How to explain a corporate credit spread

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Abstract

This paper concerns credit risk modeling in a continuous time market model and pricing of credit derivatives. We focus on default risk and describe the credit spread between a corporate bond and a government bond by explaining variables. For this purpose, we consider a specific market model consisting of four assets where the default process of the company is incorporated in a risky money market by a Cox process. The modeling in this paper is done by the intensity based approach. We show that the presented market model has a unique equivalent martingale measure and is complete. As a consequence, contingent claim valuation can be executed in the usual default free way. This is examplified in the case of a convertible bond which fits naturally in our setting. The work is concluded by some simulations under the equivalent martingale measure using different explaining variables for the (spot) spread.

As a by-product we discuss the problem of maintaining the Cox property under change of measure. It turns out that changing from the subjective measure P to the equivalent martingale measure Q preserves the Cox property in our framework.

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1 Introduction

In recent years many models and ideas concerning credit risk went public. Here, we refer to three related approaches. In the early nineties Duffie and Singleton introduced the *reduced-form model*, where default is treated as an unpredictable event governed by a hazard-rate process. A recent publication is Duffie and Singleton (1999). Jarrow, Lando and Turnbull (1997) share the *intensity based approach*, but they focus on transitions inbetween different rating classes incorporating a time continuous homogeneous Markov chain with rating classes in a discrete state space. Lando (1997) presents a technique of adding a certain set of 'explaining variables' to this model, thus the Markov chain becomes heterogeneous. A more global point of view is stated in Schönbucher (1998) where he extends the term structure model of Heath, Jarrow and Morton (1992) by a spread structure.

Naturally, discontinuous term structure models are a field strongly related to credit risk. Shirakawa (1991) investigates a bond model where the forward rate curve follows a multidimensional Poisson-Gaussian process. In this setting, he finds necessary and sufficient conditions for completeness of the financial market and derives explicitly the price of a call option. With Björk, Kabanov and Runggaldier (1997) marked point processes entered the interest rate theory as sources of discontinuities. Marked point processes are a generalization of multivariate Poisson processes.

For contingent claim pricing the existence of equivalent martingale measures is a necessary topic to study. As 'guidelines' we refer to Harrison Pliska (1981), Bardhan and Chao (1996) and Björk (1998).

In this article we formulate a model for the stochastic behavior of corporate bond prices. In this context, corporate bonds are bonds issued by public liability companies or other legal entities. A public liability company is thereby a company whose shares are traded at the stock market.

Here, we focus on default risk and present a setup for explaining the yield spread between corporate bonds and government bonds by a set of appropriate state variables. We associate with government bonds bonds that are free of default risk. This terminology can be applied to so called 'hard currency' countries like the USA or countries belonging to the EC.

As what the modeling of the default risk concerns, our model belongs to the intensity based approach. The behavior of the default events is modelled in the present paper by a Cox process, i.e. a Poisson process with stochastic intensity. It is well-known that in such a framework, being under the equivalent martingale measure, the (spot) credit spread s of a corporate bond is the product of the stochastic intensity of a Cox process and the loss rate l of the company. Further, every defaultable bond is a contingent claim and it's price process can be expressed by a conditional expectation under the equivalent martingale measure. For an overview of these results see for instance Lando (1997).

The goal of this article is to describe totally the credit spread s of a corporate bond by 'explaining factors'. Therefore we define an appropriate environment. Denote $(\mathcal{G}_t)$ the market filtration explaining the credit spread; the (spot) credit spread s is a $(\mathcal{G}_t)$-predictable process. The explainig market filtration $(\mathcal{G}_t)$ is generated by the price processes of a riskless money market account, of a riskless zero bond (both free of default risk) and of the company's stock price process. In this paper, we focus on modeling the (spot) credit spread s. Thus, a mathematical convenient choice for introducing credit risk in the market is a defaultable money market account C (see also Schönbucher (2000)). We assume that C exhibits a (negative) jump whenever a default occurs. The choice of a defautable money market account is clearly in general not consistent with reality. However, we show that our market model consisting with the four above mentioned assets has a unique martingale measure and is complete. The price process of a zero bond with default risk $v(\cdot, T)$ can be therefore derived by a contingent claim price process and the defautable money market account can be thus theoretically replaced by a linear combination of the three remaining assets and the defautable zero bond.

An other possible idea might be to replace in our market model the defautable money market account C directly by the corporate zero bond with default risk $v(\cdot, T)$. However, modeling $v(\cdot, T)$ requires a complete description of the 'spread structure' of the corporate zero bond. In our setting such a model would make barely sense since in the final result the 'spread structure' would be reduced to some time depending function of the spot spread s under the equivalent martingale measure. For comparison, a similar effect can be observed when dealing with one factor models in interest rate theory. There, the price process of a zero bond can be written as an expectation under the equivalent martingale measure, which turns out to be only a function of the short rate.

This paper is organized as follows. In Section 2 we define the terminology and describe our market model in detail. Some useful lemmata and the proper version of Girsanov's theorem are stated in Section 3. In particular, we represent all price processes and discounted price processes, respectively, as Doléans Dade exponentials. This new representation appears to be very convenient in our later computations. Finally, we discuss the behavior of the Cox process N after some change of measure. Corollary 3.6 states that changing from the subjective measure P to the equivalent measure Q preserves the Cox property of N. This result is as far as we know new and simplifies considerably the contingent claim valuation later on. In Section 4 we prove the existence and uniqueness of an equivalent martingale measure and show that consequently our market models is complete (later we will properly specify the word "complete"). Moreover, we discuss briefly the pricing of credit derivates in our setting and show that the convertible bond fits naturally in it. In section 5 we set up a special martingale model and study the applications of our framework. Finally, a short introduction to stochastic analysis terminology is given in the Appendix. For further details we refer to Protter (1995).

2 The market model

In this section we present the market model and setup and introduce some required assumptions. They are assumed to hold from now on if not stated otherwise.

The modeling takes place in an intensity based framework; i.e. default is triggered by a point process with an intensity λ. The default intensity λ can be seen as a function of certain describing variables. Examples of such variables in our setup are the short rate process $r = \{r(s) : 0 \leq s \leq T\}$ and the stock price process $S = \{S(s) : 0 \leq s \leq T\}$ of the company.

As mentioned in the introduction, we consider a market model which consists of a money market account and a zero bond (both free of default risk), a company's stock and a defaultable money market account issued by the same company. This market model is set in a filtered probability space $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \leq t \leq T}\right)$ large enough to support a two dimensional standard Brownian motion $W = \left(W^1, W^2\right)$, $W^i = \left\{W^i(t) : 0 \leq t \leq T\right\}$, $i = 1, 2$, and a point process $N = \{N(t) : 0 \leq t \leq T\}$, where $T > 0$ is some finite time horizon.

In what follows we need the information structures

$$\mathcal{F}_t \equiv \sigma\left(W^1(s), W^2(s), N(s) : 0 \leq s \leq t\right), \qquad \text{for } 0 \leq t \leq T,$$

and

$$\mathcal{G}_t \equiv \sigma\left(W^1(s), W^2(s) : 0 \leq s \leq t\right), \qquad \text{for } 0 \leq t \leq T.$$

In both cases, we always think of them as the augmentation of the natural filtration. For the exact definition of *augmentation* see Karatzas and Shreve(1991), p. 89. Naturally, for some technical reason, we take the continuous version of the Brownian motion W and the right continuous version of the point process N. This ensures that the *usual hypotheses* hold. Note that these conditions are necessary for stochastic integration with respect to semimartingales, see Protter (1995), p. 3.

It is straightforward that $(\mathcal{G}_t)_{0\leq t\leq T}$ is a subfiltration of $(\mathcal{F}_t)_{0\leq t\leq T}$, i.e. $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $0 \leq t \leq T$. $\Lambda = \{\Lambda(t) : 0 \leq t \leq T\}$ denotes in the following the compensator of N with respect to $(\mathcal{F}_t)_{0\leq t\leq T}$. Thus Λ is a $(\mathcal{F}_t)$–predictable process with paths of finite variation and $N - \Lambda$ is a local $(\mathcal{F}_t)$–martingale. For the definition of a compensator see e.g. Protter, Ch. III, p. 97.

Next we describe our market model in more detail. The term structure is given by the money market account $B = \{B(t) : 0 \leq t \leq T\}$

$$B(t) \equiv \exp\left(\int_0^t r(u)\,du\right), \qquad \text{for } 0 \leq t \leq T, \tag{2.1}$$

and the zero bond $p(\cdot, T) = \{p(t,T) : 0 \leq t \leq T\}$ with maturity $T > 0$

$$p(t,T) \equiv \exp\left(-\int_t^T f(t,u)\,du\right), \qquad \text{for } 0 \leq t \leq T. \tag{2.2}$$

Both quantities are free of default risk. Analogously to Heath, Jarrow and Morton (1986), we use a one factor model given by

$$f(s,t) \equiv f(0,t) + \int_0^s \alpha(u,t)\,du + \int_0^s \sigma(u,t)\,dW^1(u) \tag{2.3}$$

and

$$r(t) \equiv f(t,t), \qquad \text{for } 0 \leq s \leq t \leq T, \tag{2.4}$$

where we additionally define

$$A(t,T) \equiv \int_t^T \alpha(t,u)\,du \tag{2.5}$$

and

$$D(t,T) \equiv \int_t^T \sigma(t,u)\,du\,, \qquad \text{for } 0 \leq s \leq t \leq T. \tag{2.6}$$

We assume similar conditions as in Heath, Jarrow and Morton (1986).

Assumption 2.1 *(a) r is positive and*

$$\int_0^T r(u)\,du < \infty, \quad P\text{–a.s.}. \tag{2.7}$$

(b) $\alpha(\cdot, s)$ and $\sigma(\cdot, s)$ are progressively measurable with respect to the filtration $(\mathcal{G}_t)$ for $0 \leq s \leq T$, and σ is strictly positive.

(c) The objects $\alpha(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ allow to interchange the order of integration for P-a.a. $\omega \in \Omega$.

The stock price process $S = \{S(t) : 0 \leq t \leq T\}$ is defined by its initial value $S(0)$ at time zero and the stochastic differential equation

$$dS(t) \quad = \quad \mu(t)S(t)\,dt \quad + \quad \nu(t)S(t)\,dW^2(t)\,, \qquad \text{for } 0 \leq t \leq T. \tag{2.8}$$

By Itô's formula we find

$$S(t) = S(0)\,\exp\left(\int_0^t \mu(u)\,du - \frac{1}{2}\int_0^t \nu(u)^2\,du + \int_0^t \nu(u)\,dW^2(u)\right)\,, \qquad \text{for } 0 \leq t \leq T. \tag{2.9}$$

Finally, we define the price process $C = \{C(t) : 0 \leq t \leq T\}$ of the defaultable money market account by

$$C(t) \quad \equiv \quad \Pi(t)\,\exp\left(\int_0^t (r(u) + s(u))\,du\right), \qquad \text{for } 0 \leq t \leq T, \tag{2.10}$$

where $s = \{s(t) : 0 \leq t \leq T\}$ is the (spot) spread process and $\Pi = \{\Pi(t) : 0 \leq t \leq T\}$ describes the loss fraction or negative return of the invested money after default events. The process Π is modelled by

$$\Pi(t) \quad \equiv \quad \prod_{0<u\leq t} (1 - l(u)\,\Delta N(u)) \quad = \quad \prod_{n=1}^{N(t)} (1 - l(T_n))\,. \tag{2.11}$$

The jump times $(T_n)_{n\geq 1}$ of the point process N are associated with a default event of the company. The loss ratio process $l = \{l(t) : 0 \leq t \leq T\}$ takes values in the open interval $(0, 1)$. Therefore, at every default time T_n the defaultable money market account C bears a loss of $l(T_n)$ in fraction. Note that $l = 1$ is not possible in our setting. If we allowed $l = 1$ then the price process C could reach the absorbing state 0 which might cause some technical problems. In what follows we want C to be strictly positive.

In order to ensure the existence of S and C as semimartingales we have to presume some additional technical assumptions. Part (c) of the following assumption guarantees that N is a point process in the spirit of Brémaud (1981), whereas (d) leads one step further to Cox processes or so–called doubly stochastic processes (Brémaud (1981), Grandell (1997)).

Assumption 2.2 (a) *The processes μ and ν are progressively measurable with respect to $(\mathcal{G}_t)$. The process ν is a strictly positive and*

$$\int_0^T |\mu(u)|\,du \quad + \quad \int_0^T \nu(u)^2\,du \quad < \quad \infty\,, \quad P\text{–a.s.}. \tag{2.12}$$

(b) *The processes s and l are $(\mathcal{G}_t)$-predictable. The process s is strictly positive, the process l takes values in the open intervall $(0, 1)$ and*

$$\int_0^T s(u)\,du \quad < \quad \infty\,, \quad P\text{–a.s.}. \tag{2.13}$$

(c) *Λ is absolutely continuous and has the representation*

$$\Lambda(t) = \int_0^t \lambda(u)\,du\,, \qquad \textit{for } 0 \leq t \leq T, \tag{2.14}$$

where $\lambda = \{\lambda(t) : 0 \leq t \leq T\}$ is a $(\mathcal{G}_t)$-predictable and strictly positive process satisfying

$$\int_0^T \lambda(u)\,du < \infty, \quad P\text{–a.s.} \tag{2.15}$$

(d) *Moreover, we assume* Λ *is the compensator of* N *with respect to the enlarged filtration* $(\mathcal{F}_t \vee \mathcal{G}_T)_{0 \le t \le T}$*; i.e.* $N - \Lambda$ *is a local martingale with respect to the filtration* $(\mathcal{F}_t \vee \mathcal{G}_T)_{0 \le t \le T}$.

Remark 2.3 Part (c) and (d) of Assumption 2.2 for the point process N lead naturally to doubly stochastic processes introduced by Cox (1955). Brémaud (1981) shows that N on the filtered probability space $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t \vee \mathcal{G}_T)_{0 \le t \le T}\right)$ with enlarged filtration $\left(\widetilde{\mathcal{F}}_t\right)_{0 \le t \le T} \equiv (\mathcal{F}_t \vee \mathcal{G}_T)_{0 \le t \le T}$ has conditional independent increments; that is $N(t) - N(s)$ is P-independent of $\widetilde{\mathcal{F}}_s = \mathcal{F}_s \vee \mathcal{G}_T$ given $\widetilde{\mathcal{F}}_0$. Since $\mathcal{G}_T$ is the σ–algebra generated by W, we say N is driven by W. Moreover, we have for $k \in \mathbb{N}_0$ and $0 \le s \le t \le T$

$$P\left(N(t) - N(s) = k \,|\, \mathcal{F}_s \vee \mathcal{G}_T\right) = \exp\left(-\int_s^t \lambda(u)\,du\right) \frac{\left(\int_s^t \lambda(u)\,du\right)^k}{k!}. \tag{2.16}$$

Therefore N conditioned on $\mathcal{G}_T$ (in short hand notation $N|_{\mathcal{G}_T}$) can be interpreted as an inhomogeneous Poisson process.
The choice of N as a doubly stochastic process is very particular. Elliot, Jeanblanc and Yor (2000) discuss a more general approach. They consider a Brownian motion W and a single default event carried by a random time τ; i.e. $N(t) = \mathbf{1}_{\{\tau \le t\}}$. The natural filtrations in their setup are $(\mathcal{G}_t)$ of W and $(\mathcal{H}_t)$ of N, whereas the enlarged filtration is $(\mathcal{F}_t) = (\mathcal{G}_t \vee \mathcal{H}_t)$. In their article the problem of preserving the martingale property of W on the enlarged filtration $(\mathcal{F}_t)$ is studied. They introduce the hypothesis (H); that is, every square integrable $(\mathcal{G}_t)$-martingale is a square integrable $(\mathcal{F}_t)$-martingale. Whether hypothesis (H) holds or not depends on the question whether W is a martingale on the enlarged filtration $(\mathcal{F}_t)$ or not. Here, we do not consider this problem since we assume W is a Brownian Motion on the enlarged filtration $(\mathcal{F}_t)$ and hence a $(\mathcal{F}_t)$–martingale. Naturally, W is still a Brownian motion on its natural filtration $(\mathcal{G}_t)$. In other words, hypothesis (H) holds in our setting.

3 Representation lemma and a version of Girsanov's theorem

In this section we represent the discounted price processes as Doléans Dade exponentials using classical stochastic integration theory (see e.g. Protter (1995)). Moreover, a version of Girsanov's theorem is presented which is adequate for our purposes. This result can be found in Björk, Kabanov and Runggaldier (1997). Last but not least, we study the problem of maintaining the Cox property under change of measure.

We start by rewriting the actual price processes as Doléans Dade exponentials.

Lemma 3.1 *Under the assumptions 2.1 and 2.2, we have for* $0 \le t \le T$

$$B(t) = \exp\left(\int_0^t r(u)\,du\right) = \mathcal{E}(R)(t), \tag{3.1}$$
$$p(t,T) = p(0,T)\,\mathcal{E}(R_p)(t), \tag{3.2}$$
$$S(t) = S(0)\,\mathcal{E}(R_S)(t) \tag{3.3}$$

and

$$C(t) = \mathcal{E}(R_C)(t), \tag{3.4}$$

where

$$R(t) \equiv \int_0^t r(u)\,du\,, \tag{3.5}$$

$$R_p(t) \equiv R(t) - \int_0^t A(u,T)\,du - \int_0^t D(u,T)\,dW^1(u) + \frac{1}{2}\int_0^t D(u,T)^2\,du\,, \tag{3.6}$$

$$R_S(t) \equiv \int_0^t \mu(u)\,du + \int_0^t \nu(u)\,dW^2(u) \tag{3.7}$$

and

$$R_C(t) \equiv R(t) - \int_0^t l(u)\,dN(u) + \int_0^t s(u)\,du\,. \tag{3.8}$$

Proof. $B = \mathcal{E}(R)$ follows directly from (A.9) and the fact that R is a Lebesgue-Stieltjes integral. Thus, R has continuous paths. Equation (3.2) is a well-known result in interest theory, see e.g. Björk (1997). The representation of $p(\cdot,T)$ as a Doléans Dade exponential is a consequence of (A.8). $S(t) = S(0)\,\mathcal{E}(R_S)(t)$ follows from (A.2).

Therefore, it remains to prove (3.4). Since $l \cdot N$ is a process with $(l \cdot N)^c = 0$ P–a.s., we have by equation (A.10)

$$\Pi(t) = \prod_{0<u\le t} (1 - l(u)\,\Delta N(u)) = \mathcal{E}(-l\cdot N)(t)\,, \qquad \text{for } 0 \le t \le T. \tag{3.9}$$

Plugging this in equation (2.10) and applying (A.8) and (A.12) yields for $0 \le t \le T$

$$\begin{aligned} C(t) &= \Pi(t)\,\exp\left(\int_0^t (r(u)+s(u))\,du\right) \\ &= \mathcal{E}(-l\cdot N)(t)\,\mathcal{E}\left(\int_0^{\cdot} (r(u)+s(u))\,du\right)(t) \\ &= \mathcal{E}(-l\cdot N)(t)\,\mathcal{E}\left(R + \int_0^{\cdot} s(u)\,du\right)(t) \\ &= \mathcal{E}\left(R - l\cdot N + \int_0^{\cdot} s(u)\,du\right)(t) \\ &= \mathcal{E}(R_C)(t)\,, \end{aligned}$$

as we claimed before. □

Now, we define the discounted price processes.

Definition 3.2 *The processes* $Z_p = \{Z_p(t) : 0 \le t \le T\}$, $Z_S = \{Z_p(t) : 0 \le t \le T\}$ *and* $Z_C = \{Z_p(t) : 0 \le t \le T\}$ *are defined through*

$$Z_p(t) \equiv \frac{p(t,T)}{B(t)},\; Z_S(t) \equiv \frac{S(t)}{B(t)} \; \textit{and} \; Z_C(t) \equiv \frac{C(t)}{B(t)}\,, \qquad \textit{for } 0 \le t \le T.$$

The next lemma states the representation of the discounted price processes in terms of Doléans Dade exponentials.

Lemma 3.3 (Representation Lemma) *With the notation in Definition 3.2 we have*

$$\begin{aligned} Z_p(t) &= p(0,T)\,\mathcal{E}\left(Y_p\right)(t), \\ Z_S(t) &= S(0)\,\mathcal{E}\left(Y_S\right)(t) \end{aligned}$$

and

$$Z_C(t) = \mathcal{E}\left(Y_C\right)(t), \qquad \text{for } 0 \le t \le T,$$

where

$$\begin{aligned} Y_p(t) &\equiv \frac{1}{2}\int_0^t D(u,T)^2\,du - \int_0^t A(u,T)\,du - \int_0^t D(u,T)\,dW^1(u), \\ Y_S(t) &\equiv \int_0^t \mu(u)\,du - \int_0^t r(u)\,du + \int_0^t \nu(u)\,dW^2(u) \end{aligned}$$

$$Y_C(t) \equiv \int_0^t s(u)\,du - \int_0^t l(u)\,dN(u), \qquad \text{for } 0 \le t \le T.$$

Proof. We use Lemma 3.1 and some properties of the Doléans Dade exponential, to prove the desired results. First observe that

$$\frac{1}{B} = \exp\left(-\int_0^{\cdot} r(u)\,du\right) = \mathcal{E}\left(-R\right). \tag{3.10}$$

Now, we apply equation (A.12) to the representation of the price processes in Lemma 3.1. This is possible since R is a process with P–a.s. continuous paths of finite variation. We derive that

$$\begin{aligned} Z_p &= p(0,T)\,\mathcal{E}\left(R_p\right)\,\mathcal{E}\left(-R\right) &&= p(0,T)\,\mathcal{E}\left(Y_p\right), \\ Z_S &= S(0)\,\mathcal{E}\left(R_S\right)\,\mathcal{E}\left(-R\right) &&= S(0)\,\mathcal{E}\left(Y_S\right) \end{aligned}$$

and

$$Z_C = \mathcal{E}\left(R_C\right)\,\mathcal{E}\left(-R\right) = \mathcal{E}\left(R_C - R\right) = \mathcal{E}\left(Y_C\right), \tag{3.11}$$

which finishes the proof. □

The key to all needed equivalent measures lies in the following version of Girsanov's theorem (see Björk, Kabanov and Runggaldier (1997), Theorem 3.12).

Theorem 3.4 (Girsanov's theorem) *Suppose we have a point process* $N = \{N(t) : 0 \le t \le T\}$ *on a filtered probability space* $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\le t\le T}\right)$*, where* $\Lambda = \{\Lambda(t) : t \ge 0\}$ *is the compensator of* N*, i.e.* $M = \{M(t) : 0 \le t \le T\}$ *defined by the equation* $M \equiv N - \Lambda$ *is a local martingale. We assume,* N *has a predictable intensity* $\lambda = \{\lambda(t) : 0 \le t \le T\}$*; that is*

$$\Lambda(t) = \int_0^t \lambda(s)\,ds, \qquad \text{for } 0 \le t \le T.$$

Let $W = \left(W^1, ..., W^d\right)$ *be a* d*–dimensional Standard Brownian motion on* $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\le t\le T}\right)$*,* $W^k = \left\{W^k(t) : 0 \le t \le T\right\}$ *for* $k = 1, ..., d$*. We assume* $(\mathcal{F}_t)_{0\le t\le T}$ *is the* P*–extension of the*

natural filtration of (W,N), *furthermore* $\mathcal{F}=\mathcal{F}_T$.
Let $\psi=\left(\psi^1,...,\psi^d\right)$ *be a* d*–dimensional predictable process,* $\psi^k=\{\psi^k(t):0\le t\le T\}$ *for* $k=1,...,d$, *and let* $\phi=\{\phi(t):0\le t\le T\}$ *be a strictly positive predictable process satisfying*

$$\int_0^T ||\psi(s)||^2\,ds<\infty \quad \text{and} \quad \int_0^T |\phi(s)-1|\,\lambda(s)\,ds<\infty \quad P\text{–}a.s. \tag{3.12}$$

Define the process $L=\{L(t):0\le t\le T\}$ *by*

$$L\equiv\mathcal{E}\left(\sum_{k=1}^{d}\psi^k\cdot W^k+(\phi-1)\cdot M\right).$$

Suppose $\mathbf{E}_P\{L(T)\}=1$, *then we can define a measure* Q *by*

$$dQ=L(T)\,dP.$$

We then have, Q *is a probability measure equivalent to* P, $\widetilde{W}=\left(\widetilde{W}^1,...,\widetilde{W}^d\right)$ *defined by*

$$\widetilde{W}^k(t)\equiv W^k(t)-\int_0^t\psi^k(s)\,ds, \qquad \text{for } 0\le t\le T \text{ and } k=1,...,d,$$

is a Standard Brownian motion under Q *and the process* $\lambda_Q=\{\lambda_Q(t):0\le t\le T\}$, *where*

$$\lambda_Q(t)\equiv\phi(t)\,\lambda(t), \qquad \text{for } 0\le t\le T, \tag{3.13}$$

is the intensity of N *under the measure* Q.
Moreover, every probability measure Q *equivalent to* P *has the structure above.*

Remark 3.5 The present version of Girsanov's theorem is based on the martingale representation theorem for our specific filtered probability space. In Björk, Kabanov and Runggaldier (1997), Remark 3.2, the suitable martingale representation theorem is presented. The result was refered to Corollary 4.31 in Jacod and Shiryaev (1987). In fact, Björk, Kabanov and Runggaldier investigate a more complex situation by introducing a marked point process.

In Assumption 2.2 (d) we introduced the Cox property by enlarging the filtration to $(\widetilde{\mathcal{F}}_t)_{0\le t\le T}=(\mathcal{F}_t\vee\mathcal{G}_T)_{0\le t\le T}$ and assuming that $N-\Lambda$ remains a local martingale on the enlarged filtration $(\widetilde{\mathcal{F}}_t)_{0\le t\le T}$. Naturally, the question arises what kind of measure changes preserve the Cox property of N; i.e. Assumption 2.2 (d). A partial answer can be found with Girsanov-Meyer (Theorem A1.3).

Corollary 3.6 *With the notation of the preceding theorem, we assume* N *is a* P*-Cox process conditioned on* $(\mathcal{G}_t)_{0\le t\le T}$*; i.e. Assumption 2.2 (c) and (d) hold, where* $(\mathcal{G}_t)_{0\le t\le T}$ *is the completed natural filtration of the Brownian motion* W, $\mathcal{G}_t=\sigma\left(W(s):0\le s\le t\right)$, $0\le t\le T$.
If ψ *and* ϕ *are* $(\mathcal{G}_t)$*-predictable and* $\mathcal{E}\left((\phi-1)\cdot M\right)$ *is in* $\mathcal{H}^2(P)$ *(see (A.15) for the definition), then* N *is a* Q*-Cox process conditioned on* $(\mathcal{G}_t)_{0\le t\le T}$*. The* Q*-intensity of* N *is given by* $\lambda_Q=\lambda\,\phi$.

Proof. We need to show that the Q-analogon to Assumption 2.2 (c) and (d) hold. Theorem 3.4 implies that the $\left((\mathcal{F}_t),Q\right)$-intensity λ_Q of N equals $\lambda\,\phi$. The process λ_Q is $(\mathcal{G}_t)$-predictable since λ and ϕ are $(\mathcal{G}_t)$-predictable. Thus N admits the $\left((\mathcal{F}_t),Q\right)$-intensity λ_Q that is $(\mathcal{G}_t)$-predictable.

Next, we study the change of measure given by $dQ = L(T)\,dP$ on the enlarged filtration $(\widetilde{\mathcal{F}}_t)_{0\le t\le T} = (\mathcal{F}_t \vee \mathcal{G}_T)_{0\le t\le T}$. Define

$$Z(t) \equiv \mathbf{E}_P\left\{ \frac{dQ}{dP} \,\middle|\, \widetilde{\mathcal{F}}_t \right\}, \quad \text{for } 0 \le t \le T.$$

Assumption 2.2 (d) gives us the local martingale property of M on the filtration $(\widetilde{\mathcal{F}}_t)$ and equation (3.12) with Theorem 8 in Bremaud, Ch. II., gives us the local martingale property of $(\phi - 1)\cdot M$. Hence, the Doléans Dade exponential of $(\phi-1)\cdot M$ is also a $((\widetilde{\mathcal{F}}_t), P)$-local martingale due to Corollary A1.2. Morover, $\mathcal{E}\left((\phi-1)\cdot M\right)$ is a square integrable $((\widetilde{\mathcal{F}}_t), P)$-martingale, since it is in $\mathcal{H}^2(P)$. Furthermore,

$$L(T) = \mathcal{E}\left(\sum_{k=1}^{d} \psi^k \cdot W^k + (\phi-1)\cdot M\right)(T) = \mathcal{E}\left(\sum_{k=1}^{d}\psi^k\cdot W^k\right)(T)\,\mathcal{E}\left((\phi-1)\cdot M\right)(T)$$

because $[W, M] = 0$, since W has continuous paths and M has paths of finite variation a.s.. Further, the vector process ψ is $(\mathcal{G}_t)$-predictable and thus the Doléans Dade exponential of the 'integrated' Brownian motion $\psi \cdot W$ is $(\mathcal{G}_t)$-adapted. With $\mathcal{G}_T \subset \widetilde{\mathcal{F}}_0$, we find that for every $0 \le t \le T$

$$\begin{aligned}
Z(t) &= \mathbf{E}_P\left\{ L(T) \,|\, \widetilde{\mathcal{F}}_t \right\} \\
&= \mathbf{E}_P\left\{ \mathcal{E}\left(\sum_{k=1}^{d}\psi^k\cdot W^k\right)(T)\,\mathcal{E}\left((\phi-1)\cdot M\right)(T) \,\middle|\, \widetilde{\mathcal{F}}_t \right\} \\
&= \mathcal{E}\left(\sum_{k=1}^{d}\psi^k\cdot W^k\right)(T)\,\mathbf{E}_P\left\{ \mathcal{E}\left((\phi-1)\cdot M\right)(T) \,|\, \widetilde{\mathcal{F}}_t \right\} \\
&= \mathcal{E}\left(\sum_{k=1}^{d}\psi^k\cdot W^k\right)(T)\mathcal{E}\left((\phi-1)\cdot M\right)(t).
\end{aligned}$$

Hence

$$Z(t) = Z(0)\,\mathcal{E}\left((\phi-1)\cdot M\right)(t), \quad \text{for } 0 \le t \le T,$$

where $Z(0) = \mathcal{E}\left(\sum_{k=1}^{d}\psi^k\cdot W^k\right)(T)$ is a $\widetilde{\mathcal{F}}_0$-measurable random variable.
Due to $\Delta(H\cdot X) = H\Delta X$, we see $\Delta Z(t) = Z(t-)\,(\phi(t)-1)\Delta N(t)$. Therefore,

$$\frac{Z(t-)}{Z(t)} = \frac{Z(t-)}{Z(t-)+\Delta Z(t)} = \frac{Z(t-)}{Z(t-)+Z(t-)\,(\phi(t)-1)\Delta N(t)} = \frac{1}{1+\Delta N(t)(\phi(t)-1)}. \quad (3.14)$$

The jump process N is a classical semimartingale. We choose the decomposition $N = M_P - A_P$, where $M_P \equiv \phi\cdot M$ is a $((\widetilde{\mathcal{F}}_t), P)$-local martingale due to Assumption 2.2 (d) and $A_P \equiv N - \phi\cdot M$ is a P-FV process. By Theorem A1.3, we get that

$$\begin{aligned}
M_Q(t) &= M_P(t) - \int_0^t \frac{1}{Z(s)}\, d[Z, M_P](s) \\
&= M_P(t) - \int_0^t \frac{1}{Z(s)}\, d[Z(0) + Z_- \cdot ((\phi-1)\cdot M), \phi\cdot M](s) \\
&= M_P(t) - \int_0^t \frac{Z(s-)}{Z(s)}\, d[(\phi-1)\cdot M, \phi\cdot M](s)
\end{aligned}$$

$$
\begin{aligned}
&= M_P(t) - \int_0^t \frac{1}{1+\Delta N(s)(\phi(s)-1)}\,(\phi(s)-1)\,\phi(s)\,d[M,M](s) \\
&= (\phi\cdot M)(t) - \int_0^t \frac{1}{1+\Delta N(s)(\phi(s)-1)}\,(\phi(s)-1)\,\phi(s)\,dN(s) \\
&= (\phi\cdot N)(t) - (\phi\cdot\Lambda)(t) - \int_0^t \frac{(\phi(s)-1)\,\phi(s)}{1+\phi(s)-1}\,dN(s) \\
&= (\phi\cdot N)(t) - \int_0^t \phi(s)\,\lambda(s)\,ds - \int_0^t (\phi(s)-1)\,dN(s) \\
&= N(t) - \int_0^t \lambda_Q(s)\,ds
\end{aligned}
$$

is a $((\widetilde{\mathcal{F}}_t), Q)$-local martingale. Thus λ_Q is the $((\widetilde{\mathcal{F}}_t), Q)$-intensity of N. It is unique since λ_Q is predictable. This conclusion is the 'Q-equivalent' formulation of Assumption 2.2 (d). Therefore, N is a Q-Cox process conditioned on $(\mathcal{G}_t)$ with intensity λ_Q (by a combination of Theorem 4, Definition 7 and Theorem 9 in Brémaud, Ch. II., pp. 25). □

4 Completeness of the market model and contingent claim valuation

In the present section we show that the set of all equivalent martingale measures connected to our market model is a singleton. For a brief repetition, a probability measure Q is an *equivalent martingale measure* in our market model if $P \sim Q$ and the discounted price processes Z_P, Z_S and Z_C are $(Q, (\mathcal{F}_t))$-martingales. It turns out that the uniqueness of an equivalent martingale measure implies that the market model is complete. In other words, for every contingent claim we can find a self-financing trading strategy such that the payoff at maturity can be replicated. For the exact definition of completeness in our setting see Corollary 4.

Proving existence and uniqueness of an equivalent martingale measure demands some additional technical assumptions which are summarized next.

Assumption 4.1 *(a) The process $L = \{L(t) : 0 \le t \le T\}$ defined by*

$$L \equiv \mathcal{E}\left(\psi^1 \cdot W^1 + \psi^2 \cdot W^2 + (\phi - 1)\cdot M\right),$$

is a square integrable P–martingale, where the processes $\psi^k = \{\psi^k(t) : 0 \le t \le T\}$, $k = 1, 2$, and $\phi = \{\phi(t) : 0 \le t \le T\}$ are given for every $0 \le t \le T$ by

$$\psi^1(t) \equiv \frac{\frac{1}{2}D(t,T)^2 - A(t,T)}{D(t,T)}, \tag{4.1}$$

$$\psi^2(t) \equiv \frac{r(t) - \mu(t)}{\nu(t)} \tag{4.2}$$

and

$$\phi(t) \equiv \frac{s(t)}{l(t)\,\lambda(t)} \tag{4.3}$$

and satisfy the regularity conditions (3.12) in Theorem 3.4.

(b) *The process* $\mathcal{E}((\phi - 1)\cdot M)$ *is in* $\mathcal{H}^2(P)$.

(c) *The discounted price processes* Z_p, Z_S *and* Z_C *are* $\mathcal{H}^2(P)$*–semimartingales.*

Existence of the processes ψ^1, ψ^2 and ϕ is ensured by Assumptions 2.1 and 2.2. The square integrable martingale properties for L and for the Doléans Dade exponential $\mathcal{E}((\phi - 1)\cdot M)$ are needed for some technical reason in the proof of the following main result.

Theorem 4.2 *In the defined market model the set of all equivalent martingale measures* $\mathcal{Q}$ *is a singleton; i.e.* $\mathcal{Q} = \{Q\}$.

Proof. Since all defined price processes are strictly positive we know by Corollary A1.2 that the discounted price processes are local martingales if and only if their stochastic exponents are local martingales. In what follows we have to find conditions to ensure the local martingale property for the stochastic exponents given in Lemma 3.3.
Let $\mathcal{P}$ denote the set of all probability measures equivalent to the 'original' measure P. We fix a measure $P^* \in \mathcal{P}$. For the processes ψ_{P^*} and ϕ_{P^*}, satisfying the regularity conditions (3.12) in Theorem 3.4, we have for every $0 \le t \le T$

$$\begin{aligned} Y_p(t) &= \frac{1}{2}\int_0^t D(u,T)^2\,du - \int_0^t A(u,T)\,du - \int_0^t D(u,T)\,\psi^1_{P^*}(u)\,du - \int_0^t D(u,T)\,d\widetilde{W}^1(u)\,,\\ Y_S(t) &= \int_0^t \mu(u)\,du - \int_0^t r(u)\,du + \int_0^t \nu(u)\,\psi^2_{P^*}(u)\,du + \int_0^t \nu(u)\,d\widetilde{W}^2(u) \end{aligned}$$

and

$$Y_C(t) = \int_0^t s(u)\,du - \int_0^t l(u)\,\lambda(u)\,\phi_{P^*}(u)\,du - \int_0^t l(u)\,dM_Q(u)\,,$$

where we just replaced the 'old' martingales (under P) by the drift transformed 'new' local martingales $\widetilde{W}$ and M_Q with respect to P^* using Theorem 3.4. Here $M_Q = N - \int_0^{\cdot} \lambda_{P^*}(u)\,du$, and $\lambda_{P^*} = \lambda\,\phi_{P^*}$ is the intensity of N under P^*.
In each line, the last integral is a local martingale whereas all the Lebesgue integrals are continuous and hence predictable processes of finite variation. As mentioned before, our price processes are local martingales if and only if their stochastic exponents have this property. Further, by the unique decomposition of a special semimartingale, see e.g. Theorem 18, Ch. III, Protter (1995), the stochastic exponents are local martingales if and only if the Lebesgue integrals become zero. As conclusion, the equality system

$$\begin{aligned} 0 &= \frac{1}{2}D(\cdot,T)^2 - A(\cdot,T) - D(\cdot,T)\,\psi^1_{P^*}\,,\\ 0 &= \mu - r + \nu\,\psi^2_{P^*} \end{aligned}$$

and

$$0 = s - l\,\lambda\,\phi_{P^*}\,, \qquad dP\otimes dt \text{ a.s.}$$

is a necessary and sufficient condition for the local martingale property of the discounted price processes.
Straightforward calculations yield that $\psi^1(t), \psi^2(t)$ and $\phi(t)$ defined in (4.1)-(4.3) is the unique

solution $(\psi^1_{P^*}, \psi^2_{P^*}, \phi_{P^*})$ for the above equality system. Thus, due to Assumption 4.1, the measure dQ defined by

$$dQ = L_Q(T)dP\,.$$

with $L_Q(T) \equiv \mathcal{E}\left(\psi^1_Q \cdot W^1 + \psi^2_Q \cdot W^2 + (\phi_Q - 1) \cdot M\right)(T)$ is an equivalent probability measure with respect to P and the discounted price processes are local Q–martingales according to the considerations above.

Further, by Assumption 4.1, L_Q is P-square integrable and the discounted price processes are $\mathcal{H}^2(P)$–semimartingales. Therefore, Corollary A1.5 applies and the discounted price processes are Q–martingales.

We conclude that $Q \in \mathcal{Q}$, where $\mathcal{Q}$ denotes the set of equivalent martingale measures. Moreover, the derivation of the necessary and sufficient conditions yields uniqueness of Q in the sense that $P^* \in \mathcal{Q}$ implies $L_Q = L_{P^*}$ a.s., where L_{P^*} is the density of the change of measure from P to P^*. This completes the proof. □

The next result is an immediate consequence of Corollary 3.6 and Theorem 4.2.

Corollary 4.3 *N is a Q-Cox process conditioned on $(\mathcal{G}_t)$ with unique intensity λ_Q.*

Proof. Due to the definition of ψ^1, ψ^2 and ϕ, Assumption 2.1, 2.2 and 4.1, all conditions of Corollary 3.6 are satisfied and the statement follows. □

Corollary 4.4 *The underlying measure P is a martingale measure if and only if*

$$A(t,T) = \frac{1}{2}D(t,T)^2\,, \tag{4.4}$$

$$\mu(t) = r(t) \text{ and} \tag{4.5}$$

$$s(t) = l(t)\,\lambda(t)\,, \qquad \text{for } 0 \le t \le T. \tag{4.6}$$

Proof. By Theorem 3.4, $P = Q$ if and only if $\psi^1 = \psi^2 = 0$ and $\phi = 1$. □

We are now ready to show the completeness of the market model. For the notation in the next corollary see e.g. Protter (1995), p. 134.

Corollary 4.5 *The market model is complete. More precisely, for every $\mathcal{F}_T$-measurable random variable X with $\mathbf{E}_Q\left\{(X/B(T))^2\right\} < \infty$ there exists a vector process $h = (h_P, h_S, h_C)$ with $h_P \in L(Z_P), h_S \in L(Z_S)$ and $h_C \in L(Z_C)$ such that the discounted value process of X defined by $V(t) \equiv \mathbf{E}_Q\{X/B(T)\,|\mathcal{F}_t\}$ for $0 \le t \le T$ satisfies*

$$V(t) = V(0) + \int_0^t h_P(s)dZ_P(s) + \int_0^t h_S(s)dZ_S(s) + \int_0^t h_C(s)dZ_C(s), \quad 0 \le t \le T\,, \tag{4.7}$$

and $V(T) = X$.

Proof. Let X be an arbitrary random variable satisfying the assumptions of the corollary and V be the corresponding discounted value process. From the martingale representation theorem (see Remark 3.2 in Björk, Kabanov and Runggaldier) we know that V can be written as a stochastic integral with respect to $\widetilde{W}^1, \widetilde{W}^2$ and M_Q, i.e. for every $0 \le t \le T$

$$V(t) = V(0) + \int_0^t \psi^1(s)\,d\widetilde{W}^1(s) + \int_0^t \psi^2(s)\,d\widetilde{W}^2(s) + \int_0^t \phi(s)\,dM_Q(s)\,, \tag{4.8}$$

where $V(0) = \mathbf{E}_Q\{X/B(T)\}$, $\mathbf{E}_Q\{\int_0^T \|\psi(s)\|^2\,ds\} < \infty$ and $\mathbf{E}_Q\{\int_0^T |\phi(s)|^2\,\lambda(s)\,ds\} < \infty$. Note that by definition V is a uniformly and square integrable Q-martingale.

Next, we try to replace in (4.8) the integrators $d\widetilde{W}^1, d\widetilde{W}^2, dM_Q$ by dZ_P, dZ_S, dZ_C in an adequate way in order to get (4.7). For this purpose recall that from Lemma 3.3 and Corollary 3.6 the discounted price processes Z_P, Z_S and Z_C are given for every $0 \le t \le T$ under the equivalent martingale measure Q by

$$Z_P(t) = p(0,T)\mathcal{E}(D(\cdot,T)\cdot\widetilde{W}^1)(t)\,,$$
$$Z_S(t) = S(0)\mathcal{E}(\nu\cdot\widetilde{W}^2)(t)$$

and

$$Z_C(t) = \mathcal{E}(-l\cdot M_Q)(t)\,.$$

Thus in all three cases the discounted price processes have the form $Z = Z(0)\,\mathcal{E}(Y)$, where $Z(0)$ is P-a.s. constant and $Y = H\cdot U$, where $H \in L(U)$. The integrability condition holds since $D(\cdot,T)$ is continuous, $\int_0^T \nu(s)^2 ds < \infty$ a.s. and l is bounded. Further, by Proposition A1.1 we have that $Y = \left(\frac{1}{Z}\right)_- \cdot Z$ and $\left(\frac{1}{Z}\right)_- \in L(Z)$ since $\left(\frac{1}{Z}\right)_-$ is left continuous. Therefore, by Theorem 21, Protter, p. 135, we conclude that

$$U = \frac{1}{H}\cdot(H\cdot U) = \frac{1}{H}\cdot Y = \frac{1}{H}\cdot\left(\left(\frac{1}{Z}\right)_-\cdot Z\right) = \left(\frac{1}{H}\left(\frac{1}{Z}\right)_-\right)\cdot Z \equiv K\cdot Z$$

and $K \in L(Z)$.

Next note that the discounted value process V is the sum of stochastic integrals of the form $\xi\cdot U$ where $\xi \in L(U)$. Again, by Theorem 21, Protter, p. 135, we have that

$$\xi\cdot U = \xi\cdot(K\cdot Z) = (\xi\,K)\cdot Z \equiv h\cdot Z$$

and $h = \xi\,K \in L(Z)$, since $K \in L(Z)$ and $\xi \in L(U) = L(K\cdot Z)$. Every $\mathcal{F}_T$-measurable random variable X with $E(X/B(T)) < \infty$ can be thus duplicated by a self-financing strategy and the market model is complete. □

In what follows, we discuss the assumption that a bank account C of the company exists. Clearly, this is normally not in line with reality. Such an assumption might cause some problems in calibrating the model. Nevertheless, the corollary of the next proposition shows that a zero bond issued by the company – hence affected by default risk – can be seen as a contingent claim in our framework. The statement of the following proposition is more general since it shows that we can price contingent claims in the usual default free way.

Proposition 4.6 *Let the contingent claim X be an $\mathcal{F}_T$–measurable and P–square integrable random variable and let $\pi_X = \{\pi_X(t) : 0 \le t \le T\}$ be the corresponding price process of X, then*

(a) $$\pi_X(t) = \mathbf{E}_Q\left\{\exp\left(-\int_t^T r(u)\,du\right)X\,\middle|\,\mathcal{F}_t\right\}, \qquad \text{for } 0 \le t \le T.$$

(b) *Moreover, if X has the representation $X = \Pi(T)\,Y$, where Y is a $\mathcal{G}_T$–measurable and P–square integrable random variable, then*

$$\pi_X(t) = \Pi(t)\,\mathbf{E}_Q\left\{\exp\left(-\int_t^T (r(u)+s(u))\,du\right)Y\,\middle|\,\mathcal{G}_t\right\}, \qquad \text{for } 0 \le t \le T.$$

Proof. Part (a) is the standard martingale argument; where in (b) we mainly have to use the Cox property of N stated in Assumption 2.2. Corollary 4.3 implies that N is a Q-Cox process with intensity $\lambda_Q = \phi\,\lambda$. Define $M_Q \equiv N - \Lambda_Q$, where $\Lambda_Q \equiv \int_0^{\cdot} \lambda_Q(s)\,ds$. The process M_Q is a $((\widetilde{\mathcal{F}}_t), Q)$-local martingale, where $\widetilde{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{G}_T$.

Since l is bounded on $(0,1)$, the local martingale property is preserved for the process $l \cdot M_Q$. The Doléans Dade exponential of $-l \cdot M_Q$ is $Z \equiv \mathcal{E}(-l \cdot M_Q)$ and clearly a $((\widetilde{\mathcal{F}}_t), Q)$-local martingale. Note that for every $0 \le t \le T$

$$Z(t) = \mathcal{E}(-l \cdot N + l \cdot \Lambda_Q) = \mathcal{E}(-l \cdot N)(t) \exp(\int_0^t l(u)\,\lambda_Q(u)\,du) = \Pi(t)\,\exp(\int_0^t s(u)\,du)$$

because $s = \phi\,\lambda\,l = \lambda_Q\,l$ and $[N, \Lambda_Q] = 0$ since Λ_Q is continuous and of finite variation. Without loss of generality, we may assume Z is a martingale. If this is not the case find a sequence of stopping times (T_n) such that Z^{T_n} is a martingale for each n. The sequence (T_n) is a $(\mathcal{G}_t)$-stopping time and hence $\widetilde{\mathcal{F}}_0$-measurable. This is possible, since $Z > 0$ and Z is bounded by the $\mathcal{G}_T$-measurable expression $\exp(\int_0^T s(u)\,du)$. Passing the limit $n \to \infty$ yields the same results by monotone convergence.

Using the fact that Z is a martingale and s is $(\mathcal{G}_t)$-adapted and thus $\widetilde{\mathcal{F}}_0$-measurable, we derive that for every $0 \le t \le T$

$$\begin{aligned}
\mathbf{E}_Q\left\{\Pi(T)\,\middle|\,\widetilde{\mathcal{F}}_t\right\} &= \mathbf{E}_Q\left\{Z(T)\exp(-\int_0^T s(u)\,du)\,\middle|\,\widetilde{\mathcal{F}}_t\right\} \\
&= \mathbf{E}_Q\left\{Z(T)\,\middle|\,\widetilde{\mathcal{F}}_t\right\}\,\exp(-\int_0^T s(u)\,du) \\
&= Z(t)\,\exp(-\int_0^T s(u)\,du) \\
&= \Pi(t)\,\exp(-\int_t^T s(u)\,du)\,.
\end{aligned}$$

Therefore, for all $0 \le t \le T$

$$\begin{aligned}
\pi_X(t) &= \mathbf{E}_Q\left\{\exp\left(-\int_t^T r(u)\,du\right)\,Y\,\Pi(T)\,\middle|\,\mathcal{F}_t\right\} \\
&= \mathbf{E}_Q\left\{\mathbf{E}_Q\left\{\exp\left(-\int_t^T r(u)\,du\right)\,Y\,\Pi(T)\,\middle|\,\widetilde{\mathcal{F}}_t\right\}\middle|\,\mathcal{F}_t\right\} \\
&= \mathbf{E}_Q\left\{\exp\left(-\int_t^T r(u)\,du\right)\,Y\,\mathbf{E}_Q\left\{\Pi(T)\,\middle|\,\widetilde{\mathcal{F}}_t\right\}\middle|\,\mathcal{F}_t\right\} \\
&= \mathbf{E}_Q\left\{\exp\left(-\int_t^T r(u)\,du\right)\,Y\,\Pi(t)\,\exp\left(-\int_t^T s(u)\,du\right)\middle|\,\mathcal{F}_t\right\} \\
&= \Pi(t)\,\mathbf{E}_Q\left\{\exp\left(-\int_t^T (r(u)+s(u))\,du\right)\,Y\,\middle|\,\mathcal{F}_t\right\}\,.
\end{aligned}$$

Finally, we need to show that we can replace $\mathcal{F}_t$ by $\mathcal{G}_t$ in the last expression. We define

$$Z \equiv \exp\left(-\int_0^T (r(u)+s(u))\,du\right)\,Y\,.$$

The random variable Z is $\mathcal{G}_T$-measurable, in $L^1(Q)$ and

$$\pi_X(t) = \Pi(t)\,\exp\left(\int_0^t (r(u)+s(u))\,du\right)\,\mathbf{E}_Q\left\{Z\,|\,\mathcal{F}_t\right\}\,.$$

Note that the processes $\mathbf{E}_Q\{Z \mid \mathcal{F}_t\}$ and $\mathbf{E}_Q\{Z \mid \mathcal{G}_t\}$ are uniformly integrable martingales closed by the same random variable Z and $\mathbf{E}_Q\{Z \mid \mathcal{F}_T\} = Z = \mathbf{E}_Q\{Z \mid \mathcal{G}_T\}$. Thus a sufficient condition for the equality of these conditional expectation for all t is to show that $\mathbf{E}_Q\{Z \mid \mathcal{G}_t\}$ is a $\mathcal{F}_t$-martingale. However, due to the martingale representation theorem for standard Brownian motions (see e.g. Theorem 42 in Protter, p.155) we can represent $\mathbf{E}_Q\{Z \mid \mathcal{G}_t\}$ as a stochastic integral with respect to the Q-Brownian motion $\widetilde{W}$. Note, that (W^1, W^2) and $(\widetilde{W}^1, \widetilde{W}^2)$ generate both the filtration $(\mathcal{G}_t)$, since $\widetilde{W}^k = W - \int_0^{\cdot} \psi^k(t)\, dt$ and ψ^k is adapted to the internal history of W, $(\mathcal{G}_t)$. Theorem 3.4 states that $\widetilde{W}$ is a Brownian motion on $(\mathcal{F}_t)$, hence $\mathbf{E}_Q\{Z \mid \mathcal{G}_t\}$ is a $((\mathcal{F}_t, Q)$-martingale. □

Corollary 4.7 *Let $v(\cdot, T) = \{v(t,T) : 0 \le t \le T\}$ be the price process of the contingent claim $X \equiv \Pi(T)$ such that $v(\cdot, T)$ is a zero bond with default risk and maturity T. Then*

$$v(t,T) \quad = \quad \Pi(t)\, \mathbf{E}_Q \left\{ \exp\left(- \int_t^T (r(u) + s(u))\, du \right) \middle| \mathcal{G}_t \right\}, \qquad \text{for } 0 \le t \le T.$$

Proof. Directly Proposition 4.6. □

Contingent claim valuation leads in such a setting directly to the well-known problem of pricing credit derivatives. Our setup belongs to the intensity based approaches, and the credit derivative pricing has been widely studied in such frameworks, see e.g. Schönbucher (1998) and Lando (1997). In what follows we mainly focus on convertible bonds. For an introduction in convertible bonds see for instance Davis and Lischka (1999).

Convertible bonds are a combination of simple securities (bonds) and derivative securities. They are bonds which at the option of the holder can be converted into a specified number of common stock shares. They are referred to as hybrid securities since they contain both fixed income and equity components.

A convertible bond can be seen as the equivalent to the embedded corporate (default) bond plus an American option on the underlying stock with a changing strike price equal to the price of the embedded bond.

We assume that the stock pays no dividends. This is usually not restrictive since convertible bonds were originally developped for companies with poor credit. Such companies do not pay dividends. It is known that under this condition the pricing of a convertible bond simplifies to the pricing of a convertible bond which has European style, i.e. which can be only converted at the maturity $T_1 \le T$. The payoff of a European convertible bond at time T_1 is in the above spirit given by

$$\begin{aligned} X_E &= v(T_1, T) 1_{\{v(T_1,T) \ge c_0 S(T_1)\}} + c_0 S(T_1) 1_{\{v(T_1,T) < c_0 S(T_1)\}} \\ &= v(T_1, T) + (c_0 S(T_1) - v(T_1, T))_+ \,, \end{aligned} \tag{4.9}$$

where $c_0 \ge 0$ denotes the number of shares specified at time $t = 0$ that can be converted at T_1. Note that if $c_0 = 0$ then we have just the contingent claim of a defaultable bond. Pricing a defautable bond is therefore a special case of pricing a convertible bond.

The next lemma states once again the well-known property that the option of converting a convertible bond issued by a company not paying dividends before T_1 is worthless.

Lemma 4.8 *In our market model, the price process of a convertible bond with no divident payments and maturity T_1 is given for every $0 \leq t \leq T_1$ by*

$$\begin{aligned} c(t,T_1) &= \mathbf{E}_Q \left\{ \exp\left(- \int_t^{T_1} r(u)du \right) X_E \,\middle|\, \mathcal{F}_t \right\} \\ &= v(t,T) + \mathbf{E}_Q \left\{ \exp\left(- \int_t^{T_1} r(u)du \right) (c_0 S(T_1) - v(T_1,T))_+ \,\middle|\, \mathcal{F}_t \right\}. \quad (4.10) \end{aligned}$$

Proof. For completeness we briefly sketch the proof. Let $c_E(t,T_1), 0 \leq t \leq T_1$, be the price of a convertible bond without the option to convert before maturity (European style). Before proceeding, we claim that $c_E(t,T_1) \geq c_0 S(t)$ for all $0 \leq t \leq T_1$. If there exists $t \in [0,T_1]$ such that $c_E(t,T_1) < c_0 S(t)$ an arbitrage exists. To see this buy the convertible bond and sell short the stock for an initial cash flow of $c_0 S(t) - c_E(t,T_1) > 0$. Hold the position until the maturity of the bond. Convert the bond into stock and use this stock to cover the short position. The cash flow will be zero. Thus, the assumption that $c_E(t,T_1) < c_0 S(t)$ for every $t \in [0,T_1]$ gives us an initial positive cash flow with no risk of future loss. We conclude that $c_E(t,T_1) \geq c_0 S(t)$ for all $0 \leq t \leq T_1$.

Now we show that $c(t,T_1) \leq c_E(t,T_1)$ for all $0 \leq t \leq T_1$. If $c(t,T_1) > c_E(t,T_1)$ for some $0 \leq t \leq T_1$ an arbitrage exists. Buy the European convertible bond and sell the American bond for an initial cash flow of $c(t,T_1) - c_E(t,T_1) > 0$. If the counterparty converts before maturity, sell the European convertible bond and use the proceeds to purchase the stock required to cover the short position in the American convertible bond. Since we have shown that $c_E(t,T_1) \geq c_0 S(t)$ for all $0 \leq t \leq T_1$, the cash flow will be non-negative. If the counterparty holds until maturity, the two instruments are identical. Thus, the assumption that $c(t,T_1) \leq c_E(t,T_1)$ for all $0 \leq t \leq T_1$ gives us an initial positive cash flow with no risk of future loss. This is a contradiction to the assumption that the market is arbitrage free.

Finally, since the holder of the American convertible bond has all of the conversion opportunities as the holder of the European bond, it must be also that $c(t,T_1) \geq c_E(t,T_1)$ for all $0 \leq t \leq T_1$. By Proposition 4.6 and (4.9), the statement follows. □

5 A martingale model

We consider in this section an illustrative implementation of the continuous market model introduced in the previous sections under the equivalent martingale measure. Using the results which we developed so far, we compute numerically the fair price of a convertible bond.

Having evaluated theoretically the price process of a convertible bond in the last section, our objective is to compute and compare for illustrative purposes the fair price numerically for a particular martingale model which fits in our setting. We consider the market model directly under the equivalent martingale measure Q, i.e. all processes that we study are modelled under the equivalent martingale measure Q. In particular, all relations in Corollary 4.4 must hold. A very important task of the martingale modeling is the choice of the riskless spot and forward rate process in (2.4) and (2.3), respectively. We assume that the spot rate process for riskless debt is given by a Cox-Ingersoll-Ross model. In other words, $\{r(t) : 0 \leq t \leq T\}$ satisfies the stochastic differential equation

$$dr(t) = \alpha(\beta - r(t))dt + \sigma\sqrt{r(t)}dW^1(t)\,, \quad t \geq 0\,, \tag{5.1}$$

where $r(0) = r_0 > 0, \alpha > 0, \sigma > 0$ and $\beta > \sigma^2/(2\alpha)$. In contrast to the Vasicek model, $\{r(t) : 0 \leq t \leq T\}$ given by (5.1) fulfills the condition $r(t) > 0$ a.s. for any $t \geq 0$ (see Assumption 2.1(a)). Moreover, the Cox-Ingersoll-Ross model still has nice computational properties such as the existence of a so called *affine term structure* (see e.g. Baxter and Rennie (1996) or Björk (1997)). By section 5.4 in Baxter and Rennie (1996) we conclude after some straightforward but tedious calculations that the Heath-Jarrow-Morton one factor model is completely specified by (5.1) and is given for every $0 \leq t \leq T$ by

$$\sigma(t,T) = \sigma^3 \sqrt{r(t)}\,(\alpha + c(\alpha)) \left(\frac{1}{2(\alpha + c(\alpha))} - \frac{1}{c(\alpha)} \right) \exp\left((\alpha + c(\alpha))(T-t)\right) Z(t,T)^{-2} \quad (5.2)$$

and

$$D(t,T) = \int_t^T \sigma(t,u)du = \sigma \sqrt{r(t)} \left(Z(t,T)^{-1} + \frac{c(\alpha)}{\sigma^2} \right), \quad \text{respectively,} \quad (5.3)$$

where

$$Z(t,T) = -\frac{\sigma^2}{2(\alpha + \sigma^2 c(\alpha))} + \left(\frac{\sigma^2}{2(\alpha + \sigma^2 c(\alpha))} - \frac{1}{c(\alpha)} \right) \exp\left(-2(\alpha + c(\alpha))(T-t)\right)$$

and $c(\alpha) = -\alpha - \sqrt{\alpha^2 + 2\sigma^2}$. Recall once again that we are modeling under the equivalent martingale measure and hence the drift term $\alpha(t,T)$ and $A(t,T)$, for $0 \leq t \leq T$, can be easily established from (5.2) and (5.3), respectively, using Corollary 4.4. Further, by Proposition 3.5 of Björk (1997), the price process of the defautable bond can be written for every $0 \leq t \leq T$ as

$$p(t,T) = \exp\left(-\alpha\beta \int_t^T (Z(u,T)^{-1} + c(\alpha))du - (Z(t,T)^{-1} + c(\alpha))r(t) \right). \quad (5.4)$$

For the the stock price process $S = \{S(t) : 0 \leq t \leq T\}$ in (2.8) we choose $\nu(t) = \nu$ whereas $\mu(t)$ is the riskless short rate $r(t)$.

It remains to specify the intensity of the default process $\{N(t) : 0 \leq t \leq T\}$. Again because of Corollary 4.4, modeling the intensity of the Cox process is in our framework equivalent with modeling the spread $s(t)$ and the loss rate $l(t)$, $t \geq 0$. Due to our market model, the quantities $s(t)$ and $l(t)$ can be described as functions of the short term process and the stock price process, i.e.

$$s(t) = f(\{r(u), S(u) : 0 \leq u \leq t\}) \quad \text{and} \quad l(t) = g(\{r(u), S(u) : 0 \leq u \leq t\})$$

for some measurable functions f and g. The properties of the functions f and g remain to be specified. Intuitively, it is clear that if the stock price process of the company is large and the negative price changes are small then default is very unlikely and the loss rate should also be small. Moreover, if the short rate is low the company can borrow money at a low rate of interest which is of course less risky and default is again not likely. For calibrating purposes, we assume that the functions f and g should not be too complicated.

The following linear approach takes our above considerations into account. We set for every $0 \leq t \leq T$

$$\begin{aligned} s(t) &= a_0 + a_1 r(t) + a_2 f_1(S(t)) + a_3 f_2(t,S), \\ l(t) &= g(S(t)), \end{aligned} \quad (5.5)$$

where $S = \{S(u) : 0 \leq u \leq T\}, a_0, a_1, a_2, a_3 \geq 0$, $f_1 : (0,\infty) \longrightarrow (0,\infty)$ and $g : (0,\infty) \longrightarrow (0,1)$ are non-increasing and $f_2 : [0,T] \times \mathcal{C}([0,T],(0,\infty)) \longrightarrow [0,\infty)$. Examples for f_1 and g, respectively, are

$$\text{const}\,, \quad (1+x)^{-1}\,, \quad e^{-x}\,, \quad \text{or} \quad (1+\log(1+x))^{-1}\,, \quad x > 0\,,$$

whereas a natural choice for f_2 for every $0 \leq t \leq T$ and $x \in \mathcal{C}([0,T],(0,\infty))$ is for instance

$$1_{\{\inf\{\log x(u) - \log x(v)\,:\,u,v \in [0 \vee t-h, t]\} < z\}}\,, \quad \text{for } h > 0, z < 0 \text{ fixed}\,.$$

Figure 1 and 2 show numerical results for the price processes $\{c(t,T) : 0 \leq t \leq T_1\}$ and $\{v(t,T) : 0 \leq t \leq T_1\}$ (the case $c_0 = 0$) using the proposed setting for two examples. The implementation and testing of the suggested framework will be topic of a subsequent work and therefore the chosen parameters should be seen primarly as an illustration of our modeling.

The top two plots represent in both figures a simulated path of the stock price process $\{S(t) : 0 \leq t \leq T\}$ (left) and of the short rate process $\{r(t) : 0 \leq t \leq T\}$ (right). The dotted line in the right upper picture denotes the default adjusted short rate $\{r(t) + s(t) : 0 \leq t \leq T\}$. The spread $s(\cdot)$ was computed by formula (5.5) with $f_1(x) = (1+x)^{-1}$, $x > 0$, and $f_2(t,x) = 1_{\{\inf\{\log x(u) - \log x(v)\,:\,u,v \in [0 \vee t-0.4, t]\} < 0.18\}}$, $t \in [0,T]$, $x \in \mathcal{C}([0,T],(0,\infty))$. The chosen parameters $\nu, \alpha, \beta, a_0, a_1, a_2$ and a_3 are always summarized at the bottom of the plots.

The pictures in the second row in Figure 1 and 2 show the corresponding sample path of the Cox process (left) and of the stochastic intensity process (right). For simplicity, we set in both cases the recovery rate $l \equiv 0.5$.

The lower three plots in Figure 1 and 2 show the numerical results of our simulations. They indicate the corresponding price processes of a convertible bond with maturity $T_1 = 18$ and $T_1 = 19$, respectively, (solid line), of the defautable zero-bond with maturity $T = 20$ and of the contingent claim $c_0 S(T_1)$ at time T_1 (dotted lines). Note that the price process of the contingent claim $c_0 S(T_1)$ is clearly given by $\{c_0 S(t) : 0 \leq t \leq T_1\}$ because of the martingale property of the stock price process under the equivalent martingale measure. The three plots are at each case generated by a different value of c_0 - the specified amount of common stock shares which can be obtained in the case of conversion. The upper two pictures present in both figures the price process of the convertible bond in a bullish market (convertible bonds behave more like stocks, left) and in a bearish market (convertible bonds behave more like bonds, right). The last plot shows the fair price for a particular c_0 in the region between the two economic extremes.

All diffusion processes were simulated by means of the Milstein scheme with stepsize $m = 0.1$ (strong Taylor approximation of convergence order 1) and we refer to Kloeden and Platen (1992) for details. The price processes for the convertible and defautable bonds have been computed by Monte-Carlo simulation. Because of the large computation complexity we have chosen at each case only 100 simulations. However, the small number of simulations is justified by our results.

Figure 1 and 2 indicate that the sample paths of the price processes are reasonable smooth. Further, except in the bullish market, we can observe that the prices of convertible and defautable bonds occur to have negative jumps whenever the correspondig Cox-process increases. This behavior is especially good visible in Figure 2. Note also that in Figure 1 all defaults happened because of the constant low level of the stock price process. The spread in Figure 1 is dramatically increasing in time. Considering the history of the stock price process the explanation might be that there is little hope that the stock price process will improve again in the future. Further defaults of the company are therefore likely.

Figure 2 shows a different scenario. Here, the first default appeared because of large negative changes of the (logarithmical) stock prices. The difference between the two consecutive prices at the time of the first default is -0.22. In contrast to the first case the spread is not increasing in time. The stock price process has developped differently. Although the stock price process is low at the end chances are still that it recover again.

In conclusion, we may say that our modeling approach yields a lot of feasible scenarios. The simulations produce realistic results and confirm that our modeling is reasonable.

Appendix

A1 Appendix

Here, we give a short introduction to the stochastic analysis we need in this paper. For details we refer to Protter (1995).

In the following, we work on a filtered probability space $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0}\right)$ satisfying the *usual hypotheses*. The processes $X = \{X(t) : t \geq 0\}$ and $Y = \{Y(t) : t \geq 0\}$ are semimartingales with $X(0) = Y(0) = 0$. The process $H = \{H(t) : t \geq 0\}$ is predictable.

First, we remind the reader of some notation. With the expression $X(t-)$ we associate $X(t-) = \lim_{s\nearrow t} X(s)$ for $t > 0$ and $X(0-) = 0$. The jump part $\Delta X = \{\Delta X(t) : t \geq 0\}$ of a semimartingale X is defined by $\Delta X(t) = X(t) - X(t-)$. If $\sum_{0<s\leq t} |\Delta X(s)| < \infty$ a.s., each $t \geq 0$, we can define the semimartingale $X^c = \{X^c(t) : t \geq 0\}$ which is the continuous part of the semimartingale X; i.e. $X^c(t) = X(t) - \sum_{0\leq s\leq t} \Delta X(s)$.

For the stochastic Itô integral we often use an abbreviation

$$H \cdot X \quad = \quad \int_0^{\cdot} H(s)\, dX(s). \tag{A.1}$$

Now, we recapture the definition of the Doléans Dade exponential. Let us consider the stochastic integral equation

$$Z(t) \quad = \quad 1 + \int_0^t Z(s-)\, dX(s)\,, \qquad \text{for } t \geq 0, \tag{A.2}$$

for a given semimartingale X. Equation (A.2) has the unique solution for $t \geq 0$

$$Z(t) = \exp\left(X(t) - \frac{1}{2}[X,X](t)\right) \prod_{0<s\leq t} (1 + \Delta X(s)) \exp\left(-\Delta X(s) + \frac{1}{2}\left(\Delta X(s)\right)^2\right), \tag{A.3}$$

alternatively,

$$Z(t) = \exp\left(X(t) - \frac{1}{2}[X,X]^c(t)\right) \prod_{0<s\leq t} (1 + \Delta X(s)) \exp\left(-\Delta X(s)\right)\,, \tag{A.4}$$

or, if $\sum_{0<s\leq t} |\Delta X(s)| < \infty$ a.s., for each $t \geq 0$,

$$Z(t) = \exp\left(X^c(t) - \frac{1}{2}[X,X]^c(t)\right) \prod_{0<s\leq t} (1 + \Delta X(s))\,. \tag{A.5}$$

The Doléans Dade exponential is also known as the *stochastic exponential of* X, written $\mathcal{E}(X)$.

Strictly positive semimartingales have the nice property that they can be represented as a Doléans Dade exponentials.

Proposition A1.1 *A strictly positive semimartingale Z with $Z(0) = 1$ allows the representation as Doléans Dade exponential $Z = \mathcal{E}(X)$, where X is unique; i.e.*

$$X(t) = \int_0^t \left(\frac{1}{Z}\right)(s-)\, dZ(s)\,, \qquad \textit{for } t \geq 0. \tag{A.6}$$

Corollary A1.2 *With the notation of the Proposition, Z is a local martingale iff X is a local martingale.*

The proof of the proposition is just the construction suggested by (A.6). The Corollary is a consequence of the equations (A.2) and (A.6), because Z is a stochastic integral with respect to X (with a càglàd integrand) and vice versa.

Equation (A.3) has some interesting special cases.

(*a*) If the semimartingale X has P–a.s. continuous paths, then

$$\mathcal{E}(X) = \exp\left(X - \frac{1}{2}[X,X]\right)\,, \tag{A.7}$$

and

$$\exp(X) = \mathcal{E}\left(X + \frac{1}{2}[X,X]\right)\,. \tag{A.8}$$

(*b*) If the semimartingale X has P–a.s. continuous paths of finite variation, then

$$\exp(X) = \mathcal{E}(X)\,. \tag{A.9}$$

(*c*) If the semimartingale X is pure jump; i.e. $X^c = 0$, then

$$\mathcal{E}(X)(t) = \prod_{0<s\leq t} (1 + \Delta X(s))\,, \quad \text{for } t \geq 0. \tag{A.10}$$

Doléans Dade exponentials have also 'good' properties with respect to multiplication.

(*a*) For two semimartingales X and Y with $X(0) = Y(0) = 0$ we have

$$\mathcal{E}(X)\ \mathcal{E}(Y) = \mathcal{E}(X + Y + [X,Y])\,. \tag{A.11}$$

(*b*) If X has P–a.s. continuous paths and either X or Y has paths of finite variation P–a.s. then

$$\mathcal{E}(X)\ \mathcal{E}(Y) = \mathcal{E}(X + Y)\,. \tag{A.12}$$

space

A classical semimartingale X has decomposition $X = M + A$ where M is a local martingale and A is a FV process. This decomposition is in general not unique. The Girsanov-Meyer Theorem presents a possible decomposition of $X = L + C$ after a change of measure.

Theorem A1.3 (Girsanov-Meyer) *On a filtered probability space* $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0}\right)$ *let* Q *be an equivalent measure with respect to* P *and the process* $Z = \{Z(t) : t \geq 0\}$ *be defined by*

$$Z(t) \equiv \mathbf{E}_P \left\{ \frac{dQ}{dP} \middle| \mathcal{F}_t \right\}, \qquad \text{for } t \geq 0. \tag{A.13}$$

Let X *be a classical semimartingale under* P *with decomposition* $X = M + A$*, where* M *is a local martingale and* A *is a FV process. Then* X *is also a classical semimartingale under* Q *and has a decomposition* $X = L + C$*, where*

$$L(t) = M(t) - \int_0^t \frac{1}{Z(s)}\, d[Z, M](s)\,, \qquad \text{for } t \geq 0, \tag{A.14}$$

is a Q*–local martingal and* $C = X - L$ *is a* Q*–FV process.*

In the following, we consider a filtered probability space $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\leq t\leq T}\right)$ satisfying the *usual hypotheses* with a finite time horizon $T > 0$. Let X be a special semimartingale; i.e. X has a unique decomposition

$X = M + A$, where M is a local martingale, A is a predictable process with paths of finite variation P–a.s. and $M(0) = A(0) = 0$. The $\mathcal{H}^2$ norm of X on the intervall $[0, T]$ is defined by

$$||X||_{\mathcal{H}^2} = ||[M, M](T)^{1/2}||_{L^2} + ||\int_0^T |dA(s)|\ ||_{L^2}. \tag{A.15}$$

The next theorem, Theorem 5 Chapter IV, Protter (1995), has a usefull corollary.

Theorem A1.4 *Let* X *be a special* $\mathcal{H}^2$*–semimartingale. Then* $\mathbf{E}\left\{\left(\sup_{0\leq t\leq T}|X(t)|\right)^2\right\} < 8\,||X||^2_{\mathcal{H}^2}$.

Corollary A1.5 *Let* X *be a special* $\mathcal{H}^2(P)$*–semimartingale; i.e.* X *has a finite* $\mathcal{H}^2$ *norm under the measure* P*. Let* L *be a square integrable* P*–martingale defining the measure* Q *by* $dQ = L(T)\,dP$*. If* X *is a local* Q*–martingale, then* X *is a* Q*–martingale.*

Proof. By the inequality

$$\begin{aligned}
\mathbf{E}_Q\left\{\sup_{0\leq t\leq T}|X(t)|\right\} &= \mathbf{E}_P\left\{\sup_{0\leq t\leq T}|X(t)|\,L(T)\right\} \\
&\leq \frac{1}{2}\left(\mathbf{E}_P\left\{\left(\sup_{0\leq t\leq T}|X(t)|\right)^2\right\} + \mathbf{E}_P\left\{L(T)^2\right\}\right) \\
&\leq \frac{1}{2}\left(8\,||X||^2_{\mathcal{H}^2} + \mathbf{E}_P\left\{L(T)^2\right\}\right) \quad < \quad \infty
\end{aligned}$$

we have shown a sufficent condition for a local martingale to be a martingale, Theorem 47 Chapter I, Protter (1995). □

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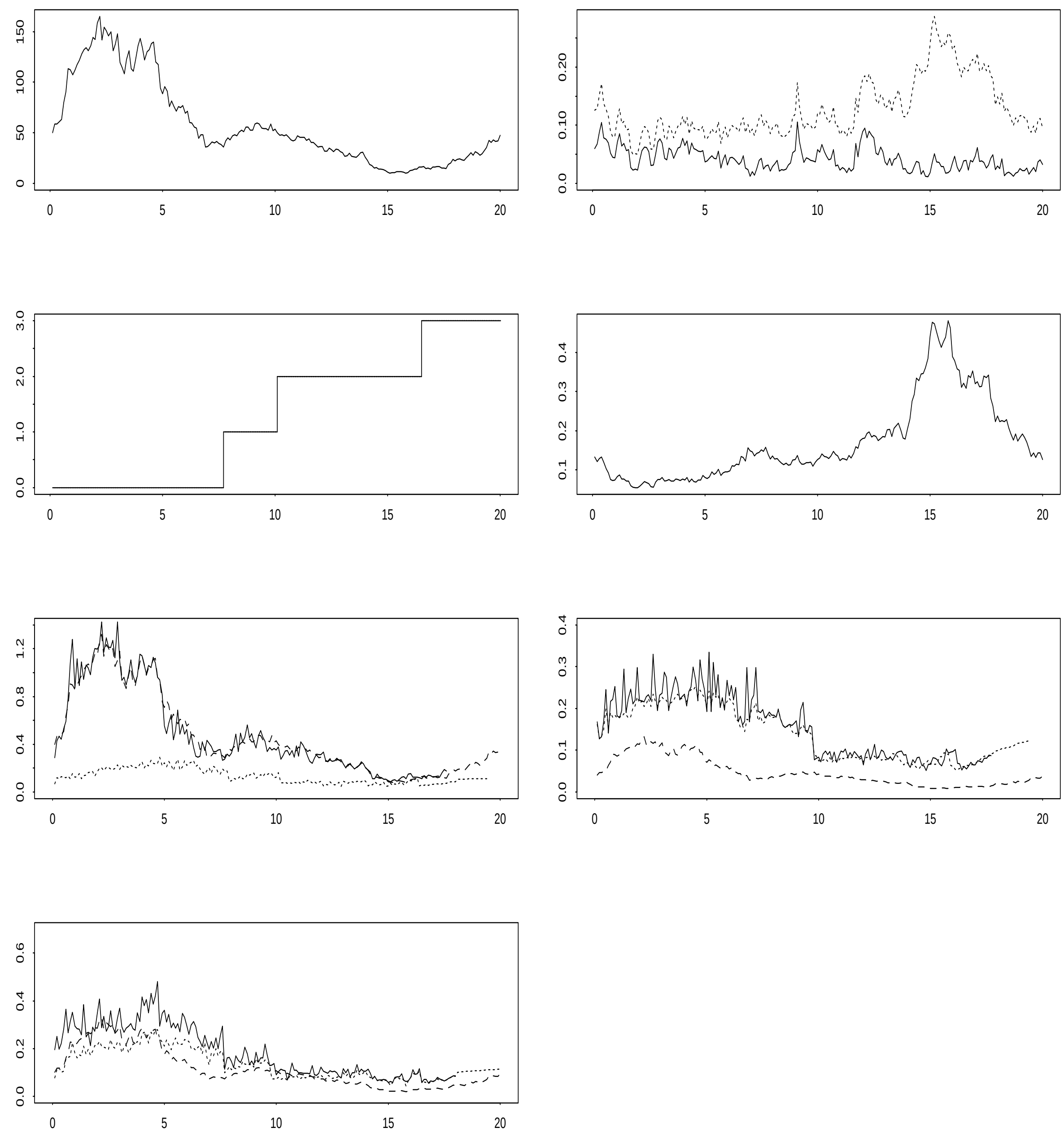

Figure 1: Chosen parameters: $\nu = 0.3$, $\alpha = -2$, $\beta = 0.1$, $\sigma = 0.2$, $a_0 = 0.005$, $a_1 = 0.3$, $a_2 = 0.5$, $a_3 = 0$, $z = -0.18$, $l = 0.5$, $N = 100$, $m = 0.1$, $T = 20$, $T_1 = 18$ and $c_0 = 0.008$ (third row,left), 0.0008 (third row,right) and 0.002 (bottom,left), respectively.

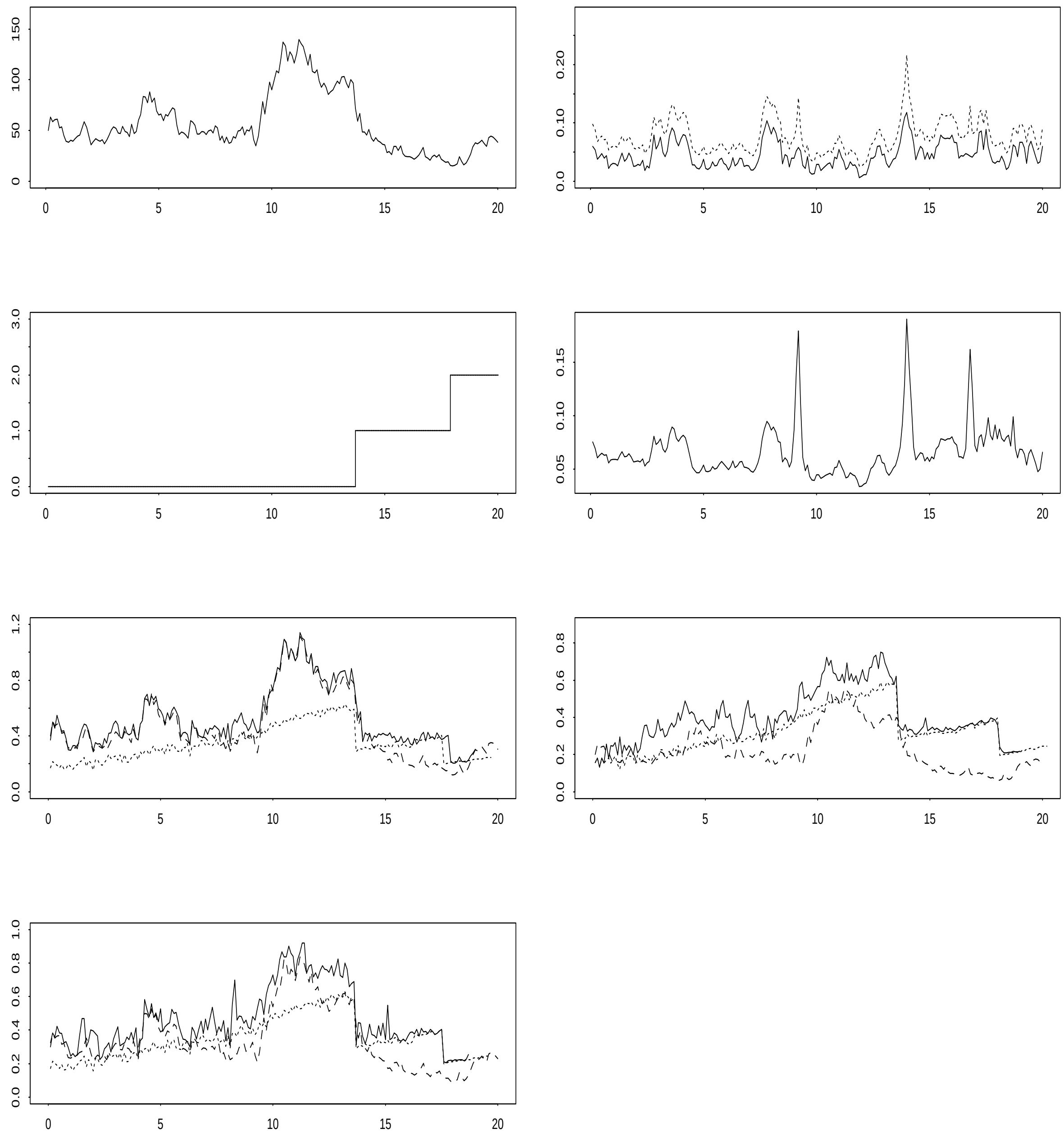

Figure 2: Chosen parameters: $\nu = 0.3$, $\alpha = -2$, $\beta = 0.1$, $\sigma = 0.3$, $a_0 = 0.005$, $a_1 = 0.2$, $a_2 = 0.3$, $a_3 = 0.08$, $z = -0.18$, $l = 0.5$, $N = 100$, $m = 0.1$, $T = 20$, $T_1 = 19$ and $c_0 = 0.008$ (third row,left), 0.002 (third row,right) and 0.004 (bottom,left), respectively.