers have to be optimized, the concavity was also shown in [6], but without studying the differentiability and not for the general case with covariance matrices.

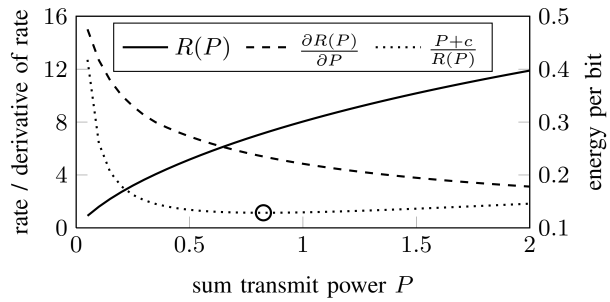


Fig. 2. Optimal rate function $R(P)$ with its first derivative and the objective function of (5) with its optimum (example without fairness constraints for $M = 10$ transmit antennas, $K = 4$ users, and $N_k = 5$ receive antennas $\forall k$).

Using the derivative $\frac{\mathrm{d}R(P)}{\mathrm{d}P}$ stated in Proposition 1 and 2, respectively, the derivative of $E_{\mathrm{b}}$ is given by

$$\frac{\mathrm{d}E_{\mathrm{b}}}{\mathrm{d}P} = \frac{R(P) - (P+c)\frac{\mathrm{d}R(P)}{\mathrm{d}P}}{R^2(P)}. \tag{12}$$

The global optimum of (5) can then be found, e.g., by finding a root of (12) using the bisection method.

Other methods to solve this kind of fractional program exist and were reviewed in [3]. An advantage of finding a root of (12) is, however, that existing implementations of the sum rate maximization in (5) can be reused. This is not the case when applying the so-called Dinkelbach method [27] as proposed, e.g., for the MIMO broadcast channel in [17] and for point-to-point systems in [3]–[5].

## V. Special Cases

The energy efficiency problem in a point-to-point MIMO channel (cf. [4]) is a special case of a MIMO broadcast channel with $K = 1$. The case of parallel orthogonal channels without fairness considerations (e.g., [5]–[7], [11]) can be obtained by setting $K = 1$ and $\boldsymbol{H}_1^{\mathrm{H}} = \mathrm{diag}\{h^{(c)}\}$, where $h^{(c)}$ is the channel coefficient on the channel $c$. Therefore, the method described above can be directly applied as an alternative solution method for these well-studied energy efficiency problems. In these cases, the function $R(P)$ is equivalent to the simpler optimal rate function discussed in [6] and [11], for which the derivative can even be calculated explicitly [11].

## VI. Non-pseudoconvex Cases

The framework described above can also be applied to other energy efficiency problems. However, the sensitivity analysis is specific to the particular considered system, and it might happen that concavity of the rate function does not hold (e.g., in broadcast channels with linear transceivers).

Nevertheless, $R(P)$ is still an increasing function due to the same argument as in the proof of Proposition 1. Therefore, the energy per bit is a ratio of monotonic functions and can be optimized using the branch-and-bound method (cf. [21]).

## VII. Numerical Results and Discussion

As the optimization methods discussed in this paper find the globally optimal solutions of the energy efficiency problems under consideration, there is no need for an experimental validation of the quality of the obtained solutions. Instead, we compare the case with fairness constraints to the case

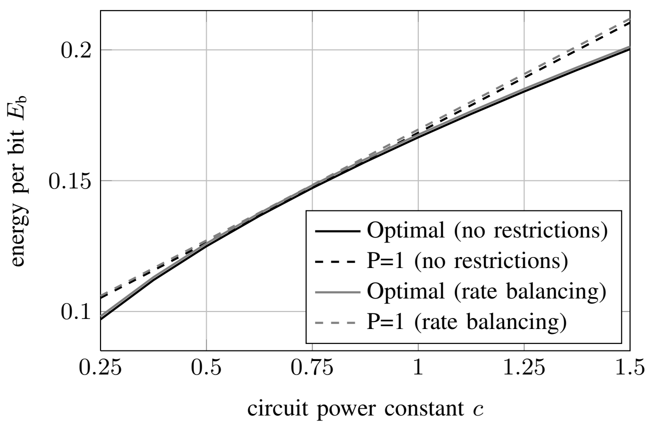


Fig. 3. Energy per bit achieved in a system with $M = 10$ transmit antennas, $K = 4$ users, and $N_k = 5$ receive antennas for all $k$.

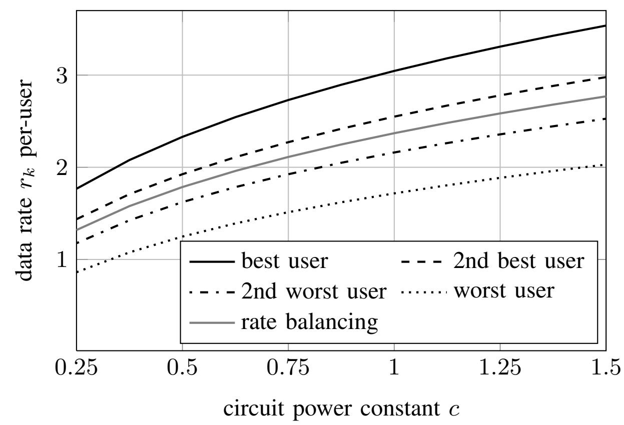


Fig. 4. Per-user rates at the energy-optimal solution in a system with $M = 10$ transmit antennas, $K = 4$ users, and $N_k = 5$ receive antennas for all $k$.

where fairness is not considered. Furthermore, the energy per bit consumed by transmission with fixed sum transmit power is compared to the optimal energy per bit.

In Fig. 3, we consider a MIMO broadcast channel with a 10-antenna base station and four 5-antenna users. The simulation results are averaged over 1000 realizations of channel matrices with i.i.d. complex Gaussian entries with zero mean and unit variance. The optimal energy per bit is plotted over the circuit power constant $c$ for the case without fairness constraints and for the case of rate balancing with $\rho_k = \frac{1}{K}\ \forall k$. As expected, the energy per bit is higher for the rate balancing case since adding new constraints can never improve the optimal value.

Fig. 3 also contains curves for a fixed transmit power $P = 1$, where only the rate with and without rate balancing constraints was optimized. This curve lies close to the optimal curve at a certain value of $c$ for which the fixed transmit power $P = 1$ is a good choice on average. However, for low values of the circuit power constant, the transmit power $P = 1$ is too high, just as it is too low in case of a high circuit power constant. By optimizing the sum transmit power instead of using a fixed value, a better energy efficiency can be achieved.

The rate of the best, the second best, the second worst, and the worst user at the optimal energy-efficient solution is shown in Fig. 4. A large gap between the rates achieved for the worst and the best user can occur. This is not true for the case of rate balancing, where the rate is the same for all users. The price in terms of higher energy consumption that has to be paid for the increased fairness (Fig. 3) is relatively small in the considered example of a MIMO broadcast channel. For unequal rate targets $\rho_k$, the penalty can be more pronounced.

## VIII. Summary

Energy efficiency problems in MIMO broadcast channels can be solved by decomposing them into an inner problem specific to the system model and an outer problem specific to the energy efficiency optimization. Conventional algorithms, which have not been developed in the context of energy efficiency, can be used to solve the inner problem. By performing a sensitivity analysis as linking element between the problems, it can be shown that the outer problem is a convex-concave fractional program and can be solved efficiently.

## Appendix A

*Proof of Proposition 1:* Let $\boldsymbol{\mu}^\star$ be the optimal vector of Lagrangian multipliers in (10) so that

$$R_{\text{fair}}(P) = \max_{\boldsymbol{r}\in\mathcal{R}_{\text{BC}}(P)} \boldsymbol{\mu}^{\star,\mathrm{T}}\boldsymbol{r}. \tag{13}$$

The weighted sum rate maximization is equivalent to

$$\max_{(\boldsymbol{Q}_k\succeq\mathbf{0})_{\forall k}} \sum_{k=1}^{K} \nu_k \log\det\left(\mathbf{I}_M + \sum_{j=1}^{k} \boldsymbol{H}_{\pi(j)}\boldsymbol{Q}_{\pi(j)}\boldsymbol{H}_{\pi(j)}^{\mathrm{H}}\right) \quad \text{s.t.} \quad \sum_{k=1}^{K}\operatorname{tr}[\boldsymbol{Q}_k] \le P \tag{14}$$

with $\nu_k = \left(\mu^\star_{\pi(k)} - \mu^\star_{\pi(k+1)}\right)$, where $\pi : \{1,\ldots,K\} \mapsto \{1,\ldots,K\}$ is a permutation such that $\mu^\star_{\pi(k)} \ge \mu^\star_{\pi(k+1)}$, and $\mu^\star_{\pi(K+1)} = 0$ by definition [28]. Note that $\pi$ corresponds to a certain choice of the DPC encoding order [23], [28].

Let us rewrite (14) in the following abstract notation:

$$R_{\text{fair}}(P) = \left(\max_{\boldsymbol{Q}\in\mathcal{Q}}\ g(\boldsymbol{Q}) \quad \text{s.t.} \quad h(\boldsymbol{Q}) \le P\right). \tag{15}$$

As relaxing the constraint $h(\boldsymbol{Q}) \le P$ can only increase the maximum, $R_{\text{fair}}(P)$ is increasing in $P$.

To proof that the function is concave, we perform a sensitivity analysis as described in [25, Section 5.6.2]:

$$\begin{aligned}
R_{\text{fair}}(P) &= \min_{\lambda\ge0}\max_{\boldsymbol{Q}\in\mathcal{Q}}\ g(\boldsymbol{Q}) - \lambda(h(\boldsymbol{Q}) - P) && (16)\\
&\le \max_{\boldsymbol{Q}\in\mathcal{Q}}\ g(\boldsymbol{Q}) - \tilde{\lambda}(h(\boldsymbol{Q}) - P) && (17)\\
&= \max_{\boldsymbol{Q}\in\mathcal{Q}}\ g(\boldsymbol{Q}) - \tilde{\lambda}(h(\boldsymbol{Q}) - \tilde{P}) + \tilde{\lambda}(P - \tilde{P}) && (18)\\
&= R_{\text{fair}}(\tilde{P}) + \tilde{\lambda}(P - \tilde{P}). && (19)
\end{aligned}$$

Eq. (16) is due to strong duality [25, Section 5.2.3] since (15) is a convex program. In (17), the inequality holds $\forall\tilde{\lambda} \ge 0$ due to the minimization in (16). Thus, it also holds if $\tilde{\lambda}$ is the optimal Lagrangian multiplier for the power $\tilde{P}$, which is assumed in (19). Eq. (19) is an upper bound to $R_{\text{fair}}(P)$ by means of a subgradient $\tilde{\lambda}$ in the point $\tilde{P}$. Since a subgradient exists in each point $\tilde{P}$, $R_{\text{fair}}(P)$ is concave [25, Section 6.5.5].

This proves that the solution of the inner problem in (10) is a concave function of $P$. The outer problem in (10) is a minimization, which preserves concavity [25, Section 3.2.3].

To prove that $R_{\text{fair}}(P)$ is differentiable, we have to show that the subgradient is a gradient, i.e., it is unique, which cannot be concluded from the sensitivity analysis.[3] The KKT conditions [25, Section 5.5.3] of (14) read as

$$\underbrace{\sum_{k=1}^{K} \nu_k \boldsymbol{H}_{\pi(k)} \left( \mathbf{I}_M + \sum_{j=1}^{k} \boldsymbol{H}_{\pi(j)} \boldsymbol{Q}_{\pi(j)} \boldsymbol{H}_{\pi(j)}^{\mathrm{H}} \right)^{-1} \boldsymbol{H}_{\pi(k)}^{\mathrm{H}}}_{\boldsymbol{A}_k} + \boldsymbol{\Lambda}_{\pi(k)} - \lambda \mathbf{I}_{N_{\pi(k)}} = \mathbf{0} \ \forall k. \quad (20)$$

$$\boldsymbol{Q}_k \succeq \mathbf{0} \ \forall k, \quad \boldsymbol{\Lambda}_k \succeq \mathbf{0} \ \forall k, \quad \boldsymbol{\Lambda}_k \boldsymbol{Q}_k = \mathbf{0} \ \forall k \quad (21)$$

$$\sum_{k=1}^{K} \operatorname{tr}[\boldsymbol{Q}_k] \leq P, \quad \lambda \geq 0, \quad \lambda \left( \sum_{k=1}^{K} \operatorname{tr}[\boldsymbol{Q}_k] - P \right) = 0 \quad (22)$$

with Lagrangian multipliers $(\boldsymbol{\Lambda}_k \in \mathbb{C}^{N_k \times N_k})_{\forall k}$ and $\lambda \geq 0$. $\boldsymbol{\Lambda}_k = \boldsymbol{U}_k \boldsymbol{\Delta}_k \boldsymbol{U}_k^{\mathrm{H}}$, $\boldsymbol{Q}_k = \boldsymbol{U}_k \boldsymbol{\Psi}_k \boldsymbol{U}_k^{\mathrm{H}}$, and $\boldsymbol{A}_k = \boldsymbol{U}_k \boldsymbol{D}_k \boldsymbol{U}_k^{\mathrm{H}}$ have the same eigenbasis $\boldsymbol{U}_k$. Consequently, for the nonnegative diagonal matrices $\boldsymbol{D}_k$, $\boldsymbol{\Delta}_k$, and $\boldsymbol{\Psi}_k$, it must hold that

$$\boldsymbol{D}_k + \boldsymbol{\Delta}_k - \lambda \mathbf{I}_{N_k} = \mathbf{0} \ \forall k \quad \text{and} \quad \boldsymbol{\Delta}_k \boldsymbol{\Psi}_k = \mathbf{0} \ \forall k \quad (23)$$

which implies that no change of $\lambda$ is possible without also changing some diagonal elements of the matrices $\boldsymbol{D}_k$, and thus, changing at least one of the matrices $\boldsymbol{Q}_k$. Therefore, for given optimal $(\boldsymbol{Q}_k)_{\forall k}$, there is a unique $\lambda$ fulfilling the KKT conditions. As the strictly concave objective function implies that there is a unique optimizer $(\boldsymbol{Q}_k)_{\forall k}$, we have a unique optimal Lagrangian multiplier $\lambda$ for given $\boldsymbol{\mu}^\star$ and $\pi$.

The solution to the rate balancing problem (8) corresponds to the intersection point of a line whose slope is determined by the vector $\boldsymbol{\rho}$ and the Pareto boundary of the capacity region $\mathcal{R}_{\text{BC}}(P)$. The optimal vector $\boldsymbol{\mu}^\star$ corresponds to the normal vector to the capacity region in this intersection point and is, thus, unique. In case that the corresponding permutation $\pi$ is ambiguous (same weight $\mu_k$ for two or more users), all possible choices lead to the same objective function in (14).

Thus, the subgradient in (19) is unique, i.e., a gradient. This proves differentiability of $R_{\text{fair}}(P)$, and the optimal $\lambda$ obtained for the optimal vector $\boldsymbol{\mu}^\star$ is the derivative. ∎

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