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Opportunities to Learn Mathematical Proofs in Geometry: Comparative Analyses of Textbooks from Germany and Taiwan

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ABSTRACT

Studies have shown that reasoning and proving is a challenge for students. This could partly be due to the way proofs are taught in schools. Mathematics textbooks are an important indicator of teaching. This study aims at investigating opportunities to learn mathematical proofs in German and Taiwanese lower secondary school (grades 7–9) textbooks. The results show that validation of a new idea plays a more important role in Germany, whereas practice with various ideas is more relevant in Taiwan.

ZUSAMMENFASSUNG

Begründen und Beweisen stellt für Schülerinnen und Schüler eine große Herausforderung dar. Dies könnte mit der Behandlung im Schulunterricht zusammenhängen. Diese Studie untersucht Lerngelegenheiten für mathematisches Beweisen in deutschen und taiwanesischen Schulbüchern der Sekundarstufe I (Kassen 7 bis 9). Die Ergebnisse deuten darauf hin, dass in Deutschland die Validierung neuer Ideen eine wesentliche Rolle spielt, während in Taiwan dem Anwenden unterschiedlicher Ideen größere Bedeutung zukommt.

TABLE OF CONTENTS

LIST OF FIGURES

Figure

LIST OF TABLES

Table

LIST OF APPENDICES

1. INTRODUCTION

Mathematics is a *proving science* (German: “beweisende Wissenschaft”; Heintz, 2000a, 2000b), and this distinguishes mathematics from all other disciplines. Proving is also fundamental in mathematics classrooms (Heinze & Reiss, 2007). However, many empirical studies indicate that students in all countries have substantial difficulties in performing school mathematical proofs (e.g., Healy & Hoyles, 2000; Heinze, 2004; Lin & Cheng, 2003; Lin, Yang, & Chen, 2004; Reiss, Hellmich, & Thomas, 2002). For example, several specific difficulties have been found in Germany and Taiwan. The reasons for these difficulties are an important research topic.

Reasoning and proving has become an increasingly important content of mathematics curricula. For example, it is one curriculum standard (*mathematics as reasoning*) in the *Curriculum and Evaluation Standards for School Mathematics* (1989) and one process standard (*reasoning and proof*) in the *Principles and Standards for School Mathematics* (2000) released by the National Council of Teachers of Mathematics [NCTM]. It is also considered as one of five necessary strands of mathematical proficiency (*adaptive reasoning*) (National Research Council, 2001). Recently, it also plays an important role in several standards for mathematical practice in the U.S. national standards initiative (e.g., *reason abstractly and quantitatively*, *construct viable arguments and critique the reasoning of others*, *look for and express regularity in repeated reasoning*) (Common Core State Standards Initiative, 2010).

The opportunities to learn mathematical proofs and reasoning might vary within and across countries, with respect to curriculum materials (e.g., textbooks), mathematical content domains (e.g., geometry, algebra), task designs, and methods of instruction. All these factors influence students’ learning and depend on each other. As textbooks mostly reflect the intended curriculum, they also mirror opportunities for students to learn mathematical proofs. Indeed, Mayer (1989) found that *learning materials* are among the most important components influencing teaching and learning, and textbooks are frequently used especially by novice teachers (Ball & Feiman-Nemser, 1988). Begle (1973) also found that textbooks have a powerful influence on what students actually learn: if a topic appears in the textbook, then students do learn it; however, if the topic does not appear in the textbook, then, on the average, students do not learn it.

Although there is agreement on the importance of mathematical reasoning and proving between academic research and curriculum, the argument of what kinds of *mathematical proofs/reasoning and proving* should be provided in schools has been discussed intensively over the past years (e.g., Balacheff, 1988; Ball, 1993; Ball & Bass, 2003; Chazan, 1993; Hanna, 1990; Knuth, 2002a; Maher & Martino, 1996; Moore, 1994; Raman, 2003; Recio & Godino, 2001; G. Stylianides, 2007; Wu, 1997). What kinds of content and activities of mathematical proofs are considered to be suitable for students might also depend on the teaching tradition or the culture of a country.

Most of previous comparison studies on mathematics textbooks focused on comparing the differences of semantic features (e.g., Herbel-Eisenmann, 2007; Herbel-Eisenmann & Wagner, 2005, 2007) or general textual presentations, and only some of them discussed the details of specific topics (e.g., Charalambous, Delaney, Hsu, & Mesa, 2010; G. Stylianides, 2005, 2007, 2008, 2009; Thompson, Senk, & Johnson, 2012). Though different analytic frameworks were developed for their special purposes, they may not be suitable for all research.

Moreover, geometry is viewed as a centered content area to introduce reasoning and proving in school. Geometry[1] involves not only *intuition* (intuitive understanding), influenced by visual figures, but also *logical reasoning* (abstraction), using rules or principles to chain different

[1] Cf. *intuitive understanding* and *abstraction* (Hilbert & Cohn-Vossen, 1952)

properties or theorems. Though geometry is not the only topic providing opportunities to learn mathematical proofs, it is particularly suitable in order to explore how students develop competence in mathematical proving from the very beginning stage to formal proof. In addition, geometry in secondary school provides students with opportunities to learn mathematics in an analytical way, for example by connecting the geometric content with algorithms for calculating unknown lengths, areas, volumes, or angles. The analytical process occasionally requires operating figure construction or decomposition of figures based on some basic properties or principles. Therefore, geometry provides a good opportunity to investigate how students start to learn mathematical proofs.

The main theme of this study is to compare the presentations of geometry content in lower secondary school in order to examine the design of mathematical proofs in German and Taiwanese textbooks. An analytic framework and principles are developed to carry out textbook comparisons from a general and a more focused perspective, respectively. This work is divided in seven chapters.

Chapter 2 describes in brief the practical background of educational circumstances and mathematics curricula in Germany and Taiwan. The school systems, national standards in mathematics, and mathematics textbooks will be broadly introduced. Though the term *textbook* is named differently in Germany ("Schulbuch": schoolbook) and Taiwan ("教科書": instructional book), both of them provide subject-matter (i.e., mathematical) knowledge in an authoritative pedagogic way (Stray, 1994). Therefore, 'textbook' is used in this study as a shared terminology.

Chapter 3 focuses on the theoretical background related to teaching and learning of mathematical proofs in lower secondary school. First, the concept of mathematical knowledge will be introduced to clarify its position in learning mathematics. Second, the role of mathematical proofs in schools will be discussed. Third, the connection between geometry to proofs in schools will be provided. Fourth, studies of mathematics textbooks will be compared in this section.

Chapter 4 raises the aims and the research questions of this study in details.

Chapter 5 demonstrates the development of the analytic framework, its application to textbook comparisons with several examples, and the development of a set of three potential principles for comparing three specific mathematical statements.

Chapter 6 introduces the research method of this study.

Chapter 7 depicts the results of the general comparison on geometry content of the German and Taiwanese textbooks and the specific comparison on three mathematical statements, the *sum of the interior angles of a triangle*, the *Thales theorem*, and the *Pythagorean theorem*.

Finally, Chapter 8 provides the discussion on the major findings and the limitations of this study. It also gives possible directions for future research studies and curriculum development.

2. PRACTICAL BACKGROUND

In this chapter, *the practical background* of the educational circumstances, especially the mathematics curricula, in the state of Bavaria in Germany and in Taiwan, will be described. A general introduction into the school systems, the goals and content of national mathematics standards, and mathematics textbooks will be presented.

2.1 School Systems

The school system in the Federal Republic of Germany may be regarded particularly complex because of the different policies and educational systems that can be found in different federal states (Bundesländer). In most states, education in primary schools (Grundschule) starts at the age of six and lasts for four years. Education in secondary schools is, in most states, differentiated into three different tracks, namely, Gymnasium, Realschule, and Hauptschule/Mittelschule. The track of Gymnasium offers eight years (ages 10–18) of further education. The tracks of Realschule and Hauptschule/Mittelschule offer six years (ages 10–16) and five years (ages 10–15) of education, respectively, followed by some years of additional part-time or full-time compulsory vocational education.

In Taiwan, school education starts at the age of six. It lasts for six years in elementary school (ages 6–12), three years in junior high school (ages 12–15), and generally another three years in senior high school or vocational high school (ages 15–18).

In Table 2.1, the specifics of both systems are presented. As the situation differs between the states in Germany, the school system in the state of Bavaria[1] is chosen as an example and is compared with the school system in Taiwan.

Table 2.1. School systems in Germany and Taiwan

<table>
<tr><th>Age (yr)</th><th>Grade</th><th colspan="3">Germany (Bavaria)</th><th colspan="2">Taiwan</th></tr>
<tr><td>5–6</td><td>K</td><td colspan="3">Kindergarten</td><td colspan="2">Kindergarten</td></tr>
<tr><td>6–7</td><td>1</td><td colspan="3" rowspan="4">Grundschule (Primary School)</td><td colspan="2" rowspan="6">Elementary School</td></tr>
<tr><td>7–8</td><td>2</td></tr>
<tr><td>8–9</td><td>3</td></tr>
<tr><td>9–10</td><td>4</td></tr>
<tr><td>10–11</td><td>5</td><td rowspan="8">Gymnasium</td><td rowspan="6">Realschule</td><td rowspan="5">Hauptschule/ Mittelschule</td></tr>
<tr><td>11–12</td><td>6</td></tr>
<tr><td>12–13</td><td>7</td><td colspan="2" rowspan="3">Junior High School</td></tr>
<tr><td>13–14</td><td>8</td></tr>
<tr><td>14–15</td><td>9</td></tr>
<tr><td>15–16</td><td>10</td><td rowspan="3"></td><td rowspan="3">Senior High School</td><td rowspan="3">Vocational High School</td></tr>
<tr><td>16–17</td><td>11</td><td rowspan="2"></td></tr>
<tr><td>17–18</td><td>12</td></tr>
</table>

This study aims at comparing how students learn the content of mathematical proofs. Therefore, the relevant grade levels are grades 7–9, the lower secondary school, which correspond to *junior high school* in Taiwan. In Germany, the content of mathematical proofs is primarily given in the *Gymnasium* track, so that the comparison will focus on this type of German school.

2.2 National Mathematics Standards

In Germany and Taiwan, there are national standards for school mathematics. In Germany, these standards are represented by the educational standards for the whole country and the state syllabi

[1] See more details under http://www.km.bayern.de/eltern/schularten.html

for each state. In Taiwan, they are represented by the curriculum guidelines. The national standards describe the learning goals and content which mirror the intended curriculum in schools. In the following, the national standards in mathematics in the state of Bavaria in Germany and in Taiwan are illustrated through some examples.

2.2.1 National standards in Germany

In 2003, the *Standing Conference of the Ministers of Education and Cultural Affairs of the Länder in the Federal Republic of Germany* (Kultusministeriumskonferenz, KMK) released the educational standards of mathematics, namely the *Educational Standards in Mathematics for Middle School Certification* (Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss), for the secondary education level (grades 5–10). The educational standards are obligatory for all federal states, however, a full implementation may still be regarded as being underway.

German educational standards are defined by Klieme and colleagues (2003/2004):

> Educational standards articulate requirements for school-based teaching and learning. They identify goals for pedagogical work, expressed as desired learning outcomes for students. […] They specify the competencies that schools must impart to their students in order to achieve certain key educational goals, and the competencies that children or teenagers are expected to have acquired by a particular grade. These competencies are described in such specific terms that they can be translated into particular tasks and, in principle, assessed by tests (p. 15).

Table 2.2 provides a close look on the main content of German educational standards in mathematics for middle (secondary) school.

Table 2.2. Table of content of German educational standards (2003)

Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss	**Educational Standards in Mathematics for Middle School Certification**
1. Der Beitrag des Faches Mathematik zur Bildung	1. The contribution of mathematics to general education
2. Allgemeine mathematische Kompetenzen im Fach Mathematik	2. General mathematical competences in mathematics
3. Standards für inhaltsbezogene mathematische Kompetenzen im Fach Mathematik	3. Standards for content-related mathematical competences in mathematics
3.1 Mathematische Leitideen	3.1 Guiding principles (content domains) of mathematics
3.2 Inhaltsbezogene mathematische Kompetenzen geordnet nach Leitideen	3.2 Content-related mathematical competences arranged by guiding principles
4. Aufgabenbeispiele	4. Tasks/Exercises
4.1 Anforderungsbereiche der allgemeinen mathematischen Kompetenzen	4.1 Levels of requirement (competence levels) of general mathematical competences
4.2 Kommentierte Aufgabenbeispiele	4.2 Commented tasks/exercises

The general statement made by Klieme et al. was transferred into the educational standards for the primary and secondary school mathematics. The educational standards therefore encompass the description of competences that students are supposed to acquire. The six general mathematical competences and five mathematical guiding principles (content domains) are fundamental to the standards, and all other issues are generated from them. The six general mathematical competences (Mathematische *Kompetenzen*) are:

(K1) Mathematical argumentation (*Mathematisch argumentieren*);
(K2) Mathematical problem solving (*Probleme mathematisch lösen*);
(K3) Mathematical modelling (*Mathematisch modellieren*);
(K4) Mathematical representations (*Mathematische Darstellungen verwenden*);
(K5) Dealing with symbolic, formal, and technical elements of mathematics (*Mit symbolischen, formalen und technischen Elementen der Mathematik umgehen*); and
(K6) Communication (*Kommunizieren*).

The requirements of these six general mathematical competences are categorized into three *levels of requirement* (*Anforderungsbereich*) for each competence (K1–6):

(I) Reproduction (*Reproduzieren*);
(II) Establishing connections (*Zusammenhänge herstellen*); and
(III) Generalization and reflection (*Verallgemeinern und Reflektieren*).

The five mathematical guiding principles (Mathematische *Leitideen*) are:

(L1) Number (*Zahl*);
(L2) Measurement (*Messen*);
(L3) Space and shape (*Raum und Form*);
(L4) Functional relations (*Funktionaler Zusammenhang*); and
(L5) Data and probability (*Daten und Zufall*).

Moreover, in each federal state, there is an obligatory syllabus (*Lehrplan*), which used to be the traditional way to implement the curriculum before the educational standards were introduced. Syllabi describe which kinds of topics and content should be taught at school and provide sometimes recommendations or even rules for instruction. Syllabi are approved by the state administration, and they are the foundation for textbook design. More precisely, to translate these syllabi into concrete teaching materials is the task of textbook authors (Howson, Keitel, & Kilpatrick, 1981; interview[2], 2011). To implement the curricula in a classroom or a lesson is the responsibility of teachers.

The state syllabi differ not only between the sixteen federal states in Germany, but also between the three school tracks within a state. Nevertheless, all syllabi list and formulate concrete missions to learn. The syllabi are provided for each grade level of each school track. For example, in the syllabi for grades 5–10 of Gymnasium track, the *required abilities*, *mathematical basic knowledge*, and *content* with the corresponding *teaching hours* are formulated for each individual grade. Therefore, the textbook authors *can* and *have to* follow the information to design their

[2] This interview was conducted in a semi-structured way. The interview guide is given in Appendix C. The interviewee is an author as well as an editor of a German textbook series, Delta (*C. C. Buchner* Verlag). Before she began her work as a publisher, she served as a mathematics teacher in Gymnasium track for more than 30 years and may be regarded a very experienced teacher.

distinctive textbooks (interview, 2011). The general requirements of learning the guiding principle (L3) *space and shape* (geometry) of the educational standards, are listed in the following.

- recognize and describe geometric structures in their environment;
- mentally operate with lines, areas, and solids;
- represent geometric figures in the Cartesian coordinate system;
- represent solids (for example, grid, oblique configurations or model) and recognize solids from their corresponding representation;
- analyze and classify two-dimensional and three-dimensional geometric objects;
- describe and give reasons for special characteristics and relations of geometric objects (like symmetry, congruence, similarity, relative position) and use them in problem-solving processes for the analysis of real-world problems;
- apply statements/propositions of the plane geometry by constructions, calculations and proof, especially the Pythagorean theorem and the Thales theorem;
- draw and construct geometric figures with adequate instruments like compass, ruler, set square, or dynamic geometry software;
- examine the solvability and various solutions of construction tasks and formulate related statements; and
- use appropriate devices for exploration and problem solving.

These missions are strongly related to the six competences of the educational standards. The *required abilities*, *mathematical basic knowledge*, and *content* with its corresponding teaching hours of the same guiding principle (geometry) of the Bavarian syllabi for grades 7–9 in Gymnasium track can be found in Appendix A.

2.2.2 National standards in Taiwan

In Taiwan, the *Grades 1–9 Curriculum Guidelines in Mathematics* (九年一貫數學領域課程綱要) was first released by the Ministry of Education [MOE] in 2003, a revised edition is available since 2008. The curriculum guidelines provide detailed information on the content of mathematics curriculum. The curriculum guidelines specify the national requirements for teachers to reflect their teaching and for textbook authors to design the textbooks.

The main content of the national standards (curriculum guidelines) of 2003[3] is presented in Table 2.3.

Table 2.3. Table of content of Taiwanese curriculum guidelines (2003)

92 年版 九年一貫數學領域課程綱要	**Grades 1–9 Curriculum Guidelines in Mathematics**
壹、基本理念	1. Basic ideas
貳、課程目標	2. Curricular goals
參、能力指標	3. Ability index
五大主題能力指標	3.1 Ability index of the *five topics* (content domains)
階段能力指標	3.2 Ability index of *stages by topics*
分年細目	3.3 Ability index of *topics by grades*
肆、能力指標與十大基本能力的關係	4. The relationship between *ability index* and *ten fundamental abilities*
伍、實施要點	5. The aspects regarding the implementation (in four domains: instruction, evaluation, textbook, computer and calculator)
陸、附錄	6. Appendix
附錄一 五大主題說明	6.1Illustration for five main topics
附錄二 分年細目詮釋	6.2 Annotation of ability index of topics by grades
附錄三「連結」能力指標之詮釋	6.3 Annotation of ability index of “connection”
附錄四 度量衡列表	6.4 Table of metric system
附錄五 標準用詞與解釋	6.5 Standard mathematical term and its explanation
附錄六 指標與細目專詞釋義	6.6 Explanation for ability index and specific terms

The *ability indices* frame the structure and content by the five *topics* (content domains), four *stages* and nine *grades*, that is, there are ability indices of the five topics, ability indices of stages by topics, and ability indices of topics by grades (e.g., Table 2.4). The five topics *(*cf. *guiding principles* in German educational standards*)* are: (1) number and magnitude; (2) geometry; (3) algebra; (4) (descriptive) statistics and probability; (5) connections (cf. NCTM, 2000). The four stages[4] are defined by the nine consecutive grades (1–9): stage 1 (grades 1–3); stage 2 (grades 4–5); stage 3 (grades 6–7); and stage 4 (grades 8–9).

[3] The textbook series adopted in this study are based on the curriculum guidelines in 2003, hence the 2003 edition is presented. There are only minor differences between the editions of 2003 and 2008.
[4] In the revised guidelines in 2008, these four stages are changed to stages 1 (grades 1–2), 2 (grades 3–4), 3 (grades 5–6) (elementary school), and 4 (grades 7–9) (junior high school).

The aims/missions in geometry for junior high school students are to:

- use the geometric properties of a figure in order to define some categorized configurations.
- indicate the configuration according to given properties.
- illustrate possible relations between elements of a complex figure.
- utilize the properties of configurations in order to solve geometric problems.
- apply calculations in order to deduce the Pythagorean theorem.
- understand parallelism and perpendicularity of two straight lines on a plane.
- finish constructions with ruler and compass by instruction.
- understand geometric properties of triangles.
- understand geometric properties of polygons.
- identify the difference between a statement and its reverse statement.
- understand the definitions and related properties of parallel lines.
- examine whether two plane figures are similar.
- apply the properties of triangular similarity to measurement.
- understand geometric properties of circles.
- utilize properties of triangles and circles for mathematical reasoning.

These missions focus on *the final learning results* of geometric rules, properties, and theorems rather than on *the processes* of how to access these final learning results in different types of settings.

2.2.3 Comparison

The significant difference between German and Taiwanese national standards might be their depth and breadth of elaboration. Though the syllabi in Germany provide more specific elaboration than the educational standards do, the instruction in specific mathematical idea in detail is not given in the syllabi. Only the goals/targets are available in German national standards. In contrast, the Taiwanese national standards provide annotations, delivered in the appendix (see Table 2.3), for explaining the targets, the mathematical concepts or possible instruction of each ability index. An example is given in Table 2.4.

Table 2.4. Example of the ‘annotation of ability index of topics by grades’ and ‘stage by topics’

Ability index of topics by grades	**Content**	**Ability index of stage by topics**
9-s-03	**The students are able to understand properties concerning the similarity of triangles**	**S-4-13**
Annotation	▪ Understand the properties/conditions of AAA (or AA) similarity, SAS similarity, SSS similarity, or parallel lines intercept triangle in different similar triangles ▪ Understand that between two similar triangles, the ratio of corresponding sides = the ratio of corresponding heights = the ratio of corresponding angle bisectors = the ratio of midlines, and the ratio of corresponding areas = the square ratio of corresponding lengths	

Note 1: **9-s-03** is an ability index of *topics by grades*. The first number stands for the school grade *9*, the second letter *s* denotes the topic, geometry; and the third number *03* is a serial number (in order).

Note 2: **S-4-13** is an ability index of *stage by topics*. The first letter *S* denotes the topic *geometry*; the second number *4* means the fourth stage (grade 8–9); the third number *13* is a serial number (in order).

To summarize, though the mathematics curricula of both countries emphasize the understanding of mathematics, the German national standards emphasize that students should be able to *appreciate* mathematics through acquiring different competences in learning mathematics, whereas the Taiwanese national standards stress that students should be able to (technically) *master* the subject. This mastery is defined by specific indices given in the standards.

2.3 Mathematics Textbooks

The national standards may be regarded as the written goals for teaching and learning. Textbooks are the design products based on them. They provide opportunities for teachers to refer when deciding on their lesson plan or to use them in the classroom.

In this section, two issues will be discussed: the general relationship between mathematics curriculum and textbooks and the specific role of mathematics textbooks in German and Taiwanese schools.

2.3.1 Mathematics curriculum and textbooks

The roles of curricula were first examined on three levels, known as the tripartite curriculum model involving *intended curriculum*, *implemented curriculum*, and *attained curriculum*, in the Second International Mathematics Study [SIMS] (Garden, 1987; Robitaille, Schmidt, Raizen, McKnight, Britton, & Nicol, 1993; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). The intended curriculum is what the society/system considers that students should be taught; the implemented curriculum is what students are actually taught in the classroom; and the attained curriculum is what students have actually learned.

There are different curriculum materials, such as teachers' guides, students' books (textbooks), or practice books, which aim at specific needs for teaching and learning. The designs of such curriculum materials, especially textbooks, and their qualities, are increasingly discussed by research studies (e.g., Project 2061, n.d.; Tamir, 1985). Herbel-Eisenmann (2007) emphasizes that the curriculum materials have an important impact on teaching and learning.

> Curriculum materials are valuable tools that can support the teacher's goal of introducing students to the practices and language of the mathematical community. Studies on curriculum materials that preceded national standards in mathematics and science education suggested that textbooks can impact both what and how teachers teach, as well as what and how students learn.
>
> (p. 345)

Furthermore, from a *research-based perspective* of the International Association for the Evaluation of Educational Achievement [IEA] (Valverde et al., 2002), ***textbooks*** are part of *the intended curriculum* because they embody specific academic goals for particular groups of students. From a *practical perspective*, ***textbooks*** represent the *implemented curriculum* because of their organized and structured content that is often used by teachers, especially by novice teachers (Ball & Feiman-Nemser, 1988), in the classroom. However, there is an argument that a textbook cannot fully reflect the implemented curriculum, because the way in which a textbook is used in the classroom depends on and varies between teachers.

Valverde and colleagues (2002) describe the role of the textbook with respect to the different types of curricula:

> [Textbooks] are the mediators between intention and implementation. Curriculum policy makers make decisions regarding instructional goals. These are shaped into instruments such as content standards, curriculum guides, frameworks, or other such documents. Unfortunately, these documents rarely spell out the operations that must take place to build instructional activities that embody the content present in the standards. However, textbooks are written to serve teachers and students in this way—the work on their behalf as the links between the ideas present in the intended curriculum and the very different world of classrooms (p. 9).

In other words, textbooks embody the ideas of the national standards (intended curriculum) with referable instructional activities designed by textbook authors, but cannot reflect the real instructional situation in the classroom.

2.3.2 Role of textbooks in German and Taiwanese schools

Textbooks are designed based on the national standards and teachers use them as a tool to write their own lesson plan or include them directly in their teaching in Germany and Taiwan. The content of textbooks is designed by a group which encompasses school teachers, researchers, (sometimes) mathematicians, and (sometimes) mathematics educators, and is finally edited by the responsible editors(s). All written textbooks are designed based on the national standards and published with the approval of the Ministry of Education/KMK. Each individual textbook is developed under the textbook editors' intentions.

The role of German and Taiwanese textbooks in the classroom[5] is briefly presented in Figure 2.1. It is adapted from the *conceptual model of curriculum and achievement* (Schmidt, McKnight, Houang, Wang, Wiley, Cogan, & Wolfe, 2001, p. 15) which clarifies the relations between elements/processes of curriculum design and students' learning and achievement based on the tripartite curriculum model.

Figure 2.1 makes clear the relationships between elements of each layer from ideal to reality and the relationships between layers (shown in three different groups of shapes: ellipses, rectangles, and rounded rectangles) from abstract elements to concrete elements.

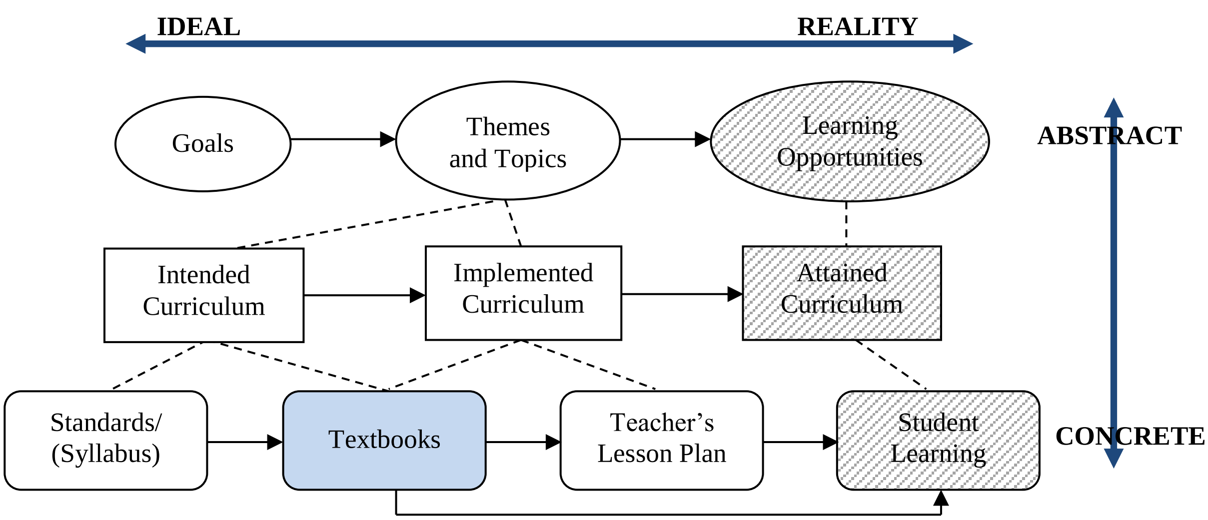


Figure 2.1. The role of German and Taiwanese textbooks (modified from Schmidt et al., 2001, p. 15)

The three layers with their respective trajectory show their elements involved. The middle layer presents the three different roles of a curriculum (the tripartite curriculum model) and provides the bridge between the abstract ideas/plans (in ellipse) and the concrete materials (in rounded rectangles).

[5] The actual situation of using textbooks in the classroom depends on the teacher. This structure presents a general situation of the classroom which adopts textbook as an instructional material.

As discussed before, textbooks could be part of the intended curriculum or the implemented curriculum. Textbooks in Germany and Taiwan are provided as the intended curricula, especially by those teachers treating the content of textbooks as the goals to achieve (e.g. using them in the design of lesson plan), and as the implemented curriculum by those teachers using textbooks directly for teaching in their class.

3. THEORETICAL BACKGROUND

This chapter deals with the theoretical background related to learning school geometry, mathematical proof, and the role of textbooks. The first section introduces the role of mathematical knowledge in learning mathematics. The second section illustrates mathematical proofs in school. The third section presents the learning content of geometry in school. The last section provides an overview of studies concerning the analyses of mathematics textbooks.

3.1 Mathematical Knowledge

The relationship between mathematical knowledge and learning is complex. The learning of mathematics is influenced by an individual's experience or background which may influence the interpretations of mathematical knowledge (Gowers, 2007). Nevertheless, learning mathematics mostly happens in schools. In order to provide a close link between mathematical knowledge and school mathematics, the following discussion will mainly focus on their connections.

3.1.1 Mathematical knowledge and learning mathematics

It is commonly accepted that epistemological rigor and validity/truth are crucial to mathematics as a subject. In order to generate mathematical knowledge, mathematicians have to make sure that a proposition[1]/statement can be proved. It is widely accepted that "mathematical knowledge consists [primarily] of a set of propositions together with their proofs" (Ernest, 1991, p. 3), which means that mathematical knowledge is in particular validated by the proofs of the propositions involved. Without this validation of propositions by proofs, mathematical knowledge can be fallible. According to G. H. Hardy, "a mathematical theorem is a proposition; a mathematical proof is clearly in some sense a collection or pattern of propositions" (Hardy, 1929, p. 3).

[1] A proposition is a statement of a theorem, and an explanation of how it can be proved (Oxford Advanced Learner's Dictionary [OALD]).

Balacheff (2010) indicated that "mathematical ideas do not exist as plain facts but as statements which are accepted only once they have been proved explicitly; before that, they cannot be instrumental either within mathematics or for any application" (p. 117). This opinion compensates what Hardy proposed and implies the importance of proving in teaching and learning mathematics. Moreover, the growth of mathematical knowledge, which seems to provide a network between mathematical concepts, is different from the growth of scientific knowledge, which seems to evolve in response to experience (which emphasizes on the importance of *observations* and *experiments*) (cf. Kitcher, 1984)[2].

In addition to emphasizing the process of learning mathematical knowledge, Balacheff (2010) also describes different 'intellectual postures' of learners in learning mathematics and suggested that getting involved into mathematics means for learners to change their postures and to become *theoreticians*. He provided two different types of shift (from the pragmatism to the theoretician) as examples in learning mathematics. One is the shift from *practical geometry* (e.g., the geometry of drawings and shapes) to *theoretical geometry* (e.g., the deductive or axiomatic geometry). The other is the shift from *symbolic arithmetic* (e.g., computation of quantities by using letters) to *algebra*. The different postures of learning mathematics involve dealing with different mathematical elements and knowledge.

3.1.2 The process of acquiring mathematical knowledge

Mathematics content and the ways of instruction might play an important role in influencing the process of acquiring mathematical knowledge. Both mathematical content and the ways of instructions and therefore also the process of acquiring mathematical knowledge might differ between countries (e.g., Kawanaka, Stigler, & Hiebert, 1999; Stigler, Gallimore, & Hiebert,

[2] Cf. Chapter 7: Mathematical change and scientific change.

2000). How to differentiate the processes of learning mathematics hence becomes an important issue. To do so, it is necessary to have a look at theories about the object of knowledge and the process of acquiring this knowledge.

Mathematical knowledge is commonly conceived as consisting of two types: *conceptual knowledge* and *procedural knowledge* (e.g., Hiebert, 1986). Conceptual knowledge is not only the knowledge encompassing mathematical facts and properties but also the knowledge being rich in relationships. Procedural knowledge is the knowledge of written symbols in the syntactic system or the set of rules and algorithms that are used to solve mathematical problems (Hiebert & Lefevre, 1986; Hiebert & Wearne, 1986).

Moreover, Sfard (1991) used two types of mathematical conception to differentiate the ways to conceive abstract mathematical notions (knowledge), the *structural conception* and the *operational conception*. Abstract notions are treated as *objects* in view of the structural conception while abstract notions are treated as *processes* in view of the operational conception.

With respect to mathematical knowledge of a specific topic, Lampert (1986) identified four types of mathematical knowledge: *intuitive*, *concrete*, *computational*, and *principled* knowledge, in teaching and learning multiplication. The different ways discerning mathematical knowledge are concerned with the tenet—*the ways of knowing mathematics* (Lampert, 1986)—and focus on two components—*process* and *object*.

With regard to the learning of mathematical knowledge in different societies, studies have identified some significant differences between Germany and Taiwan.

Blum and colleagues (1992) considered that teaching mathematics in German classrooms will "place the understanding of *structures* and *general principles* in the foreground and lead to a low importance being attached to active work through examples" (p. 114). This shows that the special emphasis on the structures and general principles influences the focal instruction in comprehending the structures of knowledge rather than the generalization or application of knowledge in Germany.

In contrast, Lin and Tsao (1999) argued that mathematics content is significantly influenced by competitive examinations in Taiwan. They proposed that Taiwanese textbooks are not developed to support knowledge construction, but rather present *a glossary of mathematical knowledge* emphasizing problem-solving algorithms that are augmented by well-chosen examples and followed by exercises. Therefore, the discrimination of the role of knowledge or the ways of acquiring knowledge can provide a systematic mode of classifying knowledge.

In addition, it is not only (objective) knowledge, which influences learning mathematics, but also students' beliefs on the certainty of mathematical knowledge.

3.1.3 Epistemological beliefs

Research in how beliefs influence students understanding mathematical knowledge provides a useful view on reflecting the instruction in classroom. In a meta-analysis study, Muis (2004) investigated students' epistemological beliefs about mathematics from some studies and concluded the ineffectual beliefs for learning, which might happen at all grade levels.

> [W]hen asked about the certainty of mathematical knowledge, students believe[d] that knowledge is unchanging. The use and existence of mathematics proofs support this notion, and students believe the goal in mathematics problem solving is to find *the* right answer. Students also believe mathematics knowledge is passively handed to them by some authority figure, typically the teacher or textbook author, and that they are incapable of learning mathematics through logic or reason. (Muis, 2004, p. 330)

Importantly, students believe that exercises from mathematics textbooks can be solved only by the methods presented in the specific section of the textbook (Garofalo, 1989). They believe that teachers and textbooks are authorities on mathematical knowledge and accept the knowledge presented to them without challenging, e.g., "conscious guessing" (Lakatos, 1976; Lampert, 1990; Schoenfeld, 1985). Students' passive views on learning mathematical knowledge might influence their opportunities to learn mathematics. However, "by understanding the nature and influence of epistemological beliefs on students' performance, instruction can be modified to encourage students to be thoughtful, persistent, and independent learners" (Schommer, Crouse, & Rhodes, 1992, p. 442). Moreover, understanding of students' epistemological beliefs may add further information about whether the designs of curriculum materials are in an appropriate and effective way to improve teaching and learning.

3.1.4 Pragmatic issues

Applying previously learned mathematical knowledge in different situations does not mean to just replicate mathematical knowledge. Rather, it involves decision-making processes to select and connect proper knowledge in solving different tasks with specific strategies. These aspects will be discussed in the following.

3.1.4.1 Relationship between mathematical reasoning and problem solving

> We need *skills* and not only *understanding*, and skills can be acquired only by practical, systematic training. The reciprocal is also sometimes forgotten. Mathematical reasoning cannot be reduced to a system of solving procedures.
>
> (Fischbein, 1994, p. 232)

The words of Fischbein emphasize that recognizing an organized set of concepts does not mean being able to solve a class of problems with the similar components, e.g., mathematical properties or rules. Fischbein stressed that *mathematical understanding* (in reasoning) and *mathematical skills* (in problem solving) are both important in learning mathematics. Glaser (1984) also suggested that the process of acquiring the *structures of knowledge and skills* can help to connect reasoning with problem solving. Therefore, mathematical reasoning and problem solving are not independent but closely related.

3.1.4.2 The mechanism of mathematical knowledge, strategies, and problem solving

Knowledge is not just a 'basket of facts' (Anderson, 1984), and the absorption of knowledge does not guarantee true understanding. The factors influencing the learning of knowledge are complex. Next to the amount of knowledge to be learned, other aspects need to be considered, such as the strategies applied in using knowledge or problem solving. The application of different strategies in solving mathematical problems usually involves the selection from different (pieces of) knowledge.

The example below presents the mechanism of problem solving applying different mathematical knowledge and strategies. A problem like the one illustrated in this example is commonly seen in Japanese classrooms in order to determine the intersected angle within a pair of parallel lines (Kawanaka et al., 1999; Stigler & Hiebert, 1999/2009; see Figure 3.1). It shows that different pieces of mathematical knowledge and strategies can be involved in the same task with different solutions.

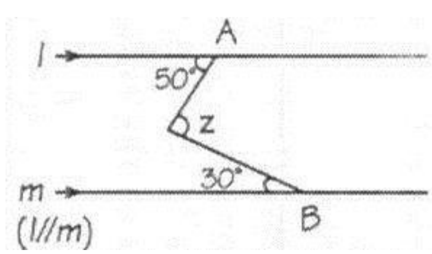


Figure 3.1. A task determining the unknown angle (Kawanaka et al., 1999, p. 98)

The two possible solutions presented in Figures 3.2 and 3.3 involve different sets of knowledge and strategies. Nevertheless, the targets of them and the grounded mathematical ideas namely the parallel postulates are the same.

Figure 3.2 shows one solution by constructing an auxiliary line, a parallel, as the strategy. The three different steps that lead to a solution and their involved knowledge are: (1) construct an auxiliary line *n*, through the point C, which parallelizes to the lines *l* and *m*; (2) use the property that the alternate interior angles of a pair of parallel lines with a transversal are congruent (parallel postulate) to find the angles 1 and 2: $\angle 1 = 30°$ and $\angle 2 = 50°$; and (3) calculate the sum of angle 1 and 2 as the answer.

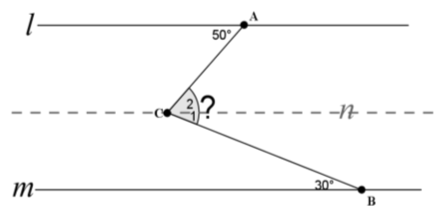


Figure 3.2. Solution 1: constructing a parallel

Another solution by extending the intersection C to the two (parallel) lines includes two strategies for each extension. Since these two extensions denote the same solution, only one extension of segment AC to *m* is given in Figure 3.3 to present its two strategies and their

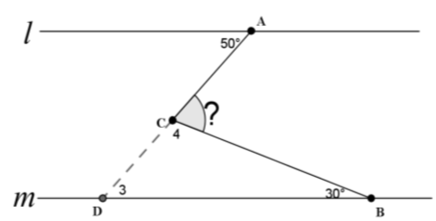


Figure 3.3. Solution 2: extending the intersection (two strategies)

involved knowledge. Two separate sets of mathematical knowledge are used in each strategy. The four different steps that lead to one strategy and their involved mathematical knowledge are: (1) construct the auxiliary line (which is the extension of segment AC) and intersect line *m* at point D; (2) use the property that the alternate interior angles of a pair of parallel lines with a transversal are congruent (parallel postulate) to find the angle 3: $\angle 3 = 50°$; (3) use the property of the sum of interior angles of a triangle to calculate the size of angle 4: $\angle 4 = 180° - (50° + 30°) = 100°$; and (4) use the property that the interior angle and exterior angle of a straight angle are supplementary (two angles together make 180°) and determine the unknown angle: $180° - 100° = 80°$. Another strategy is leaded by three steps. The pieces of involved mathematical knowledge of these three steps are: (1) construct the auxiliary line (which is the extension of segment AC) and intersect line *m* at point D; (2) use the property that the alternate interior angles of a pair of parallel lines with a transversal are congruent (parallel postulate) to find angle 3: $\angle 3 = 50°$; and (3) use the exterior angle theorem of a triangle (the sum of any two interior angles of a triangle is equal to the size of the third interior angle) to determine the unknown angle: $50° + 30° = 80°$.

Retrieving a specific set of knowledge depends primarily on the experience of using or understanding it. The example described above illustrates the mechanism applying different mathematical knowledge and strategies in solving the same problem. Though the final answer is the same, the processes of reasoning the solution are different.

3.2 Mathematical Proofs

In order to produce a mathematical proof, it is not sufficient to consider the specific forms/representations (e.g., the representation in two columns), or syntactic rules. It is even more important to provide valid evidence to the statements (cf. Jaffe & Quinn, 1993; Thurston, 1994).

In school, the rigor of mathematical proofs is not necessarily as strict as it is among mathematicians. Though there are some advocates of mathematical rigor, it might differ by subjects (e.g., mathematicians and students) or contexts.

> Valid proofs are often associated with the idea of *rigor*. In many classrooms, there is a de facto definition: a proof is rigorous if there is a reason given for each step. **Yet**, among mathematicians, rigor varies depending on time and circumstance, and few proofs in mathematics journals meet the criteria used by secondary school geometry teachers. Generally one increases the rigor only when the result does not seem to be correct. (Usiskin, 1987, p. 25)

Thereby, focusing on the validity of the mathematical statements becomes an important mission of doing mathematical proofs instead of the rigor. Specifically, "the flow of ideas and the social standard of validity" (Thurston, 1994) should be emphasized.

3.2.1 The functions of mathematical proofs

Most students "see mathematics as a collection of rules, procedures, and facts that must be remembered" (Richland, Stigler, & Holyoak, 2012, p. 191). However, mathematicians view reasoning the relationships between (mathematical) objects or processes as the central of mathematics learning. Such difference between the novices and the experts comes from their divergent understanding or experiences on mathematics learning.

Harel (1998) found that "*when students have a clear purpose for a concept, they are unlikely to misunderstand its meaning*" (p. 505). However, the purposes for a concept are complex and cannot be separated from the discussion of its functions. Weber (2002) compiled four purposes into two categories for introducing mathematical proofs in the classroom. The first category contains proofs that provide knowledge about mathematical truth. It is composed of two purposes, *convincing* and *explaining* (Hanna, 1989, 1990; Hersh, 1993). In the second category, there are proofs that justify the use of terminology and proofs that illustrate technique. It is not focused on gaining knowledge about mathematical truths, but about *why an obvious conclusion is true* and *how to prove with the assistance of other proven or unproven theorems*.

In order to differentiate the specific purposes of proof activities, de Villiers (1990, 1999) made a list of six different functions of a proof which is commonly approved and very often used by mathematics educators:

> - *Verification* (concerned with the truth of a statement);
> - *Explanation* (providing insight as to why it is true);
> - *Systematization* (the organization of various results into a deductive system of axioms, major concepts and theorems);
> - *Discovery* (the discovery or invention of new results);
> - *Communication* (the transmission of mathematical knowledge); and
> - *Intellectual challenge* (the self-realization/fulfillment derived from constructing a proof).
>
> (de Villiers, 1999, p. 5; emphasis in original)

Moreover, G. Stylianides (2009) clarified four purposes, which partially overlap de Villiers' six purposes, of proofs:

> - *Explanation*, when the proof provides insight into why a claim is true or false.
> - *Verification*, when it establishes the truth of a given claim.
> - *Falsification*, when it establishes the falseness of a given claim.
> - *Generation of new knowledge*, when it contributes to the development of new results used to describe products that solvers in a particular community add to their knowledge base as a result of constructing a proof (p. 269).

The importance of the connection between mathematical knowledge and proofs should be taken into account. Mathematical proofs do not only have the purpose of *validation*—confirming the truth of an assertion (statement), but also have to contribute more widely to *knowledge construction* (Mariotti, 2006) which corresponds to the fourth purpose G. Stylianides clarified.

3.2.2 The types of mathematical proofs

Balacheff (1988) categorized four different types of students' proofs in two groups. One group is *pragmatic proofs*, including (1) *naïve empiricism* which consists of asserting the truth of a result after verifying several cases, and (2) *crucial experiment* which refers to verifying a proposition on an instance which 'does not come for free' and asserting that 'if it works here, it will always work'. The other group is *conceptual proofs*, consisting of (3) a *generic example* which involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, and (4) *thought experiment* which invokes action by internalizing it and detaching itself from a particular representation. According to Balacheff's definitions, pragmatic proofs are "those having recourse to actual action or showings" (p. 217), while conceptual proofs are "those which do not involve action and rest on formulations of the properties in question and relations between them" (p. 217). The two types of pragmatic proofs do not establish the truth of an assertion; and the two types of conceptual proof do not mean to be a matter of 'showing' the results are true, but "concerns establishing the necessary nature of its truth by giving reasons" (p. 218). Balacheff claimed that these types form a hierarchy, and that moving from the generic example to the thought experiment requires a transfer from (physical) action to internalized action and a decontextualisation.

Harel and Sowder (1998, 2007; see also Harel, 2007) proposed a proof scheme framework composed of three different classes of proof schemes: *external conviction proof schemes*, *empirical proof schemes*, and *deductive proof schemes* (original: analytical proof scheme). The external conviction proof schemes[3] mean that proving depends on an authority (e.g., a teacher or a textbook), on strictly the appearance of the argument (e.g., a two-column format in geometry

[3] They consist of three proof schemes: The authoritarian proof scheme, the ritual proof scheme, and the non-referential symbolic proof scheme.

proof), or on symbol manipulations, with the symbols or the manipulations having no potential coherent system of referents in the eyes of the student. The empirical proof schemes[4] rely on either the evidence from example(s) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions and so forth, or perceptions. The deductive proof schemes[5] involve the processes of generality, operational thought, and logical inference.

G. Stylianides (2005; 2009) differentiated non-proof arguments from mathematical proofs. He used two criteria, *empirical argument* and *rationale*, to determine them as non-proof argument. He defined the notion of an empirical argument and presented how students engage in empirical arguments as follows:

> An empirical argument […] purports to show the truth of a mathematical claim by validating the claim in a proper subset of all the possible cases covered by the claim […] Students' engagement in empirical arguments, which are invalid, is likely to reinforce the common misconception that examples can prove general mathematical claims. (Stylianides, 2009, p. 266)

Moreover, there are at least three reasons that he considered why students should engage in rationales:

> 1. Rationales do not support the development of inaccurate understandings of proof that would have later on to be addressed by instruction.
> 2. Rationales are valid but not as developed as proofs. Rationales offer a good choice of an argument when the production of a proof is impractical (e.g., due to time constraints), impossible (e.g., due to conceptual barriers), or undesirable (e.g., due to the focus of activity being on a concept other than the justification of a claim).
> 3. Rationales are valid but less developed arguments than proofs. Rationales can be more easily accessible than proofs and, thus, have the potential to serve as transitional stage between empirical arguments and proofs. (Stylianides, p. 267)

[4] They are composed of the inductive proof schemes and the perceptual proof schemes.
[5] They include the transformational proof schemes and the axiomatic proof schemes.

He later used both of them (empirical argument and rationale) in the discussion of the hierarchy of arguments to discriminate the levels of sophistication between non-proof arguments and proofs (see Figure 3.4).

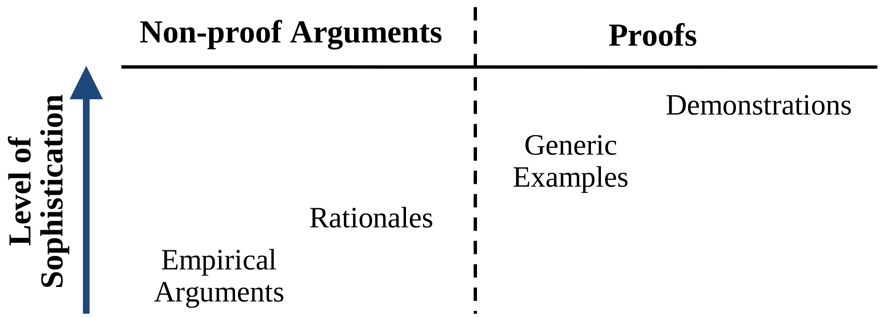


Figure 3.4. Hierarchy of arguments based on their level of mathematical sophistication (Stylianides, 2009, p. 280)

3.2.3 Teaching and learning mathematical proofs

Teaching and learning mathematical proofs are complex activities. Many different aspects need to be considered. For example, students' learning difficulties in mathematical proofs (e.g., the use of strategies in constructing proofs, the objects in the proof stand, and the lack of arguments during the construction of proofs; Boero, Garuti, Lemut, & Mariotti, 1996; Chazan, 1993; Weber, 2001; Zaslavsky, Nickerson, Stylianides, Kidron, & Wincki-Landman, 2012), as well as mathematics teachers' knowledge of mathematical proofs (Knuth, 2002a, 2002b; Ko, 2010) are widely discussed.

Moreover, there are many different functions that mathematical proofs bear and various ways to present mathematical proofs; nevertheless, the essential principle for mathematical proofs is "to *specify clearly the assumptions* made and to *provide an appropriate argument* supported by *valid reasoning* so as to draw necessary conclusions" (Hanna & de Villiers, 2008, p. 329).

A. Stylianides (2007) proposed a definition of proof in school mathematics by analyzing Ball's teaching experiments which focused on the third graders' reasons and mathematical arguments in a public elementary school in the U.S. He elaborated the elements of the conception and illustrated its applicability even in the early elementary school. He treated a proof as a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim with the following characteristics:

> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community.
>
> (A. Stylianides, 2007, p. 291)

He also gave the examples regarding these three components of a mathematical argument (see Table 3.1).

Table 3.1. Examples of the three components of a mathematical argument (A. Stylianides, 2007)

Component of an argument	**Examples**
Set of accepted statements	• Definitions, axioms, theorems, etc.
Modes of argumentation	• Application of logical rules of inference (such as modus ponens and modus tollens); • Use of definitions to derive general statements; • Systematic enumeration of all cases to which a statement is reduced (given that their number is finite); • Construction of counterexamples; • Development of a reasoning that shows that acceptance of a statement leads to a contradiction, etc.
Modes of argument representation	• Linguistic (e.g., oral language), physical, diagrammatic/pictorial, tabular, symbolic/algebraic, etc.

Ball and colleagues (2002) defined 'mathematical reasoning' as a set of practices and norms that are collective, not merely individual or idiosyncratic, and that are rooted in the discipline. Moreover, Tall and colleagues (2012) provided a *proof structure* (see Figure 3.5) which represents the hierarchical level of the development of proofs:

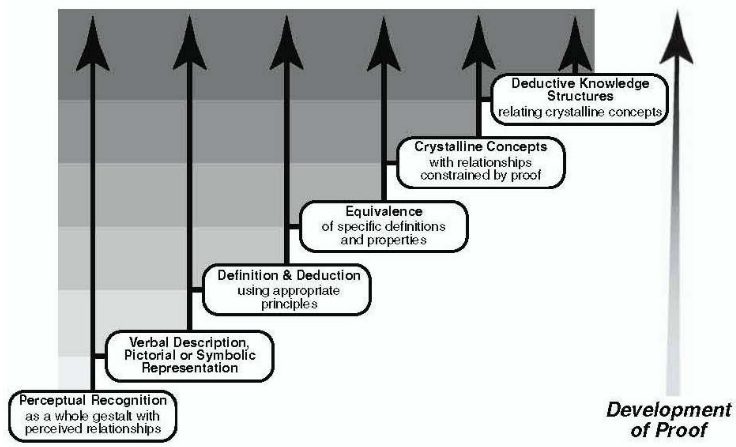


Figure 3.5. The broad maturation of proof structure (Tall et al., 2012, p. 2)

This structure shows *the successive stages* to develop and interrelate one with another, and the gradually *sophisticated knowledge structures* (shades of grey) connected together as each new stage develops and matures. Therefore, the consideration for suitable knowledge, strategies, and the gradual processes to introduce mathematical proofs to students is important for teaching and learning.

Concerning teachers' knowledge of teaching mathematical proofs, A. Stylianides (2011) used *knowledge package* to describe a cluster of related kinds of knowledge (of proof) which are important to teachers to teach effectively a particular mathematical idea (mathematical proof) in classrooms. This knowledge package is *not only* the description of teachers' concept map (Ma, 1999) or how teachers organize structure within their mathematical knowledge for teaching, *but also* a delicate description of teachers' knowledge about students' conceptions of such particular idea and teachers' pedagogical knowledge to help the practice and implementation of tasks in classrooms. To sum up, he emphasized the importance of teachers' *mathematical (subject-matter) knowledge* about proofs, *pedagogical content knowledge* about students' understanding of mathematical proofs, and *pedagogical knowledge* for teaching proofs in classrooms. He provided prospective (mathematics) teachers instructional intervention regarding the misconception that *empirical arguments are proofs*. Three joint activities along a 'learning trajectory' (cf. Stylianides & Stylianides, 2009), from a naïve empirical conception, to a crucial experiment conception, and to a non-empirical conception were given in the intervention. He found that such research-based instructional intervention is effectively helpful to expand their knowledge for teaching proof.

In view of instructional approaches to mathematical proofs, Hanna and Jahnke (2002) suggested a new approach—the application of physics (the argument from physics)—which is different from simple physical representation of mathematical concepts. They claimed that it is important to convey the concept of mathematical proof to students by presenting fresh and more attractive approaches to the teaching of proof, especially in those (Western) countries which are being away from using proofs in classroom. They presented a teaching unit with the instruction of some examples using *the principle of statics* (the lever principle) to find the center of gravity and then the Varignon theorem (given an arbitrary quadrangle, the midpoints of its sides form a parallelogram). After the instruction of the application of physics in this situation, they assigned a work to students to prove the Varignon theorem with two different methods—the application of physics and the traditional geometric proof[6] (divide the quadrangle into two triangles and then apply the similarity conditions). They found that the preferences of 25 Canadian students (grade

[6] A traditional method (purely geometry proof) of the Varignon theorem needs to divide the quadrangle into two triangles by constructing the diagonal of the quadrangle and apply the intercept theorem of similarity conditions twice to prove that two pairs of the opposite sides are parallel.

12) for these two methods were different. However, these students accepted that concepts and principles applied in physics could be used in proving mathematical theorems.

3.2.4 Mathematical proofs in school curriculum

In order to evaluate the mathematics consistency with the (national) *Curriculum Standards*, NCTM (1989) suggested that the examination of curriculum and instructional resources is necessary and should be focused on:

- goals, objectives, and mathematical contents;
- relative emphases of various topics and processes and their relationships;
- instructional approaches and activities;
- articulation across grades;
- assessment methods and instruments;
- availability of technological tools and support materials.

(NCTM, 1989, p. 241)

These points help to provide a broad overview of reviewing the quality of curriculum and instructional resources. Moreover, it is necessary to consider the further criteria when discussing different topics of mathematics. When focusing on special topics, such as geometry, data processing, or numbers, the foci of the concepts and strategies used in each topic vary from each other.

The studies of the current research indicated the complexity of the ideas of proofs and the difficulties that teachers and students face when proofs become part of mathematical activities in classroom (Mariotti, 2006). It stimulates more and more researchers and mathematics educators to re-concern the importance of proofs and its need in mathematics curriculum.

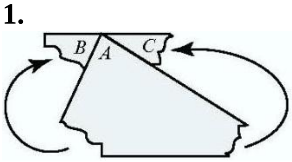
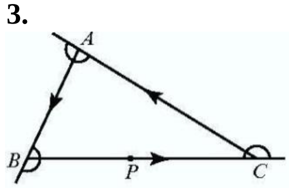
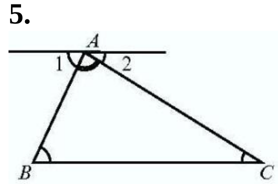
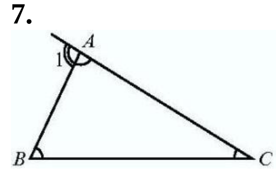
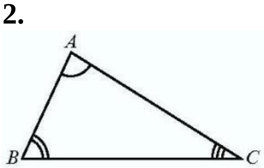
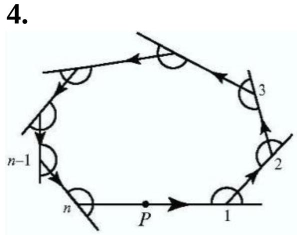
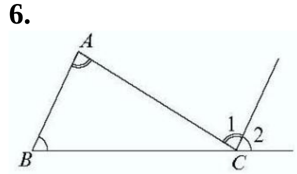
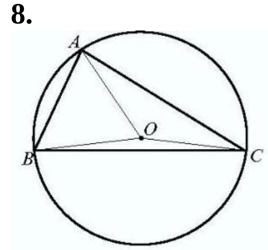
The curriculum materials in mathematical proof play an important role in schools. Research studies (Chazan, 1993; Hoyles, 1997; Healy & Hoyles, 2000; G. Stylianides, 2007) indicate that the basic features, such as content, organization, and sequencing, of the curriculum have an impact on students' conception of proofs.

Hoyles (1997) supposed that the wider influences of curriculum organization and sequencing cannot be ignored. She and colleagues conducted a nationwide research project of the conceptions of justification and proof in geometry and algebra amongst 15-year-old U.K. students. They investigated the students' understanding of proof and the proving process in mathematics and found that the students' *approach to proof* and *flavor of seeing proof* through the presentation of a selected sample of questions with some other students' responses echo their investigation on curriculum. That is, the content and the designs of curriculum are correlated to students' performances on proofs.

Moreover, the arrangements/designs of curriculum differ from country to country, and shape students' ways of thinking. Knipping (2002) compared proof and proving in geometry teaching between France and Germany. She differentiated the teaching contexts into *the introductory phases* and *the phases of exercises* to analyze the meaning and the role of proofs. She found that *the introductory phases* are essential for German teaching while *the phases of exercises* are central to French teaching. Based on the observed German lessons, she pointed out that *the discovery of theorems based on special cases* is a typical teaching pattern for proof in Germany. Its function is *to expand the students' knowledge*. In contrast to the German ways of teaching proof with an emphasis on understanding and meaning, *successful defense of claims of validity of statements* (her original terms: mathematical assertions) can be described as typical ways in the observed French lessons, that is, the teaching patterns of *justification for the problems* is typical for proof in France. Therefore, the instruction of mathematical proofs is to "understand why" in Germany, whereas to "defend why" in France (Knipping, 2002).

Furthermore, the representation for a proof should not be uniform. Relying on a linear chain of arguments to discuss a proof is not regarded an appropriate way to deal with proof. According to the special characteristics of mathematical proof, Leron (1983; 1985) suggested that proofs should not be presented as a linear chain of arguments but according to their structure. This is especially apt for comparing different representations of a proof. For example, Tall and colleagues (2012) listed eight different strategies of students in doing proofs of *the sum of interior angles of a triangle* from three different countries—Germany, Taiwan and UK (Healy & Hoyles, 1998; Lin & Cheng, 2003; Reiss, 2005). The different presentations of these eight strategies are shown in Table 3.2.

Table 3.2. Different presentations of the statement—the sum of interior angles of a triangle

1. 	**3.** 	**5.** 	**7.** 
2. 	**4.** 	**6.** 	**8.** 

These eight representations, though focusing all on "proving" the same statement, concern different aspects of mathematical knowledge. In order to discriminate the opportunities to learn these strategies in different countries, it is necessary to focus on their curriculum design respectively.

3.3 Geometry

Farrell (1987) discussed the nature of geometry by viewing its two basic aspects: *product* (e.g., defined concepts, postulates, theorems) and *process* (e.g., *deducing*: computing, hypothesizing, proving by logical rules, defining; *inducing*: conjecturing, testing, generalizing; *idealizing*:

formulating, symbolizing, abstracting). Usiskin (1987) categorized geometry in school mathematics into four major dimensions (1–4 in the following list) and two minor dimensions (5–6 in the following list), which integrate the above two aspects and the *setting*.

1. The *measurement-visualization* dimension (geometry as the visualization, construction, and measurement of figures);
2. The *physical real-world* dimension (geometry as the study of the real, physical world);
3. The *representation* dimension (geometry as a vehicle for representing other mathematical concepts);
4. The *mathematical-underpinnings* dimension (geometry as an example of a mathematical system);
5. The *sociocultural* dimension (dealing with the history and development of ideas); and
6. The *cognitive* dimension of understanding (involving one's mental images and cognition, particularly studied by psychologists).

These six dimensions lead to different didactical questions. They are presented and discussed in three issues below: curriculum, teaching and learning, and proofs.

3.3.1 Geometric curriculum

The debate about the position of geometry in the curriculum is influenced by a 'dilemmas' of geometry learning in schools (Allendoerfer, 1969; Senk, 1985; Usiskin, 1987). On the one hand, Lang and Ruane (1981) considered that, in tradition, geometry was examined in the context of *deduction*. Though some questions are given with specific measurements of lines or angles, most of them are set in a general framework which is comparable to the proof of theorems in that no measurements are involved. On the other hand, there are some opinions considering that geometry is "proofless mathematics" (mathematics without proof) (Wheeler, 1990). Another significant debate may be described by the influential slogan 'Euclid must go' from the demand on Dieudonné (1906–1992) in 1959.

Allendoerfer (1969) listed five major objectives of geometry that should be included in schools. They are:

(1) An understanding of the basic facts about geometric figures in the plane and geometric solids in space
(2) An understanding of the basic facts about geometric transformations such as reflections, rotations, and translations
(3) An appreciation of the deductive method
(4) An introduction to imaginative thinking
(5) Integration of geometric ideas with other parts of mathematics

(Allendoerfer, 1969, p. 165–166)

Besides, he suggested a reasonable curriculum for geometry in different levels for schools:

(1) Elementary school—informal plane and solid geometry and geometric transformations;
(2) Junior high—more informal geometry, use of coordinates in algebra, graphing, elements of deductive proofs;
(3) Tenth grade—formal deductive plane geometry with informal solid geometry. Possible inclusion of brief analytic geometry; and
(4) Eleventh or twelfth grade—full semester of plane and solid analytic geometry and geometric transformations in preparation for use in calculus.

(Allendoerfer, 1969, p. 169)

3.3.2 Teaching and learning geometry

The operation, calculation, and argumentation with figures make geometry a powerful topic for students. Geometry provides the best opportunities to learn how to mathematize reality, to make discoveries with geometric configurations, computation, arguments (Freudenthal, 1973; Griffiths & Howson, 1974).

3.3.2.1 Figural comprehension

Geometry mobilizes at least two multifunctional registers (Duval, 2007)—*natural language* (in order to "explain") and *geometric figure* (in order to "see"). Duval clarified that "becoming aware of the functioning of valid reasoning is absolutely essential *whenever deduction has to compensate for the limitations of vision and visualization*" (p. 160) (cf. discursive apprehension and perceptual apprehension; Duval, 1995). This emphasis on configuration is similar to the discussion of the importance of *figural representation* (Mesquita, 1998) and *figural concept* (Fischbein, 1993; Mariotti, 1997). The information borne by a configuration can be more than merely perception.

Moreover, Duval (1995) analyzed a figure with a set of cognitive apprehension to see how a heuristic figure works. He provided four different kinds of apprehension—*perceptual apprehension*, *sequential apprehension*, *discursive apprehension*, and *operative apprehension*—to differentiate the ways of viewing a drawing or an array of visual stimuli. Each apprehension has its specific laws of organizing and processing the visual figures. The resolution of geometric problems very often requires their interaction.

Reasoning with geometric figures concerns not only the perceptual configuration but also the mental decomposition of a two-dimensional figure separated into attributes of length and angle, and even young children (grades 1 and 2) can do that (Lehrer, Jenkins, & Osana, 1998). However, extracting information by decomposing figures is not only the issue of separating the components (e.g., angle and length) from original figure, it requires strategy.

3.3.2.2 Geometric calculation

Another important issue in geometry is the computation with formulae and algebraic reasoning. Hsu (2010) analyzed the differences of tasks, from curriculum materials, with geometric calculations with number (GCN) and with geometric proof (GP). She argued that Taiwanese students' competences of geometric proof are developed through working with numerous/abundant GCN tasks.

The abundant geometric calculation with number does not mean that individuals are the operators working with calculation for the final results, but the active participator searching for the strategy via the method of calculation. Geometric calculation is connected not only to the process of dealing with arithmetic or algebraic problems, but also to the application of appropriate heuristic skills, geometric knowledge (e.g., principles, formulae, properties, theorems) (Hsu, 2007; Lawson & Chinnappan, 2000; Schumann & Green, 2000).

3.3.3 Geometry proof

The study by G. Stylianides (2005) found that different contents—*number theory*, *geometry*, and *algebra*—in the Connected Mathematics Project [CMP] textbooks provide different opportunities for reasoning-and-proving. The number of proofs in *geometry* is higher than in *algebra* but lower than in *number theory*. However, the opportunities provided by geometry are very different from the other two content domains. The components, attributes and the purposes of geometry content are far from the other two.

Chinnappan and colleagues (2012) found that *geometry content knowledge* is an important factor responsible for the development of proofs. In addition, there are the other two factors involved in their research: *general problem-solving skills* and *geometry reasoning skills*. They claimed that all these three knowledge strands were necessary and influenced students' development of geometry proof.

There are studies indicating the differences between experts and students in dealing with geometry proofs (Chazan, 1993; Knuth, 2002a; Martin, McCrone, Bower, & Dindyal, 2005; Senk, 1985). Anderson, Greeno, Kline, and Neves (1981) posed an expert approach, *proof tree* (knowledge structure), to illustrate how students use this strategy with a set of geometric rules to work forward with a geometry problem, that is proof execution, in a quick and efficient way.

However, when facing a new and novel problem, including unusual configurations, with the unfamiliar features, students often fall back to a slow speed to work with it. They analogized this process in solving geometry problems as the development of playing chess, that is, “experts in geometry proof generation have simply encoded many special case rules” (p. 228). However, the strategies of students in solving geometry problems are diverse. Some students lean heavily on their prior knowledge; others try to apply the general rules of geometry directly; and still others (probably the majority) rely mostly on *past examples* to guide their problem solving and learning.

Moreover, empirical studies show that students’ competences of geometry proofs differ from country to country. A national survey with a large-scale quantitative study on proof and argumentation with 659 grade 8 students in Germany and the interviews with ten of these students indicated students’ difficulties in geometry proofs (Reiss et al., 2002; Heinze, 2004). The studies described students’ difficulties with proof and logical argumentation. It showed that there are three main difficulties: (a) insufficient knowledge of facts; (b) deficits in methodological knowledge about mathematical proofs; and (c) a lack of knowledge with respect to developing and implementing a proof strategy. In addition, it indicated that low-achieving students show their difficulties with respect to all three deficiencies and high-achieving students revealed their difficulties in developing an adequate and correct proof strategy (Heinze, 2004). Lin and colleagues conducted a national investigation of junior high school students’ conceptions and performances of mathematical proofs in Taiwan. In analyzing students’ geometry arguments, they found that the Taiwanese students were able to organize their prior knowledge learned in elementary school to solve difficult and unknown or new questions, but were hardly able to retrieve a simple principle to judge and explain why a property was true (Lin & Cheng, 2003).

3.4 Mathematics Textbooks

Stray (1994) concerned textbooks as the situated objects in that textbooks form part of processes of education. He considered that “[t]extbooks are the bearers of messages which are multiply coded […] the coded meanings of a field of knowledge (what is to be taught […]) are combined with those of pedagogy (how anything is to be taught and learned)” (p. 2).

To have a close look on how mathematics textbooks provide opportunities to transmit mathematical knowledge, this section provides first a broad view on the roles and designs of mathematics textbooks, and then presents four models of textbook analyses in analyzing different mathematical contents.

3.4.1 Roles of mathematics textbooks

The textbook is commonly considered as one of the curriculum materials (Ball & Cohen, 1996; Collopy, 2003) because of different messages it carries.

> From a traditional IEA perspective, textbooks are a part of the intended curriculum because they embody specific academic goals for specific sets of students. From a practical perspective, textbooks represent the implemented curriculum because they are most often employed in classrooms to organize, structure, and inform students’ learning experiences.
>
> (Schmidt et al., 2001, p.16)

The roles of mathematics textbook are diversely discussed. Some studies focus on its contents/presentations or positions, including linguistic issue (e.g., Dowling, 1996; Herbel-Eisenmann, 2007; Herble-Eisenmann & Wagner, 2007; Kang & Kilpatrick, 1992; Morgan, 1996); some focus on its influences on students learning mathematics (e.g., Boaler, 1998, 2002; Stein, Remillar, & Smith, 2007); and some focus on its relation to teachers’ learning (e.g., Ball & Cohen, 1996; Ball & Feiman-Nemser, 1988; Collopy, 2003; Remillard, 2005).

3.4.1.1 Textbooks as materials to transmit mathematical knowledge

Mathematics textbooks provide "a typical way of preserving mathematics knowledge" (Kang & Kilpatrick, 1992, p. 3). Textbook authors usually set students as the main readers and write the contents in teachers' positions to transmit knowledge to the readers (Kang & Kilpatrick, 1992; Herbel-Eisenmann & Wanger, 2007).

In order to investigate the contents/presentations of textbooks, Howson (1995) treated mathematics textbook as a material that supplies "teacher-free" texts in order to discern its roles as the source of problems and exercises or as the "kernels"/hard core of mathematics (van Dormolen, 1986) which contains factual knowledge, e.g., theorems, rules, definitions, procedures, notations, and conventions.

3.4.1.2 Textbooks as materials to facilitate teaching and learning

The use of textbooks and its important influences on teaching and learning in classrooms are emphasized and discussed in several studies (e.g., Collopy, 2003; Ball & Cohen, 1996; Ball & Feiman-Nemser, 1988; Haggarty & Pepin, 2002; Howson et al., 1981; Pepin, Haggarty, & Keynes, 2001; Stein et al., 2007; Stodolsky, 1988; Tamir, 1985). Although textbooks are disdained by some teachers who do not use them (Ball, 1996), they generally provide numerous and useful information for school education.

Teachers' usage of a textbook might influence students in learning mathematics (Ball, 1996). Relevant aspects are, for example, how they choose the tasks or how they apprehend and interpret the contents of a textbook. Remillard (2005) examined a teacher's use of mathematics curriculum materials (e.g., textbooks, teacher's guides) from the past decades, and developed a possible *framework of components of teacher-curriculum relationship* for characterizing and studying teachers' interactions with curriculum materials.

3.4.1.3 The quality of mathematics textbooks

"Knowledge of the characteristics of the textbook may help the teacher in deciding how other parts of the course must be modified to take advantage of the useful features and to counteract the undesirable features of the textbook" (Tamir, 1985, p. 92). Therefore, how to differentiate the useful and undesirable features of textbooks or evaluate the quality of textbooks can be considered important questions for researchers and teachers.

Project 2061 (n.d.), funded by the American Association for the Advancement of Science [AAAS], proposes three basic criteria for its evaluation of mathematics textbooks.

> *First*, good textbooks can play a central role in improving mathematics education for all students;
>
> *Second*, the quality of mathematics textbooks should be judged mainly on their effectiveness in helping students to achieve important mathematics learning goals for which there is a broad national consensus; and,
>
> *Third*, an in-depth analysis of much more than a textbook's content coverage would be required to evaluate whether there is potential for students' actually learning the desired subject matter.
>
> (Project 2061, n.d.)

Pepin and colleagues (2001; see also Haggarty & Pepin, 2002) compared the content and structure and the use of mathematics textbooks in English, French, and German classrooms. They found that the complexity and coherence of mathematics textbooks in Germany are relatively higher than the other two countries, in particular concerning the *mathematical logic* and *structure*. Yet, the representations of the contents in Germany are often given in relatively 'dry', especially in Gymnasium textbooks, that is, the contents of mathematics textbooks in Gymnasium track are often presented in a formal way.

3.4.2 Design of mathematics textbooks

The design of textbooks might differ in their representations (micro) or structure of contents (macro). This sub-section focuses on discussing the purposes and the types of texts in order to see the behind design intention.

3.4.2.1 Purposes of texts

The purpose of texts provided by mathematics textbooks is quite broad, but generally the mathematical goals can be briefly summarized as the acquisition of *concepts*, *principles*, *skills*, and *problem-solving strategies* (Shuard & Rothery, 1984). Within these goals, a particular passage of written materials may be intended to

> 1. *teach* concepts, principles, skills and problem-solving strategies;
> 2. give *practice* in the use of concepts, principles, skills and problem-solving strategies;
> 3. provide *revision* of 1 and 2 above;
> 4. *test* the acquisition of concepts, principles, skills and problem-solving strategies;
> 5. develop mathematical *language*, for instance by broadening the pupils' mathematical vocabulary and their skill in the presentation of mathematics in a written form.
>
> (Shuard & Rothery, 1984, p. 5–6)

Additionally, the purpose of texts is highly connected to the types of texts (see next discussion 3.4.2.2). For example, skills and problem-solving strategies are usually taught by means of worked examples (Shuard & Rothery, 1984).

However, these purposes provided in the texts are usually transmitted by teachers in class, that is, these texts are interpreted by teachers. Therefore, Remillard (2000) suggested that the designs of texts "need to be flexible and responsive to teachers' choices as well as incomplete without teachers' input" (p. 346). She considered that good curriculum materials should contain multiple possible routes (strategies), in that it provides space (opportunity) for teachers' decision making.

3.4.2.2 Types of texts

The definitions for types of texts are various. There are studies investigating the types of texts by surveying the *terms* used in the textbooks (e.g., Herbel-Eisenmann, 2007; Miyakawa, 2012) or the specific *features*, e.g., formulae, or geometric features, provided in the textbooks (e.g., Dowling, 1996; Fujita, Jones, & Kunimune, 2009). The above mentioned studies are highly connected to the linguistic approach (Morgan, 1996). However, the criteria to categorize the different texts apart from language issue in order to decide the unit for following analysis are not often discussed.

Shuard and Rothery (1984) summarized five different types of texts in textbooks. They suggested treating these types of text as a crude system of analysis.

> Expo — *exposition* of concepts and methods, including explanations of vocabulary, notation, and rules; summaries are included in this category;
>
> Instr — *instructions* to the reader to write, draw or do;
>
> Exer — *examples and exercises* for the reader to work on; often these are 'routine' problems involving symbols, but they also include word problems, non-routine problems and investigations;
>
> Periph— *peripheral writing*, such as introductory remarks, meta-exposition (writing about the exposition), 'jollying the reader along,' giving clues, etc.; and
>
> Sig — *signals*, e.g., headings, letters, numbers, boxes, logos.

(Shuard & Rothery, 1984, p. 9; emphasis in original)

According to their definitions of different types of texts, it seems that these types of texts bear their distinctive mathematical concepts and provide different purposes to learn.

3.4.3 Models of textbook analyses

Textbook analysis is an emerging study issue, although not all studies discussed the contents cohesively or systematically. There are several models providing comprehensive analytic frameworks for analyzing different mathematical contents, such as fraction (Charalambous et al., 2010), geometry proofs (Miyakawa, 2012), number theory, geometry, and algebra (G. Stylianides, 2005, 2007, 2008, 2009), and algebra and pre-calculus (Thompson et al., 2012). The studies mentioned above are grounded in different methods and theories in framing their specific models and schemes. The contents of the models mentioned above will be introduced briefly in the following.

3.4.3.1 Model for analyzing reasoning-and-proving

G. Stylianides (2005; 2008) developed an analytic framework in analyzing the curriculum materials of the curriculum program *Connected Mathematics Project* [CMP] funded by National Science Foundation [NSF]. His study focused on investigating (1) the opportunities provided by different tasks in three different content area (algebra, geometry, and number theory) to learn reasoning-and-proving, and (2) the ways how these tasks provide inductive and deductive modes of reasoning in middle schools (across the sixth, seventh, and eighth grade levels).

The analytic framework he offered is based on the roles and the functions of reasoning and proving and its components and purposes of different types of reasoning and proving, in different contexts. He defined ‘reasoning-and-proving’ as an activity dealing with two different issues—

making mathematical generalization and providing support to mathematical claims—and discussed these issues with three different groups of components[7]—mathematical, psychological, and pedagogical (see Figure 3.6).

	Reasoning-and-proving			
Mathematical Component	Making Mathematical Generalizations		Providing Support to Mathematical Claims	
	Identifying a Pattern	Making a Conjecture	Providing a Proof	Providing a Non-proof Argument
	• Plausible Pattern • Definite Pattern	• Conjecture	• Generic Example • Demonstration	• Empirical Argument • Rationale
Psychological Component	What is the solver's perception of the mathematical nature of a pattern / conjecture / proof / non-proof argument?			
Pedagogical Component	How does the mathematical nature of a pattern / conjecture / proof / non-proof argument compare with the solver's perception of this nature? How can the mathematical nature of a pattern / conjecture / proof / non-proof argument become transparent to the solver?			

Figure 3.6. The analytic framework of reasoning-and-proving (Stylianides, 2008, p. 10)

3.4.3.2 Model for all (dealing with a general issue)

Charalambous et al. (2010) developed an analytic framework (see Figure 3.7) which integrates two dimensions: (1) *horizontal* part (background issue) and (2) *vertical* part (content issue). The former dimension focuses on the examination of two general categories of textbook characteristics—*background information* and *overall structure*, e.g. physical appearance, the organization of the content across the book (cf. Schmidt, McKnight, Valverde, Houang, & Wiley, 1997; Stevenson & Bartsch, 1992). The latter dimension focuses on examining how textbooks treat a single mathematical concept under the environment with several criteria which are grouped into three categories—(1) (how a concept is) *communicated to students*, (2) (what is)

[7] See also the analytic framework (Stylianides, 2009) focusing on the components in two dimensions: (1) components and subcomponents of reasoning-and-proving; (2) purposes of pattern, conjecture, and proof, instead of components in three groups.

required to students (to do with a concept), and (3) *connections* (between textbooks and the learning situations). Each category of the vertical part contains its respective criteria to analyze.

HORIZONTAL ANALYSIS OF THE TEXTBOOK

Background Information	**Overall Structure**
• Title • Number of books • Pages (Number and Density) • Profile of authors and advisory committee • Publisher and year of publication • Accompanying materials (e.g., teachers' guides, resource materials)	• Number of units/lessons and average number of pages per unit/lesson • Structure of units/lessons • Topics covered • Sequencing of topics

VERTICAL ANALYSIS OF THE TEXTBOOK

Communicated to Students	**Required of Students**	**Connections**
Mathematical Content • Topic-specific construct, structure etc. (e.g. part-whole, ratio, operator, quotient, measure fraction constructs) • Definitions, rules, conventions • Illustrations-representations (irrelevant, relevant to the context but not to the mathematics, supporting the mathematics) *Mathematical Practices* • Worked examples • Modeling thinking *Attitudes* • Equity • View of mathematics	• Potential Cognitive Demands (memorization, procedures with connections, procedures without connections, doing mathematics) • Type of Response (answer only, answer and mathematical sentence, explanation, justification)	• Connecting within and between strands • Classroom instruction - textbook connections • Connecting to situations outside of school

Key: Dimension: Uppercase letters; Categories: bold; Sub-categories: italicized; Criteria: bulleted points.

Figure 3.7. The analytic framework of mathematics textbook analysis (Charalambous et al., 2010, p. 123)

They used this analytic framework to analyze the textbook contents on fraction in Cyprus (grade 4), Ireland (grade 5), and Taiwan (grade 4). Though this study focused on analyzing the content of fraction, the framework is useful in comparing other mathematical content domains. In addition, their study focused the analysis on the worked examples which might be either fully worked out or provide no final answer.

3.4.3.3 Model for analyzing proof-related reasoning

Thompson et al. (2012) built two analytic frameworks (for narratives and for exercises) for analyzing the nuances in proof-related reasoning of the written curriculum, namely textbooks from 20 different companies or curriculum development projects. These two analytic frameworks are for analyzing *narratives* and *exercises*. The narratives and exercises dealing with the topics of exponents, logarithms, and polynomials are examined in this study.

This study is based on the TIMSS curriculum framework consisting of *content* (subject matter topic), *performance expectation* (what students are expected to do with the particular content), and *perspective* (an overarching orientation to the subject matter and its place among the disciplines and in the everyday world) (Valverde et al., 2002). The performance expectations in mathematical reasoning are discussed in further six subcategories, which are mathematical and logical components: (1) *developing notation and vocabulary*; (2) *developing algorithms*; (3) *generalizing*; (4) *conjecturing*; (5) *justifying and proving*; and (6) *axiomatizing*.

3.4.3.4 Model for analyzing the nature of proof

Miyakawa (2012) conducted a cross-cultural comparative study between French and Japanese textbooks. He presented how he investigated the nature of proof by analyzing the geometry contents in two textbook series—one from France and one from Japan. In his model, he provided four steps in identifying different elements provided in the textbooks.

These four steps and the elements investigated in the study are: (1) to identify what is called 'proof' by searching the *terms* such as 'prove', 'justify', or 'explain' in the textbooks; (2) to identify the main characteristics of the *form* of proof, e.g., segment, paragraph; (3) to identify the *interrelations between geometrical objects* (geometrical properties) created by proofs; and (4) to identify the *functions* of proof in the textbook, e.g., justification, explanation. Based on the differences in analyzing these four aspect in French and Japanese textbooks, he found that the nature of (to be taught) proofs differs in France and Japan.

In summary, the ideas of the above mentioned models are practical for the studies of textbook analysis in a systematical structure. Each of them provides different advantages and shortages for achieving different targets which depend on the research aim/s and research questions of each study. For the present study, an analytic framework and a set of principles will be developed (see Chapter 5) to meet the specific requirements for addressing the aims and research questions (see Chapter 4).

4. AIMS AND RESEARCH QUESTIONS

4.1 Aims

There are two aims of this research. The first one is investigating the *opportunities to access mathematical proofs in geometry content of mathematics textbooks in Western Europe and East Asia*. More specifically, the presentations of geometry content in German and Taiwanese textbooks for grades 7–9 are compared. For this purpose, an *analytic framework* for general comparison on mathematics textbooks is developed on the basis of theoretical and practical considerations.

The second aim is focusing on the *essence of school mathematical proofs in Western European and East Asian*. Three specific mathematical statements introduced in German and Taiwanese textbooks are chosen for a systematic comparison. Three *principles* are developed to process this comparison.

4.2 Research Questions

The specific research questions are raised as follows in order to achieve the aims mentioned above.

RQ1. What kinds of elements are emphasized in textbooks to support learning of geometry proofs and how do they differ by country?

RQ2. Can representative approaches of introducing mathematical proofs be identified in German and Taiwanese mathematics textbooks?

5. ANALYTIC FRAMEWORK AND PRINCIPLES

This chapter introduces the *analytic framework* for the textbook analysis (*general comparison*) and the *principles* for specific content comparison (*specific comparison*). The first section aims at presenting the structure and components of the analytic framework and describes how to use it for the analysis of the geometry content in textbooks. The second section introduces three principles—*continuity*, *accessibility*, and *contextualization*—and how to use them to compare the contents of three specific mathematical statements selected from different textbooks. The last section presents an application in systematically analyzing students' different strategies in proving a mathematical statement.

5.1 Analytic Framework

An analytic framework (see Figure 5.1) is developed to analyze the geometry content of textbooks effectively, especially related to mathematical proof. It integrates aspects from

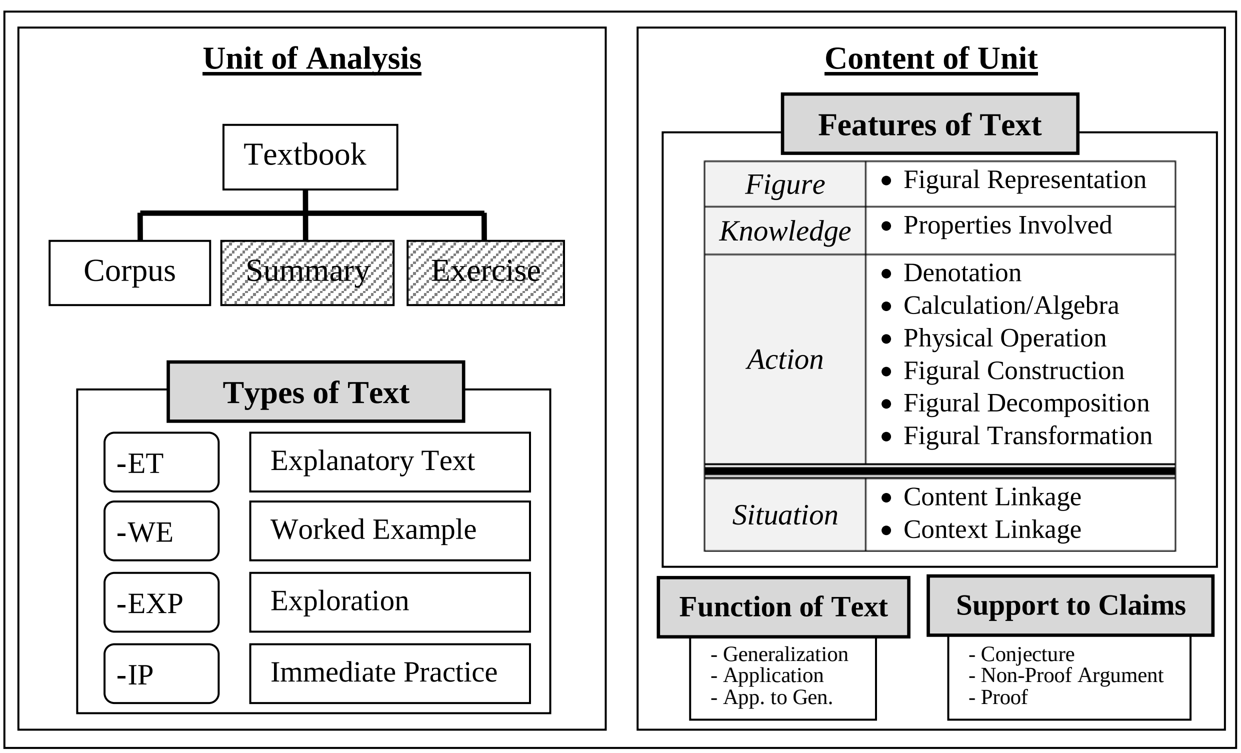


Figure 5.1. The analytic framework

previous research, mainly from the following studies: (1) the functions of texts from Shuard and Rothery (1984), (2) the functions/models/schemas of mathematical and geometry proofs from Balacheff (1988), Hanna (1989, 1990), Harel (2007), and Harel and Sowder (1998), (3) three components (*set of accepted statements*, *modes of argumentation*, and *modes of argument representation*) suggested by A. Stylianides (2007) to judge mathematical argument, (4) the analytic frameworks of textbook analyses developed by Charalambous et al. (2010) and G. Stylianides (2005, 2007, 2008, 2009).

This analytic framework aims at investigating the objective representations of the textbooks in order to minimize subjective interpretations of the materials. The framework is composed of two parts/dimensions, one refers to the *unit of analysis*, and the other is to the *content of unit*.

The purpose of selecting the unit of analysis is to define the category *types of text*. These four types of text are *explanatory text* (ET), *worked example* (WE), *exploration* (EX), and *immediate practice* (IP). A detailed discussion is presented in the next sub-section.

The content of a unit can be assigned to three categories. The first category, *features of text*, discusses the <u>cognitive functions</u> of geometry content and pertains to the four sub-categories *figure*, *knowledge*, *action*, and *situation*. These four sub-categories differentiate between the textual representations (static) and the procedures (dynamic) initiated in the textbook. In particular, how the textbooks deal with the content in different situations is considered in this category. The second category, (practical) *function of text*, provides information about the <u>practicality</u> of each unit. The third category, (the roles of) *support to claims*, also relates to the <u>cognitive functions</u>. It is similar to the first category, but focuses on the content of a unit which provide different types of support in processing reasoning and proving.

In summary, the category *types of text* discusses the forms of texts which are based on the textual patterns (e.g., Table 6.5 in Chapter 6) across the six textbook series. The categories *feature of text* and *support to claims* consider the cognitive function of geometry content (texts). The category *function of text* examines the practicality provided by texts of each unit. In order to use this analytic framework, the categorization of the types of text is the first step of the analysis. Therefore, it is possible to analyze the text according to the other three categories: features of text, function of text, and support to claims. These three categories can be treated independently to process the analyses.

The details of these four categories will be introduced in the following four sub-sections.

5.1.1 Unit of analysis: Types of text

The texts in textbook can be separated generally into three parts—*corpus* (the main body of texts), *summary* (the brief review of one section/chapter), and *exercise* (the pool of exercises related to the mathematical ideas within the section/chapter) (cf. Table 6.5). Only the corpus part is used for the analyses in this study in order to avoid repetition from summary and subjective selection without considering the continuity of mathematical ideas embedded in exercises. Within the corpus part, the texts might be presented in various forms to introduce new mathematical ideas, e.g., expository texts, worked examples, exploration activities (cf. exposition; Shuard & Rothery, 1984).

Most studies of textbook analysis are grounded on analyzing tasks of worked examples. These 'worked examples' might be provided with solution or without solution (Charalambous et al., 2010). However, this study tries to investigate *how a mathematical idea is introduced* and such introduction is usually arranged in the corpus part. How to define and categorize the units from the texts into different types to analyze will be discussed below.

To determine the unit of analysis and to decide on its types of text is the first step in using the analytic framework. A practical question is how to deal with pieces of text that are not directly relevant to the content, such as “now, we are going to introduce the next section”; “below, let us focus on some examples to see whether they belong to a similar context”. Such sentences of *peripheral writing* (Shuard & Rothery, 1984, p. 9) are commonly seen in the Taiwanese textbooks and might express the authors’ intention to inform the readers what the next steps are. If these intentions inform about the connection to specific mathematical conceptions or ideas, they will be regarded as a unit to be analyzed, otherwise, they will be excluded.

5.1.1.1 Operational definitions

The operational definitions of the four types of texts[1] are provided (see Table 5.1). They are adapted and revised from Shuard and Rothery (1984) to suit the selected textbooks of this study. The original definitions by Shuard and Rothey integrate the *functions* provided by the text and the *actions* used in the text. However, the same function or the same action might be provided or used in different types of text. In order to differentiate the functions and actions from the types of text, it is necessary to re-define the types of text. Therefore, in this study, the four types of text are defined according to the properties of activity. Their functions and actions will be described within each unit.

Table 5.1. Operational definitions of four types of text

Types of Text	Operational Definitions
Explanatory Text [ET]	Explanatory text denotes the narrative part in presenting new and prior mathematical knowledge with elaboration. For example, giving the definition of a concept (i.e. angle), illustrating and explaining the property/-ies of a theorem. Note: The length of the text is not regarded as a factor, which influences the separation of text into different units.

[1] Next to the four categorized types of text listed in Table 5.1, an additional category *brainstorming* was used initially. For practical reasons, this category can be combined with *exploration*.

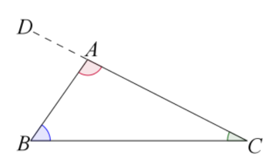
Worked Example [WE]	The basic elements of worked example include the given problem and its (commented) solution, which might be provided with other alternative methods. Worked examples are composed of both elements. If one of the elements is missing, a unit will not be categorized as worked example. Note: If it is mentioned by the textbook authors that a text is treated as a worked example, and then it will firstly be assigned to this type, otherwise, the text is judged according to the definition.
Exploration [EXP]	Exploration is an activity in which neither the specific principles/rules are provided to apply in a problem nor clear expository texts are given to elaborate the detailed mathematical concepts in this activity. Note: If it is mentioned by the textbook authors that a text is treated to initiate an exploration, it will always be assigned to this type.
Immediate Practice [IP]	Immediate practice is a given problem provided for students to practice. It precedes a new knowledge/ideas by addressing prerequisites/previously learned concepts or serves as a practice after introducing the new knowledge/ideas. Note: If it is mentioned by the textbook authors that a text is treated as an immediate practice, it will always be assigned to this type.

5.1.1.2 Decision on units

The following examples (see Figure 5.2-1 and Figure 5.2-2) are presented to explicate how to determine a unit from the texts.

The text in Figure 5.2-1 is treated as one unit, though it is separated into three

<u>Exterior Angle Theorem of a Triangle</u> (*Nan I*, vol. 4, p. 107)



Draw an arbitrary triangle ΔABC, extend $\overline{CA}$ till point D (cf. the figure on the right) $\angle BAD$ is the exterior angle of $\angle CAB$, the other two interior angles $\angle B$ and $\angle C$ are two opposite interior angles of $\angle BAD$.

Since $\angle B + \angle C + \angle CAB = 180°$, and $\angle BAD + \angle CAB = 180°$ therefore $\angle BAD = \angle B + \angle C$, that means the exterior angle of $\angle CAB$ is the sum of its opposite interior angles $\angle B$ and $\angle C$. Hence $\angle BAD > \angle B$; $\angle BAD > \angle C$

Therefore we can get:

1. The exterior angle theorem of a triangle: the exterior angle of an angle in a triangle is equal to the sum of its two opposite interior angles.
2. The exterior angle of an angle in a triangle is larger than its opposite interior angle.

Note: The texts are translated from the original texts. The symbol system does not follow the Cartesian coordinate system, that is, the angle is not generated in the counterclockwise sense.

Figure 5.2-1. A unit

paragraphs/segments which explicate different aims, namely (1) introducing the new term *opposite interior angles*; (2) providing reasons for the validity of the exterior angle theorem of a triangle and the relationships between the exterior angle and its opposite interior angles; and (3) drawing the final conclusion. Though these three paragraphs can be viewed as three independent events, the reasoning process is highly connected and ordered. In particular, the introduction of a new term is introduced in order to make the following texts comprehensible, and the last conclusion is the summary of the above arguments. Therefore, it seems reasonable to treat these three paragraphs as one unit.

Concerning the category *explanatory text*, the difficulty arises that the length of text ranges referring to this type can vary largely even within a textbook. It can encompass only one or two sentences or be as long as two pages. The criteria to identify these units of text are: the mathematical ***concepts*** and ***approaches*** (mainly) involved in the text. If the text refers to one central concept with one specific approach, it is assigned as one unit of explanatory text. If a consecutive text is connected to one central concept but more than one specific approach is involved in, the text is separated into different units according to the number of approaches. As an example, the text in Figure 5.2-2 covers nearly two pages of explanatory text and an immediate practice unit connected to it.

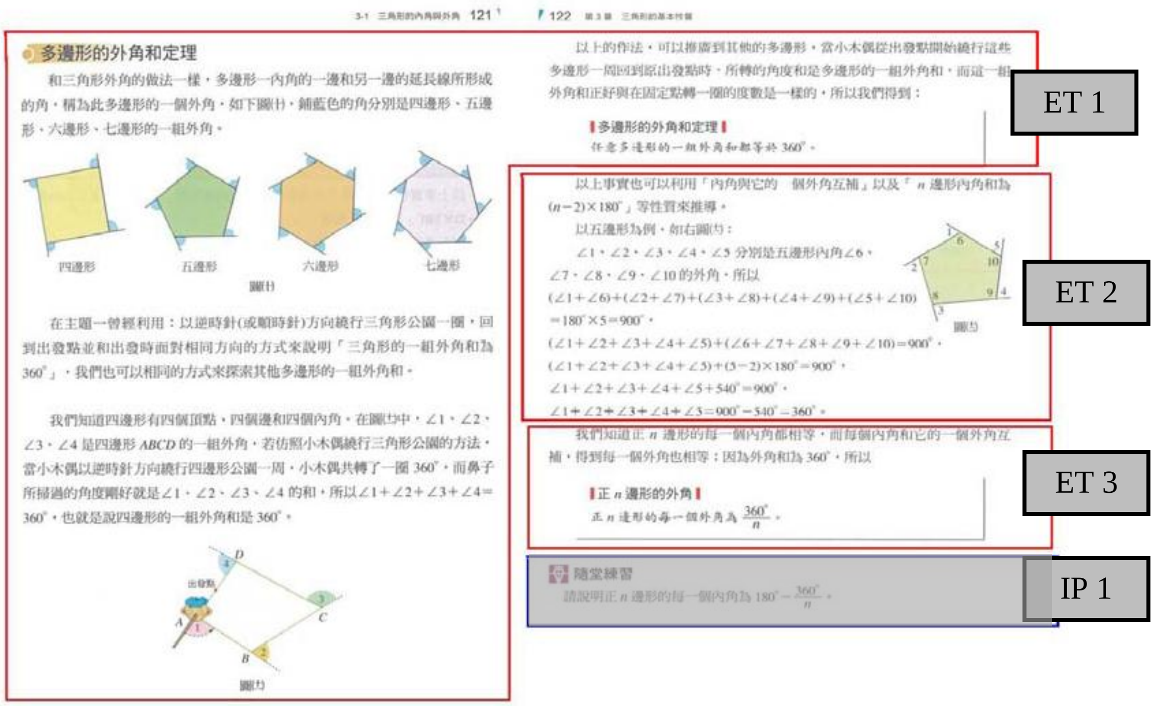


Figure 5.2-2. Deviding a long passage of text into units (*Kang Hsuan*, vol. 4, p. 121–122)

The above example of a long passage of text needs to be separated into four units. There are three units of explanatory text determined by different approaches used in each and one unit of immediate practice. These three units of explanatory text are presented in Tables 5.2, 5.3, and 5.4 to explicate the approaches involved. The unit of immediate practice is presented in Table 5.8.

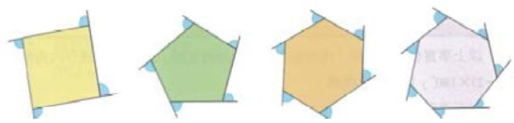
5.1.1.3 Unit of explanatory text

The upper part of Table 5.2 shows a serial expository explanation of a proposition/statement—the sum of exterior angles of a polygon—from a textbook. The lower part of Table 5.2 presents how this unit is coded into different variables, which will be illustrated later, related to the second part of the analytic framework (content of unit). This unit uses physical operation (move with the puppet) to explain the proposition with prior mathematical knowledge (a set of exterior angles of any triangle is equal to 360°, and the angle of a round circle is equal to 360°) from the specific example, the quadrangle, and then generalizes the conclusion to all polygons. It provides an empirical argument, which is not followed by a proof (cf. 5.1.4 Support to claims).

Table 5.2. Unit of explanatory text 1 (ET 1; cf. Figure 5.2-2)

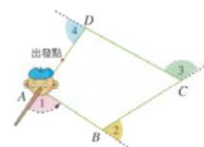
<u>Theorem of The sum of Exterior Angles of a Polygon</u>

Similar to the exterior angle of a triangle, *one side* and *the extension of the other side* of an interior angle form one exterior angle of the polygon, as shown in the figures below. The angles marked with blue color are a set of exterior angles of a quadrangle, a pentagon, a hexagon, and a heptagon respectively.



Topic one […] used—round a triangular park counterclockwise, go back to the start, and face still the same direction as the start—to illustrate that "a set of exterior angles of a triangle is equal to 360°." We can also use this similar method to explore a set of exterior angles of other polygons.

We know there are four vertices, four sides, and four interior angles of a quadrangle. In the figure below, ∠1, ∠2, ∠3, ∠4 are a set of exterior angles of a quadrangle ABCD. If we copy the method that a puppet rounds the triangular park, when the puppet rounds a quadrangular park counterclockwise, the puppet rounds a circle 360° and the swept angles by his nose correspond to the sum of ∠1, ∠2, ∠3, ∠4, therefore ∠1 + ∠2 + ∠3 + ∠4 = 360°, namely a set of exterior angles of a quadrangle is 360°.



The method used above may be generalized to other polygons, when the puppet rounds the park from the start and back to the start, the sum of the angles is the sum of a set of exterior angles of the polygon, and this sum corresponds right to the degree of rotating a circle around a fixed point, therefore, we get:

||The theorem of the sum of exterior angles of a polygon||

A set of exterior angles of an arbitrary polygon is equal to 360°.

Textbook Series	Kang Hsuan
Level	Second semester of Grade 8 (vol. 4)
Page	121-122
Type of text	Explanatory Text
Features of text	
Figure Representation	Serial figures
Involved Property	A circle is 360°
Denotation	New terms: the exterior angle theorem of a polygon
Calculation	No
Physical Operation	Yes
Figural Construction	Construct auxiliary lines
Figural Decomposition	No
Figural Transformation	No
Content Linkage	Link to previous unit - the sum of interior angles of a triangle, within this section
Context Linkage	Link only to real-life context
Function of text	Apply *the sum of interior angles of a triangle* to the sum of interior angles of a quadrangle and generalize the conclusions that *the exterior angle theorem of a polygon* [Application to generalization]
Support to claim(s)	This unit provides an empirical argument, round a park with different shapes, to the final claims, the sum of exterior angles of a polygon. [Non-proof Argument]

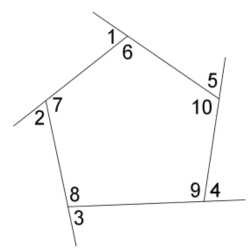
Another approach to elaborate the proposition with algebraic reasoning (calculation) is provided in Table 5.3. It involves the mathematical knowledge that an interior angle is supplementary to its exterior angle, and the formula, $(n-2) \times 180°$ of the sum of interior angles of a polygon (n-gon) in the process of algebraic reasoning. By grouping five pairs of interior angles and exterior angles in a pentagon, the unit presents the calculation of the sum of (five) straight angles and the sum of interior angles to show the validity of the proposition.

Table 5.3. Unit of explanatory text 2 (ET 2; cf. Figure 5.2-2)

The fact mentioned above may also be deduced from the properties "the interior angle is supplementary to one of its exterior angle" and "the sum of interior angles of a n-gon is $(n-2) \times 180°$".

Take the pentagon as an example (figure on the right):

$\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ are the exterior angles of the interior angles $\angle 6$, $\angle 7$, $\angle 8$, $\angle 9$, $\angle 10$ of the pentagon respectively, therefore,



$(\angle 1 + \angle 6) + (\angle 2 + \angle 7) + (\angle 3 + \angle 8) + (\angle 4 + \angle 9) + (\angle 5 + \angle 10) = 180° \times 5 = 900°$.
$(\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5) + (\angle 6 + \angle 7 + \angle 8 + \angle 9 + \angle 10) = 900°$.
$(\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5) + (5-2) \times 180° = 900°$.
$\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + 540° = 900°$.
$\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 = 900° - 540° = 360°$.

Textbook Series	Kang Hsuan
Level	Second semester of Grade 8 (vol. 4)
Page	122
Type of text	Explanatory Text
Features of text	
Figure Representation	Single figure
Involved Property	(1) An interior angle is supplementary to its exterior angle; (2) The sum of interior angles of a n-gon is $(n-2) \times 180°$
Denotation	No
Calculation	Calculation with explanation
Physical Operation	No
Figural Construction	Construct auxiliary lines (though there is no instruction in the text)
Figural Decomposition	Decompose figure into five pairs of supplementary angles
Figural Transformation	No
Content Linkage	Link to previous unit – to provide an alternative method for the sum of exterior angles
Context Linkage	Link only to mathematics context
Function of text	Apply *an interior angle is supplementary to its exterior angle* and *the sum of interior angles of a n-gon is* $(n-2) \times 180°$ to solve the problem [Application]
Support to claim(s)	This unit provides a valid reasoning (chaining) with two involved properties (warrants) to the final assertions/conclusions. It is classified as a proof unit.

Table 5.4 shows the explanatory texts by providing a proof of the statement, *the size of an exterior angle of a regular polygon*, with generalization. The text is generalized from the concepts that (1) all the interior angles of a regular polygon (*n*-gon) are equal; (2) every interior angle and its (one) exterior angle are supplementary to each other; and (3) the sum of exterior angles is equal to 360° (accepted forward). Therefore, all the exterior angles of a regular polygon can be presented as $\frac{360°}{n}$.

Table 5.4. Unit of explanatory text 3 (ET 3; cf. Figure 5.2-2)

We know that all interior angles of a regular *n*-gon are equal, and every interior angle and its one exterior angle are supplementary to each other, hence all exterior angles are equal. Because the sum of exterior angles equals to 360°, we get

||The (any) exterior angle of a regular *n*-gon||

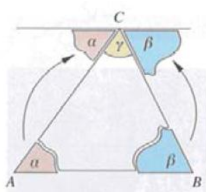
Every exterior angle of a regular *n*-gon is $\frac{360°}{n}$

Textbook Series	Kang Hsuan
Level	Second semester of Grade 8 (vol. 4)
Page	122
Type of text	Explanatory Text
Features of text	
Figure Representation	No
Involved Property	(1) All interior angles of a regular *n*-gon are equal; (2) Every interior angle is supplementary to its exterior angle (3) The sum of all exterior angles (of a regular *n*-gon) is 360°
Denotation	No
Calculation	No
Physical Operation	No
Figural Construction	No
Figural Decomposition	No
Figural Transformation	No
Content Linkage	Link to previous unit—the sum of exterior angles of a polygon is 360°
Context Linkage	Link only to mathematics context
Function of text	Generalization
Support to claim(s)	Proof

Table 5.5 presents a unit of explanatory text from a German textbook. This unit involves the physical operation (tear the paper and align the torn angles into a line) to experience the concrete knowledge, and also provides explicit exposition of the involved mathematical ideas to prove the statement (chaining the properties involved).

Table 5.5. Unit of explanatory text 4

Processing Task 2 Discover by Experiment



The angles are torn away and jointed together as shown in the figure. Another possible (figural arrangement) is addressed in Task 1 on page 43. You may assume that the angles α, β and γ compose to a straight angle. This assumption can be substantiated like this:
Think of a parallel line to [AB] which passes through C.

The original angle α and the angle α, which has been torn away and put on C are then the alternate (interior) angles from the intersection with two parallel lines. The same is true for β.

α, β and γ therefore form a straight angle, $\alpha + \beta + \gamma = 180°$

Textbook Series	Fokus
Level	Grade 7
Page	42
Type of text	Explanatory Text
Features of text	
Figure Representation	Single figure
Involved Property	(1) Alternative interior angles from the intersection of a pair of parallels are equal (parallel postulate); (2) a straight angle is 180°
Denotation	No
Calculation	No
Physical Operation	Yes
Figural Construction	No
Figural Decomposition	Decompose the (positions of three) angles of a triangle to form a straight angle
Figural Transformation	No
Content Linkage	Link to previous unit on page 41. This unit is its elaboration.
Context Linkage	Link only to mathematics context
Function of text	Use alternative interior angles and straight angle to generalize the conclusion the sum of interior angles of a triangle is 180° (Generalization).
Support to claim(s)	This is a proof unit: Chain the properties (1) and (2) to the conclusion the sum of interior angles of a triangle.

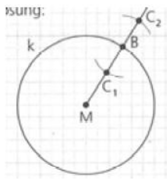
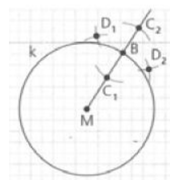
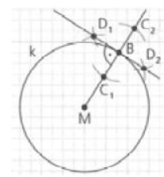
5.1.1.4 Unit of worked example

In all Taiwanese textbooks, worked examples are marked with the word *example*. In these cases, identifying the units of worked example is defined by these labels. In German textbooks, worked examples are not always marked. Table 5.6 shows an example. In this case, the text is identified as a unit of a work example based on the question and solutions provided in the unit.

Table 5.6. Unit of worked example 1

- Construct the tangent at point B of a circle k (center M, radius 3 cm). The drawn figures and the brief descriptions are provided.

  Solution:

		
Traverse a circle with (arbitrary) radius $r_1 < \overline{MB}$ on ray MB	Two circles are made from the intersectional points C_1 and C_2 as centers, with the same (arbitrary) radius $r_2 > r_1$, and which are traversed to each other (at D_1 and D_2)	The straight line of intersectional points D_1 and D_2 is the tangent of circle k at point B

Textbook Series	Delta
Level	Grade 7
Page	171
Type of text	Worked Example (note: there's a word "Example (*Beispiele*)" to denote the role of this unit)
Features of text	
Figure Representation	Serial figures
Involved Property	1. The tangent of a circle is perpendicular to its (touched) radius (p.170) 2. The construction of symmetry line (perpendicular bisector) (p.16)
Denotation	No
Calculation	No
Physical Operation	No
Figural Construction	Construct figures following instruction
Figural Decomposition	No
Figural Transformation	No
Content Linkage	Link to previous unit (the tangent of a circle is perpendicular to its radius)
Context Linkage	Link only to mathematics context
Function of text	Apply how to construct a perpendicular bisector to validate a tangent is perpendicular to its (touched) radius [Application]
Support to claim(s)	Using construction to validate the previous statement that a tangent is perpendicular to its (touched) radius. [Proof]

5.1.1.5 Unit of exploration

Based on the operational definition of exploration, it is an activity in which neither the specific principles/rules are provided to apply in solving nor a clear expository text is given to elaborate the detailed mathematical concepts in this activity. A unit from a German textbook is given as an example in Table 5.7. This unit is labeled as 'discovery by experiment'. Disregarding the label of this unit in judging the types of unit, it poses the consecutive questions/activities to the readers to be the warm-up before introducing a new issue (i.e. the sum of the interior angles of a triangle of this unit) or to retrospect to the previously learned concepts which can be linked to the introduction of a new mathematical idea.

Table 5.7. Unit of exploration 1

Discover by Experiment	**Task 2**
Draw a triangle ABC with the interior angles α, β and γ. Cut and tear the corners (angles) away and joint the angles together. Which result do you expect for the sum of the interior angles of the triangle? Explain your assumption. Use this knowledge to determine the interior angles in a quadrangle.	

Textbook Series	Fokus
Level	Grade 7
Page	41
Type of text	Exploration
Features of text	
Figure Representation	No figure
Involved Property	No (though it's implicitly related to straight angle, however, there's no figure provide as intuitive evidence in this unit)
Denotation	No
Calculation	No
Physical Operation	Cut and joint the angles together
Figural Construction	Intend the reader to draw a triangle
Figural Decomposition	Need to decompose the figure, but there's no figural representation in this unit
Figural Transformation	No figural representation and no figural transformation (transformation in this analysis goes with a complete figure and this case needs to decompose the invisible figure)
Content Linkage	New content
Context Linkage	Link only to mathematics context
Function of text	This unit provide an application of straight angle (though it's not mentioned directly)
Support to claim(s)	This unit encourage the readers to make/provide conjectures:

5.1.1.6 Unit of immediate practice

Immediate practice provides opportunities to practice after the introduction of some mathematical ideas or after the exemplification of a worked example. It is usually very close to the mathematical concepts of the previous (sometimes the following) unit and independent from the type of this previous unit. Tables 5.8 and 5.9 exemplify two units of immediate practice from a Taiwanese textbook and a German textbook, respectively.

Table 5.8. Unit of immediate practice 1 from a Taiwanese textbook (IP 1; cf. Figure 5.2-2)

Accompanying Practice in the Lesson
Please explain that every interior angle of a regular *n*-gon is $180° - \frac{360°}{n}$

Textbook Series	Kang Hsuan
Level	Second semester of Grade 8 (vol. 4)
Page	122
Type of text	Immediate Practice
Features of text	
Figure Representation	No
Involved Property	Others (it is a problem posed without mentioning which property can be used in the text; this problem can be solved in this context with the help of two methods: (1) apply the previous unit with its supplementary angle or (2) calculate the sum of interior angles and then divide by n.)
Denotation	No
Calculation	Others (it is intended the readers to calculate though the texts provide no information on calculation
Physical Operation	No
Figural Construction	No
Figural Decomposition	No
Figural Transformation	No
Content Linkage	Link to previous unit - every exterior angle of a regular *n*-gon is $\frac{360°}{n}$ or the sum of interior angles of a polygon is $(n-2) \times 180°$
Context Linkage	Link only to mathematics context
Function of text	Application to Generalization
Support to claim(s)	Others (it is obviously a proof problem, however, this unit provides no textual support to the claim, therefore, I categorized it into others.)

Table 5.9. Unit of immediate practice 2 from a German textbook (IP 2)

- How can a chord be the longest of a circle?
- How many tangents can you generate from a point P to the circle k?
- How many common tangents can two circles have?

Textbook Series	Delta
Level	Grade 7
Page	172
Type of text	Immediate Practice
Features of text	
Figure Representation	No
Involved Property	Others (no information provided from texts)
Denotation	No
Calculation	No
Physical Operation	No
Figural Construction	Others (This unit can be solved with the help of construction, but the text doesn't mention)
Figural Decomposition	No figural representation and no need for decomposition
Figural Transformation	No figural representation and no need for figural transformation
Content Linkage	Link to previous unit
Context Linkage	Link only to mathematics context
Function of text	This is a serial problems of application (though the answer can afterward be generalized to conclusions, however, there's no conclusion provided forwards or afterwards this unit) [Application]
Support to claim(s)	The claims/conjectures are made in the provided problems and intended the readers to prove to the final conclusions (not provided). Based on the limited textual information, this unit is categorized to Others.

The Taiwanese unit of immediate practice, in Table 5.8, is linked to its previous unit which introduces the mathematical knowledge of the sum of interior angles of all polygon and the prerequisites that all the interior angles are congruent in a regular polygon. Apparently, the textbook authors' intention is that readers can link the aforementioned mathematical knowledge to this unit. The German unit, in Table 5.9, provides the consecutive questions which is similar to the unit of exploration. However, the questions posed here are highly linked to the previous unit which introduces the mathematical idea of circles and lines. It provides the opportunity to practice and apply the introduced mathematical knowledge. Therefore, it is categorized as immediate practice.

5.1.2 Content of unit: Features of text

The category *features of text* of the analytic framework is composed of four sub-categories—*figure*, *knowledge*, *action*, and *situation* (see Figure 5.1). These four sub-categories provide the information that can be analyzed objectively from the written texts. They are presented in the following.

5.1.2.1 Sub-category: Figure

Geometric figures are important elements in learning geometry. The representation of a figure can be a simple geometric shape or a complex shape bearing more information. A complex figure might be decomposed into different sub-figures and this decomposition may depend on the specific and the related strategies, which involve different mathematical concepts. The use of strategies is related to the action of processing the unit. Therefore, there is a link between this sub-category and the sub-category *action*.

In this sub-category *figure*, only the variables of the representations of the figure are listed, that is, whether the unit is provided with a figure and what kind of form the figure is. There are five different variations:

(1) No figural representation, which means there are no geometric figures given in the text. If figures are connected neither to geometric shapes nor to the mathematical knowledge involved in the texts, they also fall in this variation. Examples are comics, pictures, or drawings, which do not represent the geometric (mathematical) ideas but are used for motivational reasons.
(2) No figural representation, but readers are supposed to construct a figure.
(3) A single figure is given in the text. This single figure can be a simple figure or a complex/compound figure that appears as only one configuration.
(4) A series of figures is given in the text. This means that more than one configuration is presented in the unit, even though it might be replicated in one unit.

(5) Others. This variation is given for the case that the information from the text cannot the assigned to one of the four variations described above.

5.1.2.2 Sub-category: Knowledge

This sub-category[2] is designed to investigate whether there is any property, rule, or theorem involved in the unit. The variations of this sub-category are:

(1) No mathematical property/theorem/rule is involved in the unit.
(2) At least one mathematical property/theorem/rule is involved in the unit.
(3) Others. This variation is used for the units, in which the related mathematical knowledge is not mentioned. For example, the calculation provided in the unit involves *hidden/implicit mathematical knowledge* but this knowledge is not provided.

5.1.2.3 Sub-category: Action

The sub-category *action* is composed of six different criteria and their respective variations. These criteria focus on what kinds of action are used in dealing with geometry. The criteria are *denotation*, *calculation*, *physical operation*, *figural construction*, *figural decomposition*, and *figural transformation*.

5.1.2.3.1 Denotation

Naming or giving definitions in learning geometry is an important procedure to build common mathematical language in communication. For this criterion, four different variations are created:

(1) No denotation involved.

[2] In the beginning, it was considered to differentiate all the axioms included in the geometry in grades 7–9 before defining this sub-category. However, many mathematical ideas cannot be categorized as (mathematical) axioms in lower secondary school mathematics. Moreover, making differentiations between various ‘axioms’ (mathematical ideas/knowledge) is hardly efficient for the analysis. Therefore, the differentiation of axioms is not adopted, but only the issue whether the mathematical ideas are involved in the unit is considered.

(2) Naming a new term or a new symbol.
(3) Using the specific term in problem solving/reasoning process.
(4) Others which cannot be categorized to the above three variations.

5.1.2.3.2 Calculation

Calculation involves applying specific rules between operands and operators, and operands can be numbers or symbols. Calculation in the latter case can be referred to as *algebraic calculation*.

Calculation is often necessary to solve geometric problems. It can also link visual geometry and abstract algebra. For this criterion, there are four variations:

(1) There is no calculation in the unit.
(2) There is calculation, including algebraic calculation, with detailed explanation in the unit.
(3) There is calculation, including algebraic calculation, but without any explanation in the unit.
(4) Others. This means that there is no calculation in the textual presentation, though the reader might be asked to calculate the solutions of the unit.

5.1.2.3.3 Physical operation

Physical operation (or experimental activity) means that the textual information provides concrete instruction for readers to operate/manipulate the objects, which might be material kits or figures. There are three variations:

(1) No physical operation.
(2) The physical operation is required/provided in the unit.
(3) Others, if the instruction provided in the unit is not clear enough to be classified to one of the previous two choices.

5.1.2.3.4 Figural construction

In geometry, configurations are important elements for teaching and learning. Constructing figures may help beginners to understand the basic properties step by step, and to figure out how to connect the solution to the unknown strategy (i.e. construction of auxiliary line). There are four variations:

(1) There is no construction in the unit.
(2) The unit provides the opportunity for constructing geometric shape(s) (figure(s)).
(3) The unit provides the opportunity for constructing one or more auxiliary lines.
(4) Others, if the instruction provided in the texts is not clear enough to be classified to one of the previous three variations. For example, a unit requires the construction of an auxiliary line, however, this is not mentioned in the text.

5.1.2.3.5 Figural decomposition

In this criterion, the first step is to ascertain the main figural body and then consider its sub-figures. The relevant issue is whether there is a decomposition involved. That means the main figure can be separated into different sub-figures (which still belong to the main figure). The variations are:

(1) There is no figure and no need to decompose an imagined figure in the unit.
(2) There is a figure, but no need to decompose the figure in the unit.
(3) There is a figure which needs to be decomposed in the unit.
(4) There is no figure, but there is a need to decompose an imagined figure in the unit.

5.1.2.3.6 Figural transformation

Transformation refers to the actions used on the figure(s) without changing the (topological) elements of the figure(s). That means that the transformation is based on some invariant geometric elements.

(1) There is no figure and no need to transform the figure in the unit.
(2) There is a figure(s), but no transformation involved in the unit.
(3) There is a figure(s) and the transformation is involved in the unit.

5.1.2.4 Sub-category: Situation

The sub-category *situation* considers two issues of a unit with respect to the connection of content between units (content linkage) and the setting of the unit (context linkage).

5.1.2.4.1 Content linkage

There are four different variations with respect to relations of content between units.

(1) The content of the unit is connected/refers to the content introduced earlier (prior learned mathematical concepts or strategies) beyond this section/chapter.
(2) The content of the unit is connected/refers to a previous analytic unit within this section/chapter.
(3) The content of the unit is new to the students and has not been introduced yet in the textbook.
(4) Others. It cannot be categorized to the above three choices.

5.1.2.4.2 Context linkage

There are four different variations to describe the context of a unit.

(1) The context is connected to real life.
(2) The context is connected to history.

(3) The context is connected only to mathematics.
(4) Others. It cannot be categorized to the above three choices.

5.1.3 Content of unit: Function of text

The function of text focuses on the issue of practicality especially the generalization and application handled in a unit. There are four aspects, which discriminate the different functions provided by a unit.

(1) The unit provides neither the function of generalization nor application.
(2) The unit provides the function of generalization, that is, a new property or theorem (mathematical knowledge) based on the rules is generalized from the unit, but not application.
(3) The unit provides the function of application, that is, the newly learned property/-ies or theorem(s) are applied in processing the unit, especially solving the problem posed in the unit. It does not provide generalization.
(4) The unit presents the application first and then generalizes the solution as a conclusion of new mathematical knowledge (application to generalization).

5.1.4 Content of unit: Support to claims

This category focuses on the methodological strategy provided as the *support* to the claim(s) that might be the mathematical knowledge later in a unit. To be precise, it centers on which kinds of supports (methodological strategies) are used to ensure the reliability of the mathematical knowledge (cf. Hanna & Barbeau, 2008; Rav, 1999). There are five different sub-categories to discriminate the forms of support:

(1) No support. That means the unit does not provide any evidence to support the elaborations of the texts. It may be the definition of a property without providing any support (such as

authorized knowledge: "two points can constitute a segment"), the introduction of a term, etc.

(2) (Making) Conjecture as support. The unit provides the opportunity to make conjecture(s) for the purpose of generating a claim which has not been presented as a formal mathematical knowledge in the unit (cf. Table 5.7).

(3) Non-proof argument as support (e.g., the analysis in Table 5.2). There are four criteria to judge non-proof argument:

(a) Using empirical argument

The empirical argument is adapted from G. Stylianides' (2009) definition. An empirical argument is an argument that pretends to show the "truth" of a mathematical claim by providing a proper subset of all the possible cases covered by the claim, which is similar to the crucial experiment (cf. Balacheff, 1988). An example can be found in Table 5.2.

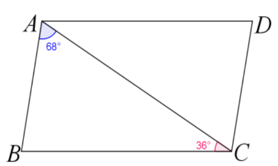
(b) Using rationale

The rationale is also adapted from G. Stylianides' (2009) definition. It is introduced to capture valid arguments for or against mathematical claims, which do not qualify as proofs. An argument counts as a rationale instead of a proof if it does not make explicit reference to the key accepted truths that it used, or if the used statements that do not belong to the set of the accepted truths of a particular community. An example is provided in Table 5.10 below.

Table 5.10. Criterion: Using Rationale

Example 2 The opposite angles of a parallelogram are equal

In the right figure, the parallelogram ABCD, given ∠BAC = 68°, ∠ACB = 36°. Find ∠D and ∠BCD.



Solution:

Since the sum of interior angles of a triangle is 180°,
∠B = 180° - ∠BAC - ∠ACB = 180° - 68° - 36° = 76°,
The opposite angles of a parallelogram are equal, then ∠D = ∠B = 76°
Moreover, since $\overline{AB}//\overline{CD}$, then ∠B + ∠BCD = 180°,
Hence, ∠BCD = 180° - ∠B = 180° - 76° = 104°

Textbook Series	Nan I
Level	Second semester of Grade 8 (vol. 4)
Page	169
Type of text	Worked Example
Features of text	
Figure Representation	Single Figure
Involved Property	(1) The sum of interior angles of a triangle; (2) the opposite angles of a parallelogram are equal; (3) two pairs of opposite sides of a parallelogram are parallel; (4) parallel postulate (consecutive interior angles are supplementary)
Denotation	Yes, use new learned term (within this section), *opposite angles*, in problem solving
Calculation	Yes
Physical Operation	No
Figural Construction	No
Figural Decomposition	Decompose the parallelogram to triangle ABC (to get angle B)
Figural Transformation	No
Content Linkage	Link to previous unit—the opposite angles of a parallelogram are equal
Context Linkage	Link only to mathematics context
Function of text	Apply learned property—the opposite angles of a parallelogram, to solve a new problem [Application]
Support to claims	This unit uses the rationale, *the sum of interior angle of a triangle* (the validity is provided with experiment in the previous unit/introduction), to calculate the angle B; the rationale, *the opposite angles of a parallelogram are equal* (a postulate), as the fact to get angle D; and the rationale of *a parallelogram* ($\overline{AB}//\overline{CD}$) to get ∠BCD. Therefore, it is categorized into a Non-Proof Argument.

Note: The texts are translated from the original texts. The symbol system does not follow the Cartesian coordinate system, that is, the angle is not generated in the counterclockwise sense.

(c) Concluded statement by analogous reasoning with example(s)

If a unit provides a statement as a new conclusion by only the means of analogously connecting it to the previously mentioned example or conclusion (the previous unit), it is also classified as a non-proof argument. An example related to this criterion is provided in Table 5.11.

Table 5.11. Criterion: Concluded Statement by Analogous Reasoning

Using the method of (Worked) Example 2, we can deduce that other pairs of corresponding angles are equal, alternate interior angles are equal, and consecutive interior angles are supplementary.

When two parallel lines are intersected by one straight line, the corresponding angles are equal, alternate interior angles are equal, and consecutive interior angles are supplementary

Textbook Series	National Academy for Educational Research
Level	Second semester of Grade 8 (vol. 4)
Page	76
Type of text	Explanatory Text
Features of text	
Figure Representation	No
Involved Property	No
Denotation	No
Calculation	No
Physical Operation	No
Figural Construction	No
Figural Decomposition	No decomposition and no figural representation
Figural Transformation	No transformation and no figural representation
Content Linkage	Link to previous Worked Example unit
Context Linkage	Link only to mathematics context
Function of text	Generalize the method of Worked Example 2 (the previous unit, p.75-76) to **other pairs** of corresponding angles are equal, alternate interior angles are equal, and consecutive interior angles are supplementary
Support to claim(s)	Using analogous reasoning (from the method of example 2: deduce *one denoted pair* of corresponding angles are equal, of alternate interior angles are equal and of consecutive interior angles are supplementary by the properties of exterior angle theorem, opposite vertical angles are equal, and straight angle) to conclude that *other pairs* stand as well. [Non-proof Argument]

(d) Authorized generalization

If a unit provides authorized generalization without offering any process of argumentation in details, it is also classified as a non-proof argument. For example, the unit presents two properties—the sum of interior angles of any triangle, and ASA congruence—and generalizes to a new property—AAS congruence—immediately without giving any further details in reasoning (see Table 5.12).

Table 5.12. Criterion: Authorized Generalization

Since the sum of interior angles of any triangle is 180°, use this property and ASA congruence, can get AAS congruence, see (Worked) Example 6.

Textbook Series	National Academy for Educational Research
Level	Second semester of Grade 8 (vol. 4)
Page	91
Type of text	Explanatory Text
Features of text	
Figure Representation	No
Involved Property	The sum of interior angles of a triangle and ASA congruence
Denotation	A new term: AAS Congruence
Calculation	No
Physical Operation	No
Figural Construction	No
Figural Decomposition	No decomposition and no figural representation
Figural Transformation	No transformation and no figural representation
Content Linkage	Link to previous unit within this section—the property of ASA congruence
Context Linkage	Link only to mathematics context
Function of text	A generalization
Support to claim(s)	Authorized generalization as a statement [Non-proof Argument]

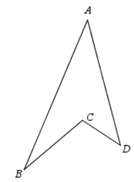
(4) Proof as support. The information of the unit provides valid reasoning to support the claim. Considering the process of chaining (Minsky, 1985) and what objects are needed to be chained helps to have a structure how proofs are processed (cf. *proof tree* proposed by Anderson et al., 1981). Therefore, this criterion of proof as a support focuses on whether the unit provides the information of proof, including both *the warrant(s)* and *the process to chain the warrants*. The warrants mean the mathematical knowledge which is used to support the original statement's validation (from the premise to the conclusion). The process of proofs may be one-step or multi-steps. Each step should include the warrant(s) and the process of chaining. Table 5.13 (see also Tables 5.3, 5.4, 5.5, and 5.6) is an example of proof as support to the claim.

Table 5.13. Criterion: Proof

<u>Example 5</u> The application of the exterior angle theorem of a triangle

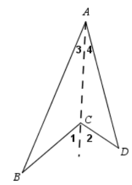
See right figures, please explain

∠BCD = ∠B + ∠A + ∠D



<u>Solution</u>:

Above figure is a concave quadrilateral. After connecting segment AC, the above figure is separated into two triangles, as right figure.



Since ∠1 is an exterior angle of △ABC,

Then, ∠1 = ∠B + ∠3

Similarly, ∠2 is an exterior angle of △ADC,

Then, ∠2 = ∠D + ∠4

Therefore, ∠BDC = ∠1 + ∠2 = ∠B + ∠3 + ∠4 + ∠D

= ∠B + ∠A + ∠D

Textbook Series	Kang Hsuan
Level	Second semester of Grade 8 (vol. 4)
Page	118
Type of text	Worked Example
Features of text	
Figure Representation	Serial figures
Involved Property	The exterior angle theorem of a triangle: the sum of any two interior angles is equal to the third interior angle's exterior angle
Denotation	No denotation involved in this unit
Calculation	Algebraic calculation with detailed explanation
Physical Operation	No
Figural Construction	Auxiliary line (ray AC)
Figural Decomposition	Decompose the original concave quadrangle into two triangles (with their exterior angles)
Figural Transformation	No transformation but with figural representation
Content Linkage	Link to previous unit within this section—the explanatory text of the exterior angle theorem
Context Linkage	Link only to mathematics context
Function of text	Application (of exterior angle theorem of a triangle)
Support to claim(s)	The solution includes the warrants for two decomposed triangles and the process to chain them. [Proof]

(5) *Others*. If the textual information provided in the unit is not enough to assign it to one of the four options above, it is classified to this option. Some examples can be found in Tables 5.8 and 5.9.

5.2 Three Principles for Specific Comparison

What principles are available to designers for scheming and evaluating the (learning and teaching) content in mathematical textbooks? Perkins (1986) identifies four elements—*purpose, structure, model,* and *argument*—which help to understand the nature of design.

(1) *Purpose*—What is the purpose of the design?
(2) *Structure*—What is the structure of the design?
(3) *Model*—In what ways can the model (of the design) be applied?
(4) *Argument*—What is the concept(s) the design is attempting to convey or reference?

These four elements provide a basic framework for analyzing the design of contents, and should thus help textbook authors to design more effective learning materials, and help teachers and students to convey and grasp the mathematical concepts being taught or studied more easily. Though these elements are useful, they are somewhat general. Perkins' scheme does not provide designers of textbook materials with any guidance on engaging students' interest in the learning content, in other words in enculturating them.

Bishop (1988) suggests five principles that a curriculum for mathematical enculturation should follow. These five principles are:

(1) *Representativeness*—it should represent the mathematical culture, in terms of both symbolic technology and values;
(2) *Formality*—it should objectify the formal level of that culture;
(3) *Accessibility*—it should be accessible to all students;
(4) *Explanatory power*—it should provide sufficient explanation of the concept(s) involved;
(5) *Broad and elementary*—it should be relatively broad and elementary rather than narrow and demanding in its conception.

The principles that he suggested provide a specific view of the notion of what and how a mathematics curriculum should present. These can be applied to assessing textbook designs in a concrete and practical way.

In order to examine the similarity and dissimilarity in the design of the introduction of specific mathematical knowledge/statement, especially mathematical proof, in German and Taiwanese textbooks, it is necessary to set principles for the analyses of different textbooks. Whether the mathematical concepts involved in the statement are the same, whether the approach to the introduction of a statement is representative or highly accessible within the country, and which kinds of settings and activities involved in introducing the statement are considered for comparison. Therefore, three principles: *continuity*, *accessibility*, and *contextualization*, drawing on the principles developed by Perkins and Bishop referred to above, have been developed and are described below. In addition, these three potential principles can also be used to explore the intentions of textbooks developed.

5.2.1 Principle one: Continuity

The principle *continuity* discusses whether the knowledge is arranged in a comprehensible order. It examines the continuity of the already learned and the new mathematical knowledge or ideas involved in the unit, or, in other words, the flow of concepts. The learned mathematical knowledge involved in the unit can be introduced and validated in preceding sections, or may simply be accepted as a fact.

Sfard (1998) uses two metaphors, the *acquisition metaphor* and the *participation metaphor*, to describe the situation of learning. Concerning the acquisition metaphor, if one discusses mathematical knowledge from a concept development perspective, it can be viewed that "concepts are to be understood as basic units of knowledge that can be accumulated, gradually refined, and combined to form ever richer cognitive structures" (p. 5). However, school mathematics learning is not meant to treat the individual as a container or a sponge to absorb all knowledge separately. Furthermore, to accept the not yet introduced or unrelated mathematical concepts in introducing a new mathematical idea is not reasonable in didactical situation. The connection and continuity of these concepts should be considered and inspected carefully in the learning situation.

In a study by Lloyd and Wilson (1998) on the impact of one teacher's conceptions on his involvement and instruction of a unit of a reformed curriculum, they validated that "teachers' comprehensive and well-organized conceptions contribute to instruction [which is] characterized by emphases on *conceptual connections*, *powerful representations*, and *meaningful discussions*" (p. 270; emphasis added). Therefore, the importance of the connection between concepts is crucial for successful learning. Examining the conceptual change can help to understand how the design provides opportunities for students to learn—how to organize the knowledge learned.

5.2.2 Principle two: Accessibility

The principle *accessibility* does not refer to the opportunity to access the materials, that is, the textbooks, but the opportunity to retrieve different strategies or methods to a statement or a new mathematical idea. It is defined as the opportunity to obtain and access the presentation which explicates the introduction of the statement. It can be treated as the opportunity provided in one textbook or different textbooks in one country.

Considering the opportunity provided in one textbook, it also endows the discussion with the connection between the *strategy to be used* and the *statement to be learned* (see Figure 5.3). Since there might be different strategies to proving the same statement and each strategy might be emphasized in the textbooks to different degrees, the accessibility of each strategy to the statement can be high, medium, or low. For example, a strong connection between strategy and statement means that the introduction to this statement is very clear. It addresses the clarity of the relationship between the final result or goal and the transparency of figural or textual representation. In this way, it can differentiate the accessibility of the presentation.

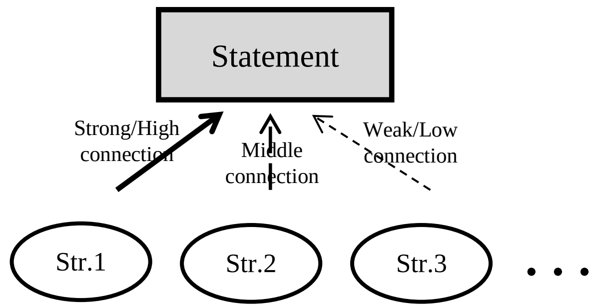


Figure 5.3. The judgement of accessibility on the connection between strategy and statement

Moreover, if one considers the opportunities between different textbooks based on this principle, it helps to examine the typical presentation introducing a statement or a mathematical idea in one country. If there is a high opportunity to obtain the similar presentation (high accessibility), the typical or representative introduction to a new mathematical idea or the stereotype of proving one specific statement in one country can be found. This principle can therefore examine the "textbook signature", postulated by Charalambous et al. (2010), in a country.

5.2.3 Principle three: Contextualization

The theory of contextualization is widely valued in the topic of probability and has proven to be useful in teaching and learning mathematics in different settings (Iversen & Nilsson, 2007; Nilsson & Ryve, 2010). It can refer to the *sociocultural perspective* which is regarded as stable physical and discursive elements of a setting in which a learning activity takes place. It can refer also to the *constructivist perspective*, meaning that the personal, cognitive context shaped by the learner's personal interpretations of an activity is regarded (Nilsson & Ryve, 2010).

This principle does not focus on varying the situations or settings of the single unit, but on the complete units involved in introducing the mathematical statement. It examines the variations of *the consecutive units* and *the contexts* surrounding the same mathematical idea. Both variations can be differentiated quickly with the assistance of the previously mentioned analytic framework (see Figure 5.1). Considering the consecutive units, it can focus on their *types of text* (explanatory text/worked example/exploration/immediate practice) together with *what kinds of support* (conjecture/non-proof argument/proof) they provide to a claim. In this way, it can judge the degree of diversity of these units. With respect to the contexts, it examines the actions used in the units and the functions provided by the units. By doing so, the complexity of the contexts can be evaluated. If the actions and functions involved in the units within the introduction of the statement are stable (similar), they can be categorized as *stable contexts*. If the actions and functions of the units within the introduction of the statement are various, they can be categorized as *diverse contexts*.

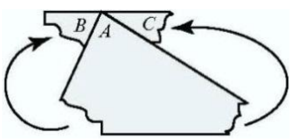
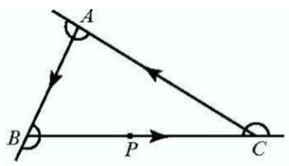
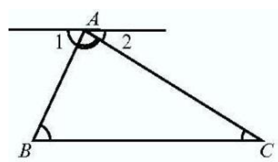
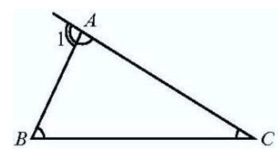
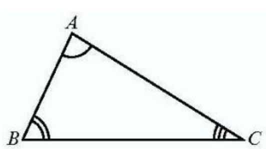
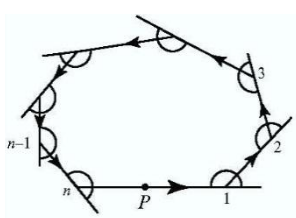
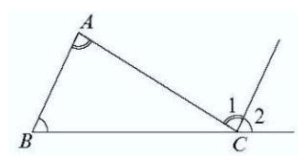
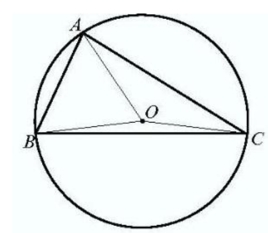
5.3 Application

The previously mentioned presentations of students' proofs (see Chapter 3) on the sum of interior angles of a triangle collected by Tall et al. (2012) can be used to exemplify their knowledge involved, action, and function of text (cf. previous analytic framework in this chapter) taken in each individual strategy. Moreover, the shift of different postures (see 3.1.1), practical geometry and theoretical geometry (Balacheff, 2010)[3], of these strategies are compared systematically based on these three factors, knowledge involved, action, and function of text. Table 5.14 shows

[3] The practical and theoretical geometry are defined in a more precise way. The practical geometry does not mean that once the unit relates to the geometry of drawings and shapes belongs to the practical geometry. Instead, if the action used in the unit is only physical or practical, then it is categorized to the practical geometry. The theoretical geometry relates to a more complex situation. The action used in the theoretical geometry is not experiment-based, but axiom-based which means the action is raised and supported by the axioms. Both can provide valid proof to students, that means can convince students, but in different degree (cf. Balaceff's *pragmatic proofs* and *conceptual/intellectual proofs*).

the different strategies that can be used in 'proving' the same statement. They stand in different positions and thereby can be used to differentiate students' approaches to learning mathematics.

Table 5.14. Examining different strategies in 'proving' the sum of interior angles of a triangle

Practical	**Practical–Theoretical**	**Theoretical**	
1. 	3. 	5. 	7. 
<u>Knowledge involved</u>	<u>Knowledge involved</u>	<u>Knowledge involved</u>	<u>Knowledge involved</u>
• Straight angle	• Straight angle	• Parallel postulate: alternate angles • Straight angle	• Exterior angle theorem • Straight angle
<u>Action</u>	<u>Action</u>	<u>Action</u>	<u>Action</u>
• Physical experiment	• Turtle-trip • Calculation	• Construction of the auxiliary line (parallel line)	• Algorithm (Calculation)
<u>Function of text</u>	<u>Function of text</u>	<u>Function of text</u>	<u>Function of text</u>
• Application (of tearing and aligning angles) to generalization	• Application (of turtle-trip) to generalization	• Generalization (based on knowledge involved)	• Generalization (based on knowledge involved)
2. 	4. 	6. 	8. 
<u>Knowledge involved</u>	<u>Knowledge involved</u>	<u>Knowledge involved</u>	<u>Knowledge involved</u>
	• $n \cdot 180° - 360°$ (fact)	• Parallel postulate: alternate angles; corresponding angles • Straight angle	• Circumferential angle = ½ central angle
<u>Action</u>	<u>Action</u>	<u>Action</u>	<u>Action</u>
• Practical measurement	• (implicit/unrequired) Turtle-trip • Calculation	• Construction of the auxiliary line (parallel line)	• Algorithm (Calculation)
<u>Function of text</u>	<u>Function of text</u>	<u>Function of text</u>	<u>Function of text</u>
• Application (of measuring angles with protractor) to generalization	• Application (of the formula of the sum of interior angles of a polygon) to generalization	• Generalization (based on knowledge involved)	• Generalization (based on knowledge involved)

6. METHOD

The purpose of this Chapter is to introduce the method used in this study. There are four sections, presenting *the research design*, the *interviews with textbook authors*, *the research materials*, and *coding and data analysis*.

6.1 Research Design

In order to achieve the two aims and answer the research questions stated in Chapter 4, this study is designed to conduct two kinds of comparisons—a general comparison on geometry content, and specific comparison on the selected statements, between the German and Taiwanese textbooks. The general structure of the design is presented in Figure 6.1. The analytic framework (see Chapter 5) is used to carry out the general comparison on geometry content; a set of principles (see also Chapter 5) is used to compare the specific statements.

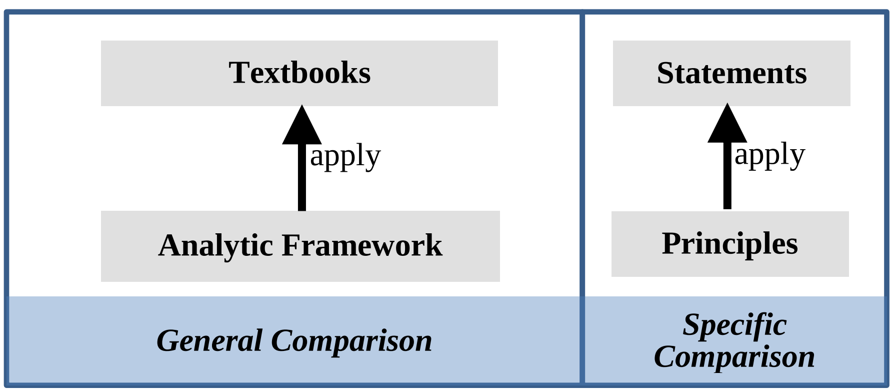


Figure 6.1. General structure of the design

The work flow of the analytic processes including different stages is presented in Figure 6.2. There are three stages, which contain different tasks in each stage, in the general comparison. At the *preliminary stage*, literature review (see Chapter 3) and national standards comparison (see Chapter 2) precede the preliminary design of the analytic framework. At the *first stage*, a preliminary design of the framework was used to process the first round of analyses. At the same time, several questionnaires (see Appendix C) were designed to conduct semi-structured interviews with the textbook authors (described in 6.2). Based on the problems collected in the first round of analyses and the interviews with the textbook authors, the analytic framework was revised to a new version. At the *second stage*, the revised framework was used to process the second round of analyses and to refine the framework to its final version. After finishing the general comparison, three principles were carried out to conduct the specific comparison on proving processes of the selected statements at the *third stage*.

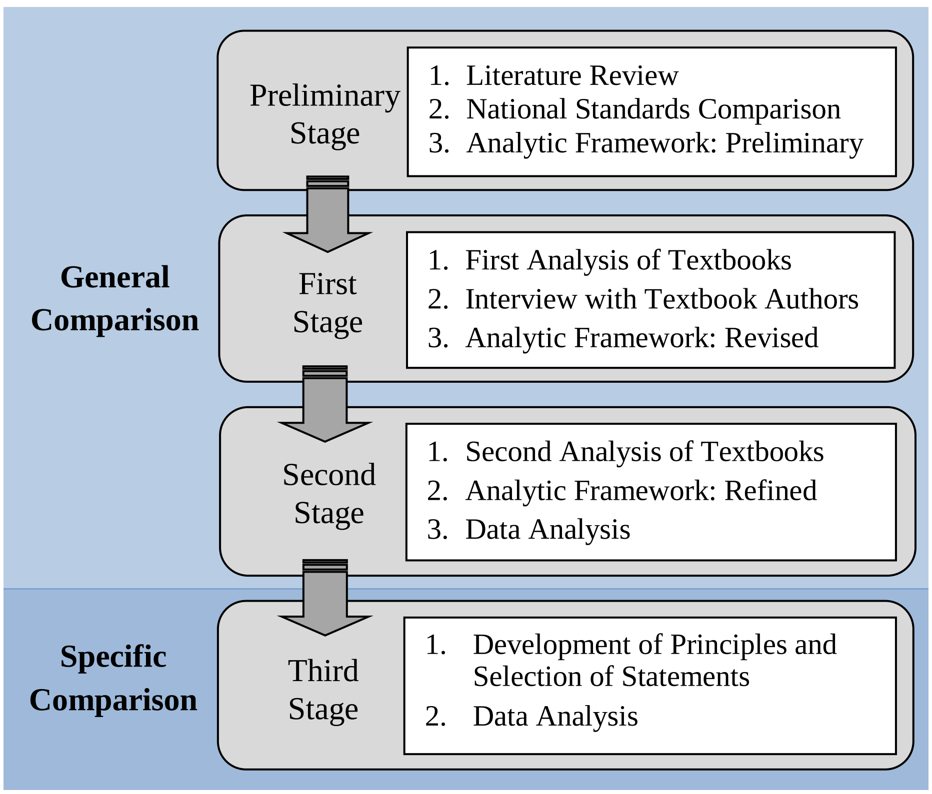


Figure 6.2. The flow of processing the study

6.2 Interviews with Textbook Authors

As mentioned above, the interviews with the textbook authors were conducted with the intention to help to refine the analytic framework. Questionnaires were designed as a semi-structured interview guide (see Appendix C) for processing the interviews with the textbook authors after finishing the initial comparison on national standards and the structures of different textbook series (see 6.1).

This interview guide is mainly focused on investigating the textbook authors' intentions on developing textbooks and meant to collect information on five dimensions. These five dimensions are: (1) basic information; (2) the author's philosophy in designing textbooks; (3) the author's goals of editing mathematics textbooks; (4) principles of the topics covered and aspects of their arrangement; (5) students' activities. Each dimension was implemented by a set of questions. The interview guide was designed and revised with three experts in mathematics education (two German and one Taiwanese) in order to collect useful interview information in both countries.

There were three textbook authors invited to the interviews. Two of them were editors, one German and one Taiwanese, who compiled the design and discussed it with a group of teachers. Bothe of them had more than ten years of experience in developing textbooks (one started in 2000 and the other in 2001) and they were experienced in teaching mathematics to students and prospective teachers. The third one was a co-author of a Taiwanese textbook series, who supervised a group of teachers how to design a specific content in the textbooks.

The interviews were conducted after finishing a first round of textbook analyses in which some problems were collected by the coders. Therefore, the interviews played an important role in collecting information for revising the framework in order to increase the workability of the analytic framework.

6.3 Research Materials

This section describes the selection process of the textbooks, the distributions in different grades and sequence of geometry content, the proportions of analyzed geometry content, and the textual patterns of the six textbook series.

6.3.1 Selection of textbooks

Six textbook series were chosen for the analysis, three from the state of Bavaria in Germany and three from Taiwan. The selection of the three German textbook series was recommended by German school teachers and university teachers; the Taiwanese textbooks were chosen from two top selling series and the NAER series. The three German textbook series[1] were *Delta* (DT), *Fokus* (FK), and *Lambacher Schweizer* (LS). The three Taiwanese textbook series[2] were *Kang Hsuan* (KH), *National Academy for Educational Research*[3] (NAER), and *Nan I* (NI). All textbooks were approved for use in schools during the school year 2009–2010.

In Germany as well as in Taiwan, textbooks have to be approved by the Ministry of Education. All German and most Taiwanese textbooks are designed and published by publishing houses. An exception is a Taiwanese textbook series which is developed by the National Academy for Educational Research [NAER]. NAER is affiliated to the Ministry of Education but also published by a publishing house.

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[1] There are five different mathematics textbooks series (publishing houses: *Bayerischer Schulbuch Verlag*, *Buchner Verlag*, *Cornelsen Verlag, Klett Verlag*, and *Schroedel Verlag*) which are designed following 2004 national standards (Educational Standards and the latest Bavarian syllabi, especially the latter) and published. Three series selected in this study: *Lambacher Schweizer* (Klett) is a well-established textbook series for decades, and *Delta* (Buchner) and *Fokus* (Cornelsen) are in their first or second edition.

[2] There are five different mathematics textbook series following the 2003 national standards and published.

[3] National Academy for Educational Research is an institute affiliated to the Ministry of Education in Taiwan. They develop a textbook series for grades 1–9. This textbook series is not market-driven but (potentially) intended to provide teachers, parents, students, and publishing houses for information. The books are also designed following the national standards, and approved by the Ministry of Education.

Since the mathematics content differs by grades or semesters in Germany and Taiwan, the textbooks were chosen by the geometry content involved as a criterion. The German textbooks are developed by year. There is one textbook for each year. However, the Taiwanese textbooks are developed by semesters, and there are two semesters every school year. Therefore, there are two textbooks for each school year. Moreover, the German textbooks provide geometry content in grades 7–9, while the Taiwanese textbooks provide geometry content neither in grade 7 nor in the second semester of grade 9 (see Table 6.1). Therefore, the selection of textbooks was focused on grades 7–9 in Germany, and grades 8 (2 semesters) and 9 (first semester) in Taiwan. That is, there were 3 (grade levels) × 3 (publishers) German textbooks and 3 (semesters) × 3 (publishers) Taiwanese textbooks, resulting in a total of 18 textbooks (see complete list of textbooks in Appendix B).

6.3.2 Distribution and sequence of geometry content in structure

As mentioned before, the geometry content differs by grades or semesters in Germany and Taiwan. The distribution of geometry content in grades 7–9 and their semesters in the both countries is shown in Table 6.1:

Table 6.1. Distribution of geometry content in grades 7–9

Content	Germany			Taiwan					
	7	8	9	7a	7b	8a	8b	9a	9b
2-D (Plane) Geometry	X	X	X			X	X	X	
3-D (Solid) Geometry			X				X		
[Circles] (tangent, intersection, angles)	X							X	
Perimeter, area and volume			X				X		
Constructions (ruler and compasses)	X						X		
Symmetry	X						X		
Parallel postulate	X						X		
Congruence	X						X		
Similarity		X						X	
Pythagorean theorem			X			X			
Slope & Trigonometry			X						

The topic *circles* is an important issue before the introduction of formal geometry proof in Taiwan, while it is not an independent topic in German curriculum, but introduced together with the special triangles. The topic *trigonometry* is introduced in the ninth grade in Germany but it is not introduced in grades 7–9 nowadays in Taiwan.

The sequence of geometry content also differs in Germany and Taiwan, as a preliminarily comparison of the six textbook series selected for this study showed (see Table 6.2). The sequence is mainly extracted from the headings of textbooks (see Appendix D for details). It is worth mentioning that the topic *solid geometry* introduces even *oblique* configurations of prism, cylinder, pyramid, and cone. Furthermore, the concept of volume is introduced with the Cavalieri's principle and the volumes of pyramid and cone are introduced in the German textbooks. These issues are not included in the Taiwanese (grades 7–9) textbooks.

Table 6.2. Sequence of geometry content in the German textbooks

Germany			
Construction 1		**Symmetry**	Perpendicularity
			(Angle) Bisector
		Angles	Parallel Postulate
Construction 2			Congruence postulates
		Special Triangles	Isosceles triangle
			Right-angled triangle
			Thales theorem
Construction 3			Special lines of triangles
			(Centers of triangles)
		Similarity	Projection
			Intercept theorem
		Pythagorean theorem	Hypotenuse-leg thm (Kathetensatz)
			Leg-leg thm (Höhensatz)
		Trigonometry	sine, cosine, tangent
Solid Geometry	***Regular*** ***Oblique***	*Prism, Cylinder,* ***Pyramid****,* ***Cone***	Angles
			Surface area
			Volumes [Cavalieri's principle]
			Length of Perimeter

Table 6.3 presents two[4] different routes of sequence of geometry content in Taiwan. The difference between these sequences is mainly the content in the second semester of grade 8 (after the topic the *Pythagorean theorem* and before the topic *similarity*).

Table 6.3. Sequence of geometry content in the Taiwanese textbooks

Taiwan			
Sequence 1		Sequence 2	
Pythagorean theorem		**Pythagorean theorem**	
Construction 1	Perpendicularity	**Angles**	Interior and exterior
	(Angle) Bisection		Parallelism
(Line) Symmetry			Parallelogram
Solid Geometry	Surface area		Perpendicularity
	Volumes [Prism, Cylinder]	Construction1	Congruence postulates
	Length of perimeter	**Side-Angel relation of triangles**	
Angles	Interior and exterior	**Parallelogram**	
Construction 2	Congruence postulates	**(Line) Symmetry**	
Side-Angle relation of triangles		**Solid Geometry**	Surface area
Parallelism	Parallelogram		Volumes [Prism, Cylinder]
Similarity	Proportional segment		Length of perimeter
	(Intercept theorem)	**Similarity**	Proportional segment
Circles			(Intercept theorem)
Geometry Proof		**Circles**	Geometry Proof

Though the sequence of geometry content differs in German and Taiwanese textbooks (Tables 6.2 and 6.3), the comparison of this study can be processed by comparing the general features of geometry (general comparison) or focusing on some specific topics (specific comparison) that are introduced in both countries.

[4] Sequence 1 is taken from the textbook series, *Kang Hsuan* and *Nan I*; and sequence 2 is taken from *National Academy for Educational Research*.

6.3.3 Analyzed pages to the geometry content

The number of textbook pages that were analyzed in the study (see the marked pages in Appendix D) is presented in Table 6.4. This table also shows their percentages in relation to the overall number of pages of geometry content[5].

Table 6.4. Pages involved in textbook analysis

	Germany			Taiwan		
Textbook Series	DT	FK	LS	KH	NAER	NI
Analyzed Pages	80	89	84	266	266	257
Pages of Geometry Content	178	192	192	332	300	320
Percentage of Analyzed GC	44.94%	46.35%	43.75%	80.12%	88.67%	80.31%

Apparently, the number of pages of geometry content and the percentage of analyzed pages differ substantially between the German and Taiwanese textbooks. In the Taiwanese textbooks geometry content is spread on more pages than in the German textbooks. One reason for this difference is a technical one: the Taiwanese textbooks offer more *large line spacing* between lines and *blank space* for practice. Another reason is that the amount of the final exercises (*exercise pool*)[6] is larger in the German textbooks than in the Taiwanese textbooks. As mentioned in Chapter 5, this study is intended to analyze only the corpus text, but not the summary or the exercise pool. However, the final exercise pool in each section is a major part in the German textbooks and there are many exercises given in this pool with different levels of difficulties. For this reason, the percentage of analyzed pages is significantly low in the German textbooks.

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[5] Trigonometry is introduced in grade 9 in most German curricula, but this topic is not introduced in junior high school in Taiwan (but in grade 10, senior high school). Therefore, the pages of trigonometry are not taken into account.

[6] The mathematics teachers in Germany usually search tasks for students to practice from this pool. Though there are exercises provided at the end of each section in the Taiwanese textbooks, the role of these exercises is different. Students are expected to finish all these exercises for recalling what is covered, i.e. what has been introduced, in this section, and these exercises are connected to the (basic) mathematical ideas introduced within the section and should not be too difficult to complete.

6.3.4 Textual pattern within chapter and its sections

The textual patterns within each chapter and each section of the six textbook series are presented in Table 6.5. The table summarizes the repeated patterns in chapters and sections in each textbook series. These patterns differ between the six textbook series. Nonetheless, the texts can be generally classified into three parts (see 5.1.1) by the functions they provide. All textbook series can be treated in the same way. These three parts are: (1) *main body* (corpus), which introduces new mathematical knowledge and presents the content with different forms of activity, such as warm-up, exploration, worked example, and practice; (2) *summary*, which integrates important parts of the section; and (3) *exercise pool*, which provides various kinds of tasks for exercise.

Table 6.5. Textual pattern within the chapter and its sections in six textbook series

Textbook Series		Within Chapter	Within Section
Germany	**DT**	• **Historic Story** (usually one page) • Several **Sections** introducing different mathematical topics (see Within Section) • **Discovery on Specific Topic** (“Themenseite”: Topic page), e.g. Dynamic Geometric Software, some discovery or advanced contents related to the former section, such as non-Euclidean Geometry (normally woven among sections) • Final section of **Complementary Tasks** (“Ergänzende Aufgaben”: Additional tasks), the ended section of each chapter • **Additional Tasks** (“explore –get more”) • **Self-Test** (“Selbsttest – Kann ich das?“: Self-test—Can I?)	• ***Warm-up***: Exploration of several tasks in the beginning • ***Working task*** according to the topic of this section (“Arbeitsaufträge”) • ***Worked Examples*** (“Beispiele”) • ***Immediate Practice*** • ***Tasks*** (“Aufgaben”)
	Note: The **Historic Story** and the **Sections** (except ***Tasks*** in each section) in each chapter are analyzed.		
	FK	• **Cover** page (one page) with some specific geometric configurations of buildings or real materials • **Arts Corner** (“Kunst-Ecke”) introducing the linkage between arts and mathematics (normally one to two pages in the beginning of the chapter) • Several **Sections** introducing different mathematical topics (see Within Section) • Final **Exercises** of this chapter (“Zeig, was du kannst!”: Show, what you can do!)	• ***Exploration*** of several tasks related to the topic of this section (“Auftrag”) • Introduce ***Formal Issues*** by processing one of the former tasks used for exploration (“Bearbeitung”) • ***Summary*** of this section (“Zusammenfassung”) • Three different levels of ***Tasks*** (“Aufgaben”): “Trainieren”: Train, “Anwenden”: Apply, and “Verknüpfen”: Link
	Note: The **Arts Corners** and **Sections** (except ***Summary*** and final ***Tasks*** in each section) in each chapter are analyzed.		
	LS	• Several **Sections** introducing different mathematical topics (see Within Section) • **Discovery on Specific Topic** (“Thema”: Topic), e.g., Dynamic Geometric Software, some advanced contents related to this chapter (designed in the final part of each chapter) • **Additional Materials** (“Lesetext”: Reading text), e.g., pieces of art related to a specific pattern (1. Designed in the final part of each chapter following the Thema; 2. Does not regularly appear in each chapter, but is arranged if regarded suitable for the topic) • **Summary** of the content of this chapter (“Rückblick”: Retrospection)	• ***Warm-up***: Exploration of several tasks in the beginning • ***Formal Issues*** of the topic in this section • ***Worked Examples*** (“Beispiel”) • ***Tasks*** (“Aufgaben”)

		• **Exercises** for practicing and reviewing (“Aufgaben zum Üben und Wiederholen”: Tasks for practice and repetition)	
	Note: The **Sections** (except ***Tasks*** in each section) in each chapter are analyzed.		
Taiwan	**KH**	• **Cover** pages (two pages) including a picture and the list of sections with a brief introduction to the chapter • Several **Sections** in introducing different mathematical topics (see Within Sections) • **Discovery on Specific Topic** (“數學櫥窗”: Math window) introducing advanced contents related to the chapter (last page of each chapter)	• ***Formal Issues*** in introducing the concepts of this section, accompanying with *Worked Examples* (“例題”), *Accompanying Practice in the Lesson* (“隨堂練習”) are interwoven • ***Summary*** (“重點整理”) • ***Exercises*** (“自我評量”)
	Note: The **Sections** (except ***Summary*** and ***Exercises*** in each section) in each chapter are analyzed.		
	NAER	• **Cover** page (only one page) includes a cartoon and the list of sections of the chapter • Several **Sections** introduce different mathematical topics (see Within Sections)	• ***Formal Issues*** in introducing the concepts of this section, accompanying with *Worked Examples* (“例題”), *Accompanying Practice in the Lesson* (“隨堂練習”) are interwoven • ***Summary*** (“摘要”) • ***Exercises*** (“自我評量”)
	Note: The **Sections** (except ***Summary*** and ***Exercises*** in each section) in each chapter are analyzed.		
	NI	• **Cover** pages (two pages) includes a picture and the list of sections with a brief introduction to the chapter • Several **Sections** in introducing different mathematical topics (see Within Sections) • **Discovery on Specific Topic** (“數學部落格”: Math blog) introducing advanced contents related to the chapter (last page of each chapter)	• ***Formal Issues*** in introducing the concepts of this section, accompanying with *Worked Examples* (“例題”), *Accompanying Practice in the Lesson* (“隨堂練習”) are interwoven • ***Summary*** (“重點整理”) • ***Exercises*** (“自我評量”)
	Note: The **Sections** (except ***Summary*** and ***Exercises*** in each section) in each chapter are analyzed.		

6.4 Coding and Data Analysis

This section describes the coding process with respect to the *general comparison* of textbook analyses and the inter-rater reliability. The *specific comparison* is a qualitative comparison processed by the researcher, and the results are directly presented in Chapter 7.

6.4.1 Coding process

The textbooks selected in this study are written in either German or Mandarin. With regard to the German textbooks, one coder, a native German speaker who is a prospective mathematics teacher for Gymnasium track, coded all contents selected by the researcher. The researcher coded around 20–50% of the contents. With respect to the Taiwanese textbooks, another coder, a native Taiwanese who is a doctoral student majoring in mathematics education coded two topics—the *angles of the triangles* and the *circles*—in all textbooks. The researcher coded all the contents.

As mentioned in section 6.1, there were two rounds of complete general comparison. Thus, there were also two round of coding. The process of coding the textbooks was a dynamic and progressive interaction between coders and a continuous development of the analytic framework. In each round, the *constant comparative* method (Glaser & Strauss, 1967) and the *consensus* among coders (Charalambous et al., 2010; Ding & Li, 2010, Roseman, Stern, & Koppal, 2010; Stylianides, 2005, 2009; Thompson et al., 2012) were applied in the coding process. That is, in designing and revising the analytic framework, the researcher had to continually compare the workability and the generation of its components; in coding the textbooks, the coders needed a consensus on the coding results. The constant comparative method was a:

> "[…] continuously growing process—each stage after a time is transformed in to the next—earlier stages do remain in operation simultaneously throughout the analysis and each provides continuous development to its successive stage until the analysis is terminated" (Glaser & Strauss, 1967, p. 193).

After the coders processed the coding individually, they met and discussed controversial issues in order to get a consensus. The problems collected during this process could also be the basis for a revision of the analytic framework.

6.4.2 Inter-rater reliability

In order to achieve high inter-rater reliability on same-lingual materials, two co-coders from the two countries were selected to work with the researcher. Cohen's *K* (see Charalambous et al., 2010) can be used as a measure of the reliability of multiple determinations on the same subjects (Landis & Koch, 1977). In this study, different coders in a group coded the same materials (textbooks) based on the same analytic framework. The coding results differed from different coders' judgements. In order to find the differences of coding and reach a consensus between coders, Cohen's *K* was adapted to calculate the inter-rater reliability as the index to reflect on the coding process.

The German co-coder was selected to code all German textbooks. After the co-coder finished the first round of coding, the researcher randomly selected sections to check the consistency between sets of coding. When the discrepancy was high, the researcher went through the whole section[7]. Moreover, the co-coder's agreement was necessary. The criteria of judging the discrepancy depended on whether there were different patterns between two co-coders' data. If the discrepancy was low, the researcher checked again with the original codebook and weighed both sets of coding with the codebook and then selected the suitable coding as the final data. After the whole circle, the inter-rater reliability in the second round of analyses was 0.85–1.0[8].

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[7] Therefore, the contents analyzed by the researcher increased from around 20% to more than 50%.

[8] Cohen's Kappa coefficient (Landis & Koch, 1977):
> .80: Greater outstanding reliability (Excellent)
.60 –.79: Substantial reliability (Good)
.49 –.59: Moderate inter-rater reliability (Satisfactory)

The Taiwanese textbooks were completely analyzed by the researcher. In order to avoid the subjective analyses, a co-coder was selected to code two topics: the *angles of triangles*, which involves some preliminary mathematical proof; and the *circles*, which is usually set before or as the introduction of formal mathematical proofs in textbooks. Before the coding process, the Taiwanese co-coder kept discussing with the researcher the criteria of codebook till a consensus was made. After finishing the coding, the researcher calculated the inter-rater reliability (which varied from 0.65 to 0.85) from the data and made further discussions with the co-coder to figure out the factors of the differences between coders. The inter-rater reliability after the final agreement between the two raters was 0.95–1.0.

7. RESULTS

This chapter presents the results and tries to provide answers of the research questions (RQ1 and RQ2) in four different sections. In the *first* section, the distributions of geometry content in each grade and the sequence of geometry content in German and Taiwanese textbooks are reviewed in brief. This is in order to give an outlook on geometry content and content sequence in Germany and Taiwan. In the *second* section, the significant findings of the textual features of geometry in German and Taiwanese textbooks are presented in several sub-sections. These findings are grounded on the analytic framework. In the *third* section, three examples of mathematical proofs are discussed with the set of three principles. It is described how three mathematical statements, the *sum of interior angles of a triangle*, the *Thales theorem*, and the *Pythagorean theorem*, are introduced in Germany and Taiwan. Lastly, the *fourth* section gives an additional overview how textbooks are developed in Germany and Taiwan. There is a short summary based on the interviews with textbook editors (authors).

7.1 Distributions and Sequence of Geometry Content

In this section, differences in the distributions of geometry content and in the sequence of geometry content are respectively discussed in sub-sections 7.1.1 and 7.1.2 in detail. It follows Section 6.3 and presents as part of results between German and Taiwanese textbooks.

7.1.1 Different distributions of geometry content

Table 6.1 (cf. Chapter 6) shows that the distributions of geometry content in grades 7 to 9 in German and Taiwanese textbooks are different. As mentioned in Chapter 6, mathematics content at the lower secondary school is not developed in the same sequence/order in the two countries.

The same content may be arranged in different grades or even introduced with different mathematical concepts.

In Germany, most geometry content is introduced in the seventh grade. This content encompasses only topics from *plane geometry* (2-D geometry) and is fundamental for preparing the students' understanding of topics following in higher grades, including *symmetry*, *parallel postulate*, *congruence*, and *figure construction*. In grade 8, only the topic of *similarity* is introduced and this topic is highly connected to ratio and proportion in solving various problems of similar figures. In the last year of this period, namely in grade 9, *solid geometry* (3-D geometry), the *Pythagorean theorem*, and *basic trigonometry* (sine, cosine, and tangent with respect to right-angled triangles) are introduced. There is no individual section or chapter introducing mathematical proof.

In Taiwan, there is no geometry content in grade seven and the second semester of grade nine. The first geometry topic, the Pythagorean theorem, is given in the first semester of grade 8. In the second semester of grade 8, the abundant geometry content dominates the whole mathematics textbook, including plane geometry—figure construction, symmetry, parallel postulate, and congruence—and solid geometry. In the first semester of grade 9, the properties generated from circles (e.g., *tangent* and *intersection* between circles), lines and circles (e.g., *tangent*, *intersection*, and *angles* in a circle), the conditions of similarity, and mathematical proof (related to formal mathematical proof), are introduced intensively.

It is notable that *circles* are introduced in Taiwanese textbooks much more intensive than in German textbooks. Though tangent and intersection between circles, and tangent, intersection, and angles in a circle are discussed in Germany, they are not treated as an independent topic. The content mentioned above is affiliated to the discussion of *special triangles* in Germany.

Furthermore, there are several problems generated from this issue (circles) and related to mathematical proofs in the Taiwanese textbooks (see *the Thales theorem* as an example in 7.3.2.2).

7.1.2 Different routes of sequence of geometry content

The sequence of geometry content summarized by headings (see details in Appendix D) of the German and Taiwanese textbooks is presented briefly in Table 6.2 and Table 6.3. The routes of geometry content are congruent among the three German textbook series. Therefore, they are presented in one single sequence (see Table 6.2). In Taiwan, the routes of geometry content are congruent between two of the three textbook series, but differ in the third textbook series. They are presented in two different sequences (see Table 6.3)

The geometry content in the German textbooks differs from that in the Taiwanese textbooks and has a strong connection with *figure construction*. There are three stages introducing geometry knowledge that goes with figure construction in the German textbooks. The *first stage* introduces the issue of *symmetry* with the construction of some basic geometric properties, such as perpendicularity and angle bisector. The *second stage* introduces *congruence postulates* to deal with and to prove the properties generated from these postulates. The postulates are introduced with the construction of congruent triangles. The *third stage* relates to the three different *centers of triangles*, that is, the circumcenter, incenter, and centroid (center of gravity) which are introduced by their respective definitions and presented with figure construction.

Another difference is that *trigonometry* is not part of the Taiwanese curricula whereas presented in the German textbooks. The basic definitions/rules, such as sine, cosine, and tangent, of triangles are covered in this topic. Furthermore, *solid geometry* is introduced later in Germany than in Taiwan. It covers not only the regular (Platonic) solids, but also the oblique solids, including prism, cylinder, pyramid, and cone. The content relates to angles, surface area, volumes (using Cavalieri's principle), and length of perimeter. Some of these topics are not introduced in the Taiwanese (junior high school) textbooks, such as the oblique solid geometry, the volumes of pyramids and cones, and the use of Cavalieri's principle when solving the problems of volume.

The two routes of sequence of geometry content concluded from the three Taiwanese textbooks (sequence 1: *Kang Hsuan* and *Nan I*; sequence 2: *National Academy for Educational Research*) are slightly different. For example, sequence 1 gives the topics *symmetry* and *solid geometry* in the early chapters after the topic *basic geometry properties* (the beginning chapters of the second semester of grade 8), while sequence 2 arranges both topics (symmetry and solid geometry) in the late chapters of the second semester of grade 8 after introducing the serial properties and rules of triangles and parallelograms.

Compared to German geometry content, the role of *figure construction* is less important in Taiwanese textbooks, especially in sequence 2 in which only the *congruence postulates* are involved with figure construction. In addition to introducing the congruence postulates with figure construction, sequence 1 also provides construction activities in introducing basic geometry properties, such as perpendicularity and angle bisector.

One significant difference between German and Taiwanese geometry content is that an independent topic *geometry proof* involving formal proofs is introduced as the last geometry topic in the Taiwanese textbooks. This topic is not explicitly included in the German textbooks.

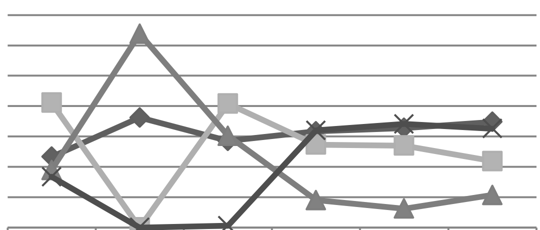
In summary, the opportunities to learn mathematical proofs are provided differently according to the sequence of geometry content in German and Taiwanese textbooks. Further comparison in their designs focusing on the features of texts and the approaches to proofs is discussed in the following sections.

7.2 The Findings of the Textual Features

This section tries to present the most relevant findings[1] concerning the textual features in seven sub-sections. The analyses are based on the analytic framework as presented in Chapter 5. In all Figures presented in this section, the number of the units analyzed within each textbook series is given in brackets under the title of each textbook series.

7.2.1 Types of text

Figure 7.1 shows the distribution of the four *types of text* (explanatory text, worked example, exploration, immediate practice; see Chapter 5) in each of the six textbook series.



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
Explanatory Text	23.30%	36.20%	28.50%	31.70%	32.80%	34.80%
Worked Example	41.10%	0.00%	40.80%	27.30%	27.00%	21.90%
Exploration	18.90%	63.80%	30.20%	9.00%	6.20%	10.70%
Immediate Practice	16.70%	0.00%	0.60%	32.00%	34.10%	32.60%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1. The distribution of types of text in the six textbook series

In two of the three German textbook series, *Delta* and *Lambacher Schweizer*, *worked examples* play an important role (both 41%). However, the third German textbook, *Fokus*, does not provide any *worked example*. Furthermore, *Fokus* and *Lambacher Schweizer* provide no or hardly any *immediate practice* whereas *Delta* includes this type of text. *Fokus* provides *exploration* to a high

[1] The complete data are provided in Appendices E and F.

extent (64%), while *Lambacher Schweizer* and *Delta* provide it to a relatively low extent (30% and 19%). Lastly, the three German textbook series provide *explanatory text* to a similar level, compared to the other three types of text: *worked example*, *exploration*, and *immediate practice* between the three German textbooks.

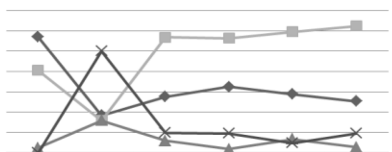
Unlike the German textbooks, *worked examples* or *explorations* do not play a significant role in all three Taiwanese textbook series. However, *worked example* is still an important type of text to some extent (around 25%). Instead, *explanatory text* and *immediate practice* are more prominent in Taiwanese textbooks (around 32% individually in each series).

To better understand the learning opportunities these types of text provide in the different textbooks, Figures 7.1-1–7.1-8 illustrate the *practical function*[2] and the *support to claims*[3] that these four separate types of text bear.

Figures 7.1-1 and 7.1-2 present the practical function and the support to claims of *explanatory text*. Explanatory text provides low or even no *application* and *application to generalization* as the practical function in two German textbooks and all three Taiwanese textbooks (see Figure 7.1-1). However, there is an exception, *Fokus*, in which explanatory text encompasses high percentages of *application to generalization* (50%) and *application* (16%). In contrast, *generalization* is provided in many units in the other five textbooks. Moreover, explanatory text of the German *Delta* series provides the highest percentage of *no practical function* (57%).

[2] There are four different functions discussed in this category: (1) no practical function; (2) generalization; (3) application; and (4) application to generalization (see Chapter 5).
[3] There are five different ways the unit can be provided as the support to claims: (1) no support/no claim of the unit; (2) conjecture provided as support; (3) non-proof argument provided as support; (4) proof provided as support; and (5) others (see Chapter 5).
Only three roles of support—conjecture, non-proof argument, proof—are discussed in the figure presented. Therefore, the proportion might not be equal to 100% within the individual textbook series. The information including the other two categories can be found completely in Appendix F.

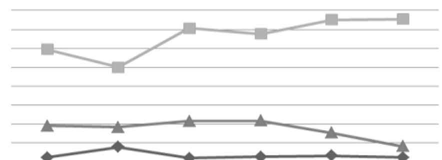
Considering the role of support provided to claims (see Figure 7.1-2), explanatory text provides *non-proof argument* to a relatively high percentage (50–76%) and *conjecture* to a very low percentage (2–8%) in all six textbook series.



	DT (42)	FK (38)	LS (51)	KH (179)	NAER (208)	NI (189)
None	57.10%	18.40%	27.50%	32.40%	28.80%	25.40%
Generalization	40.50%	15.80%	56.90%	56.40%	59.60%	62.40%
Application	2.40%	15.80%	5.90%	1.70%	6.70%	2.60%
Application to Generalization	0.00%	50.00%	9.80%	9.50%	4.80%	9.50%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1-1. The distribution of practical function of *explanatory text* in the six textbook series



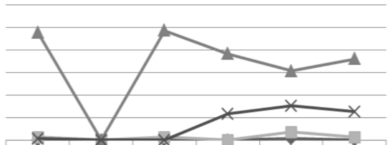
	DT (42)	FK (38)	LS (51)	KH (179)	NAER (208)	NI (189)
Conjecture	2.40%	7.90%	2.00%	2.80%	3.20%	2.40%
N-P Argument	59.50%	50.00%	70.60%	67.60%	75.10%	75.50%
Proof	19.00%	18.40%	21.60%	21.80%	15.30%	8.20%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1-2. The distribution of support to claims of *explanatory tex*t in the six textbook series

Figures 7.1-3 and 7.1-4 present the practical function and the support to claims of *worked example*. As mentioned before, there is no worked example presented in *Fokus*. Worked example provides almost exclusively *application* in the other two German textbooks (96%, 97%) (see Figure 7.1-3). *Application* is also a frequent function in all three Taiwanese textbooks (77%, 61%, 72%). Additionally, beyond application, *application to generalization* is also provided as the practical function by worked examples in all Taiwanese textbooks (23%, 30%, 25%).

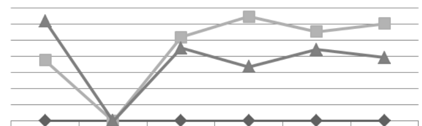
In view of the support provided to claims (see Figure 7.1-4), the two German series provide more *proof* as support to claims by worked examples (62%, 45%) than the Taiwanese series (34%, 44%, 40%). However, all three Taiwanese textbooks provide more *non-proof argument* (65%, 56%, 61%) than two German textbooks (38%, 52%). There is no *conjecture* provided by worked examples in all five textbook series.



	DT (74)	FK (0)	LS (73)	KH (154)	NAER (171)	NI (119)
None	0.00%	0.00%	0.00%	0.00%	1.20%	0.00%
Generalization	2.70%	0.00%	2.70%	0.00%	7.00%	2.50%
Application	95.90%	0.00%	97.30%	76.60%	61.40%	72.30%
Application to Generalization	1.40%	0.00%	0.00%	23.40%	30.40%	25.20%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1-3. The distribution of practical function of *worked example* in the six textbook series

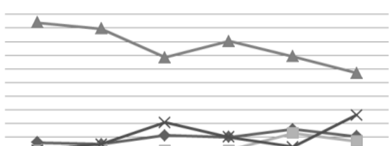


	DT (74)	FK (0)	LS (73)	KH (154)	NAER (171)	NI (119)
Conjecture	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
N-P Argument	37.80%	0.00%	52.10%	64.90%	55.60%	60.50%
Proof	62.20%	0.00%	45.20%	33.80%	44.40%	39.50%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1-4. The distribution of support to claims of *worked example* in the six textbook series

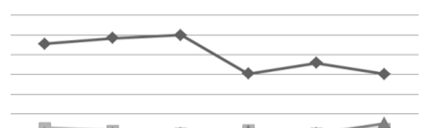
Figures 7.1-5 and 7.1-6 present the practical function and the support to claims of *exploration*. Exploration units provide frequent *application* in all six textbook series (57%–94%), especially in *Delta* (94%) and *Fokus* (90%) (see Figure 6.1-5). Though *application to generalization* is hardly provided in some textbooks, there are two exceptions, *Lambacher Schweizer* (20%) and *Nan I* (26%).



	DT (34)	FK (67)	LS (54)	KH (51)	NAER (39)	NI (58)
None	5.90%	4.50%	11.10%	9.80%	15.40%	10.30%
Generalization	0.00%	1.50%	0.00%	0.00%	12.80%	6.90%
Application	94.10%	89.60%	68.50%	80.40%	69.20%	56.90%
Application to Generalization	0.00%	4.50%	20.40%	9.80%	2.60%	25.90%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1-5. The distribution of practical function of *exploration* in the six textbook series



	DT (34)	FK (67)	LS (54)	KH (51)	NAER (39)	NI (58)
Conjecture	91.20%	97.00%	100.00%	60.80%	71.80%	60.30%
N-P Argument	5.90%	3.00%	0.00%	3.90%	0.00%	5.20%
Proof	0.00%	0.00%	0.00%	2.00%	0.00%	10.30%

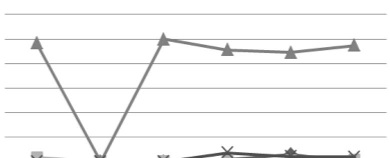
Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1-6. The distribution of support to claims of *exploration* in the six textbook series

As shown in Figure 7.1-6, *conjecture* plays an important role as support to claims in the exploration units of all six textbooks. The other two roles of support, *non-proof argument* and *proof*, are rarely provided by exploration units.

Figures 7.1-7 and 7.1-8 present the practical function and the support to claims of *immediate practice*. As mentioned in the beginning, two German textbooks, *Fokus* and *Lambacher Schweizer*, provide almost no immediate practice, therefore, the following analysis focuses on the other four textbooks. *Application* plays an important role in immediate practices of four textbooks (see Figure 7.1-7). Only three Taiwanese textbooks provide the function of *application to generalization*, though the proportion is small (7%, 4%, 4%).

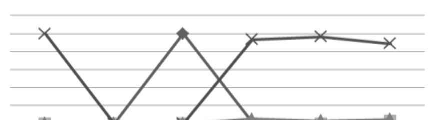
Concerning the support provided to claims (see Figure 7.1-8), most of immediate practices in the four textbooks provide undecided information (*others*) to the claims in a large proportion (89–100%). In addition, most of immediate practices are pending questions.



	DT (30)	FK (0)	LS (1)	KH (181)	NAER (216)	NI (177)
None	0.00%	0.00%	0.00%	1.70%	6.00%	0.00%
Generalization	3.30%	0.00%	0.00%	0.00%	0.90%	1.10%
Application	96.70%	0.00%	100.00%	91.20%	88.90%	94.90%
Application to Generalization	0.00%	0.00%	0.00%	7.20%	4.20%	4.00%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1-7. The distribution of practical function of *immediate practice* in the six textbook series



	DT (30)	FK (0)	LS (1)	KH (181)	NAER (216)	NI (177)
Conjecture	0.00%	0.00%	100.00%	1.10%	0.90%	3.40%
N-P Argument	0.00%	0.00%	0.00%	0.00%	0.00%	3.40%
Proof	0.00%	0.00%	0.00%	5.00%	2.80%	4.50%
Others	100.00%	0.00%	0.00%	93.40%	96.30%	88.70%

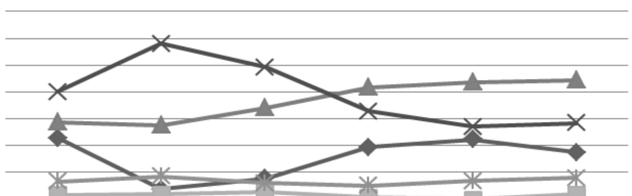
Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.1-8. The distribution of support to claims of *immediate practice* in the six textbook series

In summary, two German textbook series, *Delta* and *Lambacher Schweizer*, provide much more *worked example* than the other four textbook series, and worked examples in these two textbook series mainly provide *application* as the practical function and offer *proof* as support to claims. *Explanatory text* and *immediate practice* are two main types of text presented in the Taiwanese textbook series. They largely provide *generalization* as the practical function and offer *non-proof argument* as support to claims. *Fokus* is the only textbook series which provides neither *worked example* nor *immediate practice*, but includes more *exploration*. *Exploration* in *Fokus*, as well as in the other five textbook series, provides mainly *application* as practical function and *conjecture* as support to claims.

7.2.2 Presentations of figures in geometry

Figure 7.2 shows the distribution of figural presentation[4] for each of the six textbook series.



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
No	22.80%	3.80%	7.80%	19.50%	22.40%	17.50%
No (to Construct)	1.70%	1.90%	2.80%	0.70%	0.00%	1.80%
Single	28.90%	27.60%	34.10%	41.80%	43.80%	44.40%
Serial	40.00%	58.10%	49.20%	32.90%	27.00%	28.40%
Others	6.70%	8.60%	6.10%	5.10%	6.80%	7.90%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

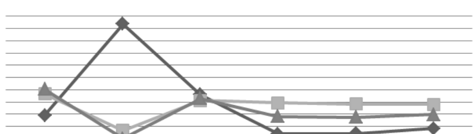
Figure 7.2. The distribution of figural presentation in the six textbook series

According to the first two aspects of figural presentation, it shows that the proportion of (analytical) unit presented without figures is particularly low in *Fokus* (6%) and *Lambacher Schweizer* (11%). *Single figure* and *serial figures* are two types of figure categorized in the unit. The three German textbook series provide higher proportions of *serial figures* and lower

[4] There are five different aspects of the figural presentation: (1) no figure presented in the unit and no need to construct; (2) no figure presented but need to construct; (3) single figure presented; (4) serial figures presented; and (5) others (see Chapter 5).

proportions of *single figure*, while it is the other way around for all three Taiwanese textbook series.

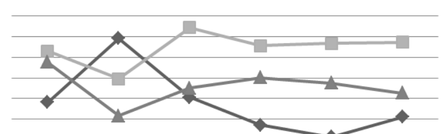
Figures 7.2-1 and 7.2-2 present the relation between the roles of *support to claims* to these two types of presentation, *single figure* and *serial figures*, respectively. The three German textbooks provide different distributions of single figures and serial figures. These two types of presentation are more strongly related to *non-proof argument* and *proof* than to *conjecture* in *Delta*. Single figures are mainly connected to *conjecture* (94%), while serial figures are connected not only to *conjecture*, but also to *non-proof argument* and *proof* in *Fokus*. Single figures are connected on an average to all three types of support (31–36%), while serial figures are connected higher to *non-proof argument* (55%) in *Lambacher Schweizer*. In view of the Taiwanese textbook series, single figures and serial figures both are connected mainly to *non-proof argument* (higher) and *proof*, especially serial figures provide higher proportions in both types of support.



	DT (52)	FK (29)	LS (52)	KH (236)	NAER (278)	NI (241)
Conjecture	19.20%	93.70%	36.10%	3.80%	4.30%	8.30%
N-P Argument	36.50%	6.90%	31.10%	29.20%	28.40%	28.20%
Proof	40.40%	0.00%	32.80%	18.20%	17.60%	19.90%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.2-1. The distribution of support to claims of single figure in the six textbook series



	DT (72)	FK (61)	LS (72)	KH (186)	NAER (171)	NI (154)
Conjecture	18.10%	49.20%	20.50%	7.00%	1.20%	11.00%
N-P Argument	43.10%	29.50%	54.50%	45.70%	46.80%	47.40%
Proof	37.50%	11.50%	25.00%	30.10%	27.50%	22.70%

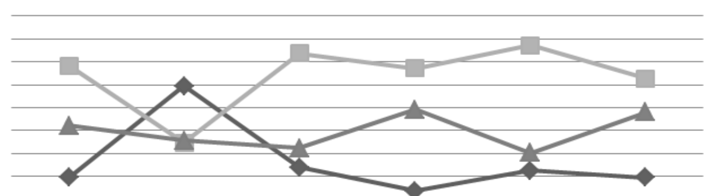
Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.2-2. The distribution of support to claims of serial figures in the six textbook series

In summary, the German textbook series *Fokus* and *Lambacher Schweizer* provide a higher proportion of figure in the text (different units). The German textbook series provide higher proportions of *serial figures* than of *single figures*, while the Taiwanese textbook series provide higher proportions of *single figures* than of *serial figures*. With respect to the connection between figural presentation and the roles of support, there is no clear pattern in the German series. However, the figural presentation of *serial figures* in the Taiwanese series is relatively more close to *non-proof argument* and *proof* than *single figures* is.

7.2.3 The connection with mathematical ideas

Figure 7.3 shows the distribution of mathematical knowledge[5] involved in the units for each of the six textbook series.



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
No	9.40%	49.50%	14.00%	3.70%	12.50%	9.40%
Yes	58.30%	24.80%	63.70%	57.20%	67.20%	52.70%
Others	32.20%	25.70%	22.30%	39.10%	20.30%	37.90%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

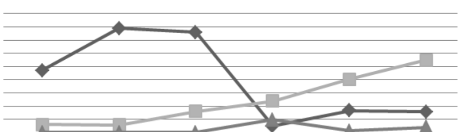
Figure 7.3. The distribution of knowledge involved in the six textbook series

In two German series, *Delta* and *Lambacher Schweizer*, and the three Taiwanese series, the proportion of unit that includes mathematical ideas is higher than that does not include mathematical ideas. Only *Fokus* provides *no mathematical ideas* in the unit to a high proportion (50%). The hidden/implicit mathematical knowledge (*others*) in the texts is more often seen in *Delta* (32%), *Kang Hsuan* (39%), and *Nan I* (38%).

[5] There are three different aspects of this issue: (1) no mathematical property/theorem/rule involved in the unit; (2) mathematical property(-ies)/theorem(s)/rule(s) involved in the unit; and (3) others (see Chapter 5).

Figures 7.3-1 and 7.3-2 present the relation between the roles of *support to claims* to two variations, *no knowledge involved* and *knowledge involved*, in the text. Units involving no knowledge in the three German textbooks provide much more *conjecture as support* (47–79%), while in the three Taiwanese textbooks more *non-proof argument as support* (24–55%) (see Figure 7.3-1) is provided.

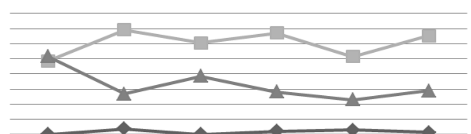
Concerning units involving knowledge, *proof* as support is provided more in two German series, *Delta* (51%) and *Lambacher Schweizer* (39%), than in the third German series, *Fokus* (27%) and the three Taiwanese series (23%, 28%, 29%). *Non-proof argument* as support to the students is used relatively often in all six textbooks (49–69%).



	DT (17)	FK (52)	LS (25)	KH (21)	NAER (79)	NI (51)
Conjecture	47.10%	78.80%	76.00%	4.80%	16.50%	15.70%
N-P Argument	5.90%	5.80%	16.00%	23.80%	40.50%	54.90%
Proof	0.00%	0.00%	0.00%	9.50%	1.30%	3.90%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.3-1. The distribution of support to claims of no knowledge involved in the six textbook series



	DT (105)	FK (26)	LS (114)	KH (323)	NAER (426)	NI (286)
Conjecture	0.00%	3.80%	0.00%	2.20%	2.80%	1.40%
N-P Argument	48.60%	69.20%	60.50%	67.20%	51.40%	65.40%
Proof	51.40%	26.90%	38.60%	28.20%	23.00%	29.00%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.3-2. The distribution of support to claims of knowledge involved in the six textbook series

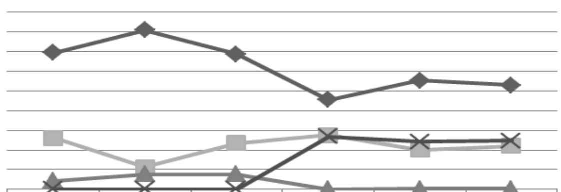
In summary, more than half of the units in two German series, *Delta* and *Lambacher Schweizer*, and in the Taiwanese series involve *mathematical ideas*. Most of these units provide *non-proof argument* (more) and also *proof* (less) as support to claims. *Fokus* is the exception, in which half of the units do not explicitly indicate which mathematical idea is concerned. Most of the units involving *no mathematical idea* provide *conjecture* as support in the three German textbooks and *non-proof argument* as support in the three Taiwanese textbooks.

7.2.4 Comparison of selected actions used in geometry

In the following, four different actions—calculation, physical operation, figural construction, and the decomposition of figures—are selected for comparison. These actions are particularly used in different as well as in similar ways in the six textbook series.

7.2.4.1 Calculation

Figure 7.4.1 shows the distribution of the process of calculation[6] provided in the units for each of the six textbook series.



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
No	69.40%	81.00%	68.70%	45.50%	55.40%	53.00%
With Explanation	26.10%	11.40%	23.50%	27.60%	20.20%	22.10%
Without Explanation	4.40%	7.60%	7.80%	0.00%	0.20%	0.20%
Others	0.00%	0.00%	0.00%	26.90%	24.30%	24.70%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

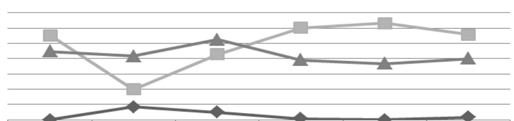
Figure 7.4.1. The distribution of calculation in the six textbook series

All six textbook series provide *no calculation* with a high proportion (46–81%), and all German series, especially *Fokus* (81%), have higher percentages than all Taiwanese series. In the Taiwanese series, a high proportion of unit is categorized as *others* (24–27%), which means that the units could not be labeled, but none of *others* in the three German series. However, two German series (*Delta* and *Lambacher Schweizer*) and the Taiwanese series provide *calculation with explanation* with a comparable ratio, around 20–28%, *Fokus* is the exception with 11%.

[6] There are four different variations related to the issue of calculation: (1) no calculation involved in the text; (2) calculation involved in the text with explanation; (3) only numeral/algebraic calculation provided in the text without explanation; (4) others (see Chapter 5).

Moreover, the three German series provide *calculation without explanation*, albeit in a low proportion (4–8%), whereas the Taiwanese series hardly provide this at all.

When focusing on the variation *calculation with explanation* involved in the units in all six textbook series, especially the German series, it plays an important role in providing *proof* as support to claims (see Figure 7.4.1-1). Moreover, providing *non-proof argument* as support is commonly seen in all three Taiwanese series and in two German series (*Delta* and *Lambacher Schweizer*).



	DT (47)	FK (12)	LS (42)	KH (156)	NAER (128)	NI (120)
Conjecture	0.00%	8.30%	4.80%	0.60%	0.00%	1.70%
N-P Argument	55.30%	20.00%	42.90%	60.30%	63.30%	55.80%
Proof	44.70%	41.70%	52.40%	39.10%	36.70%	40.00%

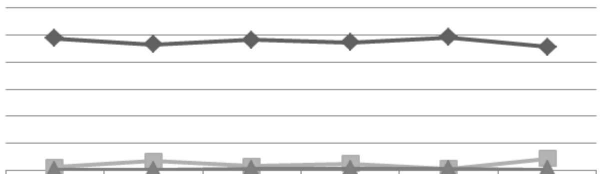
Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.4.1-1. The distribution of support to claims of calculation with explanation in the six textbook series

In summary, two German textbook series, *Delta* and *Lambacher Schweizer*, provide more *calculation* (with and without explanation) in the units than the Taiwanese textbook series and the third German series, *Fokus*, provides the lowest percentage of calculation. There are around one fourth of units in the Taiwanese series that ask for calculation, though there is no calculation presented in the texts (categorized as *others*). Units which present *calculation with explanation* provide more *proof as support* in the German series than in the Taiwanese series.

7.2.4.2 Physical operation

Figure 7.4.2 presents the distribution of physical operation[7] (experimental activity) for each of the six textbook series.



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
No	97.80%	93.30%	96.60%	94.70%	98.30%	91.20%
Yes	2.20%	6.70%	2.80%	4.80%	1.10%	8.50%
Others	0.00%	0.00%	0.60%	0.50%	0.60%	0.40%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.4.2. The distribution of physical operation in the six textbook series

Most of the units in all six textbook series do not provide physical operation (around 91–98%). Two Taiwanese series (*Kang Hsaun*, 5%, and *Nan I*, 9%) and one German series (*Fokus*, 7%) provide more operation activities than the other textbook series, though the proportion is not high. One Taiwanese series, *National Academy for Educational Research*, provides the least opportunity (1%) to work with experimental activity (physical operation).

7.2.4.3 Figural construction

Figure 7.4.3 presents the distribution of figural construction[8] for each of the six textbook series.

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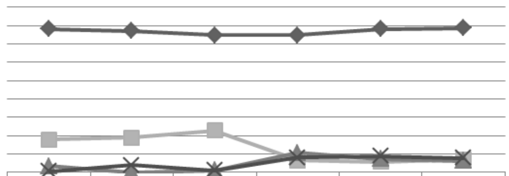
[7] There are three different categories of the physical operation: (1) there is no operation asked in the unit; (2) there is operation asked in the unit; and (3) others (see Chapter 5).

[8] There are four different categories discussed in this issue: (1) there is no construction in the unit; (2) the unit ask/provide opportunity to construct figure/s; (3) the unit ask/provide opportunity to construct auxiliary line/s; and (4) others (see Chapter 5).



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
No	78.30%	77.10%	74.90%	74.70%	78.20%	78.80%
Figure	17.80%	19.00%	22.90%	6.50%	5.80%	7.00%
Auxiliary Line	3.30%	0.00%	1.10%	10.60%	7.30%	6.30%
Others	0.60%	3.80%	1.10%	8.10%	8.70%	7.90%

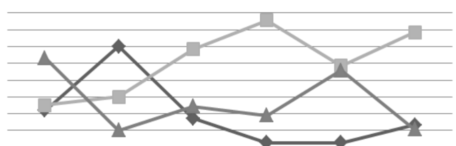
Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.4.3. The distribution of figural construction in the six textbook series

There are around 75–79% of the units which do not require to construct any figure. Focusing on the construction, the German series provide higher opportunities *to construct figures* (18–23%), while the Taiwanese series provide higher opportunities *to construct auxiliary lines* (6–11%). *Fokus* does not provide any unit which asks for constructing an auxiliary line, and the other two German series do so very rarely (3%, 1%).

Figures 7.4.3-1 and 7.4.3-2 present the relations between two types of construction and the role of support to claims. *Delta* and *National Academy for Educational Research* provide higher connection between *constructing figures* and *providing proof as support to claims* (53%, 46%; see Figure 7.4.3-1), while *Fokus* provides a high connection to *conjecture as support to claims*, *Lambacher Schweizer* and the Taiwanese series provide high connection to *non-proof argument* as support to claims.

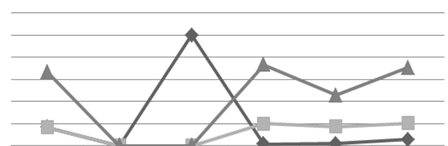
In view of *constructing auxiliary lines* (see Figure 7.4.3-2), the Taiwanese series mainly provide *proof as support to claims*. In *Delta*, it is more related to *proof* (67%), and in *Lambacher Schweizer*, it connects completely to *conjecture*. However, as mentioned before, the number of units asking for constructing auxiliary lines of these two series is very low.



	DT (32)	FK (20)	LS (41)	KH (37)	NAER (37)	NI (38)
Conjecture	21.90%	60.00%	17.10%	2.70%	2.70%	13.20%
N-P Argument	25.00%	30.00%	58.50%	75.70%	48.60%	68.40%
Proof	53.10%	10.00%	24.40%	18.90%	45.90%	10.50%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.4.3-1. The distribution of support to claims of the construction of figure(s) in the six textbook series



	DT (6)	FK (0)	LS (2)	KH (60)	NAER (46)	NI (34)
Conjecture	16.70%	0.00%	100.00%	1.70%	2.20%	5.90%
N-P Argument	16.70%	0.00%	0.00%	20.00%	17.40%	20.60%
Proof	66.70%	0.00%	0.00%	73.30%	45.70%	70.60%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

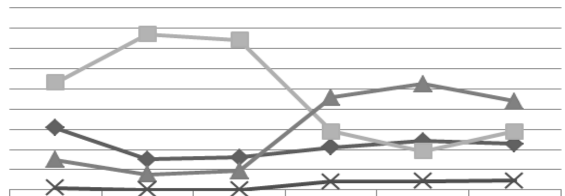
Figure 7.4.3-2. The distribution of support to claims of the construction of auxiliary line(s) in the six textbook series

In summary, *figures* are the main assignments for figural construction in the textbooks and the construction of *auxiliary lines* is hardly used in the German textbook series. Both *figures* and *auxiliary lines* are constructions used in all Taiwanese textbook series. Compared to the German series, the Taiwanese series provide a lower extent to *figures* but a relatively high extent to *auxiliary lines*. The relation between the construction of *figures* and the kind of *support to claims* is not differentiated, while the construction of *auxiliary lines* is presumably related to *proof* in the Taiwanese series.

7.2.4.4 The decomposition of figure

Figure 7.4.4 presents the distribution of decomposing figure[9] for each of the six textbook series.

[9] There are four different categories discussed in this issue: (1) no figure presented and no need to decompose the un-presented figure; (2) figure/s presented but no need to decompose the figure/s; (3) figure/s presented and it is



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
Null	30.60%	15.20%	16.20%	21.20%	24.30%	22.70%
No	53.30%	77.10%	74.30%	29.00%	19.10%	28.90%
With Figure	15.00%	7.60%	9.50%	45.70%	52.40%	43.80%
Without Figure	1.10%	0.00%	0.00%	4.10%	4.30%	4.60%

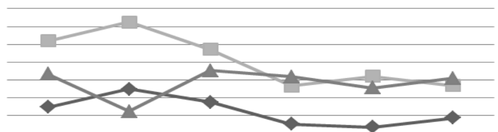
Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.4.4. The distribution of figural decomposition in the six textbook series

In the German textbook series (84%, 92%, 91%), there is a large proportion of the units that do not provide opportunities to decompose a figure, no matter whether there are figures presented or not. *Delta* is the series that provides more opportunities to decompose a figure (16%). However, the percentage is still much lower than in the Taiwanese series that provide opportunities to decompose figure (50%, 57%, 48%).

In view of those units providing decomposition with figure/s, most of them provide *non-proof argument* as support to claims in all three German series (47–63%; see Figure 6.4.4-1) and then *proof* or *conjecture*. However, most of them provide *proof* followed by *non-proof argument* in two Taiwanese series, *Kang Hsuan* and *Nan I*. Though *National Academy for Educational Research* provides more *non-proof argument* followed by *proof*, the difference between them is not large (around 7%). The Taiwanese series provide less *conjecture* in these units.

necessary to decompose the figure(s); and (4) no figure presented but it is necessary to decompose the figure (see Chapter 5).



	DT (27)	FK (8)	LS (17)	KH (258)	NAER (332)	NI (238)
Conjecture	14.80%	25.00%	17.60%	5.40%	3.60%	8.80%
N-P Argument	51.90%	62.50%	47.10%	26.70%	32.20%	26.90%
Proof	33.30%	12.50%	35.30%	31.80%	25.60%	31.10%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.4.4-1. The distribution of support to claims of decomposition with figure(s) in the six textbook series

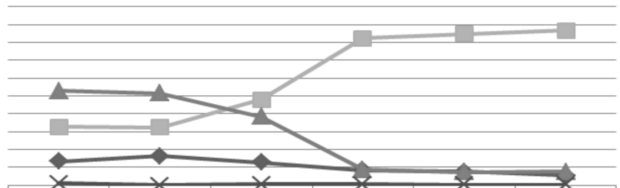
In summary, the Taiwanese textbook series provide much more units that require *decomposition with figure* than the German textbook series do. The German series provide *non-proof argument* as support to claims more frequently related to *decomposition with figure*, while the Taiwanese series provide either *proof* or *non-proof argument*.

7.2.5 The contents and the contexts of units

Figure 7.5.1 shows the distribution of content linkage[10] of each unit. Most units in the German textbook series are either *treated as an individual/new unit*, especially in *Delta* (53%) and *Fokus* (51%), or *connected to the former unit*, especially in *Lambacher Schweizer* (48%). Beyond these two ways, some units in these three series are connected also to *experience* (13%–16%).

In contrast, the Taiwanese series provide a large number of units which are *connected to the former unit/s* (82%–87%). Only a small amount of units is *connected to experience* (5%–8%) or *treated as a new unit* (7%–9%).

[10] There are four different items to discuss the relation between contents: (1) the content (of the unit) links/refers to experience; (2) the content links/refers to former analytic unit within this topic; (3) The content is completely new; and (4) others (see Chapter 5).

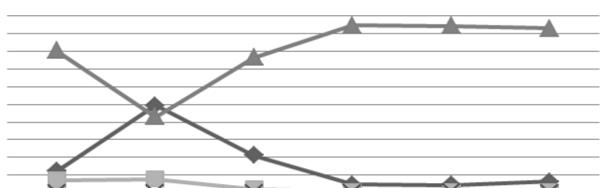


	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
Experience	13.30%	16.20%	12.80%	8.30%	7.70%	5.30%
Former	32.80%	32.40%	48.00%	82.30%	84.90%	86.70%
New	52.80%	51.40%	38.50%	8.80%	7.30%	7.70%
Others	1.10%	0.00%	0.60%	0.50%	0.20%	0.20%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.5.1. The distribution of content linkage in the six textbook series

Figure 7.5.2 presents which kinds of context[11] the unit links to. Two German series, *Delta* and *Lambacher Schweizer*, provide higher proportions of *complete mathematics context* (81%, 77%) than the third series, *Fokus* (43%). However, *Fokus* provides more *daily life context* (50%) than the other two series (12%, 21%). The German series also provide *history context* in the texts, though the proportion of unit is low.



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
Life	12.20%	49.50%	21.20%	4.60%	4.30%	6.30%
History	7.20%	7.60%	2.20%	0.70%	0.80%	0.60%
Mathematics	80.60%	42.90%	76.50%	94.70%	94.60%	93.00%
Others	0.00%	0.00%	0.00%	0.00%	0.30%	0.20%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.5.2. The distribution of context linkage in the six textbook series

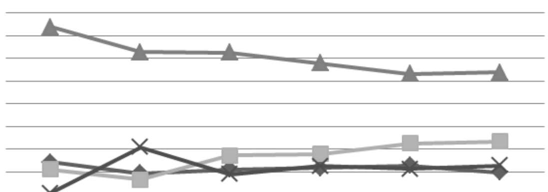
[11] There are four different items to discuss the connection of the unit to which kinds of the context: (1) the daily life; (2) the history; (3) only mathematics; and (4) others (see Chapter 5).

In view of the Taiwanese series, *complete mathematics context* dominates most units (93–95%). Though some units are involved in *daily life context* or *history context*, these two contexts are not commonly seen in the Taiwanese series (daily life context: 4–6%; history context: 1%).

In summary, two ways of content linkage, namely unit is *treated as a new unit* and unit is *connected to the former unit*, are commonly seen in the German series, while in the Taiwanese series, units are commonly *connected to the former unit*. Regarding the context linkage in the German series, *complete mathematics context* and *daily life context* are two main contexts. The Taiwanese series provide a large amount of *complete mathematics context*.

7.2.6 The practical functions in general

Figure 7.6 presents the distribution of the practical function.



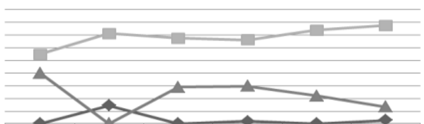
	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
None	14.40%	9.50%	11.20%	11.70%	12.80%	9.90%
Generalization	11.10%	6.70%	17.30%	17.90%	22.60%	23.40%
Application	73.90%	62.90%	62.60%	57.90%	53.30%	53.80%
Application to Generalization	0.60%	21.00%	8.90%	12.60%	11.40%	12.90%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.6. The distribution of practical function in the six textbook series

Application is the main function provided in the German textbook series (74%, 63%, 63%), as well as in the Taiwanese series.

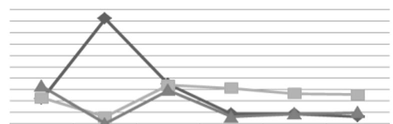
Figures 7.6-1, 7.6-2 and 7.6-3 show the relations between three practical functions and three types of support to claims. According to the three distributions, there is no clear pattern to show the relations. However, focusing on the Taiwanese series, it is found that *generalization* and *application* are closely connected to *non-proof argument* as support and *application to generalization* is closely connected to *proof* as support.



	DT (20)	FK (7)	LS (31)	KH (101)	NAER (143)	NI (127)
Conjecture	0.00%	14.30%	0.00%	2.00%	0.00%	2.80%
N-P Argument	55.00%	71.40%	67.70%	66.30%	74.00%	77.60%
Proof	40.00%	0.00%	29.00%	29.70%	22.00%	13.30%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational

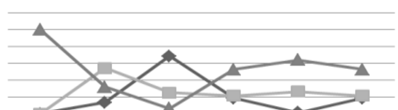
Figure 7.6-1. The distribution of support to claims of generalization in the six textbook series



	DT (133)	FK (66)	LS (112)	KH (327)	NAER (338)	NI (292)
Conjecture	21.80%	92.40%	34.80%	8.60%	8.60%	6.20%
N-P Argument	22.60%	6.10%	33.90%	31.20%	26.40%	25.70%
Proof	33.10%	0.00%	29.50%	6.10%	8.60%	9.80%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational

Figure 7.6-2. The distribution of support to claims of application in the six textbook series



	DT (1)	FK (22)	LS (16)	KH (71)	NAER (72)	NI (70)
Conjecture	0.00%	13.60%	68.80%	18.60%	1.40%	18.60%
N-P Argument	0.00%	54.50%	25.00%	21.40%	26.40%	21.40%
Proof	100.00%	31.80%	6.30%	52.90%	63.90%	52.90%

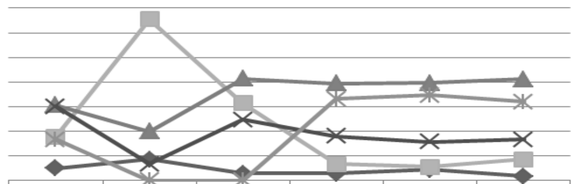
Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational

Figure 7.6-3. The distribution of support to claims of application to generalization in the six textbook series

In summary, *application* is the main function of units in all textbook series. The Taiwanese series provide the functions in a similar way, especially *Kang Hsuan* and *Nan I*, while the German series provide the functions in discrepant ways. Taiwanese units providing the functions of *generalization* and *application* serve *non-proof argument* as support to claims, and units providing the function of *application to generalization* serve *proof* as support to claims.

7.2.7 The supports to claim in general

Figure 7.7 shows the distribution of the *kinds of support provided to claims* in all textbook series.



	DT (180)	FK (105)	LS (179)	KH (565)	NAER (634)	NI (543)
None	5.00%	8.70%	2.80%	2.80%	4.40%	1.80%
Conjecture	17.80%	65.40%	31.30%	6.70%	5.50%	8.70%
N-P Argument	30.60%	20.00%	41.30%	39.50%	39.70%	41.10%
Proof	30.00%	6.70%	24.60%	17.90%	15.60%	16.60%
Others	16.70%	0.00%	0.00%	33.10%	34.70%	31.90%

Note. DT: Delta; FK: Fokus; LS: Lambacher Schweizer; KH: Kang Hsuan; NAER: National Academy for Educational Research; NI: Nan I

Figure 7.7. The distribution of support to claims in the six textbook series

The three German series lay different emphases on different types of support, however, they all provide these three kinds of support, *conjecture*, *non-proof argument*, and *proof*, in the units. *Fokus* is the series that provides the least *proof* (7%) but the most *conjecture* (65%). The three Taiwanese series provide higher *non-proof argument* (40%, 40%, 41%) and pending units (*others*) (33%, 35%, 32%) among different categories of supports. *Proof* as support is provided more frequently than *conjecture*.

Comparing the distribution in Germany and Taiwan, the German textbooks provide a higher amount of *conjecture* (especially *Fokus*) and *proof* (except *Fokus*) than the Taiwanese textbooks.

7.3 Three Examples of Geometry Designs

This section compares the designs of the three statements—(1) *the sum of the interior angles of a triangle*; (2) *the Thales theorem*; and (3) *the Pythagorean theorem*—with respect to their introduction and their proofs in the German and Taiwanese textbooks. It will also be reported whether there is a typical/representative approach in the textbooks of each country. The comparison of these statements is intended to answer RQ4.

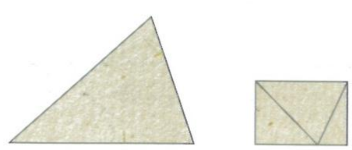
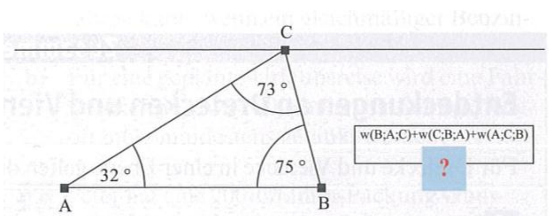
7.3.1 The sum of interior angles of a triangle

As mentioned in Chapters 3 and 5, there are different approaches to prove that the sum of the interior angles of a triangle equals 180º. Some of them present with pragmatic actions in specific cases (embodied approaches) to accept it; some apply with the known facts (the sum of the exterior angles of a triangle) to infer to it; and some use the Euclidean idea of parallel lines (parallel postulates) to prove it. However, some of these approaches cannot be viewed as an introduction but rather as an application. Therefore, in the following, the introduction to the statements is presented before the compilation of approaches is compared. The comparison between the countries is based on the aforementioned three principles[12] (see Chapter 5).

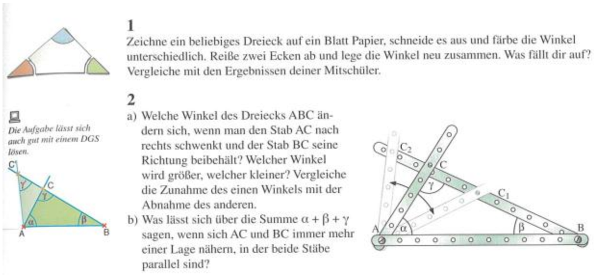
7.3.1.1 Introduction and proofs in the German textbooks

The three German textbooks provide different contexts to introduce the concept of the sum of interior angles of a triangle (see Table 7.1). These contexts are operation/experiment with figures/materials (e.g., *Delta* and *Lambacher Schweizer*) or calculation (e.g., *Fokus*).

Table 7.1. The introduction of the sum of interior angles of a triangle in German textbooks

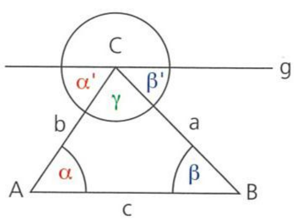
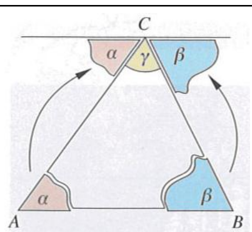
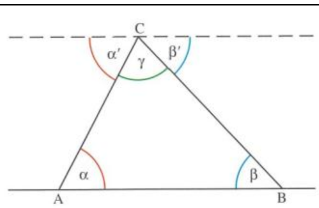
	Folding paper (*Delta*, grade 7, p. 46)
	Calculating the sum of angles (*Fokus*, grade 7, p. 41)

[12] The set of three principles are: (1) (conceptual) continuity; (2) accessibility; and (3) contextualization.

 1 Zeichne ein beliebiges Dreieck auf ein Blatt Papier, schneide es aus und färbe die Winkel unterschiedlich. Reiße zwei Ecken ab und lege die Winkel neu zusammen. Was fällt dir auf? Vergleiche mit den Ergebnissen deiner Mitschüler. 2 a) Welche Winkel des Dreiecks ABC ändern sich, wenn man den Stab AC nach rechts schwenkt und der Stab BC seine Richtung beibehält? Welcher Winkel wird größer, welcher kleiner? Vergleiche die Zunahme des einen Winkels mit der Abnahme des anderen. b) Was lässt sich über die Summe $\alpha + \beta + \gamma$ sagen, wenn sich AC und BC immer mehr einer Lage nähern, in der beide Stäbe parallel sind? Die Aufgabe lässt sich auch gut mit einem DGS lösen.	1. Tearing and aligning papers; 2. Experiencing the changes of the size of angles and length/the invariance of the sum of angles (*Lambacher Schweizer*, grade 7, p. 43)

All German textbook series provide an explicit *proof* using deductive reasoning with the parallel postulate—*the alternate (interior) angles of a pair of parallel lines with a transversal are congruent/equal*—after the introduction (see Table 7.2). The proof is chained by two mathematical ideas: *alternate interior angles are congruent* and a *straight angle* is equal to 180° (180° is not mentioned in texts). The content is presented in different units, including *explanatory texts*, *worked examples*, and *exploration*, which provide different kinds of support to their claims posed in the units.

Table 7.2. The proofs of the sum of interior angles of a triangle in German textbooks

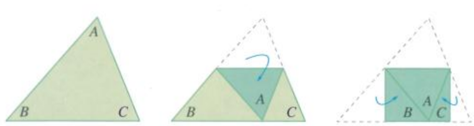
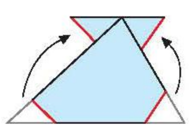
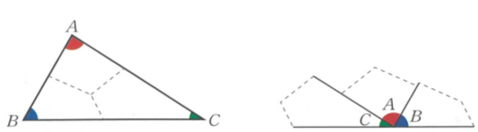
	1. The alternate interior angles are congruent; 2. Straight angle (*Delta*, grade 7, p. 46)
	1. The alternate interior angles are congruent; 2. Straight angle (*Fokus*, grade 7, p. 42)
	1. The alternate interior angles are congruent; 2. Straight angle (*Lambacher Schweizer*, grade 7, p. 43)

In summary, the German textbooks provide different introductions in the beginning but present the same approach in their proofs. They all provide deductive reasoning with the parallel postulate (proof) to validate this statement.

7.3.1.2 Introduction and proofs in the Taiwanese textbooks

The three Taiwanese textbooks provide very similar contexts to introduce the statement. All of them show an operation with the figure (experiment)—folding paper (e.g., *Kang Hsaun*) or tearing and aligning angles (e.g., *National Academy for Educational Research* and *Nan I*)—and link this experiment to the concept of a straight angle (see Table 7.3). They relate to the folding/tearing angles and aligning these angles to a straight line which implies that the sum of the angles is *180°*, though this is not explicitly mentioned in the texts. By doing this experiment, the statement is 'proved' at the same time and ready to be a fact[13] used to generalize other new mathematical ideas.

Table 7.3. The introduction of the sum of interior angles of a triangle in Taiwanese textbooks

	Folding paper (*Kang Hsuan*, grade 8, vol. 4, p. 110)
	Tearing and aligning angles (*National Academy for Educational Research*, grade 8, vol. 4, p. 42)
	Tearing and aligning angles (*Nan I*, grade 8, vol. 4, p. 100)

[13] Later the contents in the texts are introduced with this fact to apply and generalize a formula—the sum of interior angles of a polygon is $180° \cdot (n-2)$—and another statements—the sum of exterior angles of a triangle/polygon is *360°*.

Beyond such an approach of crucial experiment[14], only the textbook from *National Academy for Educational Research* gives the proof of this statement (see Figure 7.8). However, it is connected to a new statement, *the sum of the exterior angles of a triangle*, which is generalized from this fact which is accepted without proof. Obviously, this proof should be a circular reasoning and cannot be recorded a valid proof when considering its complete context.

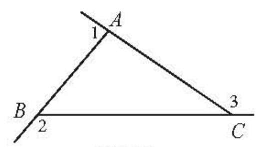


Figure 7.8. The proof of the sum of interior angles of a triangle: Calculating with three straight angles and the known fact of the sum of exterior angles (*National Academy for Educational Research*, vol. 4, p. 48)

The above mentioned contents are presented in different units, including all four types of text (mainly *explanatory texts*), and provide *non-proof argument* as support to their claims.

In summary, the Taiwanese textbooks provide only experiments to introduce and 'prove' this statement, however, there is no valid proof provided in the introduction. Though *Nan I* afterwards encompasses the valid proof of this statement (see Appendix G) in the ninth grade textbook when introducing what geometry proof is, it is not part of a consecutive process to learn this statement.

7.3.1.3 Comparison of approaches used in German and Taiwanese textbooks

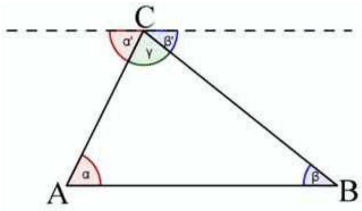
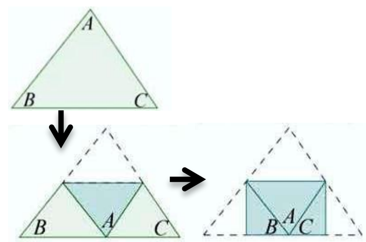
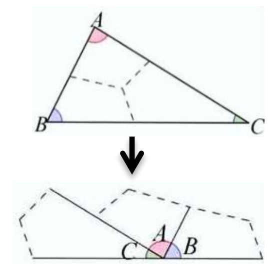
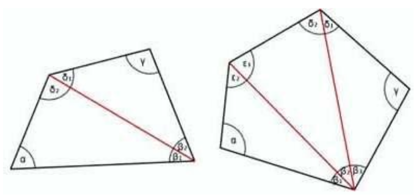
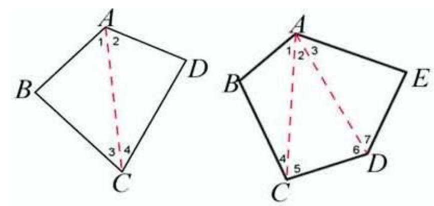
Table 7.4 compiles the main approaches provided in the two countries separately. The German textbooks introduce this statement in their specific designs. All designs try to present the content in presentations for students to immediately catch the main concepts of the texts. Nevertheless, the prior mathematical ideas are the same, namely that the alternate angles of a pair of parallel

[14] cf. Balacheff (1988), see 3.2.2.

lines with a transversal are the same/congruent. This is related to what Euclidean geometry called the *parallel postulates*. The representations of German texts use the compensate symbols and same color as the inscriptions to present the congruent angles (cf. Table 7.2). Such inscriptions provide strong visual links between the concepts and figures. After the introduction, this statement is used to generalize a formula for $180° \times (n-2)$ of the sum of interior angles of a polygon with n vertices.

The Taiwanese textbooks introduce this statement in a similar way, though in different presentations. As mentioned above, all textbooks use *tearing and aligning* to 'prove' that the sum of the interior angles is equal to the straight angle. They have the same typical introduction (cf. 5.2.2), moreover, the contexts are almost the same. After the introduction, this statement is used to generalize a formula, $180° \times (n-2)$, for the sum of interior angles of a polygon with n vertices. This is similar to the design of the German textbooks. Additionally, only the Taiwanese textbooks introduce another statement—the sum of exterior angles of a polygon—with this statement. The proof of this statement is similar to that of the *sum of the exterior angles of a triangle*, that is, application of n straight angles and the formula of the sum of interior angles of a polygon: $180° \times n - 180° \times (n-2) = 360°$.

Table 7.4. The compilation of the approaches to the sum of interior angles of a triangle in German and Taiwanese textbooks

<table>
<tr><th>German Main Approach</th><th>Taiwanese Main Approach</th></tr>
<tr><td>
<u>Former ideas</u>:
Parallel postulates and Straight angle</td><td>
or

Physical/figural experiment</td></tr>
<tr><td colspan="2">Further</td></tr>
<tr><td></td><td></td></tr>
<tr><td colspan="2">(Algorithmic) Generalization of the statement to polygons: 180°·(n-2)</td></tr>
</table>

The above comparison was analyzed based on the three principles (see Section 5.2) and the summary is presented in Table 7.5. The *conceptual continuity* of German and Taiwanese textbooks is presented in the previous exposition. All textbooks provide high *accessibility* to the typical 'proof' in each country. More precisely, the German textbooks provide high accessibility to the typical proof of the statement. The Taiwanese textbooks give high accessibility to an experiment which gives an idea of the proof (crucial experiment). The German textbooks include (1) *various units*, such as explanatory texts (mainly with deductive reasoning, that is proof as support), worked examples, and explorations; and (2) *diverse contexts*, such as folding paper, calculating the angles, physical operation (crucial experiments), and deductive reasoning, to this statement. The Taiwanese textbooks provide (1) *various units*, such as explanatory texts, worked examples, immediate practices and explorations (especially explanatory texts; mainly with non-proof argument as support); and (2) *stable contexts*, only the physical operation (crucial experiment), to introduce and 'validate' this statement.

Table 7.5. The summary of the sum of interior angles of a triangle in German and Taiwanese textbooks

Principles	**Germany**	**Taiwan**
Continuity	Main ideas: • Parallel postulate • Straight angle Further: • The sum of interior angles of a polygon	Main idea: • Straight angle Further: • The sum of interior angles of a polygon • The sum of exterior angles of a polygon
Accessibility	• High accessibility to a typical introduction	• High accessibility to a typical introduction
Contextualization	• Various tasks • Diverse contexts	• Various tasks • Stable contexts

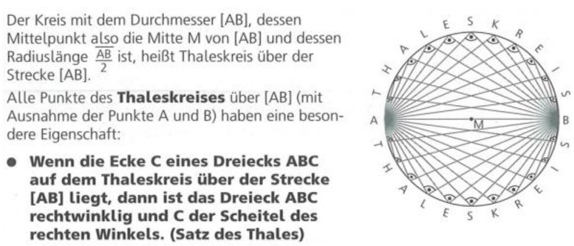
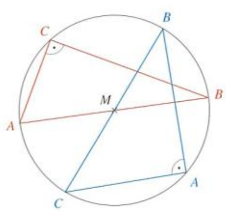
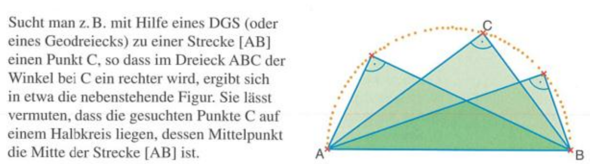
7.3.2 The Thales theorem

This section provides information how the Thales theorem is developed in the textbooks, following the same structure as in the previous sub-section.

7.3.2.1 Introduction and proofs in the German textbooks

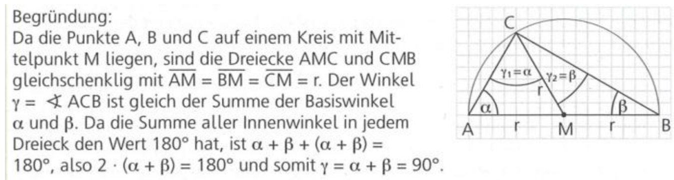
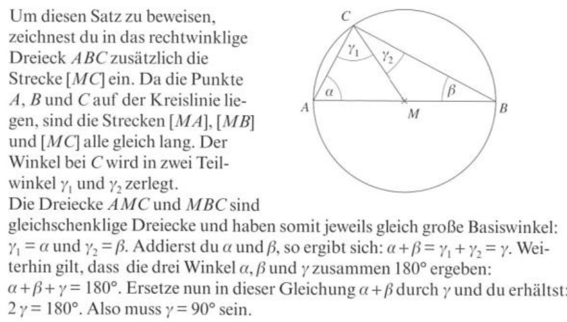
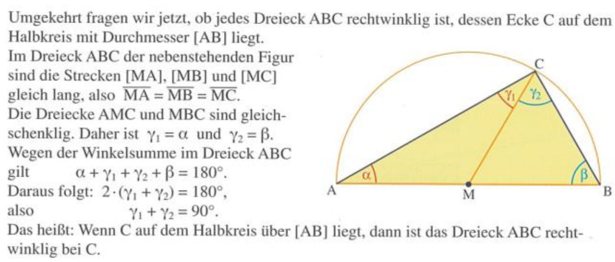
Table 7.6 shows that the German textbooks present different ways to introduce the Thales theorem. *Delta* introduces the definition of the 'Thales circle' by drawing various triangles on the circle (one side is fixed on the diameter). *Fokus* introduces the history and definition with the figure. *Lambacher Schweizer* introduces the definition by giving the instruction on using Dynamic Geometry Software [DGS]. Though the texts cannot act dynamic motions as it would be typical for a real DGS approach, the reader can use the texts as an instruction for working with DGS. It is worthwhile to mention that all three textbooks introduce the idea with *transformation* to emphasize the *invariance* of this statement. That means, once the two points (of a triangle) are located on the polar opposites of the diameter of a circle, no matter where the third point is located (on the circle), the triangle composed of these three points should be a right-angled triangle.

Table 7.6. The introduction of the Thales theorem in German textbooks

 Der Kreis mit dem Durchmesser [AB], dessen Mittelpunkt also die Mitte M von [AB] und dessen Radiuslänge $\frac{\overline{AB}}{2}$ ist, heißt Thaleskreis über der Strecke [AB]. Alle Punkte des **Thaleskreises** über [AB] (mit Ausnahme der Punkte A und B) haben eine besondere Eigenschaft: ● **Wenn die Ecke C eines Dreiecks ABC auf dem Thaleskreis über der Strecke [AB] liegt, dann ist das Dreieck ABC rechtwinklig und C der Scheitel des rechten Winkels. (Satz des Thales)**	1. Introduce with the definition of the 'Thales circle' 2. Present various/different point "C" on the circumference (A, B on the polar points of a diameter) (*Delta*, grade 7, p. 166)
	Introduce with the definition of the Thales theorem (with history) (*Fokus*, grade 7, p. 158):
 Sucht man z. B. mit Hilfe eines DGS (oder eines Geodreiecks) zu einer Strecke [AB] einen Punkt C, so dass im Dreieck ABC der Winkel bei C ein rechter wird, ergibt sich in etwa die nebenstehende Figur. Sie lässt vermuten, dass die gesuchten Punkte C auf einem Halbkreis liegen, dessen Mittelpunkt die Mitte der Strecke [AB] ist.	Introduce with DGS to draw different points on the circumference of the semicircle and observe the angle C (*Lambacher Schweizer*, grade 7, p. 152)

Though these textbooks use different approaches to introduce the Thales theorem, they provide the same method to prove it (see Table 7.7). This method involves the same mathematical concepts: (1) all radiuses of a circle are the same; (2) an isosceles triangle means that both sides of it are equal and two base angles are equal; and (3) the sum of interior angles of a triangle is *180°*. It mainly reasons with these three concepts and presents the proof to this statement with the algorithm to support it.

Table 7.7. The proofs of the Thales theorem in German textbooks

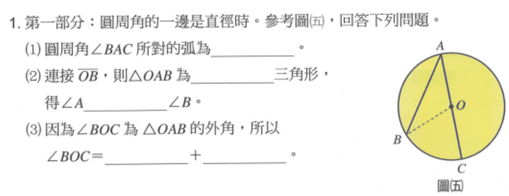
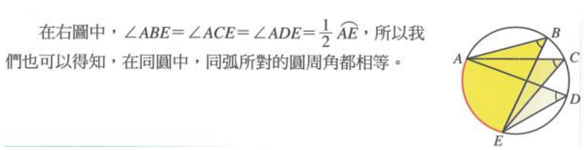
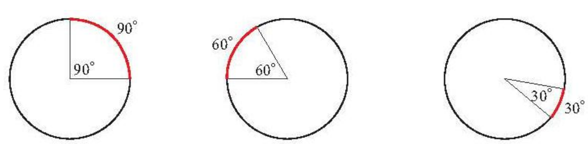
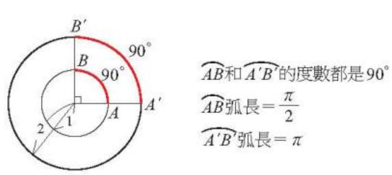
 Begründung: Da die Punkte A, B und C auf einem Kreis mit Mittelpunkt M liegen, sind die Dreiecke AMC und CMB gleichschenklig mit $\overline{AM} = \overline{BM} = \overline{CM} = r$. Der Winkel $\gamma = ∢ ACB$ ist gleich der Summe der Basiswinkel α und β. Da die Summe aller Innenwinkel in jedem Dreieck den Wert 180° hat, ist $\alpha + \beta + (\alpha + \beta) = 180°$, also $2 \cdot (\alpha + \beta) = 180°$ und somit $\gamma = \alpha + \beta = 90°$.	1. All radiuses of a circle are the same 2. Isosceles triangle (two sides and two base angles are equal) 3. The sum of interior angles of a triangle (*Delta*, grade 7, p. 166)
 Um diesen Satz zu beweisen, zeichnest du in das rechtwinklige Dreieck ABC zusätzlich die Strecke $[MC]$ ein. Da die Punkte A, B und C auf der Kreislinie liegen, sind die Strecken $[MA]$, $[MB]$ und $[MC]$ alle gleich lang. Der Winkel bei C wird in zwei Teilwinkel γ_1 und γ_2 zerlegt. Die Dreiecke AMC und MBC sind gleichschenklige Dreiecke und haben somit jeweils gleich große Basiswinkel: $\gamma_1 = \alpha$ und $\gamma_2 = \beta$. Addierst du α und β, so ergibt sich: $\alpha + \beta = \gamma_1 + \gamma_2 = \gamma$. Weiterhin gilt, dass die drei Winkel α, β und γ zusammen 180° ergeben: $\alpha + \beta + \gamma = 180°$. Ersetze nun in dieser Gleichung $\alpha + \beta$ durch γ und du erhältst: $2\gamma = 180°$. Also muss $\gamma = 90°$ sein.	1. All radiuses of a circle are the same 2. Isosceles triangle (two sides and two base angles are equal) 3. The sum of interior angles of a triangle (*Fokus*, grade 7, p. 158)
 Umgekehrt fragen wir jetzt, ob jedes Dreieck ABC rechtwinklig ist, dessen Ecke C auf dem Halbkreis mit Durchmesser [AB] liegt. Im Dreieck ABC der nebenstehenden Figur sind die Strecken [MA], [MB] und [MC] gleich lang, also $\overline{MA} = \overline{MB} = \overline{MC}$. Die Dreiecke AMC und MBC sind gleichschenklig. Daher ist $\gamma_1 = \alpha$ und $\gamma_2 = \beta$. Wegen der Winkelsumme im Dreieck ABC gilt $\alpha + \gamma_1 + \gamma_2 + \beta = 180°$. Daraus folgt: $2 \cdot (\gamma_1 + \gamma_2) = 180°$, also $\gamma_1 + \gamma_2 = 90°$. Das heißt: Wenn C auf dem Halbkreis über [AB] liegt, dann ist das Dreieck ABC rechtwinklig bei C.	1. All radiuses of a circle are the same 2. Isosceles triangle (two sides and two base angles are equal) 3. The sum of interior angles of a triangle (*Lambacher Schweizer*, grade 7, p. 152)

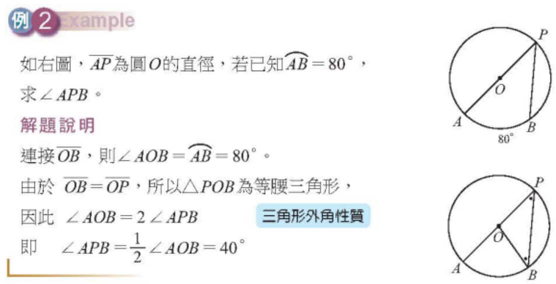
In summary, the German textbooks provide different experiences of transformation and then focus on the same route of proof.

7.3.2.2 Introduction and proofs in the Taiwanese textbooks

The Taiwanese textbooks provide a significantly different approach to introduce the Thales theorem. They start with the definition of a *circumferential angle* and its relationships to the *central angle* and the *arc*. Table 7.8 shows how these three textbooks provide the definitions and proofs of the relationships between circumferential angles, central angles, and arcs. *Kang Hsuan* and *Nan I* present a similar sequence to (1) prove the circumferential angle (details see Table 7.8) first and then (2) present the invariant relationship between the circumferential angle and the arc by presenting the transformation of different circumferential angles but refer to the same arc. However, *National Academy for Educational Research* provides a slightly different route to introduce it. It (1) starts from the definition of the relation between the central angle and the arc, and then (2) proves the relation between the central angle and the circumferential angle. The intermediary role of the three angles changes from the arc (*Kang Hsuan* and *Nan I*) to the central angle (*National Academy for Educational Research*). The specific arrangement influences the methods of proofs presented (see discussion below).

Table 7.8. The introduction/proofs of the angles of a circle in Taiwanese textbooks

	1. Circumferential angle 2. Central angle (proofs with the property: The exterior angle of an interior angle of a triangle equals the sum of the other two interior angles) (*Kang Hsuan*, grade 9, vol. 5, p. 87)
	1. Circumferential angle 2. Arc (*Kang Hsuan*, grade 9, vol. 5, p. 90)
	1. Central angle 2. Arc (*National Academy for Educational Research*, grade 9, vol. 5, p. 74)
	1. Central angle 2. Arc (*National Academy for Educational Research*, grade 9, vol. 5, p. 74)



例2 Example

如右圖，$\overline{AP}$為圓O的直徑，若已知$\widehat{AB}=80^\circ$，求$\angle APB$。

解題說明

連接$\overline{OB}$，則$\angle AOB=\widehat{AB}=80^\circ$。

由於 $\overline{OB}=\overline{OP}$，所以$\triangle POB$為等腰三角形，

因此 $\angle AOB=2\angle APB$ 三角形外角性質

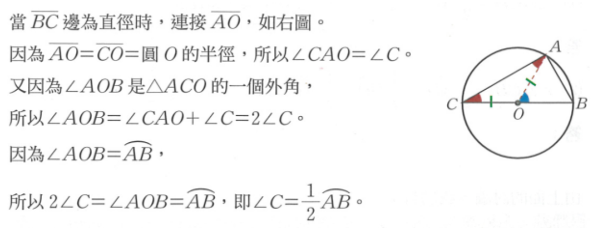
即 $\angle APB=\frac{1}{2}\angle AOB=40^\circ$

1. Circumferential angle

2. Central angle

(proofs with the property: The exterior angle of an interior angle of a triangle equals the sum of the other two interior angles)

(*National Academy for Educational Research*, grade 9, vol. 5, p. 76)



當$\overline{BC}$邊為直徑時，連接$\overline{AO}$，如右圖。

因為$\overline{AO}=\overline{CO}$=圓$O$的半徑，所以$\angle CAO=\angle C$。

又因為$\angle AOB$是$\triangle ACO$的一個外角，

所以$\angle AOB=\angle CAO+\angle C=2\angle C$。

因為$\angle AOB=\widehat{AB}$，

所以$2\angle C=\angle AOB=\widehat{AB}$，即$\angle C=\frac{1}{2}\widehat{AB}$。

1. Circumferential angle

2. Central angle

(proofs with the property: The exterior angle of an interior angle of a triangle equals the sum of the other two interior angles)

(*Nan I*, grade 9, vol. 5, p. 87)

例如右圖中：

$\angle B_1=\angle B_2=\angle B_3=\angle B_4=\frac{1}{2}\widehat{AC}$

由上面的說明可得：

同一圓中，同弧或等弧所對的圓周角的度數相等。

1. Circumferential angle

2. Arc

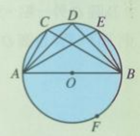
(*Nan I*, grade 9, vol. 5, p. 89)

The aforementioned approach to the introduction of the angles of a circle is the preparation for the following proof. The Taiwanese textbooks present the proof by calculating the circumferential angle (see Table 7.9), which is half of the arc, to support that the circumferential angle of a semicircle is a right angle. In addition, the *National Academy for Educational Research* makes use of the properties of an equilateral triangle. This proof is found in all German textbooks (cf. Table 7.7). It is presented after the introduction of the relation between the central angle and the arc.

Table 7.9. The poofs of the Thales theorem in Taiwanese textbooks

例 4 直徑所對的圓周角是直角

如右圖，$\overline{AB}$ 是圓 O 的直徑，C、D、E 三點均在圓周上，則$\angle ACB$、$\angle ADB$、$\angle AEB$ 的度數分別是多少？



解 ∵ $\overline{AB}$ 是圓 O 的直徑

∴半圓 $\overset{\frown}{AFB}$ 為 180°

又∵$\angle ACB$ 是圓周角

∴$\angle ACB=\frac{1}{2}\overset{\frown}{AFB}=\frac{1}{2}\times 180^\circ=90^\circ$

又$\angle ADB$、$\angle AEB$ 與$\angle ACB$ 對同弧

∴$\angle ADB=\angle AEB=90^\circ$

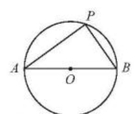
由例題 4 可發現：$\overline{AB}$ 為直徑，$\overset{\frown}{AB}$ 所對的圓周角都是直角。事實上，**直徑或半圓所對的圓周角是直角。**

The circumferential angle equals half of the arc

(*Kang Hsuan*, grade 9, vol. 5, p. 91)

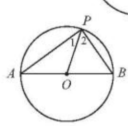
例 1 Example

如右圖，$\overline{AB}$為圓O的直徑，在圓上任取異於A、B的一點P，說明圓周角$\angle APB$必為直角。



解題說明

如右圖，作$\overline{OP}$。



由於 $\angle 1=\angle A$ △OAP為等腰三角形

$\angle 2=\angle B$ △OBP為等腰三角形

由三角形內角和為180°，得

$\angle A+\angle B+(\angle 1+\angle 2)=180^\circ$

因此 $(\angle 1+\angle 2)\times 2=180^\circ$

所以 $\angle 1+\angle 2=\frac{180^\circ}{2}=90^\circ$

1. All radiuses of a circle are the same
2. Isosceles triangle (two sides and two base angles are equal)
3. The sum of interior angles of a triangle

(*National Academy for Educational Research*, grade 9, vol. 5, p. 75–76)

同學們有沒有注意到，$\angle APB=90^\circ$，正好是$\angle APB$所對弧的弧度180°的一半（圖2-18）。這個性質對於一般圓周角也成立嗎？

要回答這個問題，我們首先討論圓周角$\angle APB$其中一邊是直徑的情況。

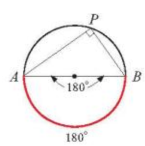


圖2-18

The circumferential angle equals half of the arc

(*National Academy for Educational Research*, grade 9, vol. 5, p. 76)

因為一周角等於 360°，半圓所對的圓心角為 180°，所以我們有：

半圓所對的圓周角都是 90°。

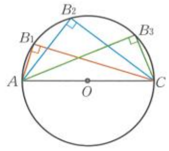
如右圖，

$\angle B_1=\angle B_2$

$=\angle B_3$

$=\frac{1}{2}\overset{\frown}{AC}$

$=90^\circ$



The circumferential angle equals half of the arc

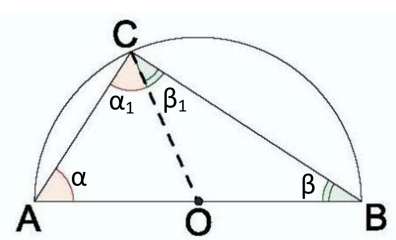
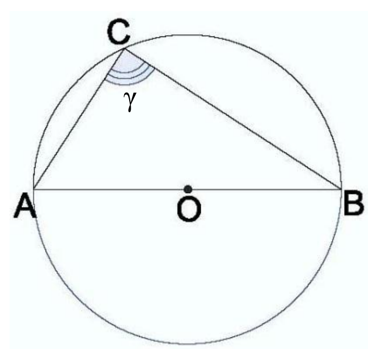
(*Nan I*, grade 9, vol. 5, p. 90)

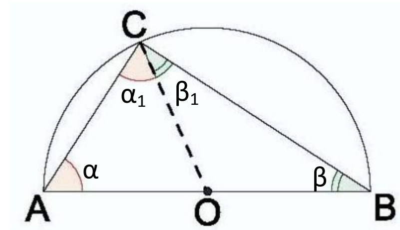
In summary, the three Taiwanese textbooks introduce the relations between different kinds of angles (circumferential angle, central angle, and arc) of a circle and apply them in the proof of this statement.

7.3.2.3 Comparison of approaches used in German and Taiwanese textbooks

The approaches to the Thales theorem in Germany and Taiwan respectively are compiled in Table 7.10. The German textbooks present very similar situations to introduce and prove the Thales theorem. Especially the sequences of contents from introduction to conclusion are nearly identical in all textbook series. The Taiwanese textbooks, especially the *Kang Hsuan* and *Nan I*, provide a similar approach to this statement. Moreover, *National Academy for Educational Research* presents an additional/alternative approach to different introduction and proof (see also Tables 7.8 and 7.9).

Table 7.10. The compilation of the approaches to proving the Thales theorem in German and Taiwanese textbooks

German Main Approach	**Taiwanese Main Approach**
	Most used approach 
Calculation: $\alpha = \alpha_1$, $\beta = \beta_1$ $\alpha + (\alpha_1 + \beta_1) + \beta = 180°$ $\alpha_1 + \beta_1 = 90°$	Calculation: $\gamma = ½ \text{ arc} = 90°$

	Alternative approach 
	Calculation: $\alpha = \alpha_1$, $\beta = \beta_1$ $\alpha + (\alpha_1 + \beta_1) + \beta = 180°$ $\alpha_1 + \beta_1 = 90°$

The summary of how the Thales theorem is introduced in the two countries is given in Table 7.11. According to the principle of *conceptual continuity*, the three German textbooks present the proof by chaining the same mathematical ideas: (1) all radiuses of a circle are congruent/the same; (2) an isosceles triangle has two equal base angles and two equal sides; and (3) the sum of interior angles of a triangle equals 180°. All three textbooks provide *high accessibility* in proving this statement with the above three mathematical ideas in the same presentation (see Table 7.10). There is a typical introduction in the German textbooks. In accordance with the instruction mentioned before, it can be found that the *contextualization* of the German textbooks makes use of *diverse contexts* in all textbooks, such as the presentation of transformation or the application of the dynamic geometric software. Moreover, there are *various tasks* (mainly explanatory texts) in every textbook which support the main idea.

The three Taiwanese textbooks, examined with the principle of *conceptual continuity*, make use of the same mathematical idea, the circumferential angle equals half of the arc, in proving this statement but the sequence in introducing this idea is not completely the same among the three books (see details in 7.3.2.2). As to the accessibility, the Taiwanese textbooks provide *high accessibility* to the typical proof of the Thales theorem. However, *National Academy for*

Educational Research provides *medium accessibility* to the proof as used in German books. If a strategy would give an independent approach to proving the statement, it would provide high accessibility. However, the textbook introduces various angles of a circle in the beginning, and focuses on the relation between the central angle and the arc of a circle in the initial introduction, but turns to this interrupted approach (medium accessibility) to prove the statement without any connection to the previous concepts. Therefore, this approach should be regarded to give medium accessibility. Regarding the principle of *contextualization*, the Taiwanese textbooks provide *various tasks*, such as explanatory texts and worked examples, within one textbook; moreover there are *diverse contexts*, such as algorithm and transformation, across the three textbooks.

Table 7.11. The summary of the Thales theorem in German and Taiwanese textbooks

Principles	**Germany**	**Taiwan**
Continuity	Main idea: • All radiuses of a circle are congruent • Isosceles triangle: Base angles are equal; two sides (except base side) are equal • The sum of interior angles of a triangle	Main idea: • Circumferential angle = ½ central angle = ½ arc Note: The alternate approach provided by the NAER involves the same ideas as the German textbooks
Accessibility	• High accessibility to a typical proof	• High accessibility to a typical proof (most used approach) and median accessibility (depends on textbook series) to different method (alternative approach)
Contextualization	• Various tasks (deductive reasoning) • Diverse contexts (transformation)	• Various tasks (of algorithm on calculating unknown angles) • Diverse contexts (algorithm, transformation)

7.3.3 The Pythagorean theorem

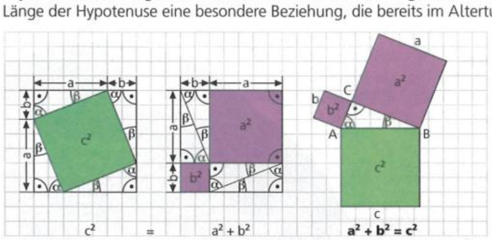
The Pythagorean theorem is a statement usually connected with the calculation of the areas with different permutations of figures. In Germany, other than in Taiwan, this theorem is not treated as a single theorem with the formula $a^2 + b^2 = c^2$ but rather as a group of theorem which includes the conventional Pythagorean theorem, the hypotenuse-leg theorem (Kathetensatz), and the leg-leg theorem (Höhensatz).

7.3.3.1 Introduction and proofs in the German textbooks

Table 7.12 provides the approach of *Delta*. In this textbook, the definition of the Pythagorean theorem is first given by presenting the relation (of areas) between figures (of squares) (in the upper Table). It is worth noting that there is no procedure of calculation given in the texts. Later, the proofs are provided for the Pythagorean theorem group—the hypotenuse-leg theorem and the leg-leg theorem (in the lower Table). These proofs are based on the conditions of similarity of the triangles.

Table 7.12. The introduction and proofs of the Pythagorean theorem in *Delta*

In jedem rechtwinkligen Dreieck besteht zwischen den Längen der Katheten und der Länge der Hypotenuse eine besondere Beziehung, die bereits im Altertum bekannt war:

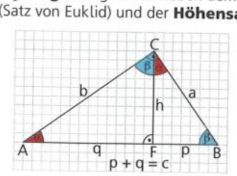


Diese Beziehung, der **Satz von Pythagoras**, lautet in Worten:
In jedem rechtwinkligen Dreieck hat das Quadrat über der Hypotenuse den gleichen Flächeninhalt wie die Quadrate über den beiden Katheten zusammen.

Definition with visual figures (areas of the squares) of the Pythagorean theorem

(*Delta*, grade 9, p. 30)

Zu der **Satzgruppe von Pythagoras** gehören neben dem Satz von Pythagoras noch der **Kathetensatz** (Satz von Euklid) und der **Höhensatz**.



Die beiden Dreiecke CAF und BCF sowie das Dreieck ABC sind zueinander ähnlich, da sie in allen drei Winkeln übereinstimmen:

$\Delta CAF \sim \Delta ABC$	$\Delta BCF \sim \Delta ABC$	$\Delta CAF \sim \Delta BCF$
$\frac{b}{q} = \frac{c}{b}; \mid \cdot bq$	$\frac{a}{p} = \frac{c}{a}; \mid \cdot ap$	$\frac{h}{q} = \frac{p}{h}; \mid \cdot hq$
$b^2 = cq$	$a^2 = cp$	$h^2 = pq$

In jedem rechtwinkligen Dreieck ist das Quadrat über jeder Kathete flächengleich dem Rechteck mit der Hypotenuse und dem dieser Kathete anliegenden Hypotenusenabschnitt als Seiten. (**Kathetensatz**)

In jedem rechtwinkligen Dreieck ist das Quadrat über der Hypotenusenhöhe flächengleich dem Rechteck mit den beiden Hypotenusenabschnitten als Seiten. (**Höhensatz**)

Proofs of the hypotenuse-leg theorem (Kathetensatz) and the leg-leg theorem (Höhensatz) with the conditions of similarity of the similar triangles

(*Delta*, grade 9, p. 38)

Table 7.13 shows the contents of *Fokus* which provide an experimental approach to the Pythagorean theorem. At first, a table is presented which enables the reader to discover and identify the relation ($a^2 + b^2 = c^2$) between the given variables (sides of the triangles; in the upper Table). After this, a proof of the Pythagorean theorem with the calculation of areas of different figures (right-angled triangles and squares) is provided (in the upper middle Table). Later, the Pythagorean theorem group is introduced with experiments (constructing and operating figures) on figures (in the lower middle and lower Table). Though there is no concrete text that points out what the related concepts of these experiments are, the concepts (the conditions of similarity) are actually embedded in the experiments. For example, (1) the figural construction in introducing the hypotenuse-leg theorem is based on the relation between areas of the squares and rectangles which implies the relation between sides; and (2) cutting and piecing together the figures which introduce the leg-leg theorem based on the conditions of similarity.

Table 7.13. The introduction and proofs of the Pythagorean theorem in *Fokus*

	Fokus-Buch			Taschenbuch			Atlas		
a	25,2	26,3	24,9	17,1	18,3	16,5	29,2	28,8	29,6
b	8,5	3,6	9,4	7,2	3,4	8,5	8,0	9,3	6,4
c	26,6	26,6	26,6	18,6	18,6	18,6	30,3	30,3	30,3
a^2	635,04	691,69	620,01	292,41	334,89	272,25	852,64	829,44	876,16
b^2	72,25	12,96	88,36	51,84	11,56	72,25	64	86,49	40,96
c^2	707,56	707,56	707,56	345,96	345,96	345,96	918,09	918,09	918,09
$a^2 + b^2$	707,29	704,65	708,37	344,25	346,45	344,50	916,64	915,93	917,12

Finding the relations between the given variables (sides of the triangles)

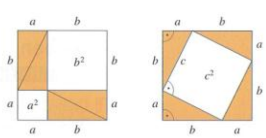
(*Fokus*, grade 9, p. 36)

Die Flächeninhalte der beiden großen Quadrate messen $A = (a + b)^2$ (siehe nebenstehende Bilder).
Der Flächeninhalt A lässt sich jeweils aus einzelnen Flächeninhalten zusammensetzen.

linken Figur: $A = a^2 + b^2 + 4 \cdot \frac{1}{2}ab = a^2 + b^2 + 2ab$

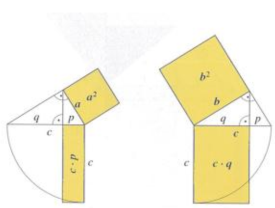
rechte Figur: $A = c^2 + 4 \cdot \frac{1}{2}ab = c^2 + 2ab$

Da beide Figuren denselben Flächeninhalt haben, gilt $a^2 + b^2 + 2ab = c^2 + 2ab$ und somit $a^2 + b^2 = c^2$.



Proof of the Pythagorean theorem by calculating the areas of figures (right-angled triangles and squares)

(*Fokus*, grade 9, p. 37)



Das Quadrat einer Kathetenlänge ist gleich dem Produkt aus der Hypotenusenlänge und der Länge des anliegenden Hypotenusenabschnitts.

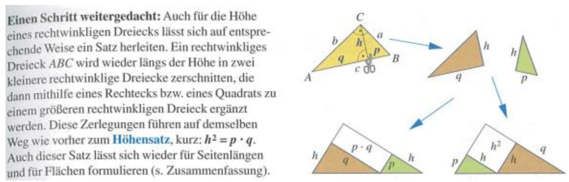
Oder als Flächensatz formuliert:

Im rechtwinkligen Dreieck hat das Quadrat über einer Kathete denselben Flächeninhalt wie ein Rechteck, dessen Seiten so lang wie die Hypotenuse und der an der Kathete anliegende Hypotenusenabschnitt sind.

In Kurzform:
$a^2 = c \cdot p$ bzw. $b^2 = c \cdot q$.

Presenting with the figural construction to introduce the hypotenuse-leg theorem (based on the concept of the relation between areas of the figures)

(*Fokus*, grade 9, p. 46)



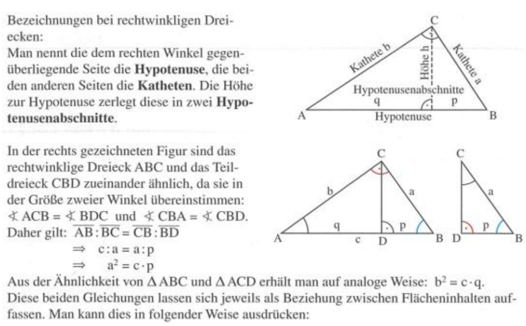
Einen Schritt weitergedacht: Auch für die Höhe eines rechtwinkligen Dreiecks lässt sich auf entsprechende Weise ein Satz herleiten. Ein rechtwinkliges Dreieck *ABC* wird wieder längs der Höhe in zwei kleinere rechtwinklige Dreiecke zerschnitten, die dann mithilfe eines Rechtecks bzw. eines Quadrats zu einem größeren rechtwinkligen Dreieck ergänzt werden. Diese Zerlegungen führen auf demselben Weg wie vorher zum **Höhensatz**, kurz: $h^2 = p \cdot q$. Auch dieser Satz lässt sich wieder für Seitenlängen und für Flächen formulieren (s. Zusammenfassung).

Presenting with cutting and piecing together the figures to introduce the leg-leg theorem (based on the conditions of similarity)

(*Fokus*, grade 9, p. 47)

Table 7.14 provides the approach taken in *Lambacher Schweizer*. It first offers the proof of the hypotenuse-leg theorem with the conditions of similarity (in the upper Table) and then provides the conclusions in different presentations which can easily be recognized by the areas of different shapes—rectangles and squares (in the upper middle Table). Later, it combines two presentations of the hypotenuse-leg theorem to prove the Pythagorean theorem deduced from two formulae of the hypotenuse-leg theorem (algebraic calculation) (in the lower middle Table). Finally, it provides the proof of the leg-leg theorem deduced from the formulae of the Pythagorean theorem and the hypotenuse-leg theorem (algebraic calculation) and the construction of figures which help to recognize the leg-leg theorem by the areas of different shapes—square and rectangle (in red; in the lower Table).

Table 7.14. The introduction and proofs of the Pythagorean theorem in *Lambacher Schweizer*

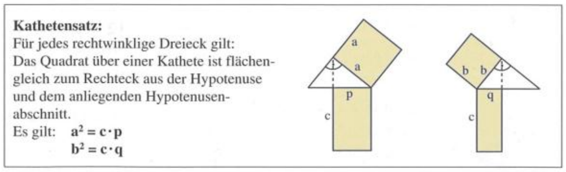


Bezeichnungen bei rechtwinkligen Dreiecken:
Man nennt die dem rechten Winkel gegenüberliegende Seite die **Hypotenuse**, die beiden anderen Seiten die **Katheten**. Die Höhe zur Hypotenuse zerlegt diese in zwei **Hypotenusenabschnitte**.

In der rechts gezeichneten Figur sind das rechtwinklige Dreieck ABC und das Teildreieck CBD zueinander ähnlich, da sie in der Größe zweier Winkel übereinstimmen:
∢ ACB = ∢ BDC und ∢ CBA = ∢ CBD.
Daher gilt: $\overline{AB} : \overline{BC} = \overline{CB} : \overline{BD}$
$\Rightarrow \quad c : a = a : p$
$\Rightarrow \quad a^2 = c \cdot p$
Aus der Ähnlichkeit von Δ ABC und Δ ACD erhält man auf analoge Weise: $b^2 = c \cdot q$.
Diese beiden Gleichungen lassen sich jeweils als Beziehung zwischen Flächeninhalten auffassen. Man kann dies in folgender Weise ausdrücken:

Proof of the hypotenuse-leg theorem with the conditions of similarity.

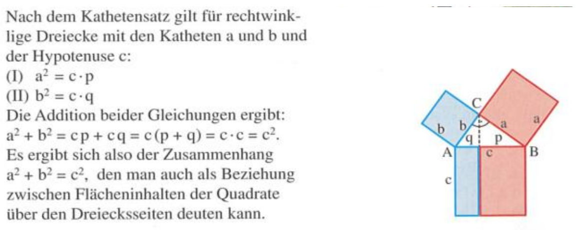
(*Lambacher Schweizer*, grade 9, p. 42)



Kathetensatz:
Für jedes rechtwinklige Dreieck gilt:
Das Quadrat über einer Kathete ist flächengleich zum Rechteck aus der Hypotenuse und dem anliegenden Hypotenusenabschnitt.
Es gilt: $a^2 = c \cdot p$
$b^2 = c \cdot q$

The conclusions and the presentations of the hypotenuse-leg theorem.

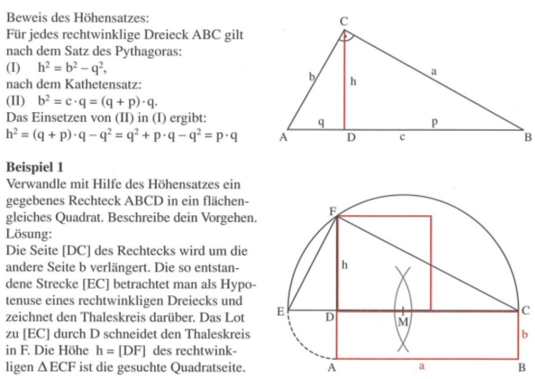
(*Lambacher Schweizer*, grade 9, p. 42)



Nach dem Kathetensatz gilt für rechtwinklige Dreiecke mit den Katheten a und b und der Hypotenuse c:
(I) $a^2 = c \cdot p$
(II) $b^2 = c \cdot q$
Die Addition beider Gleichungen ergibt:
$a^2 + b^2 = cp + cq = c(p + q) = c \cdot c = c^2$.
Es ergibt sich also der Zusammenhang
$a^2 + b^2 = c^2$, den man auch als Beziehung zwischen Flächeninhalten der Quadrate über den Dreiecksseiten deuten kann.

Proof of the Pythagorean theorem deduced from the formulae of the (two) hypotenuse-leg theorem

(*Lambacher Schweizer*, grade 9, p. 45)



Beweis des Höhensatzes:
Für jedes rechtwinklige Dreieck ABC gilt nach dem Satz des Pythagoras:
(I) $h^2 = b^2 - q^2$,
nach dem Kathetensatz:
(II) $b^2 = c \cdot q = (q + p) \cdot q$.
Das Einsetzen von (II) in (I) ergibt:
$h^2 = (q + p) \cdot q - q^2 = q^2 + p \cdot q - q^2 = p \cdot q$

Beispiel 1
Verwandle mit Hilfe des Höhensatzes ein gegebenes Rechteck ABCD in ein flächengleiches Quadrat. Beschreibe dein Vorgehen.
Lösung:
Die Seite [DC] des Rechtecks wird um die andere Seite b verlängert. Die so entstandene Strecke [EC] betrachtet man als Hypotenuse eines rechtwinkligen Dreiecks und zeichnet den Thaleskreis darüber. Das Lot zu [EC] durch D schneidet den Thaleskreis in F. Die Höhe $h = [DF]$ des rechtwinkligen ΔECF ist die gesuchte Quadratseite.

(1) Proof of the leg-leg theorem, deduced from the formulae of the Pythagorean theorem and the hypotenuse-leg theorem, and (2) its figural construction.

(*Lambacher Schweizer*, grade 9, p. 51)

In summary, the three German textbooks provide different contexts to introduce and validate the Pythagorean theorem group, but the content focuses on the same mathematical concepts, the conditions of similarity.

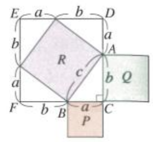
7.3.3.2 Introduction and proofs in the Taiwanese textbooks

The three Taiwanese textbooks introduce the Pythagorean theorem together with its proof. The proofs provided by these three textbooks are quite similar. They make use of algebraic calculation, which requires the skill of expanding perfect squares, $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$, learned in the previous section, with the different permutations of figures (see Table 7.15). Only *National Academy for Educational Research* provides both methods of using two perfect squares with different combinations of figures (cf. the methods provided by *Kang Hsuan* and *Nan I*) to prove this statement and concludes a synthesized figure from the figures used in different methods.

Table 7.15. The introduction and proofs of the Pythagorean theorem in Taiwanese textbooks

<table>
<tr><td>

圖(土)中，三角形 ABC 是一個直角三角形，P、Q、R 分別為以三邊的長度所畫出的正方形。假設 $\overline{BC}$ 長為 a、$\overline{AC}$ 長為 b、$\overline{AB}$ 長為 c，再取三個與三角形 ABC 一模一樣的三角形和邊長為 c 的正方形一起拼成一個邊長為 $a+b$ 的正方形$EFCD$，所以 R 的面積為



圖(土)

正方形 $EFCD$ 面積減掉 4 個三角形 ABC 面積，

即：R 的面積 $=(a+b)^2-4\times\frac{1}{2}\times a\times b$

$=a^2+2ab+b^2-2ab$

$=a^2+b^2$

$=P$ 的面積 $+Q$ 的面積

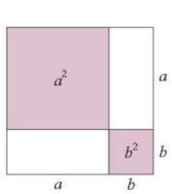
因為 R 的面積 $=c^2$，所以 $c^2=a^2+b^2$。

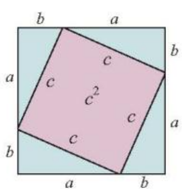
</td><td>

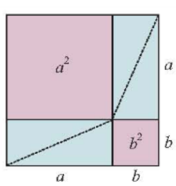
Presenting the proof of the Pythagorean theorem with algebraic calculation (perfect square) with the figure

(*Kang Hsuan*, grade 8, vol. 3, p. 90)

</td></tr>
<tr><td>







楚，首先，四個直角三角形的面積總和是

$\frac{1}{2}ab\times 4=2ab$

而由和平方公式，圖2-3或圖2-4的大正方形面積是

$(a+b)^2=a^2+2ab+b^2$ ············ ①

但大正方形分割成紫色正方形和四個直角三角形，所以

$(a+b)^2=c^2+2ab$ ············ ②

由①、②兩式，就得到

$a^2+b^2=c^2$

</td><td>

Method 1: Presenting the proof with algebraic calculation (perfect square) with the figural presentations

(*National Academy for Educational Research*, grade 8, vol. 3, p. 47)

</td></tr>
<tr><td>

底下我們再介紹另一種方法，來說明這個美妙的性質。這是東漢末年中國數學家趙爽的傑作。

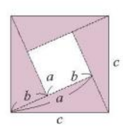


圖2-6

圖2-6是一個以直角三角形斜邊c為邊長的正方形，巧妙的在正方形中安排四個像圖2-2的直角三角形，中間所夾的小正方形邊長為$a-b$。從圖中可以很清楚看到，大正方形的面積等於小正方形面積與四個直角三角形的面積和，也就是

$c^2=(a-b)^2+(\frac{1}{2}ab)\times 4$

$=a^2-2ab+b^2+2ab$

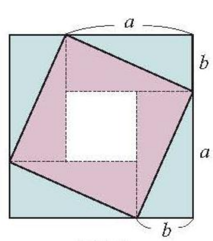
$=a^2+b^2$

</td><td>

Method 2: Presenting the proof with algebraic calculation (perfect square) with a different figural presentation

(*National Academy for Educational Research*, grade 8, vol. 3, p. 48)

</td></tr>
<tr><td>



</td><td>

Conclusion of method 1 and method 2 with another figural presentation

(*National Academy for Educational Research*, grade 8, vol. 3, p. 48)

</td></tr>
</table>

設直角三角形 ABC 兩股長分別為 a 與 b（$a \geq b$），斜邊長為 c。

(1) 將四個與圖 2-1 三角形 ABC 相同的直角三角形，與一個邊長為（$a-b$）的正方形甲拼成四邊形 $DEFG$（如圖 2-2 所示），故此四邊形 $DEFG$ 的面積為 $4 \times \frac{1}{2}ab + (a-b)^2$。

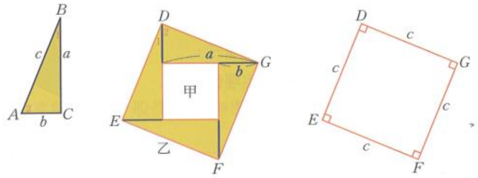


圖 2-1　　圖 2-2　　圖 2-3

(2) 在三角形 ABC 中，
因為三角形內角和為 180 度，
所以角 1 與角 2 與一個直角的和為 180 度，
即角 1 與角 2 的和為 90 度，故角 EDG＝90 度，
同理，四邊形 $DEFG$ 的四個角都是 90 度（直角），
又 $\overline{ED}$ 為三角形 ABC 的斜邊，長度為 c，
$\overline{EF}$、$\overline{FG}$、$\overline{DG}$ 也等於 c，
所以 $\overline{ED}=\overline{EF}=\overline{FG}=\overline{DG}=c$，
即四邊形 $DEFG$ 四邊等長，
因此四邊形 $DEFG$ 為一正方形，面積為 c^2。

(3) 由 (1) 與 (2)，我們知道
$$c^2 = 4 \times \frac{1}{2}ab + (a-b)^2 = 2ab + a^2 - 2ab + b^2 = a^2 + b^2$$

Presenting the proof with algebraic calculation (perfect square) with figural presentations

(*Nan I*, grade 8, vol. 3, p. 85)

In summary, the algebraic calculation with figures is the method of proving the Pythagorean theorem in the Taiwanese textbooks. Though there are no proofs of the hypotenuse-leg theorem or the leg-leg theorem, they are introduced later, together with the topic of similarity in grade 9 (see Appendix G). However, these introduction/proofs are not discussed in relation to the Pythagorean theorem.

7.3.3.3 Comparison of approaches used in German and Taiwanese textbooks

Table 7.16 compiles the main approach used in German and Taiwanese textbooks. The German textbooks introduce the Pythagorean theorem with high connection to the conditions of the similarity (high accessibility), whereas the Taiwanese textbooks introduce this statement with high connection to the calculation of the areas of different figures (high accessibility; see Table 7.16). Such differences result from the different routes of sequence of geometry content in Germany and Taiwan (see 6.3.2). The Pythagorean theorem is introduced after the topic of similarity in grade 9 in Germany, while it is introduced as the first geometry topic in the first semester of grade 8 in Taiwan.

Table 7.16. The compilation of the approaches to the Pythagorean theorem in German and Taiwanese textbooks

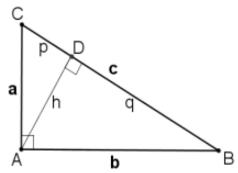
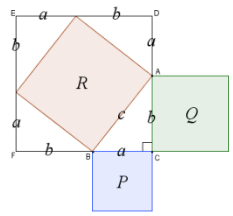
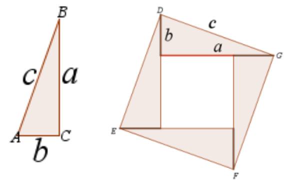
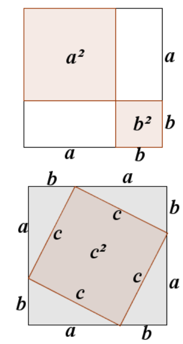
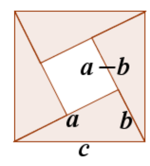
<table>
<tr><th colspan="2">German Main Approach</th><th colspan="3">Taiwanese Main Approach</th></tr>
<tr><td colspan="2">
</td><td rowspan="4">
</td><td>
</td><td rowspan="4">
</td></tr>
<tr><td colspan="2">Generic</td><td rowspan="3">
</td></tr>
<tr><td>Statements</td><td>Former ideas</td></tr>
<tr><td>$a^2 = p \times c$
$b^2 = q \times c$</td><td>Similarity:
$\triangle ACD \sim \triangle BCA$
$\triangle BAD \sim \triangle BCA$</td></tr>
<tr><td>$h^2 = p \times q$</td><td>Similarity:
$\triangle ACD \sim \triangle BAD$</td><td colspan="3">Visual-algorithmic</td></tr>
<tr><td>$a^2 + b^2 = c^2$</td><td>1. Deduced from above formulae
2. Relations between area</td><td>$c^2 = (a+b)^2 - 4 \times \left(\frac{a \times b}{2}\right) = a^2 + b^2$</td><td>$c^2 = 4 \times \left(\frac{a \times b}{2}\right) + (a-b)^2 = a^2 + b^2$</td><td>$(a+b)^2 = a^2 + 2ab + b^2 \ldots(1)$
$(a+b)^2 = c^2 + 2ab \ldots\ldots\ldots(2)$
$a^2 + b^2 = c^2$</td></tr>
</table>

Table 7.17 presents the summary of the approaches to the Pythagorean theorem in Germany and Taiwan. Though the sequences and the arrangements in introducing the Pythagorean theorem are significantly different in the six textbooks, the main idea involved is similar in the textbooks of both countries.

In view of *conceptual continuity* in the German textbooks, the main idea, the conditions of similarity (though *Fokus* does not point them out directly, the figural constructions/presentations reveal this hidden concepts; cf. Table 7.13), is used to prove the hypotenuse-leg theorem. The other relevant ideas used in the different textbooks are figural construction and the relation among areas of different figures. All textbooks provide *high accessibility* to the typical introduction of the Pythagorean theorem with the hypotenuse-leg theorem and the leg-leg theorem (generic approach; Chang, Lin, & Reiss, accepted). As to the *contextualization*, the three textbooks provide *various tasks*, mainly explanatory texts with deductive reasoning, in introducing this statement; and provide *diverse contexts*, for example, the figural construction, different permutation of figures, and algebraic calculation on figural areas, among three textbooks.

In contrast, the Taiwanese textbooks introduce the Pythagorean theorem by calculating the areas of figures. Therefore, the main ideas with respect to *conceptual continuity* are (1) the (area) formulae of right-angled triangle and square which are introduced in the elementary school (see Chang et al., accepted) and (2) the algebraic skill—perfect squares—which is introduced in the previous section (before the introduction of the Pythagorean theorem). All textbooks also provide *high accessibility* to the typical introduction. However, the typical introduction is different from that in Germany. The introduction is the stereotype of visual-algorithmic approach. Regarding the *contextualization*, the three textbooks provide *various tasks*, including four different types of texts, in introducing and applying the Pythagorean theorem; moreover, they provide a *stable context*, namely algebraic calculation with the figures.

Table 7.17. The summary of the Pythagorean theorem in German and Taiwanese textbooks

Principles	**Germany**	**Taiwan**
Continuity	Main idea: • Conditions of similarity Additional ideas: • Figural construction • Area formulae	Main idea: • Area formulae of triangle and square • (Skills: Perfect squares)
Accessibility	• High accessibility to a typical introduction	• High accessibility to a typical introduction
Contextualization	• Various tasks • Diverse contexts	• Various tasks • Stable contexts

7.4 Textbooks Development in Germany and Taiwan

In the interviews with the textbook authors, both authors mentioned that they had to develop the textbooks following the contents of the official national standards—educational standards plus syllabi in Germany and general guidelines in Taiwan (cf. Chapter 2). One practical reason to follow the national standards is that the textbooks need the final approval from the examination of the Ministries of Education and it is the basic requirement provided by the Ministry of Education to follow the national standards. Another reason is that the goals and the topics to be learned by students are clearly mentioned in the national standards. Textbook developers can focus on these goals and topics when developing the textbooks.

Though the textbooks from different publishing houses in the same country have to follow the same national standards, it does not mean all the textbooks in Germany or in Taiwan are the same. Each publishing house has to build its own characteristics of textbooks to differentiate themselves from the others. For example, the design of 'Math Blog' of *Nan I*, presents mathematics together with physical world, with history, with deeper mathematical concepts, etc. in order to present it in a not so 'dry' presentation (interview, 2011). When the textbook authors designed their textbooks, they had their specific intentions set before they started to design. For example, the German author (interview, 2011) considered some competences—mathematical argumentation, mathematical problem-solving, mathematical modeling, and reflection—to be fundamental for all mathematical topics, and therefore, integrated all these elements in the design. She also thought that other competences—communication, mathematical representations, and dealing with symbolic, formal, and technical elements of mathematics—are important for specific mathematical topics, and selectively used them in the design.

The Taiwanese editor Tso (interview, 2011) mentioned that using different examples to practice mathematical knowledge is important to students in learning mathematics. This opinion is established in Taiwanese textbook and is particularly reflected by a high percentage of immediate practice and worked example. In addition, he commented on the high competition among different textbook publishers in Taiwan and said that it forced the textbook developers to develop textbooks in a way adapted to the mathematics teachers' practice of using textbooks. It is because teachers are those influencing the decision making on the selection of textbooks for the new school year. The German editor Schätz (interview, 2011) pointed out that the textbook publishers in Germany develop their own specific features of textbooks, though they all have to develop their textbooks based on the national standards and receive the pressure from the open market which means that they also need to meet teachers' requirements. In doing so, they can differentiate their specific characteristics from the others easily and schools/teachers can choose the suitable textbooks for their teaching.

In general, most of the representations among the German textbooks show a broad variety (see Figure 7.1) whereas most of them among the three Taiwanese textbooks put an emphasis on coherent presentation (cf. Section 7.2). For example, four types of texts appearing in the Taiwanese textbooks are, ranked in descending order, explanatory text/immediate practice, worked example, and exploration. This result is in line with the comments provided by two textbook authors.

To summarize, the development of textbooks is highly connected to the national standards and textbook authors' intentions (see Figure 7.9 in summary).

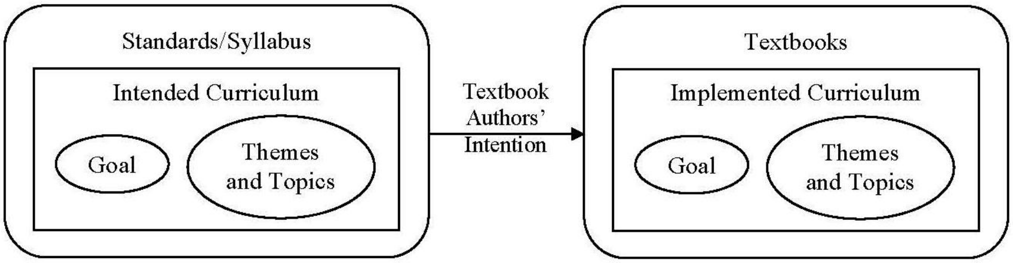


Figure 7.9. How a textbook developed in Germany and Taiwan

8. DISCUSSION

This chapter considers the major findings of the study, the limitations of the study, and directions for future research and curriculum development.

8.1 The Major Findings

This study aimed at investigating opportunities to learn mathematical proofs in geometry content of textbooks by developing an analytic framework (general comparison) and at examining the essential distinction of mathematical proofs in German and Taiwanese textbooks by building additional principles (specific comparison).

After analyzing the geometry content in different German and Taiwanese textbooks, the results showed that the *presentations* of a mathematical proof are various in different textbooks. However, the *main mathematical concepts* used in introducing some specific and well-known mathematical ideas are similar among textbooks *within* one country but not always across countries. This situation is closely connected to the sequence of mathematics content (i.e. intended curriculum). Moreover, this study found that the *purposes* of mathematical proofs are different in Germany and Taiwan. Each country has its own typical approaches to the specific mathematical proof.

This study found four major differences between the German and Taiwanese textbooks: (1) the presentations of the textbooks; (2) the actions used in the geometry design; (3) the purposes of mathematical proofs in geometry content; and (4) the typical approaches to mathematical proofs in the two countries. They are discussed in the following four sub-sections. Lastly, an outlook on using the analytic framework and principles in analyzing textbooks is discussed in another sub-section (5).

8.1.1 Presentations of the textbooks

There are two main differences in text presentations with respect to the coherence of textual content among textbooks.

First, the presentations are different among the German textbooks, whereas they are similar among the Taiwanese textbooks (see the results in Section 7.2). Furthermore, with regard to the introduction of a new mathematical idea, the German textbooks provide *diverse contexts* in various units (cf. the analytic framework), while the Taiwanese textbooks supply *stable contexts* in various units (cf. the principle of contextualization in 5.2.3; see the results in Section 7.3).

As mentioned in Chapter 5, there are different intentions held by textbook authors to design various distributions of different *types of text* (explanatory text, worked example, exploration, and immediate practice; cf. the analytic framework). The *content* of these types of text provides distinctive features, functions, and supports (cf. content of unit in the analytic framework) to convey mathematical content. The results of the most content examined with the analytic framework show that the presentations (types of text and content of text) are varying among the German textbooks and more homogeneous among the Taiwanese textbooks.

The diverse and stable contexts reveal the differing focuses of the application of mathematical concepts in the German and Taiwanese textbooks. They are found based on the principle *contextualization* (see 5.2.3). Using this principle, two of the three examples introducing distinctive mathematical statements are found to present *stable contexts* in the Taiwanese textbooks (see the results in Section 7.3), which means that the actions and functions used in different tasks for introducing and proving a specific mathematical statement are analogical to each other. The three examples in the German textbooks provide different actions and functions, namely *diverse contexts*, in introducing each mathematical statement, *except* for the process of proving the specific mathematical statement.

Both approaches have their pros and cons. On the one hand, different contexts embedded in various tasks provide rich opportunities for teachers to select and use in class (Remillard, 2000); the varieties of tasks might distract teachers/students from the main mathematical concepts that (textbooks) are intended to convey to them. On the other hand, stable and consistent actions and functions decrease the risk that tasks provide procedures without connection of their cognitive demand (Stein, Smith, Henningsen, & Silver, 2009) to students; the stable of tasks might not motivate students' learning.

Second, the processes of proving the three mathematical statements are consistent with respect to the structure of the pre-requisites of mathematical knowledge among the German textbooks. They differ with respect to the use of distinct strategies or different mathematical concepts among the Taiwanese textbooks (see conclusive discussion on typical approaches in 8.1.4).

Though the actions and functions provided in introducing the three mathematical statements are homogeneous in Taiwan and diverse in Germany, this does not apply to the situation in proving these statements. The analyses of the three mathematical statements reveal that the respective approaches (methods) of 'proving' are consistent within each country, so that they might be representative for proving in German and Taiwanese schools respectively. The German textbooks put more emphasis on the *theoretical position* (cf. Table 5.15) in proving the mathematical statements in a structural way, while the Taiwanese textbooks emphasize more the *practical position* (cf. Table 5.15) in 'proving' (accepting) the mathematical statements in order to apply it in solving the following problems. Both routes provide different insights into mathematical proofs (of the statements) for students. The German route might support the practice of *validating the statements* (proving) to students and the Taiwanese route might offer the practice of *retrieving strategies in solving different types of problems* to students. Therefore, the *rigor* of mathematical proofs seems to be more important in Germany, while the *efficiency* of the application of mathematical ideas seems to be more important in Taiwan (cf. Balacheff, 1991).

8.1.2 Actions used in the geometry design

Several actions used in introducing geometry, such as denotation, calculation, physical operation, figure construction, figure decomposition, and figure transformation, were investigated in this study. Not all show clear patterns while comparing German and Taiwanese textbooks. *Calculation*, *figure construction*, and *figure decomposition* are the three actions worth mentioning.

It was found that in both Germany and Taiwan, algorithms are commonly and similarly used in geometry content and geometry proof in textbooks. This is as expected because *calculation*, specifically algebraic and arithmetic *algorithm*, is a common action used in geometry, and also in geometry proof. Precisely, the geometric concepts, e.g., intercept theorem (usually goes with the conditions of similarity), areas, volume, often need the process of algorithm.

Figure construction provides visual images to visualize the task. It is fundamental to the introduction of basic geometric properties (e.g., the perpendicularity, angle bisector, and the congruence postulates of triangles) in both Germany and Taiwan. Moreover, in solving or proving task problems, it needs to closely connect with learned geometric properties to figure out the steps in construction. In German textbooks, it is regarded particularly important to learn geometric properties by constructing complete configuration (geometric figure). For example, three specific centers of triangles (circumcenter, incenter, and centroid) are introduced via constructing the intersectional points of perpendicular bisectors (circumcenter), angle bisectors (incenter), and medians (centroid) of a triangle. In Taiwanese textbooks, it is regarded especially important to solve or prove geometry problems with the strategy of constructing *auxiliary lines* (see the results in 7.2.4.3).

Figure decomposition plays an important role in processing mathematical proofs in Taiwan but has not the same meaning in Germany. The reason could be that there is a topic (unit), introducing mathematical proofs, which provides various tasks for proving in the Taiwanese textbooks. These tasks usually need a specific strategy to decompose the given figure to prove. Figure decomposition might not be the essential factor in mathematical proving, however, it provides opportunities to *idealize* the context, e.g., avoid using apparatus to measure the figure, and to *extract* useful information from the figure to process logical reasoning (cf. Davis & Hersh, 1981), for achieving mathematical *abstraction* (Hilbert & Cohn-Vossen, 1952).

8.1.3 Purposes of mathematical proofs in geometry content

The purposes of learning mathematical proofs in geometry content differ significantly between Germany and Taiwan. They can be summarized as follows.

The German textbooks provide mathematical proofs for *validating a new mathematical idea*. It means that mathematical proving is embedded in the content in order to 'challenge' (Lakatos, 1976; Lampert, 1990; Schoenfeld, 1985) this new mathematical idea by using rationales. These kinds of mathematical proofs provide the function of verification.

The Taiwanese textbooks introduce mathematical proofs as an individual topic with the intention to provide a more rigorous view on mathematics. Therefore, doing mathematical proofs is viewed as a mission in late geometry learning (cf. Table 6.3) in that proving can apply all learned geometric properties flexibly, namely by selecting different strategies. When introducing new mathematical knowledge, especially that is viewed basic knowledge in Taiwan, the textbooks provide crucial experiments (Balacheff, 1988) or authoritative arguments for students to accept. This might also be due to the restriction on the sequence of geometry content in Taiwan. Some proofs cannot be provided at early stages because of the too complex mathematical concepts to reason with. For example, *the parallel postulates* cannot be used in introducing *the sum of the interior angles of a triangle* because at school the first concept is introduced later than the second concept.

Moreover, the results show that the German textbooks provide opportunities to reach the theoretical position in learning mathematical proofs, whereas the Taiwanese textbooks provide opportunities to experience different positions, namely practical position, practical-theoretical position, theoretical position (cf. 5.2.4) gradually. It seems that the German textbooks arrange a '*causal sequence*' (von Wright, 1971) for students to learn deductive reasoning, whereas the Taiwanese textbooks set a '*teleological intention*' (von Wright, 1971) by gradually practicing various mathematical ideas in different tasks to achieve the formal proof (cf. Halldén, 1999).

8.1.4 Typical approaches to mathematical proofs in the two countries

In view of the distinctive approaches to introducing mathematical proofs in Germany and Taiwan, two typical approaches can be identified.

In the German textbooks, mathematical proofs are driven by *the validation of mathematical knowledge with generic concepts* (see the results in Section 7.3). Therefore, deductive reasoning with a hierarchy of learned mathematical concepts is the important approach in introducing mathematical proofs in Germany.

In the Taiwanese textbooks, mathematical proofs are driven by *non-proof arguments* (see 5.1.4). The non-proof arguments are used for introducing new mathematical ideas. Later, various kinds of mathematical knowledge (including these new ideas) are applied in mathematical proof problems. The validation[1] of the mathematical ideas is not particularly emphasized in the textbooks, especially in the beginning stage. More precisely, most mathematical ideas are authoritatively introduced and then applied in problem solving with the assistance of figures and algorithm/calculation. This finding corresponds to Lin and Tsao's (1999) observation of Taiwanese textbooks: "there is no trace of knowledge construction, but rather a glossary of mathematical knowledge that emphasizes problem solving algorithms, augmented by well-chosen examples and followed by exercises …" (p. 232).

These two findings also seem to be in line with research from the national survey studies showing that German students often lack strategies/skills in doing mathematical proofs (Reiss et al., 2002; Heinze, 2004), while Taiwanese students often lack principles in explaining why a proof is valid (the validity of proof; see Lin & Cheng, 2003).

8.1.5 Discussion on the analytic framework and principles

This study developed an analytic framework which provides a different perspective to revisit the texts of textbooks. It separates the functions of different *types of texts* and inspects the individual geometry content from *cognitive functions* and *practicality*. The cognitive functions can be adjusted according to different content domains and the practicality can be used in analyzing all content, therefore, the analytic framework can be applied to different mathematics content. Furthermore, the three established principles were used to examine the design of some specific

[1] The crucial experiment (to validate the mathematical knowledge) is excluded.

mathematical statements, how they are introduced in textbooks, with a systematical comparison of textbooks within or across countries.

Though the analytic framework cannot indicate which mathematical concepts are involved in introducing which mathematical idea (the main deficiency), a further analysis, on the specific mathematical statement, together with the set of principles can complement this deficiency. Though the set of principles applied in this study only analyzed three mathematical statements, it provided useful information and results between Germany and Taiwan. The analytic framework can investigate the general situation, while the set of principles can explore the flow of introducing a specific mathematical idea, which might help to involve teachers in and reach the core mathematical concepts of the textbooks easily.

8.2 Limitations of the study

With respect to the generalizability of the results, there are at least three limitations of this study. They will be discussed shortly in the following.

A first limitation is that the results may not be generalizable with respect to the opportunities to learn mathematical proofs in other mathematical content domains. This study focused on geometry to discuss the opportunities to learn mathematical proofs. Furthermore, it focused on discussing the content at the lower secondary level (grades 7–9) in which geometry is an important topic to learn mathematical proofs. Geometry is the suitable content to link students' concrete ideas and abstract knowledge they need to learn. However, this is not enough to generalize the results of this study to all proof situations. Geometry is of course not the only content providing opportunities to learn mathematical proofs. It is known that the content of mathematical proofs covers many domains of mathematics, especially algebra which is an important topic dealing with the proofs of patterns and theorems (e.g., Hanna, 1990, 1995; Hanna & Jahnke, 1993; Healy & Hoyles, 2000).

Second, the comparison of this study only concentrated on textbooks and cannot represent the real learning situations in the classroom. Teaching and learning mathematical proofs in classroom is a dynamic process, including teachers' instruction, students' beliefs, and interactions between teacher and students. Moreover, the alignment between designed tasks and how teachers select and use them influences the learners' engagement (Watson & Chick, 2011). Though textbooks bear an amount of content and offer some insights into teaching approaches, the content and approaches engaged in classes might not be exactly consistent with those provided in textbooks. For example, teacher's *epistemological vigilance* (Kang & Kilpatrick, 1992) is regarded an important factor influencing the effective use of textbooks.

Third, the research materials (textbooks) analyzed in this study might not be representative for the two countries, and even less for Western Europe and East Asia. The German textbooks analyzed in this study were selected from the state of Bavaria, but there are sixteen federal states in Germany. Furthermore, not all approval textbooks that may be chosen for use in the classroom were included. Nevertheless, it can be assumed that the Bavarian textbooks are probably not too different from textbooks in the other states because of the designs are following the national standards. The same holds for Taiwanese textbooks.

For the reasons discussed here, the results of this study should not be over-generalized to the situations of learning mathematical proofs in Germany and Taiwan.

8.3 Directions for future research and curriculum development

Though students have a more strong reliance on *teacher* as a source for learning mathematics (Stodolsky, Salk, & Glaessner, 1991), textbook authors develop textbook content deliberately from teachers' perspectives. Therefore, the textbooks might somehow present the ideal teaching strategy proposed by textbook authors.

This sub-section provides two directions for further research and possible ways of curriculum development generated from these two directions. One is the usage of textbooks and the approaches to mathematical proofs in classroom; the other is teachers' understanding of textbook content. How to extend these two directions for future research and the curriculum development is elaborated in the following.

The first direction is the investigation on the usage of textbooks and the approaches to mathematical proofs introduced in classroom. There are typical approaches used in introducing some specific mathematical statements in German and Taiwanese textbooks (see Section 7.3) and these approaches differ in both countries. The finding provides an opportunity to discuss whether the differences in students' performances with respect to problem solving (e.g., Tall et al., 2012) stem from different learning approaches. This is based on the hypothesis that the styles of mathematics texts provide a different opportunity for developing strategies of mathematical proofs and the comparison on textbook design has supported this hypothesis. Thus, how textbook content is instructed in classroom and students' solutions of the specific mathematical proofs is worth a further examination.

The second direction is teachers' understanding of mathematics textbook content. Again, the alignment between the designed tasks and how teachers select and use tasks influences the learners' engagement with mathematics (Watson & Chick, 2011). Moreover, textbooks are thought to offer novel tasks or concepts for teachers to construct curriculum in classroom (Remillard & Bryans, 2004). This study provides useful information for teachers to reflect on their usage of textbooks and how to compare mathematics content from different textbooks. Teachers need to have the knowledge of the characteristics of the textbook, because it may help the teachers decide "how other parts of the course must be modified to take advantage of the useful features and to counteract the undesirable features of the textbook" (Tamir, 1985, p. 92). Moreover, textbook analyses might help student teachers adapting to textbooks or even curriculum materials during teacher education training (cf. Ball & Feiman-Nemser, 1988; Lloyd, 2008). Therefore, the training for understanding how to use textbooks is important for mathematics teachers, especially for prospective and novice teachers.

The two directions suggested here can also provide useful information for curriculum developers. Since "[c]urriculum developers must find ways to guide teachers' pedagogical and mathematical decisions, not make decontextualized decisions for them" (Heaton, 1994, p. 376, cited in Remillard, 2000, p. 346).

In order to improve the qualities of instruction and textbooks, carefully examining the foci of instruction and the design of textbooks could be one possible method. Considering students' difficulties in mathematical proofs and the results of this study, it is worth re-examining and reflecting the role of mathematical proofs in designing/developing mathematics curriculum and textbooks. The goals of curriculum influence the textbook development putting more emphasis on different features, e.g., skills or the process of validation. That is, if the goals focus on the *effective practical problem-solving*, it may involve more tacit knowledge in the curriculum; if the goals focus on the *systematic and precise approach (formal approach)* which relies on explicit proposition, it may involve more logical thinking (cf. Rogoff, 1984).

The importance of the connection between pieces of knowledge should be emphasized. It is still a problem for even some tertiary students to reason out mathematical problems flexibly with the learned mathematics knowledge (Richland et al., 2012). This troubles even more the lower secondary school students (Reiss et al., 2002; Heinze, 2004). Therefore, it enlightens the importance of the connection between pieces of knowledge. How the mathematical knowledge is connected in the curriculum and how it is introduced in schools. This study found that the sequence of mathematical knowledge differ between countries. Curriculum developers should probably be more aware of the broad range of possibilities how to introduce mathematical proofs. A specific introduction is usually suggested by specific characteristics of the mathematics curriculum, however, there are important educational goals which manifest in it. Getting more insight in the different approaches might foster a presentation that is oriented towards students' understanding of mathematics.

In conclusion, this study provided an overview of the presentations and approaches used in some textbooks and gave evidences for different possible teaching styles in different cultures. Although the results may not be over-generalized, this study is a first step for a more profound and more general comparison of textbooks.

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APPENDICES

APPENDIX A

CURRICULAR GOALS OF GEOMETRY

The Bavarian Syllabi (Lehrpläne) for Grades 7–9

Grade 7

M 7.1 Figure geometry: from drawing and describing to designing and justifying

With the production of *symmetrical figures*, the students become acquainted with the culture-historical principle of *the construction with ruler and compasses*.

They learn to *analyze geometrical phenomena gradually* and to *argue and justify logically*.

M 7.1.1 Axial and point symmetrical figures (approx. 12 hr.)

On the basis of figures from their experience world, the students recognize the axial and point symmetry as a natural principle of design. They use the ideas obtained from the fundamental principles and theorems in reasoning processes under the first basic constructions. Based on the variety of the four-sided figures, the symmetry becomes accessible for them as a principle of classification.

- Axial symmetry: properties, construction of reflected point and axis
- Perpendicular bisectors; perpendicular line; angle bisector
- Point symmetry: properties, construction of reflected point and center
- Overview of symmetrical squares

M 7.1.2 Views on angles of figures (approx. 8 hr.)

The students discover the essential connections between rectilinear intersections and parallel lines with transverse, and deal with *theorems of the sum of angles*. Thereby, the difference between fundamental theorems and statements derived from them is made clear for them.

- Rectilinear intersection: vertically opposite angle and adjacent angle;
  parallel lines with transverse: corresponding angle and alternate angle
- The sum of the interior angles of a triangle and a quadrilateral

M 7.5 Figure geometry: the triangle as a basic figure

Usually, real objects can be represented well with rectilinear geometrical figures, whose analyzes come from triangles as the basic modules. Therefore, the students deal with the different faces further with the basic figure “triangle”. In order to open geometrical and experimental connections, the students use dynamic geometry software as an interactive tool and therefore they link together the nature and technology (emphasis: computer science) known as the object-oriented perspective.

M 7.5.1 Congruence (approx. 6 hr.)

The question, when two triangles are congruent, leads the students to the unique constructibility of a triangle from the given sides or angles. They learn the congruent postulates, which are used as the fundamental principles.

- Concept of the congruence of figures
- Congruent postulates of triangles and fundamental figure construction

M 7.5.2 Special triangles (approx. 14 hr.)

By congruence or symmetry considerations, the students capture the properties of the isosceles and equilateral triangles. By the example of the Thales theorem, they can experience how dynamic geometry software can facilitate to build assumptions. They understand the proof of the Thales theorem as well as its converse/anti-theorem. They recognize that it opens new possibilities for figure construction.

- The isosceles and equilateral triangle
- Right-angled triangle; the Thales theorem; Construction of tangents to the circle/s

M 7.5.3 (Figure) Construction (approx. 12 hr.)

When constructing triangles and quadrangles, students' imaginativeness and mental agility are developed. In addition, the ability to plan and document the procedures of construction is an essential goal. The students explore the questions of the *constructability* (Konstruierbarkeit) and *the variety of solutions* (Lösungsvielfalt) with the variation in *the pieces of assignments* (Bestimmungsstücke) , e.g., with the assistance of dynamic geometry software. To round off their geometry knowledge, they use their acquired skills (Fähigkeiten) in application-oriented *task formulations* (Aufgabenstellungen).

- Repetition of height, angle bisectors and perpendicular bisectors; circumcircle (Umkreis)
- Construction of triangles and quadrangles, also set the relationship among properties

Grade 8

M 8.4 Intercept theorems and similarity (approx. 15 hr.)

Students learn, by means of the intercept theorems, how geometry becomes available for many practical purposes with the application of algebraic methods. Thus the students are aware of the close connection of geometry and algebra. In particular, they practice again solving fraction equations connected with proportions. The enlargement (scale up) and shrink (scale down) leads students directly to the similarity of figures, which generalizes the already well-known term "congruence".

In the sense of rounding off the repetition and cross-linkage, the students realize thereby it also references to other content, for example to the coherences of *functional description* (funktionalen Beschreibung).

- Intercept theorems
- Similarity of triangles

Grade 9

M 9.5 The right-angled triangle

The group of the Pythagorean theorems, a central topic of this grade, is not least because of its rich reference to other content for students. In addition to the statements of these theorems about areas, the juveniles experience their practical meaning of the calculation of lengths. With the introduction of sine, cosine and tangent, further capabilities are developed with the connections of analyzing the right-angled triangle.

M 9.5.1 The group of the Pythagorean theorems (approx. 14 hr.)

The students realize that they can make calculations in right-angled triangles and construct route distances, whose measured values are square roots, with the assistance of the Pythagorean theorems. With the proofs of the group of theorems, they realize again the general structure of mathematical theorems and practice again the logical argumentation. Moreover, the various examples, from the related materials in everyday context, make the importance of the Pythagorean theorems clear to them.

- The hypotenuse-leg theorem and the leg-leg theorem, the Pythagorean theorem and its reversal
- Applications in the algebraic and geometrical context

M 9.5.2 Trigonometry of the right-angled triangle (approx.. 8 hr.)

…(omit)

M 9.6 Continuation of solid geometry (approx. 25 hr.)

The known properties from everyday life (prism, cylinder, pyramid and cone) are examined more closely. In consideration of oblique images and networks, students develop their spatial imagination further. Concerning the surface content and volumes, they strengthen their knowledge of plane and solid measurement.

The students draw or sketch perspective drawings in order to illustrate lengths and angles of the spatial figures. Supported by their algebraic knowledge, they calculate geometrical quantities; they experience again that these skills are essential to mathematical activity. As rounding off repetition and cross-linkage, the juveniles work on tasks different from either this content or the previous school year—such as trigonometry, intercept theorems or functions are needed.

- Surface area and volume of regular prism and regular cylinder
- Surface area and volume of pyramid and cone
- Considerations on geometric figures (3-D) for the analysis of route distances and angle sizes; special applications

The Taiwanese General Guidelines for Junior High School (Grades 7–9)

According to the 9-year guideline (2003) in Taiwan, the objects of learning geometry are listed as several indices. The listed indices below are geometric objects for the stage of junior high school and for each grade in details.

Geometric Objects			
Indices by *stage*		Indices by *grade*	
S-4-01	Can use the geometric properties of a shape to define some categorized shape	8-s-01	Can recognize plane figures in daily life (triangle, quadrilateral, polygon, and circle)
		8-s-02	Can know and define point, line, angle (including symbols: ∠ABC, $\overline{AB}$) of simple geometric figures
		8-s-03	Can identify the definitions and its related terminologies in a circle (center, radius, chord, diameter, arc, segment, central angle, sector)
		8-s-09	Can recognize a triangle by the least property
		8-s-17	Can understand the basic properties of a quadrilateral
		8-s-18	Can understand the definitions in particular quadrilaterals
		8-s-23	Can understand the meaning and properties of a parallelogram
		8-s-33	Can identify a solid figure by the least property
S-4-02	Can indicate the shape according to the given properties		
S-4-03	Can illustrate the possible relationship among elements of a complex shape	8-s-31	Can illustrate the possible relationship among elements of a complex plane figure
		8-s-32	Can calculate the problems related to circumference and area in complex plane figure
		8-s-34	Can illustrate the possible relationship among elements in complex solid figures
		8-s-36	Can calculate the problems related to volume and surface area in complex solid figures
S-4-04	Can utilize the properties of shapes to solve geometric problem	8-s-10	Can understand the meaning of axial symmetry of plane figures
		8-s-25	Can understand the formula of area in parallelograms
		8-s-29	Can use the properties of plane figures to solve problems of circumference
		8-s-30	Can use the properties of circles to solve the problems of area in sectors
		8-s-35	Can calculate the problems of surface area in cylinders
		8-s-36	Can calculate the problems related to volume and surface area in complex solid figures
S-4-05	Can apply the calculation of area to deduce the Pythagorean theorem	8-s-20	Can deduce the relationship between sides of a right triangle by the relationship between area
S-4-06	Can understand the parallelism and perpendicularity of two straight lines on a plane	8-s-05	Can utilize the right triangle to define the perpendicularity of two straight lines and utilize the perpendicularity in the same line to define the parallelism of two straight lines
		8-s-06	Can illustrate concretely that all distances

			between two parallel lines are equal
		8-s-08	Can know the basic properties of parallel lines
		8-s-19	Can construct squares and parallelograms
S-4-07	Can finish the construction with ruler and compass by the instructions on ruler and compass	8-s-04	Can know about construction with ruler and compass
		8-s-07	Can be proficient in basic construction with ruler and compass
		8-s-14	Can understand the meaning of congruence of two triangles by the construction with ruler and compass
		8-s-19	Can construct squares and parallelograms
S-4-08	Can understand the geometric properties of triangles	8-s-09	Can recognize a triangle by the least property
		8-s-11	Can understand the definitions of special triangles
		8-s-12	Can understand the basic properties of triangles
		8-s-13	Can understand the properties of special triangles
		8-s-14	Can understand the meaning of congruence of two triangles by the construction with ruler and compass
		8-s-15	Can understand the congruent properties of triangles
		8-s-16	Can understand the relationship between sides and angles of triangles
S-4-09	Can understand the geometric properties of polygons	8-s-17	Can understand the basic properties of a quadrilateral
		8-s-23	Can understand the meaning and properties of a parallelogram
		8-s-24	Can understand the related properties on judging a parallelogram
		8-s-26	Can understand the meaning and properties of trapezoids (including the midline property of a trapezoid)
		8-s-27	Can use the property of angle sum of interior angles in a triangle is 180 degree to solve the problems related to theorem of sum of interior angles and exterior angles in a polygon
S-4-10	Can identify the difference between a statement and its converse statement	8-s-28	Can identify the difference between a statement and its converse statement
S-4-11	Can understand the definitions and related properties of parallel lines	8-s-05	Can utilize the right triangle to define the perpendicularity of two straight lines and utilize the perpendicularity in the same line to define the parallelism of two straight lines
		8-s-21	Can understand the property of intersection of parallel lines: in two parallel lines, the corresponding angles are equal, the interior angles are supplementary, the alternate (interior) angles are equal
		8-s-22	Can understand the related properties on judging parallel lines
		9-s-01	Can reason based on the property of intersection in parallel lines
S-4-12	Can examine whether two plane figures are similar.	9-s-02	Can identify the properties, corresponding sides are in constant ratio, and corresponding

			angles are equal to each other, in simple polygons in similar.
S-4-13	Can apply the properties of triangular similarity to measure	9-s-03	Can understand the properties of similarity in triangles
		9-s-04	Can understand the property of segments of intersections in parallel lines are in constant ratio
		9-s-05	Can use the concept of the corresponding sides of similar triangles are in constant ratio to apply in physical measurement
		9-s-08	Can understand the definition and the related properties of circumcenter in a triangle
		9-s-09	Can understand the definition and the related properties of incenter in a triangle
S-4-14	Can understand the geometric properties of circles	9-s-06	Can understand the relationships between lines and circle and two circles
		9-s-07	Can understand the related properties of circles
		9-s-08	Can understand the definition and the related properties of circumcenter in a triangle
		9-s-09	Can understand the definition and the related properties of incenter in a triangle
S-4-15	Can utilize the properties of triangles and circles to reason	9-s-01	Can reason based on the property of intersection in parallel lines
		9-s-08	Can understand the definition and the related properties of circumcenter in a triangle
		9-s-09	Can understand the definition and the related properties of incenter in a triangle
		9-s-10	Can understand the definition and the related properties of barycenter in a triangle
		9-s-11	Can learn reasoning with the properties of triangles and circles

APPENDIX B

SOURCES OF TEXTBOOK SERIES

Schmid, A., & Weidig, I. (2005). *Lambacher Schweizer - Mathematik für Gymnasien 7*. Stuttgart: Ernst Klett Verlag.

Schmid, A., & Weidig, I. (2006). *Lambacher Schweizer - Mathematik für Gymnasien 8*. Stuttgart: Ernst Klett Verlag.

Schmid, A., & Weidig, I. (2007). *Lambacher Schweizer - Mathematik für Gymnasien 9*. Stuttgart: Ernst Klett Verlag.

Schätz, U., & Eisentraut, F. (2005). *Delta - Mathematik für Gymnasien 7*. Bamberg: C. C. Buchners Verlag.

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Freytag, C., Härtinger, R., Herz, A., Kammermeyer, F., Kurz, K., Sauermann, B., Sinzinger, M., & Zechel, J. (2005). *Fokus Mathematik – Gymnasium Bayern, Jahrgangsstufe 7*. Berlin: Cornelsen Verlag.

Freytag, C., Härtinger, R., Herz, A., Kammermeyer, F.,Kilian, H., Kurz, K., Sauermann, B., Schmitt, B., Sinzinger, M., Wagner, A., Zebhauser, E., Zebhauser, M., & Zechel, J. (2006). *Fokus Mathematik – Gymnasium Bayern, Jahrgangsstufe 8*. Berlin: Cornelsen Verlag.

Freytag, C., Gräupner, C., Herz, A., Kammermeyer, F., Kilian, H., Kurz, K., Sauermann, B., Schmähling, R., Schmitt, B., Sinzinger, M., Zebhauser, E., & Zebhauser, M. (2007). *Fokus Mathematik – Gymnasium Bayern, Jahrgangsstufe 9*. Berlin: Cornelsen Verlag.

Hung, Y.-C., Wang, C.-D., Lin, B.-P., Yang, R.-C., Hsu, L.-M., Ma, R.-H., & Chen, S.-Y. (2009a). *Mathematics, Grade 8* (Vol. 3). Taipei: Kang Hsuan.

Hung, Y.-C., Wang, C.-D., Lin, B.-P., Yang, R.-C., Hsu, L.-M., Ma, R.-H., & Chen, S.-Y. (2009b). *Mathematics, Grade 8* (Vol. 4). Taipei: Kang Hsuan.

Hung, Y.-C., Wang, C.-D., Lin, B.-P., Yang, R.-C., Hsu, L.-M., Ma, R.-H., & Chen, S.-Y. (2009c). *Mathematics, Grade 9* (Vol. 5). Taipei: Kang Hsuan.

Chen, M.-H., Huang, W.-D., Tso, T.-Y., Hsu, C.-M., Hao, H.-C., Hsu, C.-T., & Su, H.-Y. (2009a). *Mathematics, Grade 8* (Vol. 3). Tainan: Nan I.

Chen, M.-H., Huang, W.-D., Tso, T.-Y., Hsu, C.-M., Hao, H.-C., Hsu, C.-T., & Su, H.-Y. (2009b). *Mathematics, Grade 8* (Vol. 4). Tainan: Nan I.

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National Academy for Educational Research (2009a). *Mathematics, Grade 8* (Vol. 3). Retrieved from http://www.naer.edu.tw/bookelem/u_booklist_v1.asp?id=267&bekid=2&bemkind=1 on 07.02.2010.

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APPENDIX C

INTERVIEW GUIDE

(Delta)

Thank you for agreeing the attendance of this interview. The content and result of the interview will be confidentially treated and for research purpose only. Your name and further private details will not be publicly released.

Basic information

Before we start, could you talk about your experience in designing mathematics textbooks:

1. How many years have you been involved designing textbook?
2. How many years have you been working with C. C. BUCHNER Publishing House?

Regarding the textbook prevalence,

1. Do you know the percentage of Bavarian students (esp. 7th, 8th,and 9th graders) working with your textbook?
2. Who is the main target group (teachers, students, parents, others)?

About the editing group of mathematics textbook,

1. Please tell me how many teachers are in the group who design the book series. How many university professors? Are there others, and how many, and what are their professional backgrounds?
2. Do they work for full-time or part-time job? How many hours per week (percentage of work, %) ?
3. How do they cooperate in working?

Philosophy

Could you talk about your ideas or your philosophy for designing the textbook:

1. Why did you decide to design the textbook?
2. What does your ideal textbook look like?

From your point of view, can you talk about the role of textbook:

1. What is the function of it?
2. What does/should it stand for learning? Should it be a necessary material or a supplementary material in class?

Goals of editing mathematics textbooks

1. What do you expect students to learn from your book/book series? Please arrange the significance of *general points* and then the *details* (ref. appendix, if it is not encompassed, please write them on the empty card, one for each).
2. What do you expect teacher to teach with your book/book series? Please arrange the significance of *general points* and then the *details* (ref. appendix, if it is not encompassed, please write them on the empty card, one for each).

Focus on geometric content,

1. What's the most important idea or the specific characteristics of your design? For example, realistic problems, figural construction or operation, axioms, reasoning process…
2. If you could re-write the textbook, what kind of change would you do?

Principles of topic and content arrangement

Compared to other textbooks, could you tell me how you arrange a topic and also how to set the content:

1. How do you select the contents? What are your criteria to set these contents?
2. Do you arrange the topics according to the State Curriculum (Bayerischer Lehrplan)?

Treat geometry content as an example and compare it to other textbooks,

1. What are the parts of your textbook that make it special?

Activities arrangement

Your textbook seems to be formed from these elements:

Each chapter: Topic units, (random) Themenseite, and "Ergänzende Aufgaben - explore-get more, Kann ich das?"

Each topic unit (section): Corpus part- Arbeitsaufträge and Beispiele, and Aufgaben

Could you tell me:

1. What is the specific function behind each of these elements?
2. How do you decide this specific arrangement?

Suppose there are two teachers using your textbook, one is an experienced teacher and the other is a novice, and they are teaching the same topic unit now:

1. How will you suggest them to make use of these elements in the text?
2. From your perspective, what's the most important part of the text and which would you suggest that every student should work through it?

THANK YOU!

appendix

General points	**Details**
Texts	Explanatory/Presentational content Worked-example Exercise
Skills	Calculational/algorithmic fluency Figural construction
Competencies	Mathematical arguing (*K1, Mathematisch argumentieren*) Mathematical problem-solving (*K2, Probleme mathematisch lösen*) Mathematical modeling (*K3, Mathematisch modellieren*) Mathematical representations (*K4, Mathematische Darstellungen verwenden*) Dealing with symbolic, formal and technical elements of mathematics (*K5, Mit symbolischen, formalen und technischen Elementen der Mathematik umgehen*) Communicating (*K6, Kommunizieren*) Reflection

(Nan I)

Thank you for agreeing the attendance of this interview.

The content and result of the interview will be confidentially treated and for research purpose only. Your name and further private details will not be publicly released.

Basic information

Before we start, could you talk about your experience in designing mathematics textbooks:

3. How many years have you been involved designing textbook?
4. How many years have you been working with Nan I Publishing House?

Regarding the textbook prevalence,

3. Do you know the percentage of Taiwanese students (esp. 7^{th}, 8^{th}, and 9^{th} graders) working with your textbook?
4. Who is the main target group (teachers, students, parents, others)?

About the editing group of mathematics textbook,

4. Please tell me how many teachers are in the group who design the book series. How many university professors? Are there others, and how many, and what are their professional backgrounds?
5. Do they work for full-time or part-time job? How many hours per week (percentage of work, %) ?
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Could you talk about your ideas or your philosophy for designing the textbook:

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Focus on geometric content,

3. What's the most important idea or the specific characteristics of your design? For example, realistic problems, figural construction or operation, axioms, reasoning process…
4. If you could re-write the textbook, what kind of change would you do?

Principles of topic and content arrangement

Compared to other textbooks, could you tell me how you arrange a topic and also how to set the content:

3. How do you select the contents? What are your criteria to set these contents?
4. Do you arrange the topics according to the National Curriculum?

Treat geometry content as an example and compare it to other textbooks,

2. What are the parts of your textbook that make it special?

Activities arrangement

Your textbook seems to be formed from these elements:

Each chapter: 各章節(概念)初步簡介、各章節內容、章末的數學部落格

Each topic unit (section): 主要內容：說明式內文、例題、隨堂練習、活動、問題與討論、重點整理、自我評量

Could you tell me:

3. What is the specific function behind each of these elements?
4. How do you decide this specific arrangement?

Suppose there are two teachers using your textbook, one is an experienced teacher and the other is a novice, and they are teaching the same topic unit now:

3. How will you suggest them to make use of these elements in the text?
4. From your perspective, what's the most important part of the text and which would you suggest that every student should work through it?

THANK YOU!

appendix

General points	**Details**
Texts	Explanatory/Presentational content Worked-example Exercise
Skills	Calculational/algorithmic fluency Figural construction
Competencies	Mathematical argumentation Mathematical problem-solving Mathematical modeling Mathematical representations Dealing with symbolic, formal and technical elements of mathematics Communicating Reflection

APPENDIX D

GEOMETRY CONTENT[1]

GERMAN TEXTBOOK SERIES

	Lambacher Schweizer		Delta		Fokus	
	Heading	Page	Heading	Page	Heading	Page
Grade 7	**Symmetrie** **Symmetry**	**8-37**	**Achsen-und punktsymmetrische Figuren** **Line and point symmetry figures**	**9-36** **9**	**Symmetrische Figuren** **Symmetrical figures**	**5-30** **5-7**
	Achsensymmetrie Line symmetry	8-10 /10-11	Achsensymmetrische Figuren Line-symmetrical figures	10-11 /11-11	Achsen- und Punktsymmetrie Line and point symmetry	8-11 /12-16
	Konstruieren von Spiegelpunkten und Achse To construct from mirror points and the line	12-14 /14-15	Konstruktion von Spiegelpunkt und Achse Construction with mirror points and the line	16-17 /18-19	Symmetrische Vierecke Symmetrical quadrangles	17-19 /19-22
	Mittelsenkrechte, Winkelhalbierende und Lote Perpendicularity, angle bisector, and perpendicular lines	16-18 /18-21	Weitere Grundkonstruktionen Other basic constructions	20-22 /22-23	Grundkonstruktionen Basic constructions	23-25 /26-30
	Punktsymmetrie Point symmetry	22-23 /24-26	Punktsymmetrie Point symmetry	24-25 /25-27		
	Symmetrische Vierecke Symmetrical quadrangles	27-29 /29-31	Symmetrische Vierecke Symmetrical quadrangles	28-29 /29-31		
	Winkelbetrachtungen **Views of angle**	**38-51**	**Winkelbetrachtungen an Figuren** **Views of angle by figures**	**37-60** **37**	**Winkelbetrachtungen** **Views of angle**	**31-46** **31-32**
	Scheitelwinkel und Nebenwinkel Opposite vertical angles and adjacent angles	38-39 /39-39	Winkel an einer Geradenkreuzung Angles of a line-line intersection	38-39 /39-39	Entdeckungen an Geraden- und Doppelkreuzungen Discoveries of one and double line-line	33-35 /36-40

[1] The pages analyzed are marked in gray and with character border.

					intersection	
	Stufenwinkel und Wechselwinkel Corresponding angles and alternate angles	40-41 /41-42	Winkel an Parallelen Angles of the parallels	42-43 /43-45	Entdeckungen an Dreiecken und Vierecken Discoveries of triangles and quadrangles	41-42 /42-46
	Winkelsumme im Dreieck Angle sum of the triangle	43-44 /44-46	Winkelsumme im Dreieck Angle sum of the triangle	46-47 /47-49		
	Winkelsumme im Vieleck Angle sum of the quadrangle	47-48 /48-48	Winkelsumme im Viereck Angle sum of the quadrangle	52-53 /53-55		
	Kongruenz und Dreiecke **Congruence and triangles**	**134-167**	**Das Dreieck als Grundfigure: Kongruenz** **The triangle as a basic figure: Congruence**	**147-158** **147**	**Kongruenz** **Congruence**	**131-146** **131-132**
	Kongruente Figuren Congruent figures	134-135 /135-137	Kongruente Figuren Congruent figures	148-149 /149-150	Kongruente Figuren Congruent figures	133-134 /135-138
	Kongruenz von Dreiecken Congruence of triangles	138-140 /141-143	Kongruenzsätze für Dreiecke Congruent propositions (congruence conditions) of triangles	152-154 /154-155	Kongruenzsätze für Dreiecke Congruent propositions (congruence conditions) of triangles	139-141 /142-145
	Das gleichschenklige Dreieck Isosceles triangle	144-146 /146-148				
	Satz und Kehrsatz Proposition and "backward proposition"	149-150 /150-151				
	Das rechtwinklige Dreieck – der Satz des Thales Right-angled triangle – Thales theorem (proposition/statement)	152-154 /154-156				
	Dreieckskonstruktionen Triangular construction	157-158 /159-160				
	Besondere Linien im Dreieck und Konstruktionen **Special lines of a triangle and the constructions**	**168-187**	**Besondere Dreiecke** **Special triangles**	**159-178** **159**	**Eigenschaften von Dreiecken** **Properties of the triangle**	**147-164** **147-148**
	Mittelsenkrechten und Umkreis Perpendicular bisector	168-169 /169-169	Gleichschenklige Dreiecke Isosceles triangles	160-162 /162-163	Besondere Dreiecke Special triangles	149-152 /152-156

	and the circumcircle (circumscribed circle)					
	Winkelhalbierende Angle bisector	170-171 /172-172	Rechtwinklige Dreiecke Right-angled triangles	166-168 /168-169	Eigenschaften rechtwinkliger Dreiecke Properties of the right-angled triangle	157-159 /160-164
	Höhen Heights	173-174 /174-174	Kreis und Gerade-Kreistangenten Circle and straight line –tangents of a circle	170-172 /172-173	**Konstruktionen mithilfe von Dreiecken** **Construction with triangles**	**165-188** **165-167**
	Seitenhalbierende Side bisector	175-176 /176-176	**Konstruktionen an Dreiecken und Vierecken** **Constructions of triangles and quadrangles**	**179-196** **179**	Besondere Linien und Punkte im Dreieck Special lines and points of a triangle	168-171 /171-175
	Besondere Dreieckskonstruktionen Special triangular construction	177-178 /179-179	Mittelsenkrechte und Umkreis eines Dreiecks Perpendicular bisector and circumcircle of triangles	180-181 /182-182	Konstruktionen Construction	176-179 /179-188
	Konstruktion von Vierecken Construction of quadrangles	180-181 /181-182	Höhen eines Dreiecks Heights of a triangle	184-186 /186-186		
			Winkelhalbierende eines Dreiecks Angle bisector of a triangle	188-189 /189-190		
Grade 8	**Ähnlichkeit** **Similarity**	**132-155**	**Strahlensätze und Ähnlichkeit** **Intercept theorems and similarity**	**145-171** **145**	**Strahlensatz und Ähnlichkeit** **Intercept theorem and similarity**	**149-180** **149-151**
	Zentrische Streckungen Centric stretching	132-134 /135-136	Streckenverhältnisse Partition of line segment	146-147 /147-148	Maßtäbliches Vergrößern und Verkleinern Scales to enlarge and to narrow (Scale-up and scale-down)	152-155 /156-160
	Der Strahlensatz Intercept theorems	137-139 /139-142	Strahlensätze 1 Intercept theorems 1	150-151 /151-153	Strahlensatz Intercept theorem	161-163 /164-171
	Ähnliche Figuren Similar figures	143-144 /144-145	Strahlensätze 2 Intercept theorems 2	154-155 /155-156	Ähnlichkeit von Dreiecken Similarity of triangles	172-174 /175-179

	Ähnlichkeitssätze für Dreiecke Propositions (properties) of similarity of triangles	146-147 /147-149	Umkehrung der Strahlensätze Reverse of intercept theorems	158-159 /159-159		
			Ähnliche Figuren- Ähnlichkeit von Dreiecken Similar figures- similarity of triangles	160-162 /162-163		
Grade 9	**Die Satzgruppe des Pythagoras** **Group of pythagorean propositions (statements/theorems)**	**42-61**	**Die Satzgruppe von Pythagoras** **Groupd of Pythagorean propositions (statements/theorems)**	**29-50** **29**	**Die Satzgruppe des Pythagoras** **Group of Pythagorean propositions (statements/theorems)**	**31-52** **31-33**
	Der Kathetensatz The hypotenuse-leg theorem	42-43 /43-44	Der Satz von Pythagoras Pythagorean proposition (statement/theorem)	30-31 /32-33	Der Satz des Pythagoras Pythagorean proposition (statement/theorem)	34-37 /38-43
	Der Satz des Pythagoras Pythagorean proposition (statement/theorem)	45-46 /46-48	Die Umkehrung des Satzes von Pythagoras Reverse of Pythagorean propositions (statements/theorems)	34-35 /35-35	Weitere Flächensätze zum rechtwinkligen Dreieck Another area proposition of a right-angled triangle	44-47 /47-51
	Kehrsatz zum Satz des Pythagoras “backward proposition (theorem)” from Pythagorean theorem	49-50 /50-50	Der Kathetensatz und der Höhensatz The hypotenuse-leg theorem and leg-leg theorem	38-40 /40-41		
	Der Höhensatz The leg-leg theorem	51-52 /52-52				
	Berechnungen an Figuren und Körpern Calculations of figures and solids	53-54 /54-56				
	Trigonometry				**Geometrische Körper** **Geometric solids**	**53-70** **53-54**
	Raumgeometrie **Solid geometry**	**150-189**	**Fortführung der Raumgeometrie** **Continuation of solid geometry**	**161-197** **161**		
	Geraden und Ebenen im Raum Lines and planes of a solid	150-151 /152-152	Schrägbilder Oblique images (pictures)	162-164 /164-165	Prisma und Zylinder Prism and cylinder	55-57 /57-60

	Schrägbilder von Prisma, Zylinder, Pyramide und Kegel Oblique images of prism, cylinder, pyramid, and cone	153-155 /155-156	Das gerade Prisma Right prism	166-167 /167-171	Pyramide und Kegel Pyramid and cone	61-63 /64-69
	Volumen und Oberflächeninhalt von Prismen Volume and surface area of prisms	157-158 /158-159	Der gerade Kreiszylinder Right circular cylinder	174-175 /175-177	Trigonometry **Volumenberechnungen** **Volume calculations**	 **149-168** 149-151
	Volumen und Oberflächeninhalt von Zylindern Volum and surface area of cylinders	160-160 /161-162	Oberflächeninhalt der Pyramide Surface area of a pyramid	178-179 /179-179	Volumen von Prisma und Zylinder Volumes of prism and cylinder	152-155 /156-160
	Der Satz von Cavalieri Cavalieri's proposition (principle)	163-164 /164-164	Volumen der Pyramide Volume of a pyramid	180-181 /182-185	Volumen der Pyramide und des Kegels Volumes of pyramids and cones	161-163 /163-167
	Volumen und Oberflächeninhalt von Pyramiden Volume and surface area of pyramids	165-167 /167-170	Der gerade Kreiskegel Right circular cylinder	188-189 /190-191		
	Volumen und Oberflächeninhalt von Kegeln Volume and surface area of a cone	171-172 /172-173				

TAIWANESE TEXTBOOK SERIES

	Nan I		Kang Hsuan		National Academy for Educational Research	
	Heading	Page	Heading	Page	Heading	Page
Grade 8 Semester1	勾股定理 Pythagorean theorem	82-96 /96-98	勾股定理 Pythagorean theorem	88-100 /100-102	勾股定理 Pythagorean theorem	46-52 / 52-53 & 83-93 / 93-94
Grade 8 Semester2	**簡單幾何圖形** **Simple geometric shapes (figures)**	**38-97**	**幾何圖形與尺規作圖** **Geometric shapes (figures) and construction with ruler and compass**	**40-107**	**幾何圖形的角** **Angles of geometric figures**	**35-80**
	平面圖形 Plane shapes (figures)	40-52 /53-54	生活中的平面圖形 Plane shapes (figures)	42-57 /58-59	三角形的角 Angles of triangles	36-53 /53-55
	垂直平分與線對稱 Perpendicular bisection and line symmetry	55-63 /63-65	垂直、平分與線對稱圖形 Perpendicularity, bisection, and line-symmetrical figures	60-71 /71-74	多邊形的內角與外角 Interior angles and exterior angles of the polygon	56-67 /68-69
	尺規作圖 Construction with ruler and compass	66-75 /76-78	尺規作圖 Construction with ruler and compass	75-84 /84-86	平行與垂直 Parallelism and perpendicularity	70-79 /79-80
	生活中的立體圖形 Solid shapes (figures) in daily life	79-93 /94-96	生活中的立體圖形 Solid shapes (figures) in daily life	87-103 /104-106		
	三角形的性質 **Properties of the triangle**	**98-147**	**三角形的基本性質** **Basic properties of the triangle**	**108-173**	**三角形的基本性質** **Basic properties of the triangle**	**81-130**
	三角形的內角與外角 Interior angles and exterior angles of the triangle	100-110 /111-112	三角形的內角與外角 Interior angles and exterior angles of the triangle	110-124 /125-127	全等的概念 Concepts of congruence	82-93 /94-95
	三角形的全等性質 Congruent properties of the triangle	113-129 /129-131	三角形的全等性質 Congruent properties of the triangle	128-142 /143-144	SSS全等與尺規作圖 SSS congruence and construction with ruler and compass	96-112 /113-114
	三角形的邊角關係 Side-angle relation of the triangle	132-142 /143-145	三角形的全等性質的應用 The application of congruent properties of the triangle	145-153 /154-155	三角形的邊角關係 Side-angle relation of the triangle	115-128 /129-130
			三角形的邊角關係 Side-angle relation of the triangle	156-169 /170-172		

	平行與四邊形 **Parallelism and the quadrangle**	**148-200**	平行 **Parallelism**	**174-208**	幾何圖形 **Geometric shapes (figures)**	**131-196**
	平行線 Parallel line	150-163 /163-165	平行 Parallelism	176-186 /187-189	平行四邊形 Parallelogram	132-146 /147-148
	平行四邊形 Parallelogram	166-176 /177-179	平行四邊形與梯形 Parallelogram and trapezium	190-205 /206-207	線對稱與幾何圖形 Line symmetry and geometric shapes (figures)	149-168 /169-170
	特殊的平行四邊形與梯形 Special parallelogram and trapezium	180-196 /197-199			周長與面積 Perimeter and area	171-181 /182-183
					表面積與體積 Surface area and volume	184-195 /195-196
Grade 9 Semester1	比例線段與相似形 **Proportional segments and similar figures**	**4-53**	相似形 **Similar figures**	**4-57**	相似三角形 **Similar triangles**	**3-50**
	比例線段 Proportional segments	6-19 /19-21	相似形 Similar shapes (figures)	6-18 /18-20	縮放 Enlarging and shrinking	4-21 /22-23
	相似形 Similar shapes (figures)	22-48 /49-52	相似三角形 Similar triangles	21-39 /40-43	相似三角形 Similar triangles	24-35 /35-36
			相似三角形的應用 The application of the similar triangles	44-54 /54-56	相似形的應用 The application of the similar shapes (figures)	37-49 /49-50
	圓的性質 **Properties of the circle**	**54-105**	圓 **Circle**	**58-109**	圓 **Circle**	**51-120**
	點、直線、圓之間的關係 Relations between points, lines, and circles	56-78 /78-81	點、直線與圓的關係及兩圓的位置關係 Relations between points, lines, and circles and relations between two circles	60-79 /80-82	圓 Circles	52-72 /72-73
	圓心角、圓周角與弦切角 Central angle, circumferential angle, chord-tangent angle	82-102 /102-104	圓心角、圓周角及弦切角 Central angle, circumferential angle, and chord-tangent angle	83-104 /105-108	圓與角 Circles and angles	74-84 /85-86
					圓與多邊形 Circles and polygons	87-102 /102-103
					數學證明 Mathematical proof	104-119 /119-120
	幾何證明	**106-143**	幾何與證明	**110-152**		

	<u>Geometric proof</u>		**<u>Geometry and proof</u>**			
	學習幾何證明 To learn geometric proof	108-118 /119-121	幾何推理 Geometric reasoning	112-124 /125-128		
	三角形的心 Centers of the triangle	122-139 /140-142	三角形的外心、內心、重心 The Othocenter (circumcenter), incenter and centroid (barycenter/center of gravity)	129-148 /149-151		

APPENDIX E

One Layer Cross Tabulation: Textbook Series–*Variables*

Types of Text

Types of Text	Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						**Total**	
	KH		NI		NAER		DT		LS		FK			
	n	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*
Explanatory Text	179	31.7	189	34.8	208	32.8	42	23.3	51	28.5	38	36.2	707	32.0
Worked Example	154	27.3	119	21.9	171	27.0	74	41.1	73	40.8	0	0.0	591	26.8
Exploration	51	9.0	58	10.7	39	6.2	34	18.9	54	30.2	67	63.8	303	13.7
Immediate Practice	181	32.0	177	32.6	216	34.1	30	16.7	1	0.6	0	0.0	605	27.4
Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Features of Text – Figure Representation

Figure Representation	Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						**Total**	
	KH		NI		NAER		DT		LS		FK			
	n	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*
No	110	19.5	95	17.5	142	22.4	41	22.8	14	7.8	4	3.8	406	18.4
No (to Construct)	4	0.7	10	1.8	0	0.0	3	1.7	5	2.8	2	1.9	24	1.1
Single Figure	236	41.8	241	44.4	278	43.8	52	28.9	61	34.1	29	27.6	897	40.7
Serial Figures	186	32.9	154	28.4	171	27.0	72	40.0	88	49.2	61	58.1	732	33.2
Others	29	5.1	43	7.9	43	6.8	12	6.7	11	6.1	9	8.6	147	6.7
Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Features of Text – Geometric Knowledge

Geometric Knowledge		Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						**Total**	
		KH		NI		NAER		DT		LS		FK			
		n	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*	*n*	*%*
Properties Involved	No	21	3.7	51	9.4	79	12.5	17	9.4	25	14.0	52	49.5	245	11.1
	Yes	323	57.2	286	52.7	426	67.2	105	58.3	114	63.7	26	24.8	1280	58.0
	Others	221	39.1	206	37.9	129	20.3	58	32.2	40	22.3	27	25.7	681	30.9
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Features of Text - Action

Action		Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						**Total**	
		KH		NI		NAER		DT		LS		FK			
		n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
Denotation	No	392	69.4	360	66.3	380	59.9	146	81.1	128	71.5	73	69.5	1479	67.0
	New	87	15.4	76	14.0	90	14.2	30	16.7	40	22.3	19	18.1	342	15.5
	Used in P-S	86	15.2	102	18.8	161	25.4	3	1.7	9	5.0	13	12.4	374	17.0
	Others	0	0.0	5	0.9	3	0.5	1	0.6	2	1.1	0	0.0	11	0.5
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**
Calculation	No	257	45.5	288	53.0	351	55.4	125	69.7	123	68.7	85	81.0	1229	55.7
	With Explanation	156	27.6	120	22.1	128	20.2	47	26.1	42	23.5	12	11.4	505	22.9
	With no Expl.	0	0.0	1	0.2	1	0.2	8	4.4	14	7.8	8	7.6	32	1.5
	Others	152	26.9	134	24.7	154	24.3	0	0.0	0	0.0	0	0.0	440	19.9
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**
Physical Operation	No	535	94.7	495	91.2	623	98.3	176	97.8	173	96.6	98	93.3	2100	95.2
	Yes	27	4.8	46	8.5	7	1.1	4	2.2	5	2.8	7	6.7	96	4.4
	Others	3	0.5	2	0.4	4	0.6	0	0.0	1	0.6	0	0.0	10	0.5
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**
Figural Construction	No	422	74.7	428	78.8	496	78.2	141	78.3	134	74.9	81	77.1	1702	77.2
	Figure	37	6.5	38	7.0	37	5.8	32	17.8	41	22.9	20	19.0	205	9.3
	Auxiliary Line	60	10.6	34	6.3	46	7.3	6	3.3	2	1.1	0	0.0	148	6.7
	Others	46	8.1	43	7.9	55	8.7	1	0.6	2	1.1	4	3.8	151	6.8
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**
Figural Decomposition	Null	120	21.2	123	22.7	154	24.3	55	30.6	29	16.2	16	15.2	497	22.5
	No	164	29.0	157	28.9	121	19.1	96	53.3	133	74.3	81	77.1	752	34.1
	With Figure	258	45.7	238	43.8	332	52.4	27	15.0	17	9.5	8	7.6	880	39.9
	With no Figure	23	4.1	25	4.6	27	4.3	2	1.1	0	0.0	0	0.0	77	3.5
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**
Figural Transformation	Null	142	25.1	143	26.3	179	28.2	53	29.4	28	15.6	13	12.4	558	25.3
	No	381	67.4	356	65.6	370	58.4	107	59.4	121	67.6	73	69.5	1408	63.8
	With Figure	41	7.3	42	7.7	80	12.6	15	8.3	29	16.2	17	16.2	224	10.2
	With no Figure	1	0.2	2	0.4	5	0.8	5	2.8	1	0.6	2	1.9	16	0.7
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Features of Text - Situation

Situation		Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						Total	
		KH		NI		NAER		DT		LS		FK			
		n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
Content Linkage	Experience	47	8.3	29	5.3	49	7.7	24	13.3	23	12.8	17	16.2	189	8.6
	Former	465	82.3	471	86.7	538	84.9	59	32.8	86	48.0	34	32.4	1653	74.9
	New	50	8.8	42	7.7	46	7.3	95	52.8	69	38.5	54	51.4	356	16.1
	Others	3	0.5	1	0.2	1	0.2	2	1.1	1	0.6	0	0.0	8	0.4
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**
Context Linkage	Life	26	4.6	34	6.3	27	4.3	22	12.2	38	21.2	52	49.5	199	9.0
	History	4	0.7	3	0.6	5	0.8	13	7.2	4	2.2	8	7.6	37	1.7
	Mathematics	535	94.7	505	93.0	600	94.6	145	80.6	137	76.5	45	42.9	1967	89.2
	Others	0	0.0	1	0.2	2	0.3	0	0.0	0	0.0	0	0.0	3	0.1
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Function of Text

Practical Function	Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						Total	
	KH		NI		NAER		DT		LS		FK			
	n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
None categorized	66	11.7	54	9.9	81	12.8	26	14.4	20	11.2	10	9.5	257	11.7
Generalization	101	17.9	127	23.4	143	22.6	20	11.1	31	17.3	7	6.7	429	19.4
Application	327	57.9	292	53.8	338	53.3	133	73.9	112	62.6	66	62.9	1268	57.5
App. to Gen.	71	12.6	70	12.9	72	11.4	1	0.6	16	8.9	22	21.0	252	11.4
Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Support to Claims

Forms	Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						Total	
	KH		NI		NAER		DT		LS		FK			
	n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
No	16	2.8	10	1.8	28	4.4	9	5.0	5	2.8	9	8.6	77	3.5
Making Conjecture	38	6.7	47	8.7	35	5.5	32	17.8	56	31.3	68	64.8	276	12.5
N-P Argument	223	39.5	223	41.1	252	39.7	55	30.6	74	41.3	21	20.0	848	38.4
Proof	101	17.9	90	16.6	99	15.6	54	30.0	44	24.6	7	6.7	395	17.9
Others	187	33.1	173	31.9	220	34.7	30	16.7	0	0.0	0	0.0	610	27.7
Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

APPENDIX F

Two Layers Cross Tabulation: Textbook Series–*Variation*-Support to Claims

Textbook Series–Types of Text-Support to Claims

	Types of Text-Support to Claims	Taiwanese Textbook Series (n=1742) KH *n*	KH %	NI *n*	NI %	NAER *n*	NAER %	German Textbook Series (n=464) DT *n*	DT %	LS *n*	LS %	FK *n*	FK %	Total *n*	Total %
Explanatory Text	No	14	7.8	10	5.3	28	13.5	8	19.0	3	5.9	9	23.7	72	10.2
	Conjecture	5	2.8	6	3.2	5	2.4	1	2.4	1	2.0	3	7.9	21	3..0
	N-P Argument	121	67.6	142	75.1	157	75.5	25	59.5	36	70.6	19	50.0	500	70.7
	Proof	39	21.8	29	15.3	17	8.2	8	19.0	11	21.6	7	18.4	111	15.7
	Others	0	0.0	2	1.1	1	0.5	0	0.0	0	0.0	0	0.0	3	0.4
	Total	**179**	**100.0**	**189**	**100.0**	**208**	**100.0**	**42**	**100.0**	**51**	**100.0**	**38**	**100.0**	**707**	**100.0**
Worked Example	No	0	0.0	0	0.0	0	0.0	0	0.0	2	2.7	0		2	0.3
	Conjecture	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	0		0	0.0
	N-P Argument	100	64.9	72	60.5	95	55.6	28	37.8	38	52.1	0		333	56.3
	Proof	52	33.8	47	39.5	76	44.4	46	62.2	33	45.2	0		254	43.0
	Others	2	1.3	0	0.0	0	0.0	0	0.0	0	0.0	0		2	0.3
	Total	**154**	**100.0**	**119**	**100.0**	**171**	**100.0**	**74**	**100.0**	**73**	**100.0**	**0**		**591**	**100.0**
Exploration	No	1	2.0	0	0.0	0	0.0	1	2.9	0	0.0	0	0.0	2	0.7
	Conjecture	31	60.8	35	60.3	28	71.8	31	91.2	54	100.0	65	97.0	244	80.5
	N-P Argument	2	3.9	3	5.2	0	0.0	2	5.9	0	0.0	2	3.0	9	3.0
	Proof	1	2.0	6	10.3	0	0.0	0	0.0	0	0.0	0	0.0	7	2.3
	Others	16	31.4	14	24.1	11	28.2	0	0.0	0	0.0	0	0.0	41	13.5
	Total	**51**	**100.0**	**58**	**100.0**	**39**	**100.0**	**34**	**100.0**	**54**	**100.0**	**67**	**100.0**	**303**	**100.0**
Immediate Practice	No	1	0.6	0	0.0	0	0.0	0	0.0	0	0.0	0		1	0.2
	Conjecture	2	1.1	6	3.4	2	0.9	0	0.0	1	100.0	0		11	1.8
	N-P Argument	0	0.0	6	3.4	0	0.0	0	0.0	0	0.0	0		6	1.0
	Proof	9	5.0	8	4.5	6	2.8	0	0.0	0	0.0	0		23	3.8
	Others	169	93.4	157	88.7	208	96.3	30	100.0	0	0.0	0		564	93.2
	Total	**181**	**100.0**	**177**	**100.0**	**216**	**100.0**	**30**	**100.0**	**1**	**100.0**	**0**		**605**	**100.0**
Total	No	16	2.8	10	1.8	28	4.4	9	5.0	5	2.8	9	8.6	77	3.5
	Conjecture	38	6.7	47	8.7	35	5.5	32	17.8	56	31.3	68	64.8	276	12.5
	N-P Argument	223	39.5	223	41.1	252	39.7	55	30.6	74	41.3	21	20.0	848	38.4
	Proof	101	17.9	90	16.6	99	15.6	54	30.0	44	24.6	7	6.7	395	17.9
	Others	187	33.1	173	31.9	220	34.7	30	16.7	0	0.0	0	0.0	610	27.7
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Textbook Series–Calculation-Support to Claims

	Calculation-Support to Claims	Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						**Total**	
		KH		NI		NAER		DT		LS		FK			
		n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
No	No	16	6.2	10	3.5	28	8.0	9	7.2	5	4.1	9	10.6	77	6.3
	Conjecture	29	11.3	35	12.2	30	8.5	29	23.2	47	38.2	60	70.6	230	18.7
	N-P Argument	129	50.2	150	52.1	169	48.1	28	22.4	53	43.1	15	17.6	544	44.3
	Proof	33	12.8	37	12.8	52	14.8	31	24.8	18	14.6	1	1.2	172	14.0
	Others	50	19.5	56	19.4	72	20.5	28	22.4	0	0.0	0	0.0	206	16.8
	Total	**257**	**100.0**	**288**	**100.0**	**351**	**100.0**	**125**	**100.0**	**123**	**100.0**	**85**	**100.0**	**1229**	**100.0**
With Explanation	No	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0
	Conjecture	1	0.6	2	1.7	0	0.0	0	0.0	2	4.8	1	8.3	6	1.2
	N-P Argument	94	60.3	67	55.8	81	63.3	26	55.3	18	42.9	6	50.0	292	57.8
	Proof	61	39.1	48	40.0	47	36.7	21	44.7	22	52.4	5	41.7	204	40.4
	Others	0	0.0	3	2.5	0	0.0	0	0.0	0	0.0	0	0.0	3	0.6
	Total	**156**	**100.0**	**120**	**100.0**	**128**	**100.0**	**47**	**100.0**	**42**	**100.0**	**12**	**100.0**	**505**	**100.0**
Without Explanation	No			0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0
	Conjecture	0		0	0.0	0	0.0	3	37.5	7	50.0	7	87.5	17	53.1
	N-P Argument	0		1	100.0	1	100.0	1	12.5	3	21.4	0	0.0	6	18.8
	Proof	0		0	0.0	0	0.0	2	25.0	4	28.6	1	12.5	7	21.9
	Others	0		0	0.0	0	0.0	2	25.0	0	0.0	0	0.0	2	6.3
	Total	**0**		**1**	**100.0**	**1**	**100.0**	**8**	**100.0**	**14**	**100.0**	**8**	**100.0**	**32**	**100.0**
Others	No	0	0.0	0	0.0	0	0.0	0		0		0		0	0.0
	Conjecture	8	5.3	10	7.5	5	3.2	0		0		0		23	5.2
	N-P Argument	0	0.0	5	3.7	1	0.6	0		0		0		6	1.4
	Proof	7	4.6	5	3.7	0	0.0	0		0		0		12	2.7
	Others	137	34.3	114	85.1	148	96.1	0		0		0		399	90.7
	Total	**152**	**34.5**	**134**	**100.0**	**154**	**100.0**	**0**		**0**		**0**		**440**	**100.0**
Total	No	16	2.8	10	1.8	28	4.4	9	5.0	5	2.8	9	8.6	77	3.5
	Conjecture	38	6.7	47	8.7	35	5.5	32	17.8	56	31.3	68	64.8	276	12.5
	N-P Argument	223	39.5	223	41.1	252	39.7	55	30.6	74	41.3	21	20.0	848	38.4
	Proof	101	17.9	90	16.6	99	15.6	54	30.0	44	24.6	7	6.7	395	17.9
	Others	187	33.1	173	31.9	220	34.7	30	16.7	0	0.0	0	0.0	610	27.7
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Textbook Series–Figure Representation-Support to Claims

Figure Representation-Support to Claims		Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						**Total**	
		KH		NI		NAER		DT		LS		FK			
		n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
No	No	5	4.5	6	6.3	12	8.5	2	4.9	3	21.4	1	25.0	29	7.1
	Conjecture	13	11.8	8	8.4	16	11.3	2	4.9	2	14.3	2	50.0	43	10.6
	N-P Argument	55	50.0	52	54.7	63	44.4	4	9.8	7	50.0	1	25.0	182	44.8
	Proof	1	0.9	4	4.2	3	2.1	5	12.2	2	14.3	0	0.0	15	3.7
	Others	36	32.7	25	26.3	48	33.8	28	68.3	0	0.0	0	0.0	137	33.7
	Total	**110**	**100.0**	**95**	**100.0**	**142**	**100.0**	**41**	**100.0**	**14**	**100.0**	**4**	**100.0**	**406**	**100.0**
To Construct	No	0	0.0	0	0.0	0		0	0.0	0	0.0	0	0.0	0	0.0
	Conjecture	0	0.0	0	0.0	0		2	66.7	5	100.0	2	100.0	9	37.5
	N-P Argument	0	0.0	2	20.0	0		0	0.0	0	0.0	0	0.0	2	8.3
	Proof	0	0.0	0	0.0	0		1	33.3	0	0.0	0	0.0	1	4.2
	Others	4	100.0	8	80.0	0		0	0.0	0	0.0	0	0.0	12	50.0
	Total	**4**	**100.0**	**10**	**100.0**	**0**		**3**	**100.0**	**5**	**100.0**	**2**	**100.0**	**24**	**100.0**
Single Figure	No	4	1.7	2	0.8	5	1.8	0	0.0	0	0.0	0	0.0	11	1.2
	Conjecture	9	3.8	20	8.3	12	4.3	10	19.2	22	36.1	27	93.7	100	11.1
	N-P Argument	69	29.2	68	28.2	79	28.4	19	36.5	19	31.1	2	6.9	256	28.5
	Proof	43	18.2	48	19.9	49	17.6	21	40.4	21	32.8	0	0.0	181	20.2
	Others	111	47.0	103	42.7	133	47.8	2	3.8	2	0.0	0	0.0	349	38.9
	Total	**236**	**100.0**	**241**	**100.0**	**278**	**100.0**	**52**	**100.0**	**52**	**100.0**	**29**	**100.0**	**897**	**100.0**
Serial Figure(s)	No	7	3.8	1	0.6	5	2.9	1	1.4	1	0.0	6	9.8	20	2.7
	Conjecture	13	7.0	17	11.0	2	1.2	13	18.1	13	20.5	30	49.2	93	12.7
	N-P Argument	85	45.7	73	47.4	80	46.8	31	43.1	31	54.5	18	29.5	335	45.8
	Proof	56	30.1	35	22.7	47	27.5	27	37.5	27	25.0	7	11.5	194	26.5
	Others	25	13.4	28	18.2	37	21.6	0	0.0	0	0.0	0	0.0	90	12.3
	Total	**186**	**100.0**	**154**	**100.0**	**171**	**100.0**	**72**	**100.0**	**72**	**100.0**	**61**	**100.0**	**732**	**100.0**
Others	No	0	0.0	1	2.3	6	14.0	6	50.0	2	18.2	2	22.2	17	11.6
	Conjecture	3	10.3	2	4.7	5	11.6	5	41.7	9	81.8	7	77.8	31	21.1
	N-P Argument	14	48.3	28	65.1	30	69.8	1	8.3	0	0.0	0	0.0	73	49.7
	Proof	1	3.4	3	7.0	0	0.0	0	0.0	0	0.0	0	0.0	4	2.7
	Others	11	37.9	9	20.9	2	4.7	0	0.0	0	0.0	0	0.0	22	15.0
	Total	**29**	**100.0**	**43**	**100.0**	**43**	**100.0**	**12**	**100.0**	**11**	**100.0**	**9**	**100.0**	**147**	**100.0**
Total	No	16	2.8	10	1.8	28	4.4	9	5.0	5	2.8	9	8.6	77	3.5
	Conjecture	38	6.7	47	8.7	35	5.5	32	17.8	56	31.3	68	64.8	276	12.5
	N-P Argument	223	39.5	223	41.1	252	39.7	55	30.6	74	41.3	21	20.0	848	38.4
	Proof	101	17.9	90	16.6	99	15.6	54	30.0	44	24.6	7	6.7	395	17.9
	Others	187	33.1	173	31.9	220	34.7	30	16.7	0	0.0	0	0.0	610	27.7
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Textbook Series–Figure Construction-Support to Claims

Figure Construction-Support to Claims		Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						**Total**	
		KH		NI		NAER		DT		LS		FK			
		n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
No	No	15	3.6	10	2.3	28	5.6	9	6.4	5	3.7	9	11.1	76	4.5
	Conjecture	30	7..1	34	7.9	27	5.4	23	16.3	47	35.1	52	64.2	213	12.5
	N-P Argument	182	43.1	188	43.9	226	45.6	46	32.6	48	35.8	15	18.5	705	41.4
	Proof	49	11.6	62	14.5	61	12.3	33	23.4	34	25.4	5	6.2	244	14.3
	Others	146	34.6	134	31.3	154	31.0	30	21.3	0	0.0	0	0.0	464	27.3
	Total	**422**	**100.0**	**428**	**100.0**	**496**	**100.0**	**141**	**100.0**	**134**	**100.0**	**81**	**100.0**	**1702**	**100.0**
Construct Figure(s)	No	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0
	Conjecture	1	2.7	5	13.2	1	2.7	7	21.9	7	17.1	12	60.0	33	16.1
	N-P Argument	28	75.7	26	68.4	18	48.6	8	25.0	24	58.5	6	30.0	110	53.7
	Proof	7	18.9	4	10.5	17	45.9	17	53.1	10	24.4	2	10.0	57	27.8
	Others	1	2.7	3	7.9	1	2.7	0	0.0	0	0.0	0	0.0	5	2.4
	Total	**37**	**100.0**	**38**	**100.0**	**37**	**100.0**	**32**	**100.0**	**41**	**100.0**	**20**	**100.0**	**205**	**100.0**
Construct Auxiliary Line	No	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	0		0	0.0
	Conjecture	1	1.7	2	5.9	1	2.2	1	16.7	2	100.0	0		5	3.4
	N-P Argument	12	20.0	7	20.6	8	17.4	1	16.7	0	0.0	0		30	20.3
	Proof	44	73.3	24	70.6	21	45.7	4	66.7	0	0.0	0		93	62.8
	Others	3	5.0	1	2.9	16	34.8	0	0.0	0	0.0	0		20	13.5
	Total	**60**	**100.0**	**34**	**100.0**	**46**	**100.0**	**6**	**100.0**	**2**	**100.0**	**0**		**148**	**100.0**
Others	No	1	2.2	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	1	0.7
	Conjecture	6	13.0	6	14.0	6	10.9	1	100.0	2	100.0	4	100.0	25	16.6
	N-P Argument	1	2.2	2	4.7	0	0.0	0	0.0	0	0.0	0	0.0	3	2.0
	Proof	1	2.2	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	1	0.7
	Others	37	80.4	35	81.4	49	89.1	0	0.0	0	0.0	0	0.0	121	80.1
	Total	**46**	**100.0**	**43**	**100.0**	**55**	**100.0**	**1**	**100.0**	**2**	**100.0**	**4**	**100.0**	**151**	**100.0**
Total	No	16	2.8	10	1.8	28	4.4	9	5.0	5	2.8	9	8.6	77	3.5
	Conjecture	38	6.7	47	8.7	35	5.5	32	17.8	56	31.3	68	64.8	276	12.5
	N-P Argument	223	39.5	223	41.1	252	39.7	55	30.6	74	41.3	21	20.0	848	38.4
	Proof	101	17.9	90	16.6	99	15.6	54	30.0	44	24.6	7	6.7	395	17.9
	Others	187	33.1	173	31.9	220	34.7	30	16.7	0	0.0	0	0.0	610	27.7
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Textbook Series–Figure Decomposition-Support to Claims

	Figure Decomposition-Support to Claims	Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						Total	
		KH		NI		NAER		DT		LS		FK			
		n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
Null	No	6	5.0	7	5.7	17	11.0	8	14.5	4	13.8	4	25.0	46	9.3
	Conjecture	11	9.2	8	6.5	16	10.4	9	16.4	15	51.7	11	68.8	70	14.1
	N-P Argument	60	50.0	75	61.0	85	55.2	5	9.1	8	27.6	1	6.3	234	47.1
	Proof	2	1.7	2	1.6	2	1.3	7	12.7	2	6.9	0	0.0	15	3.0
	Others	41	34.2	31	25.2	34	25.2	26	47.3	0	0.0	0	0.0	132	26.6
	Total	**120**	**100.0**	**123**	**100.0**	**154**	**100.0**	**55**	**100.0**	**29**	**100.0**	**16**	**100.0**	**497**	**100.0**
No	No	8	4.9	3	1.9	9	1.9	1	1.0	1	0.8	5	6.2	27	3.6
	Conjecture	8	4.9	16	10.2	3	10.2	19	19.8	38	28.6	55	67.9	139	18.5
	N-P Argument	86	52.4	78	49.7	52	49.7	36	37.5	58	43.6	15	18.5	325	43.2
	Proof	17	10.4	9	5.7	12	5.7	38	39.6	36	27.1	6	7.4	118	15.7
	Others	45	27.4	51	32.5	45	32.5	2	2.1	0	0.0	0	0.0	143	19.0
	Total	**164**	**100.0**	**157**	**100.0**	**121**	**100.0**	**96**	**100.0**	**133**	**100.0**	**81**	**100.0**	**752**	**100.0**
With Figure	No	2	0.8	0	0.0	2	0.6	0	0.0	0	0.0	0	0.0	4	0.5
	Conjecture	14	5.4	21	8.8	12	3.6	4	14.8	3	17.6	2	25.0	56	6.4
	N-P Argument	69	26.7	64	26.9	107	32.2	14	51.9	8	47.1	5	62.5	267	30.3
	Proof	82	31.8	74	31.1	85	25.6	9	33.3	6	35.3	1	12.5	257	29.2
	Others	91	35.3	79	33..2	126	38.0	0	0.0	0	0.0	0	0.0	296	33.6
	Total	**258**	**100.0**	**238**	**100.0**	**332**	**100.0**	**27**	**100.0**	**17**	**100.0**	**8**	**100.0**	**880**	**100.0**
Without Figure	No	0	0.0	0	0.0	0	0.0	0	0.0	0		0	0.0	0	0.0
	Conjecture	5	21.7	2	8.0	4	14.8	0	0.0	0		0		11	14.3
	N-P Argument	8	34.8	6	24.0	8	29.6	0	0.0	0		0		22	28.6
	Proof	0	0.0	5	20.0	0	0.0	0	0.0	0		0		5	6.5
	Others	10	43.5	12	48.0	15	55.6	2	100.0	0		0		39	50.6
	Total	**23**	**100.0**	**25**	**100.0**	**27**	**100.0**	**2**	**100.0**	**0**		**0**		**77**	**100.0**
Total	No	16	2.8	10	1.8	28	4.4	9	5.0	5	2.8	9	8.6	77	3.5
	Conjecture	38	6.7	47	8.7	35	5.5	32	17.8	56	31.3	68	64.8	276	12.5
	N-P Argument	223	39.5	223	41.1	252	39.7	55	30.6	74	41.3	21	20.0	848	38.4
	Proof	101	17.9	90	16.6	99	15.6	54	30.0	44	24.6	7	6.7	395	17.9
	Others	187	33.1	173	31.9	220	34.7	30	16.7	0	0.0	0	0.0	610	27.7
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Textbook Series–Function of Text-Support to Claims

Function of Text-Support to Claims		Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						Total	
		KH		NI		NAER		DT		LS		FK			
		n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
No	No	13	19.7	7	13.0	22	27.2	8	30.8	2	10.0	7	70.0	59	23.0
	Conjecture	5	7.6	9	16.7	9	11.1	3	11.5	6	30.0	3	30.0	35	13.6
	N-P Argument	40	60.6	37	68.5	35	43.2	14	53.8	11	55.0	0	0.0	137	53.3
	Proof	3	4.5	0	0.0	1	1.2	1	3.8	1	5.0	0	0.0	6	2.3
	Others	5	7.6	1	1.9	14	17.3	0	0.0	0	0.0	0	0.0	20	7.8
	Total	**66**	**100.0**	**54**	**100.0**	**81**	**100.0**	**26**	**100.0**	**20**	**100.0**	**10**	**100.0**	**257**	**100.0**
Generalization	No	2	2.0	3	2.4	5	3.5	0	0.0	1	3.2	1	14.3	12	2.8
	Conjecture	2	2.0	0	0.0	4	2.8	0	0.0	0	0.0	1	14.3	7	1.6
	N-P Argument	67	66.3	94	74.0	111	77.6	11	55.0	21	67.7	5	71.4	309	72.0
	Proof	30	29.7	28	22.0	19	13.3	8	40.0	9	29.0	0	0.0	94	21.9
	Others	0	0.0	2	1.6	4	2.8	1	5.0	0	0.0	0	0.0	7	1.6
	Total	**101**	**100.0**	**127**	**100.0**	**143**	**100.0**	**20**	**100.0**	**31**	**100.0**	**7**	**100.0**	**429**	**100.0**
Application	No	1	0.3	0	0.0	1	0.3	1	0.8	2	1.8	1	1.5	6	0.5
	Conjecture	28	8.6	25	8.6	21	6.2	29	21.8	39	34.8	61	92.4	203	16.0
	N-P Argument	102	31.2	77	26.4	87	25.7	30	22.6	38	33.9	4	6.1	338	26.7
	Proof	20	6.1	25	8.6	33	9.8	44	33.1	33	29.5	0	0.0	155	12.2
	Others	176	53.8	165	56.5	196	58.0	29	21.8	0	0.0	0	0.0	566	44.6
	Total	**327**	**100.0**	**292**	**100.0**	**338**	**100.0**	**133**	**100.0**	**112**	**100.0**	**66**	**100.0**	**1268**	**100.0**
Application to Generalization	No	0	0.0	0.0	0.0	0	0.0	0	0.0	0	0.0	0	0.0	0	0.0
	Conjecture	3	18.6	13	18.6	1	1.4	0	0.0	11	68.8	3	13.6	31	12.3
	N-P Argument	14	21.4	15	21.4	19	26.4	0	0.0	4	25.0	12	54.5	64	25.4
	Proof	48	52.9	37	52.9	46	63.9	1	100.0	1	6.3	7	31.8	140	55.6
	Others	6	7.1	5	7.1	6	8.3	0	0.0	0	0.0	0	0.0	17	6.7
	Total	**71**	**100.0**	**70**	**100.0**	**72**	**100.0**	**1**	**100.0**	**16**	**100.0**	**22**	**100.0**	**252**	**100.0**
Total	No	16	2.8	10	1.8	28	4.4	9	5.0	5	2.8	9	8.6	77	3.5
	Conjecture	38	6.7	47	8.7	35	5.5	32	17.8	56	31.3	68	64.8	276	12.5
	N-P Argument	223	39.5	223	41.1	252	39.7	55	30.6	74	41.3	21	20.0	848	38.4
	Proof	101	17.9	90	16.6	99	15.6	54	30.0	44	24.6	7	6.7	395	17.9
	Others	187	33.1	173	31.9	220	34.7	30	16.7	0	0.0	0	0.0	610	27.7
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

Textbook Series–Properties Involved-Support to Claims

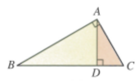
Properties Involved-Support to Claims		Taiwanese Textbook Series (n=1742)						German Textbook Series (n=464)						Total	
		KH		NI		NAER		DT		LS		FK			
		n	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%	*n*	%
No	No	9	42.9	4	7.8	18	22.8	8	47.1	2	8.0	8	15.4	49	20.0
	Conjecture	1	4.8	8	15.7	13	16.5	8	47.1	19	76.0	41	78.8	90	36.7
	N-P Argument	5	23.8	28	54.9	32	40.5	1	5.9	4	16.0	3	5.8	73	29.8
	Proof	2	9.5	2	3.9	1	1.3	0	0.0	0	0.0	0	0.0	5	2.0
	Others	4	19.0	9	17.6	15	19.0	0	0.0	0	0.0	0	0.0	28	11.4
	Total	**21**	**100.0**	**51**	**100.0**	**79**	**100.0**	**17**	**100.0**	**25**	**100.0**	**52**	**100.0**	**245**	**100.0**
Yes	No	6	1.9	6	2.1	10	2.3	0	0.0	1	0.9	0	0.0	23	1.8
	Conjecture	7	2.2	4	1.4	12	2.8	0	0.0	0	0.0	1	3.8	24	1.9
	N-P Argument	217	67.2	187	65.4	219	51.4	51	48.6	69	60.5	18	69.2	761	59.5
	Proof	91	28.2	83	29.0	98	23.0	54	51.4	44	38.6	7	26.9	377	29.5
	Others	2	0.6	6	2.1	87	20.4	0	0.0	0	0.0	0	0.0	95	7.4
	Total	**323**	**100.0**	**286**	**100.0**	**426**	**100.0**	**105**	**100.0**	**114**	**100.0**	**26**	**100.0**	**1280**	**100.0**
Others	No	1	0.5	0	0.0	0	0.0	1	1.7	2	5.0	1	3.7	5	0.7
	Conjecture	30	13.6	35	17.0	10	7.8	24	41.4	37	92.5	26	96.3	162	23.8
	N-P Argument	1	0.5	8	3.9	1	0.8	3	5.2	1	2.5	0	0.0	14	2.1
	Proof	8	3.6	5	2.4	0	0.0	0	0.0	0	0.0	0	0.0	13	1.9
	Others	181	81.9	158	76.7	118	91.5	30	51.7	0	0.0	0	0.0	487	71.5
	Total	**221**	**100.0**	**206**	**100.0**	**129**	**100.0**	**58**	**100.0**	**40**	**100.0**	**27**	**100.0**	**681**	**100.0**
Total	No	16	2.8	10	1.8	28	4.4	9	5.0	5	2.8	9	8.6	77	3.5
	Conjecture	38	6.7	47	8.7	35	5.5	32	17.8	56	31.3	68	64.8	276	12.5
	N-P Argument	223	39.5	223	41.1	252	39.7	55	30.6	74	41.3	21	20.0	848	38.4
	Proof	101	17.9	90	16.6	99	15.6	54	30.0	44	24.6	7	6.7	395	17.9
	Others	187	33.1	173	31.9	220	34.7	30	16.7	0	0.0	0	0.0	610	27.7
	Total	**565**	**100.0**	**543**	**100.0**	**634**	**100.0**	**180**	**100.0**	**179**	**100.0**	**105**	**100.0**	**2206**	**100.0**

APPENDIX G

ADDITIONAL PROOFS APART FROM ORIGINAL INTRODUCTION

若$\triangle ABC$為直角三角形，且$\angle A=90^\circ$，$\overline{AD}$為斜邊上的高，則$\overline{AD}$會將$\triangle ABC$分成兩個較小的直角三角形。這兩個較小的三角形與$\triangle ABC$有什麼關係呢？

如右圖(一)，直角三角形ABC中，$\angle BAC=90^\circ$
D在$\overline{BC}$上，且$\overline{AD}\perp\overline{BC}$
在$\triangle ABC$和$\triangle DBA$中
$\angle BAC=\angle BDA=90^\circ$，$\angle ABC=\angle ABD$
根據AA相似性質，可知$\triangle ABC\sim\triangle DBA$
因此$\overline{AB}:\overline{DB}=\overline{BC}:\overline{BA}$
故$\overline{AB}^2=\overline{BC}\times\overline{BD}$



圖(一)

隨堂練習

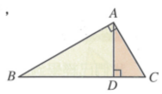
試說明上圖(一)中，(1) $\triangle ABC\sim\triangle DAC$，且$\overline{AC}^2=\overline{CB}\times\overline{CD}$。
(2) $\triangle DBA\sim\triangle DAC$，且$\overline{AD}^2=\overline{DB}\times\overline{DC}$。

Proof of the hypotenuse-leg theorem

<u>*Explanatory Text & Immediate Practice*</u>

(*Kang Hsuan*, grade 9, book 1, vol. 5, p. 44)

由上可知，$\triangle ABC$中，$\angle A=90^\circ$，$\overline{AD}$為$\overline{BC}$上的高，
則：(1) $\triangle ABC\sim\triangle DBA\sim\triangle DAC$
(2) $\overline{AB}^2=\overline{BC}\times\overline{BD}$
(3) $\overline{AC}^2=\overline{CB}\times\overline{CD}$
(4) $\overline{AD}^2=\overline{DB}\times\overline{DC}$



Conclusions of the hypotenuse-leg theorem and leg-leg theorem

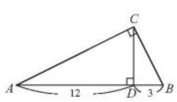
<u>*Explanatory Text (conclusion)*</u>

(*Kang Hsuan*, grade 9, book 1, vol. 5, p. 45)

Note. The Pythagorean theorem is first introduced in the first semester of grade 8 (vol. 3)

例10 Example

如右圖，直角三角形ABC中，$\angle ACB=90^\circ$，
$\overline{CD}$是斜邊上的高，已知$\overline{AD}=12$，$\overline{BD}=3$。
(1) 以AA相似性質說明$\triangle ACD\sim\triangle CBD$。
(2) 求$\overline{CD}$。



解題說明

(1) 因為 $\angle ACD+\angle A=90^\circ$ ∠D是直角
且 $\angle B+\angle A=90^\circ$ ∠ACB是直角
所以 $\angle ACD=\angle B$
又知 $\angle ADC=\angle CDB=90^\circ$
因此 $\triangle ACD\sim\triangle CBD$ AA相似性質

(2) 在(1)的相似關係中，$\triangle ACD$中的A、C、D分別對應於$\triangle CBD$中的C、B、D。
所以 $\overline{AD}:\overline{CD}=\overline{CD}:\overline{BD}$ 對應邊成比例
即 $12:\overline{CD}=\overline{CD}:3$
得$\overline{CD}^2=36$，因此$\overline{CD}=6$。

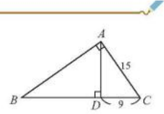
Explanation and calculation with *the conditions of similarity* (the implicit hypotenuse-leg theorem and leg-leg theorem)

<u>*Worked Example*</u>

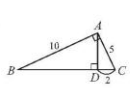
(*National Academy for Educational Research*, grade 9, book 1, vol. 5, pp. 33–34)

隨・堂・練・習

1. 如右圖，$\angle BAC=90^\circ$，$\overline{AD}$為$\overline{BC}$上的高，$\overline{AC}=15$，$\overline{DC}=9$，求$\overline{BC}$。



2. 如右圖，可不可能有一個$\triangle ABC$，$\angle BAC=90^\circ$，$\overline{AD}$為$\overline{BC}$上的高，且$\overline{AB}=10$，$\overline{AC}=5$，$\overline{DC}=2$？

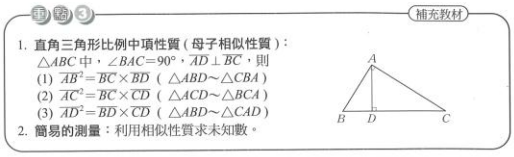


Application (to calculate)

<u>*Immediate Practice*</u>

(*National Academy for Educational Research*, grade 9, book 1, vol. 5, p. 34)

Note. The Pythagorean theorem is first introduced in the first semester of grade 8 (vol. 3)



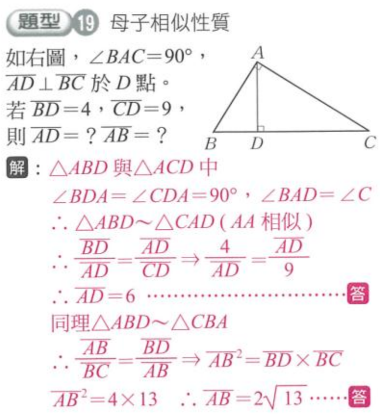
重點3 補充教材

1. 直角三角形比例中項性質（母子相似性質）：
   △ABC中，$\angle BAC=90^\circ$，$\overline{AD}\perp\overline{BC}$，則
   (1) $\overline{AB}^2=\overline{BC}\times\overline{BD}$（△ABD∼△CBA）
   (2) $\overline{AC}^2=\overline{BC}\times\overline{CD}$（△ACD∼△BCA）
   (3) $\overline{AD}^2=\overline{BD}\times\overline{CD}$（△ABD∼△CAD）
2. 簡易的測量：利用相似性質求未知數。

Conclusions of hypotenuse-leg theorem and leg-leg theorem

Appendix—additional materials

(from **Teacher's Book**, *Nan I*, grade 9, book 1, vol. 5, p. 53-6)



題型19 母子相似性質

如右圖，$\angle BAC=90^\circ$，$\overline{AD}\perp\overline{BC}$ 於 D 點。若 $\overline{BD}=4$，$\overline{CD}=9$，則 $\overline{AD}=$？$\overline{AB}=$？

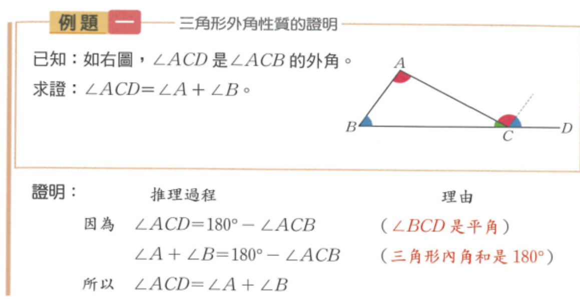
解：△ABD與△ACD中
$\angle BDA=\angle CDA=90^\circ$，$\angle BAD=\angle C$
∴ △ABD∼△CAD（AA相似）
$\therefore \frac{\overline{BD}}{\overline{AD}}=\frac{\overline{AD}}{\overline{CD}} \Rightarrow \frac{4}{\overline{AD}}=\frac{\overline{AD}}{9}$
$\therefore \overline{AD}=6$ ……答
同理△ABD∼△CBA
$\therefore \frac{\overline{AB}}{\overline{BC}}=\frac{\overline{BD}}{\overline{AB}} \Rightarrow \overline{AB}^2=\overline{BD}\times\overline{BC}$
$\overline{AB}^2=4\times 13 \quad \therefore \overline{AB}=2\sqrt{13}$ ……答

Calculation with the above conclusions

Appendix—additional materials

(from **Teacher's Book**, *Nan I*, grade 9, book 1, vol. 5, p. 53-6)

Note. The Pythagorean theorem is first introduced in the first semester of grade 8 (vol. 3)



例題一 三角形外角性質的證明

已知：如右圖，$\angle ACD$ 是 $\angle ACB$ 的外角。
求證：$\angle ACD=\angle A+\angle B$。

證明：

	推理過程	理由
因為	$\angle ACD=180^\circ-\angle ACB$	（$\angle BCD$ 是平角）
	$\angle A+\angle B=180^\circ-\angle ACB$	（三角形內角和是 180°）
所以	$\angle ACD=\angle A+\angle B$	

Proof of the sum of interior angles of a triangle

Worked Example

(*Nan I*, grade 9, book 1, vol. 5, p. 109)

Note. The sum of interior angles of a triangle is first introduced in the second semester of grade 8 (vol. 4)