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# THE LIMIT DISTRIBUTION OF THE MAXIMUM INCREMENT OF A RANDOM WALK WITH DEPENDENT REGULARLY VARYING JUMP SIZES

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ABSTRACT. We investigate the maximum increment of a random walk with heavy-tailed jump size distribution. Here heavy-tailedness is understood as regular variation of the finite-dimensional distributions. The jump sizes constitute a strictly stationary sequence. Using a continuous mapping argument acting on the point processes of the normalized jump sizes, we prove that the maximum increment of the random walk converges in distribution to a Fréchet distributed random variable.

## 1. INTRODUCTION

For several decades, the interplay between heavy tails and serial dependence in a strictly stationary sequence $(X_t)$ of random variables with common distribution $F$ has attracted a lot of attention. One of the main goals of this research is to investigate the distributional behavior of suitable functions acting on $(X_t)$ and to compare it with the corresponding behavior of an iid sequence with the same marginal distribution $F$. In a dependent sequence $(X_t)$, high/low level exceedances typically appear in clusters and significantly determine the distributional behavior of functions of $(X_t)$.

Early on, the asymptotic behavior of the maximum functional $M_n = \max_{i=1,\ldots,n} X_i$, $n \geq 1$, was studied for various classes of heavy-tailed stationary sequences. Not surprisingly, linear processes (such as ARMA) with heavy-tailed innovations were the first objects of interest: they constitute a major class of stationary processes in time series analysis. The extremes of such processes were studied first by [35] in the case of infinite variance stable innovations and later by [11] for general innovation processes with regularly varying tails. Another class of a heavy-tailed stationary processes constitute solutions to stochastic recurrence equations. Pioneering work by [23] and [16] showed that the marginal distributions of such a process have power law tails. The extreme value theory of such processes was studied in detail in [18]. We mention that the class of GARCH processes can be embedded in a natural way in some stochastic recurrence equation and therefore, under general conditions, these processes have power law tails and the asymptotic behavior of their extremes can be treated by similar methods; see [18, 26, 1, 2] in the ARCH$(1)$, GARCH$(1, 1)$ and general GARCH cases. Motivated by applications in time series analysis, [12] studied the asymptotic behavior of the sample autocovariance and sample autocorrelation functions of linear processes with innovations with regularly varying tails and infinite 4th moment. The sample autocovariance and sample autocorrelation functions of GARCH and heavy-tailed bilinear processes were studied in the papers [13, 1, 9, 26, 2].

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Due to very distinct extremal clustering behavior, the results for GARCH, heavy-tailed linear and iid processes differ significantly. The extremal clustering behavior of these sequences is well described by the point processes of the scaled points $X_t$. The results for the extremes and the sample autocovariances of a heavy-tailed sequence are then a consequence of a continuous mapping argument acting on the weakly converging sequence of these point processes. Under weak dependence conditions on the heavy-tailed stationary process $(X_t)$, the seminal paper [8] developed a general theory for the weak convergence of these point processes.

Different distributional behavior of functions acting on heavy-tailed stationary sequences can also be observed on the magnitude of the ruin probabilities $P(\sup_{n\geq 1}(S_n - c\,n) > u)$ as $u \to \infty$, for a constant $c > 0$ and assuming that the random walk $S_n = X_1 + \cdots + X_n$, $n \geq 1$, is driftless. These probabilities were studied for linear processes with regularly varying innovations, solutions to stochastic recurrence equations and infinite variance stable processes; see for example [28, 24, 29]. In this context, the study of large deviation probabilities $P(S_n > x_n) \to 0$ for suitable choices of $x_n \to \infty$ is crucial for the understanding of the very different ruin probabilities of distinct heavy-tailed processes $(X_t)$.

Distinct distributional behavior of functions acting on stationary sequencees $(X_t)$ sheds light on the dependence structure of these sequences. In a sense, these results yield qualitative and quantitative indicators which measure certain aspects of the dependence structure, far beyond covariances and correlations. This aspect is particularly important in the case of heavy-tailed sequences, where covariances and correlations are less meaningful.

As a final example of a functional acting on a heavy-tailed sequence $(X_t)$ we mention the maximum increment of a random walk $(S_n)$. For an iid sequence with regularly varying tails, [27] studied the asymptotic behavior of the quantities

$$\max_{1\leq l\leq n}(f(l))^{-1}\max_{0\leq k\leq n-l}|S_{k+l}-S_k|\,,\quad n\geq 1\,, \tag{1.1}$$

for suitable choices of non-decreasing sequences $(f(l))$. They gave conditions under which the distributions of (1.1) converge weakly to a Fréchet distribution, i.e. one of the extreme value distributions. The purpose of this paper is to investigate (1.1) and related functionals for dependent stationary sequences $(X_t)$. As in the case of maxima, sample autocovariances, ruin and large deviation probabilities, a natural candidate of a heavy-tailed stationary sequence $(X_t)$ is given by a linear process

$$X_t=\sum_{j=0}^{\infty}\psi_j\,Z_{t-j}\,,\quad t\in\mathbb{Z}\,, \tag{1.2}$$

for an iid regularly varying sequence $(Z_t)$, i.e. a generic element $Z$ of this sequence[1] satisfies the tail balance condition

$$P(Z>x)=\widetilde{p}\,x^{-\alpha}\,L(x)\quad\text{and}\quad P(Z\leq -x)=\widetilde{q}\,x^{-\alpha}\,L(x)\,,\quad x\to\infty\,, \tag{1.3}$$

where $L$ is a slowly varying function, $\alpha > 0$ is the index of regular variation and $\widetilde{p},\widetilde{q}\geq 0$, $\widetilde{p}+\widetilde{q} = 1$. We refer to $Z$ as a regularly varying random variable. It is well known (see e.g. [15]) that, under conditions on $(\psi_j)$ ensuring the a.s. convergence of the infinite series in (1.2), the relations $P(X > x) \sim c_+P(Z > x)$ and $P(X \leq -x) \sim c_-P(Z \leq -x)$ as $x \to \infty$ hold for constants $c_-$ and $c_+$ depending on $\alpha$ and $(\psi_j)$; see also Lemma 3.1 below. Using a truncation of the infinite series in (1.2) and the techniques from [27], in Section 3 we derive a Fréchet limit distribution for the maximum increment functional (1.1). For the derivation of this result we make heavy use of the linear structure of $(X_t)$. For linear processes, this approach is similar to the investigation of maxima, sample autocovariances, ruin and large deviation probabilities.

[1] Here and in what follows, we denote by $X$ a generic element of any strictly strictly stationary sequence $(X_t)$.

On the other hand, [8] provided some extreme value theory for very general strictly stationary sequences with regularly varying finite-dimensional distributions. This theory is less suited for linear processes but it is applicable to sequences which satisfy certain mixing conditions. In Section 4 we exploit this theory together with a continuous mapping argument to derive an asymptotic theory for the functional (1.1) for general strictly stationary regularly varying sequences. The limit of (1.1) will again be a Fréchet distribution. We apply these results to two standard financial time series models: the GARCH and heavy-tailed stochastic volatility models.

The paper is organized as follows. In Section 2.1 we introduce notions such as regular variation of a random vector and a regularly varying sequence. In Section 2.2 we recall a result from [27]. It describes the distributional limit behavior of the maximum increment of a random walk with iid regularly varying jump sizes. This result serves as a benchmark in the case of dependent jump sizes. In Section 3 we treat the maximum increment of a random walk with jump sizes given by a linear process with regularly varying innovations. In Section 4 we consider the maximum increment of a random walk for a general strictly stationary sequence, but we assume certain mixing conditions.

## 2. Preliminaries and notation

2.1. **Regular variation of random elements and stationary sequences.** In this paper, we describe the heavy distributional tails of a sequence of random variables by the notion of regular variation. For a real-valued random variable $Z$, this notion was made precise in (1.3). However, we will also need regular variation of random vectors: An $\mathbb{R}^d$-valued random vector $Z$ is said to be regularly varying with index $\alpha > 0$ if there exist a non-null Radon measure $\mu$ on the Borel $\sigma$-field $\mathcal{B}_0$ of $\overline{\mathbb{R}}_0^d = \overline{\mathbb{R}}^d \backslash \{0\}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ and a non-decreasing sequence $(a_n)$ of positive numbers such that

$$n\, P(a_n^{-1} Z \in \cdot) \xrightarrow{v} \mu(\cdot), \tag{2.1}$$

where $\xrightarrow{v}$ denotes vague convergence; see [22, 34] for details on vague convergence and [34, 19, 3] for more reading on regular variation. The sequence $(a_n)$ can always be chosen as

$$a_n = \inf\{x \geq 0 : P(|Z| \leq x) \geq 1 - n^{-1}\}\,, \quad n \geq 1\,,$$

and the measure $\mu$ necessarily satisfies the relation $\mu(t\cdot) = t^{-\alpha}\mu(\cdot)$ for any $t > 0$.

The strictly stationary sequence $(X_t)$ of real-valued random variables is said to be regularly varying with index $\alpha > 0$ if for every $d \geq 1$ the vector $Z_d = (X_1, \ldots, X_d)$ is regularly varying with index $\alpha$ and limiting measure $\mu_d$, where for all $d \geq 1$ the sequence $(a_n)$ is chosen such that $nP(|X| > a_n) \to 1$. A simple example of a regularly varying sequence with index $\alpha > 0$ is an iid sequence $(X_t)$, where $X$ is regularly varying with index $\alpha$. The limiting measures $\mu_d$ are then concentrated on the axes. More complicated examples of regularly varying sequences $(X_t)$ will be considered in the following sections.

2.2. **The maximum increment of a random walk with independent regularly varying jump sizes.** In [27] the limit distribution of the maximum increment of a random walk with iid regularly varying Banach-valued jump sizes was studied. We recall some of the results as a benchmark for the case of dependent jump sizes, but we restrict ourselves to real-valued random variables $X_t$.

For an iid sequence $(X_t)$ of random variables we define the corresponding random walk

$$S_0 = 0 \quad \text{and} \quad S_n = X_1 + \cdots + X_n \quad \text{for} \quad n \geq 1,$$

and $(\overline{X}_n)$ denotes the corresponding sequence of the sample means. The maximum increment of the random walk $(S_n)$ is then described by the quantities

$$\widetilde{M}_n^{(\gamma)} := \max_{1 \leq l \leq n} (f(l))^{-1} \max_{0 \leq k \leq n-l} |S_{k+l} - S_k|\,, \quad n \geq 1\,, \tag{2.2}$$

and their centered analogs

$$\widetilde{T}_n^{(\gamma)} := \max_{1\le l\le n} \big(f(l(1-l/n))\big)^{-1} \max_{0\le k\le n-l} |S_{k+l}-S_k - l\overline{X}_n|\,,\quad n\ge 1\,. \tag{2.3}$$

The functions $f$ are chosen from the class $\mathcal{F}_\gamma$, $\gamma\ge 0$, given by

$$\begin{aligned}\mathcal{F}_\gamma := \big\{f : f &\text{ is a non-decreasing function on } [0,\infty),\ f(1)=1, f(l)\ge l^\gamma \text{ for } l\ge 1 \text{ and}\\ &\text{for any increasing sequence } (d_n) \text{ of positive numbers such that } d_n^2/n\to 0,\\ &\text{the following relation holds } \lim_{n\to\infty}\inf_{1\le l\le d_n} f(l(1-l/n))/f(l)=1\big\}.\end{aligned}$$

The class $\mathcal{F}_\gamma$, $\gamma>0$, contains the functions $f(x)=x^{\gamma'}$ for $\gamma'\ge\gamma$, and $f(x)=x^\gamma\log^\beta(1+x)$ for $\beta>0$.

The main result of [27] in the case of real-valued random variables is the following.

**Theorem 2.1.** *Let $(X_t)$ be a sequence of iid random variables which are regularly varying with index $\alpha>0$. In addition, assume $EX=0$ if $E|X|<\infty$. Then, for $f\in\mathcal{F}_\gamma$, $\gamma>\max(0,0.5-\alpha^{-1})$, with the normalizing sequence $(a_n)$ chosen such that $nP(|X|>a_n)\to 1$,*

$$\lim_{x\to\infty} P(a_n^{-1}\widetilde{M}_n^{(\gamma)}\le x) = \Phi_\alpha(x)\,,\quad x>0\,,$$
$$\lim_{x\to\infty} P(a_n^{-1}\widetilde{T}_n^{(\gamma)}\le x) = \Phi_\alpha(x)\,,\quad x>0\,,$$

*where $\Phi_\alpha(x)=\exp\{-x^{-\alpha}\}$, $x>0$, denotes the Fréchet distribution function.*

**Remark 2.2.** It is known from classical extreme value theory (e.g. [15], Chapter 3) that regular variation of $|X|$ is necessary and sufficient for the convergence in distribution of the sequence of normalized partial maxima $(a_n^{-1}\max_{i=1,\dots,n}|X_i|)$ towards a $\Phi_\alpha$-distributed random variable. Hence, for $l\ge 2$ the normalized random walk increments $a_n^{-1}(f(l))^{-1}\max_{0\le k\le n-l}|S_{k+l}-S_k|$ do not contribute to the limits of $a_n^{-1}\widetilde{M}_n^{(\gamma)}$ in Theorem 2.1.

**Remark 2.3.** The results in [27] are sharp in the following sense. If $f(l)=l^\gamma$ for some $\gamma<0.5-\alpha^{-1}$ and some $\alpha>2$, then an application of the invariance principle in Hölder space yields

$$n^{-0.5+\gamma}\widetilde{M}_n^{(\gamma)} \xrightarrow{d} \sup_{s,t\in[0,1],s\ne t}\frac{|W(t)-W(s)|}{|t-s|^\gamma}\,,$$

where $W$ is a Brownian motion on $[0,1]$; see [32]. One of the original motivating ideas for considering limit theory for statistics of the type $\widetilde{M}_n^{(\gamma)}$ or $\widetilde{T}_n^{(\gamma)}$ was to use these statistics for detecting epidemic changes in a sample. We refer to the monograph [7] as a general reference to change point problems and the recent papers [31, 32] for advanced limit theory in the context of epidemic changes.

## 3. Random walks with linearly dependent jump sizes

3.1. **Preliminaries on regularly varying linear processes.** In this section, we consider a linear process $(X_t)$ defined in (1.2), where $(\psi_j)$ is a sequence of real numbers and the noise sequence $(Z_t)$ constitutes an iid real-valued sequence such that a generic element $Z$ is regularly varying with index $\alpha>0$; see Section 2.1. If $X_t$ is a genuine infinite series one needs to verify whether the series (1.2) converges a.s., and this condition has to be reconciled with the regular variation of $Z$. In the following lemma we give sufficient conditions for the a.s. convergence and regular variation of $X_t$, see e.g. [15], Section A3.3. Condition (3.1) below is taken from [28]. This condition is close to those dictated by the 3-series theorem.

**Lemma 3.1.** *Let $(Z_t)$ be an iid sequence of regularly varying random variables with index $\alpha > 0$ satisfying the tail balance condition* (1.3) *and, if $\alpha > 1$, $EZ = 0$. Moreover, assume*

$$\sum_{i=0}^{\infty} |\psi_i|^p < \infty \tag{3.1}$$

*where $p = 2$ for $\alpha > 2$ and $p = \alpha - \delta$ for some $\delta > 0$ for $\alpha \le 2$. Then the series* (1.2) *converges a.s., $X_t$ is regularly varying with index $\alpha > 0$ and the following relation holds*

$$\lim_{x\to\infty} \frac{P(X > x)}{P(|Z| > x)} = \sum_{i=1}^{\infty} \left(\widetilde{p}\,(\psi_j)_+^\alpha + \widetilde{q}\,(\psi_j)_-^\alpha\right).$$

We mention that the linear process $(X_t)$ with regularly varying innovation sequence is then also regularly varying. However, this property will not be crucial in the sequel.

3.2. **The jump size is a finite moving average.** In this section, we derive the limit distribution of the quantities $\widetilde{M}_n^{(\gamma)}$ and $\widetilde{T}_n^{(\gamma)}$ defined in (2.2) and (2.3), respectively, for a moving average of finite order $q$, i.e.

$$X_t = X_t^{(q)} = \sum_{i=1}^{q} \psi_i Z_{t-i}\,, \quad t \in \mathbb{Z}\,, \tag{3.2}$$

for an iid sequence $(Z_t)$. The following result is the analog of Theorem 2.1 for moving averages.

**Theorem 3.2.** *Let $(X_t)$ be a moving average process of order $q > 1$ with an iid noise sequence $(Z_t)$ of regularly varying random variables with index $\alpha > 0$. If $E|Z| < \infty$ we also assume $EZ = 0$. Then, for $f \in \mathcal{F}_\gamma$, $\gamma > \max(0, 0.5 - 1/\alpha)$, and $(a_n)$ chosen such that $nP(|X| > a_n) \to 1$, the following relations hold:*

$$\lim_{n\to\infty} P(a_n^{-1}\widetilde{M}_n^{(\gamma)} \le x) = \Phi_\alpha^{m_q^\alpha}(x), \quad x > 0, \tag{3.3}$$

$$\lim_{n\to\infty} P(a_n^{-1}\widetilde{T}_n^{(\gamma)} \le x) = \Phi_\alpha^{m_q^\alpha}(x), \quad x > 0, \tag{3.4}$$

*where $\Phi_\alpha$ again denotes the Fréchet distribution, $(a_n)$ is chosen such that $n\,P(|Z| > a_n) \to 1$ and*

$$m_q := \max_{1\le l\le q}\ \max_{1\le k\le q-l+1} \frac{|\psi_{k,k+l-1}|}{f(l)} \quad \textit{with} \quad \psi_{i,j} := \sum_{k=i}^{j} \psi_k, \quad \textit{for} \quad 1 \le i \le j \le q.$$

The proofs of this and the other results of this section will be given in Section 3.4.

**Remark 3.3.** The parameter $m_q$ depends on the choice of the function $f \in \mathcal{F}_\gamma$. For the ease of presentation, we suppress this dependence in the notation. The same comment applies to the $m$-parameters which will be introduced in the remainder of the present section.

**Remark 3.4.** For $\gamma \ge 1$, we have $l/f(l) \le 1$, $l \ge 1$, and then direct calculation shows that $\widetilde{M}_n^{(\gamma)} = \max_{t=1,\dots,n} |X_t|$. It follows from [11] that $\lim_{n\to\infty} P(a_n^{-1}\widetilde{M}_n^{(\gamma)} \le x) = \Phi_\alpha^{\psi^\alpha}(x)$, where $\psi = \max_{q\ge i\ge 1} |\psi_i|$. This is in agreement with (3.3), since, for $\gamma \ge 1$,

$$\max_{1\le k\le q} (f(1))^{-1}|\psi_k| \le m_q \le \max_{1\le l\le q} (l/f(l)) \max_{1\le k\le q} |\psi_k| = \max_{1\le k\le q} |\psi_k|\,.$$

A corresponding remark applies to (3.4).

It is also possible to derive results for quantities based on the one-sided increments of a random walk, for example,

$$\begin{aligned} M_n^{(\gamma)} &:= \max_{1\le l\le n} (f(l))^{-1} \max_{0\le k\le n-l} (S_{k+l}-S_k)\,, \\ r_n^{(\gamma)} &:= \min_{1\le l\le n} (f(l))^{-1} \min_{0\le k\le n-l} (S_{k+l}-S_k)\,, \\ T_n^{(\gamma)} &:= \max_{1\le l\le n} f(l(1-l/n))^{-1} \max_{1\le k\le n-l} (S_{k+l}-S_k-l\overline{X}_n)\,. \end{aligned}$$

**Theorem 3.5.** *Assume the conditions of Theorem 3.2. Choose $(b_n)$ such that $nP(Z>b_n)\to 1$, where we also assume that $\widetilde{p}>0$. Then, for $f\in\mathcal{F}_\gamma$, $\gamma>\max(0,0.5-1/\alpha)$ and $x,y>0$, the following limit relations hold*

$$\begin{aligned} \lim_{n\to\infty} P\left(b_n^{-1} r_n^{(\gamma)} \le -x\,,\ b_n^{-1} M_n^{(\gamma)} \le y\right) &= \Phi_\alpha^{(m_q^+)^\alpha}(y)\left(1-\Phi_\alpha^{(\widetilde{q}/\widetilde{p})(m_q^-)^\alpha}(x)\right), \\ \lim_{n\to\infty} P(b_n^{-1} T_n^{(\gamma)} \le x) &= \Phi_\alpha^{(m_q^+)^\alpha}(x)\,, \end{aligned}$$

*where*

$$\begin{aligned} m_q^+ &= \max_{1\le l\le q}\ \max_{1\le k\le q-l+1} \frac{(\psi_{k,k+l-1})_+}{f(l)}\,, \\ m_q^- &= \max_{1\le l\le q}\ \max_{1\le k\le q-l+1} \frac{(\psi_{k,k+l-1})_-}{f(l)}\,. \end{aligned}$$

3.3. **The jump sizes constitute a linear process.** In this section we analyze the limit distribution of the maximum increment of a random walk whose jump sizes constitute an infinite order moving average, i.e. $q=\infty$ in (3.2). Formally, the limit distributions of $\widetilde{M}_n^{(\gamma)}$ and $\widetilde{T}_n^{(\gamma)}$ can be obtained by letting $q\to\infty$ in Theorem 3.2. However, due to the more complicated dependence structure of $(X_t)$, the proof is not straightforward and additional conditions and arguments are needed.

Throughout we assume that the infinite series $X_t$ in (3.2) is finite a.s.; Lemma 3.1 provides sufficient conditions. In what follows, we assume additional conditions on the coefficients $(\psi_j)$:

$$\sum_{j=3}^{\infty} |\psi_j|\ell(j) < \infty, \tag{3.5}$$

where

$$\ell(j) = \begin{cases} (j\log\log j)^{1/2} & \text{if } \operatorname{var}(Z)<\infty \\ j^{1/p} & \text{for some } p<\alpha \text{ if } \alpha<2 \text{ or } \alpha=2 \text{ and } \operatorname{var}(Z)=\infty. \end{cases}$$

In view of Lemma 3.1, (3.5) implies the a.s. convergence of the series $X_t$ for $\alpha>1$ if in addition $EZ=0$.

**Theorem 3.6.** *Let $(X_n)$ be a linear process (1.2) with iid regularly varying noise sequence $(Z_t)$ with index $\alpha>0$. If $E|Z|<\infty$ we also assume $EZ=0$. Furthermore, assume (3.5), and if $\alpha\le 1$ also $\sum_{j=0}^\infty |\psi_j|^{\alpha-\delta}<\infty$ for some $\delta<\alpha$. Then for $f\in\mathcal{F}_\gamma$, $\gamma>\max(0,0.5-1/\alpha)$ we have*

$$\begin{aligned} \lim_{n\to\infty} P(a_n^{-1}\widetilde{M}_n^{(\gamma)} \le x) &= \Phi_\alpha^{m_\infty^\alpha}(x), \quad x>0, \\ \lim_{n\to\infty} P(a_n^{-1}\widetilde{T}_n^{(\gamma)} \le x) &= \Phi_\alpha^{m_\infty^\alpha}(x), \quad x>0\,, \end{aligned}$$

*where*

$$m_\infty := \lim_{q\to\infty} m_q = \max_{l\ge 1}\ \max_{k\ge 1} \frac{|\psi_{k,k+l-1}|}{f(l)}.$$

**Remark 3.7.** From the proofs of Theorems 3.2, 3.5 and 3.6 it is straightforward that Theorem 3.5 extends to the infinite order moving average case as well provided the conditions of Theorem 3.6 for $(\psi_j)$ are satisfied and the constants $m_q^+$ and $m_q^-$ are replaced by

$$m_\infty^+ = \lim_{q\to\infty} m_q^+ = \max_{l\ge1}\max_{k\ge1}\frac{(\psi_{k,k+l-1})_+}{f(l)} \quad \text{and} \quad m_\infty^- = \lim_{q\to\infty} m_q^- = \max_{l\ge1}\max_{k\ge1}\frac{(\psi_{k,k+l-1})_-}{f(l)},$$

respectively.

We mention in passing that [21, 33] proved functional central limit theorems in Hölder space for partial sum processes of linear processes. Results of this type yield limits for $\widetilde{M}_n^{(\gamma)}$ and $\widetilde{T}_n^{(\gamma)}$ when $\alpha > 2$ and $\gamma < 0.5-\alpha^{-1}$. The mentioned results show that the statements of Theorem 3.6 are sharp in the sense that different limit distributions appear when the normalizing functions $f(l)$ increase too slowly.

### 3.4. **Proofs.**

3.4.1. *Proof of Theorem 3.2.* The proof uses arguments similar to the ones in the proof of Theorem 2.2 in [27]. Lemma 3.8 in this paper replaces Lemma 2.4 in [27].

**Lemma 3.8.** *Assume the conditions of Theorem 3.2 hold for $(Z_t)$ and $(X_t)$. Then the following statements hold.*

(1) *For any $f \in \mathcal{F}_\gamma$, $\gamma \ge 0$ and $h \ge 1$,*

$$\lim_{n\to\infty} P\Big(a_n^{-1}\max_{1\le l\le h}(f(l))^{-1}\max_{0\le k\le n-l}|S_{k+l}-S_k| \le x\Big) = \Phi_\alpha^{m_{q,h}^\alpha}(x),$$

*where*

$$m_{q,h} := \max_{1\le l\le h\wedge q}\ \max_{1\le k\le q-l+1}\frac{|\psi_{k,k+l-1}|}{f(l)}.$$

(2) *Assume in addition that $EZ=0$ if $E|Z|<\infty$. Then, for any $\delta>0$ and $f\in\mathcal{F}_\gamma$, $\gamma > \max\{0, 0.5-\alpha^{-1}\}$,*

$$\lim_{h\to\infty}\limsup_{n\to\infty} P\left(\max_{h\le l\le n}(f(l))^{-1}\max_{0\le k\le n-l}|S_{k+l}-S_k| > \delta a_n\right) = 0.$$

**Remark 3.9.** Notice that for $h\ge q$, $m_{q,h}=m_q$ as defined in Theorem 3.2.

*Proof.* **(1)** The proof uses arguments from the proof of Lemma 2.4(1) in [27] and a continuous mapping argument for point processes, going back to [11]. Write $M_p(E)$ for the set of point measures on a Borel state space $E$ of some Euclidean space, and equip $M_p(E)$ with the vague topology. Introduce the point processes

$$\widetilde{N}_n^{(h)} := \sum_{t=1}^n \varepsilon_{a_n^{-1}(Z_{t-q},Z_{t-q+1},\dots,Z_{t+h-2})} \quad \text{and} \quad N_n^{(h)} := \sum_{t=1}^n \varepsilon_{a_n^{-1}(X_t,X_t+X_{t+1},\dots,X_t+\cdots+X_{t+h-1})},$$

on $M_p(\overline{\mathbb{R}}_0^{q+h-1})$ and $M_p(\overline{\mathbb{R}}_0^h)$, respectively. It follows from [11] that

$$\widetilde{N}_n^{(h)} \xrightarrow{d} \widetilde{N}^{(h)} := \sum_{k=1}^{q+h-1}\sum_{i=1}^\infty \varepsilon_{J_i\mathbf{e}_k}, \quad n\to\infty,$$

where $(J_i)$ are the points of a Poisson random measure on $\overline{\mathbb{R}}_0$ with intensity $\alpha|x|^{\alpha-1}[p\,I_{(0,\infty]}(x)+q\,I_{[-\infty,0)}(x)]dx$ and $\mathbf{e}_k$ is the $k$th unit vector in $\mathbb{R}^{q+h-1}$. According to the proof of Theorem 2.4 in

[11], the continuous function

$$(Z_{t-q}, Z_{t-q+1}, \ldots, Z_{t+h-2}) \mapsto \Big( \sum_{k=t}^{t} \sum_{i=1}^{q} \psi_i Z_{k-i},\ \sum_{k=t}^{t+1} \sum_{i=1}^{q} \psi_i Z_{k-i}, \ldots,\ \sum_{k=t}^{t+h-1} \sum_{i=1}^{q} \psi_i Z_{k-i} \Big)$$
$$= (X_t, X_t + X_{t+1}, \ldots, X_t + \cdots + X_{t+h-1}),$$

induces a continuous mapping on the limit relation $\widetilde{N}_n^{(h)} \xrightarrow{d} \widetilde{N}^{(h)}$, resulting in the following convergence in $M_p(\overline{\mathbb{R}}_0^h)$

$$N_n^{(h)} \xrightarrow{d} N^{(h)} = \sum_{k=2-h}^{q} \sum_{i=1}^{\infty} \varepsilon_{J_i(\psi_{1\vee k, q\wedge k},\, \psi_{1\vee k, q\wedge (k+1)},\, \cdots,\, \psi_{1\vee k, q\wedge (k+h-1)})}\,, \tag{3.6}$$

where we set $\psi_{i,j} = 0$ if $j < 1$. As in [27] we write for $l \geq 1$, $\widetilde{M}_{nl} = \max_{0 \leq k \leq n} |S_{k+l} - S_k|$ and introduce the sets

$$B_f(y) := \left\{ (x_1, \ldots, x_h) \in \mathbb{R}_0^h : |x_i| \leq y f(i) \text{ for } i = 1, \ldots, h \right\}, \quad y > 0\,.$$

Write $J = \sup_{i \geq 1} |J_i|$. It follows from (3.6) and the monotonicity of $f$ that

$$\begin{aligned}
P(a_n^{-1} \max_{1 \leq l \leq h} (f(l))^{-1} \widetilde{M}_{nl} \leq y) &= P(a_n^{-1} \widetilde{M}_{n1} \leq y f(1), \ldots, a_n^{-1} \widetilde{M}_{nh} \leq y f(h)) \\
&= P(N_n^{(h)}((B_f(y))^c) = 0) \to P(N^{(h)}((B_f(y))^c) = 0) \\
&= P\Big( J \max_{1 \leq k \leq q} |\psi_{1\vee k, q\wedge k}| \leq y f(1), J \max_{0 \leq k \leq q} |\psi_{1\vee k, q\wedge (k+1)}| \leq y f(2), \ldots, \\
&\qquad J \max_{2-h \leq k \leq q} |\psi_{1\vee k, q\wedge (k+h-1)}| \leq y f(h) \Big) \\
&= P(J \leq y / m_{q,h}) = \Phi_\alpha(y / m_{q,h}).
\end{aligned}$$

This concludes the proof of the first statement.

**(2)** Let $(\widetilde{S}_n)_{n \geq 0}$ denote the random walk generated by the iid sequence $(Z_t)$. Observe that for $1 \leq l \leq n$ and some constant $c > 0$ depending on $(\psi_j)$, using the stationarity of the iid sequence $(Z_t)$,

$$\begin{aligned}
\max_{h \leq l \leq n} \max_{k \leq n-l} |S_{k+l} - S_k| &\leq c \max_{h \leq l \leq n} \max_{k \leq n-l} \max_{1 \leq i \leq q} \Big| \sum_{j=k+1}^{k+l} Z_{j-i} \Big| \\
&\stackrel{d}{=} c \max_{h \leq l \leq n} \max_{k \leq n-l} \max_{1 \leq i \leq q} |\widetilde{S}_{k+q-i+l} - \widetilde{S}_{k+q-i}| \\
&\leq c \max_{h \leq l \leq n} \max_{0 \leq k \leq n+q-l} |\widetilde{S}_{k+l} - \widetilde{S}_k|\,.
\end{aligned}$$

An application of Lemma 2.4(2) in [27] yields that for every $\delta > 0$,

$$\begin{aligned}
&\lim_{h \to \infty} \limsup_{n \to \infty} P\Big( \max_{h \leq l \leq n} \max_{0 \leq k \leq n-l} |S_{k-l} - S_k| > \delta a_n \Big) \\
&\leq \lim_{h \to \infty} \limsup_{n \to \infty} P\Big( \max_{h \leq l \leq n+q} \max_{0 \leq k \leq n+q-l} |\widetilde{S}_{k+l} - \widetilde{S}_k| > a_n \delta \Big) = 0
\end{aligned}$$

This concludes the proof of the lemma. □

Relation (3.3) is now a straightforward consequence of Lemma 3.8. For the proof of (3.4), we can follow the arguments of Remarks 2.5 and 2.6 in [27] to show that the sequences $(a_n^{-1} \widetilde{M}_n^{(\gamma)})$ and

$$a_n^{-1} \zeta_n^{(\gamma)} := a_n^{-1} \max_{1 \leq l \leq n} (f(l))^{-1} \max_{0 \leq k \leq n-l} |S_{k+l} - S_k - l \overline{X}_n|\,, \quad n \geq 1\,,$$

have the same limits. The same arguments as in the iid case apply, using that $(X_t)$ is a finite moving average process.

3.4.2. *Proof of Theorem 3.5.* The proof is similar to the one of Theorem 3.2. We focus on the proof of the joint limit of $b_n^{-1}(r_n^{(\gamma)}, M_n^{(\gamma)})$.

Theorem 2.2 in [11] is still valid, if we substitute $a_n$ by $b_n$. The only difference is that we get a slightly different limiting Poisson random measure with mean measure $\mu$ given by $\mu(dx) = \alpha|x|^{-\alpha}[I_{(0,\infty)}(x) + (\widetilde{q}/\widetilde{p})I_{[-\infty,0)}(x)]dx$. Now we follow the proof of Lemma 3.8, replace $(a_n)$ by $(b_n)$ everywhere and use the same notation. Write

$$B_f(x,y) := \big\{(z_1,\dots,z_h) \in \mathbb{R}^h : -xf(i) \le z_i \le yf(i) \text{ for } i = 1,\dots,h\big\}, \quad x,y>0\,.$$

We obtain

$$\begin{aligned}
&P(N_n^{(h)}((B_f(x,y))^c) = 0)\\
&= P\Big(b_n^{-1} \max_{1\le l\le h}(f(l))^{-1} \max_{0\le k\le n-l}(S_{k+l}-S_k) \le y, b_n^{-1} \min_{1\le l\le h}(f(l))^{-1} \min_{0\le k\le n-l}(S_{k+l}-S_k) \ge -x\Big)\\
&\to P(N^{(h)}((B_f(x,y))^c) = 0)\\
&= e^{-\mu\big((-\infty,-x/m_{q,h}^-)\cup(y/m_{q,h}^+,\infty)\big)} = \Phi_\alpha^{\widetilde{q}/\widetilde{p}}(x/m_{q,h}^-)\,\Phi_\alpha(y/m_{q,h}^+)\,,
\end{aligned}$$

where

$$\begin{aligned}
m_{q,h}^+ &= \max_{1\le l\le h\wedge q}\ \max_{1\le k\le q-l+1} \frac{(\psi_{k,k+l-1})_+}{f(l)}\,,\\
m_{q,h}^- &= \max_{1\le l\le h\wedge q}\ \max_{1\le k\le q-l+1} \frac{(\psi_{k,k+l-1})_-}{f(l)}\,.
\end{aligned}$$

Therefore for $x,y>0$,

$$\begin{aligned}
&P\Big(b_n^{-1} \max_{1\le l\le h}(f(l))^{-1} \max_{0\le k\le n-l}(S_{k+l}-S_k) \le y, b_n^{-1} \min_{1\le l\le h}(f(l))^{-1} \min_{0\le k\le n-l}(S_{k+l}-S_k) \le -x\Big)\\
&= P\Big(b_n^{-1} \max_{1\le l\le h}(f(l))^{-1} \max_{0\le k\le n-l}(S_{k+l}-S_k) \le y\Big) -\\
&\quad P\Big(b_n^{-1} \max_{1\le l\le h}(f(l))^{-1} \max_{0\le k\le n-l}(S_{k+l}-S_k) \le y, b_n^{-1} \min_{1\le l\le h}(f(l))^{-1} \min_{0\le k\le n-l}(S_{k+l}-S_k) > -x\Big)\\
&\to \big(1-\Phi_\alpha^{\widetilde{q}/\widetilde{p}}(x/m_{q,h}^-)\big)\Phi_\alpha(y/m_{q,h}^+).
\end{aligned}$$

Finally, in view of Lemma 3.8(2), and observing that $m_{q,h}^+ = m_q^+$ and $m_{q,h}^- = m_q^-$ for large $h$, we derived the limit distribution of the sequence $(b_n^{-1}(r_n^{(\gamma)}, M_n^{(\gamma)}))$.

3.4.3. *Proof of Theorem 3.6.* For $q>1$ write

$$X_t = \sum_{j=1}^q \psi_j Z_{t-j} + \sum_{j=q+1}^\infty \psi_j Z_{t-j}.$$

In the analysis of the quantities $\widetilde{M}_n^{(\gamma)}$ and $\widetilde{T}_n^{(\gamma)}$, the first term on the right-hand side can now be controlled by Theorem 3.2 when $q\to\infty$ and the second term is handled by the following lemma.

**Lemma 3.10.** *Under the assumptions of Theorem 3.6, for any $\delta>0$ and $f\in\mathcal{F}_\gamma$, $\gamma>\max\{0,0.5-\alpha^{-1}\}$,*

$$\lim_{q\to\infty}\limsup_{n\to\infty} P\Big(\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n}\Big|\sum_{t=k+1}^{k+l}\sum_{j=q+1}^\infty \psi_j Z_{t-j}\Big| > \delta\Big) = 0.$$

*Proof.* We have

$$\begin{aligned} &\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n}\Big|\sum_{t=k+1}^{k+l}\sum_{j=q+1}^{\infty}\psi_j Z_{t-j}\Big| \\ \le\; &\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n}\Big(\sum_{j=q+1}^{3n}|\psi_j|\Big|\sum_{t=k+1}^{k+l}Z_{t-j}\Big|+\sum_{j=3n+1}^{\infty}|\psi_j|\Big|\sum_{t=k+1}^{k+l}Z_{t-j}\Big|\Big). \end{aligned} \tag{3.7}$$

The first maximum expression on the right-hand side of (3.7) is bounded by

$$R_n^{(q)} \;=\; \sum_{j=q+1}^{\infty}|\psi_j|\quad\max_{1\le l\le n}(f(l))^{-1}\max_{-3n\le k\le n}\Big|\sum_{t=k+1}^{k+l}Z_t\Big|.$$

In view of the results in [27] for the maximum increment of the iid sequence $(Z_t)$ and since $\sum_{j>q}|\psi_j|\to 0$ as $q\to\infty$ we have

$$\lim_{q\to\infty}\limsup_{n\to\infty}P(R_n^{(q)}>\delta)=0\,,\quad \delta>0\,.$$

Next we bound the second maximum term on the right-hand side of (3.7). For $n\ge 3$, $\alpha\ge 2$ and if $\operatorname{var}(Z)<\infty$ we can bound this term by

$$\begin{aligned} &\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n}\sum_{j=3n+1}^{\infty}|\psi_j|\Big|\sum_{t=k+1}^{k+l}Z_{t-j}\Big| \\ =\; &\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n}\sum_{j=3n+1}^{\infty}|\psi_j|\Big(\Big|\sum_{t=k+1-j}^{0}Z_t-\sum_{t=k+l-j+1}^{0}Z_t\Big|\Big) \\ \le\; &\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n}\sum_{j=3n+1}^{\infty}|\psi_j|\frac{\sqrt{j\log\log j}}{\sqrt{j\log\log j}}\Big(\Big|\sum_{t=k+1-j}^{0}Z_t\Big|+\Big|\sum_{t=k+l-j+1}^{0}Z_t\Big|\Big) \\ \le\; &c\sum_{j=3n+1}^{\infty}|\psi_j|\sqrt{j\log\log j}\,\sup_{r\ge 3}\frac{1}{\sqrt{r\log\log r}}\Big|\sum_{t=-r}^{0}Z_t\Big|=Q_n\,. \end{aligned}$$

By virtue of the law of the iterated logarithm the right-hand supremum is bounded a.s. Therefore and since (3.5) holds, $\lim_{n\to\infty}P(Q_n>\delta)=0$ for $\delta>0$. This proves the lemma for $\alpha\ge 2$ and $\operatorname{var}(Z)<\infty$.

In the cases $\alpha<2$ or $\alpha=2$, $\operatorname{var}(Z)=\infty$, the argument with the law of the iterated logarithm is replaced by a Marcinkiewicz-Zygmund strong law of large numbers (see e.g. Theorem 6.9 in [30]) with normalization $n^{1/p}$ for some $p<\alpha$. □

The proof of Theorem 3.6 now follows by some elementary calculations. Indeed, since

$$\begin{aligned} &\Big|\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n-l}|S_{k+l}-S_k|-\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n-l}\Big|\sum_{t=k+1}^{k+l}\sum_{j=1}^{q}\psi_j Z_{t-j}\Big|\Big| \\ \le\; &\max_{1\le l\le n}(f(l))^{-1}\max_{1\le k\le n-l}\Big|\sum_{t=k+1}^{k+l}\sum_{j=q+1}^{\infty}\psi_j Z_{t-j}\Big|, \end{aligned}$$

we can combine Lemma 3.10 and Theorem 3.2, by first letting $n\to\infty$ and then $q\to\infty$, to obtain

$$\lim_{n\to\infty}P(a_n^{-1}\widetilde{M}_n^{(\gamma)}\le x)\;=\;\lim_{q\to\infty}\Phi_\alpha(x/m_q)=\Phi_\alpha(x/m_\infty)\,.$$

This concludes the proof of the theorem.

## 4. Random walks with general dependent jump sizes

In this section we study the maximum increment of a random walk with jump sizes which constitute a general strictly stationary regularly varying sequence $(X_t)$ with index $\alpha > 0$. In contrast to the results in Section 3 we do not assume any particular dependence structure of $(X_t)$. As a compensation, we will need some mixing and anti-clustering conditions. These conditions are in general hard to verify for a regularly varying linear process with iid noise. In particular, for such a sequence mixing conditions are in general difficult to check and sometimes not true.

4.1. **General theory.** In what follows, we write $\mathcal{M}$ for the collection of Radon counting measures on $\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$, $\widetilde{\mathcal{M}}$ is the subset of those measures $\mu$ in $\mathcal{M}$ for which $\mu(\{x : |x| > 1\}) = 0$ and $\mu(\{x : |x| = 1\}) > 0$. Moreover, $\mathcal{B}(\widetilde{\mathcal{M}})$ is the Borel $\sigma$-field of $\widetilde{\mathcal{M}}$.

We recall Theorem 2.8 from [9] which is a multivariate version of Theorem 2.7 in [8]. It is based on the mixing condition $\mathcal{A}(b_n)$ of [8] which holds for $(X_t)$ if there exists a sequence of positive integers $(r_n)$ such that $r_n \to \infty$, $r_n/n \to 0$ and, as $n \to \infty$,

$$(\mathcal{A}(b_n)) \qquad E \exp\Big(-\sum_{j=1}^{n} f(X_j/b_n)\Big) - \Big(E \exp\Big(-\sum_{j=1}^{r_n} f(X_j/b_n)\Big)\Big)^{[n/r_n]} \to 0$$

for all $f \in \mathcal{F}_s$, where $\mathcal{F}_s$ is the set of all bounded and non-negative measurable step functions on $\overline{\mathbb{R}}^d \backslash \{0\}$ with bounded support and the sequence $(b_n)$ is chosen such that $n\,P(|X_1| > b_n) \to 1$.

**Theorem 4.1.** *Assume that the strictly stationary sequence $(X_t)$ of $\mathbb{R}^d$-valued random variables is regularly varying with index $\alpha > 0$ and satisfies the mixing condition $\mathcal{A}(b_n)$ and the anti-clustering condition*

$$\lim_{h\to\infty} \limsup_{n\to\infty} P\Big( \vee_{h\le |t|\le r_n} |X_t| > \delta b_n \Big| |X_0| > \delta b_n\Big) = 0\,, \quad \delta > 0\,, \tag{4.1}$$

*holds for $(r_n)$ as in $\mathcal{A}(b_n)$. Then*

$$N_n = \sum_{t=1}^{n} \varepsilon_{b_n^{-1} X_t} \stackrel{d}{\rightarrow} N = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon_{P_i Q_{ij}}\,,$$

*where $(P_i)$ are the points of a Poisson process on $(0,\infty)$ with intensity measure given by $\nu(x,\infty) = \theta_{|X|} x^{-\alpha}$ for $x > 0$, $\theta_{|X|} \in [0,1]$ is the extremal index of the sequence $(|X_t|)$ and $\sum_{j=1}^{\infty} \varepsilon_{Q_{ij}}$, $i \ge 1$, constitute an iid sequence of point processes with common distribution $Q$ on $(\widetilde{\mathcal{M}}, \mathcal{B}(\widetilde{\mathcal{M}}))$, and these point processes are independent of $(P_i)$. The distribution $Q$ is given in Theorem 2.8 in* [9].

**Remark 4.2.** The extremal index of a strictly stationary real-valued sequence is a measure of the extremal clustering in the sequence; see [25] and [15], Section 8.1.
Condition $\mathcal{A}(b_n)$ is a rather general mixing condition which is suited for the purposes of extreme value theory for dependent sequences. This condition follows for example from strong mixing of $(X_t)$ with a sufficiently fast rate $\alpha_n$; see the comment after (2.1) in [8]. To see the correctness of this statement, assume for simplicity that $k_n = n/r_n$ is an integer (the general case only requires more bookkeeping) and write $N_{n,i} = \sum_{j=(i-1)r_n+1}^{ir_n} \varepsilon_{b_n^{-1}X_j}$, $i = 1,\ldots,k_n$. Obviously, $N_n = \sum_{i=1}^{k_n} N_{n,i}$. Then the left-hand side of $\mathcal{A}(b_n)$ can be represented as a telescoping sum:

$$
\begin{aligned}
\Delta_n &= E\exp\Big(-\sum_{i=1}^{k_n}\int_{\mathbb{R}^d} f dN_{n,i}\Big) - \Big(E\exp\Big(-\int_{\mathbb{R}^d} f dN_{n,1}\Big)\Big)^{k_n} \\
&= \sum_{l=0}^{k_n-1}\Big(E\exp\Big(-\int_{\mathbb{R}^d} f dN_{n,1}\Big)\Big)^{l}\times \\
&\quad \times\Big(E\exp\Big(-\sum_{i=1}^{k_n-l}\int_{\mathbb{R}^d} f dN_{n,i}\Big) - E\exp\Big(-\int_{\mathbb{R}^d} f dN_{n,1}\Big)E\exp\Big(-\sum_{i=1}^{k_n-l-1}\int_{\mathbb{R}^d} f dN_{n,i}\Big)\Big).
\end{aligned}
$$

Let $(\ell_n)$ be an integer sequence such that $\ell_n\to\infty$ and $\ell_n = o(r_n)$. Write $N_{n,i} = \underline{N}_{n,i} + \overline{N}_{N,i}$, where $\underline{N}_{n,i} = \sum_{j=(i-1)r_n+1}^{(i-1)r_n+\ell_n} \varepsilon_{b_n^{-1}X_j}$. Then

$$
\begin{aligned}
0 &\le E\exp\Big(-\sum_{i=1}^{k_n-l-1}\int_{\mathbb{R}^d} f dN_{n,i} - \int f d\overline{N}_{n,k_n-l}\Big) - E\exp\Big(-\sum_{i=1}^{k_n-l}\int_{\mathbb{R}^d} f dN_{n,i}\Big) \\
&\le E\Big(1-\exp\Big(-\int_{\mathbb{R}^d} f d\underline{N}_{n,1}\Big)\Big) \\
&\le E\int_{\mathbb{R}^d} f d\underline{N}_{n,1} = \ell_n E f(X/b_n) \\
&\le c\,\ell_n P(|X|>b_n)\,.
\end{aligned}
$$

In the last step we used regular variation of $X$ and the fact that $f$ is bounded and has support bounded away from zero. Therefore we conclude that

$$
\begin{aligned}
|\Delta_n| &\le \sum_{l=0}^{k_n-1}\Big| E\exp\Big(-\sum_{i=1}^{k_n-l-1}\int_{\mathbb{R}^d} f dN_{n,i} - \int_{\mathbb{R}^d} f d\overline{N}_{n,k_n-l}\Big) \\
&\qquad - E\exp\Big(-\sum_{i=1}^{k_n-l-1}\int_{\mathbb{R}^d} f dN_{n,i}\Big)E\exp\Big(-\int_{\mathbb{R}^d} f d\overline{N}_{n,1}\Big)\Big| + O\big(k_n\ell_n P(|X|>b_n)\big)\,.
\end{aligned}
$$

By a classical inequality for covariances of strongly mixing sequences (see [20], Theorem 17.2.1) we get

$$
|\Delta_n| \le 4k_n\alpha_{\ell_n} + O(k_n\ell_n P(|X|>b_n)) \le c\,[k_n\alpha_{\ell_n} + \ell_n/r_n] = c\,k_n\alpha_{\ell_n} + o(1)\,. \tag{4.2}
$$

In particular, if $\alpha_n\to 0$ as $n\to\infty$ at an exponential rate (this condition is satisfied for various popular time series models; see the discussion below), then it is easily seen that the right-hand side of (4.2) converges to zero if $\ell_n = C\log n$ for a sufficiently large constant $C>0$ and $\ell_n = o(r_n)$.

Our next goal is to apply Theorem 4.1 to the lagged vector sequence $X_t^{(h)} = (X_t,\ldots,X_{t+h-1})$, $t\in\mathbb{Z}$, of a real-valued strictly stationary sequence $(X_t)$ for increasing $h\ge 1$. Instead of the normalization $(b_n)$ which would depend on the dimension $h$ we choose a sequence $(a_n)$ such that $n\,P(|X_1|>a_n)\to 1$. By regular variation, we have $b_n/a_n\to c_h$ for certain constants $c_h>0$.

**Corollary 4.3.** *Assume that* $(X_t)$ *is regularly varying with index* $\alpha > 0$*, satisfies the mixing condition* $\mathcal{A}(a_n)$ *of* [8]*, where* $(a_n)$ *is chosen such that* $n\,P(|X_1| > a_n) \to 1$*, the anti-clustering condition* (4.1) *(with* $(b_n)$ *replaced by* $(a_n)$*) and the extremal index* $\theta_{|X|}$ *is positive. Then*

$$N_n^{(h)} = \sum_{t=1}^{n} \varepsilon_{a_n^{-1} X_t^{(h)}} \stackrel{d}{\to} N^{(h)} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon_{P_i^{(h)} Q_{ij}^{(h)}}\,,$$

*where the limiting quantities were explained in Theorem 4.1. The Poisson points* $(P_i^{(h)})$ *now have an intensity measure given by* $\nu_h(x,\infty) = (\theta_{|X^{(h)}|}/c_h) x^{-\alpha}$*,* $x > 0$*.*

Now we are ready to formulate a general result.

**Theorem 4.4.** *Assume that* $(X_t)$ *is regularly varying with index* $\alpha > 0$*, satisfies the mixing condition* $\mathcal{A}(a_n)$ *of* [8]*, where* $(a_n)$ *is chosen such that* $n\,P(|X_1| > a_n) \to 1$*, the anti-clustering condition* (4.1) *and* $\theta_{|X|} > 0$*. If the condition*

$$\lim_{h\to\infty} \limsup_{n\to\infty} P\big(\max_{h\le l\le n} (f(l))^{-1} \max_{0\le k\le n-l} |S_{k+l} - S_k| > \delta a_n\big) = 0\,, \quad \delta > 0\,, \tag{4.3}$$

*holds for* $f \in \mathcal{F}_\gamma$*, some* $\gamma > 0$*, then*

$$\lim_{n\to\infty} P(a_n^{-1} \widetilde{M}_n^{(\gamma)} \le x) = \lim_{h\to\infty} P\left(\sup_{i\ge 1} P_i^{(h)} V_i^{(h)} \le x\right)\,, \quad x > 0\,, \tag{4.4}$$

*where the iid sequence* $(V_i^{(h)})$ *is defined in* (4.6) *and independent of the Poisson points* $(P_i^{(h)})$ *from Corollary 4.3. Moreover, the limit* (4.4) *can be written in the form* $\Phi_\alpha^{\xi}(x)$*,* $x > 0$*, for some constant* $\xi \ge \theta_{|X|} > 0$*.*

**Remark 4.5.** As for a moving average (see Remark 3.3 about the parameter $m_q$) the quantity $\xi$ in general depends on the choice of the function $f \in \mathcal{F}_\gamma$ and the dependence structure of the sequence $(X_t)$. We have chosen to suppress the dependence of $\xi$ on $f$.

*Proof.* A continuous mapping argument (the map acting on the points $X_t^{(h)}$ is continuous and maps zero into zero) yields that for any non-decreasing sequence of positive numbers $(f(l))$ and $h \ge 1$,

$$\begin{aligned}&\sum_{t=1}^{n} \varepsilon_{a_n^{-1}\left(X_t/f(1),(X_t+X_{t+1})/f(2),\ldots,,(X_t+\cdots+X_{t+h-1})/f(h)\right)}\\ &\stackrel{d}{\to} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon_{P_i^{(h)}\left(Q_{ij}^{(h,1)}/f(1),(Q_{ij}^{(h,1)}+Q_{ij}^{(h,2)})/f(2),\ldots,(Q_{ij}^{(h,1)}+\cdots+Q_{ij}^{(h,h)})/f(h)\right)}\,.\end{aligned}$$

Here $Q_{ij}^{(h,k)}$ denotes the $k$th component of the vector $Q_{ij}^{(h)}$. Hence we may conclude that for every $f \in \mathcal{F}_\gamma$, $\gamma \ge 0$, and $h \ge 1$,

$$\begin{aligned}&\lim_{n\to\infty} P(a_n^{-1} \max_{1\le l\le h} (f(l))^{-1} \max_{1\le k\le n} |S_{k+l} - S_k| \le x)\\ &= P(\sup_{i\ge 1} P_i^{(h)} \max_{1\le l\le h} (f(l))^{-1} \sup_{j\ge 1} |Q_{ij}^{(h,1)} + \cdots + Q_{ij}^{(h,l)}| \le x)\,, \quad x > 0\,.\end{aligned} \tag{4.5}$$

The random variables

$$V_i^{(h)} = \max_{1\le l\le h} (f(l))^{-1} \sup_{j\ge 1} |Q_{ij}^{(h,1)} + \cdots + Q_{ij}^{(h,l)}|, \quad i = 1, 2, \ldots, \tag{4.6}$$

constitute an iid sequence independent of the Poisson points $(P_i^{(h)})$. It is well known (see e.g. [17], Corollary 9.4.5) that $Y_h = \sup_{i\ge 1} P_i^{(h)} V_i^{(h)}$ has a Fréchet distribution $\Phi_\alpha^{\xi_h}$ with shape parameter $\alpha > 0$ and scale factor $\xi_h = E[(V^{(h)})^\alpha]\theta_{|X^{(h)}|}/c_h$.

In view of (4.5) we have for every $h \geq 1$,

$$Y_{nh} = a_n^{-1} \max_{1 \leq l \leq h} (f(l))^{-1} \max_{1 \leq k \leq n} |S_{k+l} - S_k| \stackrel{d}{\to} Y_h = \sup_{i \geq 1} P_i^{(h)} V_i^{(h)}$$

and the limit has a Fréchet distribution $\Phi_\alpha^{\xi_h}$. By virtue of (4.3) and Theorem 2 in [14] we conclude that $Y_h \stackrel{d}{\to} Y$ for a random variable $Y$ and $a_n^{-1} \widetilde{M}_n^{(\gamma)} \stackrel{d}{\to} Y$. Since $\lim_{h \to \infty} \Phi_\alpha^{\xi_h}$ exists, the limit $\xi = \lim_{h \to \infty} \xi_h$ exists as well and $\Phi_\alpha^\xi$ is the distribution of $Y$. Moreover, $\xi$ is positive since $a_n^{-1} \max_{t=1,\ldots,n} |X_t| \leq a_n^{-1} M_n^{(\gamma)}$ and the sequence on the left-hand side has the limiting distribution $\Phi_\alpha^{\theta_{|X|}}$ with $\theta_{|X|}$ positive by assumption; see [8].

□

4.2. **The increment process is a process with multiplicative noise.** In this section we assume that the strictly stationary real-valued process $(X_t)$ has the form

$$X_t = \sigma_t \, Z_t \,, \quad t \in \mathbb{Z} \,, \tag{4.7}$$

where $(Z_t)$ is an iid sequence and $(\sigma_t)$ is a strictly stationary sequence of non-negative random variables such that $\sigma_t$ and $Z_t$ are independent for every $t$. We mention two popular specifications of the volatility process $(\sigma_t)$.

4.2.1. *The GARCH process.* The perhaps best known process of type (4.7) is a GARCH process introduced in [5]. In this case, the squared volatility process satisfies the recursion

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i \, X_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2 \,, \quad t \in \mathbb{Z} \,. \tag{4.8}$$

Here $\alpha_i$ and $\beta_j$ are non-negative constants. In order to fix the order $(p,q)$ we also assume $\alpha_p \beta_q \neq 0$. In [2] one finds general conditions for the existence of a strictly stationary version of the process $X_t = \sigma_t Z_t$, i.e. one has to ensure that the recursion (4.8) has a strictly stationary causal solution. These conditions require that $\alpha_0 > 0$, $E \log^+ |Z_1| < \infty$ and negativity of the Lyapunov exponent of a certain matrix whose entries depend on the $Z_t^2$'s and the parameters $\alpha_i$ and $\beta_j$; see [2], Theorem 3.1(A), for details and for some simple sufficient conditions for the Lyapunov exponent to be negative. In the same result a sufficient condition for regular variation of $(X_t)$ is given: $Z_1$ has a positive density on $\mathbb{R}$ and there exists $0 < h_0 \leq \infty$ such that $E|Z_1|^h < \infty$ for $h < h_0$ and $E|Z_1|^{h_0} = \infty$. (If $h_0 = \infty$ the latter relation has the interpretation that $E|Z_1|^h \to \infty$ as $h \to \infty$.) It follows from Theorem 3.1(B) in [2] that the sequence $((X_t^2, \sigma_t^2))$ is regularly varying with some index $\alpha/2 > 0$. In [2] the result was proved under the additional condition that $\alpha/2$ is not an even integer. Following the recent approach in [6], this additional condition can be dropped. Then one may follow the proof of Corollary 3.5 in [2] to see that, under the conditions above, $((X_t, \sigma_t))_{t \in \mathbb{Z}}$ is a regularly varying sequence with index $\alpha$. We mention that, in the GARCH(1,1) case with $EZ_1 = 0$ and $EZ_1^2 = 1$, $\alpha$ is the unique positive solution to the equation $E(\alpha_1 Z_1^2 + \beta_1)^{\alpha/2} = 1$. The regular variation of the GARCH process is a consequence of [23] on the tail of solutions to stochastic recurrence equations; see also [16]. Finally, Corollary 3.5 in [2] also gives strong mixing of $((X_t, \sigma_t))_{t \in \mathbb{Z}}$ with geometric rate provided $Z_t$ is symmetric and has a density on $\mathbb{R}$. The latter condition is satisfied in most applications where $Z_t$ is assumed standard normal or $t$-distributed with variance 1. It was mentioned in Remark 4.2 that regular variation and strong mixing with geometric rate for the sequence $(X_t)$ imply the mixing condition $\mathcal{A}(a_n)$. Moreover, the anti-clustering condition (4.1) is also satisfied as proved for Theorem 2.10 in [2]. We summarize as follows.

**Theorem 4.6.** *Let $(X_t)$ be a strictly stationary* GARCH$(p,q)$ *process with an iid symmetric noise sequence $(Z_t)$ such that $EZ_1^2 = 1$. Further assume that $Z_1$ has a positive density on $\mathbb{R}$ and there* *exists $h_0 \le \infty$ such that $E|Z_1|^h < \infty$ for $h < h_0$ and $E|Z_1|^{h_0} = \infty$. Then $(X_t)$ is regularly varying with some index $\alpha > 0$ and the statement of Theorem 4.4 holds for every $\gamma > \max(0, 0.5 - \alpha^{-1})$.*

*Proof.* In view of the discussion preceding the theorem, all conditions of Theorem 4.4 but (4.3) have been verified. Thus we focus on the latter condition. We also notice that the condition $f \in \mathcal{F}_\gamma$ for some $\gamma > \max(0, 0.5 - \alpha^{-1})$ has not been used so far. It will be crucial in the sequel. For notational simplicity we restrict ourselves to the functions $f(l) = l^\gamma$.

We introduce the truncated random variables $X'_t = \sigma_t I_{\{\sigma_t \le h^\gamma a_n\}} Z_t$ (we suppress the dependence on $n$ and $h$ in the notation) and the corresponding partial sums

$$S'_k = \sum_{t=1}^k X'_t, \quad k \ge 1, \qquad S'_0 = 0.$$

Then for any $\delta > 0$,

$$\begin{aligned} &P(\max_{h \le l \le n} l^{-\gamma} \max_{0 \le k \le n-l} |S_{k+l} - S_k| > \delta a_n) \\ &\le P(\max_{k \le n} \sigma_k > h^\gamma a_n) + P(\max_{h \le l \le n} l^{-\gamma} \max_{0 \le k \le n-l} |S'_{k+l} - S'_k| > \delta a_n) \\ &\le P(\max_{k \le n} \sigma_k > h^\gamma a_n) + 2 \sum_{j=1}^{\log_2(n/h)} 2^j T_j, \end{aligned} \tag{4.9}$$

where $\log_2$ denotes logarithm with base 2 and

$$T_j = P(\max_{1 \le k \le 2n2^{-j}} |S'_k| > \delta (n2^{-j})^\gamma a_n). \tag{4.10}$$

In the last step we used Lemma 3.3 in [27]. There the proof was given for an iid sequence $(X_t)$ but the proof remains the same for a strictly stationary sequence.

It is well known that $a_n^{-1} \max_{k \le n} \sigma_k$ converges in distribution to a Fréchet $\Phi_\alpha^{\theta_\sigma}$ distributed random variable, where $\theta_\sigma > 0$ is the extremal index of the sequence $(\sigma_t)$. This fact follows e.g. from the point process convergence results in [2]. Hence

$$\lim_{h \to \infty} \lim_{n \to \infty} P(a_n^{-1} \max_{k \le n} \sigma_k > h^\gamma) = 0. \tag{4.11}$$

Since $Z_t$ is assumed symmetric an application of Lévy's maximal inequality (e.g. [30], Theorem 2.2) conditional on $(\sigma_t |Z_t|)$ yields

$$T_j \le 2P(|S'_{2N}| > \delta N^\gamma a_n), \qquad \text{where } N = [n2^{-j}]. \tag{4.12}$$

Assume $\alpha < 2$. Then by Markov's inequality and Karamata's theorem (see [4]),

$$\begin{aligned} T_j &\le c\, N^{1-2\gamma} a_n^{-2} E[Z_1^2 \sigma_1^2 I_{\{\sigma_1 \le h^\gamma a_n\}}] \sim c\, h^{2\gamma} N^{1-2\gamma} P(\sigma_1 > h^\gamma a_n) \\ &\le c h^{\gamma(2-\alpha)} n^{-1} N^{1-2\gamma}. \end{aligned}$$

Hence

$$\sum_{j=1}^{\log_2(n/h)} 2^j T_j \le c\, n^{-2\gamma} h^{\gamma(2-\alpha)} \sum_{j=1}^{\log_2(n/h)} 2^{2\gamma j} \le c\, h^{-\gamma\alpha}. \tag{4.13}$$

In sum, we conclude from (4.9)–(4.13) that (4.3) holds for every $\delta > 0$, $f(l) = l^\gamma$ and $\gamma > 0$. Now assume $\alpha \ge 2$. It follows from the assumptions that $E|Z_1|^p < \infty$ for some $p > \alpha$. Then Markov's and Burkholder's inequalities and Karamata's theorem yield

$$\begin{aligned} P(|S'_{2N}| > \delta N^\gamma a_n) &\le c\, (N^\gamma a_n)^{-p} E|S'_{2N}|^p \\ &\le c\, (N^\gamma a_n)^{-p} N^{p/2} E\sigma_1^p I_{\{\sigma_1 \le h^\gamma a_n\}} \end{aligned}$$

$$\begin{aligned} &\sim \quad cN^{p(0.5-\gamma)} h^{\gamma p} P(\sigma_1 > h^{\gamma} a_n) \\ &\sim \quad cN^{p(0.5-\gamma)} h^{\gamma (p-\alpha)} n^{-1}\,. \end{aligned} \tag{4.14}$$

Since $\gamma > 0.5 - \alpha^{-1}$,

$$\sum_{j=1}^{\log_2(n/h)} 2^j T_j \le c h^{\gamma(p-\alpha)} n^{-1+p(0.5-\gamma)} \sum_{j=1}^{\log_2(n/h)} 2^{j(1-p(0.5-\gamma))} \le c\, h^{-\alpha\gamma - 1 + 0.5p}$$

Now choose $p > \alpha$ so close to $\alpha$ that $-\alpha\gamma - 1 + 0.5p < 0$. Then (4.3) follows. □

4.2.2. *The stochastic volatility model.* Another model of the type (4.7) has attracted some attention in the financial time series literature: the stochastic volatility model. In this case, $(\sigma_t)$ is a strictly stationary sequence independent of the iid sequence $(Z_t)$. If $Z$ is regularly varying with index $\alpha > 0$ and $E\sigma^p < \infty$ for some $p > \alpha$ then $(X_t)$ is regularly varying with index $\alpha$. In this case, the limiting measures of the regularly varying sequence, i.e. the measures appearing as vague limits of $nP(a_n^{-1}(X_1, \ldots, X_h) \in \cdot)$, are concentrated on the axes; see [10]. For this reason, the stochastic volatility model $(X_t)$ has very much the same extremal behavior as an iid sequence with the marginal distribution as $X_1$. In particular, the extremal index of $(X_t)$ is one and the following point process convergence result holds: if the tail balance condition (1.3) holds and $(\log \sigma_t)$ is a linear Gaussian process, then

$$\sum_{t=1}^{n} \varepsilon_{a_n^{-1}(X_t, \ldots, X_{t+h-1})} \stackrel{d}{\to} \sum_{k=1}^{h} \sum_{i=1}^{\infty} \varepsilon_{J_i \mathbf{e}_k}\,, \tag{4.15}$$

where $\mathbf{e}_k$ is the $k$th unit vector in $\mathbb{R}^h$, $(J_i)$ are the points of a Poisson process on $\overline{\mathbb{R}}_0$ with intensity $\alpha\,|x|^{-\alpha-1}[p\, I_{(0,\infty)}(x) + q\, I_{(-\infty,0)}(x)]\, dx$ and $(a_n)$ is chosen such that $n\,P(|X_1| > a_n) \to 1$. In the case of an iid regularly varying sequence $(X_t)$ with tail balance condition (1.3) (with $Z$ replaced by $X$) relation (4.15) holds as well.

Now we are ready to formulate the following analog of Theorem 4.6.

**Theorem 4.7.** *Let $(X_t)$ be a stochastic volatility process. Assume that $Z_1$ is symmetric and has a regularly varying distribution with index $\alpha$ and $(\log \sigma_t)$ is a Gaussian linear process. Then for $f \in \mathcal{F}_\gamma$ for $\gamma > \max(0, 0.5 - \alpha^{-1})$,*

$$\lim_{n\to\infty} P(a_n^{-1} \max_{1\le l \le n} (f(l))^{-1} \max_{1\le k\le n} |S_{k+l} - S_k| \le x) = \Phi_\alpha(x)\,, \quad x > 0\,.$$

*Proof.* The point process convergence (4.15) was the starting point in [27] for proving that for any $f \in \mathcal{F}_\gamma$ and $\gamma > 0$,

$$\lim_{n\to\infty} P(a_n^{-1} \max_{1\le l \le h} (f(l))^{-1} \max_{1\le k\le n} |S_{k+l} - S_k| \le x) = \Phi_\alpha(x)\,, \quad x > 0\,.$$

The same arguments apply in the stochastic volatility case, i.e. the relation above remains valid. Now consider the truncated random variables $X_t' = \sigma_t Z_t I_{\{|Z_t| \le h^\gamma a_n\}}$ and denote their partial sums by $S_k'$, $k \ge 0$. Then the following analog of (4.9) holds:

$$\begin{aligned} &P(\max_{h\le l\le n} l^{-\gamma} \max_{0\le k\le n-l} |S_{k+l} - S_k| > \delta a_n) \\ &\le \quad P(\max_{k\le n} |Z_k| > h^\gamma a_n) + P(\max_{h\le l\le n} l^{-\gamma} \max_{0\le k\le n-l} |S'_{k+l} - S'_k| > \delta a_n) \\ &\le \quad P(\max_{k\le n} |Z_k| > h^\gamma a_n) + 2 \sum_{j=1}^{\log_2(n/h)} 2^j T_j\,, \end{aligned}$$

where $T_j$ is defined in (4.10). Since $a_n^{-1}\max_{k=1,\ldots,n}|Z_t|$ converges in distribution to a Fréchet distributed random variable,

$$\lim_{h\to\infty}\limsup_{n\to\infty} P(\max_{k\le n}|Z_k|\ge h^\gamma a_n)=0\,.$$

Now assume that $Z_1$ is also symmetric. Then, by Lévy's maximal inequality conditional on $(\sigma_t)$, relation (4.12) follows. We consider the case $\alpha<2$ first. An application of Markov's inequality and the conditional independence of the $X_t$'s yield

$$\begin{aligned}T_j &\le c\,N^{1-2\gamma}a_n^{-2}E[\sigma_1^2Z_1^2I_{\{|Z_1|\le h^\gamma a_n\}}]\sim c\,h^{2\gamma}N^{1-2\gamma}P(|Z_1|>h^\gamma a_n)\\ &\le ch^{\gamma(2-\alpha)}n^{-1}N^{1-2\gamma},\end{aligned}$$

Now one can follow the lines of the proof of Theorem 4.6. Next we consider the case $\alpha\ge 2$. Then for $p>\alpha$ the Markov and Marzinkiewicz-Zygmund inequalities conditional on $(\sigma_t)$ and the Minkowski inequality yield

$$\begin{aligned}E[P(|S'_{2N}|>\delta N^\gamma a_n\mid(\sigma_t))] &\le c\,a_n^{-p}N^{-p(\gamma-0.5)}E|Z_1|^pI_{\{|Z_1|\le h^\gamma a_n\}}E\Big(N^{-1}\sum_{t=1}^{2N}\sigma_t^2\Big)^{p/2}\\ &\le c\,a_n^{-p}N^{-p(\gamma-0.5)}E|Z_1|^pI_{\{|Z_1|\le h^\gamma a_n\}}E\sigma_1^p\,.\end{aligned}$$

Now we may follow the lines of the proof of Theorem 4.6 for $\alpha\ge 2$. □

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