

# Outcrossings of safe regions by generalized hyperbolic processes

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## Abstract

We present a simple Gaussian mixture model in space and time with generalized hyperbolic marginals. Starting with Rice's celebrated formula for level upcrossings and outcrossings of safe regions we investigate the consequences of the mean-variance mixture model on such quantities. We obtain explicit formulas and also present some explicit examples and limit results.

*Keywords:* crossings, extremal theory, generalized hyperbolic process, reliability

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## 1. Introduction

Level crossings and extreme values of physical processes are of considerable interest in engineering sciences dealing with structural reliability. If  $X = (X(t))_{t \geq 0}$  is a stationary stochastic process and  $S$  is a safe region, properties of the outcrossings of  $S$  by  $X$ , such as their expected number per unit time or the distribution of their size, are of prime interest. Such quantities have been derived in particular for Gaussian processes, but also for functions of Gaussian processes like lognormal processes (Desmond and Guy (1991)) or  $\chi^2$  processes (Lindgren (1980b)). Our goal is to prove analogous results for Gaussian variance-mean mixture models, in particular for a class of generalized hyperbolic processes.

The one-dimensional generalized hyperbolic distribution, introduced in 1977 by Barndorff-Nielsen (Barndorff-Nielsen (1977)), is a normal variance-mean mixture distribution of a very general form, including e.g. the Student's  $t$ -, the Cauchy, the variance Gamma and the skewed Laplacian distributions, and the normal and the generalized inverse Gaussians occur as limiting distributions for certain parameters tending to appropriate values. Details about their properties have been nicely summarized in (Prause, 1999, Chapter 1). All of these distributions have been applied to statistical and probabilistic modelling; for applications in finance and turbulence see e.g. Barndorff-Nielsen et al. (2004) and Prause (1999).

The extension from Gaussian to Gaussian mixture models is natural in the context of the above applications, where Gaussian mixture models have been prominent when modelling time series data. It is known that Gaussian mixture models are able to capture observed heavier tails in data, and allow for dependence modelling on high levels. In the present paper we study the effect of a simple Gaussian mixture model in space and

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time for level crossings and outcrossings of safe regions and compare with Gaussian space-time models.

Our paper is organised as follows. In Section 2 we state the definitions of the ingredients and of the generalized hyperbolic process. We also state some properties of the process, define outcrossing and recall Rice's formula for the expected number of outcrossings of a stochastic process of arbitrary dimension of a safe region. Section 3 contains the main results of our paper, presenting closed form expressions for Rice's formula for a generalized hyperbolic process in arbitrary dimension and in dimension 1. Section 4 presents the limiting Poisson process for outcrossings of a cylinder by the time and space normalized generalized hyperbolic process. In Section 5 we calculate the Palm density of  $X'$  at level crossings of  $X$ . We conclude with some explicit examples and limiting results.

## 2. Definitions and basic facts

To define the generalized hyperbolic distribution as a normal mixture, we need to introduce the generalized inverse Gaussian:

**Definition 1** (Generalized inverse Gaussian). The *generalized inverse Gaussian* distribution ( $GIG(\lambda, \chi, \psi)$ ) is given by the Lebesgue density

$$f_W(x) := \text{gig}(x, \lambda, \chi, \psi) = \frac{(\psi/\chi)^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{\psi\chi})} x^{\lambda-1} \exp\left(-\frac{1}{2}(\chi x^{-1} + \psi x)\right), \quad x > 0$$

where  $\lambda \in \mathbb{R}$ ,  $\psi, \chi \in \mathbb{R}_+$  and  $K_\lambda$  is the modified Bessel function of the third kind given by

$$K_\lambda(\omega) = \frac{1}{2} \int_0^\infty x^{\lambda-1} \exp\left(-\frac{1}{2}\omega(x + x^{-1})\right) dx, \quad x > 0.$$

For later reference we summarize some properties of the GIG distributions. Further properties of the GIG can be found in Jørgensen (1982).

We shall use the following notation throughout: For functions  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  we write  $f(x) \sim g(x)$  as  $x \rightarrow \infty$  if  $f(x)/g(x) \rightarrow 1$  as  $x \rightarrow \infty$ .

**Remark 2.** (i) The  $GIG(\lambda, \chi, \psi)$  is positively skewed for any choices of  $\lambda, \chi$  and  $\psi$ , see Nguyen et al. (2003).

(ii)  $K_\lambda(x) \sim \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}} e^{-x}$  as  $x \rightarrow \infty$  (with identity for all  $x > 0$  if  $\lambda = \frac{1}{2}$ ). This is easily seen by using standard properties of  $K_\lambda$  that can be found in e.g. (Prause, 1999, Appendix B).

(iii) The GIG distributions include the inverse Gaussian distributions ( $\lambda = -\frac{1}{2}$ ), the reciprocal inverse Gaussian distributions ( $\lambda = \frac{1}{2}$ ), the Gamma distributions ( $\Gamma(\lambda, \psi/2)$ ,  $\lambda > 0, \chi = 0$ ) and a subclass of the the inverse Gamma distributions ( $\Gamma^{-1}(\nu, \frac{1}{2})$ ,  $\chi = \nu, \lambda = -\nu/2$ ). More properties of the Gamma and inverse Gamma distributions can be found for example in Johnson et al. (1994).

We now define the generalized hyperbolic distribution as a mean-variance Gaussian mixture:

**Definition 3** (Generalized hyperbolic distribution). The  $d$ -dimensional ( $d \geq 1$ ) random variable  $X$  is *generalized hyperbolically* ( $GH(\mu, \beta, \lambda, \chi, \psi)$ ) distributed, if

$$X \sim \mu + \beta W + \sqrt{W} A Y \tag{1}$$

where  $Y \sim N_k(0, I_k)$  (a  $k$ -dimensional standard normal vector),  $\mu, \beta \in \mathbb{R}^d$ ,  $A \in \mathbb{R}^{d \times k}$  satisfying  $|AA^T| = 1$ . Moreover,  $W \sim GIG(\lambda, \chi, \psi)$  with  $Y$  and  $W$  independent.

Clearly,  $X|W = w \sim N_d(\mu + \beta w, wAA^T)$ . If  $k = d$  and  $A = I_d$ , then  $\sqrt{W}Y$  is rotationally invariant,  $\mu$  is just a location parameter, and  $\beta$  introduces some asymmetry into the model.

A simple stationary GH process can now be defined by replacing the Gaussian vector by a Gaussian process with appropriate covariance function and sample path properties (Alodat and Aludaat (2008); cf. Lindgren (1980b) for the assumptions on the Gaussian process).

**Definition 4** (GH process). Let  $Y = (Y(t))_{t \geq 0}$  be a  $d$ -dimensional Gaussian process with independent, stationary, mean-zero and unit-variance components, each having continuously differentiable sample paths. Assume further that each component  $Y_n$  of  $Y = (Y_1, \dots, Y_d)$  has covariance function  $r_n(t) = C(Y_n(s+t), Y_n(s))$  admitting the expansion  $r_n(t) = 1 - \omega_{2,n}t^2/2 + o(t^2)$ , as  $t \rightarrow \infty$ . Here  $\omega_{2,n}$  is the second spectral moment  $\text{Var}(Y_n'(t)) = -r_n''(0)$  of the  $n$ 'th Gaussian component  $Y_n$ . Let  $W$  be an independent  $GIG(\lambda, \chi, \psi)$  distributed random variable. The stochastic process

$$X(t) = \mu + \beta W + \sqrt{W}AY(t), \quad t \geq 0, \quad (2)$$

is then called GH process with parameters  $(\mu, \beta, \lambda, \chi, \psi)$ .

Before turning to the main results of our paper, we first want to summarize some properties of the GH process.

**Remark 5.** (i)  $X$  has conditionally (on  $W$ ) independent Gaussian components.  
(ii)  $X$  has GH (dependent) components with positive tail dependence coefficient, if  $\psi = 0$ , cf. Hult and Lindskog (2002).  
(iii) The sample paths of each component are continuously differentiable, so questions about *local extrema* (i.e. points where the derivative changes sign) make sense. In particular, if  $d = 1$ , then the expected number of local extrema remains for  $X$  the same as for the Gaussian process  $Y$ . This means that if  $Y$  has finite fourth spectral moment  $\omega_4$ , the expected number of local maxima of  $X$  in a unit time interval is equal to  $\frac{1}{2\pi} \sqrt{\frac{\omega_4}{\omega_2}}$ .

**Definition 6** (Outcrossing). Let  $X = (X(t))_{t \geq 0}$  be a  $d$ -dimensional process where each component has continuously differentiable sample paths and let  $S$  be a measurable set in  $\mathbb{R}^d$  with smooth boundary  $\partial S$ . Then we define the following.

- (a)  $X$  has an *outcrossing of  $S$*  at time  $t_0$  if  $X(t_0) \in \partial S$  and for some  $\varepsilon > 0$  and all  $0 < \tau < \varepsilon$ ,  $X(t_0 - \tau) \in S \setminus \partial S$  and  $X(t_0 + \tau) \notin S \cup \partial S$ .
- (b)  $N(X, S)$  denotes the *number of outcrossings of  $S$  by  $X$  in the unit interval  $(0, 1]$* . Moreover, we denote by  $\mu_+(S) = \mathbb{E}(N(X, S))$  the *expected number of outcrossings in a unit interval*.
- (c)  $N(X, S, U)$ , where  $U \subset \partial S$  is a subset of the boundary of  $S$ , denotes the *number of outcrossings of  $S$  by  $X$  through  $U$  in the unit interval  $(0, 1]$* .

We now formulate Rice's famous formula for the expected number of outcrossings. It was originally proved for  $d = 1$  by Rice (Rice (1944, 1945)). The proof in its present form for arbitrary dimension goes back to Belyaev (1968), the assumptions have been weakened in Lindgren (1980a). For textbook discussions, see Leadbetter et al. (1983); Lindgren (2012).

Here and in the following, we use the notation  $x^+ = \max(0, x)$ .

**Theorem 7** (Rice's formula for multivariate processes). *Let  $X$  and  $S$  be as in Definition 6 and  $ds(x)$  denote the Hausdorff measure of  $\partial S$ . Then*

$$\mu_+(S) = \int_{\partial S} \mathbb{E}(\langle v_x, X'(0) \rangle^+ | X(0) = x) f_{X(0)}(x) ds(x),$$

where  $v_x$  is an outgoing normal vector of  $\partial S$ .

We now introduce some notation that will be used throughout the paper. For any  $d \geq 1$ ,  $u \in \mathbb{R}_+$  and  $\mu \in \mathbb{R}^d$ , we denote by  $B_u(\mu) = \{x \in \mathbb{R}^d | \|x - \mu\| < u\}$  the open  $u$ -ball around  $\mu$  with radius  $u$ . Likewise for every  $d$ , let  $\mathcal{S}^{d-1} = \{x \in \mathbb{R}^d | \|x\| = 1\}$  denote the sphere in  $\mathbb{R}^d$ . If  $u \in \mathbb{R}$  and  $S \subset \mathbb{R}^d$  is a set, we define their product  $uS := \{x \in \mathbb{R}^d | x = uy, y \in S\}$ .

**Remark 8.** We shall calculate mean outcrossings for the GH process  $X$  with parameters  $(0, \beta, \lambda, \chi, \psi)$  from Definition 4. Note that if  $\tilde{\mu}_+(B_u(\mu)) := \mathbb{E}(\tilde{X}, B_u(\mu))$  where  $\tilde{X} = \mu + X$  is a GH process with parameters  $(\mu, \beta, \lambda, \chi, \psi)$ , then  $\tilde{\mu}_+(B_u(\mu)) = \mu + \mu_+(B_u(0))$ . Hence, it suffices to consider  $\mu = 0$ .

### 3. Expected number of outcrossings

The following result gives the expected number of outcrossings of a  $u$ -ball around 0 by the rotationally invariant and centered GH process.

**Theorem 9.** *Let  $X = (X_1, \dots, X_d)$  denote the GH process from Definition 4 with  $\mu = 0$  and  $\beta = 0$ . Then the mean number of outcrossings per unit time of  $B_u(0)$  by the process  $X$  is given by*

$$\mu_+(B_u(0)) = C_d u^{d-1} \frac{K_{\lambda + \frac{1-d}{2}}(\sqrt{(u^2 + \chi)\psi})}{K_\lambda(\sqrt{\chi\psi})} \frac{(\sqrt{u^2 + \chi})^{\lambda + \frac{1-d}{2}}}{\chi^{\frac{\lambda}{2}} \psi^{\frac{1-d}{4}}}, \quad (3)$$

where

$$C_d = (2\pi)^{-\frac{1+d}{2}} \int_{\mathcal{S}^{d-1}} \left( \sum_{n=1}^d z_n^2 \omega_{2,n} \right)^{\frac{1}{2}} ds(z). \quad (4)$$

Here  $\omega_{2,n}$  are the second spectral moments of  $X_n$  for  $n = 1, \dots, d$ . If  $\omega_{2,n} = \omega_2$  for all  $n$ , then  $C_d$  simplifies to  $C_d = \sqrt{\frac{\omega_2}{\pi}} \frac{2^{\frac{1-d}{2}}}{\Gamma(\frac{d}{2})}$ .

*Proof.* We iterate the expectation by conditioning on  $W$

$$\begin{aligned}
& \mu_+(B_u(0)) \\
&= \int_{uS^{d-1}} \mathbb{E}(\langle v_x, X'(t) \rangle^+ | X(0) = x) f_{X(0)}(x) ds(x) \\
&= \int_{uS^{d-1}} \int_0^\infty \mathbb{E}(\langle v_x, X'(t) \rangle^+ | X(0) = x, W = w) f_{X(0) | W=w}(x) f_W(w) dw ds(x) \\
&= \int_0^\infty \int_{uS^{d-1}} \mathbb{E}(\langle v_x, X'(t) \rangle^+ | W = w) f_{X(0) | W=w}(x) ds(x) f_W(w) dw \\
&= \int_0^\infty \int_{uS^{d-1}} \frac{\sqrt{w}}{u\sqrt{2\pi}} \left( \sum_{n=1}^d x_n^2 \omega_{2,n} \right)^{\frac{1}{2}} (2\pi)^{-\frac{d}{2}} w^{-\frac{d}{2}} e^{-\frac{u^2}{2w}} ds(x) f_W(w) dw \\
&= \int_0^\infty \frac{w^{\frac{1-d}{2}} u^{d-1} e^{-\frac{u^2}{2w}}}{(2\pi)^{\frac{1+d}{2}}} \int_{S^{d-1}} \left( \sum_{n=1}^d z_n^2 \omega_{2,n} \right)^{\frac{1}{2}} ds(z) f_W(w) dw \\
&= \frac{u^{d-1}}{(2\pi)^{\frac{1+d}{2}}} \int_{S^{d-1}} \left( \sum_{n=1}^d z_n^2 \omega_{2,n} \right)^{\frac{1}{2}} ds(z) \int_0^\infty w^{\frac{1-d}{2}} e^{-\frac{u^2}{2w}} f_W(w) dw.
\end{aligned}$$

Now the factors in front of the last integral is just  $u^{d-1}$  times a constant which we recognize as  $C_d$  from the formulation of the theorem. We can now write

$$\begin{aligned}
& C_d u^{d-1} \int_0^\infty w^{\frac{1-d}{2}} e^{-\frac{u^2}{2w}} f_W(w) dw \\
&= C_d u^{d-1} \frac{\chi^{-\lambda} (\sqrt{\chi\psi})^\lambda}{2K_\lambda(\sqrt{\chi\psi})} \int_0^\infty w^{(\lambda+\frac{1-d}{2})-1} \exp\left(-\frac{1}{2}((u^2 + \chi)w^{-1} + \psi w)\right) dw \\
&= C_d u^{d-1} \frac{\chi^{-\lambda} (\sqrt{\chi\psi})^\lambda}{2K_\lambda(\sqrt{\chi\psi})} \frac{2K_{\lambda+\frac{1-d}{2}}(\sqrt{(\chi+u^2)\psi})}{(\chi+u^2)^{-(\lambda+\frac{1-d}{2})} (\sqrt{(\chi+u^2)\psi})^{\lambda+\frac{1-d}{2}}} \\
&= C_d u^{d-1} \frac{K_{\lambda+\frac{1-d}{2}}(\sqrt{(\chi+u^2)\psi})}{K_\lambda(\sqrt{\chi\psi})} \frac{(\sqrt{\chi+u^2})^{\lambda+\frac{1-d}{2}}}{\chi^{\frac{\lambda}{2}} \psi^{\frac{1-d}{4}}},
\end{aligned}$$

yielding (3) with  $C_d$  as in (4). If  $\omega_{2,n} = \omega_2$  for all  $n$ , then  $C_d$  reduces to:

$$C_d := (2\pi)^{-\frac{1+d}{2}} \int_{S^{d-1}} \left( \omega_2 \sum_{n=1}^d z_n^2 \right)^{\frac{1}{2}} ds(z) = \left( \frac{\omega_2}{\pi} \right)^{\frac{1}{2}} \frac{2^{\frac{1-d}{2}}}{\Gamma(\frac{d}{2})}$$

by the formula for the area of the  $(d-1)$ -dimensional sphere.  $\square$

Let  $u > 0$  and  $S \subset S^{d-1}$  be a measurable subset of  $S^{d-1}$ , such that  $uS$  is a subset of the boundary of  $B_u(0)$ . Then, following Lindgren (1980b), the expected number  $\mathbb{E}(N(X, S, U))$  of outcrossings of  $B_u(0)$  through  $uS$  can easily be found by inspecting the above result and redefining  $C_d$  in the obvious way. We state this as a corollary.

**Corollary 10.** *Let  $X = (X_1, \dots, X_d)$  denote the GH process from Definition 4 with  $\mu = 0$  and  $\beta = 0$  and let  $S \subset S^{d-1}$  be a measurable subset. Then the mean number of outcrossings of the centered  $u$ -ball  $B_u(0)$  through  $uS$  per unit time by the process  $X$  is given by*

$$\mu_+(B_u(0), uS) = C_{d,S} u^{d-1} \frac{K_{\lambda+\frac{1-d}{2}}(\sqrt{(u^2 + \chi)\psi})}{K_\lambda(\sqrt{\chi\psi})} \frac{(\sqrt{u^2 + \chi})^{\lambda+\frac{1-d}{2}}}{\chi^{\frac{\lambda}{2}} \psi^{\frac{1-d}{4}}},$$

where

$$C_{d,S} = (2\pi)^{-\frac{1+d}{2}} \int_S \left( \sum_{n=1}^d z_n^2 \omega_{2,n} \right)^{\frac{1}{2}} \mathbf{d}s(z).$$

If  $\omega_{2,n} = \omega_2$  for all  $n$ , then  $C_{d,S}$  simplifies to  $C_{d,S} = \sqrt{\frac{\omega_2}{(2\pi)^{1+d}}} \Lambda_{d-1}(S)$  where  $\Lambda_{d-1}$  denotes the  $(d-1)$ -dimensional Hausdorff measure on the  $(d-1)$ -dimensional sphere  $\mathcal{S}^{d-1}$ .

In the univariate case, we can even calculate outcrossings for the full model (2). The  $u$ -ball around  $\mu$  reduces to  $(\mu - u, \mu + u) = \mu + (-u, u)$ . So we set  $\mu = 0$  in (2) and calculate the outcrossings of  $B_u(0) = (-u, u)$ .

**Theorem 11.** *Let  $X$  be the GH process from Definition 4 for  $d = 1$  and  $\mu = 0$ . Then the mean number of outcrossings of  $(-u, u)$  by  $X$  is:*

$$\mu_+((-u, u)) = (e^{u\beta} + e^{-u\beta}) \frac{\sqrt{\omega_2} K_\lambda(\sqrt{(\chi + u^2)(\psi + \beta^2)})}{2\pi K_\lambda(\sqrt{\chi\psi})} \left( \frac{\psi(\chi + u^2)}{(\psi + \beta^2)\chi} \right)^{\frac{\lambda}{2}},$$

where  $K_\lambda$  is the modified Bessel function of the third kind.

*Proof.* We start again with Rice's formula:

$$\mu_+(B_u(0)) = \int_{uS^0} \mathbb{E}(\langle v_x, X'(t) \rangle^+ | X(0) = x) f_{X(0)}(x) \mathbf{d}s(x).$$

Now  $uS^0$  is just  $\{-u, u\}$ ,  $\mathbf{d}s(x)$  is the counting measure and  $v_{-u}$  and  $v_u$  are just  $-1$  and  $1$ , respectively. This means that we only need to compute

$$\mathbb{E}((\pm X'(t))^+ | X(0) = \pm u) f_{X(0)}(\pm u)$$

Conditioning on  $W = w$  and using that in the Gaussian model  $Y$  and  $Y'$  are independent gives

$$\begin{aligned} & \int_0^\infty \mathbb{E}((\pm X'(t))^+ | X(0) = \pm u, W = w) f_{X(0) | W=w}(\pm u) f_W(w) \mathbf{d}w \\ &= \int_0^\infty \mathbb{E}((\pm X'(t))^+ | W = w) \frac{1}{\sqrt{2\pi w}} e^{-\frac{(\pm u - \beta u)^2}{2w}} f_W(w) \mathbf{d}w \\ &= e^{\pm u\beta} \int_0^\infty \sqrt{\frac{w\omega_2}{2\pi}} \frac{1}{\sqrt{2\pi w}} e^{-\frac{u^2}{2w} - \frac{\beta^2 w}{2}} f(w) \mathbf{d}w \\ &= e^{\pm u\beta} \frac{\sqrt{\omega_2}}{2\pi} \int_0^\infty \frac{\chi^{-\lambda} (\sqrt{\chi\psi})^\lambda}{2K_\lambda(\sqrt{\chi\psi})} w^{\lambda-1} \exp\left(-\frac{1}{2}((\chi + u^2)w^{-1} + (\psi + \beta^2)w)\right) \mathbf{d}w \\ &= e^{\pm u\beta} \frac{\sqrt{\omega_2}}{2\pi} \left( \frac{\psi(\chi + u^2)}{\chi(\psi + \beta^2)} \right)^{\frac{\lambda}{2}} \frac{K_\lambda(\sqrt{(\chi + u^2)(\psi + \beta^2)})}{K_\lambda(\sqrt{\chi\psi})}, \end{aligned}$$

and the result follows by adding the “+” and the “-” term.  $\square$

For 1-dimensional processes, the “one-sided outcrossing” of the set  $(-\infty, u)$  (or “upcrossing” of the level  $u$ ) is often of interest. The result for this case now clearly follows as an easy corollary of the proof of Theorem 11.

**Corollary 12.** *Let  $X$  be the GH process from Definition 4 for  $d = 1$  with  $\mu = 0$ . Then the mean number of outcrossings of the set  $(-\infty, u)$  by  $X$  is:*

$$\mu_+(u) := \mu_+((-\infty, u)) = e^{u\beta} \frac{\sqrt{\omega_2}}{2\pi} \frac{K_\lambda(\sqrt{(\chi + u^2)(\psi + \beta^2)})}{K_\lambda(\sqrt{\chi\psi})} \left( \frac{\psi(\chi + u^2)}{(\psi + \beta^2)\chi} \right)^{\frac{\lambda}{2}},$$

where  $K_\lambda$  is the modified Bessel function of the third kind.

**Remark 13.** (i) Corollary 12 corrects Theorem 1 of Alodat and Aludaat (2008).  
(ii) Outcrossings of the set  $(-u, u)$  are now related to Corollary 12 by observing that:

$$\mu_+((-u, u)) = \mu_+((-\infty, u)) + \hat{\mu}_+((-\infty, u)),$$

where  $\hat{\mu}_+((-\infty, u)) = \mathbb{E}(N(-X, (-\infty, u))) = \mathbb{E}(N(X, (-u, \infty)))$ .

#### 4. Asymptotic Poisson character of outcrossings

As  $X$  is assumed to be stationary, the mean number of outcrossings in any time interval of length  $T$  is just  $T\mu_+(B_0(u))$  with  $\mu_+(B_0(u))$  given by Theorem 9. Using this and the result for the Gaussian case (Lindgren (1980b)), we now prove an asymptotic result on the probability  $P(X(t) \in B_0(u))$  for all  $t \in [0, T]$  as  $T \rightarrow \infty$ .

**Theorem 14.** *Let  $X$  be the GH process from Definition 4 with  $\mu = \beta = 0$ . Assume that additionally the following holds for the covariance function of the Gaussian components of the process  $Y$ : for  $n = 1, \dots, d$ ,*

$$r_n(t) \log(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

Denote by  $N_u$  the point process of outcrossings of the cylinder  $(0, 1] \times \mathcal{S}^{d-1}$  by the time- and space-normalized process  $(t, u^{-1}X(Tt))_{t \in (0, 1]}$ . Now assume that  $T, u \rightarrow \infty$  such that for some  $\tau > 0$

$$T \sim \sqrt{\frac{2}{\pi}} \frac{\tau}{C_d} K_\lambda(\sqrt{\chi\psi}) \chi^{\frac{\lambda}{2}} \psi^{\frac{2-d}{4}} u^{1-\lambda-\frac{d}{2}} e^{u\sqrt{\psi}}. \quad (5)$$

Then  $N_u \xrightarrow{d} N$ , where  $N$  is a Poisson point process on  $(0, 1] \times \mathcal{S}^{d-1}$  with intensity  $\tau dt \times (\sum \omega_{2,n} x_n^2)^{\frac{1}{2}} ds(x)$ . Here  $dt$  is Lebesgue measure and  $ds(x)$  is Hausdorff measure normalized such that the total mass of the product measure is equal to  $\tau$ .

*Proof.* We begin by noting that if  $u, T \rightarrow \infty$  in such a way that (5) holds, then the expected number of outcrossings  $\mathbb{E}(N_u) \rightarrow \tau$ :

$$\begin{aligned} \mathbb{E}(N_u) &= T\mu_+(B_u(0)) \\ &= TC_d u^{d-1} \frac{K_{\lambda+\frac{1-d}{2}}(\sqrt{(u^2 + \chi)\psi})}{K_\lambda(\sqrt{\chi\psi})} \frac{(\sqrt{u^2 + \chi})^{\lambda+\frac{1-d}{2}}}{\chi^{\frac{\lambda}{2}} \psi^{\frac{1-d}{4}}} \\ &\sim \sqrt{\frac{2}{\pi}} \tau u^{\frac{d}{2}-\lambda} K_{\lambda+\frac{1-d}{2}}(\sqrt{(u^2 + \chi)\psi}) \psi^{\frac{1}{4}} (\sqrt{u^2 + \chi})^{\lambda+\frac{1-d}{2}} e^{u\sqrt{\psi}} \\ &\sim \tau \sqrt{\frac{2}{\pi}} u^{\frac{1}{2}} K_{\lambda+\frac{1-d}{2}}\left(u\sqrt{\left(1 + \frac{\chi}{u^2}\right)\psi}\right) \psi^{\frac{1}{4}} e^{u\sqrt{\psi}} \\ &\rightarrow \tau, \end{aligned}$$

recalling from Remark 2 that  $K_\lambda(x) \sim \sqrt{\frac{\pi}{2}}x^{-\frac{1}{2}}e^{-x}$  as  $x \rightarrow \infty$ . Let  $N_u(U)$  denote the number of outcrossings of the cylinder  $C =: (0, 1] \times \mathcal{S}^{d-1}$  through  $U \subset C$  by  $(t, u^{-1}X(Tt))$ , i.e.

$$N_u(U) = N((t, u^{-1}X(Tt), C, U),$$

cf. Definition 6. Let  $N(U)$  denote the Poisson point process  $N$  on  $U$ . Then by (Kallenberg, 1976, Theorem 4.7), it is enough to show that  $\mathbb{E}(N_u(U)) \rightarrow \mathbb{E}(N(U))$  and  $P(N_u(U) = 0) \rightarrow P(N(U) = 0)$  for any measurable subsets  $U$  of the cylinder. That  $\mathbb{E}(N_u(U)) \rightarrow \mathbb{E}(N(U))$  is trivial by construction, cf. Corollary 10. We now prove that  $P(N_u(U) = 0) \rightarrow P(N(U) = 0)$ . Since by Lindgren (1980b) this is already known for Gaussian processes, it follows by using the law of total probability on  $P(N_u(U))$  by conditioning on  $W = w$  and interchanging the order of limits:

$$\begin{aligned} P(N_u(U) = 0) &= \int_0^\infty P(N_u(U) = 0 | W = w) f_W(w) dw \\ &\rightarrow \int_0^\infty P(N(U) = 0 | W = w) f_W(w) dw \\ &= P(N(U) = 0), \end{aligned}$$

by dominated convergence as  $T, u \rightarrow \infty$  at the given rate.  $\square$

## 5. The Palm distribution

Rice's formula has been extended to include outcrossings of  $X$ , which are marked by the value of another stationary process, the derivative process  $X'$  being one of the possibilities. In the 1-dimensional Gaussian framework,  $Y(t)$  and  $Y'(t)$  are independent for fixed  $t$  (see e.g. Section 8.1.4 of Lindgren (2012)), and the Palm distribution of  $Y'$  at outcrossings of  $(-\infty, u)$  is just the Rayleigh distribution with density  $g_{Y'}(y) = \omega_2^{-1}ye^{-y^2/2\omega_2}$  for  $y > 0$ ; see e.g. Leadbetter and Spaniolo (2004).

Denote by  $N(X, u, X', y)$  the number of outcrossings of  $(-\infty, u)$  by  $X$  at which  $X'$  is less than or equal to  $y$ , cf. Corollary 12. Then the Palm distribution of the slope  $X'$  at outcrossings of  $(-\infty, u)$  by  $X$  for  $u$  fixed and positive and  $y > 0$  is given by

$$G(y) = \mathbb{E}(N(X, u, X', y)) / \mathbb{E}(N(X, (-\infty, u))), \quad y > 0. \quad (6)$$

**Theorem 15.** *Let  $X$  denote the GH process from definition 4 with  $d = 1$ . The Palm distribution of  $X'$  at outcrossings of  $(-\infty, u)$  by  $X$  has the following density:*

$$g_{X'}(y) = \frac{\sqrt{\psi} y (\chi + u^2 + \frac{y^2}{\omega_2})^{\frac{\lambda-1}{2}} K_{\lambda-1}(\sqrt{\psi(\chi + u^2 + \frac{y^2}{\omega_2})})}{\omega_2 (\chi + u^2)^{\frac{\lambda}{2}} K_\lambda(\sqrt{\psi(\chi + u^2)})}, \quad y > 0. \quad (7)$$

*Proof.* It is well-known that  $G$  is absolutely continuous with density given by the formula  $g(y) = y f_{X(0), X'(0)}(u, y) / \mu_+(u)$ , see e.g. (Leadbetter and Spaniolo, 2004,

Section 8). Now note that  $\mathbb{E}(N(X, u, X', y)) = \int_0^y z f_{X(0), X'(0)}(u, z) dz$  with

$$\begin{aligned}
& y f_{X(0), X'(0)}(u, y) \\
&= y \int_0^\infty \frac{1}{2\pi w \sqrt{\omega_2}} e^{-\frac{u^2}{2w}} e^{-\frac{y^2}{2\omega_2 w}} f(w) dw \\
&= \frac{y}{2\pi \sqrt{\omega_2}} \int_0^\infty w^{-1} \exp\left(-\frac{u^2 + \frac{y^2}{\omega_2}}{2w}\right) \frac{(\psi/\chi)^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{\psi\chi})} w^{\lambda-1} \exp\left(-\frac{1}{2}(\chi w^{-1} + \psi w)\right) dw \\
&= \frac{y}{2\pi \sqrt{\omega_2}} \frac{(\psi/\chi)^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{\psi\chi})} \int_0^\infty w^{\lambda-2} \exp\left(-\frac{1}{2}\left((\chi + u^2 + \frac{y^2}{\omega_2})w^{-1} + \psi w\right)\right) dw \\
&= \frac{y}{2\pi \sqrt{\omega_2}} \frac{(\psi/\chi)^{\frac{\lambda}{2}}}{K_\lambda(\sqrt{\psi\chi})} \frac{K_{\lambda-1}\left(\sqrt{\psi\left(\chi + u^2 + \frac{y^2}{\omega_2}\right)}\right)}{\left(\psi/\left(\chi + u^2 + \frac{y^2}{\omega_2}\right)\right)^{\frac{\lambda-1}{2}}}.
\end{aligned}$$

Dividing by  $\mu_+(u)$  from Corollary 12 now gives the result.  $\square$

## 6. Examples

In this section we will discuss special cases and present the Gaussian case as a limit by letting  $\chi, \psi \rightarrow \infty$  in such a way that  $\chi/\psi \rightarrow 1$  (cf. Barndorff-Nielsen (1978) or Prause (1999)).

**Example 16.** For specific values of  $\lambda$  the modified Bessel functions of the third kind  $K_\lambda$  have an explicit form. In that case the outcrossing results of Theorem 9 and 11 become explicit:

(1) For  $\lambda = -\frac{1}{2}$  (W inverse Gaussian) and  $d = 1$  the expected number of outcrossings of  $(-\infty, u)$ , cf. Corollary 12, can be written as:

$$\mu_+(u) = e^{u\beta} \frac{\sqrt{\omega_2}}{2\pi} \sqrt{\frac{\chi}{\chi + u^2}} \exp\left(\sqrt{\chi\psi} - \sqrt{(\chi + u^2)(\psi + \beta^2)}\right).$$

(2) For  $\lambda = \frac{1}{2}$  ( $W$  is reciprocal inverse Gaussian) and  $d = 1$  the expected number of outcrossings of  $(-\infty, u)$ , cf. again Corollary 12, can be written as:

$$\mu_+(u) = e^{u\beta} \frac{\sqrt{\omega_2}}{2\pi} \sqrt{\frac{\psi}{\psi + \beta^2}} \exp\left(\sqrt{\chi\psi} - \sqrt{(\chi + u^2)(\psi + \beta^2)}\right).$$

(3) For  $\lambda = \frac{1}{2}$  ( $W$  is reciprocal inverse Gaussian),  $d = 3$  and  $\beta = 0$ , the expected number of outcrossings of  $B_u(0)$ , cf. Theorem 9, is given by:

$$\mu_+(B_u(0)) = C_3 u^2 \sqrt{\frac{\psi}{\chi + u^2}} \exp\left(\sqrt{\chi\psi} - \sqrt{(\chi + u^2)\psi}\right),$$

where  $C_3$  is given in (4).

We now turn to limits of certain parameters.

**Example 17.** For Gaussian processes  $X$  the Palm distribution of  $X'$  at outcrossings of  $(-\infty, u)$  by  $X$  is the Rayleigh distribution with density

$$g_{X'}(y) = \frac{y}{\omega_2} e^{-\frac{y^2}{2\omega_2}}, \quad y > 0. \quad (8)$$

We show that the Palm density of the GH process (7) converges as  $\chi, \psi \rightarrow \infty$  with  $\psi/\chi \rightarrow 1$  to the Rayleigh density (8). Note first that

$$\sqrt{\psi} \frac{(\chi + u^2 + \frac{y^2}{\omega_2})^{\frac{\lambda-1}{2}}}{(\chi + u^2)^{\frac{\lambda}{2}}} = \sqrt{\frac{\psi}{(\chi + u^2 + \frac{y^2}{\omega_2})}} \left( \frac{\chi + u^2 + \frac{y^2}{\omega_2}}{\chi + u^2} \right)^{\frac{\lambda}{2}} \rightarrow 1$$

as  $\chi, \psi \rightarrow \infty, \psi/\chi \rightarrow 1$ . Now recall from Remark 2(ii) that for all  $\lambda \in \mathbb{R}$  we have that  $K_\lambda(x) \sim \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}} e^{-x}$  as  $x \rightarrow \infty$ . Consequently,

$$\frac{K_{\lambda-1}(\sqrt{\psi(\chi + u^2 + \frac{y^2}{\omega_2})})}{K_\lambda(\sqrt{\psi(\chi + u^2)})} \sim \exp\left(-\sqrt{\psi(\chi + u^2 + \frac{y^2}{\omega_2})} + \sqrt{\psi(\chi + u^2)}\right).$$

Now rewriting the exponent we get

$$\sqrt{\psi(\chi + u^2 + \frac{y^2}{\omega_2})} - \sqrt{\psi(\chi + u^2)} = \frac{\sqrt{1 + \frac{u^2}{\chi} + \frac{y^2}{\chi\omega_2}} - \sqrt{1 + \frac{u^2}{\chi}}}{(\psi\chi)^{-\frac{1}{2}}},$$

which is easily recognized as a differential quotient with limit  $\frac{y^2}{2\omega_2}$ .

**Example 18.** In view of Example 17, the outcrossing results of Theorem 9, 11 and Corollary 12 are now easily seen to satisfy the following limit laws:

$$\mu_+(u) = \frac{\sqrt{\omega_2}}{2\pi} \frac{K_\lambda(\sqrt{(u^2 + \chi)\psi})}{K_\lambda(\sqrt{\chi\psi})} \left( \frac{u^2 + \chi}{\chi} \right)^{\frac{\lambda}{2}} \rightarrow \frac{\sqrt{\omega_2}}{2\pi} e^{-\frac{u^2}{2}}$$

and

$$\mu_+(B_u(0)) = C_d u^{d-1} \frac{K_{\lambda + \frac{1-d}{2}}(\sqrt{(u^2 + \chi)\psi})}{K_\lambda(\sqrt{\chi\psi})} \frac{(\sqrt{u^2 + \chi})^{\lambda + \frac{1-d}{2}}}{\chi^{\frac{\lambda}{2}} \psi^{\frac{1-d}{4}}} \rightarrow C_d u^{d-1} e^{-\frac{u^2}{2}}$$

as  $\chi, \psi \rightarrow \infty, \psi/\chi \rightarrow 1$ , reproducing the classic Gaussian results.

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