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Random Walks in Random Environment

Random Orientations and Branching

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Contents

Introduction

In this thesis, we study Random Walks in Random Environment (RWRE) in different settings. The study of RWRE goes back to models introduced by Chernov ([Ch62]) and Temkin ([Te72]) as toy-models for the replication of DNA-chains. Later, Solomon ([So75]) gave a rigorous construction of the probability measures of the system which started an intensive study of RWRE by probabilists. Already in one dimension, RWRE show several unusual phenomena – such as sub-diffusive behaviour, aging phenomena, trapping effects, ... – which make its study so interesting and rewarding. Over the last decades, the understanding of RWRE in one dimension has reached a very high level. However in the multidimensional case, RWRE are still less understood and even many of the basic questions are still unresolved.

In Part I of this thesis (Random Walks in Random Environment), we are mainly interested in a model that lies between a one-dimensional and a multidimensional setting: We consider the movement of a particle on the two-dimensional lattice $\mathbb{Z}^2$, but the random environment is only one-dimensional, i.e. it only depends on the first component of each position. Our main question will be if the considered random walks are recurrent or transient, i.e. if they return to the origin infinitely often or not.

In Chapter 2 (Strong Recurrence of Recurrent RWRE), we start with one of the simplest extensions of the one-dimensional RWRE to a random walk on $\mathbb{Z}^2$: What happens if we consider a combination of the recurrent one-dimensional RWRE in the first coordinate and a symmetric random walk on $\mathbb{Z}$ in the second coordinate (cf. Corollary 2.6.6 and Corollary 2.6.7 for a precise description of the model)? Will the two-dimensional random walk become transient?

For an answer to this question, we start with an analysis of the return probabilities of the one-dimensional recurrent RWRE to the origin. In contrast to the symmetric random walk on $\mathbb{Z}$, the return probabilities of the RWRE do not only depend on time but also on the environment ω (cf. (1.3)). As a main tool, we will construct favourable “valleys” of the environment (cf. Figure 2.1 on page 23) which make it easier for the RWRE $(X_n)_{n\in\mathbb{N}_0}$ to return to 0 as long as it has not left this valley. Using this approach, we will get the following two main results in Theorem 2.4.1 and Theorem 2.4.2:
For P-a.e. environment ω, we have

$$\sum_{n\in\mathbb{N}} P_\omega(X_{2n}=0)\cdot n^{-\alpha}=\infty \tag{1}$$

for $0 \leq \alpha < 1$ and

$$\sum_{n\in\mathbb{N}} \Big(P_\omega(X_{2n}=0)\Big)^\alpha = \infty$$

for all $\alpha > 0$. Those two analytical statements (and the analogous statement for a combination of $d \in \mathbb{N}$ i.i.d. environments in Theorem 2.4.3) enable us to answer the question of recurrence and transience for a lot of interesting examples of random walks in different random environments (cf. Section 2.6 – Examples for Recurrent Random Walks in Random Environments). As a first result, we will recover the statement that a combination of d independent recurrent RWRE – in the same or d i.i.d. environments – is still recurrent for a.e. environment. Additionally, we can conclude that the combination of a recurrent one-dimensional RWRE and a symmetric random walk on $\mathbb{Z}$ (which we were originally interested in) is still recurrent (cf. Corollary 2.6.6 and Corollary 2.6.7).

Therefore, the question of recurrence or transience of the two-dimensional process is still interesting if we add an additional source of inhomogeneity to the random environment. To this end, we introduce the model of a RWRE with random orientations (RWRO) in Chapter 3. In this model, the one-dimensional random environment additionally contains a sequence of i.i.d. random orientations $(\alpha_x)_{x\in\mathbb{Z}}$ taking the values -1 and $+1$ each with probability $\frac{1}{2}$. In contrast to the combination of the recurrent one-dimensional RWRE with the symmetric random walk on $\mathbb{Z}$, the particle in the RWRO can only move upwards or downwards (i.e. move in the second component) in the direction of the random orientation. Thereby, the first component is still treated as in the setting of a one-dimensional RWRE. For more details on the model we refer to Section 3.2.
An analysis of a similar model containing the symmetric random walk on $\mathbb{Z}$ instead of the recurrent one-dimensional RWRE can be found in [CP03a]. Note here that the return probabilities of the recurrent one-dimensional RWRE behave very differently in comparison to the return probabilities of a symmetric random walk on $\mathbb{Z}$: For the symmetric random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$, we have for example

$$\sum_{n\in\mathbb{N}} P(S_{2n}=0)\cdot n^{-\alpha} < \infty$$

for all $\alpha > \frac{1}{2}$ in contrast to (1). Therefore, we need a different approach in comparison with [CP03a] in order to answer the question of recurrence and transience of RWRO, which goes as follows.
It is well known that the one-dimensional RWRE spends most of its time close to "bottom points" of the environment. This makes the RWRE very sensitive to local inhomogeneities as in the setting of RWRO. By using the observation that the random walk picks up a lot of orientations pointing in the same direction close to the bottom points, we will show that the RWRO is transient for a.e. environment – as in the setting of [CP03a] – even though the recurrence in the first component is stronger compared to the symmetric random walk on $\mathbb{Z}$.
By combining the results of Chapter 2 and Chapter 3, we can further answer the question of

recurrence and transience of the following process which depends on the parameter $p \in [0,1]$. Here p can be understood as the strength of the inhomogeneity of the environment in the vertical direction.
At first, we choose an environment θ which consists of the environment $\omega = (\omega_x)_{x\in\mathbb{Z}}$ of a recurrent one-dimensional RWRE and i.i.d. orientations $(\alpha_x)_{x\in\mathbb{Z}}$ taking the values -1 and $+1$ each with probability $\frac{1}{2}$ (cf. (3.4)). Then, for $0 < \delta < 1$ and $0 \le p \le 1$, we can introduce the following Markov chain $(M_n)_{n\in\mathbb{N}_0}$ with values in $\mathbb{Z}^2$ which is determined by

$$\begin{aligned}
&P_\theta\big(M_0 = (0,0)\big) = 1,\\
&P_\theta\big(M_{n+1} = (k+1,\ell)\big|M_n = (k,\ell)\big) = \delta \cdot \omega_k,\\
&P_\theta\big(M_{n+1} = (k-1,\ell)\big|M_n = (k,\ell)\big) = \delta \cdot (1-\omega_k),\\
&P_\theta\big(M_{n+1} = (k,\ell+1)\big|M_n = (k,\ell)\big) = \frac{1-\delta}{2} \cdot (1 + p\cdot\alpha_k),\\
&P_\theta\big(M_{n+1} = (k,\ell-1)\big|M_n = (k,\ell)\big) = \frac{1-\delta}{2} \cdot (1 - p\cdot\alpha_k).
\end{aligned}$$

Here, in each step, δ and $1-\delta$ are the probabilities for a movement of the particle in the first and the second component, respectively. In Section 3.6, we will see that we have the following dependence on the parameter p:

Theorem 3.6.1 *(1) For $p = 0$, the Markov chain $(M_n)_{n\in\mathbb{N}_0}$ is recurrent for* P*-a.e. environment θ.*

(2) On the contrary for $0 < p \le 1$, the Markov chain $(M_n)_{n\in\mathbb{N}_0}$ is transient for P*-a.e. environment θ.*

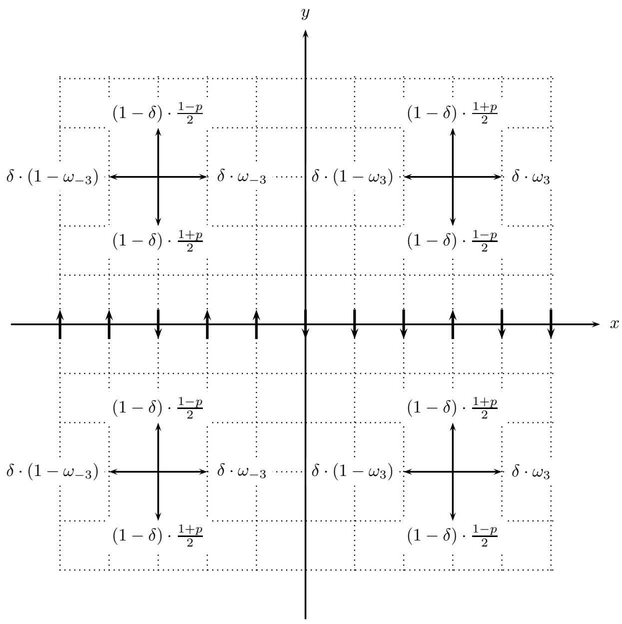

Figure 1: *A possible realization of the random orientations ↑↓ and the corresponding transition probabilities.*

In Part II (Random Walks in Random Environment with Branching), we consider a different class of models. Here, we additionally allow particles to branch which leads to the model of a Branching Random Walk in Random Environment (BRWRE). BRWRE can be used as a model for the spread of particles in inhomogeneous media. Additionally, the analysis of BRWRE is very interesting and useful from a purely mathematical point of view. For example, it turned out that the analysis of BRWRE seems to be easier than the analysis of RWRE (with only one particle) in the multidimensional case.

In Chapter 5 (Survival and Growth of a BRWRE), we analyse the behaviour of a BRWRE on $\mathbb{N}_0$ in which particles produce offspring which either stay at the same location or move one step to the right. In the considered model, we also allow the particles to produce no offspring. Therefore, our first question aims at survival and extinction of the process. For the answer, we introduce two different survival regimes – Local Survival and Global Survival – and give explicit criteria for these two types of survival.
In the following part, we answer the question of the growth of the process. We show that global survival is equivalent to the exponential growth of the expected number of particles. Further, on the event of survival the number of particles grows almost surely exponentially fast with the same growth rate as the expected number of particles.
In general, the drift parameter h_x, which gives the probability for each offspring to move one step to the right, may depend on the location x of the particles. As an interesting special case, we can choose $h_x \equiv h \in (0, 1]$ to be constant. For this case, we answer the question of survival of the process depending on the drift h. In particular, we show that if the BRWRE survives with positive probability for some $\overline{h} \in (0, 1]$, then it also survives with positive probability for all smaller drifts $0 < h \leq \overline{h}$.

The structure of this thesis is the following: As mentioned before, the thesis consists of two major parts: The first one (Random Walks in Random Environment) deals with a RWRE with one particle and in the second part (Random Walks in Random Environment with Branching) we allow the particles to branch. Each part begins with a chapter "Preliminaries", where we give an introduction to the basic setting and our general assumptions. Further, we recall some of the classical and more recent results on RWRE (in Section 1.2) and on BRWRE (in Section 4.2). This enables us to describe the context of our results in Section 1.3 and Section 4.3. The key parts of this thesis are contained in Chapter 2 (Strong Recurrence of Recurrent RWRE), Chapter 3 (RWRE with Random Orientations), and Chapter 5 (Survival and Growth of a BRWRE). Since we consider a different model with the need of a slightly different notation in each of these chapters, we left the chapters self-contained. Thus, some of the material already presented before is repeated in the introductory sections.

Part I

Random Walks in Random Environment

Chapter 1

Preliminaries

1.1 Basic Notation and General Assumptions

In the following, we analyse the behaviour of different RWRE. The construction of a RWRE always consists of two steps: In the first step, we choose an environment according to a specific distribution, and in the second step, we perform a random walk in the chosen environment.

The distribution of the environment will be denoted by P for which we make the following assumptions in this chapter if not stated otherwise:

(1) P is a measure on $([0,1]^{\mathbb{Z}}, \mathcal{F})$, where $\mathcal{F}$ denotes the corresponding product σ-field. Further, we denote the projections $[0,1]^{\mathbb{Z}} \to [0,1]$ by $\omega = (\omega_n)_{n\in\mathbb{Z}}$.

(2) $\omega = (\omega_n)_{n\in\mathbb{Z}}$ is a sequence of i.i.d. random variables with respect to P.

(3) $\mathsf{P}(0 < \omega_0 < 1) = 1$.

The expectation and the variance with respect to P will be denoted by $\mathsf{E}[\cdot]$ and $\mathsf{Var}(\cdot)$, respectively. Further, we define

$$\rho_0 = \rho_0(\omega) := \frac{1-\omega_0}{\omega_0} \ .$$

Given the environment $\omega = (\omega_n)_{n\in\mathbb{Z}}$ and a starting point $x \in \mathbb{Z}$, we can introduce the Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with values in $\mathbb{Z}$ by

$$\begin{aligned}
&P_\omega^x(X_0 = x) = 1, && \\
&P_\omega^x(X_{n+1} = y+1 | X_n = y) = \omega_y && \text{for all } y \in \mathbb{Z}, n \in \mathbb{N}_0, \\
&P_\omega^x(X_{n+1} = y-1 | X_n = y) = 1-\omega_y && \text{for all } y \in \mathbb{Z}, n \in \mathbb{N}_0.
\end{aligned}$$

For each ω, P_ω^x is a probability measure on the space of paths $(\mathbb{Z}^{\mathbb{N}_0}, \mathcal{G})$, where $\mathcal{G}$ is the corresponding product σ-field. For notational convenience, we often drop the superscript and use P_ω instead of P_ω^0. Statements involving P_ω^x are called *quenched*.

The joint distribution $\mathbb{P}^x$ of $(\omega, (X_n)_{n\in\mathbb{N}_0})$ with starting point $x \in \mathbb{Z}$ is uniquely determined by

$$\mathbb{P}^x(F \times G) := \int_F P_\theta^x(G)\mathsf{P}(d\theta)$$

for $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Again, we often use $\mathbb{P}$ instead of $\mathbb{P}^0$ for notational convenience. In contrast to above, probabilistic statements involving the second marginal distribution of $\mathbb{P}^x$, i.e. observing an event in the RWRE $(X_n)_{n\in\mathbb{N}_0}$ without observing the environment ω first, are called *annealed*.

It is important to note at this point that $(X_n)_{n\in\mathbb{N}_0}$ is a time-homogeneous Markov chain under P_ω^x but (in general) not under $\mathbb{P}^x$. On the contrary, we have

$$\mathbb{P}^x\big((X_n)_{n\in\mathbb{N}_0} - x \in \cdot\ \big) = \mathbb{P}^y\big((X_n)_{n\in\mathbb{N}_0} - y \in \cdot\ \big)$$

for all $x, y \in \mathbb{Z}$, i.e. homogeneity in space, which is (in general) not true for the quenched distributions P_ω^x and P_ω^y (for $x \neq y$).

1.2 Classical and Recent Results

The first results in the mathematical literature on RWRE are due to Solomon. His first result shows that the question of recurrence and transience of the RWRE can be answered with the help of $\mathsf{E}[\log \rho_0]$ as long as $\mathsf{E}[\log \rho_0]$ is well defined:

Theorem 1.2.1 (Recurrence and Transience of RWRE - cf. Theorem 1.7 in [So75]). *If* $\mathsf{E}[\log \rho_0]$ *is well defined, then we have the following classification:*

(1) If $\mathsf{E}[\log \rho_0] = 0$ *holds, then we* $\mathbb{P}$*-a.s. have*

$$\limsup_{n\to\infty} X_n = \infty \quad \textit{and} \quad \liminf_{n\to\infty} X_n = -\infty.$$

(2) If $\mathsf{E}[\log \rho_0] > 0$ *holds, then we* $\mathbb{P}$*-a.s. have*

$$\lim_{n\to\infty} X_n = -\infty.$$

(3) If $\mathsf{E}[\log \rho_0] < 0$ *holds, then we* $\mathbb{P}$*-a.s. have*

$$\lim_{n\to\infty} X_n = \infty.$$

In particular, $(X_n)_{n\in\mathbb{N}_0}$ *is a recurrent Markov chain for* P*-a.e. environment in case (1). On the contrary,* $(X_n)_{n\in\mathbb{N}_0}$ *is transient for* P*-a.e. environment in the cases (2) and (3).*

In the same article, Solomon also computed the linear speed of the RWRE $(X_n)_{n\in\mathbb{N}_0}$ in the transient case:

Theorem 1.2.2 (Linear Speed of transient RWRE - cf. Theorem 1.16 in [So75]). *There are the following three cases:*

(1) If $\mathsf{E}[\rho_0] < 1$ *holds, then we* $\mathbb{P}$*-a.s. have*

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1 - \mathsf{E}[\rho_0]}{1 + \mathsf{E}[\rho_0]} \,.$$

(2) If $\mathsf{E}[(\rho_0)^{-1}] < 1$ *holds, then we* $\mathbb{P}$*-a.s. have*

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1 - \mathsf{E}[(\rho_0)^{-1}]}{1 + \mathsf{E}[(\rho_0)^{-1}]} \,.$$

(3) If $(\mathsf{E}[\rho_0])^{-1} \leq 1 \leq \mathsf{E}[(\rho_0)^{-1}]$ *holds, then we* $\mathbb{P}$*-a.s. have*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0.$$

By combining the last two theorems, we see that, if we have

$$\mathsf{E}\left[\log \rho_0\right] \neq 0 \quad \text{and} \quad (\mathsf{E}[\rho_0])^{-1} \leq 1 \leq \mathsf{E}[(\rho_0)^{-1}],$$

then our RWRE $(X_n)_{n\in\mathbb{N}_0}$ is transient but the linear speed on its escape to $\{-\infty, \infty\}$ is 0. This regime is called the *sub-ballistic regime*. The occurrence of this regime is one of the first properties which were shown for the RWRE but which do not occur for a random walk in a constant environment (i.e. an environment with $\omega_n = p \in (0,1)$ for all $n \in \mathbb{Z}$).

There are many more interesting and also very recent results on transient RWRE. But since we only consider the recurrent RWRE on $\mathbb{Z}$ in the following, we restrict ourselves to the recurrent case:

As a first highlight for a recurrent RWRE, we have to mention the following result which is due to Sinai (cf. Section 1 in [Si82]). For the theorem, we assume that the following conditions are fulfilled:

$$\begin{aligned} &\mathsf{E}\left[\log \rho_0\right] = 0, \\ &\mathsf{P}(\varepsilon \leq \omega_0 \leq 1 - \varepsilon) = 1 \quad \text{for some } \varepsilon \in \left(0, \tfrac{1}{2}\right), \\ &\mathsf{Var}(\log \rho_0) > 0. \end{aligned} \tag{1.1}$$

Then, the following theorem holds for which we chose the form which is also used in Theorem 2.5.3 in [Ze04]:

Theorem 1.2.3 (Sinai's Regime). *For a RWRE* $(X_n)_{n\in\mathbb{N}_0}$ *for which the environment fulfils the assumptions in* (1.1)*, we have the following:*
There exists a sequence of random variables $(b_n)_{n\in\mathbb{N}_0}$ *- which only depends on the environment* ω *- such that*

$$\mathbb{P}\left(\left|\frac{X_n}{(\log n)^2} - b_n\right| > \eta\right) \xrightarrow{n\to\infty} 0$$

for all $\eta > 0$.

Alternatively, one can show that

$$\frac{X_n}{(\log n)^2} \xrightarrow{n\to\infty} b_\infty \quad \text{in law}$$

where the exact law of the nondegenerate limit b_∞ was computed in [Ke86] and [Go86] independently.

Again, the scaling $(\log n)^2$ is completely different from the behaviour of a random walk in a constant environment. This behaviour is often called *slowdown* of the random walk due to the random environment. Here, slowdown is to be understood in comparison with the behaviour of a random walk in a constant environment for which we have a scaling of $n^{\frac{1}{2}}$ due to the central limit theorem.

Among the more recent results, we want to present two results on the strong localization of the RWRE. Strong localization refers to the property of the RWRE to spend most of the time around the bottom point of the deepest valley it has visited so far (cf. [GPS10] and [DGPS07] for more details). For the theorems, let $(X_n)_{n\in\mathbb{N}_0}$ be a one-dimensional RWRE, and for $x \in \mathbb{Z}$, let

$$\xi(n,x) := |\{0 \le j \le n : \ X_j = x\}|$$

denote the local time of the RWRE in x. Then, the maximal local time

$$\xi^*(n) := \sup_{x\in\mathbb{Z}} \xi(n,x)$$

has the following two properties:

Theorem 1.2.4 (lim sup behaviour of $\xi^*(n)$ - cf. Theorem 1.1 and section 4 in [GPS10])**.** *Let $(X_n)_{n\in\mathbb{N}_0}$ be a one-dimensional RWRE for which the environment fulfils the assumptions in* (1.1)*. Then, there exists a constant $c \in (0,\infty)$ such that*

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n} = c$$

$\mathbb{P}$-a.s.

Further, the value of c was computed in [GPS10]. We want to remark here that the first results about the lim sup behaviour of $\xi^*(n)$ in the form

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n} \ge c$$

$\mathbb{P}$-a.s. for some $c \in (0,\infty)$ were developed in [Ré05] (p.337) and [Sh98] (Theorem 1.1).

Theorem 1.2.5 (lim inf behaviour of $\xi^*(n)$ - cf. Theorem 1.1 in [DGPS07])**.** *Let $(X_n)_{n\in\mathbb{N}_0}$ be a one-dimensional RWRE for which the environment fulfils the assumptions in* (1.1)*. Then, there exists a constant $c \in (0,\infty)$ such that*

$$\liminf_{n\to\infty} \frac{\xi^*(n)}{n/(\log\log\log n)} = c$$

$\mathbb{P}$-a.s.

Actually, the last theorem was only derived for the RWRE on the positive half-line in [DGPS07]. But it can easily be seen that the theorem can directly be extended to the RWRE on $\mathbb{Z}$ by splitting the RWRE on $\mathbb{Z}$ into the time points it spends on $\mathbb{N}_0$ and on $-\mathbb{N}$.

This strong localization property of the RWRE is important for our consideration in Chapter 3. There, the RWRE collects $+1/-1$-orientations which have been attached to the positions $x \in \mathbb{Z}$ independently and with equal probability before the RWRE starts. Since the RWRE spends most of its time in only few positions, the accumulated $+1$- and -1-orientations, which the RWRE has collected, are mainly influenced by those few positions. As a result, the RWRE has either collected a lot more $+1$-orientations or a lot more -1-orientations at most time points with only few exceptions around the time points at which the preferred sign changes from one to the other.

1.3 Context of our Results

1.3.1 Return Probabilities of the Recurrent RWRE

In [CP03b], Comets and Popov consider the return probabilities of the one-dimensional recurrent RWRE. In contrast to our setting, they consider the corresponding jump process in continuous time $(\xi_t^x)_{t\geq 0}$ started at $x \in \mathbb{Z}$ and with jump rates $(\omega_x^+, \omega_x^-)_{x\in\mathbb{Z}}$ to the right and left neighbouring sites. One advantage of this process in continuous time is that it is not periodic as the RWRE in discrete time. By considering the process $(\xi_t^x)_{t\geq 0}$ in continuous time only at the random time points of the jumps, we can recover the embedded discrete-time RWRE.

As before, let us denote the law of the environment by P which is now a product measure on $((0,\infty)\times(0,\infty))^{\mathbb{Z}}$, and again $\omega = ((\omega_x^+, \omega_x^-))_{x\in\mathbb{Z}}$ denote the projections. If we have

$$\begin{aligned}
&\mathsf{E}\left[\log\left(\frac{\omega_0^-}{\omega_0^+}\right)\right] = 0,\\
&\kappa^{-1} \leq \omega_0^+, \omega_0^- \leq \kappa \quad \mathsf{P}\text{-a.s. for some } \kappa > 0,\\
&\mathsf{Var}\left(\log\left(\frac{\omega_0^-}{\omega_0^+}\right)\right) \in (0,\infty),
\end{aligned} \tag{1.2}$$

then the following theorem holds:

Theorem 1.3.1 (Convergence of the Return Probabilities - cf. Corollary 2.1 in [CP03b]). *We have*

$$\frac{\log P_\omega(\xi_t^0 = 0)}{\log t} \xrightarrow{t\to\infty} -\widehat{a}_e$$

in law, where $\widehat{a}_e$ has the density

$$p(z) = \begin{cases} 2 - z - (z+2)\cdot e^{-2z} & \text{if } z \in (0,1) \\ \left([e^2-1]\cdot z - 2\right)\cdot e^{-2z} & \text{if } z \geq 1. \end{cases}$$

In Chapter 2, we are interested in the behaviour of the return probabilities of the discrete-time RWRE to 0 for a fixed environment ω. In contrast to [CP03b], our goal is to derive P-a.s. statements involving the return probabilities $P_\omega(X_{2n} = 0)$ for fixed environment ω.

Even though, Theorem 1.3.1 and the approach in [CP03b] to prove it are very helpful for our consideration: Since we can embed the discrete-time RWRE $(X_n)_{n\in\mathbb{N}_0}$ into the corresponding jump process in continuous time, we can expect the return probabilities to behave similarly as in the continuous setting, i.e.

$$P_\omega(X_{2n} = 0) \simeq n^{-a(\omega,n)} \tag{1.3}$$

with

$$\liminf_{n\to\infty} a(\omega, n) = 0,$$
$$\limsup_{n\to\infty} a(\omega, n) = \infty$$

for P-a.e. environment ω due to Theorem 1.3.1.

As one main result in Chapter 2, we will get the following theorems for a recurrent RWRE $(X_n)_{n\in\mathbb{N}_0}$ in discrete time for which the environment ω fulfils the assumptions in (1.1) (which are the discrete-time analogous assumptions in comparison with (1.2)):

Theorem 2.4.1 *For $0 \leq \alpha < 1$, we have*

$$\sum_{n\in\mathbb{N}} P_\omega(X_{2n} = 0) \cdot n^{-\alpha} = \infty$$

for P-a.e. environment ω.

Theorem 2.4.2 *For all $\alpha > 0$, we have*

$$\sum_{n\in\mathbb{N}} \Big(P_\omega(X_{2n} = 0)\Big)^\alpha = \infty$$

for P-a.e. environment ω.

In particular, we see that, even though for fixed environment ω there are arbitrarily large time points n for which the exponent $a(\omega, n)$ in (1.3) is arbitrarily large, there are enough time points n for which we have $a(\omega, n) \in (0, \delta]$ for every $0 < \delta < 1$ which cause the above series to diverge.

1.3.2 Random Orientations

In Chapter 3, we consider a RWRE with random orientations. Random walks on randomly oriented lattices (or with random orientations) were also considered in [CP03a] and [GKP12]. The first result deals with the symmetric random walk on $\mathbb{Z}$:

Let $(\alpha_x)_{x\in\mathbb{Z}}$ be a sequence of i.i.d. random variables with

$$\mathsf{P}(\alpha_0 = +1) = \mathsf{P}(\alpha_0 = -1) = \frac{1}{2} .$$

For a fixed environment $\alpha = (\alpha_x)_{x\in\mathbb{Z}}$, we can now introduce the following Markov chain $(X_n, Y_n)_{n\in\mathbb{N}_0}$ started in $z \in \mathbb{Z}^2$ with respect to P_α^z which is determined by

$$\begin{aligned}
&P_\alpha^z\big((X_0, Y_0) = z\big) = 1,\\
&P_\alpha^z\big((X_{n+1}, Y_{n+1}) = (x+1, y)\big|(X_n, Y_n) = (x, y)\big) = \frac{1}{3},\\
&P_\alpha^z\big((X_{n+1}, Y_{n+1}) = (x-1, y)\big|(X_n, Y_n) = (x, y)\big) = \frac{1}{3},\\
&P_\alpha^z\big((X_{n+1}, Y_{n+1}) = (x, y+1)\big|(X_n, Y_n) = (x, y)\big) = \frac{1+\alpha_x}{6},\\
&P_\alpha^z\big((X_{n+1}, Y_{n+1}) = (x, y-1)\big|(X_n, Y_n) = (x, y)\big) = \frac{1-\alpha_x}{6}
\end{aligned} \tag{1.4}$$

for all $n \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}$. Note that the last two conditional probabilities are either $\frac{1}{3}$ or 0 depending on the realisation of α_x. In particular, the random walk can only move upwards in points (x, y) with $\alpha_x = +1$ (with a positive orientation) and it can only move downwards in points (x, y) with $\alpha_x = -1$ (with a negative orientation). Further, note that the first component $(X_n)_{n\in\mathbb{N}_0}$ − only looked at when the first component has changed − behaves as a symmetric random walk on $\mathbb{Z}$ for every environment α.

Theorem 1.3.2 (Transience of the random walk on the randomly oriented lattice - cf. Theorem 1.8 in [CP03a])**.** *For almost all realisations of the environment α, the simple random walk $(X_n, Y_n)_{n\in\mathbb{N}_0}$ on the randomly vertically oriented lattice is transient with respect to P_α.*

For the reference, note that in contrast to [CP03a] we have switched the roles of the first and the second component which makes it easier to compare Theorem 1.3.2 to our results in Chapter 3. There, we want the first component $(X_n)_{n\in\mathbb{N}_0}$ − only looked at when the first component has changed − to behave as a RWRE on $\mathbb{Z}$.
An analogous result with a generalization of the transition probabilities in (1.4) can be found in Proposition 2 in [GKP12].

Another possibility for a generalization of the setting in Theorem 1.3.2 is to use an arbitrary next-neighbour Markov chain $(M_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ with transition probabilities $(p_x)_{x\in\mathbb{Z}}$ such that

$$\begin{aligned}
&P(M_{n+1} = x+1 | M_n = x) = p_x,\\
&P(M_{n+1} = x-1 | M_n = x) = 1 - p_x
\end{aligned} \tag{1.5}$$

for $n \in \mathbb{N}_0$, $x \in \mathbb{Z}$ instead of the embedded symmetric random walk in (1.4). Such a process $(X_n, Y_n)_{n\in\mathbb{N}_0}$ with values in $\mathbb{Z}^2$ for a fixed environment $\theta = (p_x, \alpha_x)_{k\in\mathbb{Z}}$, $\alpha_x \in [0,1]$ for all $x \in \mathbb{Z}$ and some $0 < \delta < 1$ is determined by

$$\begin{aligned}
&P_\theta^z\big((X_0, Y_0) = z\big) = 1,\\
&P_\theta^z\big((X_{n+1}, Y_{n+1}) = (x+1, y)\big|(X_n, Y_n) = (x, y)\big) = \delta \cdot p_x,\\
&P_\theta^z\big((X_{n+1}, Y_{n+1}) = (x-1, y)\big|(X_n, Y_n) = (x, y)\big) = \delta \cdot (1-p_x),\\
&P_\theta^z\big((X_{n+1}, Y_{n+1}) = (x, y+1)\big|(X_n, Y_n) = (x, y)\big) = (1-\delta) \cdot \alpha_x,\\
&P_\theta^z\big((X_{n+1}, Y_{n+1}) = (x, y-1)\big|(X_n, Y_n) = (x, y)\big) = (1-\delta) \cdot (1-\alpha_x)
\end{aligned} \tag{1.6}$$

for all $n \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}$. Here, δ (and $1-\delta$) is an additional parameter which gives the probability for a movement in the first (second) component. If the transition probabilities are chosen such that the Markov chain $(M_n)_{n\in\mathbb{N}_0}$ in (1.5) is positive recurrent, then we have the following characterization of recurrence and transience:

Theorem 1.3.3 (Recurrence and Transience on the randomly oriented lattice - cf. Proposition 3 in [GKP12])**.** *If $(p_x)_{x\in\mathbb{Z}}$ is fixed, such that the Markov chain $(M_n)_{n\in\mathbb{N}_0}$ in (1.5) is positive recurrent with invariant probability measure $\mu = (\mu(x))_{x\in\mathbb{Z}}$ and $(\alpha_x)_{x\in\mathbb{Z}}$ is an arbitrary sequence with values in $[0,1]$. Then, the Markov chain $(X_n, Y_n)_{n\in\mathbb{N}_0}$ is recurrent with respect to P_θ iff*

$$Z := \sum_{x\in\mathbb{Z}} \mu(x) \cdot \beta_x = 0$$

where $\beta_x = 2\alpha_x - 1$.
If $\alpha = (\alpha_x)_{x\in\mathbb{Z}}$ are i.i.d. random variables with values in [0,1] with respect to some probability measure P, $\mathsf{E}[\alpha_0] = \frac{1}{2}$, $\mathsf{Var}(\alpha_0) > 0$, and $\mu(x) > 0$ holds for all $x \in \mathbb{Z}$, then

$$\mathsf{P}\big(\alpha : \ (X_n, Y_n)_{n\in\mathbb{N}_0} \textit{ is recurrent with respect to } P_\theta\big) = \mathsf{P}\big(Z = 0\big) = 0.$$

In the final result of Chapter 3, we embed a one-dimensional RWRE into the above setting. For this we make the following choice for $(p_x)_{x\in\mathbb{Z}}$ and $(\alpha_x)_{x\in\mathbb{Z}}$:
Let $(\omega_x)_{x\in\mathbb{Z}}$ and $(\alpha_x)_{x\in\mathbb{Z}}$ be two independent sequences of i.i.d random variables with respect to the law of the environment P. The sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is a sequence of i.i.d. random variables taking values in $(0,1)$, and we assume that the following assumptions hold

$$\begin{aligned}
&\mathsf{E}[\log \rho_0] = 0,\\
&\mathsf{P}(\varepsilon \le \omega_0 \le 1-\varepsilon) = 1 \text{ for some } \varepsilon \in \left(0, \tfrac{1}{2}\right),\\
&\mathsf{Var}(\log \rho_0) > 0,
\end{aligned}$$

where

$$\rho_0 = \rho_0(\omega) := \frac{1-\omega_0}{\omega_0}$$

as usual. Note that $(\omega_x)_{x\in\mathbb{Z}}$ corresponds to the random environment of a one-dimensional RWRE and that the assumptions ensure that the one-dimensional RWRE is recurrent.

The second part of the random environment $(\alpha_x)_{x\in\mathbb{Z}}$ determines a random orientation for every position $x \in \mathbb{Z}$. More precisely, $(\alpha_x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence with

$$\mathsf{P}(\alpha_0 = +1) = \frac{1}{2} = \mathsf{P}(\alpha_0 = -1)$$

which is further independent of $(\omega_x)_{x\in\mathbb{Z}}$. Finally, our random environment is given by

$$\theta := (\theta_x)_{x\in\mathbb{Z}} := (\omega_x, \alpha_x)_{x\in\mathbb{Z}}.$$

By choosing $p_x = \omega_x$ for $x \in \mathbb{Z}$ and using

$$\frac{1 + p \cdot \alpha_x}{2} \quad \text{instead of} \quad \alpha_x$$

for some $p \in [0, 1]$ in (1.6), we come to the following (transition) probabilities

$$\begin{aligned}
&P^z_\theta\big((X_0, Y_0) = z\big) = 1,\\
&P^z_\theta\big((X_{n+1}, Y_{n+1}) = (x+1, y)\big|(X_n, Y_n) = (x, y)\big) = \delta \cdot \omega_x,\\
&P^z_\theta\big((X_{n+1}, Y_{n+1}) = (x-1, y)\big|(X_n, Y_n) = (x, y)\big) = \delta \cdot (1 - \omega_x),\\
&P^z_\theta\big((X_{n+1}, Y_{n+1}) = (x, y+1)\big|(X_n, Y_n) = (x, y)\big) = \frac{1-\delta}{2} \cdot (1 + p \cdot \alpha_x),\\
&P^z_\theta\big((X_{n+1}, Y_{n+1}) = (x, y-1)\big|(X_n, Y_n) = (x, y)\big) = \frac{1-\delta}{2} \cdot (1 - p \cdot \alpha_x)
\end{aligned}$$

for all $n \in \mathbb{N}_0$, $x, y \in \mathbb{Z}$ and $z \in \mathbb{Z}^2$.
We derive the following characterization of recurrence and transience for the Markov chain $(M_n)_{n\in\mathbb{N}_0} = (X_n, Y_n)_{n\in\mathbb{N}_0}$ in

Theorem 3.6.1 *(1) For $p = 0$, the Markov chain $(M_n)_{n\in\mathbb{N}_0}$ is recurrent for P-a.e. environment θ.*

(2) On the contrary for $0 < p \leq 1$, the Markov chain $(M_n)_{n\in\mathbb{N}_0}$ is transient for P-a.e. environment θ.

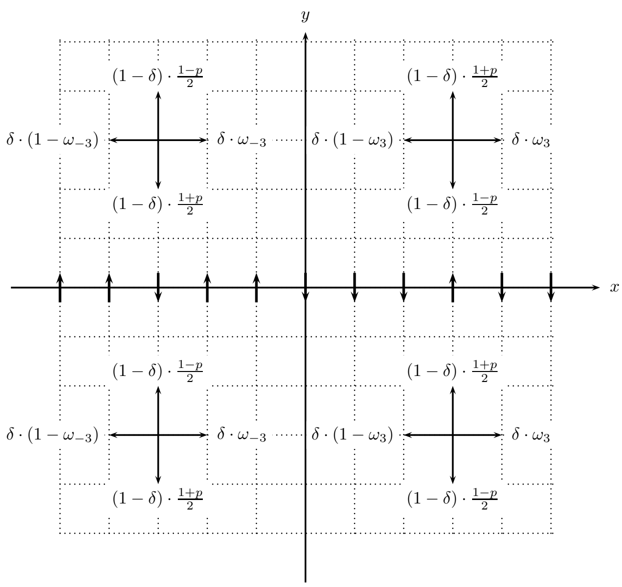

Figure 1.1: *A possible realization of the random orientations ↑↓ and the corresponding transition probabilities.*

1.3.3 RWRE in Random Scenery

Random Walk in Random Scenery (RWRS) is a family of stationary random processes which possess long-range correlations. The general configuration is the following:
Let $(M_n)_{n\in\mathbb{N}_0}$ be a Markov chain on some state space $\mathcal{S}$ and $(\alpha_x)_{x\in\mathcal{S}}$ be a sequence of i.i.d. random variables indexed by $\mathcal{S}$. Then, $(\alpha_x)_{x\in\mathcal{S}}$ is referred to as the random scenery. As the random walk $(M_n)_{n\in\mathbb{N}_0}$ moves in its state space, it observes the random scenery at its location. Therefore, we define

$$S_n := \sum_{i=0}^{n} \alpha_{S_i}$$

as the RWRS. Such a process was first introduced by Kesten and Spitzer in [KS79]. Since we only consider results on random scenery as by-product, we concentrate on results for RWRE in random scenery (RWRERS) in the following. Therefore at this point, we only refer to [KS79] for the first results on RWRS and to [HS06] for more results and recent developments.

There are only few results on problems which combine RWRE and random scenery. A first question about the behaviour of a RWRE on $\mathbb{Z}$ in a random scenery appeared in form of a conjecture of Révész ([Ré05], p.353) which is motivated by the strong localization property of the RWRE. Since the form of the conjecture in [Ré05] was still unfinished, we use the form of [Zi08]:

Conjecture 1.3.4. *Let $(X_n)_{n\in\mathbb{N}_0}$ be a one-dimensional RWRE for which the environment ω fulfils our assumptions in* (1.1)*. Further, let $(\alpha_x)_{x\in\mathbb{Z}}$ be an i.i.d. sequence with respect to some probability measure Q. Does the assumption $a := \operatorname{ess\,sup} \alpha_0 < \infty$ imply*

$$\limsup_{n\to\infty} \frac{S_n}{n} = a \quad \mathbb{P}\otimes Q - a.s.?$$

Here we use

$$S_n := \sum_{i=0}^{n} \alpha_{S_i}$$

as above. Further, the product measure $\mathbb{P}\otimes Q$ is used to indicate that we want the random scenery to be independent of $(\omega, (X_n)_{n\in\mathbb{N}_0})$. In the same work, Zindy proved the following result which shows that the conjecture is only true under further assumptions:

Theorem 1.3.5 (lim sup behaviour of RWRERS - cf. Theorem 1.1 in [Zi08])**.** *Assume that the environment ω fulfils the assumptions in* (1.1) *and $a := \operatorname{ess\,sup} \alpha_0 < \infty$.*

(1) If $Q(\alpha_0^- > \lambda) \le \frac{1}{(\log\lambda)^{2+\varepsilon}}$ for some $\varepsilon > 0$ and all large λ, then

$$\mathbb{P}\otimes Q\left(\limsup_{n\to\infty} \frac{S_n}{n} = a\right) = 1.$$

(2) If $Q(\alpha_0^- > \lambda) \geq \frac{1}{(\log \lambda)^{2-\varepsilon}}$ for some $\varepsilon > 0$ and all large λ, then

$$\mathbb{P} \otimes Q \left(\limsup_{n\to\infty} \frac{S_n}{n} = -\infty \right) = 1.$$

Here, we use the convention $\alpha_0^- = \max\{-\alpha_0, 0\}$. In particular the case $\varepsilon = 0$ is still open.

In Chapter 3, we are able to derive a first statement about the lim inf behaviour of a particular RWRERS. As mentioned before, we consider a model in which the RWRE collects $+1/-1$-orientations, which have been attached to the positions $x \in \mathbb{Z}$ independently and with equal probability before the RWRE starts. This setting can also be understood as a RWRERS, where $(\alpha_x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence with

$$\mathsf{P}(\alpha_0 = +1) = \mathsf{P}(\alpha_0 = -1) = \frac{1}{2}$$

which is further independent of the environment ω. In our notation, we use P (instead of $\mathsf{P} \otimes Q$) as the corresponding probability measure of the extended environment

$$\theta = (\omega_x, \alpha_x)_{x\in\mathbb{Z}}.$$

Using Theorem 1.3.5, we see that (expressed as a quenched statement) we have

$$\limsup_{n\to\infty} \frac{S_n}{n} = 1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{-S_n}{n} = 1$$

P_θ-a.s. for P-a.e. environment θ, where we use

$$S_n := \sum_{i=0}^{n-1} \alpha_{X_i}$$

for periodic reasons. (This statement could also be directly derived by known concentration properties of the RWRE since we have $\operatorname{ess\,sup} |\alpha_0| = 1 < \infty$ (cf. Section 1.3 in [Zi08] and Theorem 1.3 in [An07])).

But what can we say about S_{ν_n}, where $(\nu_n)_{n\in\mathbb{N}_0}$ denotes the sequence of the successive return times of the RWRE $(X_n)_{n\in\mathbb{N}_0}$ to 0?

Note here that $(X_n)_{n\in\mathbb{N}_0}$ is recurrent for P-a.e. environment by assumption on the environment. In Chapter 3, we get the following proposition, where we assume that the environment ω of the RWRE $(X_n)_{n\in\mathbb{N}_0}$ fulfils the assumptions in (1.1):

Proposition 3.5.1 *For $0 < \vartheta < 1$ we have*

$$\liminf_{n\to\infty} \frac{|S_{\nu_n}|}{\nu_n \cdot \exp\Big(- \big(\log(\nu_n)\big)^{\vartheta}\Big)} = \infty \tag{1.7}$$

P_θ-a.s. for P-a.e. environment θ.

In particular for all n large enough, the RWRERS has either collected a lot more $+1$- orientations or a lot more -1-orientations when the RWRE $(X_n)_{n\in\mathbb{N}_0}$ returns to 0 at time ν_n.

Chapter 2

Strong Recurrence of Recurrent RWRE

2.1 Overview

In this chapter, we consider a RWRE on $\mathbb{Z}$ for which we assume that we are in the recurrent regime. Our main results are three analytical results on the quenched return probabilities of the RWRE to the origin.

The structure of this chapter is the following: In Section 2.2, we introduce the model of a RWRE on $\mathbb{Z}$ together with the notation which we use in this chapter. Then we collect some useful equalities and inequalities in the context of RWRE in Section 2.3 before we state our main result in Section 2.4. Section 2.5 contains the proofs of our main results. The main tool for our proofs is a careful analysis of the corresponding potential of the RWRE (cf. (2.5)). To this end, we introduce favourable "valleys" (cf. Figure 2.1 on page 23), which help us to derive lower bounds for the quenched return probabilities of the RWRE to the origin. In the last Section 2.6, we give some examples for recurrent random walks in some random environment. In particular Corollary 2.6.2 and Corollary 2.6.3 give reason for the part "strong recurrence" in the title of this chapter when we compare the behaviour of RWRE with the behaviour of a symmetric random walk on $\mathbb{Z}^d$.

2.2 Model and Notation

Let us first introduce the notation for a random walk in random environment (RWRE) as it is usually done in the literature:
At first, let $\omega = (\omega_x)_{x\in\mathbb{Z}}$ be a sequence of i.i.d. random variables taking values in $(0,1)$ with respect to some probability measure P. For $i \in \mathbb{Z}$ we define

$$\rho_i = \rho_i(\omega) := \frac{1-\omega_i}{\omega_i}.$$

In the following, we will assume that

$$\mathsf{E}[\log \rho_0] = 0, \tag{2.1}$$

$$\mathsf{P}(\varepsilon \le \omega_0 \le 1-\varepsilon) = 1 \text{ for some } \varepsilon \in \left(0, \tfrac{1}{2}\right), \tag{2.2}$$

$$\mathsf{Var}(\log \rho_0) > 0. \tag{2.3}$$

Here, (2.1) ensures that the one-dimensional RWRE is recurrent. The second assumption is a common technical condition in the context of RWRE. Further, the third assumption excludes the case of a symmetric random walk on $\mathbb{Z}$.

For each environment ω, we can introduce the random walk $(X_n)_{n\in\mathbb{N}_0}$ whose transition probabilities are determined by $(\omega_x)_{x\in\mathbb{Z}}$. More precisely for every $x \in \mathbb{Z}$, $(X_n)_{n\in\mathbb{N}_0}$ is a Markov chain with respect to P_ω^x determined by

$$P_\omega^x(X_0 = x) = 1,$$

$$P_\omega^x(X_{n+1} = y+1 | X_n = y) = \omega_y = 1 - P_\omega^x(X_{n+1} = y-1 | X_n = y) \quad \forall y \in \mathbb{Z}. \tag{2.4}$$

As usual, we use P_ω^o instead of P_ω^0 and will even drop the superscript o where no confusion is to be expected. We can now define the potential V (cf. Section 2 in [SZ07]) as

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i & \text{for } x = 1, 2, \dots \\ 0 & \text{for } x = 0 \\ \sum_{i=x+1}^{0} \log(\rho_i)^{-1} & \text{for } x = -1, -2, \dots \end{cases} \tag{2.5}$$

Note that $V(x)$ is a sum of i.i.d. random variables which are centred and which are bounded by $C := \log(1-\varepsilon) - \log \varepsilon > 0$ due to the assumptions (2.1) and (2.2). One of the most useful properties of the RWRE is the observation that, for fixed ω, the random walk is a reversible Markov chain and can therefore be described as an electrical network (cf. [DGPS07]). The conductances are given by

$$C_{(x,x+1)}(\omega) = e^{-V(x)} = \begin{cases} \prod_{i=1}^{x} (\rho_i)^{-1} & \text{for } x = 1, 2, \dots \\ 1 & \text{for } x = 0 \\ \prod_{i=x+1}^{0} \rho_i & \text{for } x = -1, -2, \dots \end{cases}$$

and the reversible measure which is unique up to multiplication by a constant is given by

$$\mu_\omega(x) = e^{-V(x)} + e^{-V(x-1)} = \begin{cases} \prod_{i=1}^{x-1} \frac{\omega_i}{1-\omega_i} \cdot \frac{1}{1-\omega_x} & \text{for } x = 1, 2, \dots \\ \frac{1}{\omega_0} & \text{for } x = 0 \\ \prod_{i=x+1}^{0} \frac{1-\omega_i}{\omega_i} \cdot \frac{1}{\omega_x} & \text{for } x = -1, -2, \dots \end{cases} \tag{2.6}$$

As a consequence of the reversibility, we conclude that we have

$$\mu_\omega(x) \cdot P_\omega^x(X_n = y) = \mu_\omega(y) \cdot P_\omega^y(X_n = x) \tag{2.7}$$

for all $n \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}$.

2.3 Preliminaries

In the following, we collect some useful properties of the RWRE. For the random time of the first arrival in x

$$\tau(x) := \inf\{n \geq 0 : \ X_n = x\}, \tag{2.8}$$

the interpretation of the RWRE as an electrical network helps us to compute the following probability for $x < y < z$ (for a proof see for example formula (2.1.4) in [Ze04]):

$$P_\omega^y(\tau(z) < \tau(x)) = \frac{\sum_{j=x}^{y-1} e^{V(j)}}{\sum_{j=x}^{z-1} e^{V(j)}} \tag{2.9}$$

Further (cf. (2.4) and (2.5) in [SZ07] and Lemma 7 in [Go84]), we have for $k \in \mathbb{N}$ and $y < z$

$$P_\omega^y(\tau(z) < k) \leq k \cdot \exp\left(-\max_{y \leq i < z} \left[V(z-1) - V(i)\right]\right) \tag{2.10}$$

and similarly for $x < y$

$$P_\omega^y(\tau(x) < k) \leq k \cdot \exp\left(-\max_{x < i \leq y} \left[V(x+1) - V(i)\right]\right). \tag{2.11}$$

To get bounds for large values of $\tau(\cdot)$, we can use that for $x < y < z$ we have (cf. Lemma 2.1 in [SZ07])

$$E_\omega^y[\tau(z) \cdot \mathbb{1}_{\{\tau(z) < \tau(x)\}}] \leq (z-x)^2 \cdot \exp\left(\max_{x \leq i \leq j \leq z} \big(V(j) - V(i)\big)\right). \tag{2.12}$$

Further, the Komlós-Major-Tusnády strong approximation theorem (cf. Theorem 1 in [KMT76], see also formula (2) in [CP03b]) will help us to compare the shape of the potential with the paths of a two-sided Brownian motion:

Theorem 2.3.1. *In a possibly enlarged probability space, there exists a version of our environment process ω and a two-sided Brownian motion $(B(t))_{t \in \mathbb{R}}$ with diffusion constant $\sigma := \ (\mathsf{Var}(\log \rho_0))^{\frac{1}{2}}$ (i.e. $Var(B(t)) = \sigma^2 |t|$) such that for some $K > 0$ we have*

$$\mathsf{P}\left(\limsup_{x \to \pm\infty} \frac{|V(x) - B(x)|}{\log |x|} \leq K\right) = 1. \tag{2.13}$$

2.4 Results

Let us consider a RWRE $(X_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ where the law of the environment $\omega = (\omega_x)_{x\in\mathbb{Z}}$ fulfils the assumptions (2.1), (2.2), and (2.3). Then, the following two theorems hold:

Theorem 2.4.1. *For* $0 \le \alpha < 1$, *we have*

$$\sum_{n\in\mathbb{N}} P_\omega(X_{2n} = 0) \cdot n^{-\alpha} = \infty \tag{2.14}$$

for P*-a.e. environment* ω.

Theorem 2.4.2. *For all* $\alpha > 0$, *we have*

$$\sum_{n\in\mathbb{N}} \Big(P_\omega(X_{2n} = 0)\Big)^\alpha = \infty \tag{2.15}$$

for P*-a.e. environment* ω.

For the last theorem we consider a combination of d environments:

Theorem 2.4.3. *For* $d \in \mathbb{N}$, *consider* d *i.i.d. environments* $\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(d)}$ *which all fulfil the assumptions* (2.1), (2.2), *and* (2.3). *Then, we have*

$$\sum_{n\in\mathbb{N}} \prod_{k=1}^{d} P_{\omega^{(k)}}(X_{2n} = 0) = \infty \tag{2.16}$$

for $\mathsf{P}^{\otimes d}$*-a.e. environment* $(\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(d)})$.

2.5 Proofs

Let us first introduce the sets $\Gamma^+(L,\delta)$ and $\Gamma^-(L,\delta)$ of environments for $L \in \mathbb{N}$ and $0 < \delta < 1$ defined by

$$\Gamma^+(L,\delta) := \{R_1^+(L) \le \delta L,\, R_2^+(L) \le \delta L,\, T^+(L) \le L^2\},$$

$$\Gamma^-(L,\delta) := \{R_1^-(L) \le \delta L,\, R_2^-(L) \le \delta L,\, -T^-(L) \le L^2\},$$

where

$$T^+(L) := \inf\{n \ge 0 :\ V(n) - \min_{0\le k\le n} V(k) \ge L\},$$

$$T^-(L) := \sup\{n \le 0 :\ V(n) - \min_{n\le k\le 0} V(k) \ge L\},$$

$$R_1^+(L) := -\min_{0\le k\le T^+(L)} V(k),$$

$$
\begin{aligned}
R_1^-(L) &:= -\min_{T^-(L)\le k\le 0} V(k),\\
T_b^+(L) &:= \inf\{n \ge 0:\ V(n) = -R_1^+(L)\},\\
T_b^-(L) &:= \sup\{n \le 0:\ V(n) = -R_1^-(L)\},\\
R_2^+(L) &:= \max_{0\le k\le T_b^+(L)} V(k),\\
R_2^-(L) &:= \max_{T_b^-(L)\le k\le 0} V(k).
\end{aligned}
$$

Here, the $+$-sign and the $-$-sign indicate whether we deal with properties of the valley on the positive or negative half-line, respectively. Note that the definition of the sets $\Gamma^+(L,\delta)$ and $\Gamma^-(L,\delta)$ is compatible with the scaling of a Brownian motion in space and time (cf. (2.31)).

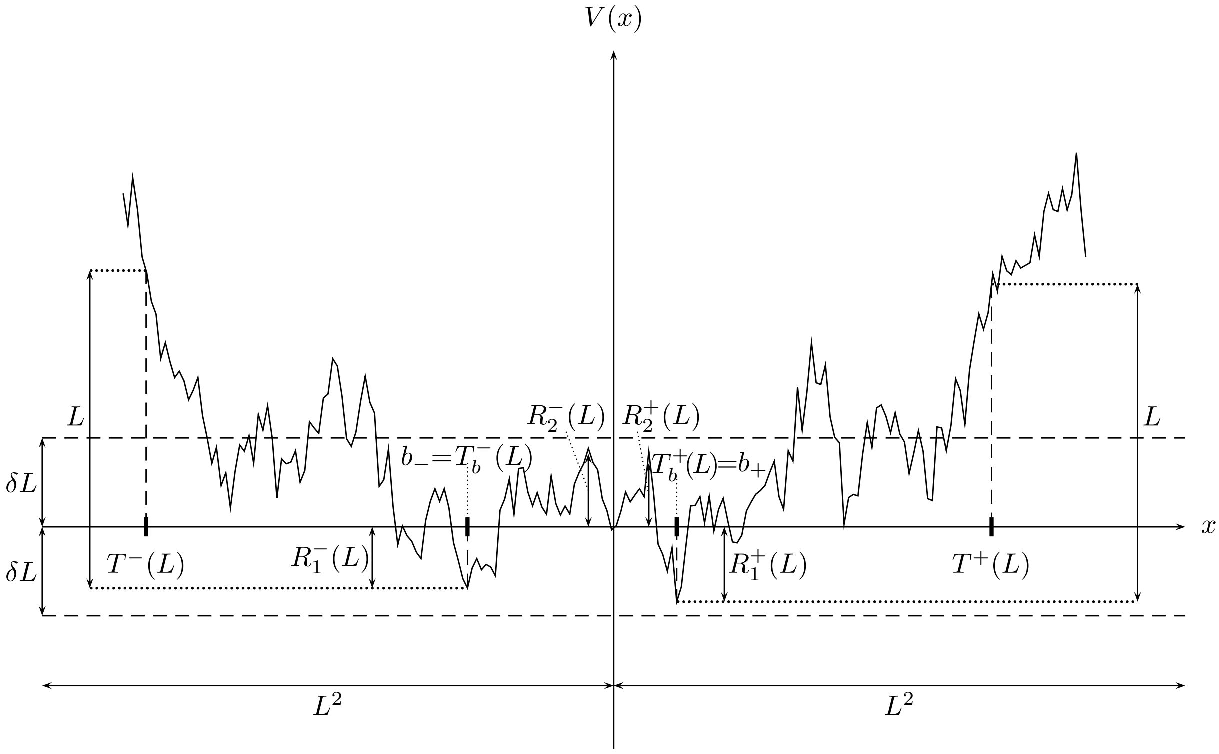

Figure 2.1: *Shape of a valley of an environment in* $\Gamma(L,\delta) := \Gamma^+(L,\delta)\cap\Gamma^-(L,\delta)$

Remark 2.5.1. We have constructed the valleys in such a way that the return probability of the random walk to the origin is high (or bounded from below as we will see) for even time points as long as the random walk has not left the valley. For $\omega \in \Gamma^+(L,\delta)\cap\Gamma^-(L,\delta)$, we have the following behaviour for the random walk $(X_n)_{n\in\mathbb{N}_0}$ in the environment ω:

(1) Since we have $V(T^-(L))-V(T_b^-(L)) \ge L$ and $V(T^+(L))-V(T_b^+(L)) \ge L$, the random walk $(X_n)_{n\in\mathbb{N}_0}$ stays within $\{T^-(L), T^-(L)+1, \dots, T^+(L)\}$ for (approximately) at least $\exp(L)$ steps (cf. (2.23)).

(2) Within the area $\{T^-(L), T^-(L)+1, \dots, T^+(L)\}$, the random walk prefers to stay at positions x with a small potential $V(x)$, i.e. at positions close to the bottom points

at $T_b^-(L)$ and $T_b^+(L)$.

(3) The return probability for the random walk from the positions $T_b^-(L)$ and $T_b^+(L)$ to the origin is mainly influenced by the potential differences $R_2^-(L)+R_1^-(L) \leq 2\delta L$ and $R_2^+(L)+R_1^+(L) \leq 2\delta L$ respectively, i.e. by the height of the climb the random walk has to trespass from the bottom points back to the origin (cf. (2.19)).

Proposition 2.5.2. *For $\omega \in \Gamma(L,\delta) := \Gamma^+(L,\delta) \cap \Gamma^-(L,\delta)$ with $0 < \delta < \frac{1}{5}$, we have*

$$P_\omega(X_{2n}=0) \geq C \cdot \exp(-3\delta L) \tag{2.17}$$

for

$$\exp(3\delta L) \leq n \leq \exp\big((1-2\delta)L\big),$$

where the constant $C = C(\delta)$ does not depend on L.

Proof of Proposition 2.5.2. The construction of "valleys" has been useful for the proofs of many theorems in the context of RWRE. Our construction uses some ideas from [CP03b], where it is shown that the transition probabilities of a RWRE in continuous time converge in distribution. Since we deal with a RWRE in discrete time and we want to have lower estimates for the return probabilities for a fixed environment in Proposition 2.5.2, we will have to adapt the construction to our setting:

The return probability to the origin for the time points of interest is mainly influenced by the shape of the "valley" of the environment ω between $T^-(L)$ and $T^+(L)$. For the positions of the two deepest bottom points of this valley on the positive and negative side, we write

$$b_+ := T_b^+(L) \quad \text{and} \quad b_- := T_b^-(L)$$

and we assume for the following proof that we have (cf. (2.8) for the definition of $\tau(\cdot)$)

$$P_\omega^o\big(\tau(b_+) < \tau(b_-)\big) \geq \frac{1}{2}. \tag{2.18}$$

(Due to the symmetry of the RWRE, the proof also works in the opposite case if we switch the roles of b_+ and b_-). We have

$$\begin{aligned}
&P_\omega^o(X_{2n}=0) \geq P_\omega^o\left(X_{2n}=0,\ \tau(b_+) \leq \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) \\
&\geq P_\omega^o\left(\tau(b_+) \leq \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) \cdot \widehat{\inf}_{\ell \in \left\{\left\lceil \frac{4n}{3}\right\rceil,\dots,2n\right\}} P_\omega^{b_+}(X_\ell = 0) \\
&= P_\omega^o\left(\tau(b_+) \leq \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) \cdot \frac{\mu_\omega(0)}{\mu_\omega(b_+)} \cdot \widehat{\inf}_{\ell \in \left\{\left\lceil \frac{4n}{3}\right\rceil,\dots,2n\right\}} P_\omega^o(X_\ell = b_+)
\end{aligned} \tag{2.19}$$

where we used (2.7) in the third step. Here, for $x, y \in \mathbb{Z}$,

$$\widehat{\inf}_{\ell \in \left\{\left\lceil \frac{4n}{3}\right\rceil,\dots,2n\right\}} P_\omega^x(X_\ell = y)$$

is the short notation for

$$\inf_{\ell \in \left\{ \left\lceil \frac{4n}{3} \right\rceil, \ldots, 2n \right\} \cap \left(2\mathbb{Z} + (x+y)\right)} P_\omega^x(X_\ell = y)$$

since we have to take care of the periodicity of the random walk.
Let us now have a closer look at the factors in the lower bound in (2.19) separately:

First factor in (2.19):
We can bound the first factor from below by

$$\begin{aligned}
&P_\omega^o\left(\tau(b_+) \le \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) \\
&\ge 1 - P_\omega^o\left(\tau(b_+) > \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) - P_\omega^o\big(\tau(b_+) \ge \tau(b_-)\big) \\
&\ge 1 - \tfrac{3}{2n} \cdot E_\omega^o\left[\tau(b_+) \cdot \mathbb{1}_{\{\tau(b_+) < \tau(b_-)\}}\right] - P_\omega^o\big(\tau(b_+) \ge \tau(b_-)\big) \\
&\ge 1 - \tfrac{3}{2n} \cdot (b_+ - b_-)^2 \cdot \exp\left(\max_{b_- \le i \le j \le b_+} \big(V(j) - V(i)\big)\right) - \frac{1}{2}\,,
\end{aligned}$$

where we used (2.12) and assumption (2.18) for the last step. Therefore, we get for $\omega \in \Gamma(L,\delta)$ and $\exp(3\delta L) \le n$ that

$$P_\omega^o\left(\tau(b_+) \le \tfrac{2n}{3},\ \tau(b_+) < \tau(b_-)\right) \ge \frac{1}{2} - \frac{3 \cdot 4 \cdot L^4}{2 \cdot \exp(3\delta L)} \cdot \exp(2\delta L) = \frac{1}{2} - 6 \cdot L^4 \cdot \exp(-\delta L). \tag{2.20}$$

Second factor in (2.19):
By using assumption (2.2) and the relation in (2.6), we get for $\omega \in \Gamma(L,\delta)$:

$$\begin{aligned}
\frac{\mu_\omega(0)}{\mu_\omega(b_+)} &= \frac{\frac{1}{\omega_0}}{e^{-V(b_+)} + e^{-V(b_+ - 1)}} = \frac{\frac{1}{\omega_0}}{e^{-V(b_+)} \cdot (1 + \rho_{b_+})} \\
&\ge \frac{\frac{1}{1-\varepsilon}}{1 + \frac{1-\varepsilon}{\varepsilon}} \cdot e^{V(b_+)} = \frac{\varepsilon}{1-\varepsilon} \cdot e^{V(b_+)} \ge \frac{\varepsilon}{1-\varepsilon} \cdot \exp(-\delta L).
\end{aligned} \tag{2.21}$$

Here we used that $V(b_+) \ge -\delta L$ holds for $\omega \in \Gamma(L,\delta)$.

Third factor in (2.19):
For the last factor in (2.19), we can compare the RWRE with the process $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ which behaves as the original RWRE but is reflected at the positions $T^- := T^-(L)$ and $T^+ := T^+(L)$, i.e. we have for $x \in \{T^-, T^- + 1, \ldots, T^+\}$

$$P_\omega^x(\widetilde{X}_0 = x) = 1,$$

$$P_\omega^x(\widetilde{X}_{n+1} = y \pm 1 | \widetilde{X}_n = y) = P_\omega^x(X_{n+1} = y \pm 1 | X_n = y),$$

$$\forall y \in \{T^- + 1, T^- + 2, \ldots, T^+ - 1\},$$

$$P_\omega^x(\widetilde{X}_{n+1} = y+1 | \widetilde{X}_n = y) = 1 \qquad \text{for } y = T^-,$$

$$P_\omega^x(\widetilde{X}_{n+1} = y-1 | \widetilde{X}_n = y) = 1 \qquad \text{for } y = T^+.$$

Therefore, we have for $\ell \in \left\{ \left\lceil \frac{4n}{3} \right\rceil, \dots, 2n \right\} \cap \left(2\mathbb{Z} + b_+ \right)$

$$\begin{aligned}
&P_\omega^o(X_\ell = b_+) \\
&\geq P_\omega^o(X_\ell = b_+,\ \min\{\tau(T^-), \tau(T^+)\} > 2n) \\
&= P_\omega^o(\widetilde{X}_\ell = b_+) - P_\omega^o(\widetilde{X}_\ell = b_+,\ \min\{\tau(T^-), \tau(T^+)\} \leq 2n) \\
&\geq P_\omega^o(\widetilde{X}_\ell = b_+) - P_\omega^o(\min\{\tau(T^-), \tau(T^+)\} \leq 2n) \\
&\geq P_\omega^o\big(\widetilde{X}_\ell = b_+,\ \tau(b_+) \leq \tfrac{\ell}{2},\ \tau(b_+) < \tau(b_-)\big) - P_\omega^o(\min\{\tau(T^-), \tau(T^+)\} \leq 2n) \\
&\geq P_\omega^o\big(\tau(b_+) \leq \tfrac{\ell}{2},\ \tau(b_+) < \tau(b_-)\big) \cdot \widehat{\inf}_{k \in \left\{ \left\lceil \frac{\ell}{2} \right\rceil, \dots, \ell \right\}} P_\omega^{b_+}(X_k = b_+) \\
&\quad - P_\omega^o(\min\{\tau(T^-), \tau(T^+)\} \leq 2n).
\end{aligned} \tag{2.22}$$

Using (2.10) and (2.11), we see that the last term in (2.22) with the negative sign decreases exponentially for $n \leq \exp\big((1-2\delta)L\big)$, i.e.

$$\begin{aligned}
&P_\omega^o(\min\{\tau(T^-), \tau(T^+)\} \leq 2n) \leq P_\omega^o\left(\min\{\tau(T^-), \tau(T^+)\} \leq 2 \cdot \exp\big((1-2\delta)L\big)\right) \\
&\leq P_\omega^o\left(\tau(T^-) \leq 2 \cdot \exp\big((1-2\delta)L\big)\right) + P_\omega^o\left(\tau(T^+) \leq 2 \cdot \exp\big((1-2\delta)L\big)\right) \\
&\leq 4 \cdot \exp\big((1-2\delta)L\big) \cdot \exp\big(-L\big) = 4 \cdot \exp\big(-2\delta L\big).
\end{aligned} \tag{2.23}$$

In order to derive a lower bound for the first term in (2.22), we first notice that the analogous calculation as in (2.20) shows for $\omega \in \Gamma(L, \delta)$ that

$$\begin{aligned}
&P_\omega^o\left(\tau(b_+) \leq \tfrac{\ell}{2},\ \tau(b_+) < \tau(b_-)\right) \geq 1 - \frac{2}{\ell} \cdot 4 \cdot L^4 \cdot \exp(2\delta L) - \frac{1}{2} \\
&\geq \frac{1}{2} - 6 \cdot L^4 \cdot \exp(-\delta L)
\end{aligned} \tag{2.24}$$

since $\ell \geq \left\lceil \frac{4n}{3} \right\rceil \geq \frac{4}{3} \cdot \exp(3\delta L)$ for $n \geq \exp(3\delta L)$. For the second factor, we show the following

Lemma 2.5.3. *For $\omega \in \Gamma(L, \delta)$ and for all $\ell \in 2\mathbb{N}$, we have*

$$P_\omega^{b_+}(\widetilde{X}_\ell = b_+) \geq \frac{1}{2} \cdot \frac{1}{|T^-| + T^+ + 1} \cdot \exp\big(-\delta L\big).$$

Proof of Lemma 2.5.3. Using the reversibility (cf. (2.7)) of $(\widetilde{X}_\ell)_{\ell \in \mathbb{N}_0}$, we get

$$P_\omega^{b_+}(\widetilde{X}_\ell = b_+)$$

$$
\begin{aligned}
&= \sum_{x=T^-}^{T^+} P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x) \cdot P_\omega^x(\widetilde{X}_{\ell/2} = b_+) \\
&= \sum_{x=T^-}^{T^+} P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x) \cdot \frac{\widetilde{\mu}_\omega(b_+)}{\widetilde{\mu}_\omega(x)} \cdot P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x), \qquad (2.25)
\end{aligned}
$$

where $\widetilde{\mu}_\omega(\cdot)$ denotes a reversible measure of the reflected random walk $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ which is unique up to multiplication by a constant. To see that $(\widetilde{X}_\ell)_{\ell\in\mathbb{N}_0}$ is also reversible, it is enough to note that $(\widetilde{X}_\ell)_{\ell\in\mathbb{N}_0}$ can again be described as an electrical network with the following conductances:

$$
\widetilde{C}_{(x,x+1)}(\omega) = \begin{cases} C_{(x,x+1)}(\omega) = e^{-V(x)} & \text{for } x = T^-, T^- + 1, \dots, T^+ - 1 \\ 0 & \text{for } x = T^- - 1, T^+ \end{cases}
$$

Therefore, a reversible measure for the reflected random walk is given by (cf. (2.6))

$$
\widetilde{\mu}_\omega(x) = \begin{cases} \mu_\omega(x) = e^{-V(x)} + e^{-V(x-1)} & \text{for } x = T^- + 1, T^- + 2, \dots, T^+ - 1, \\ e^{-V(T^-)} & \text{for } x = T^-, \\ e^{-V(T^+-1)} & \text{for } x = T^+. \end{cases}
$$

This implies, since $0 \le b_+ < T^+$,

$$
\begin{aligned}
&\frac{\widetilde{\mu}_\omega(b_+)}{\widetilde{\mu}_\omega(x)} \ge \frac{e^{-V(b_+)} + e^{-V(b_+-1)}}{e^{-V(x)} + e^{-V(x-1)}} \\
&\ge \frac{e^{-V(b_+)}}{2 \cdot e^{(-\min\{V(b_+),V(b_-)\})}} \ge \frac{1}{2} \cdot \exp(-\delta L) \qquad (2.26)
\end{aligned}
$$

for $T^- \le x \le T^+$ and for $\omega \in \Gamma(L, \delta)$. By applying (2.26) to (2.25), we get

$$
\begin{aligned}
&P_\omega^{b_+}(\widetilde{X}_\ell = b_+) \\
&\ge \frac{1}{2} \cdot \sum_{x=T^-}^{T^+} \left(P_\omega^{b_+}(\widetilde{X}_{\ell/2} = x) \right)^2 \cdot \exp(-\delta L) \\
&\ge \frac{1}{2} \cdot \sum_{x=T^-}^{T^+} \left(\frac{1}{|T^-| + T^+ + 1} \right)^2 \cdot \exp(-\delta L) \\
&= \frac{1}{2} \cdot \frac{1}{|T^-| + T^+ + 1} \cdot \exp(-\delta L). \qquad (2.27)
\end{aligned}
$$

Here, we used that we have

$$
\sum_{x=T^-}^{T^+} (a_x)^2 \ge \sum_{x=T^-}^{T^+} \left(\frac{1}{|T^-| + T^+ + 1} \right)^2
$$

for every sequence $(a_x)_x$ with $\sum_{x=T^-}^{T^+} a_x = 1$. $\square$

We can now return to the proof of Proposition 2.5.2 and finish our lower bound for the third factor in (2.19). By applying (2.23), (2.24) and Lemma 2.5.3 to (2.22), we get for $\exp(3\delta L) \leq n \leq \exp((1-2\delta)L)$ and $\omega \in \Gamma(L,\delta)$, i.e. $|T^-|, T^+ \leq L^2$,

$$\begin{aligned}
&\widehat{\inf}_{\ell \in \{\lceil \frac{4n}{3} \rceil, \dots, 2n\}} P^o_\omega(X_\ell = b_+) \\
\geq\ &\left(\frac{1}{2} - 6 \cdot L^4 \cdot \exp(-\delta L)\right) \cdot \frac{1}{2} \cdot \frac{1}{2L^2+1} \cdot \exp(-\delta L) - 4 \cdot \exp\big(-2\delta L\big) \\
\geq\ &\exp\left(-\tfrac{3}{2}\delta L\right) \qquad (2.28)
\end{aligned}$$

for all $L = L(\delta)$ large enough.

To finish the proof of Proposition 2.5.2, we can collect our lower bounds in (2.20), (2.21), and (2.28) and conclude with (2.19) that for $\exp(3\delta L) \leq n \leq \exp\big((1-2\delta)L\big)$ and for $\omega \in \Gamma(L,\delta)$ we have

$$\begin{aligned}
&P_\omega(X_{2n} = 0) \\
\geq\ &\left(\frac{1}{2} - 6 \cdot L^4 \exp(-\delta L)\right) \cdot \frac{\varepsilon}{1-\varepsilon} \exp(-\delta L) \cdot \exp\left(-\tfrac{3}{2}\delta L\right) \\
\geq\ &\exp(-3\delta L)
\end{aligned}$$

for all $L = L(\delta)$ large enough. This shows (2.17) since we have $P_\omega(X_{2n} = 0) \geq \varepsilon^{2n} > 0$ for all $n \in \mathbb{N}$ due to assumption (2.2). ■

Proposition 2.5.4. *For* $0 < \delta < 1$*, we have*

$$\mathsf{P}(\omega: \ \omega \in \Gamma(L,\delta) \text{ for infinitely many } L) = 1. \qquad (2.29)$$

Proof of Proposition 2.5.4. Let $(B(t))_{t\in\mathbb{R}}$ be the two-sided Brownian motion from Theorem 2.3.1 and let us choose some $0 < \delta < \frac{1}{2}$. For $y \in \mathbb{R}$ we define

$$\begin{aligned}
\widehat{T}^+(y) &:= \inf\{t \geq 0: \ B(t) = y\}, \\
\widehat{T}^-(y) &:= \sup\{t \leq 0: \ B(t) = y\}
\end{aligned}$$

as the first hitting times of y on the positive and negative side of the origin, respectively. Additionally, for $L \in \mathbb{N}$, $i \in \mathbb{N}$, $y \in \mathbb{R}$, we can introduce the following sets

$$\begin{aligned}
F^+_L(y) &:= \{\widehat{T}^+(y \cdot L) < \widehat{T}^+(-y \cdot L)\}, \\
F^-_L(y) &:= \{\widehat{T}^-(y \cdot L) < \widehat{T}^-(-y \cdot L)\}
\end{aligned}$$

on which the Brownian motion reaches the value $y \cdot L$ before $-y \cdot L$. Further we define

$$\begin{aligned}
G^+_L(i) &:= \left\{B(t) \geq (2i-1) \cdot \tfrac{\delta}{4} \cdot L \quad \text{for} \quad \widehat{T}^+\big(2i \cdot \tfrac{\delta}{4} \cdot L\big) \leq t \leq \widehat{T}^+\big((2i+2) \cdot \tfrac{\delta}{4} \cdot L\big)\right\}, \\
G^-_L(i) &:= \left\{B(t) \geq (2i-1) \cdot \tfrac{\delta}{4} \cdot L \quad \text{for} \quad \widehat{T}^-\big((2i+2) \cdot \tfrac{\delta}{4} \cdot L\big) \leq t \leq \widehat{T}^-\big(2i \cdot \tfrac{\delta}{4} \cdot L\big)\right\}
\end{aligned}$$

on which the Brownian motion does not decrease much between the first hitting time of the two levels of interest. Using these sets, we can define the sets

$$\begin{aligned}
A^+(L,\delta) &:= F_L^+(\delta) \cap \left\{ \widehat{T}^+(1.1\cdot L) \le L^2, \min_{\widehat{T}^+(\delta\cdot L)\le t\le \widehat{T}^+(1.1\cdot L)} B(t) \ge \frac{\delta}{4}\cdot L \right\}, \\
A^-(L,\delta) &:= F_L^-(\delta) \cap \left\{ -\widehat{T}^-(1.1\cdot L) \le L^2, \min_{\widehat{T}^-(1.1\cdot L)\le t\le \widehat{T}^-(\delta\cdot L)} B(t) \ge \frac{\delta}{4}\cdot L \right\}, \\
D^+(L,\delta) &:= G_L^+(0) \cap G_L^+(1) \cap G_L^+(2) \\
&\qquad \cap \left\{ \widehat{T}^+(1.2\cdot L) \le 0.9\cdot L^2, \min_{\widehat{T}^+\left(\frac{3\cdot\delta}{2}\cdot L\right)\le t\le \widehat{T}^+(1.2\cdot L)} B(t) \ge \frac{3\delta}{4}\cdot L \right\}, \\
D^-(L,\delta) &:= G_L^-(0) \cap G_L^-(1) \cap G_L^-(2) \\
&\qquad \cap \left\{ -\widehat{T}^-(1.2\cdot L) \le 0.9\cdot L^2, \min_{\widehat{T}^-(1.2\cdot L)\le t\le \widehat{T}^-\left(\frac{3\delta}{2}\cdot L\right)} B(t) \ge \frac{3\delta}{4}\cdot L \right\}
\end{aligned}$$

which which will be used for an approximation of our previously constructed valleys ω belonging to $\Gamma(L,\delta)$ which we illustrated in Figure 2.1 on page 23. Here, we added the factors 1.1, 1.2 and 0.9 in contrast to the construction before in order to have some space for the approximation. For the Brownian motion, we can directly compute that we have

$$\mathsf{P}\big(D^+(1,\delta) \cap D^-(1,\delta)\big) > 0. \tag{2.30}$$

Thereby, the scaling property of the Brownian motion, i.e. the property that for $L \in \mathbb{N}$

$$\left(\frac{1}{L}B(L^2\cdot t)\right)_{t\in\mathbb{R}} \tag{2.31}$$

is again a two-sided Brownian motion with diffusion constant σ, implies

$$\mathsf{P}\big(D^+(L,\delta) \cap D^-(L,\delta)\big) = \mathsf{P}\big(D^+(1,\delta) \cap D^-(1,\delta)\big) > 0 \tag{2.32}$$

for all $L \in \mathbb{N}$.

At first, we notice that for $L_0 \in \mathbb{N}$ we have

$$\mathsf{P}\left(\bigcap_{L=L_0}^{\infty} \Big(A^+(L,\delta)\cap A^-(L,\delta)\Big)^c \right) \le \mathsf{P}\left(\bigcap_{k=\ell+1}^{\infty} \Big(A^+(L_k,\delta)\cap A^-(L_k,\delta)\Big)^c \right) \tag{2.33}$$

for arbitrary $\ell \in \mathbb{N}_0$, where we define

$$L_k := \max\left\{10, \left\lceil \tfrac{2}{\delta} \right\rceil\right\} \cdot (L_{k-1})^2$$

for $k \in \mathbb{N}$ inductively. Note that for $n > \ell + 1$ with

$$\mathcal{F}_n := \sigma\left(\big(B(t)\big)_{-(L_{n-1})^2 \le t \le (L_{n-1})^2} \right),$$

the following holds:

$$
\begin{aligned}
&\mathsf{P}\left(\bigcap_{k=\ell+1}^{n}\Big(A^+(L_k,\delta)\cap A^-(L_k,\delta)\Big)^c\right)\\
&\leq \mathsf{E}\left[\prod_{k=\ell+1}^{n-1}\mathbb{1}_{\left(A^+(L_k,\delta)\cap A^-(L_k,\delta)\right)^c}\cdot\mathbb{1}_{\left\{\max\limits_{-(L_{n-1})^2\leq t\leq (L_{n-1})^2}|B(t)|<(L_{n-1})^2\right\}}\right.\\
&\quad\left.\cdot\ \mathsf{E}\left[\mathbb{1}_{\left\{\left(B(t+(L_{n-1})^2)-B((L_{n-1})^2)\right)_{t\in\mathbb{R}}\notin D^+(L_n,\delta)\right\}\cup\left\{\left(B(t-(L_{n-1})^2)-B(-(L_{n-1})^2)\right)_{t\in\mathbb{R}}\notin D^-(L_n,\delta)\right\}}\,\middle|\,\mathcal{F}_n\right]\right]\\
&\quad+\mathsf{P}\left(\max_{-(L_{n-1})^2\leq t\leq (L_{n-1})^2}|B(t)|\geq (L_{n-1})^2\right)\\
&\leq \Big(1-\mathsf{P}\Big(D^+\left(L_n,\delta\right)\cap D^-\left(L_n,\delta\right)\Big)\Big)\cdot\mathsf{P}\left(\bigcap_{k=\ell+1}^{n-1}\Big(A^+(L_k,\delta)\cap A^-(L_k,\delta)\Big)^c\right)\\
&\quad+\mathsf{P}\left(\max_{-(L_{n-1})^2\leq t\leq (L_{n-1})^2}|B(t)|\geq (L_{n-1})^2\right)\\
&\leq \Big(1-\mathsf{P}\Big(D^+\left(1,\delta\right)\cap D^-\left(1,\delta\right)\Big)\Big)^{n-\ell}+\sum_{k=\ell+1}^{n}\mathsf{P}\left(\max_{-(L_{k-1})^2\leq t\leq (L_{k-1})^2}|B(t)|\geq (L_{k-1})^2\right).
\end{aligned}
\tag{2.34}
$$

To see that the first step holds, note that for

$$
\begin{aligned}
\omega\in&\left\{\max_{-(L_{n-1})^2\leq t\leq (L_{n-1})^2)}|B(t)|<(L_{n-1})^2\right\}\\
&\cap\left\{\left(B(t+(L_{n-1})^2)-B((L_{n-1})^2)\right)_{t\in\mathbb{R}}\in D^+\left(L_n,\delta\right)\right\}
\end{aligned}
\tag{2.35}
$$

we have

$$
\begin{aligned}
\min_{0\leq t\leq (L_n)^2}B(t)\geq&\min_{0\leq t\leq (L_{n-1})^2}B(t)+\min_{(L_{n-1})^2\leq t\leq (L_n)^2}B(t+(L_{n-1})^2)-B((L_{n-1})^2)\\
>&-(L_{n-1})^2-\frac{\delta}{4}\cdot L_n>-\delta\cdot L_n,
\end{aligned}
$$

and

$$
\begin{aligned}
\max_{0\leq t\leq (L_n)^2}B(t)\geq&\ B\big((L_{n-1})^2\big)+\max_{(L_{n-1})^2\leq t\leq (L_n)^2-(L_{n-1})^2}B(t+(L_{n-1})^2)-B((L_{n-1})^2)\\
\geq&-(L_{n-1})^2+1.2\cdot L_n\geq 1.1\cdot L_n.
\end{aligned}
$$

In particular, we have $\widehat{T}^+(\delta\cdot L)<\widehat{T}^+(-\delta\cdot L)$ and $\widehat{T}^+(1.1\cdot L)\leq L^2$ on the considered set. Similarly, again on the set in (2.35), we see that we have

$$
\widehat{T}^+(\delta\cdot L)>\inf\{t\geq (L_{n-1})^2:\ (B(t+(L_{n-1})^2)-B((L_{n-1})^2)\geq \tfrac{\delta}{2}\cdot L_n\},
$$

$$\widehat{T}^{+}(\delta \cdot L) < \inf\{t \geq (L_{n-1})^2 : \ (B(t + (L_{n-1})^2) - B((L_{n-1})^2) \geq \tfrac{3\cdot\delta}{2} \cdot L_n\},$$

which implies

$$\min_{\widehat{T}^{+}(\delta\cdot L)\leq t\leq \widehat{T}^{+}(1.1\cdot L)} B(t) \geq \frac{\delta}{4} \cdot L_n$$

by construction of $D^{+}(L_n, \delta)$. Altogether, we can conclude that $\omega \in A^{+}(L_n, \delta)$ holds for our choice of ω in (2.35). The argument for the negative part runs completely analogously. Further in (2.34), we used the Markov property of the Brownian motion in the second step. Additionally, we iterated the first two steps $n - \ell - 1$ times and used (2.32) for the last step. To control the last sum in (2.34), let us recall the standard upper bound

$$\mathsf{P}\left(Z \geq x\right) \leq \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right) \quad \text{for } x > 0$$

for a random variable $Z \sim \mathcal{N}(0,1)$, which can be found for example in Lemma 12.9 in Appendix B of [MP10]. By using this upper bound, we can conclude that

$$\sum_{k=\ell+1}^{n} \mathsf{P}\left(\max_{-(L_{k-1})^2\leq t\leq (L_{k-1})^2} |B(t)| \geq (L_{k-1})^2\right) \leq 4 \cdot \sum_{k=\ell+1}^{n} \mathsf{P}\left(\max_{0\leq t\leq (L_{k-1})^2} \frac{B(t)}{\sigma \cdot L_{k-1}} \geq \frac{L_{k-1}}{\sigma}\right)$$
$$\leq 4 \cdot \sum_{k=\ell+1}^{\infty} \frac{\sigma}{L_{k-1}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(L_{k-1})^2}{2\sigma^2}\right) \xrightarrow{\ell\to\infty} 0. \tag{2.36}$$

Here, we used that

$$\max_{0\leq t\leq (L_{k-1})^2} \frac{B(t)}{\sigma \cdot L_{k-1}} \sim |Z|$$

for all $k \in \mathbb{N}$, where $Z \sim \mathcal{N}(0,1)$. By combining the upper bounds in (2.33), (2.34), and (2.36), we get for all $\ell \in \mathbb{N}_0$

$$\mathsf{P}\left(\omega \notin \left(A^{+}(L,\delta) \cap A^{-}(L,\delta)\right) \text{ for all } L \geq L_0\right)$$
$$\leq \lim_{n\to\infty} \left(1 - \mathsf{P}\left(D^{+}\left(1, \tfrac{\delta}{2}\right) \cap D^{-}\left(1, \tfrac{\delta}{2}\right)\right)\right)^{n-\ell}$$
$$+ \sum_{k=\ell+1}^{\infty} \mathsf{P}\left(\max_{-(L_{k-1})^2\leq t\leq (L_{k-1})^2} |B(t)| \geq (L_{k-1})^2\right) \xrightarrow{\ell\to\infty} 0.$$

Since $L_0 \in \mathbb{N}$ was chosen arbitrarily, we can conclude that for $0 < \delta < \frac{1}{2}$ we have

$$\mathsf{P}\left(\omega : \ \omega \in \left(A^{+}(L,\delta) \cap A^{-}(L,\delta)\right) \text{ for infinitely many } L\right) = 1.$$

Using the Komlós-Major-Tusnády strong approximation Theorem (cf. Theorem 2.3.1), we see that for $0 < \delta < \frac{1}{2}$ we have

$$\left\{\omega : \ \omega \in \left(A^{+}(L,\delta) \cap A^{-}(L,\delta)\right) \text{ for infinitely many } L\right\}$$
$$\subseteq \left\{\omega : \ \omega \in \Gamma(L, 2\delta) \text{ for infinitely many } L\right\},$$

which is enough to conlude that (2.29) holds for all $0 < \delta < 1$. ■

With the help of Proposition 2.5.2 and Proposition 2.5.4, we can now turn to the proofs of our Theorems 2.4.1 – 2.4.3:

Proof of Theorem 2.4.1. For a fixed $0 \le \alpha < 1$, we choose $0 < \delta < \frac{1}{6}$ such that

$$\alpha < \frac{1-5\delta}{1-2\delta}\ . \tag{2.37}$$

For $\omega \in \Gamma(L,\delta)$, the inequality in (2.17) implies that

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} P_\omega(X_{2n}=0)\cdot n^{-\alpha} \ge \sum_{\lceil \exp(3\delta L)\rceil \le n \le \lfloor \exp((1-2\delta)L)\rfloor} P_\omega(X_{2n}=0)\cdot n^{-\alpha} \\
&\ge \Big(\exp\big((1-2\delta)L\big) - \exp(3\delta L) - 1\Big)\cdot C\cdot \exp(-3\delta L)\cdot \big(\exp\big((1-2\delta)L\big)\big)^{-\alpha} \\
&= C\cdot \exp(-3\delta L)\cdot \exp(3\delta L)\cdot \Big(\exp\big((1-5\delta)L\big) - 1 - \exp(-3\delta L)\Big)\cdot \exp\big(-\alpha(1-2\delta)L\big) \\
&\xrightarrow{L\to\infty} \infty.
\end{aligned}$$

Since Proposition 2.5.4 shows that for P-a.e. environment ω we find L arbitrarily large such that $\omega \in \Gamma(L,\delta)$, we can conclude that (2.14) holds for P-a.e. environment ω. ■

Proof of Theorem 2.4.2. For fixed $\alpha > 0$, we choose δ such that

$$0 < \delta < \min\left\{\frac{1}{2+3\alpha}, \frac{1}{5}\right\},$$

which yields

$$1 - 2\delta - 3\alpha\delta > 0 \quad \text{and} \quad 1 - 2\delta > 3\delta.$$

For $\omega \in \Gamma(L,\delta)$, the inequality in (2.17) implies

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} \Big(P_\omega(X_{2n}=0)\Big)^\alpha \ge \sum_{\lceil \exp(3\delta L)\rceil \le n \le \lfloor \exp((1-2\delta)L)\rfloor} \Big(P_\omega(X_{2n}=0)\Big)^\alpha \\
&\ge \Big(\exp\big((1-2\delta)L\big) - \exp(3\delta L) - 1\Big)\cdot \big(C\cdot\exp(-3\delta L)\big)^\alpha \\
&= C^\alpha \cdot \exp(-3\alpha\delta L)\cdot \exp(3\alpha\delta L) \\
&\qquad \cdot \Big(\exp\big((1-2\delta-3\alpha\delta)L\big) - \exp\big((3\delta-3\alpha\delta)L\big) - \exp(-3\alpha\delta L)\Big) \\
&\xrightarrow{L\to\infty} \infty.
\end{aligned}$$

Again since Proposition 2.5.4 shows that for P-a.e. environment ω we find L arbitrarily large such that $\omega \in \Gamma(L, \delta)$, we can conclude that (2.15) holds for P-a.e. environment ω. ■

Proof of Theorem 2.4.3. Due to the independence of the environments $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(d)}$, we can extend the proof of Proposition 2.5.4 to get

$$\mathsf{P}^{\otimes d}\Big(\text{For infinitely many } L \in \mathbb{N} \text{ we have}: \ \omega^{(i)} \in \Gamma(L,\delta) \text{ for } i = 1,2,\ldots d\Big) = 1 \tag{2.38}$$

for all $0 < \delta < 1$.

Thereby due to Proposition 2.5.2, we have for $(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(d)})$ with $\omega^{(i)} \in \Gamma(L, \delta)$ for $i = 1, 2, \ldots d$

$$\sum_{n \in \mathbb{N}} \prod_{k=1}^{d} P_{\omega^{(k)}}(X_{2n} = 0) \geq \sum_{\lceil \exp(3\delta L) \rceil \leq n \leq \lfloor \exp((1-2\delta)L) \rfloor} \prod_{k=1}^{d} P_{\omega^{(k)}}(X_{2n} = 0)$$

$$\geq \Big(\exp\big((1-2\delta)L\big) - \exp(3\delta L) - 1 \Big) \cdot C^d \cdot \exp(-3\delta d L)$$

$$= C^d \cdot \exp(-3\delta d L) \cdot \exp(3\delta d L)$$

$$\cdot \Big(\exp\big((1 - 2\delta - 3\delta d)L\big) - \exp\big((3\delta - 3\delta d)L\big) - \exp(-3\delta d L) \Big)$$

$$\xrightarrow{L \to \infty} \infty$$

for

$$0 < \delta < \frac{1}{2 + 3d} \; .$$

Since (2.38) holds for arbitrarily small δ, we can conclude that (2.16) holds for $\mathsf{P}^{\otimes d}$-a.e. environment $(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(d)})$. ■

2.6 Examples for Recurrent Random Walks in Random Environments

Remark 2.6.1. Consider a RWRE $(X_n)_{n \in \mathbb{N}_0}$ for which the environment ω fulfils the assumptions (2.1), (2.2), and (2.3). By an application of Theorem 2.4.1 for $\alpha = 0$, we get

$$\sum_{n \in \mathbb{N}} P_\omega(X_{2n} = 0) = \infty$$

for P-a.e. environment ω. From this, we can conclude that the random walk is recurrent for P-a.e. environment ω.

Corollary 2.6.2 (d particles in the same random environment)**.** *Let us first choose a random environment $\omega = (\omega_x)_{x\in\mathbb{Z}}$ which fulfils the assumptions* (2.1), (2.2), *and* (2.3)*. For fixed ω, we can now consider d independent random walks $(X_n^{(i)})_{n\in\mathbb{N}_0}$ for $i = 1, 2, \dots, d$ where every random walk $(X_n^{(i)})_{n\in\mathbb{N}_0}$ is a usual RWRE in the environment ω in the sense of* (2.4)*. Then, for arbitrary d, the d-dimensional process*

$$\big(X_n^{(1)}, X_n^{(2)}, \dots, X_n^{(d)}\big)_{n\in\mathbb{N}_0}$$

is recurrent for P-a.e. environment ω.

<u>*Proof of Corollary 2.6.2.*</u> First of all, we notice that for fixed ω

$$\big(X_n^{(1)}, X_n^{(2)}, \dots, X_n^{(d)}\big)_{n\in\mathbb{N}_0}$$

is a Markov chain. For the expected amounts of returns to 0, we get by applying Theorem 2.4.2 with $\alpha = d$

$$\sum_{n\in\mathbb{N}} P_\omega\Big(\big(X_{2n}^{(1)}, X_{2n}^{(2)}, \dots, X_{2n}^{(d)}\big) = \big(0, 0, \dots, 0\big)\Big) = \sum_{n\in\mathbb{N}} \Big(P_\omega(X_{2n}^{(1)} = 0)\Big)^d = \infty$$

for P-a.e. environment ω. This implies the recurrence. $\square$

Corollary 2.6.3 (d particles in d i.i.d. random environments)**.** *For arbitrary $d \in \mathbb{N}$, we choose d i.i.d. environments $\omega^{(i)} = (\omega_x^{(i)})_{x\in\mathbb{Z}}$ which all fulfil the assumptions* (2.1), (2.2), *and* (2.3) *for $i = 1, 2, \dots, d$. For fixed $\vec{\omega} := (\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(d)})$, we consider d independent RWRE $(X_n^{(i)})_{n\in\mathbb{N}_0}$, where $(X_n^{(i)})_{n\in\mathbb{N}_0}$ is a usual RWRE in the environment $\omega^{(i)}$ in the sense of* (2.4)*. In this case, the d-dimensional process*

$$\big(X_n^{(1)}, X_n^{(2)}, \dots, X_n^{(d)}\big)_{n\in\mathbb{N}_0}$$

is recurrent for $\mathsf{P}^{\otimes d}$-a.e. environment $\vec{\omega}$.

<u>*Proof of Corollary 2.6.3.*</u> Due to the independence of the processes and the environments in every component, we get

$$\sum_{n\in\mathbb{N}} P_{\vec{\omega}}\Big(\big(X_{2n}^{(1)}, X_{2n}^{(2)}, \dots, X_{2n}^{(d)}\big) = \big(0, 0, \dots, 0\big)\Big) = \sum_{n\in\mathbb{N}} \prod_{i=1}^{d} P_{\omega^{(i)}}(X_{2n}^{(i)} = 0) = \infty$$

due to Theorem 2.4.3 for $\mathsf{P}^{\otimes d}$-a.e. environment $\vec{\omega}$. $\square$

Remark 2.6.4. An alternative proof of Corollary 2.6.3 can be found in [Ze04] after Lemma A.2. The proof there uses the Nash-Williams inequality in the context of electrical networks.

Remark 2.6.5. Corollary 2.6.2 and 2.6.3 show that the recurrence of a RWRE is indeed "stronger" than the recurrence of the symmetric random walk on $\mathbb{Z}$. Note that d particles performing a one-dimensional symmetric random walk will only meet finitely often for $d \geq 3$.

Corollary 2.6.6 (Symmetric Random Walk combined with RWRE - Version 1)**.** *We first choose an environment ω which fulfils the assumptions* (2.1)*,* (2.2)*, and* (2.3)*. For a fixed ω, let $(X_n, Y_n)_{n\in\mathbb{N}_0}$ be a 2-dimensional process where the process $(X_n)_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$ are independent with respect to P_ω, $(X_n)_{n\in\mathbb{N}_0}$ is a RWRE in the sense of* (2.4) *and $(Y_n)_{n\in\mathbb{N}_0}$ a symmetric random walk on $\mathbb{Z}$. Then, $(X_n, Y_n)_{n\in\mathbb{N}_0}$ is recurrent for* P*-a.e. environment ω.*

Proof of Corollary 2.6.6. Due to the independence of the two components, we get

$$\begin{aligned}\sum_{n\in\mathbb{N}} P_\omega\big((X_{2n}, Y_{2n}) = (0,0)\big) &= \sum_{n\in\mathbb{N}} P_\omega\big(X_{2n} = 0\big) \cdot P_\omega\big(Y_{2n} = 0\big)\\ \geq C \cdot \sum_{n\in\mathbb{N}} P_\omega\big((X_{2n} = 0\big) \cdot n^{-\frac{1}{2}} = \infty.\end{aligned}$$

Here, we used the lower bound

$$P_\omega\big(Y_{2n} = 0\big) \geq C \cdot n^{-\frac{1}{2}} \tag{2.39}$$

for the return probabilities of the symmetric random walk on $\mathbb{Z}$ with some constant $C > 0$ (cf. Section 2.18.4 in [Gut05]) and Theorem 2.4.1 with $\alpha = \frac{1}{2}$ for the last two steps. Again, we can conclude the recurrence of the process $(X_n, Y_n)_{n\in\mathbb{N}_0}$ for P-a.e. environment ω. □

Corollary 2.6.7 (Symmetric Random Walk combined with RWRE - Version 2)**.** *We first choose an environment ω which fulfils the assumptions* (2.1)*,* (2.2)*, and* (2.3) *and some $0 < \delta < 1$. For a fixed environment ω, let $(X_n, Y_n)_{n\in\mathbb{N}_0}$ be a Markov chain with values in $\mathbb{Z}^2$ which is determined by*

$$\begin{aligned}&P_\omega\big((X_0, Y_0) = (0,0)\big) = 1,\\ &P_\omega\big((X_{n+1}, Y_{n+1}) = (x+1, y)\big|(X_n, Y_n) = (x,y)\big) = \delta \cdot \omega_x,\\ &P_\omega\big((X_{n+1}, Y_{n+1}) = (x-1, y)\big|(X_n, Y_n) = (x,y)\big) = \delta \cdot (1-\omega_x),\\ &P_\omega\big((X_{n+1}, Y_{n+1}) = (x, y\pm 1)\big|(X_n, Y_n) = (x,y)\big) = \frac{1-\delta}{2}\;.\end{aligned}$$

Again, $(X_n, Y_n)_{n\in\mathbb{N}_0}$ is recurrent for P*-a.e. environment ω.*

Remark 2.6.8. In the situation of Corollary 2.6.7, we first choose the first (or second) component for the next step with probability δ (or $1-\delta$). If we choose the first component, then we change the first component by ± 1 as in the setting of a RWRE, otherwise we change the second component by ± 1 with probability $\frac{1}{2}$ as in the case of a symmetric random walk on $\mathbb{Z}$.

Proof of Corollary 2.6.7. For the proof, it is enough to look at the process $(X_n, Y_n)_{n\in\mathbb{N}_0}$ whenever it has moved in the first component. For this, we define inductively

$$\begin{aligned}\tau_0 &:= 0 && \text{and}\\ \tau_k &:= \inf\big\{n > \tau_{k-1} :\ X_n \neq X_{\tau_{k-1}}\big\} && \text{for } k \geq 1.\end{aligned}$$

Additionally, we define

$$\widetilde{X}_n := X_{\tau_n} \qquad \text{for } n \in \mathbb{N}_0,$$
$$\widetilde{Y}_n := Y_{\tau_n} \qquad \text{for } n \in \mathbb{N}_0.$$

Note that $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ is a usual RWRE on $\mathbb{Z}$ for which the environment ω fulfils our assumptions (2.1), (2.2), and (2.3). Further, we have

$$\widetilde{Y}_n \stackrel{d}{=} S(\tau_n - n), \tag{2.40}$$

where $(S(n))_{n\in\mathbb{N}_0}$ denotes a symmetric random walk on $\mathbb{Z}$ which is independent of $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$, $(\tau_n)_{n\in\mathbb{N}_0}$, and the environment ω. Note here that we can decompose τ_n into the increments

$$\tau_n = \sum_{i=1}^{n}(\tau_i - \tau_{i-1}), \tag{2.41}$$

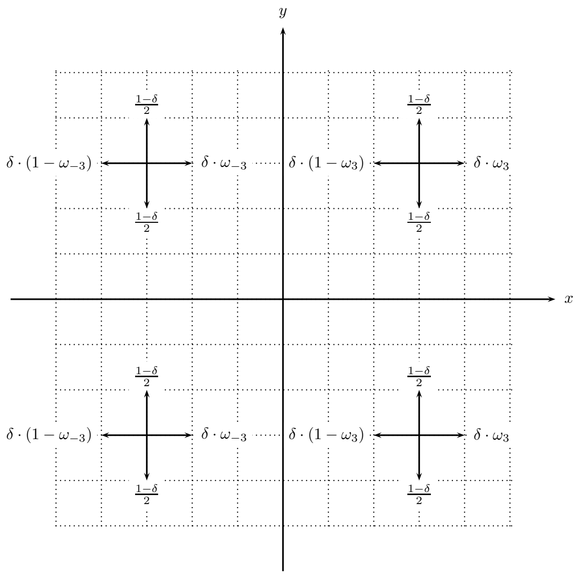

Figure 2.2: *Transition probabilities for the considered process in Corollary 2.6.7*

where $(\tau_i - \tau_{i-1})_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with a geometric distribution with parameter δ and expectation $\frac{1}{\delta}$.

Let us fix an arbitrary $\gamma > 0$. Due to (2.41), an application of Cramer's theorem implies that we have

$$P_\omega\left(\tau_n > \left(\tfrac{1}{\delta} + \gamma\right)\cdot n\right) \le \exp(-n\cdot I)$$

for some constant $I{=}I(\gamma) > 0$. Therefore, the Borel-Cantelli lemma implies that we have

$$P_\omega\left(\liminf_{n\to\infty}\left\{n \le \tau_n \le \left(\tfrac{1}{\delta}+\gamma\right)\cdot n\right\}\right) = 1$$

for every environment ω. Notice here that we have $\tau_n \ge n$ by definition. Due to the continuity of P_ω, we can therefore conclude that

$$\lim_{n\to\infty} P_\omega\Big(n \le \tau_n \le \left(\tfrac{1}{\delta}+\gamma\right)\cdot n\Big) = 1. \tag{2.42}$$

Since we are interested in the returns of the random walk to 0, we have to distinguish between the cases in which τ_n is even or odd. Only for even values of τ_n our random walk $(\widetilde{X}_{2n}, \widetilde{Y}_{2n}) = (X_{\tau_{2n}}, Y_{\tau_{2n}})$ can reach the point $(0,0)$. For this, we note that τ_n has a negative binomial distribution with parameters n and δ and therefore has the following properties:

$$\begin{aligned} &P_\omega(\tau_n = k) \le P_\omega(\tau_n = k+1) \quad \text{for } n \le k \le \frac{n-1}{\delta}, \\ &P_\omega(\tau_n = k) \ge P_\omega(\tau_n = k+1) \quad \text{for } k \ge \max\left\{\frac{n-1}{\delta}, n\right\}, \\ &\max_{k\ge n} P_\omega(\tau_n = k) \xrightarrow{n\to\infty} 0. \end{aligned} \tag{2.43}$$

Thus, a combination of (2.42) and (2.43) implies that in the limit, for $n \to \infty$, the probability for the even and odd part is the same, i.e.

$$\lim_{n\to\infty} P_\omega\Big(n \le \tau_n \le \big(\tfrac{1}{\delta}+\gamma\big)\cdot n,\ \tau_n \in 2\mathbb{N}_0\Big) = \frac{1}{2}.$$

Since due to our choice $\gamma > 0$ we have

$$P_\omega\Big(n \le \tau_n \le \big(\tfrac{1}{\delta}+\gamma\big)\cdot n,\ \tau_n \in 2\mathbb{N}_0\Big) > 0$$

for all $n \in \mathbb{N}$, a combination of the last two in-/equalities implies that there exists some constant $C_2 > 0$ such that

$$P_\omega\Big(n \le \tau_n \le \big(\tfrac{1}{\delta}+\gamma\big)\cdot n,\ \tau_n \in 2\mathbb{N}_0\Big) \ge C_2 > 0 \tag{2.44}$$

for all $n \in \mathbb{N}$ and for every environment ω. Using the independence of $(\widetilde{X}_{2n})_{n\in\mathbb{N}_0}$ and $(\widetilde{Y}_{2n})_{n\in\mathbb{N}_0}$, we therefore get the following lower bound:

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} P_\omega\big((\widetilde{X}_{2n},\widetilde{Y}_{2n}) = (0,0)\big) = \sum_{n\in\mathbb{N}} P_\omega(\widetilde{X}_{2n}=0)\cdot P_\omega(\widetilde{Y}_{2n}=0)\\
&\ge \sum_{n\in\mathbb{N}} P_\omega(\widetilde{X}_{2n}=0)\cdot \sum_{\substack{i=2n\\ i\in 2\mathbb{N}_0}}^{\lfloor(\frac{1}{\delta}+\gamma)\cdot 2n\rfloor} P_\omega(\widetilde{Y}_{2n}=0,\tau_{2n}=i)\\
&\ge \sum_{n\in\mathbb{N}} P_\omega(\widetilde{X}_{2n}=0)\cdot \sum_{\substack{i=2n\\ i\in 2\mathbb{N}_0}}^{\lfloor(\frac{1}{\delta}+\gamma)\cdot 2n\rfloor} P_\omega\big(S(i-2n)=0\big)\cdot P_\omega(\tau_{2n}=i)\\
&\ge \sum_{n\in\mathbb{N}} P_\omega(\widetilde{X}_{2n}=0)\cdot \Bigg(P_\omega(\tau_{2n}=2n) + \sum_{\substack{i=2n+2\\ i\in 2\mathbb{N}_0}}^{\lfloor(\frac{1}{\delta}+\gamma)\cdot 2n\rfloor} C\cdot(i-2n)^{-\frac{1}{2}}\cdot P_\omega(\tau_{2n}=i)\Bigg)
\end{aligned}$$

Here, we used (2.40) in the third line and the usual lower bound for the return probabilities of the symmetric random walk on $\mathbb{Z}$ (cf. (2.39)), i.e.

$$P_\omega\big(S(i-2n)=0\big) \ge C\cdot(i-2n)^{-\frac{1}{2}}$$

for $i \in 2\mathbb{N}$, $i \ge 2n+2$ and with some constant $C > 0$, in the fourth line. From this, we get

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} P_\omega\big((\widetilde{X}_{2n},\widetilde{Y}_{2n} = (0,0)\big)\\
&\ge C\cdot\sum_{n\in\mathbb{N}} P_\omega(\widetilde{X}_{2n}=0)\cdot\big(2\cdot\big(\tfrac{1}{\delta}+\gamma-1\big)\big)^{-\frac{1}{2}}\cdot n^{-\frac{1}{2}}\cdot \sum_{\substack{i=2n\\ i\in 2\mathbb{N}_0}}^{\lfloor(\frac{1}{\delta}+\gamma)\cdot 2n\rfloor} P_\omega(\tau_{2n}=i)
\end{aligned}$$

$$\geq C \cdot C_2 \cdot \big(2 \cdot \big(\tfrac{1}{\delta} + \gamma - 1\big)\big)^{-\frac{1}{2}} \cdot \sum_{n \in \mathbb{N}} P_\omega(\widetilde{X}_{2n} = 0) \cdot n^{-\frac{1}{2}} = \infty$$

for P-a.e. environment ω. Here, we additionally made use of (2.44) and Theorem 2.4.1 (applied for $\alpha = \frac{1}{2}$) in the last line. This implies that the process

$$(\widetilde{X}_n, \widetilde{Y}_n)_{n \in \mathbb{N}_0}$$

is recurrent for P-a.e. environment ω. Finally, this obviously implies that our process

$$(X_n, Y_n)_{n \in \mathbb{N}_0}$$

is also recurrent for P-a.e. environment ω since we can embed the paths of the process $(\widetilde{X}_n, \widetilde{Y}_n)_{n \in \mathbb{N}_0}$ into the paths of the process $(X_n, Y_n)_{n \in \mathbb{N}_0}$. □

Chapter 3

RWRE with Random Orientations

3.1 Overview

In this chapter, we consider a RWRE with random orientations (RWRO) which is a random walk on $\mathbb{Z}^2$ with a one-dimensional random environment. As the main result of this chapter, we show that the RWRO is transient for a.e. random environment.

The structure of this chapter is the following: In Section 3.2, we start with a formal description of the model and then state our main results. Before we can finally prove these results, we need some preparation: In a first step in Section 3.3, we construct two sequences of valleys with the help of a coupled two-sided Brownian motion and derive some properties of these valleys. Afterwards, we collect and extend several statements about the so-called strong localization of the RWRE with regard to our constructed valleys in Section 3.4. With the help of our preparation, we are then able to prove our main theorems in Section 3.5. Finally in Section 3.6, we consider an extended model of the RWRO, for which we introduce a drift p as an additional parameter. Here, we answer the question of recurrence and transience of the extended model depending on the drift p. This further connects our results from Chapter 2 and Section 3.2.

3.2 Model and Results

We first introduce the random environment: Let $(\omega_x)_{x\in\mathbb{Z}}$ and $(\alpha_x)_{x\in\mathbb{Z}}$ be two independent sequences of i.i.d random variables with respect to the law of the environment P. The sequence $(\omega_x)_{x\in\mathbb{Z}}$ is a sequence of i.i.d. random variables taking values in $(0,1)$ and we assume that the following assumptions hold

$$\mathsf{E}[\log \rho_0] = 0, \tag{3.1}$$

$$\mathsf{P}(\varepsilon \leq \omega_0 \leq 1-\varepsilon) = 1 \quad \text{for some } \varepsilon \in \left(0, \tfrac{1}{2}\right), \tag{3.2}$$

$$\mathsf{Var}(\log \rho_0) > 0, \tag{3.3}$$

where

$$\rho_0 = \rho_0(\omega) := \frac{1-\omega_0}{\omega_0}$$

as usual. Note that $(\omega_x)_{x\in\mathbb{Z}}$ corresponds to the random environment of a one-dimensional RWRE and that (3.1) ensures that the one-dimensional RWRE is recurrent. The second assumption is a common technical condition in the context of RWRE. Additionally, the third assumption excludes the case of a symmetric random walk on $\mathbb{Z}$.

The second part of the random environment, $(\alpha_x)_{x\in\mathbb{Z}}$, determines a random orientation for every position $x \in \mathbb{Z}$. More precisely, $(\alpha_x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence with

$$\mathsf{P}(\alpha_0 = +1) = \frac{1}{2} = \mathsf{P}(\alpha_0 = -1)$$

which is further independent of $(\omega_x)_{x\in\mathbb{Z}}$. Finally, the random environment is given by

$$\theta := (\theta_x)_{x\in\mathbb{Z}} := (\omega_x, \alpha_x)_{x\in\mathbb{Z}}. \tag{3.4}$$

For a fixed environment θ and some $0 < \delta < 1$, we can now introduce the associated random walk $(Z_n)_{n\in\mathbb{N}_0}$ whose transition probabilities are determined by θ: For every $z = (z_1, z_2) \in \mathbb{Z}^2$, $(Z_n)_{n\in\mathbb{N}_0}$ is a Markov chain with respect to P_θ^z determined by

$$P_\theta^z\big(Z_0 = (z_1, z_2)\big) = 1,$$

$$P_\theta^z\big(Z_{n+1} = (k+1, \ell)\big|Z_n = (k,\ell)\big) = \delta \cdot \omega_k,$$

$$P_\theta^z\big(Z_{n+1} = (k-1, \ell)\big|Z_n = (k,\ell)\big) = \delta \cdot (1-\omega_k),$$

$$P_\theta^z\big(Z_{n+1} = (k, \ell+1)\big|Z_n = (k,\ell)\big) = \frac{1-\delta}{2} \cdot (1+\alpha_k),$$

$$P_\theta^z\big(Z_{n+1} = (k, \ell-1)\big|Z_n = (k,\ell)\big) = \frac{1-\delta}{2} \cdot (1-\alpha_k)$$

for $n \in \mathbb{N}_0$, $k, \ell \in \mathbb{Z}$. Here, δ and $1-\delta$ are the probabilities for a movement in the first and the second component, respectively.

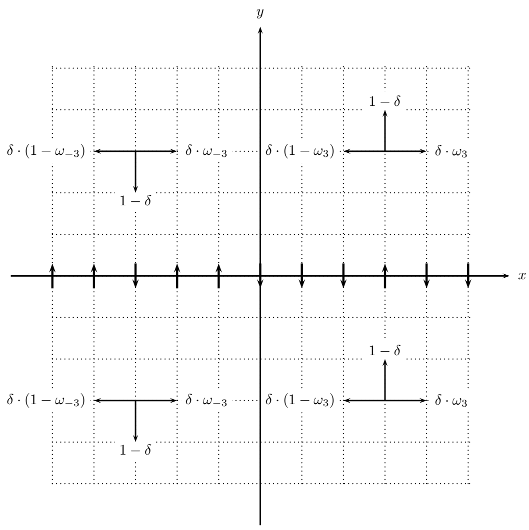

Figure 3.1: *A possible realization of the random orientations* $\uparrow\downarrow$ *and the corresponding transition probabilities.*

Note that $(Z_n)_{n\in\mathbb{N}_0}$ can only move upwards in points with a positive orientation and downwards in points with a negative orientation. For notational convenience, we will often use P_θ without the superscript 0 instead of P_θ^0 when we consider the random walk which is started at the origin.

In order to define a joint distribution of the environment θ and the random walk $(Z_n)_{n\in\mathbb{N}_0}$, we can equip the set of environments

$$\Omega := \big((0,1)\times\{-1,1\}\big)^{\mathbb{Z}}$$

with its product σ-field $\mathcal{F}$ (where we consider the natural product σ-field in every component), and we equip the set of all paths $(\mathbb{Z}^2)^{\mathbb{N}_0}$ with the product σ-field $\mathcal{G}$. The joint distribution $\mathbb{P}^z$ of $(\theta,(Z_n)_{n\in\mathbb{N}_0})$ with starting point $z\in\mathbb{Z}^2$ is uniquely determined by

$$\mathbb{P}^z(F\times G) := \int_F P^z_\theta(G)\mathsf{P}(d\theta)$$

for $F\in\mathcal{F}$ and $G\in\mathcal{G}$. Again, we will use $\mathbb{P}$ instead of $\mathbb{P}^0$ for notational convenience. Our main result is the following:

Theorem 3.2.1. *The Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ is transient for P-a.e. environment θ. In particular we have*

$$\mathbb{P}\left(Z_n = (0,0) \text{ for infinitely many } n\right) = 0.$$

Following [CP03a], we can decompose the two-dimensional random walk

$$(Z_n)_{n\in\mathbb{N}_0} =: (\widetilde{X}_n,\widetilde{Y}_n)_{n\in\mathbb{N}_0} \tag{3.5}$$

into a skeleton one-dimensional random walk (the horizontal walk $(X_n)_{n\in\mathbb{N}_0}$), an embedded one-dimensional random walk with unbounded jumps (the vertical walk $(Y_n)_{n\in\mathbb{N}_0}$), and a sequence of waiting times $(\iota_n)_{n\in\mathbb{N}_0}$. For the decomposition, we introduce the following random times inductively:

$$\begin{aligned}
\tau_0 &:= 0,\\
\tau_k &:= \inf\{n>\tau_{k-1}:\ \widetilde{X}_n\neq\widetilde{X}_{\tau_{k-1}}\} \qquad \text{for } k\geq 1.
\end{aligned} \tag{3.6}$$

With the help of the random times of the movement in the horizontal direction, we can now define for $n\in\mathbb{N}$

$$\begin{aligned}
X_n &:= \widetilde{X}_{\tau_n},\\
Y_n &:= \widetilde{Y}_{\tau_n},\\
\iota_{n-1} &:= \tau_n-\tau_{n-1}
\end{aligned} \tag{3.7}$$

where $X_0 := \widetilde{X}_0,\ Y_0 := \widetilde{Y}_0$. Note that $(X_n)_{n\in\mathbb{N}_0}$ is a RWRE on $\mathbb{Z}$. As a by-product, we get the following result for a RWRE in a random scenery (RWRERS):

Theorem 3.2.2. *For the RWRE $(X_n)_{n\in\mathbb{Z}}$ and the RWRERS $(S_n)_{n\in\mathbb{N}_0}$ with*

$$S_n := \sum_{i=0}^{n-1}\alpha_{X_i}, \tag{3.8}$$

we have that $(X_n,S_n)_{n\in\mathbb{N}_0}$ is transient for P-a.e. environment θ. In particular, we have

$$\mathbb{P}\big((X_n,S_n)=(0,0) \text{ for infinitely many } n\big)=0.$$

3.3 Construction of the Valleys

As a tool for the proofs, we construct valleys of the environment as it is often done in the context of RWRE. Note that only the first part of the environment, namely $(\omega_x)_{x\in\mathbb{Z}}$, is used for the definition of the valleys. At first, we introduce the potential V as it is usually done in the literature (cf. Section 2 in [SZ07]). With

$$\rho_x(\theta) := \frac{1-\omega_x}{\omega_x}$$

for $x \in \mathbb{Z}$, we can define the potential $V = \big(V(x)\big)_{x\in\mathbb{Z}}$ by

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i & \text{for } x = 1, 2, \dots \\ 0 & \text{for } x = 0 \\ \sum_{i=x+1}^{0} \log(\rho_i)^{-1} & \text{for } x = -1, -2, \dots \ . \end{cases}$$

Note that $V(x)$ is a sum of i.i.d. random variables which are centred and which are bounded by $C := \log(1-\varepsilon) - \log\varepsilon > 0$ due to the assumptions (3.1) and (3.2). One of the most useful properties of the RWRE is the observation that (for fixed environment θ) the random walk is a reversible Markov chain and can therefore be described as an electrical network (cf. [DGPS07]). The conductances are given by

$$C_{(x,x+1)}(\theta) = e^{-V(x)} = \begin{cases} \prod_{i=1}^{x} (\rho_i)^{-1} & \text{for } x = 1, 2, \dots \\ 1 & \text{for } x = 0 \\ \prod_{i=x+1}^{0} \rho_i & \text{for } x = -1, -2, \dots \end{cases}$$

and the reversible measure (which is unique up to multiplication by a constant) is given by

$$\mu_\theta(x) = e^{-V(x)} + e^{-V(x-1)} = \begin{cases} \prod_{i=1}^{x-1} \frac{\omega_i}{1-\omega_i} \cdot \frac{1}{1-\omega_x} & \text{for } x = 1, 2, \dots \\ \frac{1}{\omega_0} & \text{for } x = 0 \\ \prod_{i=x+1}^{0} \frac{1-\omega_i}{\omega_i} \cdot \frac{1}{\omega_x} & \text{for } x = -1, -2, \dots \ . \end{cases} \tag{3.9}$$

3.3.1 Consideration of a two-sided Brownian Motion

Since the explicit distribution of the potential V is often hard to describe, we use the Komlós-Major-Tusnády strong approximation theorem (cf. Theorem 1 in [KMT76], see also formula (2) in [CP03b]) to define the valleys with the help of a two-sided Brownian motion:

Theorem 3.3.1. *In a possibly enlarged probability space, there exists a version of the environment $\theta = (\omega, \alpha)$ and a two-sided Brownian motion $(B(t))_{t\in\mathbb{R}}$ with diffusion constant $\sigma := (\mathsf{Var}(\log \rho_0))^{\frac{1}{2}}$ (i.e. $Var(B(t)) = \sigma^2|t|$) such that for some $K > 0$ we have*

$$\mathsf{P}\left(\limsup_{x\to\pm\infty} \frac{|V(x) - B(x)|}{\log |x|} \leq K\right) = 1. \tag{3.10}$$

Remark: As in the situation of inequality (1.7) in [HS98], our assumptions (3.1), (3.2), and (3.3) imply that the approximation theorem may be applied to the potential $V(\cdot)$. Here, it is important that the moment generating function of $V(1)$ exists in some neighbourhood of 0 which is ensured by assumption (3.2). Therefore, we can start with the construction of the first component of the environment ω, then apply the Komlós-Major-Tusnády strong approximation theorem, and in a last step add the second part of the environment α independently of $(\omega, (B(t))_{t\in\mathbb{R}})$.

From now on, we consider the possibly enlarged probability space on which Theorem 3.3.1 holds for the two-sided Brownian motion $(B(t))_{t\in\mathbb{R}}$ and the potential $V = \big(V(x)\big)_{x\in\mathbb{Z}}$ with respect to the law of the environment P.

With the help of $(B(t))_{t\in\mathbb{R}}$, we can define the following sequence of valleys $(\widehat{\mathcal{V}}_k)_{k\in\mathbb{N}}$ and their deeper parts $(\widehat{\mathcal{D}}_k)_{k\in\mathbb{N}}$ which is motivated by the construction of valleys in [DGPS07]. There, the RWRE is restricted to the positive half-line by a barrier in 0. Further, our goal is to control the returns of the RWRE to 0 in contrast to [DGPS07]. This is why we have to modify the construction.

Since for the RWRE the points at which the potential $V(\cdot)$ attains its minima and its maxima in some neighbourhood play a key role, we will construct our valleys with the help of two sequences $(\widehat{\ell}_k)_{k\in\mathbb{N}_0}$ and $(\widehat{r}_k)_{k\in\mathbb{N}_0}$ on the left and the right side of the origin at which the Brownian motion attains its maxima in some connected and open neighbourhood of these points which further contains 0. Further, $(\widehat{b}_k)_{k\in\mathbb{N}_0}$ will be the sequence at which the Brownian motion attains its minimum in between. For the construction, we start with the Brownian motion at $B(0) = 0$. We always have to look a little further to the left and right side of our candidates for extrema to be sure that they have their extremal property in some open neighbourhood.

Note here that we use $\widehat{\cdot}$ for all quantities which are directly constructed with the help of the two-sided Brownian motion. In contrast, all quantities which are directly connected to the potential $V(\cdot)$ will be denoted without $\widehat{\cdot}$.

We start with $k = 0$ and we define:

$$\widehat{\eta}_0^+ := \inf\{t > 0 : \ B(t) = 1\},$$

$$\widehat{\eta}_0^- := \sup\{t < 0 : \ B(t) = 1\},$$

$$\widehat{b}_0^+ := \sup\left\{0 \leq t \leq \widehat{\eta}_0^+ : \ B(t) = \min_{0\leq s\leq \widehat{\eta}_0^+} B(s)\right\},$$

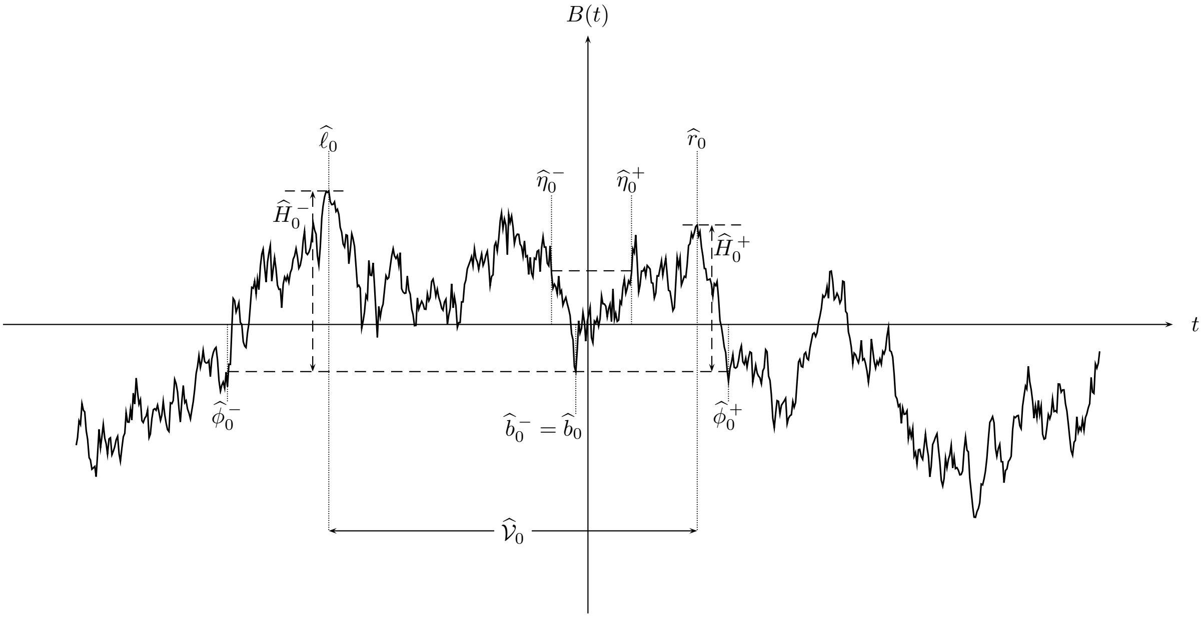

Figure 3.2: *Sampled path of the two-sided Brownian motion and the corresponding valley* $\widehat{\mathcal{V}}_0$.

$$
\begin{aligned}
\widehat{b}_0^- &:= \inf\left\{\widehat{\eta}_0^- \le t \le 0 :\ B(t) = \min_{\widehat{\eta}_0^- \le s \le 0} B(s)\right\},\\
\widehat{b}_0 &:= \begin{cases} \widehat{b}_0^+ & \text{if } B(\widehat{b}_0^+) \le B(\widehat{b}_0^-)\\ \widehat{b}_0^- & \text{if } B(\widehat{b}_0^-) < B(\widehat{b}_0^+),\end{cases}\\
\widehat{\phi}_0^+ &:= \inf\{t > \widehat{\eta}_0^+ :\ B(t) = B(\widehat{b}_0)\},\\
\widehat{\phi}_0^- &:= \sup\{t < \widehat{\eta}_0^- :\ B(t) = B(\widehat{b}_0)\},\\
\widehat{\ell}_0 &:= \inf\left\{\widehat{\phi}_0^- \le t \le \widehat{b}_0 :\ B(t) = \max_{\widehat{\phi}_0^- \le s \le \widehat{b}_0} B(s)\right\},\\
\widehat{m}_0 &:= \widehat{h}_0 := 0,\\
\widehat{r}_0 &:= \sup\left\{\widehat{b}_0 \le t \le \widehat{\phi}_0^+ :\ B(t) = \max_{\widehat{b}_0 \le s \le \widehat{\phi}_0^+} B(s)\right\},\\
\widehat{\mathcal{V}}_0 &:= [\widehat{\ell}_0, \widehat{r}_0],\\
\widehat{\mathcal{D}}_0 &:= \begin{cases} [\widehat{m}_0, \widehat{r}_0] & \text{if } \widehat{b}_0 \ge 0\\ [\widehat{\ell}_0, \widehat{m}_0] & \text{if } \widehat{b}_0 < 0,\end{cases}\\
\widehat{H}_0^+ &:= B(\widehat{r}_0) - B(\widehat{b}_0),\\
\widehat{H}_0^- &:= B(\widehat{\ell}_0) - B(\widehat{b}_0).
\end{aligned}
$$

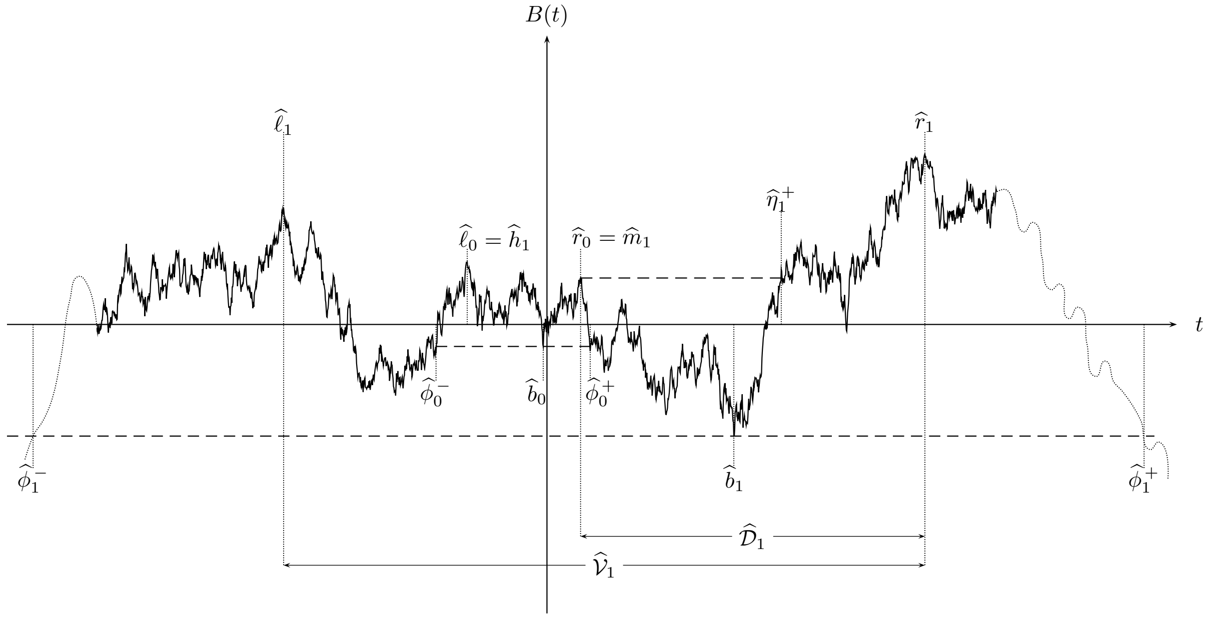

Figure 3.3: *Sampled path of the two-sided Brownian motion and the corresponding valleys $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{V}}_1$.*

For $k \geq 1$ we distinguish between two cases. If we have $\widehat{H}_{k-1}^+ \leq \widehat{H}_{k-1}^-$, then we define inductively:

$$\widehat{\eta}_k^+ := \inf\{t > \widehat{\phi}_{k-1}^+ : \; B(t) = B(\widehat{r}_{k-1})\},$$

$$\widehat{\eta}_k^- := \widehat{\phi}_{k-1}^-,$$

$$\widehat{b}_k := \sup\left\{t < \widehat{\eta}_k^+ : \; B(t) = \min_{0 \leq s \leq \widehat{\eta}_k^+} B(s)\right\},$$

$$\widehat{\phi}_k^+ := \inf\{t > \widehat{\eta}_k^+ : \; B(t) = B(\widehat{b}_k)\},$$

$$\widehat{\phi}_k^- := \sup\{t < \widehat{\eta}_k^- : \; B(t) = B(\widehat{b}_k)\},$$

$$\widehat{\ell}_k := \inf\left\{\widehat{\phi}_k^- \leq t \leq \widehat{b}_k : \; B(t) = \max_{\widehat{\phi}_k^- \leq s \leq \widehat{b}_k} B(s)\right\},$$

$$\widehat{m}_k := \widehat{r}_{k-1},$$

$$\widehat{h}_k := \widehat{\ell}_{k-1},$$

$$\widehat{r}_k := \sup\left\{\widehat{b}_k \leq t \leq \widehat{\phi}_k^+ : \; B(t) = \max_{\widehat{b}_k \leq s \leq \widehat{\phi}_k^+} B(s)\right\},$$

$$\widehat{D}_k^+ := B(\widehat{r}_k) - B(\widehat{b}_k),$$

$$\widehat{D}_k^- := B(\widehat{m}_k) - B(\widehat{b}_k).$$

In the other case, i.e. if $\widehat{H}^+_{k-1} > \widehat{H}^-_{k-1}$, we define in symmetry with the first case:

$$\begin{aligned}
\widehat{\eta}^-_k &:= \sup\{t < \widehat{\phi}^-_{k-1} : \ B(t) = B(\widehat{\ell}_{k-1})\},\\
\widehat{\eta}^+_k &:= \widehat{\phi}^+_{k-1},\\
\widehat{b}_k &:= \inf\left\{t > \widehat{\eta}^-_k : \ B(t) = \min_{\widehat{\eta}^-_k \le s \le 0} B(s)\right\},\\
\widehat{\phi}^-_k &:= \sup\{t < \widehat{\eta}^-_k : \ B(t) = B(\widehat{b}_k)\},\\
\widehat{\phi}^+_k &:= \inf\{t > \widehat{\eta}^+_k : \ B(t) = B(\widehat{b}_k)\},\\
\widehat{\ell}_k &:= \inf\left\{\widehat{\phi}^-_k \le t \le \widehat{b}_k : \ B(t) = \max_{\widehat{\phi}^-_k \le s \le \widehat{b}_k} B(s)\right\},\\
\widehat{m}_k &:= \widehat{\ell}_{k-1},\\
\widehat{h}_k &:= \widehat{r}_{k-1},\\
\widehat{r}_k &:= \inf\left\{\widehat{b}_k \le t \le \widehat{\phi}^+_k : \ B(t) = \max_{\widehat{b}_k \le s \le \widehat{\phi}^+_k} B(s)\right\},\\
\widehat{D}^-_k &:= B(\widehat{\ell}_k) - B(\widehat{b}_k),\\
\widehat{D}^+_k &:= B(\widehat{m}_k) - B(\widehat{b}_k).
\end{aligned}$$

Now we can define our valley $\widehat{\mathcal{V}}_k$ and its deeper part $\widehat{\mathcal{D}}_k$

$$\begin{aligned}
\widehat{\mathcal{V}}_k &:= [\widehat{\ell}_k, \widehat{r}_k],\\
\widehat{\mathcal{D}}_k &:= \begin{cases} [\widehat{m}_k, \widehat{r}_k] & \text{if } \widehat{H}^+_{k-1} \le \widehat{H}^-_{k-1} \\ [\widehat{\ell}_k, \widehat{m}_k] & \text{if } \widehat{H}^+_{k-1} > \widehat{H}^-_{k-1}, \end{cases}
\end{aligned}$$

the left and the right height of the valley $\widehat{\mathcal{V}}_k$

$$\begin{aligned}
\widehat{H}^-_k &:= B(\widehat{\ell}_k) - B(\widehat{b}_k),\\
\widehat{H}^+_k &:= B(\widehat{r}_k) - B(\widehat{b}_k),
\end{aligned}$$

and further the heights of the valley $\widehat{\mathcal{V}}_k$ and its deeper part $\widehat{\mathcal{D}}_k$

$$\begin{aligned}
\widehat{D}_k &:= \min\{\widehat{D}^-_k, \widehat{D}^+_k\},\\
\widehat{H}_k &:= \min\{\widehat{H}^-_k, \widehat{H}^+_k\}.
\end{aligned}$$

Remark 3.3.2. In words, we can say the following about our constructed valleys:

(1) For every $k \in \mathbb{N}_0$, we have constructed two valleys $\widehat{\mathcal{V}}_k = [\widehat{\ell}_k, \widehat{r}_k]$ and $\widehat{\mathcal{D}}_k \subseteq \widehat{\mathcal{V}}_k$. By construction, we have $\widehat{b}_k \in \widehat{\mathcal{D}}_k \subseteq \widehat{\mathcal{V}}_k$, $0 \in \widehat{\mathcal{V}}_k$ but $0 \notin \widehat{\mathcal{D}}_k$ for $k \geq 1$.

(2) Here, the name *valley* refers to the property

$$\begin{aligned} B(\widehat{\ell}_k) &= \max_{\widehat{\ell}_k \leq s \leq \widehat{b}_k} B(s), \\ B(\widehat{b}_k) &= \min_{\widehat{\ell}_k \leq s \leq \widehat{r}_k} B(s), \\ B(\widehat{r}_k) &= \max_{\widehat{b}_k \leq s \leq \widehat{r}_k} B(s) \end{aligned}$$

for the left end point $\widehat{\ell}_k$, the bottom point $\widehat{b}_k$, and the right end point $\widehat{r}_k$ of the valley $\widehat{\mathcal{V}}_k$. The valley $\widehat{\mathcal{D}}_k$ has the same property if we replace $\widehat{\ell}_k$ and $\widehat{r}_k$ by the left end right end point of $\widehat{\mathcal{D}}_k$.

(3) The reason for the distinction between the two cases $\widehat{H}^+_{k-1} \leq \widehat{H}^-_{k-1}$ and $\widehat{H}^+_{k-1} > \widehat{H}^-_{k-1}$ is the following: Due to the approximation theorem (Theorem 3.3.1), the path of the two-sided Brownian motion is connected to the potential $(V(x))_{x\in\mathbb{Z}}$ of our RWRE. The RWRE prefers paths on which the potential is not very large in between. Therefore, it is more likely that the RWRE leaves a valley to the side with the lower height.

(4) Similarly to the construction in [DGPS07], note that $(\widehat{\eta}^+_k)_{k\in\mathbb{N}_0}$ and $(\widehat{\phi}^+_k)_{k\in\mathbb{N}_0}$ are stopping times with respect to the filtration $(\mathcal{F}^+_t)_{t\geq 0}$, where

$$\mathcal{F}^+_t := \sigma\big(\{B_s:\ -\infty < s \leq t\}\big) \quad \text{for } t \geq 0.$$

The same holds for $(\widehat{\eta}^-_k)_{k\in\mathbb{N}_0}$ and $(\widehat{\phi}^-_k)_{k\in\mathbb{N}_0}$ with respect to the filtration $(\mathcal{F}^-_t)_{t\leq 0}$, where

$$\mathcal{F}^-_t := \sigma\big(\{B_s:\ t \leq s < \infty\}\big) \quad \text{for } t \leq 0.$$

In contrast, $(\widehat{\ell}_k)_{k\in\mathbb{N}_0}$, $(\widehat{r}_k)_{k\in\mathbb{N}_0}$, $(\widehat{m}_k)_{k\in\mathbb{N}_0}$, and $(\widehat{b}_k)_{k\in\mathbb{N}_0}$ are not stopping times with respect to any reasonable choice for the filtration.

In the following, we collect some properties of our constructed valleys $(\widehat{\mathcal{V}}_k)_{k\in\mathbb{N}_0}$ and $(\widehat{\mathcal{D}}_k)_{k\in\mathbb{N}_0}$ which will be useful to describe the behaviour of the RWRO later on.

The motivation for Proposition 3.3.3 – 3.3.8 is the following: Since we will often make use of the Borel-Cantelli lemma to derive the typical behaviour of the RWRE for large time points n, we need some quantity growing to infinity for our estimates. For this quantity, we use the heights $\widehat{H}_k$ and $\widehat{D}_k$ of our constructed valleys which turn out to grow at least exponentially in k (cf. Proposition 3.3.3). Further, we need some lower scale for the error terms which we can neglect in comparison with $\widehat{H}_k$ and $\widehat{D}_k$ for large k. For this scale, we use terms of the form $(\widehat{H}_k)^{1-\gamma}$ and $(\widehat{D}_k)^{1-\gamma}$ for some $0 < \gamma < 1$:

Proposition 3.3.3. *For $0 < \beta < 1$, we P-a.s. have*

$$\widehat{H}_k \geq \widehat{D}_k \geq \exp(\beta \cdot k) \tag{3.11}$$

for all k large enough.

Proposition 3.3.4. *For $\vartheta > 0$, we P-a.s. have*

$$\widehat{H}_k \leq k^{1+\vartheta} \cdot \widehat{H}_{k-1} \tag{3.12}$$

for all k large enough.

Proposition 3.3.5. *For $0 < \gamma < 1$, we P-a.s. have*

$$(\widehat{H}_{k-1})^{1-\gamma} \leq |\widehat{H}_k^+ - \widehat{H}_k^-| = |B(\widehat{r}_k) - B(\widehat{\ell}_k)| \leq (\widehat{H}_{k-1})^{1+\gamma} \tag{3.13}$$

for all k large enough.

Proposition 3.3.6. *For $\vartheta > 0$, we P-a.s. have*

$$|\widehat{\phi}_k^-| \leq (\widehat{H}_k)^{2+\vartheta} \quad \text{and} \quad |\widehat{\phi}_k^+| \leq (\widehat{H}_k)^{2+\vartheta} \tag{3.14}$$

for all k large enough. In particular, this implies that we P-a.s. have

$$|\widehat{\ell}_k| \leq (\widehat{H}_k)^{2+\vartheta} \quad \text{and} \quad |\widehat{r}_k| \leq (\widehat{H}_k)^{2+\vartheta} \tag{3.15}$$

for all k large enough.

Proposition 3.3.7. *For $0 < \gamma < 1$, we P-a.s. have*

$$\begin{aligned}
&B(\widehat{b}_k) - B(\widehat{b}_{k-1}) \leq -(\widehat{H}_{k-1})^{1-\gamma}, && && (3.16)\\
&B(\widehat{\ell}_k) - B(\widehat{\ell}_{k-1}) \geq (\widehat{H}_{k-1})^{1-\gamma} && \text{on } \{\widehat{b}_k < 0\}, && (3.17)\\
&B(\widehat{r}_k) - B(\widehat{r}_{k-1}) \geq (\widehat{H}_{k-1})^{1-\gamma} && \text{on } \{\widehat{b}_k > 0\} && (3.18)
\end{aligned}$$

for all k large enough.

In view of the last proposition, it makes sense to introduce the following two random times for $0 < \gamma < 1$ and $k \in \mathbb{N}$:

$$\begin{aligned}
\widehat{\phi}_{k,\gamma}^- &:= \sup\{t \leq \widehat{\eta}_k^- : \ B(t) = B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma}\},\\
\widehat{\phi}_{k,\gamma}^+ &:= \inf\{t \geq \widehat{\eta}_k^+ : \ B(t) = B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma}\}.
\end{aligned} \tag{3.19}$$

$\widehat{\phi}_{k,\gamma}^-$ and $\widehat{\phi}_{k,\gamma}^+$ give the locations at which the Brownian motion has almost returned to the minimal position $B(\widehat{b}_k)$ at the random times $\widehat{\phi}_k^-$ and $\widehat{\phi}_k^+$ again. Thereby, Proposition 3.3.7 ensures that we P-a.s. have

$$\widehat{\phi}_{k,\gamma}^- < \widehat{\eta}_k^-,$$

$$\widehat{\phi}^+_{k,\gamma} > \widehat{\eta}^+_k$$

for all k large enough. The next Proposition ensures that we even P-a.s. have

$$\begin{aligned}\widehat{\phi}^-_k < \widehat{\phi}^-_{k,\gamma} < \widehat{\ell}_k,\\ \widehat{\phi}^+_k > \widehat{\phi}^+_{k,\gamma} > \widehat{r}_k\end{aligned}$$

for all k large enough:

Proposition 3.3.8. *For $0 < \gamma < 1$, we P-a.s. have*

$$\begin{aligned}\max_{\widehat{\phi}^-_k \le t \le \widehat{\phi}^-_{k,\gamma}} B(t) \le B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}},\\ \max_{\widehat{\phi}^+_{k,\gamma} \le t \le \widehat{\phi}^+_k} B(t) \le B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}\end{aligned} \tag{3.20}$$

for all k large enough.

Proofs:
For the proofs, let

$$\big(\widehat{B}(t)\big)_{t\ge 0} \stackrel{d}{=} \big(B(t)\big)_{t\ge 0}$$

denote another Brownian motion which is independent of $\Big(\big(B(t)\big)_{t\in\mathbb{R}}, \big(V(x)\big)_{x\in\mathbb{Z}}\Big)$ (on a possibly enlarged probability space). Further, we define for $x \in \mathbb{R}$

$$\widehat{T}(x) := \inf\{t \ge 0 : \ \widehat{B}(t) = x\}$$

as the first hitting time of x of the Brownian motion $\big(\widehat{B}(t)\big)_{t\ge 0}$.

Proof of Proposition 3.3.3. This proposition is motivated by Lemma 2.1 in [DGPS07]. Therefore, we use a similar approach:

For $k \in \mathbb{N}$, we define

$$\widehat{A}_k := \widehat{D}_k - \widehat{H}_{k-1} \ge 0$$

as the difference of the heights of $\widehat{\mathcal{D}}_k$ and $\widehat{\mathcal{V}}_{k-1}$.

The key observation is the following:
At first, notice that on the set $\{\widehat{H}^+_{k-1} \le \widehat{H}^-_{k-1}\}$ we have $\widehat{H}_{k-1} = \widehat{H}^+_{k-1}$ and further that the valley $\widehat{\mathcal{D}}_k$ is located on the right side of $\widehat{\mathcal{V}}_{k-1}$. Using the strong Markov property of the Brownian motion for the stopping time $\widehat{\phi}^+_{k-1}$ (cf. Remark 3.3.2), we see that on the set $\{\widehat{H}^+_{k-1} \le \widehat{H}^-_{k-1}\}$ we have for $c > 0$ that

$$\begin{aligned}&\mathsf{P}\big(\widehat{A}_k \ge c \cdot \widehat{H}_{k-1} \big| \widehat{H}^+_{k-1}, \widehat{H}^-_{k-1}\big) = \mathsf{P}\big(\widehat{T}(-c \cdot \widehat{H}_{k-1}) < \widehat{T}(\widehat{H}_{k-1}) \big| \widehat{H}^+_{k-1}, \widehat{H}^-_{k-1}\big)\\ &= \frac{\widehat{H}_{k-1}}{(1+c)\cdot \widehat{H}_{k-1}} = \frac{1}{1+c}\ ,\end{aligned} \tag{3.21}$$

i.e. the conditional probability can be computed with the help of ruin probabilities for the Brownian motion. Analogously, we can use the strong Markov property at $\widehat{\phi}_{k-1}^{-}$ to get for $c > 0$ on the set $\{\widehat{H}_{k-1}^{+} > \widehat{H}_{k-1}^{-}\}$

$$\mathsf{P}\big(\widehat{A}_k \geq c \cdot \widehat{H}_{k-1} \big| \widehat{H}_{k-1}^{+}, \widehat{H}_{k-1}^{-}\big) = \frac{1}{1+c} \ . \tag{3.22}$$

A combination of (3.21) and (3.22) now implies that for $c > 1$ we have

$$\mathsf{P}\left(\frac{\widehat{H}_{k-1}}{\widehat{A}_k + \widehat{H}_{k-1}} \leq \frac{1}{c}\right) = \mathsf{P}\left(\frac{1}{c} \cdot \widehat{A}_k \geq \left(1 - \frac{1}{c}\right) \cdot \widehat{H}_{k-1}\right) = \frac{1}{c} \ .$$

Therefore, we can conclude that for $k \in \mathbb{N}$

$$\frac{\widehat{H}_{k-1}}{\widehat{A}_k + \widehat{H}_{k-1}}$$

has a uniform distribution on $(0, 1)$.

Since k was arbitrary and again using the strong Markov property, we further notice that

$$\left(\frac{\widehat{H}_{k-1}}{\widehat{D}_k}\right)_{k\in\mathbb{N}} = \left(\frac{\widehat{H}_{k-1}}{\widehat{A}_k + \widehat{H}_{k-1}}\right)_{k\in\mathbb{N}} \tag{3.23}$$

is a sequence of i.i.d. random variables with a uniform distribution on $(0, 1)$.

By an iteration, we get

$$\log \widehat{D}_k = \log(\widehat{A}_k + \widehat{H}_{k-1}) = \log\left(\frac{\widehat{A}_k + \widehat{H}_{k-1}}{\widehat{H}_{k-1}}\right) + \log \widehat{H}_{k-1}$$

$$\geq \ \log\left(\frac{\widehat{A}_k + \widehat{H}_{k-1}}{\widehat{H}_{k-1}}\right) + \log \widehat{D}_{k-1} \geq \sum_{i=2}^{k} \log\left(\frac{\widehat{A}_i + \widehat{H}_{i-1}}{\widehat{H}_{i-1}}\right) + \log \widehat{D}_1.$$

Together with the strong law of large numbers, this P-a.s. implies

$$\liminf_{k\to\infty} \frac{1}{k} \log \widehat{D}_k \geq \lim_{k\to\infty} \frac{1}{k} \sum_{i=2}^{k} \log\left(\frac{\widehat{A}_i + \widehat{H}_{i-1}}{\widehat{H}_{i-1}}\right) = \mathsf{E}\left[\log\left(\frac{\widehat{A}_2 + \widehat{H}_1}{\widehat{H}_1}\right)\right]$$
$$= -\int_0^1 \log(x)dx = 1,$$

where we used that

$$\frac{\widehat{H}_1}{\widehat{A}_2 + \widehat{H}_1}$$

has a uniform distribution on $(0, 1)$ for the last step. In particular, we can conclude that, for $0 < \beta < 1$, we P-a.s. have

$$\widehat{D}_k \geq \exp\left(\beta \cdot k\right)$$

for all k large enough. ■

Proof of Proposition 3.3.4. For $k \in \mathbb{N}$, we define

$$\widehat{C}_k := \max\{\widehat{D}_k^-, \widehat{D}_k^+\} - \min\{\widehat{D}_k^-, \widehat{D}_k^+\}$$

as the height difference between the higher and lower side of the valley $\widehat{\mathcal{D}}_k$. With an analogous calculation as in (3.21) and (3.22), we see that

$$\left(\frac{\min\{\widehat{D}_k^-, \widehat{D}_k^+\}}{\max\{\widehat{D}_k^-, \widehat{D}_k^+\}}\right)_{k\in\mathbb{N}} = \left(\frac{\widehat{D}_k}{\widehat{D}_k + \widehat{C}_k}\right)_{k\in\mathbb{N}}$$

is again a sequence of i.i.d. random variables with a uniform distribution on $(0, 1)$. For the computation, we can use the strong Markov property of the Brownian motion for the stopping times $\widehat{\eta}_k^+$ and $\widehat{\eta}_k^-$ (cf. Remark 3.3.2). Again using the strong Markov property, we see that

$$\frac{\widehat{H}_{k-1}}{\widehat{D}_k} \quad \text{and} \quad \frac{\widehat{D}_k}{\widehat{D}_k + C_k}$$

are independent for each $k \in \mathbb{N}$. Further, we have seen above and in (3.23) that both random variables have a uniform distribution on $(0, 1)$. Therefore, we can conclude that for $c > 1$ (using $\widehat{H}_k \leq \max\{\widehat{D}_k^-, \widehat{D}_k^+\}$)

$$\mathsf{P}\left(\widehat{H}_k > c \cdot \widehat{H}_{k-1}\right) \leq \mathsf{P}\left(\max\{\widehat{D}_k^-, \widehat{D}_k^+\} > c \cdot \widehat{H}_{k-1}\right)$$

$$= \mathsf{P}\left(\frac{\widehat{D}_k + \widehat{C}_k}{\widehat{D}_k} > c \cdot \frac{\widehat{H}_{k-1}}{\widehat{D}_k}\right) = \int_0^1 \int_0^1 \mathbb{1}_{\left\{\frac{1}{y} > c \cdot x\right\}} \, dx \, dy$$

$$= \int_0^1 \min\left\{\frac{1}{y \cdot c}, 1\right\} \, dy = \frac{\log(c) + 1}{c} \; .$$

Finally, an application of the Borel-Cantelli lemma finishes the proof since

$$\sum_{k=0}^{\infty} \frac{\log(k^{1+\vartheta}) + 1}{k^{1+\vartheta}} < \infty$$

for $\vartheta > 0$. ■

Proof of Proposition 3.3.5. For the proof, we fix $\gamma > 0$. The intuition behind this proposition is the following: Since the parts of the Brownian motion on the left and the right side of the origin are independent, it is very unlikely that the maxima $B(\widehat{r}_k)$ and $B(\widehat{\ell}_k)$ only differ by the small distance $(\widehat{H}_{k-1})^{1-\gamma}$. On the other hand, their distance is with high probability not larger than the large distance $(\widehat{H}_{k-1})^{1+\gamma}$. Here, small and large are to be understood in comparison with the height $\widehat{H}_{k-1}$.

More precisely, we again have the following connection to the ruin probabilities of the Brownian motion: Using the strong Markov property of the Brownian motion at $\widehat{\eta}_k^+$ (cf. Remark 3.3.2), we get on the set

$$\{\widehat{b}_k > 0\} = \{B(\widehat{\ell}_{k-1}) \geq B(\widehat{r}_{k-1})\}$$

(i.e. $\widehat{\mathcal{D}}_k$ is located on the right side of the origin) and with

$$\mathcal{A} := \sigma\big(B(\widehat{\eta}_k^-), B(\widehat{b}_k), B(\widehat{\eta}_k^+), B(\widehat{\ell}_{k-1}), B(\widehat{r}_{k-1}), \widehat{H}_{k-1}\big)$$

that

$$\begin{aligned}
&\mathsf{P}\left(-(\widehat{H}_{k-1})^{1-\gamma} \leq B(\widehat{r}_k) - B(\widehat{\ell}_k) \leq (\widehat{H}_{k-1})^{1-\gamma} \,\middle|\, \mathcal{A}, B(\widehat{\ell}_k)\right) \\
=\ &\mathsf{P}\left(B(\widehat{r}_k) - B(\widehat{\eta}_k^+) \geq B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) - (\widehat{H}_{k-1})^{1-\gamma} \,\middle|\, \mathcal{A}, B(\widehat{\ell}_k)\right) \\
&- \mathsf{P}\left(B(\widehat{r}_k) - B(\widehat{\eta}_k^+) > B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) + (\widehat{H}_{k-1})^{1-\gamma} \,\middle|\, \mathcal{A}, B(\widehat{\ell}_k)\right) \\
=\ &\mathsf{P}\left(\widehat{T}\big(\max\{0, B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) - (\widehat{H}_{k-1})^{1-\gamma}\}\big) \leq \widehat{T}\big(-[B(\widehat{\eta}_k^+) - B(\widehat{b}_k)]\big) \,\middle|\, \mathcal{A}, B(\widehat{\ell}_k)\right) \\
&- \mathsf{P}\left(\widehat{T}\big(B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) + (\widehat{H}_{k-1})^{1-\gamma}\big) < \widehat{T}\big(-[B(\widehat{\eta}_k^+) - B(\widehat{b}_k)]\big) \,\middle|\, \mathcal{A}, B(\widehat{\ell}_k)\right) \\
=\ &\left\{\begin{array}{l}
1 - \dfrac{B(\widehat{\eta}_k^+) - B(\widehat{b}_k)}{B(\widehat{\ell}_k) - B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma}} \\
\hfill \text{if } B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) - (\widehat{H}_{k-1})^{1-\gamma} \leq 0 \\
\dfrac{B(\widehat{\eta}_k^+) - B(\widehat{b}_k)}{B(\widehat{\ell}_k) - B(\widehat{b}_k) - (\widehat{H}_{k-1})^{1-\gamma}} - \dfrac{B(\widehat{\eta}_k^+) - B(\widehat{b}_k)}{B(\widehat{\ell}_k) - B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma}} \\
\hfill \text{if } B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) - (\widehat{H}_{k-1})^{1-\gamma} > 0
\end{array}\right\} \\
=\ &\left\{\begin{array}{l}
\dfrac{B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) + (\widehat{H}_{k-1})^{1-\gamma}}{B(\widehat{\ell}_k) - B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma}} \leq \dfrac{2 \cdot (\widehat{H}_{k-1})^{1-\gamma}}{\widehat{H}_{k-1} + (\widehat{H}_{k-1})^{1-\gamma}} \\
\hfill \text{if } B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) - (\widehat{H}_{k-1})^{1-\gamma} \leq 0 \\
\dfrac{2 \cdot (\widehat{H}_{k-1})^{1-\gamma} \cdot \big(B(\widehat{\eta}_k^+) - B(\widehat{b}_k)\big)}{\big(B(\widehat{\ell}_k) - B(\widehat{b}_k)\big)^2 - (\widehat{H}_{k-1})^{2\cdot(1-\gamma)}} \leq \dfrac{2 \cdot (\widehat{H}_{k-1})^{1-\gamma}}{\widehat{H}_{k-1} - (\widehat{H}_{k-1})^{1-2\gamma}} \\
\hfill \text{if } B(\widehat{\ell}_k) - B(\widehat{\eta}_k^+) - (\widehat{H}_{k-1})^{1-\gamma} > 0
\end{array}\right\} \\
\leq\ &\frac{2}{(\widehat{H}_{k-1})^{\gamma} - 1}\ .
\end{aligned}$$

By a symmetric consideration, we can further derive the analogous upper bound on the set $\{\widehat{b}_k < 0\}$ by using the strong Markov property of the Brownian motion at $\widehat{\eta}_k^-$. Since the last upper bound is P-a.s. summable in k due to the exponential growth of $\widehat{H}_{k-1}$ (cf. (3.11)), we can conclude by the Borel-Cantelli lemma that we P-a.s. have

$$(\widehat{H}_{k-1})^{1-\gamma} \leq |B(\widehat{r}_k) - B(\widehat{\ell}_k)|$$

for all k large enough. This shows the lower bound in (3.13).

For the upper bound, we show the following three properties: We P-a.s. have

$$\begin{aligned} B(\widehat{b}_k) &\geq -(\widehat{H}_{k-1})^{1+\frac{\gamma}{2}}, \\ B(\widehat{r}_k) &\leq (\widehat{H}_{k-1})^{1+\gamma}, \\ B(\widehat{\ell}_k) &\leq (\widehat{H}_{k-1})^{1+\gamma} \end{aligned} \tag{3.24}$$

for all k large enough. A combination of the last two inequalities in particular implies that we P-a.s. have

$$|B(\widehat{r}_k) - B(\widehat{\ell}_k)| \leq (\widehat{H}_{k-1})^{1+\gamma}$$

for all k large enough, i.e. the upper bound in (3.13). Note here that we have

$$B(\widehat{r}_k), B(\widehat{\ell}_k) \geq 0$$

by definition. For the first relation in (3.24), we can use the following connection to the ruin probabilities of the Brownian motion: Using the strong Markov property of the Brownian motion at $\widehat{\phi}_{k-1}^+$, we get on the set $\{\widehat{b}_k > 0\} = \{B(\widehat{\ell}_{k-1}) \geq B(\widehat{r}_{k-1})\}$

$$\begin{aligned} &\mathsf{P}\left(B(\widehat{b}_k) < -(\widehat{H}_{k-1})^{1+\frac{\gamma}{2}} \,\middle|\, B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1}), B(\widehat{b}_{k-1})\right) \\ &= \mathsf{P}\left(\widehat{T}\big(-(\widehat{H}_{k-1})^{1+\frac{\gamma}{2}} - B(\widehat{b}_{k-1})\big) < \widehat{T}\big(\widehat{H}_{k-1}\big) \,\middle|\, B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1}), B(\widehat{b}_{k-1})\right) \\ &= \frac{\widehat{H}_{k-1}}{\widehat{H}_{k-1} + (\widehat{H}_{k-1})^{1+\frac{\gamma}{2}} + B(\widehat{b}_{k-1})} \leq \frac{\widehat{H}_{k-1}}{(\widehat{H}_{k-1})^{1+\frac{\gamma}{2}}} = \frac{1}{(\widehat{H}_{k-1})^{\frac{\gamma}{2}}} . \end{aligned}$$

Due to the symmetry, we can show the same upper bound on the set $\{\widehat{b}_k < 0\}$ by an analogous argument. Since the last upper bound is P-a.s. summable in k due to the exponential growth of $\widehat{H}_{k-1}$ (cf. (3.11)), we can again conclude by the Borel-Cantelli lemma that we P-a.s. have

$$B(\widehat{b}_k) \geq -(\widehat{H}_{k-1})^{1+\frac{\gamma}{2}}$$

for all k large enough which is the first statement in (3.24).

For the second relation in (3.24), notice that on the set $\{B(\widehat{r}_k) > (\widehat{H}_{k-1})^{1+\gamma}\}$ we P-a.s. have

$$B(\widehat{r}_k) = \max_{0 \leq t \leq \widehat{\phi}_k^+} B(t) \gg (\widehat{H}_{k-1})^{1+\frac{\gamma}{2}} \geq -B(b_k) \geq - \min_{0 \leq t \leq \widehat{\phi}_k^+} B(t)$$

for all k large enough, i.e. the maximum of the Brownian up to time $\widehat{\phi}_k^+$ is of larger order than the minimum which is very unlikely. More precisely, we observe for arbitrary $0 < \beta < 1$ – using the first relation in (3.24) and the exponential growth of $\widehat{H}_{k-1}$ in k (cf. (3.11)) – that the following inclusion P-a.s. holds:

$$\begin{aligned} &\limsup_{k\to\infty} \{B(\widehat{r}_k) > (\widehat{H}_{k-1})^{1+\gamma}\} \\ \subseteq\ &\limsup_{k\to\infty} \Big(\{B(\widehat{r}_k) > (\widehat{H}_{k-1})^{1+\gamma}\} \cap \{B(\widehat{b}_k) \geq -(\widehat{H}_{k-1})^{1+\frac{\gamma}{2}}\} \cap \{\widehat{H}_{k-1} \geq \exp\big(\beta \cdot (k-1)\big)\}\Big) \end{aligned}$$

$$\subseteq \limsup_{k\to\infty} \left(\bigcup_{n\geq k} \left\{ B(\widehat{r}_k) > \big(\exp\big(\beta\cdot(n-1)\big)\big)^{1+\gamma} \right\} \cap \left\{ B(\widehat{b}_k) \geq -\big(\exp(\beta\cdot n)\big)^{1+\frac{\gamma}{2}} \right\} \right) \tag{3.25}$$

For the last step, notice that we have $\exp(\beta\cdot(n-1)) \leq \widehat{H}_{k-1} \leq \exp(\beta\cdot n)$ for some $n\geq k$ on $\{\widehat{H}_{k-1} \geq \exp(\beta\cdot(k-1))\}$. Thereby, we have

$$\begin{aligned}
&\sum_{n\geq k} \mathsf{P}\left(\left\{ B(\widehat{r}_k) > \big(\exp\big(\beta\cdot(n-1)\big)\big)^{1+\gamma} \right\} \cap \left\{ B(\widehat{b}_k) \geq -\big(\exp(\beta\cdot n)\big)^{1+\frac{\gamma}{2}} \right\} \right) \\
\leq\ &\sum_{n\geq k} \mathsf{P}\left(\widehat{T}\left(\big(\exp\big(\beta\cdot(n-1)\big)\big)^{1+\gamma} \right) < \widehat{T}\left(-\big(\exp(\beta\cdot n)\big)^{1+\frac{\gamma}{2}} \right) \right) \\
=\ &\sum_{n\geq k} \frac{\big(\exp(\beta\cdot n)\big)^{1+\frac{\gamma}{2}}}{\big(\exp\big(\beta\cdot(n-1)\big)\big)^{1+\gamma} + \big(\exp(\beta\cdot n)\big)^{1+\frac{\gamma}{2}}} \\
\leq\ &\sum_{n\geq k} \frac{1}{\big(\exp\big(\beta\cdot(n-1)\big)\big)^{\frac{\gamma}{2}}} = \exp\Big(-\tfrac{\beta\cdot\gamma}{2}\cdot(k-1)\Big)\cdot \frac{1}{1-\exp\big(-\frac{\beta\cdot\gamma}{2}\big)}
\end{aligned}$$

for all k large enough.

Since the last upper bound is summable in k, we can conclude by the Borel-Cantelli lemma and (3.25) that we P-a.s. have

$$B(\widehat{r}_k) \leq (\widehat{H}_{k-1})^{1+\gamma}$$

for all k large enough which is the second relation in (3.24).

The third relation in (3.24) can be shown by an analogous consideration for $\widehat{\ell}_k$ instead of $\widehat{r}_k$ due to the symmetry. ■

Proof of Proposition 3.3.6. Here, we can use a similar approach as in [DGPS07] (cf. Section 2.3):

For the proof, we fix $\vartheta > 0$ and recall Chung's law of the iterated logarithm (cf. Theorem in [JP75]). It states that we have

$$\liminf_{t\to\infty} \frac{\max_{0\leq s\leq t} |B_s|}{\left(\dfrac{t}{\log\log t}\right)^{\frac{1}{2}}} = \sigma\cdot\frac{\pi}{\sqrt{8}} > 0, \qquad \mathsf{P}\text{-a.s.} \tag{3.26}$$

Recall here that $0 < \sigma^2 = \mathrm{Var}(\log \rho_0)$ denotes the variance of B_1.

For the valley $\widehat{\mathcal{V}}_k$, note that we have

$$\max_{\widehat{\phi}_k^- \leq s<t\leq 0} \big(B(t)-B(s)\big) = \widehat{H}_k^-,$$
$$\max_{0\leq s<t\leq \widehat{\phi}_k^+} \big(B(s)-B(t)\big) = \widehat{H}_k^+$$

by construction. In particular, we can conclude that we have

$$\max_{\widehat{\phi}_k^- \leq s \leq 0} |B(s)| \leq \widehat{H}_k^-,$$
$$\max_{0 \leq s \leq \widehat{\phi}_k^+} |B(s)| \leq \widehat{H}_k^+.$$

Thereby (3.26), i.e. Chung's law of the iterated logarithm, implies that for $0 < \gamma < 1$ we P-a.s. have

$$\widehat{H}_k^- \geq \max_{\widehat{\phi}_k^- \leq s \leq 0} |B(s)| \geq |\widehat{\phi}_k^-|^{\frac{1}{2}\gamma},$$
$$\widehat{H}_k^+ \geq \max_{0 \leq s \leq \widehat{\phi}_k^+} |B(s)| \geq |\widehat{\phi}_k^+|^{\frac{1}{2}\gamma}$$

for all k large enough. From this, we can conclude that we P-a.s. have

$$\left.\begin{aligned} |\widehat{\phi}_k^-| &\leq \big(\widehat{H}_k^-\big)^{2\frac{1}{\gamma}} \\ |\widehat{\phi}_k^+| &\leq \big(\widehat{H}_k^+\big)^{2\frac{1}{\gamma}} \end{aligned}\right\} \leq \big(\widehat{H}_k + (\widehat{H}_k)^{1+\frac{\vartheta}{4}}\big)^{2\frac{1}{\gamma}} \leq \Big(2 \cdot (\widehat{H}_k)^{2+\frac{\vartheta}{2}}\Big)^{\frac{1}{\gamma}} \leq (\widehat{H}_k)^{2+\vartheta}$$

for all k large enough and $\gamma = \gamma(\vartheta)$ close enough to 1. Here, we applied the upper bound in (3.13) for $\frac{\vartheta}{4}$ in the second step. Further, we used that $\widehat{H}_k$ grows a.s. exponentially in k (cf. (3.11)) for the last step. ∎

<u>*Proof of Proposition 3.3.7.*</u> For the proof, we fix $0 < \gamma < 1$. Further, we assume for the beginning that we have $\widehat{b}_k > 0$ for some $k \in \mathbb{N}$, i.e. the valley $\widehat{\mathcal{D}}_k$ is located in the right side of the origin. Then, we can make the following connection between the increase $B(\widehat{b}_k) - B(\widehat{b}_{k-1})$ and the ruin probability of the Brownian motion:
Using the strong Markov property of the Brownian motion at $\widehat{\phi}_{k-1}^+$ (cf. Remark 3.3.2), we get the following relation on the set $\{\widehat{b}_k > 0\} = \{B(\widehat{r}_{k-1}) \leq B(\widehat{\ell}_{k-1})\}$

$$\begin{aligned} &\mathsf{P}\big(B(\widehat{b}_k) - B(\widehat{b}_{k-1}) \geq -(\widehat{H}_{k-1})^{1-\gamma} \big| B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1}), B(\widehat{b}_{k-1})\big) \\ &= \mathsf{P}\big(\widehat{T}(\widehat{H}_{k-1}) < \widehat{T}(-(\widehat{H}_{k-1})^{1-\gamma}) \big| \widehat{H}_{k-1}\big) = \frac{(\widehat{H}_{k-1})^{1-\gamma}}{\widehat{H}_{k-1} + (\widehat{H}_{k-1})^{1-\gamma}} = \frac{1}{(\widehat{H}_{k-1})^{\gamma} + 1} \,. \end{aligned}$$

By a symmetric analogous consideration, we can derive the same result on the set $\{\widehat{b}_k < 0\}$. Since the last upper bound is P-a.s. summable in k due to the exponential growth of $\widehat{H}_{k-1}$ in k (cf. (3.11)), the Borel-Cantelli lemma implies that we P-a.s. have

$$B(\widehat{b}_k) - B(\widehat{b}_{k-1}) < -(\widehat{H}_{k-1})^{1-\gamma}$$

for all k large enough. This shows (3.16).

The remaining two inequalities in (3.17) and (3.18) are symmetric analogues of each other. Therefore, we only show (3.18):

Similarly to above, we can use the strong Markov property of the Brownian motion at $\widehat{\eta}_k^+$ to derive the following upper bound on the set $\{\widehat{b}_k > 0\}$

$$\begin{aligned}
&\mathsf{P}\big(B(\widehat{r}_k) - B(\widehat{r}_{k-1}) \le (\widehat{H}_{k-1})^{1-\gamma} \big| B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1}), B(\widehat{b}_{k-1}), B(\widehat{b}_k)\big) \\
&= \mathsf{P}\big(\widehat{T}\big((\widehat{H}_{k-1})^{1-\gamma}\big) > \widehat{T}\big(-\widehat{H}_{k-1} + B(\widehat{b}_k) - B(\widehat{b}_{k-1})\big) \big| B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1}), B(\widehat{b}_{k-1}), B(\widehat{b}_k)\big) \\
&= \frac{(\widehat{H}_{k-1})^{1-\gamma}}{(\widehat{H}_{k-1})^{1-\gamma} + \widehat{H}_{k-1} - B(\widehat{b}_k) + B(\widehat{b}_{k-1})} \le \frac{(\widehat{H}_{k-1})^{1-\gamma}}{\widehat{H}_{k-1} + (\widehat{H}_{k-1})^{1-\gamma}} = \frac{1}{(\widehat{H}_{k-1})^{\gamma} + 1} \; .
\end{aligned}$$

Again, the Borel-Cantelli lemma finishes the proof of (3.18). ■

Proof of Proposition 3.3.8. For the proof, we fix $0 < \gamma < 1$. At first, we observe that for all $k \in \mathbb{N}$

$$\widehat{\phi}_{k,\gamma}^+ \stackrel{\text{def}}{=} \inf\{t > \widehat{\eta}_k^+ :\; B(t) = B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma}\}$$

is a stopping time with respect to the filtration $(\mathcal{F}_t^+)_{t\ge 0}$ since $\widehat{\eta}_k^+$ is a stopping time with respect to this filtration (cf. Remark 3.3.2) for all k.
Now, we can make the following connection to the ruin probabilities of the Brownian motion: Using the strong Markov property of the Brownian motion at $\widehat{\phi}_{k,\gamma}^+$, we get on the set $\{B(\widehat{b}_k) - B(\widehat{b}_{k-1}) \le -(\widehat{H}_{k-1})^{1-\gamma}\}$

$$\begin{aligned}
&\mathsf{P}\left(\max_{\widehat{\phi}_{k,\gamma}^+ \le t \le \widehat{\phi}_k^+} B(t) > B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}} \,\middle|\, B(\widehat{b}_{k-1}), B(\widehat{b}_k), \widehat{H}_{k-1}\right) \\
&= \mathsf{P}\left(\widehat{T}\big((\widehat{H}_{k-1})^{1-\frac{\gamma}{2}} - (\widehat{H}_{k-1})^{1-\gamma}\big) < \widehat{T}\big(-(\widehat{H}_{k-1})^{1-\gamma}\big) \,\middle|\, \widehat{H}_{k-1}\right) \\
&= \frac{(\widehat{H}_{k-1})^{1-\gamma}}{(\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}} = \frac{1}{(\widehat{H}_{k-1})^{\frac{\gamma}{2}}} \; .
\end{aligned}$$

Since the last expression is summable in k due to the exponential growth of $\widehat{H}_{k-1}$ in k (cf. (3.11)) and since we P-a.s. have $B(\widehat{b}_k) - B(\widehat{b}_{k-1}) \le -(\widehat{H}_{k-1})^{1-\gamma}$ for all k large enough according to (3.16), we can conclude by the Borel-Cantelli lemma that we P-a.s. have

$$\max_{\widehat{\phi}_{k,\gamma}^+ \le t \le \widehat{\phi}_k^+} B(t) \le B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}$$

for all k large enough. The argument for $\widehat{\phi}_{k,\gamma}^-$ runs completely analogously with the help of the symmetric analogues of all appearing quantities. ■

3.3.2 Associated Valleys of the Potential and their Properties

Since the potential $(V(x))_{x\in\mathbb{Z}}$ is only defined for integers, we have to discretize the valleys which we constructed in the previous section. For this, we define for $k \in \mathbb{N}_0$

$$\ell_k := \min\left\{x \in \mathbb{Z},\ \widehat{\phi}_k^- \le x \le 0 :\ V(x) = \max_{\widehat{\phi}_k^- \le y \le 0} V(y)\right\},$$

$$r_k := \max\left\{x \in \mathbb{Z},\ 0 \le x \le \widehat{\phi}_k^+ :\ V(x) = \max_{0 \le y \le \widehat{\phi}_k^+} V(y)\right\},$$

$$b_k := \min\left\{x \in \mathbb{Z},\ \ell_k \le x \le r_k :\ V(x) = \min_{\ell_k \le y \le r_k} V(y)\right\} \tag{3.27}$$

as the left end point, the right end point, and the position of the bottom point of the valley which we will consider in the following. Further, we define $m_0 := 0$ and for $k \ge 1$

$$m_k := \begin{cases} \max\left\{x \in \mathbb{Z},\ b_{k-1} \le x \le b_k :\ V(x) = \max\limits_{b_{k-1} \le y \le b_k} V(y)\right\} & \text{if } b_{k-1} \le b_k \\ \\ \min\left\{x \in \mathbb{Z},\ b_k \le x \le b_{k-1} :\ V(x) = \max\limits_{b_{k-1} \le y \le b_k} V(y)\right\} & \text{if } b_{k-1} > b_k \end{cases}$$

as the position with the highest potential between the successive bottom points at b_{k-1} and b_k. Additionally, we define $h_0 := 0$ and for $k \ge 1$

$$h_k := \begin{cases} \ell_{k-1} & \text{if } b_k \ge 0 \\ r_{k-1} & \text{if } b_k < 0 \end{cases} \tag{3.28}$$

as the point out of $\{\ell_{k-1}, r_{k-1}\}$ which is not on the same side of the origin as b_k and which turns out to have the higher potential for all k large enough, i.e. (cf. (3.36))

$$V(h_k) = \max\{V(\ell_{k-1}), V(r_{k-1})\}.$$

Analogously to the construction with the help of the two-sided Brownian motion from the previous section, we define the sequence of valleys $(\mathcal{V}_k)_{k\ge1}$ as

$$\mathcal{V}_k := \{\ell_k, \ell_k + 1, \dots, r_k\}$$

and the sequence of deeper parts $(\mathcal{D}_k)_{k\ge1}$ as

$$\mathcal{D}_k := \begin{cases} \{\ell_k, \ell_k + 1, \dots, m_k\} & \text{if } b_k < m_k \\ \{m_k, m_k + 1, \dots, r_k\} & \text{if } b_k \ge m_k. \end{cases} \tag{3.29}$$

From the definition, it would be possible to have $\mathcal{D}_k \not\subseteq \mathcal{V}_k$. But in Proposition 3.3.9, we show that this can only happen for finitely many k.
Additionally, we define the right and left heights of our valleys $\mathcal{D}_k$ and $\mathcal{V}_k$ for $k \in \mathbb{N}_0$ by

$$D_k^+ := \begin{cases} V(r_k) - V(b_k) & \text{if } b_k \ge m_k \\ V(m_k) - V(b_k) & \text{if } b_k < m_k, \end{cases}$$

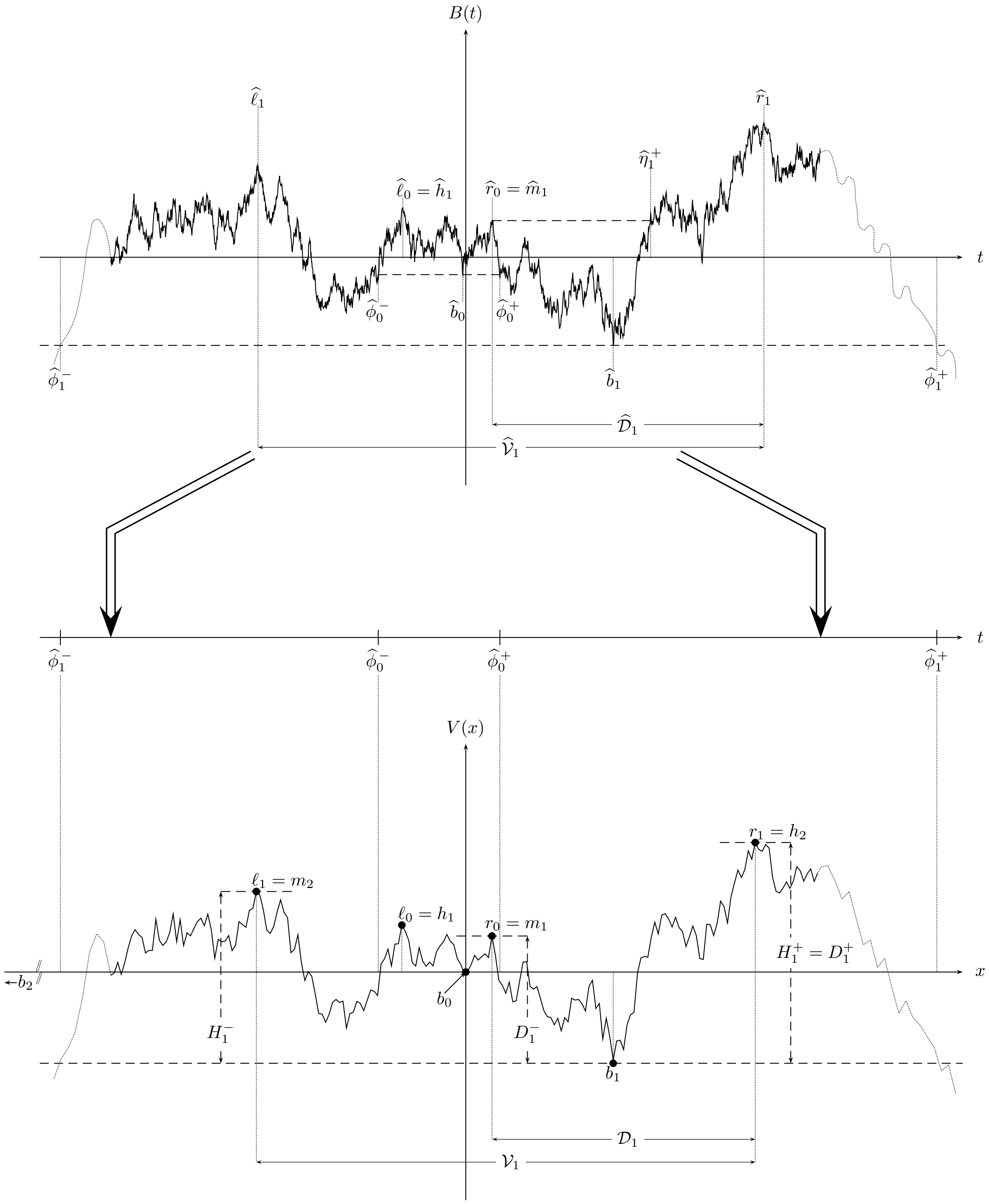

Figure 3.4: *Construction of our valleys $\mathcal{D}_1$ and $\mathcal{V}_1$ with the help of the coupled Brownian motion: Note here that the potential $V(\cdot)$ is only defined for integers and that $\ell_1, b_1, r_1, \dots$ all denote the x-coordinate of the points which are emphasized by the black circles. Further note that we cannot be sure about the positions of b_2, m_2 and h_2 alone from the shown part of the potential $V(x)$. Anyhow, we have attached them to their qualitatively correct positions where they are at least located for large indices k since here we have $V(\ell_1) < V(r_1)$.*

$$\begin{aligned} D_k^- &:= \begin{cases} V(m_k) - V(b_k) & \text{if } b_k \geq m_k \\ V(\ell_k) - V(b_k) & \text{if } b_k < m_k, \end{cases} \\ H_k^+ &:= V(r_k) - V(b_k), \\ H_k^- &:= V(\ell_k) - V(b_k), \end{aligned}$$

and further the depth as

$$\begin{aligned} D_k &:= \min\{D_k^+, D_k^-\}, \\ H_k &:= \min\{H_k^+, H_k^-\}. \end{aligned}$$

In order to make sure that one upcoming definition is well defined (cf. (3.81)), we finally define

$$b_{-1} := H_{-1} := 0.$$

Using the strong approximation theorem (Theorem 3.3.1), which gives us an upper bound for the differences between $B(x)$ and $V(x)$ for large $|x|$ ($x \in \mathbb{Z}$), we can now derive the analogous results from the previous section:

Proposition 3.3.9. *For* P*-a.e. environment* θ*, we have* $b_{k+1} \neq 0$ *and further*

$$m_{k+1} = \begin{cases} r_k > 0 & \textit{if } b_{k+1} > 0 \\ \ell_k < 0 & \textit{if } b_{k+1} < 0 \end{cases} \tag{3.30}$$

for all k *large enough. In particular, this implies that we* P*-a.s. have*

$$V(m_{k+1}) = \begin{cases} \max\limits_{0 \leq j \leq b_{k+1}-1} V(j) & \textit{if } b_{k+1} > 0 \\ \max\limits_{b_{k+1} \leq j \leq -1} V(j) & \textit{if } b_{k+1} < 0, \end{cases} \tag{3.31}$$

$$V(m_{k+1}) - V(b_{k+1}) = D_{k+1} \tag{3.32}$$

$$V(m_{k+1}) - V(b_k) = H_k \tag{3.33}$$

for all k *large enough.*

Proposition 3.3.10. *For* $0 < \beta, \gamma < 1$ *and* $0 < \vartheta$*, we* P*-a.s. have*

$$D_{k+1} \geq H_k \geq D_k \geq \exp(\beta \cdot k), \tag{3.34}$$

$$H_k \leq k^{1+\vartheta} \cdot H_{k-1}, \tag{3.35}$$

$$V(h_{k+1}) - V(m_{k+1}) \geq (H_{k-1})^{1-\gamma}, \tag{3.36}$$

$$|\ell_k| \leq (H_k)^{2+\vartheta}, \tag{3.37}$$

$$|r_k| \leq (H_k)^{2+\vartheta}, \tag{3.38}$$

$$|b_k| \leq (H_k)^{2+\vartheta}, \tag{3.39}$$

$$|m_{k+1}| \le (H_k)^{2+\vartheta}, \tag{3.40}$$
$$|h_{k+1}| \le (H_k)^{2+\vartheta} \tag{3.41}$$

for all k large enough.

Proposition 3.3.11. *For $0 < \gamma < 1$, we* P*-a.s. have*

$$\min_{x \in \mathcal{V}_k \setminus \mathcal{D}_k} V(x) \ge V(b_k) + (H_{k-1})^{1-\gamma} \tag{3.42}$$

for all k large enough. In particular, we P*-a.s. have*

$$D_k \ge H_{k-1} + (H_{k-1})^{1-\gamma} \tag{3.43}$$

for all k large enough.

For the last two propositions, we need one more definition: For $k \in \mathbb{N}$ we define

$$\begin{aligned} \eta_k^+ &:= \inf\{n \ge b_\ell : \ V(n) \ge V(m_\ell)\}, \\ \eta_k^- &:= \sup\{n \le b_\ell : \ V(n) \ge V(m_\ell)\} \end{aligned} \tag{3.44}$$

as the closest position to the bottom point at b_ℓ at which the potential $V(\cdot)$ reaches the level $V(m_\ell)$ (again).

Proposition 3.3.12. *For $0 < \gamma < 1$, we* P*-a.s. have*

$$\max_{x,y \in \mathcal{D}_k:\ x<y<b_k} V(y) - V(x) \le D_k - (H_{k-1})^{1-\gamma} \qquad \text{on } \{b_k > 0\}, \tag{3.45}$$
$$\max_{x,y \in \mathcal{D}_k:\ b_k<x<y} V(x) - V(y) \le D_k - (H_{k-1})^{1-\gamma} \qquad \text{on } \{b_k < 0\},$$
$$\max_{x,y \in \mathcal{D}_k:\ b_k<y<x<\eta_k^+} V(y) - V(x) \le D_k - (H_{k-1})^{1-\gamma} \qquad \text{on } \{b_k > 0\}, \tag{3.46}$$
$$\max_{x,y \in \mathcal{D}_k:\ \eta_k^-<y<x<b_k} V(x) - V(y) \le D_k - (H_{k-1})^{1-\gamma} \qquad \text{on } \{b_k < 0\}$$

for all k large enough.

For the next proposition recall the following definition (cf. (3.19)) for $0 < \gamma < 1$ and $k \in \mathbb{N}$

$$\begin{aligned} \widehat{\phi}_{k,\gamma}^+ &\stackrel{\text{def}}{=} \inf\{t \ge \widehat{\eta}_k^+ : \ B(t) = B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma}\}, \\ \widehat{\phi}_{k,\gamma}^- &\stackrel{\text{def}}{=} \sup\{t \le \widehat{\eta}_k^- : \ B(t) = B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma}\}. \end{aligned}$$

Proposition 3.3.13. *For $0 < \gamma < 1$, we* P*-a.s. have*

$$\min_{\eta_k^+ \le x \le r_k} V(x) \ge V(b_k) + (D_k)^{1-\gamma} \qquad \text{on } \{b_k > 0\}, \tag{3.47}$$
$$\min_{\ell_k \le x \le \eta_k^-} V(x) \ge V(b_k) + (D_k)^{1-\gamma} \qquad \text{on } \{b_k < 0\},$$

$$\max_{\widehat{\phi}^+_{k,\gamma}\le x\le b_{k+1}} V(x)-V(m_{k+1})\le -(H_k)^{1-\gamma} \qquad \textit{on } \{b_{k+1}>0\}, \tag{3.48}$$
$$\max_{b_{k+1}\le x\le \widehat{\phi}^-_{k,\gamma}} V(x)-V(m_{k+1})\le -(H_k)^{1-\gamma} \qquad \textit{on } \{b_{k+1}<0\}$$

for all k large enough.

For the proofs of Proposition 3.3.9 – 3.3.13 , the following lemmata are helpful:
The first lemma yields an upper bound for the fluctuations of the Brownian motion between two integers:

Lemma 3.3.14. *We* P*-a.s. have*

$$\max_{0\le t\le 1}\big|B(n+t)-B(n)\big| < \big(\log(n)\big)^2 \quad \textit{and}$$
$$\max_{0\le t\le 1}\big|B(-n-t)-B(-n)\big| < \big(\log(n)\big)^2 \tag{3.49}$$

for all n large enough.

The second lemma shows that the position of our valleys $\mathcal{V}_k$ and $\mathcal{D}_k$ are closely connected to the positions of our auxiliary valleys $\widehat{\mathcal{V}}_k$ and $\widehat{\mathcal{D}}_k$ which we constructed with the help of the coupled two-sided Brownian motion:

Lemma 3.3.15. *For $0<\gamma<1$, we* P*-a.s. have*

$$\ell_k \in \begin{cases} [\widehat{\phi}^-_{k,\gamma}, 0] & \textit{if } \widehat{b}_k > 0 \\ [\widehat{\phi}^-_{k,\gamma}, \widehat{\eta}^-_k] & \textit{if } \widehat{b}_k < 0, \end{cases} \tag{3.50}$$

$$b_k \in \begin{cases} [\widehat{\phi}^+_{k-1}, \widehat{\eta}^+_k] & \textit{if } \widehat{b}_k > 0 \\ [\widehat{\eta}^-_k, \widehat{\phi}^-_{k-1}] & \textit{if } \widehat{b}_k < 0, \end{cases} \tag{3.51}$$

$$r_k \in \begin{cases} [\widehat{\eta}^+_k, \widehat{\phi}^+_{k,\gamma}] & \textit{if } \widehat{b}_k > 0 \\ [0, \widehat{\phi}^+_{k,\gamma}] & \textit{if } \widehat{b}_k < 0 \end{cases} \tag{3.52}$$

for all k large enough. In particular, we P*-a.s. have*

$$B(\widehat{b}_k) \le B(b_k),$$
$$V(b_k) \le V(\lfloor \widehat{b}_k \rfloor) \tag{3.53}$$

for all k large enough.

Proofs: The main tool for the upcoming proofs is always the same one: We will use the strong approximation theorem (Theorem 3.3.1) for a comparison between the potential $V(\cdot)$ and the coupled Brownian motion $B(\cdot)$. But in each situation, we still have to adapt our argument to the considered situation. One of the main difficulties, which we have to take care of here, is the following: The strong approximation theorem states that the differences of the potential $V(n)$ and the coupled Brownian motion $B(n)$ are of order $O(|n|)$ for large $|n|$. In particular, the upper bounds coming from the approximation theorem depend on the position n we are considering. On the other hand, the propositions are mostly stated in terms of the increasing heights of the valleys $\widehat{H}_k$, $\widehat{D}_k$, and H_k, D_k. One important tool for us is therefore Proposition 3.3.6 which connects the height of the valleys and the positions of their end points.

For reasons of completeness, we will present all proofs in the following:

Proof of Lemma 3.3.14. Consider a random variable $Z \sim \mathcal{N}(0, \sigma^2)$ and recall the standard upper bound

$$\mathsf{P}\left(\frac{Z}{\sigma} > x\right) \le \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right) \quad \text{for } x > 0$$

for tails of the standard normal distribution which can be found for example in Lemma 12.9 in Appendix B of [MP10]. By choosing $x = 2 \cdot \sqrt{\log(n)}$, we see that

$$\mathsf{P}\left(\frac{|Z|}{\sigma} \ge 2 \cdot \sqrt{\log(n)}\right) \le 2 \cdot n^{-2} \tag{3.54}$$

for all n large enough. Since for all $n \in \mathbb{N}_0$

$$\max_{0 \le t \le 1} \big(B(n+t) - B(n)\big) \quad \text{and} \quad \max_{0 \le t \le 1} \big(B(-n-t) - B(-n)\big)$$

have the same distribution as $|Z|$, we can conclude with the help of (3.54) that

$$\begin{aligned} \sum_{n=0}^{\infty} \Bigg(& \mathsf{P}\left(\max_{0 \le t \le 1} \big|B(n+t) - B(n)\big| \ge \big(\log(n)\big)^2\right) \\ + \quad & \mathsf{P}\left(\max_{0 \le t \le 1} \big|B(-n-t) - B(-n)\big| \ge \big(\log(n)\big)^2\right)\Bigg) < \infty \end{aligned}$$

holds (which is not the best possible statement, but the form in which we will use it). Therefore, the Borel-Cantelli lemma implies (3.49). □

Proof of Lemma 3.3.15. We start the proof with the following remark which will be useful for all remaining proofs in this section:

Remark 3.3.16. A combination of the Komlós-Major-Tusnády approximation Theorem 3.3.1 (which describes the coupling between the potential $(V(x))_{x \in \mathbb{Z}}$ and the two-sided Brownian motion $(B(t))_{t \in \mathbb{R}}$) and Lemma 3.3.14 (which controls the fluctuations of the Brownian motion between two integers) implies for example that we P-a.s. have

$$\begin{aligned} \max_{0 \le x \le s} V(x) &\le \max_{0 \le x \le s} B(x) + 2 \cdot (\log|s|)^2, \\ \max_{r \le x \le s} V(x) &\ge \max_{r \le x \le s} B(x) - 2 \cdot (\log|r|)^2 - 2 \cdot (\log|s|)^2, \\ \min_{0 \le x \le s} V(x) &\le \min_{0 \le x \le s} B(x) + 2 \cdot (\log|s|)^2, \\ \min_{r \le x \le s} V(x) &\ge \min_{r \le x \le s} B(x) - 2 \cdot (\log|r|)^2 - 2 \cdot (\log|s|)^2 \end{aligned} \tag{3.55}$$

for $r, s \in \mathbb{R}$ with $r < 0 < s$ and $|r|, |s|$ large enough. Note here that $V(x)$ is only defined for $x \in \mathbb{Z}$ whereas $B(x)$ is well defined for all $x \in \mathbb{R}$. As long as we only compare $B(\cdot)$ and $V(\cdot)$ for integers, we can even drop the factor 2 in the above estimates. Inequalities of these types will be useful for the upcoming estimates.

Using this connection between the potential $(V(x))_{x\in\mathbb{Z}}$ and the Brownian motion $(B(t))_{t\in\mathbb{R}}$, we can continue the proof of Lemma 3.3.15:

For the proof, we fix $0 < \gamma < 1$. Further, we assume that $\widehat{b_k} > 0$, i.e. the valley $\widehat{\mathcal{D}}_k$ is located on the right side of the origin. In the other case, i.e. $\widehat{b_k} < 0$, we only have to use the symmetric analogues of all appearing quantities.

Let us start with the consideration of ℓ_k: Due to (3.55) and (3.20), we P-a.s. have

$$\begin{aligned}\max_{\widehat{\phi}_k^- \le x \le \widehat{\phi}_{k,\gamma}^-} V(x) &\le \max_{\widehat{\phi}_k^- \le x \le \widehat{\phi}_{k,\gamma}^-} B(x) + 2\cdot\big(\log|\widehat{\phi}_k^-|\big)^2 \le B(\widehat{b}_k) + (\widehat{H}_k)^{1-\frac{\gamma}{2}} + 2\cdot\big(\log|\widehat{\phi}_k^-|\big)^2 \\ &\le B(\widehat{b}_{k-1}) + 2\cdot\Big(\log\big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 \le -(\widehat{H}_{k-2})^{1-\gamma} + 2\cdot\Big(\log\big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 < 0\end{aligned}$$

for all k large enough. Here, we further used (3.14) for the third step, (3.16) (applied for $\frac{\gamma}{2}$) for the third step, and again (3.16) (applied for γ) together with $B(\widehat{b}_{k-2}) \le 0$ for the fourth step. Additionally, we made use of (3.12) and (3.11) for the last step. Since we have $V(0) = 0$, we can immediately conclude that we have

$$\ell_k \in [\widehat{\phi}_{k,\gamma}^-, 0].$$

To see where r_k is located, we notice that, P-a.s.,

$$\begin{aligned}\max_{0\le x\le \widehat{\eta}_k^+} V(x) &\le \max_{0\le x\le \widehat{\eta}_k^+} B(x) + 2\cdot\big(\log(\widehat{r}_k)\big)^2 \le B(\widehat{r}_{k-1}) + 2\cdot\Big(\log\big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 \\ &\le B(\widehat{r}_{k-1}) + \Big(\log\big((k^{1+\gamma}\cdot\widehat{H}_{k-1})^{2+\gamma}\big)\Big)^2 \le B(\widehat{r}_{k-1}) + \frac{1}{4}\cdot(\widehat{H}_{k-1})^{1-\gamma}\end{aligned}$$

for all k large enough. Here, we used (3.15) for the second step, (3.12) for the third step, and (3.11) for the last step. Due to (3.20), analogously to above, we further P-a.s. have

$$\begin{aligned}\max_{\widehat{\phi}_{k,\gamma}^+ \le x \le \widehat{\phi}_k^+} V(x) &\le \max_{\widehat{\phi}_{k,\gamma}^+ \le x \le \widehat{\phi}_k^+} B(x) + 2\cdot\big(\log(\widehat{\phi}_k^+)\big)^2 \le B(\widehat{b}_k) + (\widehat{H}_k)^{1-\frac{\gamma}{2}} + 2\cdot\big(\log(\widehat{\phi}_k^+)\big)^2 \\ &\le B(\widehat{b}_{k-1}) + 2\cdot\Big(\log\big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 \le -(\widehat{H}_{k-2})^{1-\gamma} + 2\cdot\Big(\log\big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 < 0\end{aligned}$$

for all k large enough, where we again used (3.16) (applied for $\frac{\gamma}{2}$) for the third step and (applied for γ) for the fourth step. In the last step, we again made use of (3.12) and (3.11). On the other hand, we P-a.s. have

$$\begin{aligned}\max_{\widehat{\eta}_k^+ \le x \le \widehat{\phi}_{k,\gamma}^+} V(x) &\ge \max_{\widehat{\eta}_k^+ \le x \le \widehat{\phi}_{k,\gamma}^+} B(x) - 2\cdot\big(\log(\widehat{\phi}_k^+)\big)^2 = B(\widehat{r}_k) - 2\cdot\big(\log(\widehat{\phi}_k^+)\big)^2 \\ &\ge B(\widehat{r}_{k-1}) + (\widehat{H}_{k-1})^{1-\gamma} - 2\cdot\Big(\log\big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 \\ &\ge B(\widehat{r}_{k-1}) + (\widehat{H}_{k-1})^{1-\gamma} - 2\cdot\Big(\log\big((k^{1+\gamma}\cdot\widehat{H}_{k-1})^{2+\gamma}\big)\Big)^2 \\ &\ge B(\widehat{r}_{k-1}) + \frac{1}{2}\cdot(\widehat{H}_{k-1})^{1-\gamma}\end{aligned}$$

for all k large enough. Here, we used (3.20) for the second step. Further, we made use of (3.14) and (3.18) for the third step, (3.12) for the fourth step, and (3.11) for the last step. By combining the last three upper bounds, we see that we P-a.s. have

$$\max_{\widehat{\eta}_k^+ \leq x \leq \widehat{\phi}_{k,\gamma}^+} V(x) > \max\left\{ \max_{0 \leq x \leq \widehat{\eta}_k^+} V(x), \max_{\widehat{\phi}_{k,\gamma}^+ \leq x \leq \widehat{\phi}_k^+} V(x) \right\}$$

for all k large enough which implies

$$r_k \in [\widehat{\eta}_k^+, \widehat{\phi}_{k,\gamma}^+]$$

due to the definition of r_k.

The proof for the location of b_k works very similarly. Here, we are interested in which region the potential $V(\cdot)$ attains its minimum. Again, we assume that we have $\widehat{b}_k > 0$:
Using $\ell_k \in [\widehat{\phi}_{k,\gamma}^-, 0]$ for all k large enough (which we have just shown), we can conclude that we P-a.s. have

$$\begin{aligned}\min_{\ell_k \leq x \leq \widehat{\phi}_{k-1}^+} V(x) \geq \min_{\widehat{\phi}_{k,\gamma}^- \leq x \leq \widehat{\phi}_{k-1}^+} V(x) &\geq \min_{\widehat{\phi}_{k,\gamma}^- \leq x \leq \widehat{\phi}_{k-1}^+} B(x) - 2 \cdot \left(\log(\widehat{\phi}_k^-)\right)^2 - 2 \cdot \left(\log(\widehat{\phi}_k^+)\right)^2 \\ &\geq B(\widehat{b}_k) + (\widehat{H}_k)^{1-\gamma} - 4 \cdot \left(\log\left((\widehat{H}_k)^{2+\gamma}\right)\right)^2 \geq B(\widehat{b}_k) + \frac{1}{2} \cdot (\widehat{H}_k)^{1-\gamma}\end{aligned}$$

for all k large enough. Here, we used the definition of $\widehat{\phi}_{k,\gamma}^-$, $\widehat{\phi}_{k-1}^+$, (3.16), and (3.14) for the third step. For the last inequality, we further used (3.11). Similarly, we P-a.s. have

$$\begin{aligned}\min_{\widehat{\eta}_k^+ \leq x \leq r_k} V(x) \geq \min_{\widehat{\eta}_k^+ \leq x \leq \widehat{\phi}_{k,\gamma}^+} V(x) &\geq \min_{\widehat{\eta}_k^+ \leq x \leq \widehat{\phi}_{k,\gamma}^+} B(x) - 2 \cdot \left(\log(\widehat{\phi}_k^+)\right)^2 \\ &\geq B(\widehat{b}_k) + (\widehat{H}_k)^{1-\gamma} - 2 \cdot \left(\log\left((\widehat{H}_k)^{2+\gamma}\right)\right)^2 \geq B(\widehat{b}_k) + \frac{1}{2} \cdot (\widehat{H}_k)^{1-\gamma}\end{aligned}$$

for all k large enough. Here, we again used the definition of $\widehat{\phi}_{k,\gamma}^+$ and (3.14) for the third step and (3.11) for the last step. On the other hand, we P-a.s. have (using $\widehat{b}_k > 0$, i.e. $\widehat{b}_k \in [\widehat{\phi}_{k-1}^+, \widehat{\eta}_k^+]$)

$$\begin{aligned}\min_{\widehat{\phi}_{k-1}^+ \leq x \leq \widehat{\eta}_k^+} V(x) \leq V(\lfloor \widehat{b}_k \rfloor) \leq B(\widehat{b}_k) + 2 \cdot \left(\log |\widehat{b}_k|\right)^2 &\leq B(\widehat{b}_k) + 2 \cdot \left(\log\left((\widehat{H}_k)^{2+\gamma}\right)\right)^2 \\ &\leq B(\widehat{b}_k) + \frac{1}{4} \cdot (\widehat{H}_k)^{1-\gamma}\end{aligned}$$

for all k large enough. Here, we again used (3.15) for the third step and (3.11) for the last step. A combination of the last three inequalities finally implies that, P-a.s.,

$$\min_{\widehat{\phi}_{k-1}^+ \leq x \leq \widehat{\eta}_k^+} V(x) < \min\left\{ \min_{\ell_k \leq x \leq \widehat{\phi}_{k-1}^+} V(x), \min_{\widehat{\eta}_k^+ \leq x \leq r_k} V(x) \right\}$$

for all k large enough. Therefore, we P-a.s. have

$$b_k \in [\widehat{\phi}_{k-1}^+, \widehat{\eta}_k^+] \qquad \text{on } \{\widehat{b}_k > 0\}$$

for all k large enough due to the definition of b_k.

An analogous proof shows that we also P-a.s. have

$$b_k \in [\widehat{\eta}_k^-, \widehat{\phi}_{k-1}^-] \qquad \text{on } \{\widehat{b}_k < 0\}$$

for all k large enough. Overall, we can conclude (3.51).

For the proof of (3.53), note that by definition of $\widehat{b}_k$ we have

$$B(\widehat{b}_k) = \begin{cases} \min\limits_{\widehat{\phi}_{k-1}^+ \le s \le \widehat{\eta}_k^+} B(s) & \text{if } \widehat{b}_k > 0 \\ \min\limits_{\widehat{\eta}_k^- \le s \le \widehat{\phi}_{k-1}^-} B(s) & \text{if } \widehat{b}_k < 0. \end{cases}$$

On the other hand, we have just shown in (3.51) that we P-a.s. have

$$V(b_k) = \begin{cases} \min\limits_{\widehat{\phi}_{k-1}^+ \le x \le \widehat{\eta}_k^+} V(x) & \text{if } \widehat{b}_k > 0 \\ \min\limits_{\widehat{\eta}_k^- \le x \le \widehat{\phi}_{k-1}^-} V(x) & \text{if } \widehat{b}_k < 0 \end{cases}$$

for all k large enough. □

Together with the help of Lemma 3.3.14 and 3.3.15, we can now start to prove that our constructed valleys have various properties, which we formulated in Proposition 3.3.9 – 3.3.13:

Proof of Proposition 3.3.9. For the proof, we fix $0 < \gamma < 1$. Further, we assume that we have $\widehat{b}_{k+1} > 0$, i.e. the valley $\widehat{\mathcal{D}}_{k+1}$ is located on the right side of the origin. In the other case, i.e. $\widehat{b}_{k+1} < 0$, we only have to use the symmetric analogues of all appearing quantities. Thereby, we P-a.s. have $\widehat{b}_{k+1} \neq 0$ for all k by definition of the bottom points $(\widehat{b}_k)_{k\in\mathbb{N}_0}$ since we P-a.s. have

$$\min_{-\frac{1}{n} \le t \le \frac{1}{n}} B(t) < 0$$

for all $n \in \mathbb{N}$.

First of all, we notice that according to (3.51) we P-a.s. have

$$b_{k+1} > 0$$

for all k large enough. In a first step, we show that in this case we also P-a.s. have

$$m_{k+1} > 0$$

for all k large enough. By definition of m_{k+1}, there is only something to show if we have

$$b_k < 0$$

what we will assume for the next few lines. Again using (3.51), this implies that we also P-a.s. have

$$\widehat{b}_k < 0$$

for all k large enough. Let us assume that we have $m_{k+1} \leq 0$:
In this case, due to the definition of m_{k+1}, we P-a.s. have

$$V(m_{k+1}) = \max_{b_k \leq x \leq 0} V(x) \leq \max_{\widehat{\eta}_k^- \leq x \leq 0} V(x) \leq \max_{\widehat{\eta}_k^- \leq x \leq 0} B(x) + 2 \cdot \left(\log|\widehat{\ell}_k|\right)^2$$

for all k large enough, where we further used (3.51) and (3.55). This yields that, P-a.s.,

$$V(m_{k+1}) \leq B(\widehat{\ell}_{k-1}) + 2 \cdot \left(\log|\ell_k|\right)^2 \leq B(\widehat{r}_{k-1}) - (\widehat{H}_{k-2})^{1-\gamma} + 2 \cdot \left(\log\left((\widehat{H}_k)^{2+\gamma}\right)\right)^2$$

for all k large enough, where we used the definition of $\widehat{\ell}_{k-1}$ for the first step and (3.13) and (3.15) for the last step. Notice here that $\widehat{b}_k < 0$ implies $B(\widehat{\ell}_{k-1}) < B(\widehat{r}_{k-1})$ by construction. Due to (3.11) and (3.12), the last inequality P-a.s. implies

$$V(m_{k+1}) \leq B(\widehat{r}_{k-1}) - \frac{1}{2} \cdot (\widehat{H}_{k-2})^{1-\gamma} \tag{3.56}$$

for all k large enough. On the other hand, we notice that we P-a.s. have

$$0 < \widehat{r}_{k-1} < b_{k+1}$$

for all k large enough due to (3.51). Thereby, we P-a.s. have

$$V(\lfloor \widehat{r}_{k-1} \rfloor) \geq B(\widehat{r}_{k-1}) - 2 \cdot \left(\log|\widehat{r}_{k-1}|\right)^2 \geq B(\widehat{r}_{k-1}) - 2 \cdot \left(\log\left((\widehat{H}_{k-1})^{2+\gamma}\right)\right)^2 > V(m_{k+1})$$

for all k large enough, where we used (3.15) for the second step and (3.56) together with (3.11) and (3.12) for the last step. This a contradiction to the definition of m_{k+1} as the point with the highest potential between b_k and b_{k+1}. In particular, we can conclude that we P-a.s. have

$$m_{k+1} > 0 \tag{3.57}$$

for all k large enough if also $b_{k+1} > 0$ holds.

Our claim is that we actually have

$$m_{k+1} = r_k$$

in our situation, i.e. $b_{k+1} > 0$. (Now we allow $b_k > 0$ and also $b_k < 0$.) By definition, we have

$$V(r_k) = \max_{0 \leq x \leq \widehat{\phi}_k^+} V(x). \tag{3.58}$$

Therefore, we still have to show that, P-a.s., we cannot have

$$m_{k+1} > \widehat{\phi}_k^+ \geq \widehat{r}_k$$

for arbitrary large k. For $m_{k+1} > \widehat{\phi}_k^+ \geq \widehat{r}_k$, an application of (3.55) and the definition of m_{k+1} yields that, P-a.s.,

$$\begin{aligned}
B(m_{k+1}) &\geq V(m_{k+1}) - \big(\log|m_{k+1}|\big)^2 \geq V(\lfloor \widehat{r}_k \rfloor) - \big(\log|m_{k+1}|\big)^2 \\
&\geq B(\lfloor \widehat{r}_k \rfloor) - 2\cdot\big(\log|m_{k+1}|\big)^2 \geq B(\widehat{r}_k) - 3\cdot\Big(\log\big((\widehat{H}_{k+1})^{2+\gamma}\big)\Big)^2 \\
&\geq B(\widehat{r}_k) - 3\cdot\Big(\log\big((k^{1+\gamma}\cdot\widehat{H}_k)^{2+\gamma}\big)\Big)^2 \geq B(\widehat{r}_k) - (\widehat{H}_k)^{1-\gamma}
\end{aligned}$$

for all k large enough. Here, we further used (3.14), (3.12) and the exponential growth of $\widehat{H}_k$ in k (cf. (3.11)). In terms of stopping times for the Brownian motion, this implies that we P-a.s. have

$$\begin{aligned}
&\limsup_{k\to\infty}\Big(\Big\{m_{k+1} > \widehat{\phi}_k^+\Big\}\cap\{b_{k+1}>0\}\Big) \\
\subseteq\ &\limsup_{k\to\infty}\Big(\Big\{\widehat{\phi}_k^+ < \widehat{\lambda}_k < \widehat{\phi}_k^{(2)} < \widehat{\eta}_{k+1}^+\Big\}\cap\Big\{\widehat{b}_{k+1}>0\Big\}\Big), \qquad (3.59)
\end{aligned}$$

where, for $k\in\mathbb{N}_0$,

$$\begin{aligned}
\widehat{\lambda}_k &:= \inf\{t \geq \widehat{\phi}_k^+ :\ B(\widehat{r}_k) - (\widehat{H}_k)^{1-\gamma}\}, \\
\widehat{\phi}_k^{(2)} &:= \inf\{t \geq \widehat{\lambda}_k :\ B(\widehat{b}_k)\}.
\end{aligned}$$

For the relation in (3.59), notice that the sign of b_{k+1} and $\widehat{b}_{k+1}$ P-a.s. coincides for all k large enough (cf. (3.51)). Further we have $m_k \leq b_{k+1} \leq \widehat{\eta_k}^+$ for all k large enough due to the definition of m_{k+1} and again (3.51).

On the set $\{\widehat{b}_{k+1} > 0\}$, we have the following connection to the ruin probabilities of the Brownian motion using the strong Markov property at $\widehat{\lambda}_k$

$$\begin{aligned}
&\mathsf{P}\Big(\widehat{\phi}_k^+ < \widehat{\lambda}_k < \widehat{\phi}_k^{(2)} < \widehat{\eta}_{k+1}^+\Big| B(\widehat{r}_k), B(\widehat{\ell}_k), B(\widehat{b}_k)\Big) \\
\leq\ &\mathsf{P}\Big(\widehat{T}\big(-\widehat{H}_k + (\widehat{H}_k)^{1-\gamma}\big) < \widehat{T}\big((\widehat{H}_k)^{1-\gamma}\big)\Big| B(\widehat{r}_k), B(\widehat{\ell}_k), B(\widehat{b}_k)\Big) \\
=\ &\frac{(\widehat{H}_k)^{1-\gamma}}{\widehat{H}_k} = \frac{1}{(\widehat{H}_k)^{\gamma}}\ .
\end{aligned}$$

Since this last upper bound is P-a.s. summable in k due to the exponential growth of $\widehat{H}_k$ in k (cf. (3.11)), the Borel-Cantelli lemma and (3.59) imply that on the set $b_{k+1} > 0$ we P-a.s. have

$$m_{k+1} \leq \widehat{\phi}_k^+$$

for all k large enough. In combination with (3.57) and (3.58), we can conclude that we P-a.s. have

$$m_{k+1} = r_k$$

for all k large enough if $b_{k+1} > 0$.

Finally, (3.31) is a direct consequence of (3.30) and the definition of m_{k+1}. Further, a combination of (3.30) and the definition of D_k P-a.s. yields

$$\begin{aligned} D_{k+1} &= \begin{Bmatrix} \min\{V(r_k)-V(b_{k+1}), V(r_{k+1})-V(b_{k+1})\} = V(r_k)-V(b_{k+1}) & \text{if } b_{k+1}>0 \\ \min\{V(\ell_{k+1})-V(b_{k+1}), V(\ell_k)-V(b_{k+1})\} = V(\ell_k)-V(b_{k+1}) & \text{if } b_{k+1}<0 \end{Bmatrix} \\ &= V(m_{k+1})-V(b_{k+1}) \end{aligned}$$

for all k large enough which shows (3.32). For the proof of (3.33), we notice that we P-a.s. have

$$\begin{aligned} H_k =&\ \min\{V(\ell_k)-V(b_k), V(r_k)-V(b_k)\} = \min\{V(h_{k+1})-V(b_k), V(m_{k+1})-V(b_k)\} \\ =&\ V(m_{k+1})-V(b_k) \end{aligned}$$

for all k large enough. Here, we used the definition of h_{k+1} and (3.30) for the second step. The argument for the last step, i.e. $V(m_{k+1}) \le V(h_{k+1})$ for all k large enough, is postponed to the proof of (3.36). ■

Proof of Proposition 3.3.10. Due to Proposition 3.3.9, we P-a.s. have

$$m_k = \begin{cases} r_{k-1} > 0 & \text{if } b_k > 0 \\ \ell_{k-1} < 0 & \text{if } b_k < 0 \end{cases}$$

for all k large enough. Therefore, we also P-a.s. have

$$\begin{aligned} D_k &= \begin{cases} \min\{V(r_k)-V(b_k), V(\ell_{k-1})-V(b_k)\} & \text{if } b_k>0 \\ \min\{V(\ell_k)-V(b_k), V(r_{k-1})-V(b_k)\} & \text{if } b_k<0, \end{cases} \\ H_k &= \min\{V(r_k)-V(b_k), V(\ell_k)-V(b_k)\}, \\ D_{k+1} &= \begin{cases} \min\{V(r_{k+1})-V(b_{k+1}), V(\ell_k)-V(b_{k+1})\} & \text{if } b_{k+1}>0 \\ \min\{V(\ell_{k+1})-V(b_{k+1}), V(r_k)-V(b_{k+1})\} & \text{if } b_{k+1}<0 \end{cases} \end{aligned}$$

for all k large enough by definition, where further

$$\begin{aligned} &V(r_{k+1}) \ge V(r_k) \ge V(r_{k-1}), \\ &V(\ell_{k+1}) \ge V(\ell_k) \ge V(\ell_{k-1}), \\ &V(b_{k+1}) \le V(b_k) \end{aligned}$$

also holds by definition. Overall, we can conclude that we P-a.s. have

$$D_k \le H_k \le D_{k+1} \tag{3.60}$$

for all k large enough. This shows the first part of (3.34). To see that the last inequality holds, we will compare H_k and $\widehat{H}_k$:
Let us assume that $V(r_k) \le V(\ell_k)$ holds. In the other case, we can use the same proof with the symmetrically analogous quantities. Due to the assumption, for $\vartheta > 0$, we P-a.s. have

$$H_k = V(r_k) - V(b_k) \ge V(\lfloor \widehat{r}_k \rfloor) - V(\lfloor \widehat{b}_k \rfloor)$$

$$\begin{aligned}
&\geq B(\lfloor\widehat{r}_k\rfloor) - B(\lfloor\widehat{b}_k\rfloor) - \big(\log|\widehat{r}_k|\big)^2 - \big(\log|\widehat{\ell}_k|\big)^2 \\
&\geq B(\widehat{r}_k) - B(\widehat{b}_k) - 4\cdot\Big(\log\big((\widehat{H}_k)^{2+\vartheta}\big)\Big)^2 \geq \widehat{H}_k - 4\cdot\Big(\log\big((\widehat{H}_k)^{2+\vartheta}\big)\Big)^2 \qquad (3.61)
\end{aligned}$$

for all k large enough. Thereby, we made use of the definition of r_k and (3.53) in the second step. Further, we used (3.55) and (3.15) for the remaining steps. In particular, (3.11) together with (3.60) implies that for every $0 < \beta, \lambda < 1$ we P-a.s. have

$$D_{k+1} \geq H_k \geq (1-\lambda)\cdot\exp(\beta\cdot k)$$

for all k large enough. Since this property holds for every $0 < \beta, \lambda < 1$, we can conclude that, for $0 < \beta < 1$, we even P-a.s. have

$$D_{k+1} \geq H_k \geq \exp\big(\beta\cdot(k+1)\big)$$

for all k large enough. This finishes the proof of (3.34).

For the proof of (3.35), we first notice that analogous arguments as in (3.61) show that we can derive the analogous upper bound, and therefore, for $\vartheta > 0$, we P-a.s. have

$$\widehat{H}_k - 4\cdot\Big(\log\big((\widehat{H}_k)^{2+\vartheta}\big)\Big)^2 \leq H_k \leq \widehat{H}_k + 4\cdot\Big(\log\big((\widehat{H}_k)^{2+\vartheta}\big)\Big)^2 \qquad (3.62)$$

for all k large enough. Since we have seen in Proposition 3.3.4 that for $\vartheta > 0$ we P-a.s. have

$$\widehat{H}_k \leq k^{1+\frac{\vartheta}{2}}\cdot\widehat{H}_{k-1}$$

for all k large enough, a combination of the last two inequalities show that we also have (3.35), i.e. for $\vartheta > 0$ we P-a.s. have

$$H_k \leq k^{1+\vartheta}\cdot H_{k-1}$$

for all k large enough where we further used the exponential growth of $(\widehat{H}_k)_k$ in k (cf. (3.11)).

For the proof of (3.36), we first consider the difference $V(\ell_k) - V(r_k)$ for $k \in \mathbb{N}_0$. For $0 < \gamma < 1$, we P-a.s. have

$$\begin{aligned}
V(\ell_k) - V(r_k) &\geq V(\lfloor\widehat{\ell}_k\rfloor) - B(r_k) - \big(\log(r_k)\big)^2 \\
&\geq B(\lfloor\widehat{\ell}_k\rfloor) - B(r_k) - \big(\log(r_k)\big)^2 - \big(\log(\lfloor\widehat{\ell}_k\rfloor)\big)^2 \\
&\geq B(\widehat{\ell}_k) - B(\widehat{r}_k) - 3\cdot\Big(\log\big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 \qquad (3.63)
\end{aligned}$$

for all k large enough. Here, we used the strong approximation Theorem and Lemma 3.3.14 several times (cf. Remark 3.3.16). Further, we used (3.15) for the third step. Recall that due to (3.30) and the definition of h_{k+1} we P-a.s. have

$$m_{k+1} = \begin{cases} r_k > 0 & \text{if } b_{k+1} > 0 \\ \ell_k < 0 & \text{if } b_{k+1} < 0, \end{cases}$$

$$h_{k+1} = \begin{cases} \ell_k > 0 & \text{if } b_{k+1} > 0 \\ r_k < 0 & \text{if } b_{k+1} < 0 \end{cases}$$

for all k large enough where further, P-a.s.,

$$b_{k+1} > 0 \quad \text{iff} \quad \widehat{b}_{k+1} > 0$$

for all k large enough due to (3.51). Therefore, we can conclude that on the set $\{b_{k+1} > 0\}$ we P-a.s. have, using (3.63),

$$V(h_{k+1}) - V(m_{k+1}) = V(\ell_k) - V(r_k) \geq B(\widehat{\ell}_k) - B(\widehat{r}_k) - 3 \cdot \left(\log\left((\widehat{H}_k)^{2+\gamma}\right)\right)^2$$

$$= B(\widehat{h}_{k+1}) - B(\widehat{m}_{k+1}) - 3 \cdot \left(\log\left((\widehat{H}_k)^{2+\gamma}\right)\right)^2 \geq (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}} - 3 \cdot \left(\log\left((\widehat{H}_k)^{2+\gamma}\right)\right)^2$$

$$\geq \left(H_{k-1} - 4 \cdot \left(\log\left((\widehat{H}_k)^{2+\gamma}\right)\right)^2\right)^{1-\frac{3\gamma}{4}} \geq (H_{k-1})^{1-\gamma}$$

for all k large enough. Here, we used (3.63) for the second step. Further, we used (3.13) (for $\frac{\gamma}{2}$) for the fourth step, where we have $B(\widehat{h}_{k+1}) - B(\widehat{m}_{k+1}) \geq 0$ by definition. Additionally, we used (3.62), (3.35), and the exponential growth of H_k in k (cf. (3.34)) for the last two steps.

The argument on the set $\{b_{k+1} < 0\}$ works just in the same way if we use the symmetric analogues of all appearing quantities. Overall, we have shown (3.36).

For the remaining inequalities, we can use (3.50) and (3.52) to conclude that, for $\vartheta > 0$, we P-a.s. have

$$|\ell_k| \leq |\widehat{\phi}_k^-| \leq (\widehat{H}_k)^{2+\frac{\vartheta}{2}} \leq (H_k)^{2+\vartheta},$$
$$|r_k| \leq |\widehat{\phi}_k^+| \leq (\widehat{H}_k)^{2+\frac{\vartheta}{2}} \leq (H_k)^{2+\vartheta}$$

for all k large enough, where we first applied (3.14) for $\frac{\vartheta}{2}$ and then used (3.62). This shows (3.37) and (3.38). Further, (3.39), (3.40), and (3.41) are direct consequences since we have

$$\ell_k \leq b_k \leq r_k \qquad \text{and}$$
$$h_{k+1} \in \{\ell_k, r_k\}$$

by definition and, P-a.s.,

$$m_{k+1} \in \{\ell_k, r_k\}$$

for all k large enough due to Proposition 3.3.9. ■

Proof of Proposition 3.3.11. For the proof, we fix $0 < \gamma < 1$. Further, we assume that we have $\widehat{b}_k > 0$, i.e. the valley $\widehat{\mathcal{D}}_k$ is located on the right side of the origin. In the other case, i.e. $\widehat{b}_k < 0$, we only have to use the symmetric analogues of all appearing quantities.

Due to our assumption and (3.51), we also P-a.s. have $b_k > 0$ for all k large enough. In particular, this implies

$$\mathcal{V}_k \backslash \mathcal{D}_k = \{\ell_k, \ell_k + 1, \ldots, m_k - 1\} = \{\ell_k, \ell_k + 1, \ldots, \ell_{k-1} - 1\} \cup \{\ell_{k-1}, \ell_{k-1} + 1, \ldots, r_{k-1} - 1\}$$

for all k large enough, where we further used (3.30) for the last step. According to (3.50), we therefore P-a.s. have

$$\begin{aligned} &\{\ell_k, \ell_k + 1, \ldots, \ell_{k-1} - 1\} \subseteq [\widehat{\phi}^-_{k,\gamma}, 0], \\ &\{\ell_{k-1}, \ell_{k-1} + 1, \ldots, r_{k-1} - 1\} \subseteq [\widehat{\phi}^-_{k-1,\gamma}, \widehat{\phi}^+_{k-1,\gamma}] \end{aligned}$$

for all k large enough. Thereby, we can conclude with the help of (3.50) and (3.55) that we P-a.s. have

$$\begin{aligned} \min_{\ell_k \le x \le \ell_{k-1}-1} V(x) &\ge \min_{\widehat{\phi}^-_{k,\gamma} \le x \le 0} V(x) \ge \min_{\widehat{\phi}^-_{k,\gamma} \le x \le 0} B(x) - 2 \cdot \big(\log |\widehat{\phi}^-_{k,\gamma}|\big)^2 \\ &= \min \left\{ \min_{\widehat{\phi}^-_{k,\gamma} \le x \le \widehat{\phi}^-_{k-1}} B(x), \min_{\widehat{\phi}^-_{k-1} \le x \le 0} B(x) \right\} - 2 \cdot \big(\log |\widehat{\phi}^-_{k,\gamma}|\big)^2 \\ &\ge B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma} - 2 \cdot \Big(\log \big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 \end{aligned}$$

for all k large enough. Here, we used the definition of $\widehat{\phi}^-_{k,\gamma}$ (cf. (3.19)), (3.16) (note that we have $\widehat{\phi}^-_{k-1} = \widehat{\eta}^-_k$ since $\widehat{b}_k > 0$ for all k large enough), and (3.14). Similarly, we P-a.s. get

$$\begin{aligned} \min_{\ell_{k-1} \le x \le r_{k-1}-1} V(x) &\ge \min_{\widehat{\phi}^-_{k-1,\gamma} \le x \le \widehat{\phi}^+_{k-1,\gamma}} V(x) \\ &\ge \min_{\widehat{\phi}^-_{k-1,\gamma} \le x \le \widehat{\phi}^+_{k-1,\gamma}} B(x) - 2 \cdot \big(\log |\widehat{\phi}^-_{k-1,\gamma}|\big)^2 - 2 \cdot \big(\log |\widehat{\phi}^+_{k-1,\gamma}|\big)^2 \\ &\ge B(\widehat{b}_{k-1}) - 4 \cdot \Big(\log \big((\widehat{H}_{k-1})^{2+\gamma}\big)\Big)^2 \\ &\ge B(\widehat{b}_k) + (\widehat{H}_{k-1})^{1-\gamma} - 4 \cdot \Big(\log \big((\widehat{H}_{k-1})^{2+\gamma}\big)\Big)^2 \end{aligned}$$

for all k large enough, where we again used (3.14) and further (3.16) for the last step. On the other hand, due to (3.53), we P-a.s. have

$$V(b_k) \le V(\lfloor \widehat{b}_k \rfloor) \le B(\widehat{b}_k) + 2 \cdot \big(\log |\widehat{b}_k|\big)^2 \le B(\widehat{b}_k) + 2 \cdot \Big(\log \big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2$$

for all k large enough, where we applied (3.55) in the second step and (3.15) in the last step. A combination of the last three inequalities in particular implies due to the exponential growth of $(\widehat{H}_{k-1})_k$ in k (cf. (3.11)) that we P-a.s. have

$$\min_{x \in \mathcal{V}_k \backslash \mathcal{D}_k} V(x) \ge V(b_k) + \frac{3}{4}(\widehat{H}_{k-1})^{1-\gamma} \ge V(b_k) + \frac{1}{2}(H_{k-1})^{1-\gamma} \tag{3.64}$$

for all k large enough. Here, we additionally used that, P-a.s.,

$$\widehat{H}_{k-1} - 4 \cdot \left(\log\left((\widehat{H}_{k-1})^{2+\gamma}\right)\right)^2 \leq H_{k-1} \leq \widehat{H}_{k-1} + 4 \cdot \left(\log\left((\widehat{H}_{k-1})^{2+\gamma}\right)\right)^2$$

for all k large enough which we have shown in (3.62).

Since (3.64) holds for every $0 < \gamma < 1$ and since H_{k-1} grows at least exponentially in k due to (3.34), we can finally conclude that we even P-a.s. have

$$\min_{x \in \mathcal{V}_k \setminus \mathcal{D}_k} V(x) \geq V(b_k) + (H_{k-1})^{1-\gamma} \tag{3.65}$$

for all k large enough.

For the proof of (3.43), recall (3.30) which states that, P-a.s.,

$$m_k = \begin{cases} r_{k-1} > 0 & \text{if } b_k > 0 \\ \ell_{k-1} < 0 & \text{if } b_k < 0 \end{cases}$$

for all k large enough. In particular we P-a.s. have

$$b_{k-1} \in \mathcal{V}_k \setminus \mathcal{D}_k$$

for all k large enough, where further, P-a.s.,

$$\begin{aligned} D_k &= V(m_k) - V(b_k) \quad \text{and} \\ H_{k-1} &= V(m_k) - V(b_{k-1}) \end{aligned}$$

for all k large enough (cf. (3.33)). ∎

Proof of Proposition 3.3.12. For the proof, it is helpful to have the following picture in mind: The maximal potential difference we are interested in here can be understood as a largest neighbouring spike within the valley $\mathcal{D}_k$ on the two sides of the bottom point b_k. Since this neighbouring spike cannot stick out of the valley, we will see that is very unlikely that the neighbouring spike is almost as high as the depth of the valley D_k.

For the proof, we fix $0 < \gamma < 1$. Further, we assume that we have $b_k > 0$, i.e. the valley $\mathcal{D}_k$ is located on the right side of the origin. In the other case, i.e. $b_k < 0$, we only have to use the symmetric analogues of all appearing quantities.

Let us define

$$M_k := \max_{x,y \in \mathcal{D}_k:\ x<y<b_k} V(y) - V(x).$$

Note in a first step that we P-a.s. have

$$\limsup_{k\to\infty} \left(\left\{ M_k > D_k - (H_{k-1})^{1-\gamma} \right\} \cap \{b_k > 0\} \right) \subseteq \limsup_{k\to\infty} \left(F_k \cap \{b_k > 0\} \right),$$

where, for $k \in \mathbb{N}_0$,

$$F_k := \left\{ \exists\, m_k < x < y < b_k : V(x) \leq V(m_k) - D_k + (H_{k-1})^{1-\gamma}, V(y) \geq V(m_k) - (H_{k-1})^{1-\gamma} \right\}.$$

Thereby, we P-a.s. have for x with

$$m_k < x < b_k \quad \text{and} \quad V(x) \le V(m_k) - D_k + (H_{k-1})^{1-\gamma}$$

on the set $\{b_k > 0\}$ that

$$\begin{aligned}
B(x) &\le V(x) + \big(\log(x)\big)^2 \le V(m_k) - D_k + (H_{k-1})^{1-\gamma} + \Big(\log\big((H_k)^{2+\gamma}\big)\Big)^2 \\
&\le V(b_k) + (H_{k-1})^{1-\gamma} + \Big(\log\big((k^{1+\gamma} \cdot H_{k-1})^{2+\gamma}\big)\Big)^2 \\
&\le V(b_{k-1}) - \frac{1}{2}(H_{k-1})^{1-\frac{\gamma}{2}} \le V(\lfloor \widehat{b}_{k-1} \rfloor) - \frac{1}{2}(H_{k-1})^{1-\frac{\gamma}{2}} \\
&\le B(\widehat{b}_{k-1}) + 2 \cdot \big(\log |\widehat{b}_{k-1}|\big)^2 - \frac{1}{2}(H_{k-1})^{1-\frac{\gamma}{2}} \\
&\le B(\widehat{b}_{k-1}) + 2 \cdot \Big(\log\big((\widehat{H}_{k-1})^{2+\gamma}\big)\Big)^2 - \frac{1}{2}(H_{k-1})^{1-\frac{\gamma}{2}} < B(\widehat{b}_{k-1})
\end{aligned}$$

holds for all k large enough. Here, we used the Komlós-Major-Tusnády strong approximation theorem (cf. Remark 3.55) for the first step. Further, we used (3.39) and (3.35) in the second and third step. Additionally, we used (3.42) with $\frac{\gamma}{2}$ – which implies $V(b_{k-1}) \ge V(b_k) + (H_{k-1})^{1-\frac{\gamma}{2}}$ for all k large enough – and the exponential growth of H_k (cf. (3.34)) in the third line. In the next two steps, we made use of (3.53) and the approximation theorem again. Finally, we can apply (3.62) – which connects $\widehat{H}_{k-1}$ and H_{k-1} – together with the exponential growth of $\widehat{H}_k$ to see that the last inequality holds. The last inequality, which we just derived, in particular implies that for the considered x we P-a.s. have

$$x > \widehat{\phi}^+_{k-1}$$

for all k large enough. Similarly, we P-a.s. have for y with

$$m_k < y < b_k \quad \text{and} \quad V(y) \ge V(m_k) - (H_{k-1})^{1-\gamma}$$

on the set $\{b_k > 0\}$

$$\begin{aligned}
B(y) &\ge V(y) - \big(\log(y)\big)^2 \ge V(m_k) - (H_{k-1})^{1-\gamma} - \big(\log(y)\big)^2 \\
&\ge V(\lfloor \widehat{m}_k \rfloor) - (H_{k-1})^{1-\gamma} - \Big(\log\big((H_k)^{2+\gamma}\big)\Big)^2 \ge B(\widehat{m}_k) - (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}
\end{aligned}$$

for all k large enough where we used the same inequalities as above. Additionally, we used that we P-a.s. have $m_k = r_{k-1}$ due to our assumption for all k large enough (cf. (3.30)) from which we know that the potential $V(\cdot)$ attains a maximum in some neighbourhood at $r_{k-1} = m_k$ which we used in the third step. For the last step, we made use of (3.62) again to connect $\widehat{H}_{k-1}$ and H_{k-1}. On the other hand, we P-a.s. have

$$\begin{aligned}
V(b_k) &\le V(\lfloor \widehat{b}_k \rfloor) \le B(\widehat{b}_k) + 2 \cdot \big(\log(\widehat{b}_k)\big)^2 \\
&\le B(\widehat{b}_{k-1}) - (\widehat{H}_{k-1})^{1-\gamma} + 2 \cdot \Big(\log\big((\widehat{H}_k)^{2+\gamma}\big)\Big)^2 \le B(\widehat{b}_{k-1})
\end{aligned}$$

for all k large enough. Here, we used (3.53) for the first step, (3.55) for the second step, and (3.16) for the third step. For the last step, we again used the exponential growth of $\widehat{H}_{k-1}$ in k (cf. (3.11)) together with (3.12).

In terms of stopping times for the Brownian motion with

$$\begin{aligned}\widehat{\lambda}_{k-1} &:= \inf\{t \geq \widehat{\phi}^+_{k-1} :\ B(t) = B(\widehat{m}_k) - (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}\},\\ \widehat{\phi}^*_{k-1} &:= \inf\{t \geq \widehat{\lambda}_{k-1} :\ B(t) = B(\widehat{b}_{k-1})\},\end{aligned}$$

we can conclude by the above argument that we P-a.s. have

$$\begin{aligned}&\limsup_{k\to\infty} \left(\{M_k > D_k - (H_{k-1})^{1-\gamma}\} \cap \{b_k > 0\}\right)\\ \subseteq\ &\limsup_{k\to\infty} \left(\left\{\widehat{\phi}^+_{k-1} < \widehat{\lambda}_{k-1} < \widehat{\phi}^*_{k-1} < \widehat{\eta}^+_k\right\} \cap \left\{\widehat{b}_k > 0\right\}\right).\end{aligned}$$

Here, we additionally used that (3.51) implies that we P-a.s. have $b_k \leq \widehat{\eta}^+_k$ and that the sign of b_k and $\widehat{b}_k$ coincides for all k large enough. Now we can compute the probability for the dominating event with the help of the ruin probabilities of the Brownian motion again. More precisely, we have the following upper bound on the set $\{\widehat{b}_k > 0\}$ using the strong Markov property of the Brownian motion at $\widehat{\lambda}_{k-1}$ (cf. Remark 3.3.2):

$$\begin{aligned}&\mathsf{P}\left(\widehat{\phi}^+_{k-1} < \widehat{\lambda}_{k-1} < \widehat{\phi}^*_{k-1} < \widehat{\eta}^+_k \,\middle|\, \widehat{H}_{k-1}, B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1})\right)\\ \leq\ &\mathsf{P}\left(\widehat{T}\left(-\widehat{H}_{k-1} + (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}\right) < \widehat{T}\left((\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}\right) \,\middle|\, \widehat{H}_{k-1}, B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1})\right)\\ =\ &\frac{(\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}}{\widehat{H}_{k-1}} = \frac{1}{(\widehat{H}_{k-1})^{\frac{\gamma}{2}}}\ .\end{aligned}$$

Since the last expression is P-a.s. summable in k due to the exponential growth of $\widehat{H}_{k-1}$ in k (cf. (3.34)), the Borel-Cantelli lemma yields that on the set $\{b_k > 0\}$ we P-a.s. have

$$M_k \overset{\text{def}}{=} \max_{x,y\in\mathcal{D}_k:\ x<y<b_k} V(y) - V(x) \leq D_k - (H_{k-1})^{1-\gamma}$$

for all k large enough. This shows (3.45).

The proof of (3.46) is similar to above, where we again assume $b_k > 0$. In order to have a high neighbouring spike within the valley $\mathcal{D}_k$, the potential has to reach a very large value without reaching $V(m_k)$ first, then a very small value without attaining a new minimum before the potential reaches a value of at least $V(m_k)$ at time η^+_k again. In comparison with above, all we have to do here is to use

$$\begin{aligned}M^{(1)}_k &:= \max_{x,y\in\mathcal{D}_k:\ b_k<x<y<\eta^+_k} V(x) - V(y),\\ F^*_k &:= \left\{\exists\, b_k < x < y < \eta^+_k : V(x) \geq V(m_k) - (H_{k-1})^{1-\gamma}, V(y) \leq V(m_k) - D_k + (H_{k-1})^{1-\gamma}\right\}\end{aligned}$$

instead of M and F_k. For such x and y we P-a.s. have on $\{b_k > 0\}$

$$B(x) \geq B(\widehat{m}_k) - (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}},$$

$$B(y) \leq B(\widehat{b}_{k-1})$$

for all k large enough due to the same arguments as above. With

$$\widehat{\lambda}_{k-1} \stackrel{\text{def}}{=} \inf\{t \geq \widehat{\phi}^+_{k-1} : \; B(t) = B(\widehat{m}_k) - (\widehat{H}_{k-1})^{1-\frac{\gamma}{2}}\},$$
$$\widehat{\phi}^*_{k-1} \stackrel{\text{def}}{=} \inf\{t \geq \widehat{\lambda}_{k-1} : \; B(t) = B(\widehat{b}_{k-1})\}$$

as above, this again implies

$$\begin{aligned} &\limsup_{k\to\infty} \left(\left\{ M_k^{(1)} > D_k - (H_{k-1})^{1-\gamma} \right\} \cap \{b_k > 0\} \right) \\ \subseteq\; &\limsup_{k\to\infty} \left(\left\{ \widehat{\phi}^+_{k-1} < \widehat{\lambda}_{k-1} < \widehat{\phi}^*_{k-1} < \widehat{\eta}^+_k \right\} \cap \left\{ \widehat{b}_k > 0 \right\} \right). \end{aligned}$$

Here, we used that we P-a.s. have $b_k \geq \widehat{\phi}^+_{k-1}$ for all k large enough (cf. (3.51)). An application of the Borel-Cantelli lemma therefore yields that on the set $\{b_k > 0\}$ we also P-a.s. have

$$M_k^{(1)} \stackrel{\text{def}}{=} \max_{x,y\in\mathcal{D}_k:\; b_k<x<y<\eta^+_k} V(x) - V(y) \leq D_k - (H_{k-1})^{1-\gamma}$$

for all k large enough. This finishes the proof of (3.46). ■

Proof of Proposition 3.3.13. The proof runs very similarly to the proof of Proposition 3.3.12. Again, we fix $0 < \gamma < 1$ and only consider the case $b_k > 0$ ($b_{k+1} > 0$), i.e. the case in which the valley $\mathcal{D}_k$ ($\mathcal{D}_{k+1}$) is located on the right side of the origin. In the other case, i.e. $b_k < 0$ ($b_{k+1} < 0$), we only have to use the symmetric analogues of all appearing quantities.

Using the same approach as above, we see that for

$$M_k^{(2)} := \min_{\eta^+_k \leq x \leq r_k} V(x),$$
$$M_k^{(3)} := \max_{\widehat{\phi}^+_{k,\gamma} \leq x \leq b_{k+1}} V(x) - V(m_{k+1})$$

we have

$$\begin{aligned} &\limsup_{k\to\infty} \left(\left\{ M_k^{(2)} < V(b_k) + (D_k)^{1-\gamma} \right\} \cap \{b_k > 0\} \right) \\ \subseteq\; &\limsup_{k\to\infty} \left(\left\{ \widehat{\eta}^+_k < \widehat{\lambda}^{(2)}_k < \widehat{\eta}^{(2)}_k < \widehat{\phi}^+_k \right\} \cap \left\{ \widehat{b}_k > 0 \right\} \right) \end{aligned}$$

and

$$\begin{aligned} &\limsup_{k\to\infty} \left(\left\{ M_k^{(3)} > -(H_k)^{1-\gamma} \right\} \cap \{b_{k+1} > 0\} \right) \\ \subseteq\; &\limsup_{k\to\infty} \left(\left\{ \widehat{\phi}^+_k < \widehat{\lambda}^{(3)}_k < \widehat{\phi}^{(3)}_k < \widehat{\eta}^+_{k+1} \right\} \cap \left\{ \widehat{b}_{k+1} > 0 \right\} \right) \end{aligned}$$

P-a.s., where, for $k \in \mathbb{N}_0$,

$$\widehat{\lambda}^{(2)}_k := \inf\{t \geq \widehat{\eta}^+_k : \; B(t) = B(\widehat{b}_k) + (\widehat{D}_k)^{1-\frac{\gamma}{2}}\},$$

$$\begin{aligned}
\widehat{\eta}_k^{(2)} &:= \inf\{t \geq \widehat{\lambda}_k^{(2)} :\ B(t) = B(\widehat{m}_k) - (\widehat{D}_k)^{1-\frac{\gamma}{2}}\},\\
\widehat{\lambda}_k^{(3)} &:= \inf\{t \geq \widehat{\phi}_k^{+} :\ B(t) = B(\widehat{m}_{k+1}) - (\widehat{H}_k)^{1-\frac{\gamma}{2}}\},\\
\widehat{\phi}_k^{(3)} &:= \inf\{t \geq \widehat{\lambda}_k^{(3)} :\ B(t) = B(\widehat{b}_k)\}.
\end{aligned}$$

Again using the strong Markov property of the Brownian motion at $\widehat{\lambda}_k^{(2)}$ and $\widehat{\lambda}_k^{(3)}$ respectively (cf. Remark 3.3.2), we can compute the probability of the dominating events: On the set $\{\widehat{b}_k > 0\}$, we have the following connection to the ruin probability of the Brownian motion (using the strong Markov property at $\widehat{\lambda}_k^{(2)}$)

$$\begin{aligned}
&\mathsf{P}\left(\widehat{\eta}_k^{+} < \widehat{\lambda}_k^{(2)} < \widehat{\eta}_k^{(2)} < \widehat{\phi}_k^{+} \,\middle|\, \widehat{D}_k, B(\widehat{b}_k), B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1})\right)\\
\leq\ &\mathsf{P}\left(\widehat{T}\left(\widehat{D}_k - 2\cdot(\widehat{D}_k)^{1-\frac{\gamma}{2}}\right) < \widehat{T}\left(-(\widehat{D}_k)^{1-\frac{\gamma}{2}}\right) \,\middle|\, \widehat{D}_k, B(\widehat{b}_k), B(\widehat{r}_{k-1}), B(\widehat{\ell}_{k-1})\right)\\
=\ &\frac{(\widehat{D}_k)^{1-\frac{\gamma}{2}}}{\widehat{D}_k - (\widehat{D}_k)^{1-\frac{\gamma}{2}}} = \frac{1}{(\widehat{D}_k)^{\frac{\gamma}{2}} - 1}
\end{aligned}$$

and similarly on the set $\{\widehat{b}_{k+1} > 0\}$ (using the strong Markov property at $\widehat{\lambda}_k^{(3)}$)

$$\begin{aligned}
&\mathsf{P}\left(\widehat{\phi}_k^{+} < \widehat{\lambda}_k^{(3)} < \widehat{\phi}_k^{(3)} < \widehat{\eta}_{k+1}^{+} \,\middle|\, \widehat{H}_k, B(\widehat{b}_k), B(\widehat{r}_k), B(\widehat{\ell}_k)\right)\\
\leq\ &\mathsf{P}\left(\widehat{T}\left(-\widehat{H}_k + (\widehat{H}_k)^{1-\frac{\gamma}{2}}\right) < \widehat{T}\left((\widehat{H}_k)^{1-\frac{\gamma}{2}}\right) \,\middle|\, \widehat{H}_k, B(\widehat{b}_k), B(\widehat{r}_k), B(\widehat{\ell}_k)\right)\\
=\ &\frac{(\widehat{H}_k)^{1-\frac{\gamma}{2}}}{\widehat{H}_k} = \frac{1}{(\widehat{H}_k)^{\frac{\gamma}{2}}}\ .
\end{aligned}$$

Since the last two upper bounds are P-a.s. summable in k due to the exponential growth of $\widehat{D}_k$, $\widehat{H}_k$ in k (cf. (3.34)), we can conclude with the help of the Borel-Cantelli lemma that we P-a.s. have

$$\begin{aligned}
M_k^{(2)} &\stackrel{\text{def}}{=} \min_{\eta_k^{+} \leq x \leq r_k} V(x) \geq V(b_k) + (D_k)^{1-\gamma} && \text{on } \{b_k > 0\},\\
M_k^{(3)} &\stackrel{\text{def}}{=} \max_{\widehat{\phi}_{k,\gamma}^{+} \leq x \leq b_{k+1}} V(x) - V(m_{k+1}) \leq -(H_k)^{1-\gamma} && \text{on } \{b_{k+1} > 0\}
\end{aligned}$$

for all k large enough. This shows (3.47) and (3.48). ■

3.4 Preparation for the proofs

3.4.1 Preliminaries

In the following, we collect some useful properties of the RWRE. For the random time of the first arrival in x

$$T(x) := \inf\{n \geq 0 : \ X_n = x\},$$

the interpretation of the RWRE as an electrical network helps us to compute the following probability for $x < y < z$ (for a proof see for example formula (2.1.4) in [Ze04]):

$$P_\theta^y(T(z) < T(x)) = \frac{\sum\limits_{j=x}^{y-1} e^{V(j)}}{\sum\limits_{j=x}^{z-1} e^{V(j)}} \tag{3.66}$$

Further (cf. (2.4) and (2.5) in [SZ07] and Lemma 7 in [Go84]), we have for $k \in \mathbb{N}$ and $y < z$

$$P_\theta^y(T(z) < k) \leq k \cdot \exp\left(-\max_{y \leq i < z}\left[V(z-1) - V(i)\right]\right) \tag{3.67}$$

and similarly for $x < y$

$$P_\theta^y(T(x) < k) \leq k \cdot \exp\left(-\max_{x < i \leq y}\left[V(x+1) - V(i)\right]\right). \tag{3.68}$$

To get bounds for large values of $T(\cdot)$, we can use the following estimate for $x < y < z$ (cf. Lemma 2.1 in [SZ07]):

$$E_\theta^y[T(z) \cdot \mathbb{1}_{\{T(z)<T(x)\}}] \leq (z-x)^2 \cdot \exp\left(\max_{x \leq i \leq j \leq z}\big(V(j) - V(i)\big)\right) \tag{3.69}$$

3.4.2 Notation

In the following, we introduce some more notation which we need for our proofs:
Recall that $(X_n)_{n \in \mathbb{N}_0}$ is a RWRE on $\mathbb{Z}$. For $x \in \mathbb{Z}$, let

$$\xi(n, x) := |\{0 \leq j \leq n : \ X_j = x\}|$$

denote the local time of the RWRE in x. As in [DGPS07] and [GPS10], it is helpful to decompose the RWRE into excursions away from the bottom points $(b_k)_{k \in \mathbb{N}_0}$. For this decomposition, the following successive return times to $(b_k)_{k \in \mathbb{N}_0}$ are helpful. For $k, \ell \in \mathbb{N}_0$, we define inductively

$$T_\ell^{(k)} := \inf\{i > T_{\ell-1}^{(k)} : \ X_i = b_k\} \quad \text{for } \ell \geq 1 \tag{3.70}$$

where $T_0^{(k)} := 0$ (and $T_1^{(k)} = T(b_k)$) for all k. For $x \in \mathbb{Z}$, we can decompose $\xi(n,x)$ into the number of visits to x before the first arrival in b_k and the visits belonging to the different excursions away from b_k. For this, we define for every $x \in \mathbb{Z}$

$$\begin{aligned}\mathcal{Y}_{b_k,x}^{(0)} &:= |\{0 \le j \le T_1^{(k)} : \ X_j = x\}|,\\ \mathcal{Y}_{b_k,x}^{(n)} &:= |\{T_n^{(k)} < j \le T_{n+1}^{(k)} : \ X_j = x\}| \quad \text{for } n \in \mathbb{N}.\end{aligned}$$

Then, we have for $k, n \in \mathbb{N}_0$, $x \in \mathbb{Z}$

$$\xi(n,x) = \mathcal{Y}_{b_k,x}^{(0)} + \sum_{\ell=1}^{\tau(n,k)} \mathcal{Y}_{b_k,x}^{(\ell)} + |\{T_{\tau(n,k)}^{(k)} < j \le n : \ X_j = x\}| \tag{3.71}$$

where

$$\tau(n,k) := \sup\{\ell \in \mathbb{N}_0 : \ T_\ell^{(k)} \le n\}$$

denotes the number of visits to b_k up to time n. Further, note that $(\mathcal{Y}_{b_k,x}^{(n)})_{n\ge1}$ are i.i.d. random variables with respect to P_θ for all environments θ, $x \in \mathbb{Z}$ and $k \in \mathbb{N}_0$.

For the proofs, the expectation and the variance of the random variable $\mathcal{Y}_{b_k,x}^{(1)}$ for $x \neq b_k$ will play a crucial role. Similarly to Section 3.2 in [DGPS07], we observe that the distribution of $\mathcal{Y}_{b_k,x}^{(1)}$ is almost geometric. More precisely, we have

$$P_\theta(\mathcal{Y}_{b_k,x}^{(1)} = \ell) = \begin{cases} \lambda \cdot (1-\beta)^{\ell-1} \cdot \beta & \ell = 1,2,3,\dots \\ 1-\lambda & \ell = 0, \end{cases}$$

where

$$\begin{aligned}\lambda = \lambda(b_k,x) &:= P_\theta^{b_k}\big(T^+(x) < T^+(b_k)\big),\\ \beta = \beta(b_k,x) &:= P_\theta^{x}\big(T^+(b_k) < T^+(x)\big)\end{aligned}$$

with

$$T^+(x) := \inf\{n > 0 : \ X_n = x\}.$$

In particular, we have

$$E_\theta\big[\mathcal{Y}_{b_k,x}^{(1)}\big] = \frac{\lambda}{\beta} = \frac{\mu_\theta(x)}{\mu_\theta(b_k)} = \frac{e^{-V(x)} + e^{-V(x-1)}}{e^{-V(b_k)} + e^{-V(b_k-1)}}\ , \tag{3.72}$$

where μ_θ is a reversible measure for the Markov chain $(X_n)_{n\in\mathbb{N}_0}$ (cf. (3.9)). Further, we have

$$\mathrm{Var}\big(\mathcal{Y}_{b_k,x}^{(1)}\big) = \frac{\lambda \cdot (2-\beta-\lambda)}{\beta^2} \le \frac{2}{\beta} \cdot \frac{\mu_\theta(x)}{\mu_\theta(b_k)}\ .$$

To get useful upper bounds for the variance, we observe that

$$\beta = \begin{cases} (1-\omega_x) \cdot P_\theta^{x-1}\big(T(b_k) < T(b_x)\big) = (1-\omega_x) \cdot \left(\sum_{y=b_k}^{x-1} e^{V(y)-V(x-1)}\right)^{-1} & \text{if } x > b_k \\ \omega_x \cdot P_\theta^{x+1}\big(T(b_k) < T(b_x)\big) = \omega_x \cdot \left(\sum_{y=x}^{b_k-1} e^{V(y)-V(x)}\right)^{-1} & \text{if } x < b_k, \end{cases}$$

where we used (3.66) for the second step. Using (3.2), this implies

$$\operatorname{Var}\big(\mathcal{Y}^{(1)}_{b_k,x}\big) \le \left\{ \begin{array}{ll} C\cdot e^{-[V(x)-V(b_k)]}\cdot\left(\sum\limits_{y=b_k}^{x-1} e^{V(y)-V(x-1)}\right) & \text{if } x>b_k \\ C\cdot e^{-[V(x)-V(b_k)]}\cdot\left(\sum\limits_{y=x}^{b_k-1} e^{V(y)-V(x)}\right) & \text{if } x<b_k \end{array}\right\}$$
$$\le \begin{cases} C\cdot e^{-[V(x)-V(b_k)]}\cdot(x-b_k)\cdot\exp\left(\max\limits_{b_k\le y\le x-1}\big(V(y)-V(x-1)\big)\right) & \text{if } x>b_k \\ C\cdot e^{-[V(x)-V(b_k)]}\cdot(b_k-x)\cdot\exp\left(\max\limits_{x\le y\le b_k-1}\big(V(y)-V(x)\big)\right) & \text{if } x<b_k \end{cases} \tag{3.73}$$

for some constant $C>0$.

Using the decomposition of the path of the random walk $(X_n)_{n\in\mathbb{N}_0}$ into the different excursions from the bottom point at b_k back to b_k for some $k\in\mathbb{N}_0$, we introduce

$$\begin{aligned} \mathcal{Z}^{(k)}_0 &:= \sum_{x\in\mathcal{D}_k}\alpha_x\cdot\mathcal{Y}^{(0)}_{b_k,x}, \\ \mathcal{Z}^{(k)}_j &:= \sum_{x\in\mathcal{D}_k}\alpha_x\cdot\mathcal{Y}^{(j)}_{b_k,x} \quad \text{for } j\in\mathbb{N} \end{aligned} \tag{3.74}$$

as the accumulated positive and negative orientations which the RWRE collects within the valley $\mathcal{D}_k$ on one excursion. With the help of the random variables $(\mathcal{Z}^{(k)}_j)_{j\in\mathbb{N}_0}$ we have the following decomposition

$$G^{(k)}_n := \mathcal{Z}^{(k)}_0 + \sum_{j=1}^{\tau(n,k)}\mathcal{Z}^{(k)}_j + \Delta^{(k)}_n \tag{3.75}$$

where

$$\Delta^{(k)}_n := \sum_{x\in\mathcal{D}_k}\alpha_x\cdot\big|\{T^{(k)}_{\tau(n,k)}<j\le n:\ X_j=x\}\big|$$

denotes the remainder, i.e. the collected orientations of the last excursion which has not been finished yet. Similarly to the effective width from [DGPS07], we can introduce, for $k\in\mathbb{N}_0$,

$$\Phi^{(k)} := E_\theta\big[\mathcal{Z}^{(k)}_1\big]$$

as the expected accumulated orientations which the RWRE collects on one excursion within the valley $\mathcal{D}_k$. We further define

$$s^{(k)} := \begin{cases} +1 & \text{if } \Phi^{(k)}\ge 0 \\ -1 & \text{if } \Phi^{(k)}<0 \end{cases}$$

as the "sign" of $\Phi^{(k)}$.
Finally, we define

$$R^{(k)} = \inf\{\ell > T^{(k)}_1:\ X_\ell = m_k\} \tag{3.76}$$

as the first time the RWRE leaves the k-th valley via m_k after it has reached the bottom at b_k.

3.4.3 Behaviour of the Drift within the Valleys

Recall that we have introduced

$$\Phi^{(k)} = E_\theta\big[\mathcal{Z}_1^{(k)}\big] = E_\theta\left[\sum_{x\in\mathcal{D}_k} \alpha_x \cdot \mathcal{Y}_{b_k,x}^{(1)}\right] = \sum_{x\in\mathcal{D}_k} \alpha_x \cdot \frac{\mu_\theta(x)}{\mu_\theta(b_k)}$$

as the expected accumulated orientations which the RWRE collects on one excursion away from the bottom point at b_k in the valley $\mathcal{D}_k$. This is why we can think of the quantity $\Phi^{(k)}$ as a drift the environment θ possesses in the valley $\mathcal{D}_k$. In particular, it is more likely that the RWRO collects more positive orientations as long as it moves in a valley $\mathcal{D}_k$ with drift $\Phi^{(k)} > 0$ and more negative orientations if we have $\Phi^{(k)} < 0$. In our next proposition, we show that, for all ℓ large enough, each of the valleys $(\mathcal{D}_k)_{k\geq\ell}$ possesses a drift which is different from 0 due to the inhomogeneity of our valleys. Additionally, the drift $\Phi^{(k)}$ does not decrease to 0 too fast in k which will be useful for the proofs of Theorem 3.2.1 and 3.2.2.

Proposition 3.4.1. *Let $0 < \gamma < 1$. Then, we have for P-a.e. environment $\theta = (\omega_x, \alpha_x)_{x\in\mathbb{Z}}$*

$$\big|\Phi^{(k)}\big| = \left|\sum_{x\in\mathcal{D}_k} \frac{\mu_\theta(x)}{\mu_\theta(b_k)}\alpha_x\right| \geq \exp\big(-(D_k)^\gamma\big) \tag{3.77}$$

for all $k = k(\theta,\gamma)$ large enough.

Proof of Proposition 3.4.1. At first, we fix $0 < \gamma < 1$. For an arbitrary $k \in \mathbb{N}$, we now decompose the k-th valley $\mathcal{D}_k$ into disjoint sets depending on how "deep" the positions lie in the valley. For this, we define

$$I_j^{(k)} := \left\{x \in \mathcal{D}_k :\ \exp\left(-j\cdot(D_k)^{\frac{\gamma}{2}}\right) \leq \frac{\mu_\theta(x)}{\mu_\theta(b_k)} < \exp\left(-(j-1)\cdot(D_k)^{\frac{\gamma}{2}}\right)\right\} \quad \text{for } j\in\mathbb{N},$$

$$I_0^{(k)} := \mathcal{D}_k \cap \left(\bigcup_{j=1}^{\infty} I_j^{(k)}\right)^c.$$

The idea for the proof is the following: Since the contribution of positions in $I_j^{(k)}$ to $\Phi^{(k)}$ decays exponentially in j, our goal is to find some $j \in \mathbb{N}$ such that

$$\left|\sum_{x\in\bigcup_{\ell=0}^{j} I_\ell^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)}\alpha_x\right| > \exp\left(-(j-1)\cdot(D_k)^{\frac{\gamma}{2}}\right) - \sum_{x\in\bigcup_{\ell=j+1}^{\infty} I_\ell^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)},$$

i.e. the advance of a positive or negative drift of the lower part of $\mathcal{D}_k$ is large enough such that it cannot be compensated again by the upper part of $\mathcal{D}_k$ for large k. Therefore, we define for $j \in \mathbb{N}$

$$s_j := \begin{cases} +1 & \text{if } \displaystyle\sum_{x\in\bigcup_{\ell=0}^{j} I_\ell^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)}\alpha_x \geq 0 \\ -1 & \text{if } \displaystyle\sum_{x\in\bigcup_{\ell=0}^{j} I_\ell^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)}\alpha_x < 0 \end{cases}$$

as the sign of the drift of the lower part of $\mathcal{D}_k$ up to $I_j^{(k)}$.

Further, we define for $j \in 2\mathbb{N}$

$$A_j^{(k)} = \left\{ \theta : \; s_{j-1} \cdot \sum_{x \in I_j^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x \geq \exp\left(-j \cdot (D_k)^{\frac{\gamma}{2}} \right) \right\} \cap \left\{ \theta : \; s_{j-1} \cdot \sum_{x \in I_{j+1}^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x \geq 0 \right\}$$

on which the contribution of $I_j^{(k)}$ and $I_{j+1}^{(k)}$ to the drift in particular has the same direction as s_{j-1}. We first notice that due to assumption (3.2) and (3.34) we have for P-a.e. $(\omega_x)_{x \in \mathbb{Z}}$ and all $k = k(\gamma, \theta)$ large enough that $I_j^{(k)} \neq \emptyset$ holds for $1 \leq j \leq 2 \cdot \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor$. Due to the symmetry of the sequence $(\alpha_x)_{x \in \mathbb{Z}}$ with regard to 0 and using that $(\alpha_x)_{x \in \mathbb{Z}}$ and $(\omega_x)_{x \in \mathbb{Z}}$ are two independent sequences of i.i.d. random variables, we therefore have for $1 \leq j \leq 2 \cdot \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor - 1$

$$\mathsf{P}\left(\left\{ \theta : \; s_{j-1} \cdot \sum_{x \in I_{j+1}^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x \geq 0 \right\} \middle| \, (\omega_x)_{x \in \mathbb{Z}} \right) \geq \frac{1}{2} \tag{3.78}$$

$$\text{on } \left\{ (\omega_x)_{x \in \mathbb{Z}} : \; I_j^{(k)} \neq \emptyset \text{ for } 1 \leq j \leq 2 \cdot \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor \right\}.$$

Further, let $i_j^{(k)}$ be the smallest element of $I_j^{(k)}$ (as long as $I_j^{(k)} \neq \emptyset$). Then, we have for $1 \leq j \leq 2 \cdot \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor$

$$\begin{aligned}
&\mathsf{P}\left(\left\{ \theta : \; s_{j-1} \cdot \sum_{x \in I_j^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x \geq \exp\left(-j \cdot (D_k)^{\frac{\gamma}{2}} \right) \right\} \middle| \, (\omega_x)_{x \in \mathbb{Z}} \right) \\
\geq \; &\mathsf{P}\left(\mathsf{P}\left(\{\alpha_{i_j^{(k)}} = s_{j-1}\} \cap \left\{ \theta : \; s_{j-1} \cdot \sum_{x \in I_j^{(k)} \setminus \{i_j^{(k)}\}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x \geq 0 \right\} \middle| \, s_{j-1}, (\omega_x)_{x \in \mathbb{Z}} \right) \middle| \, (\omega_x)_{x \in \mathbb{Z}} \right) \\
\geq \; &\frac{1}{4}
\end{aligned} \tag{3.79}$$

$$\text{on } \left\{ (\omega_x)_{x \in \mathbb{Z}} : \; I_j^{(k)} \neq \emptyset \text{ for } 1 \leq j \leq 2 \cdot \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor \right\}.$$

A combination of (3.78) and (3.79) and again using that $(\alpha_x)_{x \in \mathbb{Z}}$ and $(\omega_x)_{x \in \mathbb{Z}}$ are two independent sequences of i.i.d. random variables now yields

$$\begin{aligned}
&\mathsf{P}\left(\bigcap_{j=1,\ldots,\lfloor (D_k)^{\frac{\gamma}{4}} \rfloor} \left(A_{2j}^{(k)} \right)^c \middle| \, (\omega_x)_{x \in \mathbb{Z}} \right) \\
= \; &\mathsf{P}\left(\mathsf{P}\left(\bigcap_{j=1,\ldots,\lfloor (D_k)^{\frac{\gamma}{4}} \rfloor} \left(A_{2j}^{(k)} \right)^c \middle| \, (\alpha_x)_{x \in I_j^{(k)} \text{ for } j \leq 2 \cdot \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor - 2}, (\omega_x)_{x \in \mathbb{Z}} \right) \middle| \, (\omega_x)_{x \in \mathbb{Z}} \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\frac{1}{8}\right) \cdot \mathsf{P}\left(\bigcap_{j=1,\dots,\lfloor (D_k)^{\frac{\gamma}{4}} \rfloor -1} \left(A_{2j}^{(k)}\right)^c \middle| (\omega_x)_{x\in\mathbb{Z}} \right) \\
&\leq \left(\frac{7}{8}\right)^{\lfloor (D_k)^{\frac{\gamma}{4}} \rfloor} && (3.80)
\end{aligned}
$$

$$
\text{on } \left\{ (\omega_x)_{x\in\mathbb{Z}} : \ I_j^{(k)} \neq \emptyset \text{ for } 1 \leq j \leq 2\cdot \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor \right\}
$$

where we iterated the first two steps $\lfloor (D_k)^{\frac{\gamma}{4}} \rfloor - 1$-times for the last step.

Note here that on the set

$$
A_{2j}^{(k)} \cap \left\{ (\omega_x)_{x\in\mathbb{Z}} : \ I_j^{(k)} \neq \emptyset \text{ for } 1 \leq j \leq 2\cdot \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor \right\}
$$

we have by construction for $1 \leq j \leq \lfloor (D_k)^{\frac{\gamma}{4}} \rfloor$

$$
\begin{aligned}
\left| \sum_{x\in\mathcal{D}_k} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x \right| &= \left| \sum_{x\in \bigcup_{\ell=0}^{2j-1} I_j^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x + \sum_{x\in \bigcup_{\ell=2j}^{2j+1} I_j^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x + \sum_{x\in \bigcup_{\ell=2j+2}^{\infty} I_j^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x \right| \\
&\geq \left| \sum_{x\in I_{2j}^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \alpha_x \right| - \sum_{x\in \bigcup_{\ell=2j+2}^{\infty} I_j^{(k)}} \frac{\mu_\theta(x)}{\mu_\theta(b_k)} \\
&\geq \exp\left(-2j\cdot (D_k)^{\frac{\gamma}{2}}\right) - |\mathcal{D}_k| \cdot \exp\left(-(2j+1)\cdot (D_k)^{\frac{\gamma}{2}}\right) \\
&\geq \exp\left(-2j\cdot (D_k)^{\frac{\gamma}{2}}\right) \cdot \left(1 - (H_k)^{2+\gamma}\cdot \exp\left(-(D_k)^{\frac{\gamma}{2}}\right)\right) \\
&\geq \exp\left(-(D_k)^{\gamma}\right)
\end{aligned}
$$

for P-a.e. environment θ and $k = k(\gamma,\theta)$ large enough. Here, we applied (3.37) and (3.38) for the fourth step. Further, we used (3.35) together with the exponential growth if D_k in k (cf. (3.34)) for the last step.

Therefore, (3.80) and the Borel-Cantelli lemma finally imply (3.77) since the upper bound in (3.80) is summable in k for P-a.e. θ again due to the exponential growth of D_k in k. ■

3.4.4 Strong Localization of the RWRE

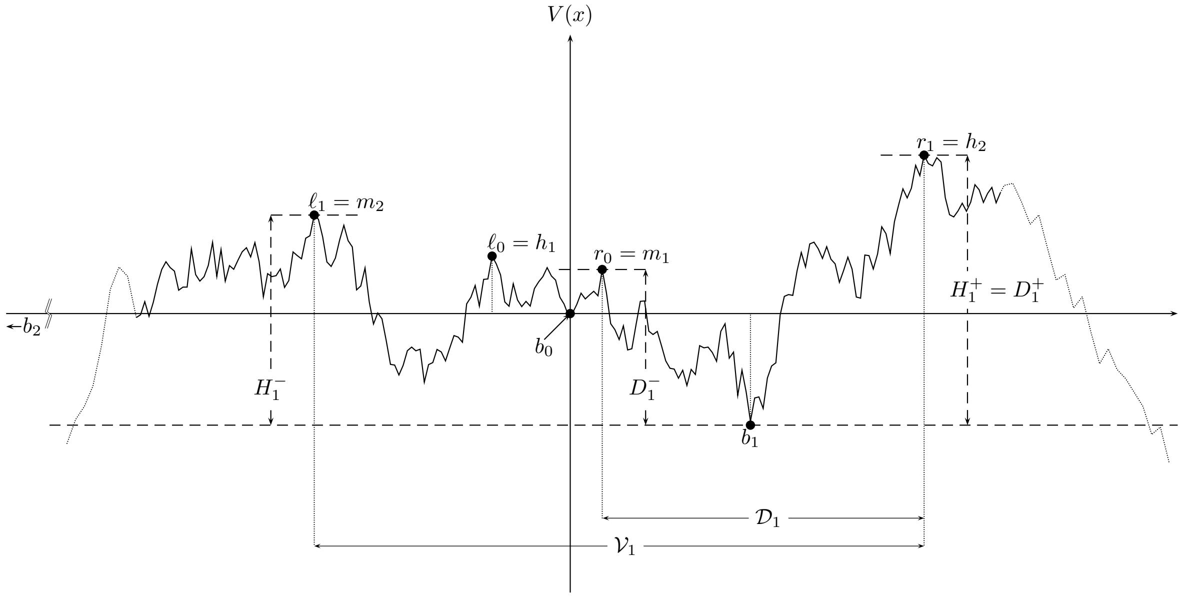

Figure 3.5: *Sampled potential $V(\cdot)$ and the corresponding location of the valleys $\mathcal{V}_1$ and $\mathcal{D}_1$ from Figure 3.4 on page 58.*

In the following, we derive some results on the strong localization of the RWRE in the form that we will use for the proofs of Theorem 3.2.1 and 3.2.2. Strong localization refers to the property of the RWRE to spend most of the time around the bottom point of the deepest valley it has visited so far. Therefore, we introduce, for $n \in \mathbb{N}_0$,

$$N_n := \begin{cases} \sup\{\ell \in \mathbb{N}_0 : \ X_j = b_\ell \text{ for some } j \le n\} & \text{if } \{\ell \in \mathbb{N}_0 : \ X_j = b_\ell \text{ for some } j \le n\} \neq \emptyset \\ -1 & \text{else} \end{cases} \tag{3.81}$$

as the random index of the deepest bottom point $(b_k)_{k\in\mathbb{N}_0}$ the RWRE has visited up to time n. Now, we can introduce the following sets which turn out to describe the typical behaviour of the RWRE for large time points. For arbitrary $0 < \beta, \gamma < 1$, which we choose later (cf. (3.141)), we define (cf. (3.70), (3.76), and (3.75) for the definition of $T_1^{(k)}$, $R^{(k)}$, and $G_n^{(k)}$):

$$B_1^{(k)} := \left\{\forall n \ge k : \ H_{N_n-1} \le \frac{1}{1-\beta}\log n\right\} \cap \left\{N_n \xrightarrow{n\to\infty} \infty\right\},$$

$$B_2^{(k)} := \left\{\forall n \le T_1^{(k+1)} : \sum_{x\in\mathbb{Z}\setminus(\mathcal{V}_k\cup\mathcal{D}_{k+1})} \xi(n,x) = 0\right\},$$

$$B_3^{(k)} := \left\{\forall n \ge R^{(k)} : \sum_{x\in\mathcal{V}_k\setminus\mathcal{D}_k} \xi(n,x) \le \exp\left(-(H_{k-1})^{1-\gamma}\right)\cdot\xi(n,b_k)\right\},$$

$$B_4^{(k)} := \left\{\forall n \le T_1^{(k+1)} \text{ with } X_n \notin \mathcal{D}_{k+1} : \sum_{x\in\mathcal{D}_{k+1}} \xi(n,x) \le \exp\left(-(H_{k-1})^{1-\gamma}\right)\cdot\xi(n,b_k)\right\},$$

$$B_5^{(k)} := \left\{ \forall n \geq R^{(k)} :\ 12 \cdot (H_k)^{2+\gamma} \cdot \xi(n, b_k) \geq \sum_{x \in \mathcal{V}_k} \xi(n, x) \right\},$$

$$B_6^{(k)} := \left\{ \forall n \geq R^{(k)} :\ (1-\beta) \cdot s^{(k)} \cdot \Phi^{(k)} \cdot \xi(n, b_k) < s^{(k)} \cdot G_n^{(k)} \right\},$$

$$F^{(k)} := \bigcap_{\ell=k}^{\infty} \left(B_1^{(\ell)} \cap B_2^{(\ell)} \cap B_3^{(\ell)} \cap B_4^{(\ell)} \cap B_5^{(\ell)} \cap B_6^{(\ell)} \right).$$

Here, $B_1^{(k)}$ gives us an upper bound for the height of the valley $\mathcal{V}_{N_n-1}$ – which is the second deepest of the valleys $(\mathcal{V}_k)_{k\in\mathbb{N}_0}$ in which the RWRE has visited the bottom point b_k up to time n – after time k. On the intersection of $(B_2^{(k)})_{k\geq\ell}$, $(B_3^{(k)})_{k\geq\ell}$, and $(B_4^{(k)})_{k\geq\ell}$, the RWRE prefers to spend most of the time in the valley $\mathcal{D}_{N_n}$ for large n, i.e. the deepest of the valleys $(\mathcal{D}_k)_{k\in\mathbb{N}_0}$ in which the RWRE has visited the bottom point b_k up to time n. Since the bottom point b_k plays a key role in our consideration, it will be useful to know how often the RWRE has visited the bottom point b_k of some valley $\mathcal{D}_k$ before it leaves $\mathcal{D}_k$ via m_k at time $R^{(k)}$ again. A lower bound for this behaviour is contained in $B_5^{(k)}$. Finally, the behaviour on $B_6^{(k)}$ tells us that the accumulated $+1/-1$-orientations $G_n^{(k)}$, which the RWRE collects in the valley $\mathcal{D}_k$ before it leaves it via m_k at time $R^{(k)}$ again, behaves – at least in the lower bound – almost like the number of visits to the bottom point b_k times the expected accumulated $+1/-1$-orientations during one excursion away from b_k to b_k which is denoted by $\Phi^{(k)}$.

One key tool for our proofs of Theorem 3.2.1 and 3.2.2 is the following proposition which tells us that the RWRE will have the properties of the above introduced sets for all but finitely many k:

Proposition 3.4.2. *For* P*-a.e. environment* θ*, we have*

$$P_\theta\left(\liminf_{k\to\infty} F^{(k)}\right) = \lim_{k\to\infty} P_\theta\left(F^{(k)}\right) = 1. \tag{3.82}$$

Remark 3.4.3. Proposition 3.4.2 could easily be extended to show that for large time points n our RWRE has spent most of its time in the valleys $\mathcal{D}_{N_n}$ and $\mathcal{D}_{N_n-1}$ which are (for large n) the last two valleys of the sequence $(D_k)_{k\in\mathbb{N}_0}$ the RWRE has visited up to time n. A similar statement for the RWRE on the positive half-line has been shown in Theorem 3.4 in [DGPS07]. For our proof, we need a slightly different information: Due to the definition of $B_3^{(k)}$ for $k\in\mathbb{N}_0$, we see that the RWRE – when returning to 0 at some large time point n – has spent most of its time in the single valley $\mathcal{D}_{N_n} \subsetneq \mathcal{V}_{N_n}$, i.e. the deepest valley of the sequence $(D_k)_{k\in\mathbb{N}_0}$ in which the RWRE has also visited the bottom point b_{N_n}. Note for this that according to Proposition 3.3.9 we a.s. have $0 \notin \mathcal{D}_k$ for all k large enough which implies that the RWRE has to leave the valley $\mathcal{D}_{N_n}$ at time $R^{(N_n)}$ again before it can return to 0.

Proof of Proposition 3.4.2. For the proof, we show the following lemmata:

Lemma 3.4.4. *For* $\ell \in \mathbb{N}_0$*,* $\beta > 0$*, and* $0<\gamma<1$*, we have for* P*-a.e. environment* θ

$$P_\theta\Big(T(b_\ell) < \exp\left((1-\beta)\cdot H_{\ell-1}\right)\Big) \leq 2\cdot\exp\left(-(H_{\ell-3})^{1-\gamma}\right) \tag{3.83}$$

for all $\ell = \ell(\beta, \gamma, \theta)$ large enough. In particular, we have

$$\lim_{k\to\infty} P_\theta\left(B_1^{(k)}\right) = P_\theta\left(\liminf_{k\to\infty} B_1^{(k)}\right) = 1 \tag{3.84}$$

for P*-a.e. environment* θ*.*

Lemma 3.4.5. *For $\ell \in \mathbb{N}_0$ and $0 < \gamma < 1$, we have for* P*-a.e. environment* θ

$$P_\theta\big(T(b_{\ell+1}) \geq T(h_{\ell+1})\big) \leq \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \tag{3.85}$$

for all $\ell = \ell(\gamma, \theta)$ large enough. In particular, we have

$$\lim_{k\to\infty} P_\theta\left(\bigcap_{\ell=k}^{\infty} B_2^{(k)}\right) = P_\theta\left(\liminf_{k\to\infty} B_2^{(k)}\right) = 1 \tag{3.86}$$

for P*-a.e. environment* θ*.*

Lemma 3.4.6. *For $\ell \in \mathbb{N}_0$, $x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell$, and $0 < \gamma < 1$, we have for* P*-a.e. environment* θ

$$\begin{aligned} &P_\theta\Big(\exists n \geq R^{(\ell)} :\ \xi(n,x) > \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \cdot \xi(n, b_\ell)\Big) \\ &\leq\ \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \end{aligned} \tag{3.87}$$

for all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). In particular, we have

$$\lim_{k\to\infty} P_\theta\left(\bigcap_{\ell=k}^{\infty} B_3^{(\ell)}\right) = P_\theta\left(\liminf_{k\to\infty} B_3^{(k)}\right) = 1 \tag{3.88}$$

for P*-a.e. environment* θ*.*

Lemma 3.4.7. *For $\ell \in \mathbb{N}_0$, $x \in \mathcal{D}_{\ell+1}$, and $0 < \gamma < 1$, we have for* P*-a.e. environment* θ

$$\begin{aligned} &P_\theta\left(\exists n \leq T_1^{(\ell+1)} :\ X_n \notin \mathcal{D}_{\ell+1},\ \xi(n,x) > \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \cdot \xi(n, b_\ell)\right) \\ &\leq\ \exp\big(-(H_{\ell-2})^{1-\gamma}\big) \end{aligned} \tag{3.89}$$

for all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). In particular, we have

$$\lim_{k\to\infty} P_\theta\left(\bigcap_{\ell=k}^{\infty} B_4^{(k)}\right) = P_\theta\left(\liminf_{k\to\infty} B_4^{(k)}\right) = 1 \tag{3.90}$$

for P*-a.e. environment* θ*.*

Lemma 3.4.8. *For $\ell \in \mathbb{N}_0$, $x \in \mathcal{V}_\ell$, and $0 < \beta, \gamma < 1$, we have for* P*-a.e. environment* θ

$$P_\theta\big(\exists n \geq R^{(\ell)} :\ \xi(n,x) \geq 6 \cdot \xi(n, b_\ell)\big) \leq \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \tag{3.91}$$

for all $\ell = \ell(\beta, \gamma, \theta)$ large enough (uniformly in x). In particular, we have

$$\lim_{k\to\infty} P_\theta\left(\bigcap_{\ell=k}^{\infty} B_5^{(\ell)}\right) = P_\theta\left(\liminf_{k\to\infty} B_5^{(k)}\right) = 1 \tag{3.92}$$

for P*-a.e. environment* θ*.*

Lemma 3.4.9. *For $\ell \in \mathbb{N}_0$ and $0 < \beta, \gamma < 1$, we have for* P*-a.e. environment* θ

$$P_\theta\left(\left\{\exists n \geq R^{(\ell)} :\ (1-\beta)\cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \xi(n, b_\ell) \geq s^{(\ell)} \cdot G_n^{(\ell)}\right\}\right) \leq \exp\left(-\frac{1}{4}(H_{\ell-1})^{1-\gamma}\right) \tag{3.93}$$

for all $\ell = \ell(\beta, \gamma, \theta)$ large enough. In particular, we have

$$\lim_{k\to\infty} P_\theta\left(\bigcap_{\ell=k}^{\infty} B_6^{(\ell)}\right) = P_\theta\left(\liminf_{k\to\infty} B_6^{(k)}\right) = 1 \tag{3.94}$$

for P*-a.e. environment* θ*.*

Proof of Lemma 3.4.4. At first we fix $0 < \beta, \gamma < 1$. Further, notice that, for P-a.e. environment θ, we have

$$\lim_{n\to\infty} N_n = \infty \tag{3.95}$$

P_θ-a.s. since $(X_n)_{n\in\mathbb{N}_0}$ is recurrent for P-a.e. environment θ due to assumption (3.1).

For the proof of (3.83), we notice that

$$\begin{aligned} &P_\theta\Big(T(b_\ell) < \exp\big((1-\beta)\cdot H_{\ell-1}\big)\Big) \\ &\leq P_\theta^{b_{\ell-1}}\Big(T(b_\ell) < \exp\big((1-\beta)\cdot H_{\ell-1}\big)\Big) + P_\theta\Big(T(b_{\ell-1}) \geq T(m_\ell)\Big). \end{aligned} \tag{3.96}$$

for P-a.e. environment θ and all $\ell = \ell(\theta)$ large enough due to (3.30). For the first summand, we get

$$\begin{aligned} &P_\theta^{b_{\ell-1}}\Big(T(b_\ell) < \exp\big((1-\beta)\cdot H_{\ell-1}\big)\Big) \leq P_\theta^{b_{\ell-1}}\Big(T(m_\ell) < \exp\big((1-\beta)\cdot H_{\ell-1}\big)\Big) \\ &\leq \left\{\begin{array}{ll} \exp\big((1-\beta)\cdot H_{\ell-1}\big)\cdot\exp\left(-\max\limits_{b_{\ell-1}\leq i<m_\ell}\big(V(m_\ell-1)-V(i)\big)\right) & \text{if } b_\ell > 0 \\ \exp\big((1-\beta)\cdot H_{\ell-1}\big)\cdot\exp\left(-\max\limits_{m_\ell<i\leq b_{\ell-1}}\big(V(m_\ell+1)-V(i)\big)\right) & \text{if } b_\ell < 0 \end{array}\right\} \\ &\leq \exp\big((1-\beta)\cdot H_{\ell-1}\big)\cdot\frac{1}{\varepsilon}\cdot\exp\big(-\big(V(m_\ell)-V(b_{\ell-1})\big)\big) \\ &= \frac{1}{\varepsilon}\cdot\exp\big(-\beta\cdot H_{\ell-1}\big) \end{aligned} \tag{3.97}$$

for P-a.e. environment θ and all $\ell = \ell(\theta)$ large enough. Here, we used the definition of m_ℓ for the first step. In the second line, we made use of (3.67) and (3.68), where we have $b_{\ell-1} < m_\ell < b_\ell$ if $b_\ell > 0$ and $b_\ell < m_\ell < b_{\ell-1}$ if $b_\ell < 0$ (for P-a.e. environment θ and for all $\ell = \ell(\theta)$ large enough). Further, we applied (3.2) and (3.33) in the last two lines.

Now we turn to the second summand in (3.96): First of all, we notice that

$$P_\theta\Big(T(b_{\ell-1}) \geq T(m_\ell)\Big) = \left\{\begin{array}{ll} 0 & \text{if } 0 < b_{\ell-1} < m_\ell \\ 0 & \text{if } m_\ell > b_{\ell-1} > 0. \end{array}\right.$$

Therefore, we only have to consider the remaining possibilities which are left for all ℓ large enough (cf. (3.30)). By using (3.66), we get

$$\begin{aligned}
P_\theta\Big(T(b_{\ell-1}) \geq T(m_\ell)\Big) &= \left\{ \begin{array}{ll} \dfrac{\sum_{j=b_{\ell-1}}^{-1} e^{V(j)}}{\sum_{j=b_{\ell-1}}^{m_\ell-1} e^{V(j)}} & \text{if } b_{\ell-1} < 0 < m_\ell \\ \dfrac{\sum_{j=0}^{b_{\ell-1}-1} e^{V(j)}}{\sum_{j=m_\ell}^{b_{\ell-1}-1} e^{V(j)}} & \text{if } b_{\ell-1} > 0 > m_\ell \end{array} \right\} \\
&\leq \left\{ \begin{array}{ll} -b_{\ell-1} \cdot \exp\Big(\max_{b_{\ell-1} \leq j \leq -1} V(j)\Big) \cdot \exp\Big(-V(m_\ell - 1)\Big) & \text{if } b_{\ell-1} < 0 < m_\ell \\ b_{\ell-1} \cdot \exp\Big(\max_{0 \leq j \leq b_{\ell-1}-1} V(j)\Big) \cdot \exp\Big(-V(m_\ell)\Big) & \text{if } b_{\ell-1} > 0 > m_\ell \end{array} \right\} \\
&\leq \left\{ \begin{array}{ll} \frac{1}{\varepsilon} \cdot |b_{\ell-1}| \cdot \exp\Big(V(m_{\ell-1}) - V(r_{\ell-1})\Big) & \text{if } b_{\ell-1} < 0 < m_\ell \\ |b_{\ell-1}| \cdot \exp\Big(V(m_{\ell-1}) - V(\ell_{\ell-1})\Big) & \text{if } b_{\ell-1} > 0 > m_\ell \end{array} \right\} \\
&\leq \frac{1}{\varepsilon} \cdot |b_{\ell-1}| \cdot \exp\Big(V(m_{\ell-1}) - V(h_{\ell-1})\Big) \leq \frac{1}{\varepsilon} \cdot (H_{\ell-1})^{2+\gamma} \cdot \exp\big(-(H_{\ell-3})^{1-\frac{\gamma}{2}}\big) \\
&\leq \frac{1}{\varepsilon} \cdot \big((\ell-1)^{1+\gamma} \cdot (\ell-2)^{1+\gamma} \cdot H_{\ell-3}\big)^{2+\gamma} \cdot \exp\big(-(H_{\ell-3})^{1-\frac{\gamma}{2}}\big) \\
&\leq \exp\big(-(H_{\ell-3})^{1-\gamma}\big)
\end{aligned} \tag{3.98}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough. Here, we used (3.30), (3.31), and (3.2) in the third line. For the fourth line, notice that $V(r_{\ell-1}) \geq V(r_{\ell-2})$ and $V(\ell_{\ell-1}) \geq V(\ell_{\ell-2})$, where further $h_{\ell-1} = r_{\ell-2}$ if $b_{\ell-1} < 0$ and $h_{\ell-1} = \ell_{\ell-2}$ if $b_{\ell-1} > 0$ by definition. For the second step in this line, we applied (3.39) and further (3.36) for $\frac{\gamma}{2}$. Finally, we used (3.35) in the last line and the exponential growth of H_ℓ in ℓ (cf. (3.34)) for the last inequality.

By applying the two upper bounds in (3.97) and (3.98) to (3.96), we see that

$$\begin{aligned}
&P_\theta\Big(T(b_\ell) < \exp\big((1-\beta) \cdot H_{\ell-1}\big)\Big) \\
&\leq \frac{1}{\varepsilon} \cdot \exp\big(-\beta \cdot H_{\ell-1}\big) + \exp\big(-(H_{\ell-3})^{1-\gamma}\big) \leq 2 \cdot \exp\big(-(H_{\ell-3})^{1-\gamma}\big)
\end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\beta, \gamma, \theta)$ large enough. This shows (3.83).

Since the last upper bound is summable in ℓ for P-a.e. environment θ (again due to the exponential growth of H_ℓ in ℓ), we can apply the Borel-Cantelli lemma to conclude that

$$\begin{aligned}
1 &= P_\theta\Big(\liminf_{\ell\to\infty} \big\{T(b_\ell) \geq \exp\big((1-\beta) \cdot H_{\ell-1}\big)\big\}\Big) \\
&= \lim_{\ell\to\infty} P_\theta\big(\big\{\forall n \geq \ell:\ T(b_n) \geq \exp\big((1-\beta) \cdot H_{n-1}\big)\big\}\big)
\end{aligned}$$

$$\leq \lim_{k\to\infty} P_\theta\left(\left\{\forall n \geq k: \; H_{N_n-1} \leq \frac{1}{1-\beta}\log n\right\}\right)$$

holds for P-a.e. environment θ. The last step can be seen by assuming $H_{N_n-1} > \frac{1}{1-\beta}\log n$ which implies $T(b_{N_n}) \leq n < \exp\left((1-\beta)\cdot H_{N_n-1}\right)$, where further $N_n \to \infty$ P_θ-a.s. for P-a.e. environment θ (cf. (3.95)). In particular, we have shown (3.84). $\square$

Proof of Lemma 3.4.5. At first we fix $0 < \gamma < 1$. Then, we recall (cf. Proposition 3.3.9) that we have

$$m_{\ell+1} = \begin{cases} r_\ell & \text{if } b_{\ell+1} > 0 \\ \ell_\ell & \text{if } b_{\ell+1} < 0 \end{cases}$$

for P-a.e. environment θ and all $\ell = \ell(\theta)$ large enough. Further, we have by definition (cf. (3.28))

$$h_{\ell+1} = \begin{cases} \ell_\ell & \text{if } b_{\ell+1} > 0 \\ r_\ell & \text{if } b_{\ell+1} < 0. \end{cases}$$

Therefore, we conclude using the definition of $\mathcal{V}_\ell$ and $\mathcal{D}_{\ell+1}$ (cf. (3.29)) that

$$\mathcal{V}_\ell \cup \mathcal{D}_{\ell+1} = \begin{cases} \{\ell_\ell = h_{\ell+1}, h_{\ell+1}+1, \dots, m_{\ell+1}, \dots, b_{\ell+1}, \dots, r_{\ell+1}\} & \text{if } b_{\ell+1} > 0 \\ \{\ell_{\ell+1}, \ell_{\ell+1}+1, \dots, b_{\ell+1}, \dots, m_{\ell+1}, \dots, r_\ell = h_{\ell+1}\} & \text{if } b_{\ell+1} < 0 \end{cases}$$

for P-a.e. environment θ and all $\ell = \ell(\theta)$ large enough. This yields that

$$P_\theta\left(\left(B_2^{(\ell)}\right)^c\right) \leq P_\theta\big(T(b_{\ell+1}) \geq T(h_{\ell+1})\big) = \left\{\begin{matrix} \dfrac{\sum\limits_{j=b_{\ell+1}}^{-1} e^{V(j)}}{\sum\limits_{j=b_{\ell+1}}^{h_{\ell+1}-1} e^{V(j)}} & \text{if } b_{\ell+1} < 0 < h_{\ell+1} \\ \dfrac{\sum\limits_{j=0}^{b_{\ell+1}-1} e^{V(j)}}{\sum\limits_{j=h_{\ell+1}}^{b_{\ell+1}-1} e^{V(j)}} & \text{if } b_{\ell+1} > 0 > h_{\ell+1} \end{matrix}\right\}$$

$$\leq \left\{\begin{matrix} -b_{\ell+1}\cdot\exp\left(\max\limits_{b_{\ell+1}\leq j\leq -1} V(j)\right)\cdot\exp\big(-V(h_{\ell+1}-1)\big) & \text{if } b_{\ell+1} < 0 < h_{\ell+1} \\ b_{\ell+1}\cdot\exp\left(\max\limits_{0\leq j\leq b_{\ell+1}-1} V(j)\right)\cdot\exp\big(-V(h_{\ell+1})\big) & \text{if } b_{\ell+1} > 0 > h_{\ell+1} \end{matrix}\right\}$$

$$\leq \frac{1}{\varepsilon}\cdot(H_{\ell+1})^{2+\gamma}\cdot\exp\big(V(m_{\ell+1}) - V(h_{\ell+1})\big) \leq \frac{1}{\varepsilon}\cdot(H_{\ell+1})^{2+\gamma}\cdot\exp\left(-(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)$$

$$\leq \frac{1}{\varepsilon}\cdot\left((\ell+1)^{1+\gamma}\cdot\ell^{1+\gamma}\cdot H_{\ell-1}\right)^{2+\gamma}\cdot\exp\left(-(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \leq \exp\left(-(H_{\ell-1})^{1-\gamma}\right)$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough. Here in the third line, we used (3.31), (3.39), and (3.2). Further we made use of (3.36) (applied for $\frac{\gamma}{2}$). Finally, we used (3.34) and (3.35) for the last line. In particular, this shows (3.85).

To finish the proof, we can use the last inequality to conclude that, for P-a.e. environment θ and all $k = k(\gamma, \theta)$ large enough, we have

$$1 - P_\theta\left(\bigcap_{\ell=k}^{\infty} B_2^{(\ell)}\right) = P_\theta\left(\bigcup_{\ell=k}^{\infty}\left(B_2^{(\ell)}\right)^c\right) \leq \sum_{\ell=k}^{\infty} \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \xrightarrow{k\to\infty} 0,$$

where we used the exponential growth of H_ℓ in ℓ (cf. (3.34)) for the convergence for $k \to \infty$.
$\square$

Proof of Lemma 3.4.6. Here, we are in similar situation as in Lemma 4.3 in [DGPS07], and therefore we can use a similar approach:

We first fix $0 < \gamma < 1$ and an environment θ. Then, we choose an arbitrary $x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell$. Using the decomposition of the visits to x with respect to the excursions of the RWRE away from b_ℓ (cf. (3.71)), we see that we have

$$\xi(n, x) \leq \mathcal{Y}_{b_\ell, x}^{(0)} + \sum_{j=1}^{\tau(n,\ell)+1} \mathcal{Y}_{b_\ell, x}^{(j)}.$$

Recall that we have (cf. (3.73)), for some constant $C > 0$,

$$\begin{aligned}\mathrm{Var}\big(\mathcal{Y}_{b_\ell,x}^{(1)}\big) &\leq \left\{\begin{array}{ll} C \cdot e^{-[V(x)-V(b_\ell)]} \cdot (x - b_\ell) \cdot \exp\left(\max\limits_{b_\ell \leq y \leq x-1}\big(V(y) - V(x-1)\big)\right) & \text{if } x > b_\ell \\ C \cdot e^{-[V(x)-V(b_\ell)]} \cdot (b_\ell - x) \cdot \exp\left(\max\limits_{x \leq y \leq b_\ell - 1}\big(V(y) - V(x)\big)\right) & \text{if } x < b_\ell \end{array}\right\} \\ &= C \cdot e^{-[V(x)-V(b_\ell)]} \cdot |x - b_\ell| \cdot \exp\big(f_\ell(x)\big), \end{aligned} \tag{3.99}$$

where

$$f_\ell(x) := \begin{cases} \max\limits_{b_\ell \leq y \leq x-1}\big(V(y) - V(x-1)\big) & \text{if } x > b_\ell \\ \max\limits_{x \leq y \leq b_\ell - 1}\big(V(y) - V(x)\big) & \text{if } x < b_\ell. \end{cases} \tag{3.100}$$

$\exp\big(f_\ell(x)\big)$ is the factor of the variance which we have to control in the following. It measures how hard it is for the RWRE to get from x to b_ℓ.

At this point we have to distinguish between two cases, namely, if we have

$$\begin{aligned} &(1)\ f_\ell(x) \leq D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}} \quad \text{or} \\ &(2)\ f_\ell(x) > D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}. \end{aligned}$$

Let us start with the first case, i.e. we assume that $f_\ell(x) \leq D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}$. Then, we define

$$g := \left\lceil \exp\left(D_\ell - \tfrac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil, c := \left\lceil \exp\left(D_\ell - \tfrac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil, \text{ and } k_r := g \cdot 2^r$$

for $r \in \mathbb{N}_0$. Observe that we have

$$\begin{aligned}
& P_\theta\left(\exists n \geq R^{(\ell)}: \ \xi(n,x) > \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \cdot \xi(n,b_\ell)\right) \\
\leq \ & P_\theta\big(\xi(T(b_\ell),x) > c\big) + P_\theta\left(\xi(R^{(\ell)},b_\ell) < g\right) \\
& + \sum_{r=0}^{\infty} P_\theta\left(\sum_{j=1}^{k_{r+1}} \mathcal{Y}^{(j)}_{b_\ell,x} > \frac{1}{2} \cdot k_r \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)\right) \\
=: \ & I_1 + I_2 + I_3
\end{aligned} \tag{3.101}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x). Here, the first summand controls the unlikely event that the RWRE visits x very often before the RWRE reaches b_ℓ for the first time. The second summand is the probability of the event that the number of finished excursions away from the bottom point b_ℓ before the RWRE reaches m_ℓ again after the first visit to b_ℓ is unlikely low. Finally, the third summand controls the event that there are a lot of visits to x up to the k_{r+1}-th excursion for some $r \in \mathbb{N}$. To see that the inequality really holds, note that on the set

$$\{\xi(T(b_\ell),x) \leq c\} \cap \{\xi(R^{(\ell)},b_\ell) \geq g\} \cap \bigcap_{r=0}^{\infty} \left\{\sum_{j=1}^{k_{r+1}} \mathcal{Y}^{(j)}_{b_\ell,x} \leq \frac{1}{2} \cdot k_r \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)\right\} \tag{3.102}$$

we have $k_r \leq \xi(n,b_\ell) < k_{r+1}$ for some $r \in \mathbb{N}_0$. Thereby, again on the set in (3.102), we have for every $r \in \mathbb{N}_0$ with $k_r \leq \xi(n,b_\ell) < k_{r+1}$ that

$$\xi(n,x) \leq c + \frac{1}{2} \cdot k_r \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \leq \xi(n,b_\ell) \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)$$

holds for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x). Here, we further use that we have $H_\ell \to \infty$ (for $\ell \to \infty$) for P-a.e. environment θ due to (3.34).

To get an upper bound for I_1, let L denote the number of excursions from x to x made by the walk during the time interval $[0, T(b_\ell))$ when we start in x. Then, $L+1$ has a geometric distribution (with values $1, 2, \ldots$) with parameter

$$\begin{aligned}
q_\ell := & \left\{\begin{array}{ll}
(1-\omega_x) \cdot P_\theta^{x-1}\big(T(b_\ell) < T(x)\big) = (1-\omega_x) \cdot \dfrac{e^{V(x-1)}}{\sum\limits_{i=b_\ell}^{x-1} e^{V(i)}} & \text{if } x > b_\ell \\[2em]
\omega_x \cdot P_\theta^{x+1}\big(T(b_\ell) < T(x)\big) = \omega_x \cdot \dfrac{e^{V(x)}}{\sum\limits_{i=x}^{b_\ell - 1} e^{V(i)}} & \text{if } x < b_\ell
\end{array}\right\} \\
\geq \ & \varepsilon \cdot \frac{1}{|x-b_\ell|} \cdot \exp\big(-f_\ell(x)\big) \geq \varepsilon \cdot \frac{1}{2 \cdot (H_\ell)^{2+\gamma}} \cdot \exp\big(-f_\ell(x)\big)
\end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we used (3.2), (3.37), (3.38), and the definition of $f_\ell(x)$ (cf. (3.99)) for the last two steps. Therefore, we can conclude, using the definition of c, that

$$\begin{aligned} I_1 &\le P_\theta^x\big(\xi(T(b_\ell), x) > c\big) = P_\theta^x(L + 1 > c) = (1 - q_\ell)^c \le \exp(-q_\ell \cdot c) \\ &\le \exp\left(-\varepsilon \cdot \frac{1}{2 \cdot (H_\ell)^{2+\gamma}} \cdot \exp\left(\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right) \\ &\le \exp\left(-\varepsilon \cdot \frac{1}{2 \cdot \big(\ell^{1+\gamma} \cdot H_{\ell-1}\big)^{2+\gamma}} \cdot \exp\left(\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right) \\ &\le \frac{1}{3} \cdot \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \end{aligned} \tag{3.103}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we applied (3.35) in the third line. For the last line, note that $H_{\ell-1}$ grows exponentially in ℓ due to (3.34).

Now, we can turn to I_2: Similarly to above, let K denote the number of excursions away from b_ℓ to b_ℓ made by the walk during the time interval $[T(b_\ell), R^{(\ell)})$. Then, $K + 1$ has a geometric distribution with parameter

$$\begin{aligned} p_\ell &:= \left\{\begin{array}{ll} \omega_{b_\ell} \cdot P_\theta^{b_\ell+1}\big(T(m_\ell) < T(b_\ell)\big) = \omega_{b_\ell} \cdot \dfrac{e^{V(b_\ell)}}{\sum\limits_{i=b_\ell}^{m_\ell-1} e^{V(i)}} & \text{if } m_\ell > b_\ell \\ (1-\omega_{b_\ell}) \cdot P_\theta^{b_\ell-1}\big(T(m_\ell) < T(b_\ell)\big) = (1-\omega_{b_\ell}) \cdot \dfrac{e^{V(b_\ell-1)}}{\sum\limits_{i=m_\ell}^{b_\ell-1} e^{V(i)}} & \text{if } m_\ell < b_\ell \end{array}\right\} \\ &\le \left\{\begin{array}{ll} \exp\big(V(m_\ell) - V(m_\ell - 1)\big) \cdot \exp\big(V(b_\ell) - V(m_\ell)\big) & \text{if } m_\ell > b_\ell \\ \exp\big(V(b_\ell - 1) - V(b_\ell)\big) \cdot \exp\big(V(b_\ell) - V(m_\ell)\big) & \text{if } m_\ell < b_\ell \end{array}\right\} \\ &\le \frac{1}{\varepsilon} \cdot \exp\big(-D_\ell\big) \end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we used (3.2) and (3.32) for the last step. Therefore, we have using the definition of g

$$\begin{aligned} I_2 &= P_\theta(K + 1 < g) \le 1 - (1 - p_\ell)^g \le p_\ell \cdot g \le \frac{2}{\varepsilon} \cdot \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \\ &\le \frac{1}{3} \cdot \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \end{aligned} \tag{3.104}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we again used the exponential growth of $H_{\ell-1}$ in ℓ for the last step.

For the upper bound for I_3, we first note that due to (3.72) we have for all $j \in \mathbb{N}$

$$E_\theta[\mathcal{Y}_{b_\ell,x}^{(j)}] = \frac{\mu_\theta(x)}{\mu_\theta(b_\ell)} = \frac{e^{-V(x)} + e^{-V(x-1)}}{e^{-V(b_\ell)} + e^{-V(b_\ell-1)}} \le \frac{1}{\varepsilon} \cdot \exp\big(-\big(V(x) - V(b_\ell)\big)\big)$$

$$\leq \frac{1}{\varepsilon} \cdot \exp\left(-(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \ll \exp\left(-(H_{\ell-1})^{1-\gamma}\right)$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we used (3.2) for the third step. Further, we made use of (3.42) for $\frac{\gamma}{2}$ (which implies $V(x) - V(b_\ell) \geq (H_{\ell-1})^{1-\frac{\gamma}{2}}$ since $x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell$) and again the exponential growth of $H_{\ell-1}$ in ℓ for the last two steps. This yields, using that $(\mathcal{Y}^{(j)}_{b_\ell,x})_{j\in\mathbb{N}}$ are i.i.d. random variables together with Chebyshev's inequality,

$$\begin{aligned}
I_3 &\leq \sum_{r=0}^{\infty} P_\theta\left(\sum_{j=1}^{k_{r+1}} \mathcal{Y}^{(j)}_{b_\ell,x} - E_\theta[\mathcal{Y}^{(j)}_{b_\ell,x}] > \frac{1}{3} \cdot k_r \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)\right) \\
&\leq \sum_{r=0}^{\infty} \frac{9 \cdot k_{r+1} \cdot \mathrm{Var}_\theta\left(\mathcal{Y}^{(1)}_{b_\ell,x}\right)}{(k_r)^2 \cdot \exp\big(-2 \cdot (H_{\ell-1})^{1-\gamma}\big)} = \frac{18 \cdot \mathrm{Var}_\theta\left(\mathcal{Y}^{(1)}_{b_\ell,x}\right)}{\exp\big(-2 \cdot (H_{\ell-1})^{1-\gamma}\big)} \cdot \sum_{r=0}^{\infty} \frac{1}{k_r} \\
&\leq \frac{36 \cdot C \cdot \exp\big(-\big(V(x) - V(b_\ell)\big)\big) \cdot |x - b_\ell| \cdot \exp\big(f_\ell(x)\big)}{\exp\big(-2 \cdot (H_{\ell-1})^{1-\gamma}\big) \cdot g} \\
&\leq \frac{36 \cdot C \cdot \exp\big(-\big(V(x) - V(b_\ell)\big)\big) \cdot 2 \cdot (H_\ell)^{2+\gamma} \cdot \exp\big(-\frac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\big)}{\exp\big(-2 \cdot (H_{\ell-1})^{1-\gamma}\big)} \\
&\leq 72 \cdot C \cdot \left(\ell^{1+\gamma} \cdot H_{\ell-1}\right)^{2+\gamma} \cdot \exp\left(-\frac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \leq \frac{1}{3} \cdot \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \qquad (3.105)
\end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we used $\sum_{r=0}^{\infty} \frac{1}{k_r} = \frac{2}{g}$ and (3.99) in the third line, where C denotes the constant which is also used in (3.99). We further applied (3.42) for $\frac{\gamma}{2}$ and (3.35) in the fifth line. For the last step, we made use of the exponential growth of H_ℓ in ℓ again (cf. (3.34)).

A combination of the three upper bounds in (3.103), (3.104), and (3.105) now implies

$$\begin{aligned}
&P_\theta\left(\exists n \geq R^{(\ell)} :\ \xi(n, x) > \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \cdot \xi(n, b_\ell)\right) \\
&\leq I_1 + I_2 + I_3 \leq \exp\big(-(H_{\ell-1})^{1-\gamma}\big),
\end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). This shows (3.87) for $x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell$ with $f_\ell(x) \leq D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}$.

Now we turn to the second case, i.e. positions $x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell$ with $f_\ell(x) > D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}$. Here it turns out that the RWRE visits b_ℓ before x with very large probability. Similarly to above, we define

$$\overline{g} := \left\lceil \exp\left(f_\ell(x) + V(x) - V(b_\ell) - \tfrac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil \text{ and } \overline{k}_r := \overline{g} \cdot 2^r$$

for $r \in \mathbb{N}_0$. Here, we have

$$P_\theta\left(\exists n \geq R^{(\ell)} :\ \xi(n, x) > \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \cdot \xi(n, b_\ell)\right)$$

$$
\begin{aligned}
&\le\ P_\theta\big(T(x) < T(b_\ell)\big) + P_\theta^{b_\ell}\big(\xi(T(x), b_\ell) < \overline{g}\big) \\
&\qquad + \sum_{r=0}^{\infty} P_\theta\left(\sum_{j=1}^{\overline{k}_{r+1}} \mathcal{Y}^{(j)}_{b_\ell,x} > \overline{k}_r \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)\right) \\
&=: \overline{I}_1 + \overline{I}_2 + \overline{I}_3
\end{aligned} \tag{3.106}
$$

for all ℓ. Note here (in contrast to (3.101)) that on the set $\{T(x) \ge T(b_\ell)\}$ we have

$$
\xi(n, x) = 0
$$

for $R^{(\ell)} \le n \le T(x)$.

For first summand $\overline{I}_1$, we observe that we have

$$
\begin{aligned}
f_\ell(x) &\overset{\text{def}}{=} \left\{ \begin{array}{ll} \max\limits_{b_\ell \le y \le x-1} \big(V(y) - V(x-1)\big) & \text{if } 0 > x > b_\ell \\ \max\limits_{x \le y \le b_\ell - 1} \big(V(y) - V(x)\big) & \text{if } 0 < x < b_\ell \end{array} \right\} \le V(m_\ell) - V(b_{\ell-1}) \\
&= H_{\ell-1} \ll D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}
\end{aligned}
$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we made use of (3.31) and (3.33) in the first line. Further, we used (3.43) (applied for $\frac{\gamma}{4}$) together with the exponential growth of $H_{\ell-1}$ in ℓ for the last step. Therefore, due to our assumption on $f_\ell(x)$, we only have to consider the remaining possibilities for the location of x and b_ℓ: We have

$$
\begin{aligned}
\overline{I}_1 &= P_\theta\big(T(x) < T(b_\ell)\big) \\
&\le \left\{ \begin{array}{ll}
\dfrac{\sum\limits_{j=b_\ell}^{-1} e^{V(j)}}{\sum\limits_{j=b_\ell}^{x-1} e^{V(j)}} \le \dfrac{-b_\ell \cdot \exp\left(\max\limits_{b_\ell \le j \le -1} V(j)\right)}{\exp\left(\max\limits_{b_\ell \le j \le x-1} V(j)\right)} = \dfrac{-b_\ell \cdot \exp\big(V(m_\ell)\big)}{\exp\big(f_\ell(x) + V(x-1)\big)} & \text{if } x > 0 > b_\ell \\
\dfrac{\sum\limits_{j=0}^{b_\ell-1} e^{V(j)}}{\sum\limits_{j=x}^{b_\ell-1} e^{V(j)}} \le \dfrac{b_\ell \cdot \exp\left(\max\limits_{0 \le j \le b_\ell-1} V(j)\right)}{\exp\left(\max\limits_{x \le j \le b_\ell-1} V(j)\right)} = \dfrac{b_\ell \cdot \exp\big(V(m_\ell)\big)}{\exp\big(f_\ell(x) + V(x)\big)} & \text{if } x < 0 < b_\ell
\end{array} \right\} \\
&\le (H_\ell)^{2+\gamma} \cdot \exp\Big(V(m_\ell) - V(b_\ell) - (H_{\ell-1})^{1-\frac{\gamma}{4}} - f_\ell(x)\Big) \\
&\le \big(\ell^{1+\gamma} \cdot H_{\ell-1}\big)^{2+\gamma} \cdot \exp\Big(-(H_{\ell-1})^{1-\frac{\gamma}{4}} + (H_{\ell-1})^{1-\frac{\gamma}{2}}\Big) \le \frac{1}{3} \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)
\end{aligned} \tag{3.107}
$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we made use of (3.31) and the definition of $f_\ell(x)$ in the second line. Further, we applied (3.39) and (3.42) (for $\frac{\gamma}{4}$) in the third line. In the last line, we used (3.32), (3.35), and the assumption on $f_\ell(x)$. Finally, the last step holds due to the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)).

Now, we can turn to $\overline{I}_2$: Similarly to above, let K denote the number of excursions away from b_ℓ to b_ℓ made by the walk during the time interval $[0, T(x))$ when we start in b_ℓ. Again, $K+1$ has a geometric distribution with parameter

$$
\begin{aligned}
p_\ell &:= \left\{ \begin{array}{ll} \omega_{b_\ell} \cdot P_\theta^{b_\ell+1}\big(T(x) < T(b_\ell)\big) = \omega_{b_\ell} \cdot \dfrac{e^{V(b_\ell)}}{\sum\limits_{i=b_\ell}^{x-1} e^{V(i)}} & \text{if } x > b_\ell \\ (1-\omega_{b_\ell}) \cdot P_\theta^{b_\ell-1}\big(T(x) < T(b_\ell)\big) = (1-\omega_{b_\ell}) \cdot \dfrac{e^{V(b_\ell-1)}}{\sum\limits_{i=x}^{b_\ell-1} e^{V(i)}} & \text{if } x < b_\ell \end{array} \right\} \\
&\le \left\{ \begin{array}{ll} e^{V(x)-V(x-1)} \cdot e^{V(b_\ell)-V(x)} \cdot \exp\left(- \max\limits_{b_\ell \le i \le x-1} \big(V(i) - V(x-1)\big)\right) & \text{if } x > b_\ell \\ e^{V(b_\ell-1)-V(b_\ell)} \cdot e^{V(b_\ell)-V(x)} \cdot \exp\left(- \max\limits_{x \le i \le b_\ell-1} \big(V(i) - V(x)\big)\right) & \text{if } x < b_\ell \end{array} \right\} \\
&\le \frac{1}{\varepsilon} \cdot e^{V(b_\ell)-V(x)} \cdot \exp\big(-f_\ell(x)\big),
\end{aligned}
$$

where we used (3.2) and the definition of $f_\ell(x)$ for the last step. Therefore, we have using the definition of $\overline{g}$

$$
\begin{aligned}
\overline{I}_2 = P_\theta^{b_\ell}(K+1 < \overline{g}) \le 1 - (1-p_\ell)^{\overline{g}} \le p_\ell \cdot \overline{g} \le \frac{2}{\varepsilon} \cdot \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \\
\le \frac{1}{3} \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)
\end{aligned} \tag{3.108}
$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we used the exponential growth of $H_{\ell-1}$ in ℓ in the last line.

The upper bound of $\overline{I}_3$ can be derived almost in the same way as the upper bound of I_3. Note that the factor $f_\ell(x)$ has to be treated differently here, which is compensated by the larger choice of $\overline{g}$. By using the same argument as in the first three lines in (3.105), we get

$$
\begin{aligned}
\overline{I}_3 &\le \frac{36 \cdot C \cdot \exp\big(-\big(V(x) - V(b_\ell)\big)\big) \cdot |x - b_\ell| \cdot \exp\big(f_\ell(x)\big)}{\exp\big(-2 \cdot (H_{\ell-1})^{1-\gamma}\big) \cdot \overline{g}} \\
&\le \frac{36 \cdot C \cdot \exp\big(-(H_{\ell-1})^{1-\frac{\gamma}{2}}\big) \cdot 2 \cdot (H_\ell)^{2+\gamma} \cdot \exp\big(\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\big)}{\exp\big(-2 \cdot (H_{\ell-1})^{1-\gamma}\big)} \\
&\le 72 \cdot C \cdot \big(\ell^{1+\gamma} \cdot H_{\ell-1}\big)^{2+\gamma} \cdot \exp\left(-\frac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}} + 2 \cdot (H_{\ell-1})^{1-\gamma}\right) \\
&\le \frac{1}{3} \cdot \exp\big(-(H_{\ell-1})^{1-\gamma}\big)
\end{aligned} \tag{3.109}
$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x).

Again, a combination of the three upper bounds in (3.107), (3.108), and (3.109) implies

$$
P_\theta\big(\exists n \ge R^{(\ell)} : \ \xi(n, x) > \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \cdot \xi(n, b_\ell)\big)
$$

$$\leq \overline{I}_1 + \overline{I}_2 + \overline{I}_3 \leq \exp\big(- (H_{\ell-1})^{1-\gamma}\big),$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). This yields (3.87) for $x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell$ with $f_\ell(x) > D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}$.

To finish the proof, note that for P-a.e. environment θ we have

$$\begin{aligned}\left(B_3^{(\ell)}\right)^c &= \left\{\forall n \geq R^{(\ell)} : \sum_{x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell} \xi(n,x) \leq \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \cdot \xi(n, b_\ell)\right\}^c \\ &\subseteq \bigcup_{x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell} \left\{\exists n \geq R^{(\ell)} : \xi(n,x) > \exp\left(-(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \cdot \xi(n, b_\ell)\right\}\end{aligned}$$

for all $\ell = \ell(\gamma, \theta)$ large enough. Here, for the last step, we used that due to (3.37) and (3.38) we have

$$|\mathcal{V}_\ell \backslash \mathcal{D}_\ell| \leq 2 \cdot (H_\ell)^{2+\gamma}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough. By applying (3.87), this implies that, for P-a.e. environment θ and all $k = k(\gamma, \theta)$ large enough, we have

$$\begin{aligned} 1 - P_\theta\left(\bigcap_{\ell=k}^{\infty} B_3^{(\ell)}\right) &= P_\theta\left(\bigcup_{\ell=k}^{\infty} \left(B_3^{(\ell)}\right)^c\right) \\ \leq \sum_{\ell=k}^{\infty} |\mathcal{V}_\ell \backslash \mathcal{D}_\ell| \cdot \exp\left(-\big(H_{\ell-1}\big)^{1-\frac{\gamma}{2}}\right) &\leq \sum_{\ell=k}^{\infty} 2 \cdot (H_\ell)^{2+\gamma} \cdot \exp\left(-\big(H_{\ell-1}\big)^{1-\frac{\gamma}{2}}\right) \xrightarrow{k \to \infty} 0, \end{aligned}$$

where we used (3.37) and (3.38) for the second step and (3.34) and (3.35) for the convergence for $k \to \infty$. □

<u>*Proof of Lemma 3.4.7.*</u> We first fix $0 < \gamma < 1$. In order to make it a little easier to read the proof, we assume in the following that we have

$$b_\ell > 0.$$

In the other case, i.e. $b_\ell < 0$, the proof works completely in the same way if we just consider the symmetric analogues of all appearing quantities (with respect to the origin). Note here that we a.s. have $b_\ell \neq 0$ for all $\ell = \ell(\theta)$ large enough (cf. Proposition 3.3.9).

For the proof of (3.89), we distinguish between two cases for which we recall (cf. (3.19)) that

$$\begin{aligned} \widehat{\phi}^-_{\ell,\frac{\gamma}{2}} &\stackrel{\text{def}}{=} \sup\{t \leq \widehat{\eta}^-_\ell : \ B(t) = B(\widehat{b}_\ell) + (\widehat{H}_{\ell-1})^{1-\frac{\gamma}{2}}\} \qquad \text{and} \\ \widehat{\phi}^+_{\ell,\frac{\gamma}{2}} &\stackrel{\text{def}}{=} \inf\{t \geq \widehat{\eta}^+_\ell : \ B(t) = B(\widehat{b}_\ell) + (\widehat{H}_{\ell-1})^{1-\frac{\gamma}{2}}\}. \end{aligned}$$

In the first case, we consider positions $x \in \mathcal{D}_{\ell+1}$ with

$$\begin{aligned} m_{\ell+1} \leq x \leq \widehat{\phi}^+_{\ell,\frac{\gamma}{2}} &\quad \text{if } b_{\ell+1} > 0 \qquad \text{or} \\ \widehat{\phi}^-_{\ell,\frac{\gamma}{2}} \leq x \leq m_{\ell+1} &\quad \text{if } b_{\ell+1} < 0. \end{aligned}$$

Those positions x therefore do not lie very deep in $\mathcal{D}_{\ell+1}$, which makes it unlikely that the RWRE spends a lot of time in x without returning to b_ℓ. The remaining positions $x \in \mathcal{D}_{\ell+1}$ with

$$\begin{aligned} &\widehat{\phi}^+_{\ell,\frac{\gamma}{2}} < x < b_{\ell+1} \quad \text{if } b_{\ell+1} > 0 \qquad \text{or} \\ &b_{\ell+1} < x < \widehat{\phi}^-_{\ell,\frac{\gamma}{2}} \quad \text{if } b_{\ell+1} < 0 \end{aligned}$$

on the other hand are very deep in $\mathcal{D}_{\ell+1}$. In this case, it is unlikely that the RWRE leaves the valley $\mathcal{D}_{\ell+1}$ again before it reaches the bottom point $b_{\ell+1}$.
Further, note that the RWRE cannot reach the remaining positions within $\mathcal{D}_{\ell+1}$ without reaching $b_{\ell+1}$ first.

Let us start with the first group, i.e. $x \in \mathcal{D}_{\ell+1}$ with

$$\begin{aligned} &m_{\ell+1} \leq x \leq \widehat{\phi}^+_{\ell,\frac{\gamma}{2}} \quad \text{if } b_{\ell+1} > 0 \qquad \text{or} \\ &\widehat{\phi}^-_{\ell,\frac{\gamma}{2}} \leq x \leq m_{\ell+1} \quad \text{if } b_{\ell+1} < 0. \end{aligned}$$

Here, we are almost in the same situation as in the proof of Lemma 3.4.6. Due to our choice for x, we have for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough that

$$\begin{aligned} V(x) &\geq B(x) - \max\Big\{\big(\log(\widehat{\phi}^+_{\ell,\frac{\gamma}{2}})\big)^2, \big(\log(\widehat{\phi}^-_{\ell,\frac{\gamma}{2}})\big)^2\Big\} \\ &\geq B(\widehat{b}_\ell) + (\widehat{H}_{\ell-1})^{1-\frac{\gamma}{2}} - \max\Big\{\big(\log(\widehat{\phi}^+_{\ell,\frac{\gamma}{2}})\big)^2, \big(\log(\widehat{\phi}^-_{\ell,\frac{\gamma}{2}})\big)^2\Big\} \\ &\geq V(\lfloor \widehat{b}_\ell \rfloor) + (\widehat{H}_{\ell-1})^{1-\frac{\gamma}{2}} - 3 \cdot \max\Big\{\big(\log(\widehat{\phi}^+_{\ell,\frac{\gamma}{2}})\big)^2, \big(\log(\widehat{\phi}^-_{\ell,\frac{\gamma}{2}})\big)^2\Big\} \\ &\geq V(b_\ell) + (\widehat{H}_{\ell-1})^{1-\frac{\gamma}{2}} - 3 \cdot \Big(\log\big((\widehat{H}_\ell)^{2+\gamma}\big)\Big)^2 \\ &\geq V(b_\ell) + (\widehat{H}_{\ell-1})^{1-\frac{\gamma}{2}} - 3 \cdot \Big(\log\big((\ell^{1+\gamma} \cdot \widehat{H}_{\ell-1})^{2+\gamma}\big)\Big)^2 \\ &\geq V(b_\ell) + \frac{3}{4} \cdot (\widehat{H}_{\ell-1})^{1-\frac{\gamma}{2}} \geq V(b_\ell) + \frac{2}{3} \cdot (H_{\ell-1})^{1-\frac{\gamma}{2}}, \end{aligned} \tag{3.110}$$

i.e. $V(x) \gg V(b_\ell)$, which makes it unlikely to observe more visits of the RWRE to x than to b_ℓ. Here, we need to choose ℓ large enough such that we may apply Lemma 3.3.15 to ensure that the Brownian motion $B(\cdot)$ and the potential $V(\cdot)$ attain their minima at $\widehat{b}_\ell$ and b_ℓ close to each other. Further, we applied the approximation theorem (cf. Remark 3.55) in the first and the third step, the definition of $\widehat{\phi}^+_{\ell,\frac{\gamma}{2}}$ and $\widehat{\phi}^-_{\ell,\frac{\gamma}{2}}$ in the second step, and (3.14) in the fourth step. Additionally, we used (3.12) and the exponential growth of $\widehat{H}_{\ell-1}$ in ℓ (cf. (3.11)) in the last two lines. Finally, we applied (3.62) – which connects $H_{\ell-1}$ and $\widehat{H}_{\ell-1}$ – for the last step.

Similarly to the approach in (3.106), we define

$$\overline{g} := \left\lceil \exp\left(f_\ell(x) + V(x) - V(b_\ell) - \tfrac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil \text{ and } \overline{k}_r := \overline{g} \cdot 2^r$$

for $r \in \mathbb{N}_0$, where

$$f_\ell(x) := \begin{cases} \max\limits_{b_\ell \leq y \leq x-1} \big(V(y) - V(x-1)\big) & \text{if } x > b_\ell \\ \max\limits_{x \leq y \leq b_\ell - 1} \big(V(y) - V(x)\big) & \text{if } x < b_\ell, \end{cases}$$

which measures how hard it is to get from x to b_ℓ. Then, we have

$$
\begin{aligned}
&P_\theta\left(\exists n \le T_1^{(\ell+1)} :\ X_n \notin \mathcal{D}_{\ell+1},\ \xi(n,x) > \exp\left(-\big(H_{\ell-1}\big)^{1-\gamma}\right)\cdot \xi(n,b_\ell)\right)\\
&\le\ P_\theta\big(T(x) < T(b_\ell)\big) + P_\theta^{b_\ell}\big(\xi(T(x),b_\ell) < \overline{g}\big)\\
&\quad + \sum_{r=0}^{\infty} P_\theta\left(\sum_{j=1}^{\overline{k}_{r+1}} \mathcal{Y}_{b_\ell,x}^{(j)} > \overline{k}_r \cdot \exp\left(-\big(H_{\ell-1}\big)^{1-\gamma}\right)\right)\\
&=:\ \widetilde{I}_1 + \overline{I}_2 + \overline{I}_3
\end{aligned}
\tag{3.111}
$$

for all ℓ.

Let us start with $\widetilde{I}_1$:

$$
\begin{aligned}
\widetilde{I}_1 &= P_\theta\big(T(x) < T(b_\ell)\big)\\
&\le \left\{\begin{array}{ll} 0 & \text{if } b_{\ell+1} > 0\\ \dfrac{\sum_{j=0}^{b_\ell-1} e^{V(j)}}{\sum_{j=x}^{b_\ell-1} e^{V(j)}} \le \dfrac{b_\ell \cdot \exp\left(\max_{0\le j\le b_\ell-1} V(j)\right)}{\exp\left(\max_{x\le j\le 0} V(j)\right)} & \text{if } b_{\ell+1} < 0 \end{array}\right\}\\
&\le (H_\ell)^{2+\gamma}\cdot \exp\big(V(m_\ell) - V(h_\ell)\big) \le \big(\ell^{1+\gamma}\cdot(\ell-1)^{1+\gamma}\cdot H_{\ell-2}\big)^{2+\gamma}\cdot \exp\left(-(H_{\ell-2})^{1-\frac{\gamma}{2}}\right)\\
&\le \frac{1}{3}\cdot \exp\big(-(H_{\ell-2})^{1-\gamma}\big)
\end{aligned}
\tag{3.112}
$$

for P-a.e. environment θ and $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x). Here in the third line, we used (3.39) and (3.31) together with the fact that $x < h_\ell < 0$ (for all ℓ large enough) if we are in the situation with $b_\ell > 0$ and $b_{\ell+1} < 0$. Additionally, we applied (3.35) and (3.36) for $\frac{\gamma}{2}$ in the next step. Finally, we used the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)) for the last step.

For $\overline{I}_2$, we can use the same approach as in (3.108): In particular we have

$$
\overline{I}_2 \le \frac{1}{3}\cdot \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \tag{3.113}
$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x).

The upper bound for $\overline{I}_3$ can be shown almost as in (3.109). The only difference is that we have to use the upper bound $|x - b_\ell| \le 2\cdot (H_{\ell+1})^{2+\gamma}$ in contrast to (3.109) and for $V(x) - V(b_\ell)$ we have to use our lower bound in (3.110): In particular we have

$$
\overline{I}_3 \le \frac{36\cdot C\cdot \exp\big(-\big(V(x) - V(b_\ell)\big)\big)\cdot |x - b_\ell|\cdot \exp\big(f_\ell(x)\big)}{\exp\big(-2\cdot (H_{\ell-1})^{1-\gamma}\big)\cdot \overline{g}}
$$

$$\begin{aligned}
&\le \frac{36 \cdot C \cdot \exp\big(-\frac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\big) \cdot 2 \cdot (H_{\ell+1})^{2+\gamma} \cdot \exp\big(\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\big)}{\exp\big(-2 \cdot (H_{\ell-1})^{1-\gamma}\big)} \\
&\le 72 \cdot C \cdot \big((\ell+1)^{1+\gamma} \cdot \ell^{1+\gamma} \cdot H_{\ell-1}\big)^{2+\gamma} \cdot \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}} + 2 \cdot (H_{\ell-1})^{1-\gamma}\right) \\
&\le \frac{1}{3} \cdot \exp\big(-(H_{\ell-1})^{1-\gamma}\big) \qquad (3.114)
\end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here in the third line, we used (3.35). Further, we made use of the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)) in the last step.

Finally, we can collect our upper bounds in (3.111), (3.112), (3.113), and (3.114) to conclude that we have

$$\begin{aligned}
&P_\theta\left(\exists n \le T_1^{(\ell+1)} :\ X_n \notin \mathcal{D}_{\ell+1},\ \xi(n,x) > \exp\left(-\big(H_{\ell-1}\big)^{1-\gamma}\right) \cdot \xi(n, b_\ell)\right) \\
&\le \widetilde{I}_1 + \overline{I}_2 + \overline{I}_3 \le \exp\big(-(H_{\ell-2})^{1-\gamma}\big). \qquad (3.115)
\end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). This shows (3.89) for the first group (of choices for x).

Now we turn to the second group, i.e. $x \in \mathcal{D}_{\ell+1}$ with

$$\begin{aligned}
\widehat{\phi}^+_{\ell,\frac{\gamma}{2}} < x < b_{\ell+1} \quad &\text{if } b_{\ell+1} > 0 \qquad \text{or} \\
b_{\ell+1} < x < \widehat{\phi}^-_{\ell,\frac{\gamma}{2}} \quad &\text{if } b_{\ell+1} < 0.
\end{aligned}$$

For

$$\phi := \begin{cases} \lceil \widehat{\phi}^+_{\ell,\frac{\gamma}{2}} \rceil & \text{if } b_{\ell+1} > 0 \\ \lfloor \widehat{\phi}^-_{\ell,\frac{\gamma}{2}} \rfloor & \text{if } b_{\ell+1} < 0, \end{cases}$$

we have

$$\begin{aligned}
&P_\theta\left(\exists n \le T_1^{(\ell+1)} :\ X_n \notin \mathcal{D}_{\ell+1},\ \xi(n,x) > \xi(n, b_\ell) \cdot \exp\left(-\big(H_{\ell-1}\big)^{1-\gamma}\right)\right) \\
&\le P_\theta^\phi\big(T(m_{\ell+1}) < T(b_{\ell+1})\big).
\end{aligned}$$

Note here that the RWRE can only leave the valley $\mathcal{D}_{\ell+1}$ via ϕ and $m_{\ell+1}$ after visiting x for the first time without visiting $b_{\ell+1}$ in between.

Thereby, we have

$$P_\theta^\phi\big(T(m_{\ell+1}) < T(b_{\ell+1})\big) = \left\{\begin{array}{ll} \dfrac{\sum\limits_{i=\phi}^{b_{\ell+1}-1} e^{V(i)}}{\sum\limits_{i=m_{\ell+1}}^{b_{\ell+1}-1} e^{V(i)}} & \text{if } b_{\ell+1} > 0 \\[4ex] \dfrac{\sum\limits_{i=b_{\ell+1}}^{\phi-1} e^{V(i)}}{\sum\limits_{i=b_{\ell+1}}^{m_{\ell+1}-1} e^{V(i)}} & \text{if } b_{\ell+1} < 0 \end{array}\right\}$$

$$\leq \left\{ \begin{array}{ll} (b_{\ell+1}-\phi)\cdot \exp\left(\max\limits_{\widehat{\phi}^{+}_{\ell,\frac{\gamma}{2}} \leq i \leq b_{\ell+1}-1} V(i) - V(m_{\ell+1}) \right) & \text{if } b_{\ell+1} > 0 \\ (\phi - b_{\ell+1})\cdot \exp\left(\max\limits_{b_{\ell+1} \leq i \leq \widehat{\phi}^{-}_{\ell,\frac{\gamma}{2}}-1} V(i) - V(m_{\ell+1}) \right) \cdot e^{V(m_{\ell+1})-V(m_{\ell+1}-1)} & \text{if } b_{\ell+1} < 0 \end{array} \right\}$$

$$\leq \frac{1}{\varepsilon}\cdot (H_{\ell+1})^{2+\gamma}\cdot \exp\big(-(H_\ell)^{1-\frac{\gamma}{2}}\big) \leq \exp\big(-(H_\ell)^{1-\gamma}\big) \tag{3.116}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough. Here in the last line, we applied (3.2) and (3.48) for $\frac{\gamma}{2}$. Further, we used the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)) together with (3.35) for the last step.

By comparing the upper bounds in (3.115) and (3.116), we conclude that overall we have for every $x \in \mathcal{D}_{\ell+1}$ that

$$P_\theta\left(\exists n \leq T_1^{(\ell+1)} :\ X_n \notin \mathcal{D}_{\ell+1},\ \xi(n,x) > \exp\left(-\big(H_{\ell-1}\big)^{1-\gamma}\right)\cdot \xi(n,b_\ell)\right) \leq \exp\left(-(H_{\ell-2})^{1-\gamma}\right)$$

holds for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x) which shows (3.89).

To finish the proof, we note that for P-a.e. environment θ we have

$$\left(B_4^{(\ell)}\right)^c = \left\{\forall n \leq T_1^{(\ell+1)} \text{ with } X_n \notin \mathcal{D}_{\ell+1} :\ \sum_{x\in\mathcal{D}_{\ell+1}} \xi(n,x) \leq \exp\left(-(H_{\ell-1})^{1-\gamma}\right)\cdot \xi(n,b_\ell)\right\}^c$$
$$\subseteq \bigcup_{x\in\mathcal{D}_{\ell+1}} \left\{\exists n \leq T_1^{(\ell+1)} :\ X_n \notin \mathcal{D}_{\ell+1},\ \xi(n,x) > \exp\left(-\big(H_{\ell-1}\big)^{1-\frac{\gamma}{2}}\right)\cdot \xi(n,b_\ell)\right\}$$

for all $\ell = \ell(\gamma,\theta)$ large enough due to the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)), where further

$$|\mathcal{D}_{\ell+1}| \leq (H_{\ell+1})^{2+\gamma}$$

for P-a.e. environment θ and all $\ell = \ell(\theta,\gamma)$ large enough due to (3.30), (3.37), and (3.38). Therefore, for P-a.e. environment θ and all $k = k(\gamma,\theta)$ large enough, we have

$$\begin{aligned} 1 - P_\theta\left(\bigcap_{\ell=k}^{\infty} B_4^{(\ell)}\right) &= P_\theta\left(\bigcup_{\ell=k}^{\infty}\left(B_4^{(\ell)}\right)^c\right) \\ \leq \sum_{\ell=k}^{\infty} |\mathcal{D}_{\ell+1}|\cdot \exp\left(-(H_{\ell-2})^{1-\frac{\gamma}{2}}\right) &\leq \sum_{\ell=k}^{\infty} (H_{\ell+1})^{2+\gamma}\cdot \exp\left(-(H_{\ell-2})^{1-\frac{\gamma}{2}}\right) \\ \leq \sum_{\ell=k}^{\infty} \left((\ell+1)^{1+\gamma}\cdot \ell^{1+\gamma}\cdot (\ell-1)^{1+\gamma}\cdot H_{\ell-2}\right)^{2+\gamma} &\cdot \exp\left(-(H_{\ell-2})^{1-\frac{\gamma}{2}}\right) \xrightarrow{k\to\infty} 0, \end{aligned}$$

where we used (3.35) in the last line and again the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)) for the convergence for $k \to \infty$. □

Proof of Lemma 3.4.8. For the proof we first fix $0 < \gamma < 1$. In view of (3.87), there is nothing left to show for $x \in \mathcal{V}_\ell \backslash \mathcal{D}_\ell$. Therefore, we only consider positions $x \in \mathcal{D}_\ell$. We further treat the following two cases separately:

$$\begin{array}{llll} (1) & m_\ell \le x \le \eta_\ell^+ & \text{if } b_\ell > 0, & \\ & \eta_\ell^- \le x \le m_\ell & \text{if } b_\ell < 0, & \text{or} \\ (2) & \eta_\ell^+ < x \le r_\ell & \text{if } b_\ell > 0, & \\ & \ell_\ell \le x < \eta_\ell^- & \text{if } b_\ell < 0. & \end{array}$$

Notice that in the first case the potential $V(x)$ can be close to $V(b_\ell)$ whereas in the second case we have $V(x) \gg V(b_\ell)$ (cf. (3.47)). Let us start with the first case, i.e.

$$\begin{array}{ll} m_\ell \le x \le \eta_\ell^+ & \text{if } b_\ell > 0, \\ \eta_\ell^- \le x \le m_\ell & \text{if } b_\ell < 0. \end{array}$$

Similarly to the approach in (3.101), we define

$$g := \left\lceil \exp\left(D_\ell - \tfrac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil, c := \left\lceil \exp\left(D_\ell - \tfrac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil, \text{ and } k_r := g \cdot 2^r$$

for $r \in \mathbb{N}_0$ and observe that we have

$$\begin{aligned} & P_\theta\big(\exists n \ge R^{(\ell)} :\ \xi(n,x) \ge 6 \cdot \xi(n,b_\ell)\big) \\ \le\ & P_\theta\big(\xi(T(b_\ell),x) > c\big) + P_\theta\big(\xi(R^{(\ell)},b_\ell) < g\big) + \sum_{r=0}^{\infty} P_\theta\left(\sum_{j=1}^{k_{r+1}} \mathcal{Y}^{(j)}_{b_\ell,x} > 5 \cdot k_r\right) \\ =:\ & I_1 + I_2 + I_3' \end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x).

For I_1 and I_2, we can copy the argument from (3.103) and (3.104) to get

$$I_1 \le \frac{1}{3}\exp\left(-(H_{\ell-1})^{1-\gamma}\right) \quad \text{and} \quad I_2 \le \frac{1}{3}\exp\left(-(H_{\ell-1})^{1-\gamma}\right) \tag{3.117}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x). Note here that we even have $I_1 = 0$ for positions $x \in \mathcal{D}_\ell$ which the RWRE cannot reach before visiting b_ℓ first.

Now, we turn to I_3': Let us again define

$$f_\ell(x) := \begin{cases} \max\limits_{b_\ell \le y \le x-1} \big(V(y) - V(x-1)\big) & \text{if } x > b_\ell \\ \max\limits_{x \le y \le b_\ell - 1} \big(V(y) - V(x)\big) & \text{if } x < b_\ell \end{cases}$$

as before. First of all, we notice that

$$E_\theta[\mathcal{Y}^{(1)}_{b_\ell,x}] = \frac{\mu_\theta(x)}{\mu_\theta(b_\ell)} = \frac{e^{-V(x)} + e^{-V(x-1)}}{e^{-V(b_\ell)} + e^{-V(b_\ell-1)}} \le 2.$$

Therefore, we can conclude by using Chebyshev's inequality that we have

$$
\begin{aligned}
I_3' &\le \sum_{r=0}^{\infty} P_\theta\left(\sum_{j=1}^{k_{r+1}}\left(\mathcal{Y}_{b_\ell,x}^{(j)} - E_\theta[\mathcal{Y}_{b_\ell,x}^{(j)}]\right) > k_r\right) \le \sum_{r=0}^{\infty} \frac{k_{r+1}\cdot \mathrm{Var}_\theta\left(\mathcal{Y}_{b_\ell,x}^{(1)}\right)}{(k_r)^2} \\
&\le 2\cdot C\cdot \exp\big(V(x)-V(b_\ell)\big)\cdot |x-b_\ell|\cdot \exp\big(f_\ell(x)\big)\cdot \sum_{r=0}^{\infty}\frac{1}{k_r} \\
&\le 4\cdot C\cdot (H_\ell)^{2+\gamma}\cdot \frac{\exp\big(-\big(V(x)-V(b_\ell)\big)\big)\cdot \exp\big(f_\ell(x)\big)}{g} \\
&\le 4\cdot C\cdot (H_\ell)^{2+\gamma}\cdot \exp\left(-\tfrac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \le \frac{1}{3}\cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)
\end{aligned}
\tag{3.118}
$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x). Here, $C > 0$ denotes the constant from our upper bound for $\mathrm{Var}_\theta(\mathcal{Y}_{b_\ell,x}^{(1)})$(cf. (3.73)), which we used in the second line. For the next step, notice again that $\sum_{r=0}^{\infty}\frac{1}{k_r} = \frac{2}{g}$. Further, we can apply (3.45) (for $|x| < |b_\ell|$) or (3.46) (for $b_\ell < x \le \eta_\ell^+$ and $\eta_\ell^- \le x \le b_\ell$ depending on the sign of b_ℓ) for $\frac{\gamma}{2}$ in the fourth line. The last step holds again due to (3.35) and due to the exponential growth of H_ℓ in ℓ (cf. (3.34)).

By collecting our bounds from (3.117) and (3.118), we get

$$
P_\theta\big(\exists n \ge R^{(\ell)}:\ \xi(n,x) \ge 6\cdot \xi(n,b_\ell)\big) \le I_1 + I_2 + I_3' \le \exp\left(-(H_{\ell-1})^{1-\gamma}\right)
$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in x). This shows (3.91) in the first case.

Now we turn to the second case, i.e.

$$
\begin{aligned}
\eta_\ell^+ < x \le r_\ell \qquad &\text{if } b_\ell > 0,\\
\ell_\ell \le x < \eta_\ell^- \qquad &\text{if } b_\ell < 0.
\end{aligned}
$$

In the following, we further assume that we have $b_\ell > 0$. For the case $b_\ell < 0$, we only have to consider the symmetric analogues of all appearing quantities.

Similarly to the consideration in (3.106), we define

$$
\overline{g} := \left\lceil \exp\left(f_\ell(x) + V(x) - V(b_\ell) - \tfrac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil \text{ and } \overline{k}_r := \overline{g}\cdot 2^r
$$

for $r \in \mathbb{N}_0$ and observe that we have

$$
\begin{aligned}
&P_\theta\big(\exists n \ge R^{(\ell)}:\ \xi(n,x) \ge 6\cdot \xi(n,b_\ell)\big) \\
\le\ & P_\theta\big(\xi(T(x),b_\ell) < \overline{g}\big) + \sum_{r=0}^{\infty} P_\theta\left(\sum_{j=1}^{\overline{k}_{r+1}} \mathcal{Y}_{b_\ell,x}^{(j)} > 6\cdot k_r\right) \\
=:\ & \overline{I}_2 + \overline{I}_3
\end{aligned}
$$

for all ℓ. Notice here, that the RWRE cannot reach x without reaching b_ℓ first.

Let us start with $\overline{I}_2$: Using the same estimates as in (3.108) we get

$$\overline{I}_2 \leq \frac{1}{2} \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \tag{3.119}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x).

For $\overline{I}_3$, we have similarly to (3.118)

$$\begin{aligned}
\overline{I}_3 &\leq \sum_{r=0}^{\infty} P_\theta\left(\sum_{j=1}^{k_{r+1}}\left(\mathcal{Y}_{b_\ell,x}^{(j)} - E_\theta[\mathcal{Y}_{b_\ell,x}^{(j)}]\right) > 2 \cdot k_r\right) \leq \sum_{r=0}^{\infty} \frac{k_{r+1} \cdot \mathrm{Var}_\theta\left(\mathcal{Y}_{b_\ell,x}^{(1)}\right)}{4 \cdot (k_r)^2} \\
&\leq \frac{C \cdot \exp\left(-\left(V(x) - V(b_\ell)\right)\right) \cdot |x - b_\ell| \cdot \exp\left(f_\ell(x)\right)}{\overline{g}} \\
&\leq C \cdot \exp\left(-(D_\ell)^{1-\frac{\gamma}{2}}\right) \cdot (H_\ell)^{2+\gamma} \cdot \exp\left(+\tfrac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}} - (D_\ell)^{1-\frac{\gamma}{2}}\right) \\
&\leq C \cdot \left(\ell^{1+\gamma} \cdot H_\ell\right)^{2+\gamma} \cdot \exp\left(-(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \leq \frac{1}{2} \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)
\end{aligned} \tag{3.120}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). Here, we additionally used (3.47) (applied for $\frac{\gamma}{2}$) in the third line.

By collecting our bounds from (3.119) and (3.120), we get

$$P_\theta\left(\exists n \geq R^{(\ell)} :\ \xi(n, x) \geq 6 \cdot \xi(n, b_\ell)\right) \leq \overline{I}_2 + \overline{I}_3 \leq \exp\left(-(H_{\ell-1})^{1-\gamma}\right)$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in x). This shows (3.91) in the second case.

For the proof of (3.92), notice that for P-a.e. environment θ we have

$$\begin{aligned}
\left(B_5^{(\ell)}\right)^c &= \left\{\forall n \geq R^{(\ell)} :\ 12 \cdot (H_\ell)^{2+\gamma} \cdot \xi(n, b_\ell) \geq \sum_{x \in \mathcal{V}_\ell} \xi(n, x)\right\}^c \\
&\subseteq \bigcup_{x \in \mathcal{V}_\ell} \left\{\exists n \geq R^{(\ell)} :\ \xi(n, x) \geq 6 \cdot \xi(n, b_\ell)\right\}
\end{aligned}$$

for all $\ell = \ell(\gamma, \theta)$ large enough. Here, we further used that

$$|\mathcal{V}_\ell| \leq 2 \cdot (H_\ell)^{2+\gamma}$$

for P-a.e. environment θ and all $\ell = \ell(\theta, \gamma)$ large enough due to (3.37) and (3.38). Therefore, we have for P-a.e. environment θ and all $k = k(\gamma, \theta)$ large enough

$$\begin{aligned}
1 - P_\theta\left(\bigcap_{\ell=k}^{\infty} B_5^{(\ell)}\right) &= P_\theta\left(\bigcup_{\ell=k}^{\infty}\left(B_5^{(\ell)}\right)^c\right) \\
&\leq \sum_{\ell=k}^{\infty} |\mathcal{V}_\ell| \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \leq \sum_{\ell=k}^{\infty} 2 \cdot (H_\ell)^{2+\gamma} \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right)
\end{aligned}$$

$$\leq \sum_{\ell=k}^{\infty} \left(\ell^{1+\gamma} \cdot H_{\ell-1}\right)^{2+\gamma} \cdot \exp\left(-(H_{\ell-1})^{1-\gamma}\right) \xrightarrow{k\to\infty} 0,$$

where we used (3.35) in the last line and again the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)) for the convergence for $k \to \infty$. $\square$

Proof of Lemma 3.4.9. Here, we are in similar situation as in Lemma 4.3 in [DGPS07], and therefore we can use a similar approach. Since in our case, we do not only have to count the number of visits to different positions $x \in \mathbb{Z}$, but we also have to take care of the orientations α_x for $x \in \mathbb{Z}$, we have to be slightly more careful in our upper bounds:

For the proof, we first fix $0 < \beta, \gamma < 1$. In order to make it a little easier to read the proof, we assume in the following that we have

$$m_\ell < b_\ell.$$

In the other case, i.e. $m_\ell > b_\ell$, the proof works completely in the same way if we just consider the symmetric analogues (with respect to the origin) of all appearing quantities.

The main tool for the proof is again to decompose the RWRE into excursions away from the bottom point b_ℓ. Since it might happen that the variance of $Y^{(1)}_{b_\ell,x}$ is relatively big, we have to further decompose our valley $\mathcal{D}_\ell$ into two parts with the help of (cf. (3.44))

$$\eta_\ell := \eta_\ell^+ \stackrel{\text{def}}{=} \inf\{n \geq b_\ell : \ V(n) \geq V(m_\ell)\}.$$

Note that in the considered case we have

$$\eta_\ell \leq r_\ell$$

for P-a.e. environment θ and all $\ell = \ell(\theta)$ large enough since we have $m_\ell = r_{l-1}$ (cf. (3.30)) for large enough $\ell = \ell(\theta)$ and further $r_\ell \geq b_\ell$ with $V(r_\ell) \geq V(r_{\ell-1})$ by definition.

The key observation here is that the RWRE has to reach $\eta_\ell + i$ first before it can reach $\eta_\ell + i + 1$. Since $V(\eta_\ell) - V(b_\ell)$ is large (for large ℓ), we can expect the RWRE to spend a lot of time within the valley $\mathcal{D}_\ell$ before it reaches the positions $\eta_\ell + 1, \eta_\ell + 2, \ldots, r_\ell$, where the time spent within the valley further increases with the increase of the position. Using this approach, we get the following upper bound

$$\begin{aligned}
&P_\theta\left(\exists n \geq R^{(\ell)} : \ (1-\beta) \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \xi(n, b_\ell) \geq s^{(\ell)} \cdot G_n^{(\ell)}\right) \\
= \ & P_\theta\left(\exists n \in [R^{(\ell)}, T(\eta_\ell)) : \ (1-\beta) \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \xi(n, b_\ell) \geq s^{(\ell)} \cdot G_n^{(\ell)}\right) \\
&+ \sum_{i=\eta_\ell}^{r_\ell - 1} P_\theta\left(\exists n \in [T(i), T(i+1)) : \ (1-\beta) \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \xi(n, b_\ell) \geq s^{(\ell)} \cdot G_n^{(\ell)}\right) \\
&+ P_\theta\left(\exists n \geq T(r_\ell) : \ (1-\beta) \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \xi(n, b_\ell) \geq s^{(\ell)} \cdot G_n^{(\ell)}\right) \\
=: \ & I_1 + \sum_{i=\eta_\ell}^{r_\ell - 1} I_2(i) + I_3. \qquad (3.121)
\end{aligned}$$

The way we can derive upper bounds for I_1, I_3 and $I_2(i)$ with $\eta_\ell \le i \le r_\ell - 1$ runs very similarly. The main difference is that we can expect that the random walk has spent more time in the valley $\mathcal{D}_\ell$ before it reaches larger positions i. Recall (cf. (3.75)) the decomposition

$$G_n^{(\ell)} := \mathcal{Z}_0^{(\ell)} + \sum_{j=1}^{\tau(n,\ell)} \sum_{x\in\mathcal{D}_\ell} \alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(j)} + \sum_{x\in\mathcal{D}_\ell} \alpha_x \cdot |\{T_{\tau(n,\ell)}^{(\ell)} < j \le n:\ X_j = x\}|$$

of the accumulated orientations which the RWRE collects on the different excursions it makes away from the bottom point at b_ℓ. Thereby, we have for $g(I) \in \mathbb{N}$, whose precise value we choose later (cf. (3.124)),

$$h := \left\lceil g(I) \cdot \exp\left(-\tfrac{1}{2}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil \quad \text{and } k_v = k_v(I) = \left\lceil g(I) \cdot \exp\left(-\tfrac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \cdot v^2\right\rceil$$

for $v \in \mathbb{N}$, and for $I \in \{I_1, I_3\} \cup \{I_2(i):\ \eta_\ell \le i \le r_\ell - 1\}$ that

$$\begin{aligned}
I \le\ & P_\theta\Big(\xi(T(I), b_\ell) < g(I)\Big) + P_\theta\Big(\big|\mathcal{Z}_0^{(\ell)}\big| > h\Big) \\
&+ \sum_{v=\left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor}^{\infty} \Bigg[P_\theta\left(s^{(\ell)} \cdot \sum_{i=1}^{k_{v+1}} \sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(i)} \le \left(1 - \frac{\beta}{2}\right) \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot k_{v+1}\right) \\
&\qquad + P_\theta\left(\sum_{i=k_v}^{k_{v+1}} \sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \mathcal{Y}_{b_\ell,x}^{(i)} > \frac{\beta}{3} \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot k_{v+1}\right)\Bigg] \\
=:\ & J_1(I) + J_2 + \sum_{v=\left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor}^{\infty} \big(J_3(I,v) + J_4(I,v)\big) \qquad (3.122)
\end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\beta, \gamma, \theta)$ large enough (uniformly in I). Here,

$$T(I) := \begin{cases} R^{(\ell)} & \text{for } I = I_1 \\ T(r_\ell) & \text{for } I = I_3 \\ T(i) & \text{for } I = I_2(i) \end{cases}$$

at which the random walk reaches the following location

$$L(I) := \begin{cases} m_\ell & \text{for } I = I_1 \\ r_\ell & \text{for } I = I_3 \\ i & \text{for } I = I_2(i). \end{cases}$$

In the above sum, the first summand controls the unlikely event that the RWRE makes only a few excursions from b_ℓ to b_ℓ before it reaches the position $L(I)$ (again for $I = I_1$). The second summand gives an upper bound for the probability of the event that the RWRE spends a lot of time in $\mathcal{D}_\ell$ before it reaches the bottom point b_ℓ for the first time. Additionally, the term $J_3(I,v)$ controls the event that the absolute value of the accumulated orientations which the RWRE collects up to the k_{v+1}-th excursion is unlikely low. Finally, the term $J_4(I,v)$ controls the probability of the event on which the time between the k_v-th and the k_{v+1}-th excursion is unlikely long. To see that the inequality really holds, note that we will choose (cf. (3.125))

$$g(I) \xrightarrow{\ell\to\infty} \infty \text{ (uniformly in } I\text{) such that } h, k_0 \gg 0 \text{ for large } \ell.$$

Therefore, we have for P-a.e. environment θ on the complement of the union of all sets which appear in the above inequality (i.e. on the intersection of all complements)

$$(1-\beta)\cdot s^{(\ell)}\cdot \Phi^{(\ell)}\cdot \xi(n,b_\ell) < s^{(\ell)}\cdot G_n^{(\ell)}$$

for all $\ell = \ell(\beta,\gamma,\theta)$ large enough and $n \in [R^{(\ell)}, T(\eta_\ell))$, $n \in [T(i), T(i+1))$ and $n \geq T(r_\ell)$, respectively. Here, we can use that we have

$$k_v \leq \xi(n,b_\ell) < k_{v+1}$$

for some $v \geq \left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor$ on the considered set. Further, recall Proposition 3.4.1 in which we have shown that for $0<\gamma<1$ and $v \geq \left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor$ we have

$$h \leq 2\cdot g(I)\cdot \exp\left(-\tfrac{1}{2}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \leq 2\cdot k_{v+1}(I)\cdot \exp\left(-\tfrac{1}{2}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \ll \frac{\beta}{6}\cdot s^{(\ell)}\cdot\Phi^{(\ell)}\cdot k_{v+1}(I)$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in I) since $|\Phi^{(\ell)}|$ does not decrease to 0 too fast.

We start with $J_1(I)$: Let $K = K(I)$ denote the number of excursions away from b_ℓ to b_ℓ made by the random walk during the time interval $[T(b_\ell), T(I))$. Then $K(I)+1$ has a geometric distribution with parameter

$$\begin{aligned}
p_\ell = p_\ell(I) :=& \left\{\begin{array}{ll} (1-\omega_{b_\ell})\cdot P_\theta^{b_\ell-1}\big(T(L(I)) < T(b_\ell)\big) = (1-\omega_{b_\ell})\cdot \dfrac{e^{V(b_\ell-1)}}{\sum_{i=L(I)}^{b_\ell-1} e^{V(i)}} & \text{if } L(I) < b_\ell \\ \omega_{b_\ell}\cdot P_\theta^{b_\ell+1}\big(T(L(I)) < T(b_\ell)\big) = \omega_{b_\ell}\cdot \dfrac{e^{V(b_\ell)}}{\sum_{i=b_\ell}^{L(I)-1} e^{V(i)}} & \text{if } L(I) > b_\ell \end{array}\right\} \\
\leq& \left\{\begin{array}{ll} e^{V(b_\ell-1)-V(b_\ell)}\cdot \exp\left(-\left(\max\limits_{L(I)\leq i\leq b_\ell-1}\big(V(i)-V(b_\ell)\big)\right)\right) & \text{if } L(I) < b_\ell \\ \exp\left(-\left(\max\limits_{b_\ell\leq i\leq L(I)-1}\big(V(i)-V(b_\ell)\big)\right)\right) & \text{if } L(I) > b_\ell \end{array}\right\} \\
\leq& \left\{\begin{array}{ll} \dfrac{1}{\varepsilon}\cdot \exp\left(-\left(\max\limits_{L(I)\leq i\leq b_\ell-1}\big(V(i)-V(b_\ell)\big)\right)\right) & \text{if } L(I) < b_\ell \\ \exp\left(-\left(\max\limits_{b_\ell\leq i\leq L(I)-1}\big(V(i)-V(b_\ell)\big)\right)\right) & \text{if } L(I) > b_\ell, \end{array}\right.
\end{aligned} \tag{3.123}$$

where we used (3.2) for the last step. Since we have

$$\left.\begin{array}{ll} \max\limits_{L(I)\leq i\leq b_\ell-1}\big(V(i)-V(b_\ell)\big) & \text{if } L(I) < b_\ell \\ \max\limits_{b_\ell\leq i\leq L(I)-1}\big(V(i)-V(b_\ell)\big) & \text{if } L(I) > b_\ell \end{array}\right\} \geq V(m_\ell) - V(b_\ell) = D_\ell$$

for all ℓ large enough (uniformly in I) by definition and due to (3.32), we can choose

$$g(I) := \left\lfloor \big(p_\ell(I)\big)^{-1} \cdot \exp\left(-\frac{1}{3}\big(H_{\ell-1}\big)^{1-\frac{\gamma}{2}}\right)\right\rfloor \tag{3.124}$$

such that we have

$$g(I) \geq \frac{1}{2} \cdot \exp(D_\ell) \cdot \exp\left(-\frac{1}{3}\big(H_{\ell-1}\big)^{1-\frac{\gamma}{2}}\right) \gg 1 \tag{3.125}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in I). For this choice of $g(I)$, we further have

$$\begin{aligned} J_1(I) = P_\theta\big(K(I)+1 < g(I)\big) &\leq 1 - \big(1-p_\ell(I)\big)^{g(I)} \leq p_\ell(I)\cdot g(I) \\ &\leq \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \leq \frac{1}{3}\cdot \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\gamma}\right) \end{aligned} \tag{3.126}$$

again for P-a.e. environment θ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in I).

To get an upper bound for J_2, we first notice that (cf. (3.74))

$$\mathcal{Z}_0^{(\ell)} = \sum_{x\in\mathcal{D}_\ell} \alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(0)} = \sum_{x=m_\ell}^{b_\ell-1} \alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(0)}$$

i.e. $\mathcal{Z}_0^{(\ell)}$ consists of the collected orientations in $\mathcal{D}_\ell$ up to the first visit in b_ℓ. Due to our assumption, we only have to take care of the number of visits to $m_\ell, m_\ell+1,\dots,b_\ell-1$ here. In particular, we can compare our RWRE with a RWRE $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ in an environment $\widetilde{\omega}$ in which the random walk is reflected in $m_\ell-1$ but has the same transition probabilities otherwise. More precisely, we consider the process $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ which is determined by the following properties:

$$\begin{aligned} &P_{\widetilde{\omega}}^{m_\ell-1}\big(\widetilde{X}_0 = m_\ell-1\big) = 1, \\ &P_{\widetilde{\omega}}^{m_\ell-1}\big(\widetilde{X}_{n+1} = \widetilde{X}_n+1 \big| \widetilde{X}_n = x\big) = \omega_x && \text{for } x\geq m_\ell,\ n\in\mathbb{N}_0, \\ &P_{\widetilde{\omega}}^{m_\ell-1}\big(\widetilde{X}_{n+1} = \widetilde{X}_n-1 \big| \widetilde{X}_n = x\big) = 1-\omega_x && \text{for } x\geq m_\ell,\ n\in\mathbb{N}_0, \\ &P_{\widetilde{\omega}}^{m_\ell-1}\big(\widetilde{X}_{n+1} = \widetilde{X}_n+1 \big| \widetilde{X}_n = x\big) = 1 && \text{for } x = m_\ell-1,\ n\in\mathbb{N}_0. \end{aligned}$$

This reflected RWRE in particular fulfills the requirements of the inequality in (A.1) in [Go84] (see also inequality (3.1) in [DGPS07]) which gives

$$E_{\widetilde{\omega}}^{m_\ell-1}\big[\widetilde{T}(x)\big] \leq (x-m_\ell+1)^2 \cdot \exp\left(\max_{m_\ell-1\leq i\leq j<x} \big(V(j)-V(i)\big)\right) \tag{3.127}$$

for $x \geq m_\ell$. Here, $\widetilde{T}(x)$ denotes the first hitting time of x of the random walk $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$. Thereby, we have

$$J_2 = P_\theta\left(\big|\mathcal{Z}_0^{(\ell)}\big| > h\right) \leq P_\theta\left(\sum_{x=m_\ell}^{b_\ell-1} \mathcal{Y}_{b_\ell,x}^{(0)} > h\right) \leq P_{\widetilde{\omega}}^{m_\ell-1}\big(\widetilde{T}(b_\ell) > h\big)$$

$$\leq \frac{E_{\widetilde{\omega}}^{m_\ell - 1}\big[\widetilde{T}(b_\ell)\big]}{h} \leq \frac{(b_\ell - m_\ell + 1)^2}{h} \cdot \exp\left(\max_{m_\ell - 1 \leq i \leq j < b_\ell} \big(V(j) - V(i)\big)\right)$$

$$\leq \frac{(H_\ell)^{4+2\gamma}}{h} \cdot \exp\left(D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \tag{3.128}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough. Here, we used (3.127) for the second line and further (3.39) and (3.45) (applied for $\frac{\gamma}{2}$) for the last step. In particular, we get using the definition of h and (3.125)

$$J_2 \leq \frac{(H_\ell)^{4+2\gamma}}{\left\lceil g(I) \cdot \exp\left(-\frac{1}{2}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rceil} \cdot \exp\left(D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}\right)$$
$$\leq 2 \cdot (H_\ell)^{4+2\gamma} \cdot \exp\left(-\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \leq \frac{1}{3} \cdot \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\gamma}\right) \tag{3.129}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in I), where we further used (3.34) and (3.35) for the last step.

Now, we turn to $J_3(I, v)$ and $J_4(I, v)$. Note that for $i \in \mathbb{N}$ we have

$$s^{(\ell)} \cdot \sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} E_\theta\big[\alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(i)}\big] \geq s^{(\ell)} \cdot \sum_{x=m_\ell}^{r_\ell} E_\theta\big[\alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(i)}\big] - \sum_{x=\eta_\ell+1}^{r_\ell} E_\theta\big[\mathcal{Y}_{b_\ell,x}^{(i)}\big]$$
$$= s^{(\ell)} \cdot \Phi^{(\ell)} - \sum_{x=\eta_\ell+1}^{r_\ell} \frac{\mu_\theta(x)}{\mu_\theta(b_\ell)} = s^{(\ell)} \cdot \Phi^{(\ell)} - \sum_{x=\eta_\ell+1}^{r_\ell} \frac{e^{-V(x)} + e^{-V(x-1)}}{e^{-V(b_\ell)} + e^{-V(b_\ell - 1)}}$$
$$\geq s^{(\ell)} \cdot \Phi^{(\ell)} - 2 \cdot (r_\ell - \eta_\ell) \cdot \exp\left(-\min_{\eta_\ell \leq x \leq r_\ell} \big(V(x) - V(b_\ell)\big)\right)$$
$$\geq s^{(\ell)} \cdot \Phi^{(\ell)} - 2 \cdot (H_\ell)^{2+\gamma} \cdot \exp\left(-(D_\ell)^{1-\gamma}\right) \geq \left(1 - \frac{\beta}{4}\right) \cdot s^{(\ell)} \cdot \Phi^{(\ell)}$$

for P-a.e. environment θ and $\ell = \ell(\beta, \gamma, \theta)$ large enough (uniformly in I), where we used (3.38) and (3.47) in the first part of the last line. Additionally, the exponential growth of D_ℓ (cf. (3.34)) and the slow decrease of $\Phi^{(\ell)}$ to 0 (cf (3.77)) explain the last step.

Using the last inequality, we can conclude with help of Chebyshev's inequality and the independence of the different excursions that we have

$$J_3(I, v) = P_\theta\left(s^{(\ell)} \cdot \sum_{i=1}^{k_{v+1}} \sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(i)} \leq \left(1 - \frac{\beta}{2}\right) \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot k_{v+1}\right)$$
$$\leq P_\theta\left(s^{(\ell)} \cdot \sum_{i=1}^{k_{v+1}} \sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \left(\alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(i)} - E_\theta\big[\alpha_x \cdot \mathcal{Y}_{b_\ell,x}^{(i)}\big]\right) \leq -\frac{\beta}{4} \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot k_{v+1}\right)$$

$$\leq \frac{\operatorname{Var}_\theta\left(\sum_{x=m_\ell}^{\max\{\eta_\ell,L(I)\}} \alpha_x \cdot \mathcal{Y}^{(1)}_{b_\ell,x}\right) \cdot k_{v+1}}{\left(\frac{\beta}{4} \cdot \Phi^{(\ell)} \cdot k_{v+1}\right)^2} = \frac{\operatorname{Var}_\theta\left(\sum_{x=m_\ell}^{\max\{\eta_\ell,L(I)\}} \alpha_x \cdot \mathcal{Y}^{(1)}_{b_\ell,x}\right)}{\left(\frac{\beta}{4} \cdot \Phi^{(\ell)}\right)^2 \cdot k_{v+1}} \tag{3.130}$$

for P-a.e. environment θ and all $\ell = \ell(\beta,\gamma,\theta)$ large enough (uniformly in I). Similarly, we notice that we have

$$\sum_{x=m_\ell}^{\max\{\eta_\ell,L(I)\}} E_\theta\big[\mathcal{Y}^{(i)}_{b_\ell,x}\big] \leq \sum_{x=m_\ell}^{r_\ell} E_\theta\big[\mathcal{Y}^{(i)}_{b_\ell,x}\big] = \sum_{x=m_\ell}^{r_\ell} \frac{\mu_\theta(x)}{\mu_\theta(b_\ell)} = \sum_{x=m_\ell}^{r_\ell} \frac{e^{-V(x)} + e^{-V(x-1)}}{e^{-V(b_\ell)} + e^{-V(b_\ell-1)}}$$

$$\leq 2 \cdot |\mathcal{D}_\ell| \leq 2 \cdot (H_\ell)^{2+\gamma} \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \exp\left((D_\ell)^{\frac{\gamma}{8}}\right)$$

$$\leq 2 \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \left(\ell^{1+\gamma} \cdot H_{\ell-1}\right)^{2+\gamma} \cdot \exp\left(\left(\ell^{1+\gamma} \cdot H_{\ell-1}\right)^{\frac{\gamma}{8}}\right)$$

$$\leq s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \exp\left((H_{\ell-1})^{\frac{\gamma}{4}}\right)$$

for P-a.e. environment θ and $\ell = \ell(\gamma,\theta)$ large enough (uniformly in I). Here, we used (3.38) and (3.77) (applied for $\frac{\gamma}{8}$) in the second line. Further, we made of use of (3.34) and (3.35) in the last two lines. Therefore, we can conclude – using

$$k_{v+1} - k_v + 1 \simeq (2v+1) \cdot g(I) \cdot \exp\left(-\tfrac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) < 3 \cdot k_{v+1} \cdot \exp\left(-\tfrac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)$$

for $v \geq \left\lfloor \exp\left(\tfrac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor$ and all $\ell = \ell(\gamma,\theta)$ large enough (uniformly in I) – that we have

$$\begin{aligned}
J_4(I,v) &= P_\theta\left(\sum_{i=k_v}^{k_{v+1}} \sum_{x=m_\ell}^{\max\{\eta_\ell,L(I)\}} \mathcal{Y}^{(i)}_{b_\ell,x} > \frac{\beta}{3} \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot k_{v+1}\right) \\
&\leq P_\theta\left(\sum_{i=k_v}^{k_{v+1}} \left(\sum_{x=m_\ell}^{\max\{\eta_\ell,L(I)\}} \mathcal{Y}^{(i)}_{b_\ell,x} - E_\theta\big[\mathcal{Y}^{(1)}_{b_\ell,x}\big]\right) > \frac{\beta}{6} \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot k_{v+1}\right) \\
&\leq \frac{\operatorname{Var}_\theta\left(\sum_{x=m_\ell}^{\max\{\eta_\ell,L(I)\}} \mathcal{Y}^{(1)}_{b_\ell,x}\right) \cdot (k_{v+1} - k_v + 1)}{\left(\frac{\beta}{6} \cdot \Phi^{(\ell)} \cdot k_{v+1}\right)^2} \\
&\leq \frac{\operatorname{Var}_\theta\left(\sum_{x=m_\ell}^{\max\{\eta_\ell,L(I)\}} \mathcal{Y}^{(1)}_{b_\ell,x}\right)}{\left(\frac{\beta}{6} \cdot \Phi^{(\ell)}\right)^2 \cdot k_{v+1}}
\end{aligned} \tag{3.131}$$

for P-a.e. environment θ, all $\ell = \ell(\beta, \gamma, \theta)$ large enough (uniformly in I) and

$$v \geq \left\lfloor \exp\left(\tfrac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor .$$

Using the inequality

$$\mathrm{Var}_\theta\left(\sum_{x=1}^{n} \mathcal{Y}^{(1)}_{b_\ell,x}\right) \leq n \cdot \sum_{x=1}^{n} \mathrm{Var}_\theta(\mathcal{Y}^{(1)}_{b_\ell,x}),$$

we therefore have, depending on the choice of I,

$$\mathrm{Var}_\theta\left(\sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \alpha_x \cdot \mathcal{Y}^{(1)}_{b_\ell,x}\right) \leq |\mathcal{D}_\ell| \cdot \sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \mathrm{Var}_\theta\left(\mathcal{Y}^{(1)}_{b_\ell,x}\right) =: \mathrm{Var}_\theta(I) \tag{3.132}$$

and

$$\mathrm{Var}_\theta\left(\sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \mathcal{Y}^{(1)}_{b_\ell,x}\right) \leq |\mathcal{D}_\ell| \cdot \sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \mathrm{Var}_\theta\left(\mathcal{Y}^{(1)}_{b_\ell,x}\right) = \mathrm{Var}_\theta(I). \tag{3.133}$$

Recall that we have the following upper bound for some constant $C > 0$ (cf. (3.73)):

$$\mathrm{Var}_\theta\left(\mathcal{Y}^{(1)}_{b_\ell,x}\right) \leq \begin{cases} C \cdot \exp\Big(-\big(V(x) - V(b_\ell)\big)\Big) \cdot |\mathcal{D}_\ell| \cdot \exp\left(\max\limits_{b_\ell \leq y < x} \big(V(y) - V(x-1)\big)\right) & \\ & \text{if } x > b_\ell \\ 0 & \text{if } x = b_\ell \\ C \cdot \exp\Big(-\big(V(x) - V(b_\ell)\big)\Big) \cdot |\mathcal{D}_\ell| \cdot \exp\left(\max\limits_{x \leq y \leq b_\ell - 1} \big(V(y) - V(x)\big)\right) & \\ & \text{if } x < b_\ell \end{cases}$$

In particular, this implies – using $V(x) - V(b_\ell) \geq 0$ for $x \in \mathcal{D}_\ell$, (3.45), and (3.46) – that we have

$$\sum_{x=m_\ell}^{\eta_\ell} \mathrm{Var}_\theta\left(\mathcal{Y}^{(1)}_{b_\ell,x}\right) \leq \big(\eta_\ell - m_\ell + 1\big) \cdot C \cdot |\mathcal{D}_\ell| \cdot \exp\left(D_\ell - (H_{\ell-1})^{1-\frac{\gamma}{2}}\right)$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough. Similarly for $L(I) \geq \eta_\ell + 1$, we can use (3.47) – applied for $\frac{\gamma}{2}$ together with $H_{\ell-1} \leq D_\ell$ for all ℓ large enough (cf. (3.34)) – to conclude that

$$\begin{aligned} &\sum_{x=\eta_\ell+1}^{L(I)} \mathrm{Var}_\theta\left(\mathcal{Y}^{(1)}_{b_\ell,x}\right) \\ &\leq \big(L(I) - \eta_\ell\big) \cdot C \cdot \exp\left(-(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \cdot |\mathcal{D}_\ell| \cdot \exp\left(\max_{b_\ell \leq y < L(I)-1} \big(V(y) - V(b_\ell)\big)\right) \end{aligned}$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in I). Overall, this yields using (3.123) and (3.125) together with the definition of $g(I)$ that for each choice of

$I \in \{I_1, I_3\} \cup \{I_2(i) : \eta_\ell \le i \le r_\ell - 1\}$ we have

$$\frac{\mathrm{Var}_\theta(I)}{g(I)} = \frac{|\mathcal{D}_\ell| \cdot \left(\sum_{x=m_\ell}^{\max\{\eta_\ell, L(I)\}} \mathrm{Var}_\theta\left(\mathcal{Y}_{b_\ell,x}^{(1)}\right)\right)}{g(I)} \le |\mathcal{D}_\ell|^3 \cdot \frac{2C}{\varepsilon} \cdot \exp\left(-\frac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)$$
$$\le \left(H_{\ell-1} \cdot \ell^{1+\gamma}\right)^{6+3\gamma} \cdot \frac{2C}{\varepsilon} \cdot \exp\left(-\frac{2}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \le \exp\left(-\frac{1}{2}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)$$

for P-a.e. environment θ and all $\ell = \ell(\gamma, \theta)$ large enough (uniformly in I). Here, we used (3.38) and (3.35) for the third step and the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)) for the last step.

Therefore, we find by the means of (3.130) and (3.132) that we have for P-a.e. environment θ and all $\ell = \ell(\beta, \gamma, \theta)$ large enough (uniformly in I)

$$\begin{aligned}
&\sum_{v=\left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor}^{\infty} J_3(I, v) \\
\le\ & \sum_{v=\left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor}^{\infty} \frac{\mathrm{Var}_\theta(I)}{\left(\frac{\beta}{4} \cdot \Phi^{(\ell)}\right)^2 \cdot \left\lceil g(I) \cdot \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \cdot (v+1)^2 \right\rceil} \\
\le\ & \frac{\exp\left(\left(-\frac{1}{2} + \frac{1}{3}\right) \cdot (H_{\ell-1})^{1-\frac{\gamma}{2}}\right)}{\left(\frac{\beta}{4}\right)^2 \cdot \exp\left(-(D_\ell)^{\frac{\gamma}{8}}\right)} \cdot \sum_{v=1}^{\infty} \frac{1}{(v+1)^2} \\
\le\ & \frac{\exp\left(-\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)}{\left(\frac{\beta}{4}\right)^2 \cdot \exp\left(-\left(\ell^{1+\gamma} \cdot H_{\ell-1}\right)^{\frac{\gamma}{8}}\right)} \cdot \sum_{v=1}^{\infty} \frac{1}{(v+1)^2} \\
\le\ & \frac{1}{6} \cdot \exp\left(-\frac{1}{3}\left(H_{\ell-1}\right)^{1-\gamma}\right),
\end{aligned} \tag{3.134}$$

where we further applied Proposition 3.4.1 for $\frac{\gamma}{16}$ in the second step and (3.35) in the third step. Similarly, an application of (3.131) and (3.133) yields

$$\begin{aligned}
&\sum_{v=\left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor}^{\infty} J_4(I, v) \\
\le\ & \sum_{v=\left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor}^{\infty} \frac{\mathrm{Var}_\theta(I)}{\left(\frac{\beta}{6} \cdot \Phi^{(\ell)}\right)^2 \cdot \left\lceil g(I) \cdot \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right) \cdot (v+1)^2 \right\rceil} \\
\le\ & \frac{\exp\left(-\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)}{\left(\frac{\beta}{6}\right)^2 \cdot \exp\left(-(D_\ell)^{\frac{\gamma}{8}}\right)} \cdot \sum_{v=1}^{\infty} \frac{1}{(v+1)^2} \le \frac{1}{6} \cdot \exp\left(-\frac{1}{3}\left(H_{\ell-1}\right)^{1-\gamma}\right)
\end{aligned} \tag{3.135}$$

for P-a.e. environment θ and all $\ell = \ell(\beta, \gamma, \theta)$ large enough (uniformly in I). By combining (3.122) with (3.126), (3.129), (3.134), and (3.135), we therefore get for each $I \in \{I_1, I_3\} \cup \{I_2(i) : \eta_\ell \leq i \leq r_\ell - 1\}$

$$
\begin{aligned}
I &\leq J_1(I) + J_2 + \sum_{v=\left\lfloor \exp\left(\frac{1}{6}(H_{\ell-1})^{1-\frac{\gamma}{2}}\right)\right\rfloor}^{\infty} \big(J_3(I,v) + J_4(I,v)\big) \\
&\leq 2 \cdot \frac{1}{3} \cdot \exp\left(-\frac{1}{3}(H_{\ell-1})^{1-\gamma}\right) + 2 \cdot \frac{1}{6} \cdot \exp\left(-\frac{1}{3}\big(H_{\ell-1}\big)^{1-\gamma}\right) && (3.136) \\
&= \exp\left(-\frac{1}{3}\big(H_{\ell-1}\big)^{1-\gamma}\right) && (3.137)
\end{aligned}
$$

for P-a.e. environment θ and all $\ell = \ell(\beta, \gamma, \theta)$ large (uniformly in I). Finally, we can apply (3.137) to (3.121) to get

$$
\begin{aligned}
&P_\theta\left(\exists n \geq R^{(\ell)} :\ (1-\beta) \cdot s^{(\ell)} \cdot \Phi^{(\ell)} \cdot \xi(n, b_\ell) \geq s^{(\ell)} \cdot G_n^{(\ell)}\right) \\
&\leq I_1 + \sum_{i=\eta_\ell}^{r_\ell - 1} I_2(i) + I_3 \leq (r_\ell - \eta_\ell + 2) \cdot \exp\left(-\frac{1}{3}\big(H_{\ell-1}\big)^{1-\gamma}\right) \\
&\leq (H_\ell)^{2+\gamma} \cdot \exp\left(-\frac{1}{3}\big(H_{\ell-1}\big)^{1-\gamma}\right) \leq \exp\left(-\frac{1}{4}\big(H_{\ell-1}\big)^{1-\gamma}\right)
\end{aligned}
$$

for P-a.e. environment θ and all $\ell = \ell(\beta, \gamma, \theta)$ large enough. Here, we used (3.38) in the third line and (3.35) together with the exponential growth of $H_{\ell-1}$ in ℓ (cf. (3.34)) for the last step. This shows (3.93).

To finish the proof, we note that, for P-a.e. environment θ and all $k = k(\beta, \gamma, \theta)$ large enough, we have

$$
\begin{aligned}
1 - P_\theta\left(\bigcap_{\ell=k}^{\infty} B_6^{(\ell)}\right) &= P_\theta\left(\bigcup_{\ell=k}^{\infty} \left(B_6^{(\ell)}\right)^c\right) \\
&\leq \sum_{\ell=k}^{\infty} \exp\left(-\frac{1}{4}\big(H_{\ell-1}\big)^{1-\gamma}\right) \xrightarrow{k\to\infty} 0,
\end{aligned}
$$

where we used (3.34) for the convergence for $k \to \infty$. □

A combination of Lemma 3.4.4, 3.4.5, 3.4.7, 3.4.6, 3.4.8, and 3.4.9 finally implies (3.82) in Proposition 3.4.2. ■

3.5 Proofs

After the construction of our valleys and the description of the typical behaviour of the RWRE for large time points n, we are now able to prove Theorem 3.2.1 and Theorem 3.2.2. For the following proposition, we let $(\nu_n)_{n\in\mathbb{N}_0}$ denote the sequence of the successive return times of the RWRE $(X_n)_{n\in\mathbb{N}_0}$ to 0. Note here that $(X_n)_{n\in\mathbb{N}_0}$ is recurrent for P-a.e. environment due to assumption (3.1).

At these return times $(\nu_n)_{n\in\mathbb{N}_0}$, we look at the accumulated orientations S_{ν_n} (cf. (3.8)) with

$$S_{\nu_n} \overset{\text{def}}{=} \sum_{i=0}^{\nu_n-1} \alpha_{X_i}$$

which the RWRE $(X_n)_{n\in\mathbb{N}_0}$ has collected before time ν_n. For large n, the RWRE will have either collected a lot more $+1$-orientations or a lot more -1-orientations when it returns to 0 at time ν_n:

Proposition 3.5.1. *For $0 < \vartheta < 1$, we have*

$$\liminf_{n\to\infty} \frac{|S_{\nu_n}|}{\nu_n \cdot \exp\Big(- \big(\log(\nu_n)\big)^{\vartheta}\Big)} = \infty \tag{3.138}$$

P_θ-a.s. for P-a.e. environment θ.

Proof of Proposition 3.5.1. For the proof, we fix an arbitrary $0 < \beta < 1$ and some $0 < \gamma < \frac{1}{4}$, whose precise value we choose later (cf. (3.141)). In a first step, we need to make sure that the valleys of the environment, which we consider, behave "typically" starting from the $(k-1)$-th valley. For this, we define:

$$A_1^{(k)} := \Big\{\theta : \ |\Phi^{(k)}| \geq \exp\big(- (D_k)^{\gamma}\big)\Big\},$$

$$A_2^{(k)} := \Big\{\theta : \ H_k \geq D_k \geq \exp(\beta \cdot k)\Big\},$$

$$A_3^{(k)} := \Big\{\theta : \ H_k \leq k^{1+\gamma} \cdot H_{k-1}\Big\},$$

$$A_4^{(k)} := \Big\{\theta : \ 0 \notin \mathcal{D}_k\Big\},$$

$$E^{(k)} := \bigcap_{\ell=k}^{\infty} \Big(A_1^{(\ell)} \cap A_2^{(\ell)} \cap A_3^{(\ell)} \cap A_4^{(\ell)}\Big).$$

Note that due to Proposition 3.4.1, (3.34),(3.35) and Proposition 3.3.9, we have

$$\mathsf{P}\left(\liminf_{k\to\infty} E^{(k)}\right) = 1. \tag{3.139}$$

In a second step, we need to make sure that (for fixed environment θ) the RWRE $(X_n)_{n\in\mathbb{N}_0}$ behaves "typically" starting from the random time point $R^{(k)}$. To this end, we introduced the following sets in the last section

$$
\begin{aligned}
B_1^{(k)} &:= \left\{\forall n \geq k:\ H_{N_n-1} \leq \frac{1}{1-\beta}\log n\right\} \cap \left\{N_n \xrightarrow{n\to\infty} \infty\right\},\\
B_2^{(k)} &:= \left\{\forall n \leq T_1^{(k+1)}:\ \sum_{x\in\mathbb{Z}\setminus(\mathcal{V}_k\cup\mathcal{D}_{k+1})} \xi(n,x) = 0\right\},\\
B_3^{(k)} &:= \left\{\forall n \geq R^{(k)}:\ \sum_{x\in\mathcal{V}_k\setminus\mathcal{D}_k} \xi(n,x) \leq \exp\left(-(H_{k-1})^{1-\gamma}\right)\cdot\xi(n,b_k)\right\},\\
B_4^{(k)} &:= \left\{\forall n \leq T_1^{(k+1)} \text{ with } X_n \notin \mathcal{D}_{k+1}:\ \sum_{x\in\mathcal{D}_{k+1}} \xi(n,x) \leq \exp\left(-(H_{k-1})^{1-\gamma}\right)\cdot\xi(n,b_k)\right\},\\
B_5^{(k)} &:= \left\{\forall n \geq R^{(k)}:\ 12\cdot(H_k)^{2+\gamma}\cdot\xi(n,b_k) \geq \sum_{x\in\mathcal{V}_k}\xi(n,x)\right\},\\
B_6^{(k)} &:= \left\{\forall n \geq R^{(k)}:\ (1-\beta)\cdot s^{(k)}\cdot\Phi^{(k)}\cdot\xi(n,b_k) < s^{(k)}\cdot G_n^{(k)}\right\},\\
F^{(k)} &:= \bigcap_{\ell=k}^{\infty}\left(B_1^{(\ell)}\cap B_2^{(\ell)}\cap B_3^{(\ell)}\cap B_4^{(\ell)}\cap B_5^{(\ell)}\cap B_6^{(\ell)}\right).
\end{aligned}
$$

Note that $F^{(k)}$ is an increasing sequence of sets by definition for which we have

$$
P_\theta\left(\liminf_{k\to\infty} F^{(k)}\right) = 1 \tag{3.140}
$$

for P-a.e. environment θ due to Proposition 3.4.2.

Now, we can decompose S_{ν_n} into the contribution from the valley $\mathcal{D}_{N_{\nu_n}}$, i.e. from the deepest of the valleys $(\mathcal{D}_k)_{k\in\mathbb{N}_0}$ in which the RWRE has reached the bottom point b_k before time ν_n, and the remaining positions $x \in \mathbb{Z}\setminus\mathcal{D}_{N_{\nu_n}}$. Notice (cf. Remark 3.4.3) that for large ν_n the RWRE has spent most of its time in the valley $\mathcal{D}_{N_{\nu_n}}$ when it returns to 0 at time ν_n. Therefore, this valley has the main influence on the accumulated orientations S_{ν_n}. Note here that for fixed environment $\theta \in E^{(k)}$, we have

$$
\nu_n \geq R^{(N_{\nu_n})}
$$

for all n with $\nu_n \geq R^{(k)}$. This holds since we know that $0 \notin \mathcal{D}_\ell$ for all $\ell \geq k$ due to our choice of the environment $\theta \in E^{(k)}$. Therefore, the RWRE has to leave the valley $\mathcal{D}_{N_{\nu_n}}$ again at time $R^{(N_{\nu_n})}$ – after it has reached the bottom point $b_{N_{\nu_n}}$ – before it can return to 0. Overall, we get for fixed environment $\theta \in E^{(k-1)}$ and $\nu_n \geq R^{(k)}$ on the set $F^{(k)}$

$$
\left|S_{\nu_n}\right| = \left|G_{\nu_n}^{(N_{\nu_n})} + \sum_{x\in\mathbb{Z}\setminus\mathcal{D}_{N_{\nu_n}}} \alpha_x\cdot\xi(\nu_n-1,x)\right| \geq \left|G_{\nu_n}^{(N_{\nu_n})}\right| - \sum_{x\in\mathbb{Z}\setminus\mathcal{D}_{N_{\nu_n}}} \xi(\nu_n-1,x)
$$

$$\geq \Big((1-\beta)\cdot s^{(N_{\nu_n})}\cdot \Phi^{(N_{\nu_n})} - 2\cdot\exp\left(-(H_{N_{\nu_n}-1})^{1-\gamma}\right)\Big)\cdot \xi(\nu_n, b_{N_{\nu_n}})$$

$$\geq \Big((1-\beta)\cdot \exp\left(-(D_{N_{\nu_n}})^{\gamma}\right) - 2\cdot\exp\left(-(H_{N_{\nu_n}-1})^{1-\gamma}\right)\Big)\cdot \xi(\nu_n, b_{N_{\nu_n}})$$

$$\geq \Big((1-\beta)\cdot \exp\left(-\left((N_{\nu_n})^{1+\gamma}\cdot H_{N_{\nu_n}-1}\right)^{\gamma}\right) - 2\cdot\exp\left(-(H_{N_{\nu_n}-1})^{1-\gamma}\right)\Big)$$

$$\cdot\, \xi(\nu_n, b_{N_{\nu_n}})$$

for all $n = n(\beta,\gamma,\theta)$ large enough. Note that, due to the exponential growth of $H_k \geq D_k$ in k and our choice $0 < \gamma < \frac{1}{4}$, the difference in the first of the last two lines is positive for all n large enough (and with this $\nu_n \geq 2n$ and N_{ν_n} large enough). Therefore, we can conclude that

$$\begin{aligned}
\left|S_{\nu_n}\right| &\geq \exp\Big(-\left(H_{N_{\nu_n}-1}\right)^{2\gamma}\Big)\cdot \frac{\sum\limits_{x\in\mathcal{V}_{N_{\nu_n}}} \xi(\nu_n,x)}{12\cdot (H_{N_{\nu_n}})^{2+\gamma}} \\
&\geq \exp\Big(-\left(H_{N_{\nu_n}-1}\right)^{2\gamma}\Big)\cdot \frac{\nu_n - \sum\limits_{x\in\mathbb{Z}\setminus\mathcal{V}_{N_{\nu_n}}} \xi(\nu_n,x)}{12\cdot \left((N_{\nu_n})^{1+\gamma}\cdot (H_{N_{\nu_n}-1})\right)^{2+\gamma}} \\
&\geq \exp\Big(-\left(H_{N_{\nu_n}-1}\right)^{3\gamma}\Big)\cdot \frac{\nu_n}{2} \geq \exp\left(-\left(\frac{1}{1-\beta}\right)^{3\gamma}\cdot \left(\log(\nu_n)\right)^{3\gamma}\right)\cdot\frac{\nu_n}{2} \\
&\geq \exp\Big(-\left(\log(\nu_n)\right)^{4\gamma}\Big)\cdot \nu_n
\end{aligned}$$

for all $n = n(\beta,\gamma,\theta)$ large enough. By choosing

$$\gamma = \frac{\vartheta}{4}, \tag{3.141}$$

we get for $\theta \in E^{(k-1)}$ on the set $F^{(k)}$

$$\left|S_{\nu_n}\right| \geq \exp\Big(-\left(\log(\nu_n)\right)^{\vartheta}\Big)\cdot \nu_n \tag{3.142}$$

for all $n = n(\vartheta,\theta)$ large enough.

Since (3.142) holds for all $0 < \vartheta < 1$, we can conclude by using (3.139) and (3.140) that (3.138) holds. ■

Proof of Theorem 3.2.2. Theorem 3.2.2 is a direct consequence of Proposition 3.5.1. ■

Proof of Theorem 3.2.1. We choose $\gamma = \frac{3}{4}$ and define the sets

$$M_n := \left\{\left|S_{\nu_n}\right| \geq \left(\nu_n\right)^{\frac{3}{4}}\right\}$$

for $n \in \mathbb{N}$. Due to (3.138), we have

$$P_\theta\left(\liminf_{n\to\infty} M_n\right) = 1 \tag{3.143}$$

for P-a.e. environment θ. Recall the definition of $(\tau_n)_{n\in\mathbb{N}_0}$ from (3.6): τ_n denotes the first random time point at which the random walk $(Z_\ell)_{\ell\in\mathbb{N}_0} = (\widetilde{X}_\ell, \widetilde{Y}_\ell)_{\ell\in\mathbb{N}_0}$ has moved in the first component for the n-th time. Notice that $(\tau_{n+1}-\tau_n)_{n\in\mathbb{N}_0}$ is a sequence of i.i.d. random variables with a geometric distribution with parameter δ which are independent of $(X_n)_{n\in\mathbb{N}_0}$ and the environment θ. Further, recall that we have introduced $(X_n, Y_n)_{n\in\mathbb{N}_0}$ as the embedded process for which we only consider the positions of the process $(Z_n)_{n\in\mathbb{N}_0} = (\widetilde{X}_n, \widetilde{Y}_n)_{n\in\mathbb{N}_0}$ right after this second process has moved in the first component. Thereby, we have the following decomposition of the second component for the embedded process:

$$\begin{aligned} Y_n &= \sum_{k=0}^{n-1} \alpha_{X_k} \cdot \big(\tau_{k+1} - \tau_k\big) \\ &\overset{d}{=} \sum_{k=1}^{\frac{n+S_n}{2}} \chi_k^{(+)} - \sum_{k=1}^{\frac{n-S_n}{2}} \chi_k^{(-)} \end{aligned}$$

Here, $(\chi_k^{(+)})_{k\in\mathbb{N}_0}$ and $(\chi_k^{(-)})_{k\in\mathbb{N}_0}$ denote two independent sequences of i.i.d. random variables with a geometric distribution with parameter δ, where $(\chi_k^{(+)})_{k\in\mathbb{N}_0}$ and $(\chi_k^{(-)})_{k\in\mathbb{N}_0}$ are further independent of $(X_n)_{n\in\mathbb{N}_0}$ and the environment θ.

In a first step, we show that Y_{ν_n} will not be close to 0 with high probability. Therefore, we start with

$$\begin{aligned} &P_\theta\big(- c\cdot \log(n) \le Y_{\nu_n} \le c \cdot \log(n) \big| \nu_n, S_{\nu_n}\big) \\ &= P_\theta\left(-c\cdot\log(n) \le \sum_{k=1}^{\frac{\nu_n+S_{\nu_n}}{2}} \chi_k^{(+)} - \sum_{k=1}^{\frac{\nu_n-S_{\nu_n}}{2}} \chi_k^{(-)} \le c\cdot\log(n) \,\middle|\, \nu_n, S_{\nu_n}\right) \\ &\le P_\theta\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}} \chi_k^{(+)} - \sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}} \chi_k^{(-)} \le c\cdot\log(n) \,\middle|\, \nu_n, S_{\nu_n}\right) \\ &\le P_\theta\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}} \chi_k^{(+)} \le \frac{\nu_n}{2\delta} + (\nu_n)^{\frac{2}{3}} + c\cdot\log(n) \,\middle|\, \nu_n, S_{\nu_n}\right) \\ &\quad + P_\theta\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}} \chi_k^{(-)} \ge \frac{\nu_n}{2\delta} - (\nu_n)^{\frac{2}{3}} \,\middle|\, \nu_n, S_{\nu_n}\right), \end{aligned} \tag{3.144}$$

where $c > 0$ denotes a constant which we choose later (cf. (3.149)). Note that the additional term $\pm(\nu_n)^{\frac{2}{3}}$ is not necessary for our proof here but it enables us to directly extend our proof of Theorem 3.2.1 to the more general setting in Theorem 3.6.1.

For the forthcoming inequalities, the common moment generating function $\phi(t)$ of $\chi_1^{(+)}$ and $\chi_1^{(-)}$ is helpful. Since $\chi_1^{(+)}$ and $\chi_1^{(-)}$ both have a geometric distribution, $\phi(t)$ is finite for $t < -\log(1-\delta)$. In particular, $\phi(t)$ is finite in a neighbourhood of 0, and therefore we have (cf. Theorem 4.8.3 (iii) in [Gut05])

$$\begin{aligned}\phi(t) &:= E_\theta\left[\exp\left(t\cdot\chi_1^{(+)}\right)\right] = 1 + E_\theta[\chi_1^{(+)}]\cdot t + E_\theta[(\chi_1^{(+)})^2]\cdot\frac{t^2}{2} + O(t^3)\\ &= 1 + \frac{t}{\delta} + \frac{2-\delta}{\delta^2}\cdot\frac{t^2}{2} + O(t^3)\ .\end{aligned}$$

By using the inequality $1 + x \le \exp(x)$ for all $x \in \mathbb{R}$, this implies that we have

$$\phi(t) \le \exp\left(\frac{t}{\delta} + \frac{2-\delta}{2\cdot\delta^2}\cdot t^2 + C\cdot|t^3|\right) \tag{3.145}$$

for all $-1 \le t \le -\frac{1}{2}\log(1-\delta)$, where $C > 0$ denotes a suitable constant. For the first summand in (3.144), we can derive the following upper bound for arbitrary

$$-1 < t = t(n) < 0$$

by applying Markov's inequality and using $\nu_n \ge 2n$

$$\begin{aligned}&P_\theta\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)} \le \frac{\nu_n}{2\delta} + (\nu_n)^{\frac{2}{3}} + c\cdot\log(n)\,\middle|\,\nu_n, S_{\nu_n}\right)\\ \le\ & P_\theta\left(\exp\left(t\cdot\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right) \ge \exp\left(t\cdot\left(\frac{\nu_n}{2\delta} + 2\cdot(\nu_n)^{\frac{2}{3}}\right)\right)\,\middle|\,\nu_n, S_{\nu_n}\right)\\ \le\ & \exp\left(-t\cdot\left(\frac{\nu_n}{2\delta} + 2\cdot(\nu_n)^{\frac{2}{3}}\right)\right)\cdot\left(\phi(t)\right)^{\frac{\nu_n+|S_{\nu_n}|}{2}}\\ \le\ & \exp\left(-t\cdot\left(\frac{\nu_n}{2\delta} + 2\cdot(\nu_n)^{\frac{2}{3}}\right)\right)\cdot\exp\left(\frac{\nu_n+|S_{\nu_n}|}{2}\cdot\left(\frac{t}{\delta} + \frac{2-\delta}{2\cdot\delta^2}\cdot t^2 + C\cdot|t^3|\right)\right)\\ =\ & \exp\left(t\cdot\left(\frac{|S_{\nu_n}|}{2\delta} - 2\cdot(\nu_n)^{\frac{2}{3}}\right) + t^2\cdot\frac{2-\delta}{2\cdot\delta^2}\cdot\frac{\nu_n+|S_{\nu_n}|}{2} + |t^3|\cdot C\cdot\frac{\nu_n+|S_{\nu_n}|}{2}\right)\end{aligned}$$

for all n large enough. Here, we used (3.145) for the third step. By choosing

$$t = t(n) := -\left(\nu_n\right)^{-\frac{2}{3}},$$

we therefore get on the set $M_n = \left\{\left|S_{\nu_n}\right| \ge \left(\nu_n\right)^{\frac{3}{4}}\right\}$

$$P_\theta\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)} \le \frac{\nu_n}{2\delta} + (\nu_n)^{\frac{2}{3}} + c\cdot\log(n)\,\middle|\,\nu_n, S_{\nu_n}\right) \le \exp\left(-n^{\frac{1}{13}}\right) \tag{3.146}$$

for all n and with this $\nu_n \ge 2n$ large enough.

Now, we can turn to the second summand in (3.144). Similarly for

$$0 < s = s(n) < -\tfrac{1}{2}\log(1-\delta),$$

we get by applying Markov's inequality and again using $\nu_n \geq 2n$

$$\begin{aligned}
&P_\theta\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}} \chi_k^{(-)} \geq \frac{\nu_n}{2\delta} - (\nu_n)^{\frac{2}{3}} \middle| \nu_n, S_{\nu_n}\right)\\
\leq\ & P_\theta\left(\exp\left(s\cdot\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}} \chi_k^{(-)}\right) \geq \exp\left(s\cdot\left(\frac{\nu_n}{2\delta} - (\nu_n)^{\frac{2}{3}}\right)\right) \middle| \nu_n, S_{\nu_n}\right)\\
\leq\ & \exp\left(-s\cdot\left(\frac{\nu_n}{2\delta} - (\nu_n)^{\frac{2}{3}}\right)\right)\cdot\left(\phi(s)\right)^{\frac{\nu_n-|S_{\nu_n}|}{2}}\\
\leq\ & \exp\left(-s\cdot\left(\frac{\nu_n}{2\delta} - (\nu_n)^{\frac{2}{3}}\right)\right)\cdot\exp\left(\frac{\nu_n-|S_{\nu_n}|}{2}\cdot\left(\frac{s}{\delta}+\frac{2-\delta}{2\cdot\delta^2}\cdot s^2 + C\cdot|s^3|\right)\right)\\
=\ & \exp\left(s\cdot\left(-\frac{|S_{\nu_n}|}{2\delta} + (\nu_n)^{\frac{2}{3}}\right) + s^2\cdot\frac{2-\delta}{2\cdot\delta^2}\cdot\frac{\nu_n-|S_{\nu_n}|}{2} + |s^3|\cdot C\cdot\frac{\nu_n-|S_{\nu_n}|}{2}\right)
\end{aligned}$$

for all n large enough, where we again used (3.145) for the third step. By choosing

$$s = s(n) := \left(\nu_n\right)^{-\frac{2}{3}},$$

we therefore get on the set $M_n = \left\{\left|S_{\nu_n}\right| \geq \left(\nu_n\right)^{\frac{3}{4}}\right\}$

$$P_\theta\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}} \chi_k^{(-)} \geq \frac{\nu_n}{2\delta} - (\nu_n)^{\frac{2}{3}} \middle| \nu_n, S_{\nu_n}\right) \leq \exp\left(-n^{\frac{1}{13}}\right) \tag{3.147}$$

for all n and with this $\nu_n \geq 2n$ large enough.

Inserting (3.146) and (3.147) in (3.144) now yields

$$P_\theta\big(\{-c\cdot\log(n) \leq Y_{\nu_n} \leq c\cdot\log(n)\}\cap M_n\big) \leq 2\cdot\exp\left(-n^{\frac{1}{13}}\right)$$

for all n large enough. Therefore, the Borel-Cantelli lemma and (3.143) imply

$$P_\theta\left(Y_{\nu_n} \in [-c\cdot\log(n), c\cdot\log(n)] \text{ for infinitely many } n\right) = 0 \tag{3.148}$$

for P-a.e. environment θ and all $c > 0$. This in particular shows the transience of the process $(Z_{\tau_n})_{n\in\mathbb{N}_0} = (X_n, Y_n)_{n\in\mathbb{N}_0}$ (cf. (3.7)) for which we only consider the time points at which the first coordinate of the process $(Z_n)_{n\in\mathbb{N}_0}$ has changed in the last step.

In a last step, we only need to make sure that the process $(Z_n)_{n\in\mathbb{N}_0} = (\widetilde{X}_n, \widetilde{Y}_n)_{n\in\mathbb{N}_0}$ – which we are mainly interested in – does not make too many steps in the second component without moving in the first component. For this, we note that for $n\in\mathbb{N}_0$ we have

$$P_\theta\big(\tau_{\nu_n+1} - \tau_{\nu_n} \geq c\cdot\log(n)\big) = (1-\delta)^{\lfloor c\cdot\log(n)\rfloor} \leq n^{c\cdot\log(1-\delta)}\cdot\frac{1}{1-\delta}\ .$$

Here, $\tau_{\nu_n+1} - \tau_{\nu_n}$ denotes the number of movements of $(Z_n)_{n\in\mathbb{N}_0}$ in the second component after returning to 0 in the first component at time ν_n up to leaving 0 again in the first component at time $\nu_n + 1$. By choosing $c = c(\delta)$ large enough such that

$$c \cdot \log(1-\delta) < -1, \tag{3.149}$$

we can therefore conclude with the help of the Borel-Cantelli lemma that

$$P_\theta\big(\tau_{\nu_n+1} - \tau_{\nu_n} \geq c \cdot \log(n) \text{ for infinitely many } n\big) = 0 \tag{3.150}$$

holds for P-a.e. environment θ. Finally, we can combine (3.148) and (3.150) to conclude that we have

$$P_\theta\big(Z_n = (0,0) \text{ for infinitely many } n\big) = 0$$

for P-a.e. environment θ. For the last conclusion, we use that, as long as

$$\tau_{\nu_n+1} - \tau_{\nu_n} < c \cdot \log(n)$$

holds for some $n \in \mathbb{N}_0$, the random walk $(Z_\ell)_{\ell\in\mathbb{N}_0}$ cannot reach $(0,0)$ between the time points $\ell = \nu_n$ and $\ell = \nu_n + 1$ as long as $Y_{\nu_n} \notin [-c \cdot \log(n), c \cdot \log(n)]$, which was left to show.

■

3.6 A Final Extension of RWRE with Random Orientations

As one possible extension of the RWRE with random orientations, we can look at the following process which combines our results from Chapter 2 and Section 3.2:
We still use the same random environment θ as introduced in Section 3.2. The only difference is that we need one more parameter $p \in [0,1]$. For every environment θ, we can now look at the following Markov chain $(M_n)_{n\in\mathbb{N}_0}$ with values in $\mathbb{Z}^2$ which is determined by

$$\begin{aligned}
&P_\theta^z\big(M_0 = (z_1, z_2)\big) = 1,\\
&P_\theta^z\big(M_{n+1} = (k+1, \ell)\big| M_n = (k,\ell)\big) = \delta \cdot \omega_k,\\
&P_\theta^z\big(M_{n+1} = (k-1, \ell)\big| M_n = (k,\ell)\big) = \delta \cdot (1-\omega_k),\\
&P_\theta^z\big(M_{n+1} = (k, \ell+1)\big| M_n = (k,\ell)\big) = \frac{1-\delta}{2} \cdot (1 + p \cdot \alpha_k),\\
&P_\theta^z\big(M_{n+1} = (k, \ell-1)\big| M_n = (k,\ell)\big) = \frac{1-\delta}{2} \cdot (1 - p \cdot \alpha_k)
\end{aligned}$$

for $z = (z_1, z_2) \in \mathbb{Z}^2$.

The process can be understood as a combination of a one-dimensional RWRE with an asymmetric random walk on $\mathbb{Z}$ for $0 < p \leq 1$ and a symmetric random walk for $p = 0$. In the case $0 < p \leq 1$, the random orientation α_k contained in the random environment θ indicates if the Markov chain at time n prefers to move upwards or downwards in the second component given $M_n = (k, \cdot)$ for some $k \in \mathbb{Z}$. Then, we have the following dependency of the parameter $p \in [0, 1]$:

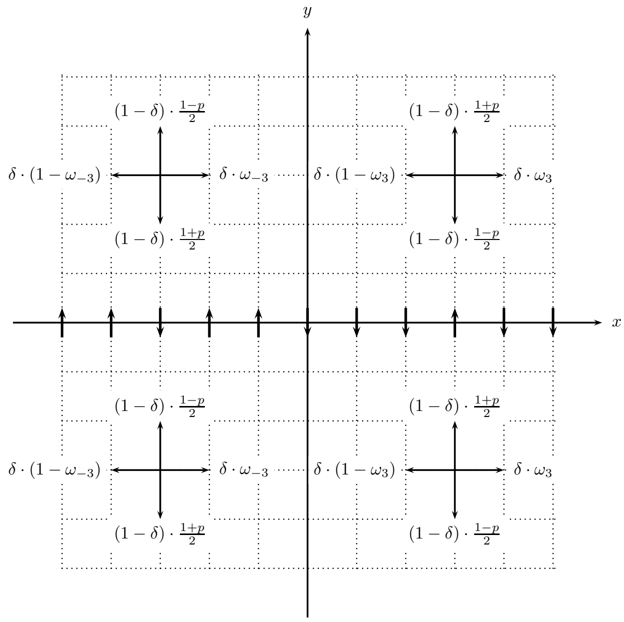

Figure 3.6: *A possible realization of the random orientations ↑↓ and the corresponding transition probabilities for the extended model.*

Theorem 3.6.1. *(1) For $p = 0$, the Markov chain $(M_n)_{n\in\mathbb{N}_0}$ is recurrent for P-a.e. environment θ.*

(2) On the contrary for $0 < p \leq 1$, the Markov chain $(M_n)_{n\in\mathbb{N}_0}$ is transient for P-a.e. environment θ.

Proof of Theorem 3.6.1. In the first case, we are exactly in the situation of Corollary 2.6.7. Therefore, there is nothing left to show.

In the second case, i.e. $0 < p \leq 1$, the proof runs almost as the proof of Theorem 3.2.1 and we can adjust our argument to the new situation:

Let us denote the second component of $(M_n)_{n\in\mathbb{N}_0}$ by

$$\widetilde{Y}_n^* := \mathrm{pr}_2(M_n) \qquad \text{for } n \in \mathbb{N}_0,$$

and note that the first component

$$\widetilde{X}_n := \mathrm{pr}_1(M_n) \qquad \text{for } n \in \mathbb{N}_0$$

for each choice of $0 < p \leq 1$ still behaves as $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ which was introduced in (3.5). As in (3.6), we can introduce

$$\begin{aligned} \tau_0 &:= 0, \\ \tau_k &:= \inf\{n > \tau_{k-1} : \ \widetilde{X}_n \neq \widetilde{X}_{\tau_{k-1}}\} \qquad \text{for } k \geq 1 \end{aligned}$$

inductively as the time points of the movement in the first component. Now, we can again define

$$X_n := \widetilde{X}_{\tau_n},$$

$$\overline{Y}_n := \widetilde{Y}^*_{\tau_n}$$

for $n \in \mathbb{N}_0$. As before, $(X_n)_{n\in\mathbb{N}_0}$ behaves as a RWRE on $\mathbb{Z}$.

At the random successive return times $(\nu_n)_{n\in\mathbb{N}_0}$ of the first component $(X_n)_{n\in\mathbb{N}_0}$ to 0, we now have the following decomposition of the movement in the second component $(\overline{Y}_n)_{n\in\mathbb{N}_0}$:

$$\begin{aligned}\overline{Y}_{\nu_n} &\overset{d}{=} \sum_{k=0}^{\nu_n-1} \alpha_{X_k} \cdot \Upsilon^{(k)}(\tau_{k+1}-\tau_k)\\ &\overset{d}{=} \Upsilon^{(1)}\left(\sum_{k=1}^{\frac{\nu_n+S_{\nu_n}}{2}} \chi_k^{(+)}\right) - \Upsilon^{(2)}\left(\sum_{k=1}^{\frac{\nu_n-S_{\nu_n}}{2}} \chi_k^{(-)}\right) \end{aligned} \tag{3.151}$$

Here, $(\chi_k^{(+)})_{k\in\mathbb{N}_0}$ and $(\chi_k^{(-)})_{k\in\mathbb{N}_0}$ again denote two independent sequences of i.i.d. random variables with a geometric distribution with parameter δ, where $(\chi_k^{(+)})_{k\in\mathbb{N}_0}$ and $(\chi_k^{(-)})_{k\in\mathbb{N}_0}$ are further independent of $(X_n)_{n\in\mathbb{N}_0}$ and the environment θ. Additionally, $(\Upsilon^{(0)}(\ell))_{\ell\in\mathbb{N}_0}$, $(\Upsilon^{(1)}(\ell))_{\ell\in\mathbb{N}_0}$,$(\Upsilon^{(2)}(\ell))_{\ell\in\mathbb{N}_0}$, ... denote i.i.d. asymmetric next neighbour random walks on $\mathbb{Z}$ with

$$\begin{aligned}&P_\theta\big(\Upsilon^{(1)}(0)=0\big)=1,\\ &P_\theta\big(\Upsilon^{(1)}(\ell+1)=y+1\big|\Upsilon^{(1)}(\ell)=y\big)=\frac{1+p}{2} \quad \text{for all } y\in\mathbb{Z},\ \ell\in\mathbb{N}_0,\\ &P_\theta\big(\Upsilon^{(1)}(\ell+1)=y-1\big|\Upsilon^{(1)}(\ell)=y\big)=\frac{1-p}{2} \quad \text{for all } y\in\mathbb{Z},\ \ell\in\mathbb{N}_0\end{aligned}$$

for every environment θ, where the sequence $(\Upsilon^{(0)}(\ell))_{\ell\in\mathbb{N}_0}$, $(\Upsilon^{(1)}(\ell))_{\ell\in\mathbb{N}_0}$,$(\Upsilon^{(2)}(\ell))_{\ell\in\mathbb{N}_0}$, ... is further independent of $(\chi_k^{(+)})_{k\in\mathbb{N}_0}$ and $(\chi_k^{(-)})_{k\in\mathbb{N}_0}$, $(X_n)_{n\in\mathbb{N}_0}$, and the environment θ. As before, the moment generating function will be helpful for our consideration. For $t\in\mathbb{R}$, we define

$$\begin{aligned}\psi(t) &:= E_\theta\left[\exp\left(t\cdot\Upsilon^{(1)}(1)\right)\right] = 1 + E_\theta\left[\Upsilon^{(1)}(1)\right]\cdot t + E_\theta\left[\left(\Upsilon^{(1)}(1)\right)^2\right]\cdot\frac{t^2}{2} + O(t^3)\\ &= 1 + p\cdot t + \frac{t^2}{2} + O(t^3),\end{aligned}$$

where the second relation holds since $\psi(t)<\infty$ holds in a neighbourhood of 0 (which is $\mathbb{R}$ here). Again using the inequality $1+x\le\exp(x)$ for $x\in\mathbb{R}$, we get for $-1\le t\le 1$ that

$$\psi(t) \le \exp\left(p\cdot t + \frac{t^2}{2} + C\cdot|t^3|\right) \tag{3.152}$$

for some suitable constant $C>0$. Similarly to the proof of Theorem 3.2.1, we get the following for arbitrary $c>0$ by using (3.151):

$$P_\theta\big(-c\cdot\log(n) \le \overline{Y}_{\nu_n} \le c\cdot\log(n)\big)$$

$$= P_\theta\left(-c\cdot\log(n) \le \Upsilon^{(1)}\left(\sum_{k=1}^{\frac{\nu_n+S_{\nu_n}}{2}}\chi_k^{(+)}\right) - \Upsilon^{(2)}\left(\sum_{k=1}^{\frac{\nu_n-S_{\nu_n}}{2}}\chi_k^{(-)}\right) \le c\cdot\log(n)\right)$$
$$\le P_\theta\left(\Upsilon^{(1)}\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right) - \Upsilon^{(2)}\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right) \le c\cdot\log(n)\right)$$

Recall that a combination of (3.143), (3.146), and (3.147) implies that we have

$$\mathsf{P}_\theta\left(\liminf_{k\to\infty}\left\{\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)} \ge \frac{\nu_n}{2\delta} + (\nu_n)^{\frac{2}{3}}\right\}\right) = 1$$
$$\mathsf{P}_\theta\left(\liminf_{k\to\infty}\left\{\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)} \le \frac{\nu_n}{2\delta} - (\nu_n)^{\frac{2}{3}}\right\}\right) = 1 \tag{3.153}$$

for P-a.e. environment θ.

In comparison with the upper bound in (3.144), here we also need upper bounds for the atypical behaviour of the asymmetric random walks $(\Upsilon^{(1)}(\ell))_{\ell\in\mathbb{N}_0}$ and $(\Upsilon^{(2)}(\ell))_{\ell\in\mathbb{N}_0}$. Thereby, we have:

$$P_\theta\left(\Upsilon^{(1)}\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right) - \Upsilon^{(2)}\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right) \le c\cdot\log(n)\,\middle|\,\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)},\ \sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right)$$
$$\le P_\theta\left(\Upsilon^{(1)}\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right) \le \frac{p\cdot\nu_n}{2\delta} + c\cdot\log(n)\,\middle|\,\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)},\ \sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right)$$
$$+ P_\theta\left(\Upsilon^{(2)}\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right) \ge \frac{p\cdot\nu_n}{2\delta}\,\middle|\,\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)},\ \sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right) \tag{3.154}$$

Notice that there exists $\lambda = \lambda(p) \in [-1, 0)$ such that

$$\psi(t) \stackrel{\text{def}}{=} E_\theta\left[\exp\left(t\cdot\Upsilon^{(1)}(1)\right)\right] = \frac{1+p}{2}\cdot\exp(t) + \frac{1-p}{2}\cdot\exp(-t) < 1$$

for all $\lambda(p) \le t < 0$. Therefore, on the set

$$A_n^{(1)} := \left\{\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)} \ge \frac{\nu_n}{2\delta} + (\nu_n)^{\frac{2}{3}}\right\},$$

and for $\lambda(p) \le t = t(n) < 0$, we get for the first summand in (3.154) by using Markov's inequality

$$P_\theta\left(\Upsilon^{(1)}\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right) \le \frac{p\cdot\nu_n}{2\delta} + c\cdot\log(n)\,\middle|\,\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)},\ \sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right)$$

$$= P_\theta\left(\exp\left(t\cdot\Upsilon^{(1)}\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right)\right)\geq \exp\left(t\cdot\left(\frac{p\cdot\nu_n}{2\delta}+c\cdot\log(n)\right)\right)\,\middle|\,\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right)$$

$$\leq \exp\left(-t\cdot\left(\frac{p\cdot\nu_n}{2\delta}+c\cdot\log(n)\right)\right)\cdot\exp\left(\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right)\cdot\log\left(\psi(t)\right)\right)$$

$$\leq \exp\left(-t\cdot\left(\frac{p\cdot\nu_n}{2\delta}+c\cdot\log(n)\right)\right)\cdot\exp\left(\left(\frac{\nu_n}{2\delta}+(\nu_n)^{\frac{2}{3}}\right)\cdot\left(t\cdot p+\frac{t^2}{2}+C\cdot|t^3|\right)\right)$$

$$= \exp\left(t\cdot\left(-c\cdot\log(n)+p\cdot(\nu_n)^{\frac{2}{3}}\right)+\frac{t^2}{2}\cdot\left(\frac{\nu_n}{2\delta}+(\nu_n)^{\frac{2}{3}}\right)+|t^3|\cdot C\cdot\left(\frac{\nu_n}{2\delta}+(\nu_n)^{\frac{2}{3}}\right)\right).$$

Here in the third line, we used that, for $\ell\in\mathbb{N}_0$, $\Upsilon^{(1)}(\ell)$ has the same distribution as the sum of ℓ i.i.d. copies of $\Upsilon^{(1)}(1)$. For the fourth line, we used (3.152) and $\psi(t)<1$ for $\lambda(p)\leq t<0$. By choosing $t=t(n):=-(\nu_n)^{-\frac{1}{2}}$, the last upper bound implies that on the set $A_n^{(1)}$ we have

$$P_\theta\left(\Upsilon^{(1)}\left(\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)}\right)\leq\frac{p\cdot\nu_n}{2\delta}+c\cdot\log(n)\,\middle|\,\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)},\ \sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right)\leq\exp\left(-n^{\frac{1}{7}}\right) \tag{3.155}$$

for all n large enough, where we further used $\nu_n\geq 2n$ for the last step. For the second summand in the upper bound in (3.154), we similarly get on the set

$$A_n^{(2)}:=\left\{\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\leq\frac{\nu_n}{2\delta}-(\nu_n)^{\frac{2}{3}}\right\}$$

for $0<s=s(n)\leq 1$ again using Markov's inequality

$$P_\theta\left(\Upsilon^{(2)}\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right)\geq\frac{p\cdot\nu_n}{2\delta}\,\middle|\,\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)},\ \sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right)$$

$$= P_\theta\left(\exp\left(s\cdot\Upsilon^{(2)}\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right)\right)\geq\exp\left(s\cdot\frac{p\cdot\nu_n}{2\delta}\right)\,\middle|\,\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right)$$

$$\leq\exp\left(-s\cdot\frac{p\cdot\nu_n}{2\delta}\right)\cdot\exp\left(\left(\frac{\nu_n}{2\delta}-(\nu_n)^{\frac{2}{3}}\right)\cdot\left(s\cdot p+\frac{s^2}{2}+C\cdot|s^3|\right)\right)$$

$$=\exp\left(-s\cdot p\cdot(\nu_n)^{\frac{2}{3}}+s^2\cdot\frac{1}{2}\cdot\left(\frac{\nu_n}{2\delta}-(\nu_n)^{\frac{2}{3}}\right)+|s^3|\cdot C\cdot\left(\frac{\nu_n}{2\delta}-(\nu_n)^{\frac{2}{3}}\right)\right),$$

where we further used $\psi(s) > 1$ for $s > 0$ for the second step. By choosing

$$s = s(n) := (\nu_n)^{-\frac{1}{2}},$$

the last upper bound implies that on the set $A_n^{(2)}$ we have

$$P_\theta\left(\Upsilon^{(2)}\left(\sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right) \geq \frac{p\cdot\nu_n}{2\delta}\,\middle|\,\sum_{k=1}^{\frac{\nu_n+|S_{\nu_n}|}{2}}\chi_k^{(+)},\ \sum_{k=1}^{\frac{\nu_n-|S_{\nu_n}|}{2}}\chi_k^{(-)}\right) \leq \exp\left(-n^{\frac{1}{7}}\right) \tag{3.156}$$

for all n large enough, where we again used $\nu_n \geq 2n$. By combining (3.153), (3.155), and (3.156), we can in particular conclude with the help of the Borel-Cantelli lemma that for all $c > 0$ we have

$$P_\theta\left(\overline{Y}_{\nu_n} \in [-c\cdot\log(n), c\cdot\log(n)] \text{ for infinitely many } n\right) = 0$$

for P-a.e. environment θ.

For the rest of the proof, we can copy the end of our argument from the proof of Theorem 3.2.1 to ensure that for P-a.e. environment θ our process $(M_\ell)_{\ell\in\mathbb{N}_0}$ P_θ-a.s. does not reach the point $(0,0)$ between time ν_n and $\nu_n + 1$ for infinitely many $n \in \mathbb{N}_0$. ■

Part II

Random Walks in Random Environment with Branching

Chapter 4

Preliminaries

4.1 Basic Notation and General Assumptions

In the following chapter, we do not only consider the movement of one particle but we allow the particles to produce a random number of offspring in each step. Models which combine those two components – random reproduction and movement of particles – are called *Branching Random Walks* (BRW). As before, we additionally assume that the environment – which determines the probabilities for the reproduction and the movement of the particles – is random itself. Altogether, this leads to the model of a *Branching Random Walk in Random Environment* (BRWRE).

The construction of a BRWRE also consists of two steps: In a first step, we choose an environment according to a specific distribution P, and in a second step, we perform a BRW in the chosen environment. The expectation with respect to P will be denoted by $\mathsf{E}[\cdot]$ as in Part I.

4.2 Classical Results

Let us first collect some basic results on *Branching Processes in Random Environment* (BPRE) from [Ta77]:
Since we are only interested in the (random) number of all existing particles in this setting, the distribution of the environment P can be chosen as a probability distribution on $(\mathcal{M}^{\mathbb{N}_0}, \mathcal{A}^{\mathbb{N}_0})$, where

$$\mathcal{M} := \left\{ (p_i)_{i \in \mathbb{N}_0} : p_i \geq 0, \sum_{i=0}^{\infty} p_i = 1 \right\}$$

is the set of all offspring distributions (i.e. probability measures on $\mathbb{N}_0$) and $\mathcal{A}$ is some suitable σ-algebra on $\mathcal{M}$ (for example we can equip $\mathcal{M}$ with the induced topology of $(\mathbb{R}^{\mathbb{N}_0}, \mathcal{B}^{\mathbb{N}_0})$ and then consider the generated Borel σ-algebra on $\mathcal{M}$).

Let $\mu = (\mu_n)_{n\in\mathbb{N}_0}$ denote a sequence in $\mathcal{M}^{\mathbb{N}_0}$ with distribution P. Then, μ is referred to as the random environment and we write P_μ for the *quenched* probability measure given μ. For fixed μ, we can now introduce the following inhomogeneous branching process $(Z_n)_{n\in\mathbb{N}_0}$ with respect to P_μ:
At time 0, we start with $Z_0 = 1$ particle. Afterwards at every time $n \in \mathbb{N}_0$, all existing particles Z_n produce offspring according to the distribution μ_n independently of all other existing particles and then die. The amount of all newly produced offspring then gives the size Z_{n+1} of the $(n+1)$-th generation. For a more detailed description of the model, we refer to Section 1 in [Ta77].

For the next three theorems from [Ta77], we have to assume that the environment $\mu=(\mu_n)_{n\in\mathbb{N}_0}$ is stationary and ergodic with respect to P. In particular, we may apply the results if $\mu = (\mu_n)_{n\in\mathbb{N}_0}$ is an i.i.d. sequence. Further, using the (random) probability distribution μ_0 on $\mathbb{N}_0$, we write

$$\begin{aligned} p_0(\mu_0) &:= \mu_0(\{0\}), \quad p_1(\mu_0) := \mu_0(\{1\}), \\ m_0 = m_0(\mu_0) &:= \sum_{k=0}^{\infty} k\mu_0(\{k\}) \end{aligned}$$

for the (random) probability to have 0 or 1 descendant at time 0, respectively, and for the (random) mean number of offspring at time 0.
Due to Tanny, we know the following sufficient criteria for survival and extinction of $(Z_n)_{n\in\mathbb{N}_0}$:

Theorem 4.2.1 (Sufficient Condition for Survival - cf. Corollary 6.3 in [Ta77]). *If*

$$\mathsf{E}\left[\left|\log\left(1 - p_0(\mu_0)\right)\right|\right] < \infty \quad \textit{and} \quad \mathsf{E}\big[\log(m_0)\big] > 0, \tag{4.1}$$

then we have

$$P_\mu(Z_n \to 0) < 1$$

for P*-a.e. environment* μ.

Here, $\mathsf{E}[\log(m_0)] > 0$ means in particular that $\mathsf{E}[\log(m_0)]$ is well-defined.

Theorem 4.2.2 (Sufficient Condition for Extinction - cf. Theorem 5.5 in [Ta77]). *If one of the conditions*

(1) $\mathsf{E}\big[\log(m_0)\big] < 0$,

(2) $\mathsf{P}\big(p_1(\mu_0) = 1\big) < 1$ *and* $\mathsf{E}\big[\log(m_0)\big] = 0$

holds, then we have

$$P_\mu(Z_n \to 0) = 1$$

for P*-a.e. environment* μ.

We remark here that the conditions in the last two theorems are in general not necessary for survival/extinction of a BPRE as Tanny showed in Section 7 in [Ta77]. If $\mu = (\mu_n)_{n\in\mathbb{N}_0}$ is an i.i.d. sequence, then we can say more: If

$$\mathsf{E}[|\log(m_0)|] < \infty \quad \text{and} \quad \mathsf{P}\big(p_1(\mu_0) = 1\big) < 1$$

holds, then the conditions in (4.1) are also necessary for survival of a BPRE in an i.i.d. environment (cf. Theorem 1 in [Sm68] and Theorem 3.1 in [SW69]).

Additionally to the above conditions for survival and extinction, Tanny also answered the question of the growth of the BPRE in the case of survival:

Theorem 4.2.3 (Growth of a Surviving BPRE - cf. Theorem 5.5 (iii) and Theorem 5.3 in [Ta77])**.** *For $\mathsf{E}\big[\log(m_0)\big] > 0$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n = \mathsf{E}\big[\log(m_0)\big]$$

P_μ-a.s. on $\{Z_n \not\to 0\}$ for P-a.e. environment μ.

In particular, the BPRE grows exponentially fast if it survives.

4.3 Context of our Results

4.3.1 Survival of BRWRE

In Chapter 5, we answer the question under which conditions and in which way a special BRWRE on $\mathbb{N}_0$ survives. The same question for a related model was answered in [GMPV08]. Let us therefore quickly recall the setting of the model, which is introduced in Section 2 in [GMPV08], as a motivation for our considered model. Further, we are able to compare our results to the results in the [GMPV08] at the end of this section:

Define $\mathcal{U}$:= {-1,0,1},

$$\mathcal{V} := \{v = (v_x, x \in \mathcal{U}) : \ v_x \in \mathbb{N}_0 \ \forall x \in \mathcal{U}\}$$

and for $v \in \mathcal{V}$ put $|v| = \sum_{x\in\mathcal{U}} v_x$. Furthermore, let $\mathcal{M}$ be the set of all probability measures ω on $\mathcal{V}$, i.e.

$$\mathcal{M} := \left\{ \omega = (\omega(v), v \in \mathcal{V}) : \ \omega(v) \geq 0 \text{ for all } v \in \mathcal{V}, \ \sum_{v\in\mathcal{V}} \omega(v) = 1 \right\}.$$

Then, suppose that $\omega := (\omega_x \in \mathcal{M}, x \in \mathbb{Z})$ is an i.i.d. sequence with values in $\mathcal{M}$ with respect to some probability measure P. As before, $\omega = (\omega_x \in \mathcal{M}, x \in \mathbb{Z})$ is called the *environment*. Given the environment ω, the evolution of the process is described in the following way: start with one particle at some fixed site of $\mathbb{Z}$. At each integer time the particles branch independently in the following way: for a particle at site $x \in \mathbb{Z}$, a random element $v = (v_y,\ y \in \mathcal{U})$ is chosen with probability $\omega_x(v)$ and then the particle is substituted by v_y particles in $x + y,\ y \in \mathcal{U}$.

Remark 4.3.1. In many models (and also in our model introduced in Chapter 5) on BRWRE, the reproduction and the movement of the particles takes place in two subsequent steps: At every discrete time point $n \in \mathbb{N}_0$, a particle in x first produces i offspring with probability $\mu_x(i)$ independently of all other particles and then dies. Then, each of the offspring jumps to $x + y$ with probability $p(x, y)$ independently off all other offspring. Thereby, the pairs $((\mu_x(i))_{i\in\mathbb{N}_0}, (p(x, y))_{y\in\mathcal{U}})$ are chosen according to some i.i.d. field on $\mathbb{Z}$. In the above notation introduced in [GMPV08], this case is included in their setup if we choose

$$\omega_x(\cdot) = \sum_{i\in\mathbb{N}_0} \mu_x(i)\mathsf{Mult}(i, p(x, y), y \in \mathcal{U})(\cdot).$$

Let us continue the description of the model: Denote

$$\mu_x^- := \sum_{v\in\mathcal{V}} \omega_x(v) \cdot v_{-1}, \quad \mu_x^0 := \sum_{v\in\mathcal{V}} \omega_x(v) \cdot v_0, \quad \text{and} \quad \mu_x^+ := \sum_{v\in\mathcal{V}} \omega_x(v) \cdot v_1,$$

i.e., given the environment ω, μ_x^- is the mean number of offspring sent by a particle from position x to $x - 1$, μ_x^+ is the mean number of offspring sent by a particle from position x to $x + 1$, and μ_x^0 is the mean number of offspring which stay at x. Further, we assume that the following two conditions hold:

$$\begin{aligned} &(1)\ \mathsf{P}(\min\{\mu_0^-, \mu_0^+\} > 0) = 1. \\ &(2)\ \text{There exists } v \in \mathcal{V} \text{ with } |v| \geq 2 \text{ such that } \mathsf{P}(\omega_0(v) > 0) > 0. \end{aligned} \tag{4.2}$$

Here, the first condition ensures that the process is irreducible in the sense that for any $x, y \in \mathbb{Z}$ a particle from x can have descendants at y. The second condition states that there are positions where particles are able to branch.

In the context of BRW, we can distinguish between different survival regimes:

Definition 4.3.2. Given the environment ω we say that

(1) there is *Global Survival* (GS) if

$$P_\omega^0\big(Z_n \to 0\big) < 1,$$

(2) there is *Local Survival* (LS) if

$$P_\omega^0\big(\eta_n(x) \to 0\big) < 1$$

for all $x \in \mathbb{N}_0$.

We say that for a given ω there is local (global) extinction if there is no local (no global) survival. Alone from the above definition, GS and LS could depend on the starting point and on the realization of the environment ω. But the next theorem shows that, for an i.i.d. random environment which fulfils the assumptions in (4.2), there is no such dependence:

Theorem 4.3.3 (cf. Theorem 2.4 and Theorem 2.9 in [GMPV08]). *Local survival and global survival do not depend on the starting point of the BRWRE. Further, there is either local survival (global survival) for P-a.e. environment ω or there is local extinction (global extinction) for P-a.e. environment ω.*

The next theorem answers the question on local survival:

Theorem 4.3.4 (Local Survival - cf. Theorem 2.6 in [GMPV08]). *There is local extinction iff there exists $\lambda > 0$ such that*

$$\mu_0^- \lambda^{-1} + \mu_0^0 + \mu_0^+ \lambda \leq 1$$

for P-a.e. environment ω.

For the next theorem on global survival, we have to assume that also the following condition holds:

Condition S. $\mathsf{E}[|\log(\omega_0(V_1))|] < \infty$ and $\mathsf{E}[|\log(\omega_0(V_{-1}))|] < \infty$.

Here, $V_1 := \{v \in \mathcal{V} : \ v_1 \geq 1\}$ and $V_{-1} := \{v \in \mathcal{V} : \ v_{-1} \geq 1\}$ denote the set of all offspring configurations which send at least one offspring one step to the right or to the left, respectively.

Further, we need to define two so-called Lyapunov exponents: For $k \in \mathbb{N}$ denote

$$A_k := \begin{pmatrix} \frac{1-\mu_k^0}{\mu_k^+} & -\frac{\mu_k^-}{\mu_k^+} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{A}_k := \begin{pmatrix} \frac{1-\mu_k^0}{\mu_k^-} & -\frac{\mu_k^+}{\mu_k^-} \\ 1 & 0 \end{pmatrix}$$

and define

$$\gamma_1 := \lim_{n\to\infty} \frac{1}{n} \mathsf{E}\big[\log \|A_n A_{n-1} \cdots A_1\|\big] \quad \text{and} \quad \widetilde{\gamma}_1 := \lim_{n\to\infty} \frac{1}{n} \mathsf{E}\big[\log \|\widetilde{A}_n \widetilde{A}_{n-1} \cdots \widetilde{A}_1\|\big],$$

where $\|\cdot\|$ is any matrix norm. Then, γ_1 and $\widetilde{\gamma}_1$ exist due to the assumptions in [GMPV08].

Theorem 4.3.5 (Global Survival - cf. Theorem 2.9 in [GMPV08]). *Suppose that condition S holds. Assume also that there is local extinction. Then, the following holds:*

(1) If there is some $\lambda > 1$ such that $\mu_0^- \lambda^{-1} + \mu_0^0 + \mu_0^+ \lambda \leq 1$ P-a.s., then there is global survival iff

$$\gamma_1 < \mathsf{E}\left[\log\left(\frac{\mu_0^-}{\mu_0^+}\right)\right].$$

(2) If there is some $\lambda < 1$ *such that* $\mu_0^- \lambda^{-1} + \mu_0^0 + \mu_0^+ \lambda \leq 1$ P*-a.s., then there is global survival iff*

$$\widetilde{\gamma}_1 < \mathsf{E}\left[\log\left(\frac{\mu_0^+}{\mu_0^-}\right)\right].$$

Even though the last two theorems give a complete answer to the question on global survival, the condition in Theorem 4.3.5 is not very explicit. In particular if we just consider a combination of two offspring configurations, we cannot answer the question on global survival without a longer and involved computation in general.

In the next chapter, we consider an easier model in which particles located at $x \in \mathbb{N}_0$ can only produce offspring to the same position $x \in \mathbb{N}_0$ and to the right neighbouring site $x+1$. This case is excluded by the assumptions in (4.2). The easier configuration of the model enables us to derive a very simple condition on global survival in

Theorem 5.3.2 *Suppose* $\Lambda \leq 1$.
There is either GS for P*-a.e.* ω *or there is no GS for* P*-a.e.* ω.
There is GS for P*-a.e.* ω *iff*

$$\mathsf{E}\left[\log\left(\frac{m_0 h_0}{1 - m_0(1-h_0)}\right)\right] > 0.$$

Here, m_0 denotes the mean number of offspring produced by one particle located at 0. Afterwards, every newly produced offspring moves to $+1$ (or stays at 0) with probability h_0 (or $1 - h_0$) independently of all other existing particles (cf. Section 5.2 for a precise description of the considered model).

Further, $\Lambda := \operatorname{ess\,sup}(m_0(1-h_0))$ (cf. (5.2)) which is a helpful quantity to characterize local survival:

Theorem 5.3.1 *There is either LS for* P*-a.e.* ω *or there is no LS for* P*-a.e.* ω.
There is LS for P*-a.e.* ω *iff*

$$\Lambda > 1.$$

Remark 4.3.6. Let us shortly summarize some aspects of Theorem 5.3.2:

(1) If we choose $h_0 \equiv 1$, then the condition for global survival reduces to the condition for survival of a BPRE in Theorem 4.2.1 by Tanny. Note for this that our assumptions in (5.1) ensure that in our setting the first condition in Theorem 4.2.1 is always fulfilled.

(2) If we choose $h_0 \equiv h$ to be constant, an analysis of the function

$$\varphi(h) := \mathsf{E}\left[\log\left(\frac{m_0 h}{1 - m_0(1-h)}\right)\right]$$

allows us to answer the dependence of survival on the drift h (cf. Theorem 5.3.7). We show that global survival for some drift $\overline{h}$ always implies that we also have global survival for all drifts h such that $0 < h \leq \overline{h}$, i.e. for the considered BRWRE, it is easier to survive for a smaller drift h.

4.3.2 Growth of BRWRE

The questions of the growth rates of the BRWRE in Chapter 5 are motivated by a series of papers by Baillon, Clement, Greven and den Hollander, see [BCGH93], [BCGH94], [GH91], [GH92] and [GH94] in which the authors consider the *Population Growth in Random Media*.

In Remark 5.4.2, we will see that there is strong connection between our model which we consider in Chapter 5 and the model which is considered in [GH91]. So, let us shortly repeat the setting of the model which is introduced in the latter article (cf. 0.2 Section Model in [GH91]):

(1) For each $x \in \mathbb{Z}$, F_x is a random probability measure on the nonnegative integers $\mathbb{N}_0$ called the offspring distribution at site x. The sequence $F = (F_x)_{x\in\mathbb{Z}}$ is i.i.d. with common distribution α. Here, F plays the role of the random medium.

(2) For fixed F, define a discrete-time Markov process $(\eta_n)_{n\geq 0}$ on $(\mathbb{N}_0)^{\mathbb{Z}}$ with

$$\begin{aligned} &\eta_n = (\eta_n(x))_{x\in\mathbb{Z}}, \\ &\eta_n(x) = \text{number of particles at site } x \text{ at time } n \end{aligned}$$

by specifying its one-step transition mechanism as follows: Given the state η_n at time n,

   (a) each particle is independently replaced by a new generation. The size of the new generation descending from a particle at site x has distribution F_x, i.e., it consists of k new particles with probability $F_x(k)$ $(k = 0, 1, 2, \dots)$. Also, particles at the same site branch independently.

   (b) Immediately after creation, each new particle at site x independently decides to either stay at x with probability $1 - h$ or to jump to $x + 1$ with probability h. The parameter $h \in [0, 1]$ is the drift and is the same for all x.

(3) The resulting sequence of particle numbers after steps (1) and (2) make up the state η_{n+1} at time $n + 1$. F stays fixed during the evolution.

(4) We start with $\eta_0(x) = 1$, i.e. one particle at each site $x \in \mathbb{Z}$ at time 0 (which corresponds to Case I in [GH91]).

In Equation (0.3) in [GH91], the quantity

$$d_n(x, F) := E_F[\eta_n(x)]$$

is introduced as the average particle density at site x at time n for fixed F. Here we use the usual "quenched" notation in the context of BRWRE (alternatively one can define $d_n(x, F) := \mathbb{E}[\eta_n(x)|F]$ as in [GH91] where $\mathbb{E}$ denotes the expectation with respect to the joint distribution of $(F, (\eta_n)_{n\in\mathbb{N}_0})$).

In Theorem 1 and Theorem 2 I. in [GH91], Greven and den Hollander describe the limit

$$\lim_{n\to\infty} \tfrac{1}{n} \log d_n(0, F) =: \lambda(h)$$

as a function of the drift h by an implicit formula which we want to shortly repeat here to emphasize the explicitness of our results:
Let

$$b_x := \sum_{k=0}^{\infty} kF_x(k)$$

denote the mean offspring of a particle located at site x and let β denote the distribution of b_x (which does not depend on x since $F = (F_x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence) and we assume

$$0 < \inf_x b_x < \sup_x b_x < \infty \quad \beta\text{-a.s.}$$

Further, we define

$$\begin{aligned}
M_{\theta,\beta} &:= \left\{ \nu \in \mathcal{P}(\mathbb{N} \times \text{supp } \beta) : \sum_{i,j} i\nu(i,j) = \theta^{-1}, \ \sum_i \nu(i,j) = \beta(j) \text{ for all } j \right\}, \\
f(\nu) &:= \sum_{i,j} \nu(i,j) i \log j, \\
I_{\theta,\beta}(\nu) &:= \sum_{i,j} \nu(i,j) \log\left(\frac{\nu(i,j)}{\pi_\theta(i)\beta(j)} \right), \\
I_h(\theta) &:= \theta \log\left(\frac{\theta}{h}\right) + (1-\theta)\log\left(\frac{1-\theta}{1-h}\right), \\
\pi_\theta(i) &:= \theta(1-\theta)^{i-1}.
\end{aligned}$$

The notation is for the situation where suppβ is countable. For more details on the definitions, we refer to Section 0.3 in [GH91]. Then, we have the following:

Theorem 4.3.7 (cf. Theorem 1 in [GH91])**.** *For $h \in (0,1)$, we have*

$$\lambda(h) \stackrel{\text{def}}{=} \lim_{n\to\infty} \tfrac{1}{n} \log d_n(0,F) = \lambda(\beta, h; 0) \quad F\text{-a.s.},$$

where

$$\begin{aligned}
\lambda(\beta, h; 0) &:= \sup_{\theta\in(0,1]} [J_\beta(\theta) - I_h(\theta)], \\
J_\beta(\theta) &:= \theta \sup_{\nu\in M_{\theta,\beta}} [f(\nu) - I_{\theta,\beta}(\nu)] \quad \text{for } \theta \in (0,1].
\end{aligned}$$

The two suprema in the last theorem have a nice interpretation as the best strategy for particles to move within the random medium (cf. Section 0.5 in [GH91]). Further, the variational expressions in Theorem 4.3.7 can be solved, i.e. there are maximizers for both expressions (cf. (0.21) and (0.22) in [GH91]). The solution leads to the following theorem which gives a very complete description of $\lambda(h)$ from a qualitative point of view:

Theorem 4.3.8 (cf. Theorem 2 I. and Corollary 2 I. in [GH91]). *With $M := \operatorname{ess\,sup} b_0$, we have*

$$\lambda(\beta, h; 0) = \log[M(1-h)] + r^*,$$

where r^* can be described by an implicit formula (cf. Section 0.4 in [GH91] for more details). Further, we have the following properties:

(1) $h \mapsto \lambda(\beta, h; 0)$ is continuous and strictly decreasing on (0,1). Further, $\lambda(\beta, 0; 0) = \log M$ and $\lambda(\beta, 1; 0) = \sum_j \beta(j) \log(j)$.

(2) In particular if $\log M > 0 > \sum_j \beta(j) \log(j)$, then $\lambda(\beta, h; 0)$ as function of h changes sign at $h = h_c^*$ which is the unique solution of $\lambda(\beta, h; 0) = 0$.

In the next chapter, we introduce the model of a BRWRE on $\mathbb{Z}$ in which every particle located at $x \in \mathbb{N}_0$ can only produce offspring to the same position $x \in \mathbb{N}_0$ and to the right neighbouring position $x+1$. The total number of particles at time n will be denoted by Z_n (cf. (5.4)). In our setting, the drift h_x (for a single offspring to move one step to the right) may also depend on the location x of the parental particles. In the case of a constant drift (i.e. $h_x \equiv h \in (0,1]$ for all $x \in \mathbb{N}_0$), we show in Remark 5.4.2 that the expected global population size $E_\omega[Z_n]$ corresponds to $d_n(0, F)$ in the notation of [GH91].

In order to make our results on global survival comparable to the results on the growth of the process in [GH91], we show that global survival of our BRWRE in Chapter 5 is equivalent to the exponential growth of $E_\omega[Z_n]$ (cf. Definition 4.3.2 for the definition of global survival):

Theorem 5.3.5 *The following assertions are equivalent:*

(1) $\lim\limits_{n\to\infty} \frac{1}{n} \log E_\omega\big[Z_n\big] > 0$ holds for P-a.e. ω.

(2) There is GS for P-a.e. ω.

So, let us recall the condition for global survival (rewritten for the case of a constant drift $h_0 \equiv h$) which we already mentioned above:

Theorem 5.3.2 *Suppose $\Lambda \overset{def}{=} \operatorname{ess\,sup}(m_0(1-h)) \leq 1$.*
There is either GS for P-a.e. ω or there is no GS for P-a.e. ω.
There is GS for P-a.e. ω iff

$$\varphi(h) \overset{\text{def}}{=} \mathsf{E}\left[\log\left(\frac{m_0 h}{1 - m_0(1-h)}\right)\right] > 0.$$

By combining the last two theorems, we are able to describe the location of the critical parameter $h = h_c^*$ in Theorem 4.3.8 by just using one implicit formula, i.e. the solution of $0 = \varphi(h)$ (uniqueness – if a solution exists – will be shown in Theorem 5.3.7). Note here that the critical parameter $h = h_c^*$ in Theorem 4.3.8 is only given by an implicit formula in which the quantity r^* itself is only available through a second implicit formula. The easier criterion from Theorem 5.3.2 in particular enables us to directly treat the cases in which we deal with a combination of just two offspring distributions. Here, we can describe the phases in which we have global survival (i.e. exponential growth of the population) and no global survival (i.e. no exponential growth of the population) without any further computation. Two examples for this configuration are included in Section 5.6.

Chapter 5

Survival and Growth of a BRWRE

5.1 Overview

The chapter consists of the article *Survival and Growth of a Branching Random Walk in Random Environment* by Christian Bartsch, Nina Gantert, and Michael Kochler ([BGK09]). In order to keep this chapter self-contained, this article has been left relatively unchanged.

In the article we considered a particular Branching Random Walk in Random Environment (BRWRE) on $\mathbb{N}_0$ started with one particle at the origin. Particles reproduce according to an offspring distribution (which depends on the location) and move either one step to the right (with a probability in $(0, 1]$ which also depends on the location) or stay in the same place. We give criteria for local and global survival and show that global survival is equivalent to exponential growth of the moments. Further, on the event of survival the number of particles grows almost surely exponentially fast with the same growth rate as the moments.

The chapter is organized as follows: In Section 2, we give a formal description of our model. Section 3 contains the results, Section 4 some remarks and Section 5 the proofs. At last, in Section 6, we provide examples and pictures.

5.2 Formal Description of the Model

The considered BRWRE will be constructed in two steps, namely we first choose an environment and then let the particles reproduce and move in this environment.

Step I (Choice of the environment)

First, define

$$\mathcal{M} := \left\{ (p_i)_{i \in \mathbb{N}_0} : p_i \geq 0, \sum_{i=0}^{\infty} p_i = 1 \right\}$$

as the set of all offspring distributions (i.e. probability measures on $\mathbb{N}_0$). Then, define

$$\Omega := \mathcal{M} \times (0, 1]$$

as the set of all possible choices for the local environment, now also containing the local drift parameter. Let α be a probability measure on Ω satisfying

$$\begin{aligned} \alpha\left(\left\{\left((p_i)_{i\in\mathbb{N}_0}, h\right) \in \Omega : p_1 = 1\right\}\right) &< 1, \\ \alpha\left(\left\{\left((p_i)_{i\in\mathbb{N}_0}, h\right) \in \Omega : p_0 \leq 1-\delta,\, h \in [\delta, 1]\right\}\right) &= 1 \end{aligned} \tag{5.1}$$

for some $\delta > 0$. The first property ensures that the branching is non-trivial and the second property is a common ellipticity condition which comes up in the context of survival of branching processes in random environment.

Let $\omega = (\omega_x)_{x\in\mathbb{N}_0} = (\mu_x, h_x)_{x\in\mathbb{N}_0}$ be an i.i.d. random sequence in Ω with distribution $\alpha^{\mathbb{N}_0} = \bigotimes_{x\in\mathbb{N}_0} \alpha$. We write $\mathsf{P} := \alpha^{\mathbb{N}_0}$ and E for the associated expectation. In the following, ω is referred to as the random environment containing the offspring distributions μ_x and the drift parameters h_x. Let

$$m_x = m_x(\omega) := \sum_{k=0}^{\infty} k\mu_x\big(\{k\}\big)$$

be the mean offspring at location $x \in \mathbb{N}_0$. We denote the essential supremum of m_0 by

$$M := \operatorname{ess\,sup} m_0$$

and furthermore we define

$$\Lambda := \operatorname{ess\,sup}\big(m_0(1-h_0)\big). \tag{5.2}$$

Step II (Evolution of the cloud of particles)

Given the randomly chosen environment $(\omega_x)_{x\in\mathbb{N}_0} = (\mu_x, h_x)_{x\in\mathbb{N}_0}$, the cloud of particles evolves at every time $n \in \mathbb{N}_0$. First, each existing particle at some site $x \in \mathbb{N}_0$ produces offspring according to the distribution μ_x independently of all other particles and dies. Then, the newly produced particles move independently according to an underlying Markov chain starting at position x. The transition probabilities are also given by the environment. We will only consider a particular type of Markov chain on $\mathbb{N}_0$ that we may call "movement to the right with (random) delay". This Markov chain is determined by the following transition probabilities:

$$p_\omega(x, y) = \begin{cases} h_x & y = x+1 \\ 1-h_x & y = x \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

Note that due to the ellipticity condition (5.1), h_x is bounded away from 0 by some positive δ. Later, we consider the case that $\mathsf{P}(h_0 = h) = 1$ for some $h \in (0, 1]$ where the drift parameter is constant and analyse different survival regimes depending on the drift parameter

h, see Theorem 5.3.7.
For $n \in \mathbb{N}_0$ and $x \in \mathbb{N}_0$, let us denote the number of particles at location x at time n by $\eta_n(x)$, and furthermore, let

$$Z_n := \sum_{x \in \mathbb{N}_0} \eta_n(x) \tag{5.4}$$

be the total number of particles at time n.
We denote the probability and the expectation for the process in the fixed environment ω started with one particle at x by P_ω^x and E_ω^x, respectively. P_ω^x and E_ω^x are often referred to as "quenched" probability and expectation.

Now we define two survival regimes:
Definition 5.2.1. Given ω, we say that

(i) there is *Global Survival* (GS) if

$$P_\omega^0\big(Z_n \to 0\big) < 1,$$

(ii) there is *Local Survival* (LS) if

$$P_\omega^0\big(\eta_n(x) \to 0\big) < 1$$

for some $x \in \mathbb{N}_0$.

Remarks 5.2.2. (i) For fixed ω, LS is equivalent to

$$P_\omega^0\big(\eta_n(x) \to 0 \;\forall\, x \in \mathbb{N}_0\big) < 1.$$

(ii) Since the drift parameter is always positive, it is easy to see that for fixed ω LS and GS do not depend on the starting point in Definition 5.2.1. Thus we will always assume that our process starts at 0. For convenience we will omit the superscript 0 and use P_ω and E_ω instead.

5.3 Results

The following results characterize the different survival regimes. As in [GMPV08] (cf. Theorem 4.3.3 and Theorem 4.3.5), local and global survival do not depend on the realization of the environment but only on its law.

Theorem 5.3.1. *There is either LS for* P*-a.e.* ω *or there is no LS for* P*-a.e.* ω.
There is LS for P*-a.e.* ω *iff*

$$\Lambda > 1.$$

Theorem 5.3.2. *Suppose $\Lambda \leq 1$.*
There is either GS for P-a.e. ω or there is no GS for P-a.e. ω.
There is GS for P-a.e. ω iff

$$\mathsf{E}\left[\log\left(\frac{m_0 h_0}{1 - m_0(1-h_0)}\right)\right] > 0.$$

We now consider the local and the global growth of the moments $E_\omega[\eta_n(x)]$ and $E_\omega[Z_n]$. For Theorems 5.3.3 – 5.3.6, we need the following stronger condition

$$\begin{aligned} \alpha\left(\left\{\left((p_i)_{i\in\mathbb{N}_0}, h\right) \in \Omega : p_1 = 1\right\}\right) &< 1, \\ \alpha\left(\left\{\left((p_i)_{i\in\mathbb{N}_0}, h\right) \in \Omega : p_0 \leq 1-\delta,\, h \in [\delta, 1-\delta]\right\}\right) &= 1 \end{aligned} \tag{5.5}$$

for some $\delta > 0$. In addition, for those theorems we assume $M < \infty$.

Theorem 5.3.3. *There exists a unique, deterministic, continuous, and concave function $\beta : [0,1] \longrightarrow \mathbb{R}$ such that for every $\gamma > 0$ we have for P-a.e. ω*

$$\lim_{n\to\infty} \max_{x \in n[\gamma,1]\cap\mathbb{N}} \left| \tfrac{1}{n} \log E_\omega\big[\eta_n(x)\big] - \beta(\tfrac{x}{n}) \right| = 0.$$

Additionally, it holds that $\beta(0) = \log\big(\Lambda\big)$ and $\beta(1) = \mathsf{E}[\log(m_0 h_0)]$.

Theorem 5.3.4. *We have*

$$\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega\big[Z_n\big] = \max_{x\in[0,1]} \beta(x) \quad \textit{for } \mathsf{P}\textit{-a.e. } \omega.$$

The next theorem shows that GS is equivalent to exponential growth of the moments $E_\omega[Z_n]$:

Theorem 5.3.5. *The following assertions are equivalent:*

(i) $\lim\limits_{n\to\infty} \tfrac{1}{n} \log E_\omega\big[Z_n\big] > 0$ *holds for P-a.e. ω.*

(ii) There is GS for P-a.e. ω.

In the following theorem we consider the growth of the population Z_n on the event of survival:

Theorem 5.3.6. *If there is GS we have for P-a.e. ω*

$$\lim_{n\to\infty} \tfrac{1}{n} \log Z_n = \max_{x\in[0,1]} \beta(x) > 0 \quad P_\omega\text{-a.s.} \quad \textit{on} \quad \{Z_n \not\to 0\}.$$

As already announced above, we now analyse the case of constant drift parameter, i.e. $\mathsf{P}(h_0 = h) = 1$ for some $h \in (0, 1]$. As it is easy to see from Theorem 5.3.1 in this case, we have LS iff

$$h < h_{LS} := \begin{cases} 1 - \frac{1}{M} & \text{if } M \in (1, \infty] \\ 0 & \text{if } M \in (0, 1] \end{cases} .$$

To analyse the dependence of GS on h we define

$$\varphi(h) := \mathsf{E}\left[\log\left(\frac{m_0 h}{1 - m_0(1-h)}\right)\right].$$

Theorem 5.3.7. *Suppose $h \geq h_{LS}$.*

(i) If $M \leq 1$, then we have $\varphi(h) \leq 0$ for all $h \in (0, 1]$ and thus there is a.s. no GS.

(ii) Assume $M > 1$.

(a) If $\varphi(h_{LS}) \geq 0$ and $\varphi(1) \leq 0$, then there is a unique $h_{GS} \in [h_{LS}, 1]$ with $\varphi(h_{GS}) = 0$. In this case, we have a.s. GS for $h \in (0, h_{GS})$ and a.s. no GS for $h \in [h_{GS}, 1]$.

(b) If $\varphi(h_{LS}) < 0$, then $\varphi(h) < 0$ for all $h \in [h_{LS}, 1]$. Thus, we have a.s. GS for $h \in (0, h_{LS})$ and a.s. no GS for $h \in [h_{LS}, 1]$. In this case, we define $h_{GS} := h_{LS}$.

(c) If $\varphi(1) > 0$, then $\varphi(h) > 0$ for all $h \in [h_{LS}, 1]$. Thus, there is a.s. GS for all $h \in (0, 1]$. In this case we define $h_{GS} := \infty$.

Hence, we have a unique $h_{GS} \in [h_{LS}, 1] \cup \{\infty\}$ such that there is a.s. GS for $h < h_{GS}$ and a.s. no GS for $h \geq h_{GS}$.

5.4 Remarks

The following remarks apply to the case of constant drift:

Remarks 5.4.1. (i) Since $\varphi(1) = \mathsf{E}[\log m_0]$, our results can be seen as an extension of the well-known condition for possible survival of branching processes in a random environment (cf. Theorem 4.2.1 and Theorem 4.2.2 by Tanny, recalling that we assume condition (5.1)). In fact, our proofs rely on this result.

(ii) If $M < \infty$ and $\varphi(h_{LS}) \in (0, \infty]$, then due to the continuity of φ there exists $z > 0$ such that there is a.s. GS but a.s. no LS for every $h \in [h_{LS}, h_{LS} + z)$. In particular, this is the case if $\mathsf{P}(m_0 = M) > 0$, since then $\varphi(h_{LS}) = \infty$.

(iii) We provide an example for a setting in which the condition of Theorem 5.3.7 *(ii)(b)* holds. In this case, there is a.s. LS for $h \in (0, h_{LS})$ and a.s. no GS for $h \in [h_{LS}, 1]$ for some $h_{LS} \in (0, 1)$. (See Section 5.6.)

Remark 5.4.2. The expected global population size $E_\omega[Z_n]$ corresponds to $d_n(0,F)$ in the notation of [GH91]. They describe the limit

$$\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] = \lim_{n\to\infty} \tfrac{1}{n} \log d_n =: \lambda(h)$$

as a function of the drift h by an implicit formula (cf. Theorem 4.3.7).
To see this correspondence, let $(S_n)_{n\in\mathbb{N}_0}$ be a random walk with (non-random) transition probabilities $(p_h(x,y))_{x,y\in\mathbb{N}_0}$ starting in 0 where the transition probabilities are defined by

$$p_h(x,y) := \begin{cases} h & y = x+1 \\ 1-h & y = x \\ 0 & \text{otherwise} \end{cases}$$

and let E_h be the associated expectation. We denote the local times of $(S_n)_{n\in\mathbb{N}_0}$ by $l_n(x)$, that is

$$l_n(x) := |\{0 \le i \le n : S_i = x\}| \quad \text{for } x \ge 0, n \ge 0.$$

For $x = 0$, we now have

$$E_\omega[\eta_n(0)] = (1-h)^n \cdot m_0(\omega)^n = E_h\left[\prod_{i=0}^{n-1} m_{S_i}(\omega) \cdot \mathbb{1}_{\{S_n=0\}}\right].$$

For $x \ge 1$, we have

$$E_\omega[\eta_n(x)] = h \cdot m_{x-1}(\omega) \cdot E_\omega\big[\eta_{n-1}(x-1)\big] + (1-h) \cdot m_x(\omega) \cdot E_\omega\big[\eta_{n-1}(x)\big]$$

which yields

$$E_\omega[\eta_n(x)] = E_h\left[\prod_{i=0}^{n-1} m_{S_i}(\omega) \cdot \mathbb{1}_{\{S_n=x\}}\right]$$

for all $x \ge 1$ by induction. Finally, we get

$$E_\omega[Z_n] = \sum_{x=0}^{\infty} E_\omega[\eta_n(x)] = E_h\left[\prod_{i=0}^{n-1} m_{S_i}(\omega)\right] = E_h\left[\prod_{x=0}^{n-1} m_x(\omega)^{l_n(x)}\right].$$

Since we can extend the environment $\omega = (\omega_x)_{x\in\mathbb{N}_0}$ to an i.i.d. environment $(\omega_x)_{x\in\mathbb{Z}}$ and since $(\omega_x)_{x\in\mathbb{Z}}$ and $(\omega_{-x})_{x\in\mathbb{Z}}$ have the same distribution with respect to P, formula (1.8) and Theorem 4.3.7 by Greven and den Hollander show that there exists a deterministic $c \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] = c \quad \text{for } \mathsf{P}\text{-a.e. } \omega.$$

In our notation this limit coincides with $\max_{x\in[0,1]} \beta(x)$.
The connection between the two models enables us to characterize the critical drift parameter at which the function $h \mapsto \lambda(h)$ in Theorem 4.3.8 (2) by Greven and den Hollander changes its sign using an easier criterion, see Theorem 5.3.7.

5.5 Proofs

Proof of Theorem 5.3.1. First, we observe that the descendants of a particle at location x that stay at x form a Galton-Watson process with mean offspring $m_x(1-h_x)$. Given ω, we therefore have

$$P_\omega^x\big(\eta_n(x) \to 0\big) < 1 \qquad \Leftrightarrow \qquad m_x(\omega)(1-h_x(\omega)) > 1.$$

Now assume $\Lambda > 1$. Thus, there is some $\lambda > 1$ such that

$$\mathsf{P}(m_0(1-h_0) \geq \lambda) > \varepsilon > 0$$

for some $\varepsilon > 0$ and using the Borel-Cantelli lemma we obtain that P-a.s. for infinitely many locations x we have

$$m_x(1-h_x) > 1.$$

Let $x_0 = x_0(\omega)$ be a location satisfying $m_{x_0}(1-h_{x_0}) > 1$.
For P-a.e. ω, we see

$$\begin{aligned} & P_\omega(\eta_{x_0}(x_0) \geq 1) \\ \geq\ & \big(1-\mu_0(\{0\})\big)h_0 \cdot \big(1-\mu_1(\{0\})\big)h_1 \cdot \ldots \cdot \big(1-\mu_{x_0-1}(\{0\})\big)h_{x_0-1} \\ >\ & 0 \end{aligned}$$

whereas the second inequality uses condition (5.1).
We obtain for P-a.e. ω

$$\begin{aligned} & P_\omega(\eta_n(x_0) \to \infty) \\ \geq\ & P_\omega(\eta_{x_0}(x_0) \geq 1) \cdot P_\omega^{x_0}(\eta_n(x_0) \to \infty) \\ >\ & 0 \end{aligned}$$

and thus LS.

Now assume $\Lambda \leq 1$. As mentioned above, for every $x \in \mathbb{N}_0$ and P-a.e. ω, the descendants of a particle at location x that stay at x form a subcritical or critical Galton-Watson process. Thus, for a given ω we have

$$\eta_n(0) \to 0 \quad P_\omega\text{-a.s.}$$

and the total number of particles that move from 0 to 1 is therefore P_ω-a.s. finite. Inductively, we conclude for every $x \in \mathbb{N}_0$ that the total number of particles that reach location x from $x-1$ is finite. By assumption, each of those particles starts a subcritical or critical Galton-Watson process at location x which dies out P_ω-a.s.. This implies

$$P_\omega(\eta_n(x) \to 0) = 1 \quad \forall\, x \in \mathbb{N}_0$$

which completes the proof. ■

Proof of Theorem 5.3.2. Since $\Lambda \leq 1$ by assumption, there is P-a.s. no LS according to Theorem 5.3.1. In other words, we have for all $x \in \mathbb{N}_0$

$$P_\omega(\eta_n(x) \to 0) = 1 \quad \text{for } \mathsf{P}\text{-a.e. } \omega.$$

We now define a branching process in random environment $(\xi_n)_{n\in\mathbb{N}_0}$ that is embedded in the considered BRWRE. After starting with one particle at 0, we freeze all particles that reach 1 and keep those particles frozen until all existing particles have reached 1. This will happen a.s. after a finite time because the number of particles at 0 constitutes a subcritical or critical Galton-Watson process that dies out with probability 1. We now denote the total number of particles frozen in 1 by ξ_1. Then, we release all particles, let them reproduce and move according to the BRWRE and freeze all particles that hit 2. Let ξ_2 be the total number of particles frozen at 2. We repeat this procedure and with $\xi_0 := 1$ we obtain the process $(\xi_n)_{n\in\mathbb{N}_0}$ which is a branching process in an i.i.d. environment.
Another way to construct $(\xi_n)_{n\in\mathbb{N}_0}$ is to think of ancestral lines. Each particle has a unique ancestral line leading back to the first particle starting from the origin. Then, ξ_k is the total number of particles which are the first particles that reach position k among the particles in their particular ancestral lines.
We observe that GS of $(Z_n)_{n\in\mathbb{N}_0}$ is equivalent to survival of $(\xi_n)_{n\in\mathbb{N}_0}$.
Due to Theorem 4.2.1 and Theorem 4.2.2 by Tanny (taking into account condition (5.1)), the process $(\xi_n)_{n\in\mathbb{N}_0}$ survives with positive probability for P-a.e. environment ω iff

$$\int \log\left(E_\omega[\xi_1]\right) \mathsf{P}(d\omega) > 0.$$

Computing the expectation $E_\omega[\xi_1]$ now completes our proof. First, we define $\xi_1^{(k)}$ as the number of particles which move from position 0 to 1 at time k. Using this notation, we may write

$$\xi_1 = \sum_{k=0}^{\infty} \xi_1^{(k)}$$

and obtain

$$E_\omega[\xi_1] = \sum_{k=0}^{\infty} E_\omega\left[\xi_1^{(k)}\right].$$

To calculate $E_\omega\left[\xi_1^{(k)}\right]$, we observe that (with respect to P_ω) the expected number of particles at position 0 at time k equals $\left(m_0(\omega)\cdot(1-h_0(\omega))\right)^k$. Each of those particles contributes $m_0(\omega)\cdot h_0(\omega)$ to $E_\omega\left[\xi_1^{(k)}\right]$. This yields

$$\begin{aligned} E_\omega[\xi_1] &= \sum_{k=0}^{\infty}\left(m_0(\omega)\cdot(1-h_0(\omega))\right)^k \cdot m_0(\omega)\cdot h_0(\omega) \\ &= \frac{m_0(\omega)\cdot h_0(\omega)}{1-m_0(\omega)\cdot(1-h_0(\omega))} \end{aligned} \tag{5.6}$$

which is defined as ∞ if $m_0(\omega)\cdot(1-h_0(\omega)) = 1$. ■

Remark 5.5.1. Alternatively, equation (5.6) can be obtained using generating functions. The crucial observation is that the generating function $f_x(s) := E_\omega[s^{\xi_{x+1}}|\xi_x = 1]$ is a solution of the equation

$$f_x(s) = g_x\big((1-h_x)f_x(s) + h_x s\big)$$

where $g_x(s) := \sum_{k=0}^{\infty} \mu_x\big(\{k\}\big)s^k$. Then, $E_\omega[\xi_1] = f_0'(1)$, leading to (5.6) .

Proof of Theorem 5.3.3. Following the ideas of [CP07], we introduce the function β to investigate the local growth rates.

(i) First, we show that β can be defined as a concave function on $(0,1] \cap \mathbb{Q}$ such that

$$\lim_{n\to\infty} \tfrac{1}{sn} \log E_\omega\big[\eta_{sn}(rn)\big] = \beta\left(\tfrac{r}{s}\right) \tag{5.7}$$

holds for all $r, s \in \mathbb{N}$ with $r \leq s$ and for P-a.e. ω.
To see this, fix $r, s \in \mathbb{N}$ with $r \leq s$. We define

$$S_{m,n}(\omega) := \tfrac{1}{s} \log E_\omega^{rm}\big[\eta_{s(n-m)}(rn)\big]$$

for $0 \leq m \leq n$ which is integrable due to (5.5) and $M < \infty$. Using this definition, we have

$$S_{m+1,n+1}(\omega) = S_{m,n} \circ \Theta(\omega) \tag{5.8}$$

where $\Theta(\omega) := \theta^r(\omega)$ with θ denoting the shift operator as usual, i.e. $(\theta\,\omega)_i = \omega_{i+1}$. Furthermore, we have

$$S_{0,n}(\omega) \geq S_{0,m}(\omega) + S_{m,n}(\omega)$$

since

$$E_\omega^0\big[\eta_{sn}(rn)\big] \geq E_\omega^0\big[\eta_{sm}(rm)\big] \cdot E_\omega^{rm}\big[\eta_{s(n-m)}(rn)\big]. \tag{5.9}$$

With the properties (5.8) and (5.9), we are able to apply the subadditive ergodic theorem to $(S_{m,n})$ and we obtain that

$$\lim_{n\to\infty} \tfrac{1}{n} S_{0,n}(\omega) = \lim_{n\to\infty} \tfrac{1}{sn} \log E_\omega\big[\eta_{sn}(rn)\big] =: \beta\left(\tfrac{r}{s}\right)$$

exists for P-a.e. ω. Clearly, the limit only depends on $\frac{r}{s}$. Whereas it is P-a.s. constant since P is i.i.d..

(ii) We now show that β is concave on $(0,1] \cap \mathbb{Q}$. Fix $a, b, t \in (0,1] \cap \mathbb{Q}$ with $t \neq 1$ and let $s := a' \cdot b' \cdot t'$ be the product of the denominators of the reduced fractions of a, b, t. Due to (5.9), we have

$$\begin{aligned}
&\tfrac{1}{sn} \log E_\omega\Big[\eta_{sn}\big(s(ta + (1-t)b)n\big)\Big] \\
\geq\ & t\tfrac{1}{stn} \log E_\omega\Big[\eta_{stn}\big(stan\big)\Big] \\
&\quad + (1-t)\tfrac{1}{s(1-t)n} \log E_\omega^{stan}\Big[\eta_{s(1-t)n}\big(s(ta + (1-t)b)n\big)\Big]
\end{aligned}$$

$$\begin{aligned} &= t\tfrac{1}{stn}\log E_\omega\Big[\eta_{stn}\big(stan\big)\Big] \\ &\quad + (1-t)\tfrac{1}{s(1-t)n}\log E_{\theta^{stan}\omega}\Big[\eta_{s(1-t)n}\big(s(1-t)bn\big)\Big]. \end{aligned} \tag{5.10}$$

We observe that for all $n \in \mathbb{N}_0$

$$E_{\theta^{stan}\omega}\Big[\eta_{s(1-t)n}\big(s(1-t)bn\big)\Big] \overset{d}{=} E_\omega\Big[\eta_{s(1-t)n}\big(s(1-t)bn\big)\Big].$$

Due to (5.7) and since β is P-a.s. constant, this implies

$$(1-t)\tfrac{1}{s(1-t)n}\log E_{\theta^{stan}\omega}\Big[\eta_{s(1-t)n}\big(s(1-t)bn\big)\Big] \xrightarrow[n\to\infty]{} (1-t)\beta(b)$$

in probability. Therefore there exists a subsequence such that we have P-a.s. convergence in (5.10) and this yields

$$\beta(ta+(1-t)b) \geq t\beta(a) + (1-t)\beta(b).$$

We observe that β is bounded with $2\log\delta + \log(1-\delta) \leq \beta(x) \leq \log M$ and thus it can be uniquely extended to a continuous and concave function $\beta : (0,1) \longrightarrow \mathbb{R}$.

(iii) We now investigate the behaviour of β for $x \downarrow 0$ and show that

$$\lim_{x\downarrow 0}\beta(x) = \log(\Lambda).$$

Fix $\varepsilon > 0$ and $a \in \mathbb{Q}\cap(0,\varepsilon]$. Let a' be the denominator of the reduced fraction of a. For P-a.e. ω there exists $y = y(\omega)$ with

$$m_{y(\omega)}(1-h_{y(\omega)}) > \Lambda - \varepsilon.$$

With

$$k := \max\{l \in \mathbb{N} : l \leq (1-\varepsilon)a'n\},$$

we get for large n such that $k \geq y(\omega)$

$$\begin{aligned} E_\omega\big[\eta_{a'n}(a'an)\big] &\geq E_\omega\big[\eta_k(y(\omega))\big]\cdot E_\omega^{y(\omega)}\big[\eta_{a'n-k}(a'an)\big] \\ &\geq \delta_0^{y(\omega)}\cdot(\Lambda-\varepsilon)^{k-y(\omega)}\cdot\delta_0^{a'n-k} \quad \text{for } \mathsf{P}\text{-a.e. } \omega \end{aligned}$$

whereas $\delta_0 := \delta^2\cdot(1-\delta)$. Taking $n\to\infty$ and $\varepsilon\to 0$, we conclude

$$\liminf_{x\downarrow 0}\beta(x) \geq \log(\Lambda).$$

To get the other inequality, we notice that for $n_1, n_2 \in \mathbb{N}$ we have

$$E_\omega\big[\eta_{n_1\cdot n_2}(n_2)\big] \leq \tbinom{n_1\cdot n_2}{n_2}\cdot\Lambda^{(n_1-1)\cdot n_2}\cdot M^{n_2} \quad \text{for } \mathsf{P}\text{-a.e. } \omega. \tag{5.11}$$

Since

$$\tfrac{1}{n_1\cdot n_2}\log\tbinom{n_1\cdot n_2}{n_2} \xrightarrow[n_2\to\infty]{} \tfrac{n_1-1}{n_1}\log\big(\tfrac{n_1}{n_1-1}\big) + \tfrac{1}{n_1}\log(n_1) \xrightarrow[n_1\to\infty]{} 0,$$

(5.11) yields for P-a.e. ω

$$\begin{aligned}\tfrac{1}{n_1\cdot n_2}\log E_\omega\big[\eta_{n_1\cdot n_2}(n_2)\big] &\leq (o(n_2)+o(n_1))+\tfrac{n_1-1}{n_1}\log(\Lambda)+\tfrac{1}{n_1}\log(M)\\ &\xrightarrow[n_2\to\infty]{} \tfrac{n_1-1}{n_1}\log(\Lambda)+o(n_1).\end{aligned}$$

This implies

$$\limsup_{n\to\infty}\beta\left(\tfrac{1}{n}\right)\leq\log(\Lambda),$$

and due to the continuity of β on $(0,1)$, we conclude

$$\limsup_{x\downarrow 0}\beta(x)\leq\log(\Lambda).$$

(iv) Since $(\eta_n(n))_{n\in\mathbb{N}_0}$ is a branching process in an i.i.d. environment satisfying

$$E_\omega[\eta_1(1)] = \ m_0h_0,$$

we have

$$\beta(1)=\mathsf{E}\big[\log(m_0h_0)\big].$$

The continuity of β in 1 can be shown with similar arguments as in part (iii).

(v) Fix $\gamma>0$ and $\varepsilon>0$. We now show that for P-a.e. ω

$$\liminf_{n\to\infty}\min_{x\in n[\gamma,1]\cap\mathbb{N}}\Big(\tfrac{1}{n}\log E_\omega[\eta_n(x)]-\beta(\tfrac{x}{n})\Big)\geq 0. \tag{5.12}$$

To see this, we observe that there is a finite set $\{a_1,\dots,a_l\}\subset(0,1)\cap\mathbb{Q}$ satisfying the following condition:

$$\forall\, b\in[\gamma,1]\ \exists\, i,j\in\{1,\dots,l\}:\ |b-a_i|<\varepsilon\,,\ a_i\leq b \text{ and } |b-a_j|<\varepsilon\,,\ a_j\geq b\,.$$

Let a_i' be the denominator of the reduced fraction of a_i. We define

$$k_i:=\max\{l\in\mathbb{N}: a_i'l\leq(1-\varepsilon)n\}.$$

By definition of k_i, for large n, it holds that

$$(1-2\varepsilon)n<(1-\varepsilon)n-a_i'<a_i'k_i\leq(1-\varepsilon)n. \tag{5.13}$$

Furthermore, for large n and for all $i\in\{1,\dots,l\}$, we have

$$\tfrac{1}{a_i'k_i}\log E_\omega\big[\eta_{a_i'k_i}(a_i'a_ik_i)\big]\geq\beta(a_i)-\varepsilon \tag{5.14}$$

for P-a.e. ω due to (5.7).
Now let $y\in n[\gamma,1]\cap\mathbb{N}$. Then, there is $a_i\leq\frac{y}{n}$ with $|\frac{y}{n}-a_i|<\varepsilon$ and we have

$$a_i'a_ik_i\leq(1-\varepsilon)na_i\leq(1-\varepsilon)y\leq y. \tag{5.15}$$

If $\beta(a_i)-\varepsilon \geq 0$ due to (5.13), (5.14), and (5.15), we have

$$
\begin{aligned}
& E_\omega\big[\eta_n(y)\big] \\
\geq\ & E_\omega\big[\eta_{a_i'k_i}(a_i'a_ik_i)\big]\cdot E_\omega^{a_i'a_ik_i}\big[\eta_{n-a_i'k_i}(y)\big] \\
\geq\ & \exp\big(a_i'k_i\cdot(\beta(a_i)-\varepsilon)\big)\cdot\delta_0^{n-a_i'k_i} \\
=\ & \exp\big(\underbrace{a_i'k_i}_{\geq(1-2\varepsilon)n}\cdot(\beta(a_i)-\varepsilon)-\underbrace{(n-a_i'k_i)}_{\leq 2\varepsilon n}\cdot\log(\delta_0^{-1})\big) \\
\geq\ & \exp\Big(n\big((1-2\varepsilon)\cdot(\beta(a_i)-\varepsilon)-2\varepsilon\cdot\log(\delta_0^{-1})\big)\Big)
\end{aligned}
$$

for P-a.e. ω and for all large n, again with $\delta_0 := \delta^2\cdot(1-\delta)$. This yields for P-a.e. ω

$$
\begin{aligned}
& \tfrac{1}{n}\log E_\omega\big[\eta_n(y)\big] \\
\geq\ & (1-2\varepsilon)\cdot(\beta(a_i)-\varepsilon)-2\varepsilon\cdot\log(\delta_0^{-1}).
\end{aligned}
\tag{5.16}
$$

If $\beta(a_i)-\varepsilon<0$, we conclude in the same way that for P-a.e. ω

$$
\begin{aligned}
& E_\omega\big[\eta_n(y)\big] \\
\geq\ & \exp\Big(n\big((1-\varepsilon)\cdot(\beta(a_i)-\varepsilon)-2\varepsilon\cdot\log(\delta_0^{-1})\big)\Big).
\end{aligned}
\tag{5.17}
$$

Since $|a_i-\frac{y}{n}|<\varepsilon$ and since β is uniformly continuous on $[\gamma,1]$, (5.16) and (5.17) imply (5.12) as $n\to\infty$ and $\varepsilon\to 0$.

(vi) To complete the proof, we now have to show that for P-a.e. ω

$$
\limsup_{n\to\infty}\max_{x\in n[\gamma,1]\cap\mathbb{N}}\Big(\tfrac{1}{n}\log E_\omega\big[\eta_n(x)\big]-\beta(\tfrac{x}{n})\Big)\leq 0. \tag{5.18}
$$

So we assume that (5.18) does not hold and thus for infinitely many $n\in\mathbb{N}$ there exists $y\in n[\gamma,1]\cap\mathbb{N}$ such that

$$
\tfrac{1}{n}\log E_\omega\big[\eta_n(y)\big]\geq\beta(\tfrac{y}{n})+\varepsilon \tag{5.19}
$$

holds with positive probability. As in (v), associated with y, there exists $a_j\geq\frac{y}{n}$ with $|\frac{y}{n}-a_j|<\varepsilon$. We define

$$
k_j' := \max\{l\in\mathbb{N} : a_j'l\leq(1+\varepsilon)n\}.
$$

Then, (5.7) implies

$$
E_\omega\big[\eta_{a_j'k_j'}(a_j'a_jk_j')\big]<\exp\big(a_j'k_j'\cdot(\beta(a_j)+\varepsilon)\big) \tag{5.20}
$$

for P-a.e. ω and for all large n. At the same time due to (5.19), we have with positive probability

$$
\begin{aligned}
& E_\omega\big[\eta_{a_j'k_j'}(a_j'a_jk_j')\big] \\
\geq\ & E_\omega\big[\eta_n(y)\big]\cdot E_\omega^y\big[\eta_{a_j'k_j'-n}(a_j'a_jk_j')\big] \\
\geq\ & \exp\big(n(\beta(\tfrac{y}{n})+\varepsilon)\big)\cdot\delta_0^{a_j'k_j'-n}
\end{aligned}
$$

since $a'_j k'_j - n > 0$ and $a'_j a_j k'_j \geq (n + \varepsilon n - a'_j) a_j \geq n a_j \geq y$ for large n. This yields a contradiction to (5.20) and hence completes the proof of the theorem. ■

Proof of Theorem 5.3.4. For any $\varepsilon > 0$, there exists $x_0 \in \mathbb{Q} \cap (0,1]$ such that

$$\beta(x_0) \geq \max_{x \in [0,1]} \beta(x) - \varepsilon.$$

Let $x'_0 \in \mathbb{N}$ be the denominator of the reduced fraction of x_0. Then, we have for P-a.e. ω

$$\begin{aligned} \liminf_{n\to\infty} \tfrac{1}{nx'_0} \log E_\omega\big[Z_{nx'_0}\big] &\geq \liminf_{n\to\infty} \tfrac{1}{nx'_0} \log E_\omega\big[\eta_{nx'_0}(nx'_0 \cdot x_0)\big] \\ &= \beta(x_0) \geq \max_{x\in[0,1]} \beta(x) - \varepsilon, \end{aligned}$$

and because of the ellipticity condition (5.5),

$$E_\omega\big[Z_{nx'_0+r}\big] \geq \delta_0^r \cdot E_\omega\big[Z_{nx'_0}\big]$$

for $r \in \{0, 1, \ldots, x'_0 - 1\}$ and for P-a.e. ω. We conclude for $\varepsilon \to 0$ that for P-a.e. ω

$$\liminf_{n\to\infty} \tfrac{1}{n} \log E_\omega\big[Z_n\big] \geq \max_{x\in[0,1]} \beta(x). \tag{5.21}$$

To get the other inequality, we first state the following

Lemma 5.5.2. *For $\varepsilon > 0$, there is $\gamma > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\tfrac{1}{n} \log E_\omega\big[\eta_n(y)\big] \leq \log(\Lambda + \varepsilon) \quad \text{for } \mathsf{P}\text{-a.e. } \omega$$

for all $y \in n[0,\gamma] \cap \mathbb{N}_0$.

Proof of Lemma 5.5.2. For $\frac{1}{2} > \gamma > 0$ and $y < \gamma n$, we have

$$E_\omega[\eta_n(y)] \leq \tbinom{n}{y} \cdot \Lambda^{n-y} \cdot M^y \quad \text{for } \mathsf{P}\text{-a.e. } \omega.$$

Since

$$\tfrac{1}{n} \log \tbinom{n}{y} \leq \tfrac{1}{n} \log \tbinom{n}{\lfloor \gamma n \rfloor} \to 0$$

for $\gamma \to 0$ uniformly in n, we get for P-a.e. ω

$$\tfrac{1}{n} \log E_\omega[\eta_n(y)] \leq o(\gamma) + \tfrac{n-y}{n} \log(\Lambda) + \tfrac{y}{n} \log(M) \leq \log(\Lambda + \varepsilon)$$

for $\gamma > 0$ small enough. □

For an arbitrary $\varepsilon > 0$, we now choose $\gamma > 0$ as in Lemma 5.5.2. Then, by Theorem 5.3.3 and Lemma 5.5.2, we get

$$\begin{aligned} &\limsup_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] \\ = \; &\limsup_{n\to\infty} \tfrac{1}{n} \log E_\omega \left(\sum_{y=0}^{\lfloor \gamma n \rfloor - 1} \eta_n(y) + \sum_{y=\lfloor \gamma n \rfloor}^{n} \eta_n(y) \right) \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{n\to\infty} \tfrac{1}{n} \log \Big(\gamma n \cdot \big(\Lambda + \varepsilon\big)^n + n \cdot \exp\Big(n \cdot \big(\max_{x\in[0,1]} \beta(x) + o(n)\big)\Big)\Big) \\
&\leq \max_{x\in[0,1]} \beta(x) + \varepsilon
\end{aligned}$$

for P-a.e. ω since $\beta(0) = \log(\Lambda)$. For $\varepsilon \to 0$, this yields for P-a.e. ω

$$\limsup_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] \leq \max_{x\in[0,1]} \beta(x)\,,$$

which, together with (5.21), proves the claim. ■

Proof of Theorem 5.3.5. We start by proving that *(ii)* implies *(i)*.
First, assume that there is P-a.s. LS. As shown in the proof of Theorem 5.3.1, for P-a.e. ω there is a location x such that the descendants of a particle at x that stay at x form a supercritical Galton-Watson process. Let $x = x(\omega)$ be such a location, i.e. $m_x(1-h_x) > 1$. Then, we have for P-a.e. ω and for $n \geq x$

$$\begin{aligned}
E_\omega[Z_n] &\geq E_\omega[\eta_n(x)] \\
&\geq \big(1-\mu_0(\{0\})\big)h_0 \cdot \ldots \cdot \big(1-\mu_{x-1}(\{0\})\big)h_{x-1} \cdot \big(m_x(1-h_x)\big)^{n-x} \\
&\geq \big(\delta^{2x} \cdot \big(m_x(1-h_x)\big)\big)^{n-x}
\end{aligned}$$

where we used condition (5.1) for the last inequality. Due to Theorem 5.3.4, we obtain for P-a.e. ω

$$\begin{aligned}
\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] &\geq \limsup_{n\to\infty} \tfrac{1}{n} \log \big(\delta^{2x} \cdot (m_x(1-h_x))^{n-x}\big) \\
&= \log\left(m_x(1-h_x)\right) \\
&> 0.
\end{aligned}$$

Now, let us assume that there is P-a.s. no LS, which is according to Theorem 5.3.1 equivalent to $\Lambda \leq 1$. Again, we use the process $(\xi_n)_{n\in\mathbb{N}_0}$ defined in the proof of Theorem 5.3.2.
Since there is GS for P-a.e. ω, the process $(\xi_n)_{n\in\mathbb{N}_0}$ has a positive probability of survival for P-a.e. ω. Thus, we have

$$\int \log\big(E_\omega[\xi_1]\big) \mathsf{P}(d\omega) > 0 \tag{5.22}$$

due to Theorem 4.2.1 and Theorem 4.2.2 by Tanny. For $T \in \mathbb{N}$, we now introduce a slightly modified embedded branching process $(\xi_n^T)_{n\in\mathbb{N}_0}$. For $k \in \mathbb{N}$, we define ξ_k^T as the total number of all particles that move from position $k-1$ to k within T time units after they were released at position $k-1$. The left over particles are no longer considered. With $\xi_0^T := 1$, we observe that $(\xi_n^T)_{n\in\mathbb{N}_0}$ is a branching process in an i.i.d. environment. By the monotone convergence theorem and (5.22), there exists some T such that

$$\int \log\left(E_\omega\left[\xi_1^T\right]\right) \mathsf{P}(d\omega) > 0. \tag{5.23}$$

By construction of $(\xi_n^T)_{n\in\mathbb{N}_0}$, we obtain

$$\xi_n^T \;\leq\; Z_n + Z_{n+1} + \ldots + Z_{nT}. \tag{5.24}$$

Using the strong law of large numbers and taking into account that ω is an i.i.d. sequence, we have

$$\begin{aligned}
&\lim_{n\to\infty} \tfrac{1}{n} \log E_\omega\left[\xi_n^T\right] \\
= \;& \lim_{n\to\infty} \tfrac{1}{n} \log \prod_{i=0}^{n} E_{\theta^n\omega}\left[\xi_1^T\right] \\
= \;& \lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n} \log E_{\theta^n\omega}\left[\xi_1^T\right] \\
= \;& \int \log\left(E_\omega\left[\xi_1^T\right]\right) \mathsf{P}(d\omega) \quad \text{for } \mathsf{P}\text{-a.e. } \omega.
\end{aligned} \tag{5.25}$$

Here, θ again denotes the shift operator as usual, i.e. $(\theta\,\omega)_i = \omega_{i+1}$. Together with (5.23) and (5.24), this yields for P-a.e. ω

$$\liminf_{n\to\infty} \tfrac{1}{n} \log E_\omega\big[Z_n + Z_{n+1} + \ldots + Z_{nT}\big] > 0. \tag{5.26}$$

Now, we conclude using Theorem 5.3.4 that for P-a.e. ω

$$\max_{x\in[0,1]} \beta(x) = \lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[Z_n] > 0$$

because otherwise there would be a contradiction to (5.26). This shows that *(ii)* implies *(i)*.

To show that *(i)* implies *(ii)*, we first notice that *(ii)* obviously holds if there is LS for P-a.e. ω. Therefore, we may assume $\Lambda \le 1$ for the rest of the proof.
Now, label every particle of the entire branching process and let Γ denote the set of all produced particles. Write $\sigma \prec \tau$ for two particles $\sigma \neq \tau$ if σ is an ancestor of τ and denote by $|\sigma|$ the generation in which the particle σ is produced. Furthermore, for every $\sigma \in \Gamma$ let X_σ be the random location of the particle σ. Using these notations, we define

$$G_i := \{\tau \in \Gamma :\, X_\tau = i,\, X_\sigma < i \text{ for all } \sigma \in \Gamma,\, \sigma \prec \tau\} \tag{5.27}$$

for every $i \in \mathbb{N}_0$. Therefore, G_i is for $i \neq 0$ the set of all the particles τ that move from position $i-1$ to position i and hence the particles in G_i are the first particles at position i in their particular ancestral lines. We observe that the process $(|G_n|)_{n\in\mathbb{N}_0}$ coincides with $(\xi_n)_{n\in\mathbb{N}_0}$. Further, define for every $\sigma \in \Gamma$ and $n \in \mathbb{N}_0$

$$H_n^\sigma := |\{\tau \in \Gamma :\, \sigma \preceq \tau,\, |\tau| = n,\, X_\tau = X_\sigma\}|$$

as the number of descendants of the particle σ in generation n which are still at the same location as the particle σ. This enables us to decompose Z_n in the following way:

$$Z_n = \sum_{i=1}^{n} \sum_{\sigma\in G_i} H_{n-|\sigma|}^\sigma \tag{5.28}$$

Since by assumption there is no LS, we have for P-a.e. ω

$$E_\omega[H_n^\sigma \mid \sigma \in \Gamma,\, X_\sigma = i] \le 1 \tag{5.29}$$

because for any existing particle σ its progeny which stays at the location of σ forms a Galton-Watson process which eventually dies out. By (5.28) and (5.29), we conclude that for P-a.e. ω we have

$$E_\omega[Z_n] \le \sum_{i=1}^{n} E_\omega[|G_i|].$$

Therefore, due to (i), we get

$$\limsup_{n\to\infty} \tfrac{1}{n} \log E_\omega[|G_n|] > 0 \quad \text{for } \mathsf{P}\text{-a.e. } \omega.$$

Since $(|G_n|)_{n\in\mathbb{N}_0}$ coincides with the branching process in random environment $(\xi_n)_{n\in\mathbb{N}_0}$, we obtain

$$\int \log\big(E_\omega[\xi_1]\big)\,\mathsf{P}(d\omega) = \lim_{n\to\infty} \tfrac{1}{n} \log E_\omega[|G_n|] > 0 \quad \text{for } \mathsf{P}\text{-a.e. } \omega$$

as in (5.25). But then again, we have GS for P-a.e. ω since $(\xi_n)_{n\in\mathbb{N}_0}$ survives with positive probability for P-a.e. ω. ■

Proof of Theorem 5.3.6. In this proof we use the expression "a.s." in the sense of "P_ω-a.s. for P-a.e. ω".

Part 1. In the first part of the proof we show in three steps that we have a.s.

$$\limsup_{n\to\infty} \tfrac{1}{n} \log Z_n \le \max_{x\in[0,1]} \beta(x). \tag{5.30}$$

(i) To obtain (5.30), we start by showing that for all $\gamma > 0$ we have a.s.

$$\limsup_{n\to\infty} \max_{x\in n[\gamma,1]\cap\mathbb{N}} \big(\tfrac{1}{n} \log \eta_n(x) - \beta(\tfrac{x}{n})\big) \le 0. \tag{5.31}$$

To see this, fix $\gamma > 0$ and $\varepsilon > 0$.
Then, by Theorem 5.3.3, for P-a.e. ω there exists $N = N(\omega,\gamma,\varepsilon)$ such that for all $n \ge N$ and for all $y \in n[\gamma,1]\cap\mathbb{N}$ we have

$$E_\omega[\eta_n(y)] \le \exp\big(n\cdot(\beta(\tfrac{y}{n}) + \varepsilon)\big).$$

Thus, for P-a.e. ω we obtain for large n and for all $y \in n[\gamma,1]\cap\mathbb{N}$

$$P_\omega\Big(\eta_n(y) \ge \exp\big(n\cdot(\beta(\tfrac{y}{n}) + 2\varepsilon)\big)\Big) \le \frac{E_\omega[\eta_n(y)]}{\exp(n\cdot(\beta(\frac{y}{n}) + 2\varepsilon))} = \exp(-\varepsilon n).$$

Using the Borel-Cantelli lemma and taking into account that $\big|n[\gamma,1]\cap\mathbb{N}\big| \le n$, this yields that a.s. we have

$$\limsup_{n\to\infty} \max_{x\in n[\gamma,1]\cap\mathbb{N}} \big(\tfrac{1}{n} \log \eta_n(y) - \beta(\tfrac{y}{n})\big) < 2\varepsilon.$$

Since ε is arbitrarily small, this proves (5.31).

(ii) Secondly, we show that for every $\varepsilon > 0$ there exists $\gamma = \gamma(\varepsilon) > 0$ such that a.s. we have

$$\limsup_{n\to\infty} \max_{x\in n[0,\gamma]\cap\mathbb{N}} \left(\tfrac{1}{n}\log \eta_n(x) - \beta(0) - \varepsilon\right) \le 0. \tag{5.32}$$

To see this, we observe that according to Lemma 5.5.2 for every $\varepsilon > 0$ there is $\gamma = \gamma(\varepsilon) > 0$ such that

$$\tfrac{1}{n}\log E_\omega\big[\eta_n(y)\big] \le \log(\Lambda + \varepsilon) \overset{(\Lambda>1)}{\le} \log(\Lambda) + \varepsilon = \beta(0) + \varepsilon$$

for P-a.e. ω and for $0 \le y \le \gamma n$. Therefore, the same argument as in (i) yields (5.32).

(iii) We now combine (i) and (ii) to obtain (5.30). For an arbitrary $\varepsilon > 0$, choose $\gamma > 0$ as in (ii). Then, (5.31) and (5.32) imply that a.s. we have

$$\begin{aligned}
&\limsup_{n\to\infty} \tfrac{1}{n}\log Z_n \\
=\;& \limsup_{n\to\infty} \tfrac{1}{n}\log\left(\sum_{y=0}^{\lfloor\gamma n\rfloor-1}\eta_n(y) + \sum_{y=\lfloor\gamma n\rfloor}^{n}\eta_n(y)\right) \\
\le\;& \limsup_{n\to\infty} \tfrac{1}{n}\log\Big(\gamma n\cdot\exp\big(n\cdot(\beta(0)+\varepsilon)\big) + n\cdot\exp\Big(n\cdot\big(\max_{x\in[0,1]}\beta(x) + o(n)\big)\Big)\Big) \\
\le\;& \max_{x\in[0,1]}\beta(x) + \varepsilon.
\end{aligned}$$

For $\varepsilon \to 0$, this implies (5.30) and thus the first part of the proof is complete.

Part 2. In the second part of the proof we show that

$$P_\omega\left(\liminf_{n\to\infty}\tfrac{1}{n}\log Z_n \ge \max_{x\in[0,1]}\beta(x) \;\middle|\; Z_n \not\to 0\right) = 1 \quad \text{for } \mathsf{P}\text{-a.e. } \omega. \tag{5.33}$$

We start by stating the following

Lemma 5.5.3. *For all $\varepsilon > 0$ and $r, s \in \mathbb{N}$ with $r \le s$ and $\beta(\frac{r}{s}) - \varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for P-a.e. ω we have*

$$P_\omega\left(\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log\eta_{nsN_0}(nrN_0) \ge \beta(\tfrac{r}{s}) - \varepsilon\right) > 0.$$

Proof of Lemma 5.5.3. Define

$$M_N := \left\{\omega \in \Omega : \tfrac{1}{sN}\log E_\omega[\eta_{sN}(rN)] \ge \beta(\tfrac{r}{s}) - \tfrac{\varepsilon}{2}\right\}.$$

Then, for every $\varepsilon_0 > 0$ there exists $N_0 = N_0(\varepsilon_0)$ such that

$$\mathsf{P}\big(M_{N_0}\big) \ge 1 - \varepsilon_0$$

and thus for sufficiently small ε_0 and the corresponding $N_0(\varepsilon_0)$ we have

$$\int \log E_\omega\big[\eta_{sN_0}(rN_0)\big]\,\mathsf{P}(d\omega)$$

$$
\begin{aligned}
&\geq\ sN_0(\beta(\tfrac{r}{s})-\tfrac{\varepsilon}{2})(1-\varepsilon_0)\\
&\geq\ sN_0(\beta(\tfrac{r}{s})-\varepsilon)+sN_0(\tfrac{\varepsilon}{2}-\beta(\tfrac{r}{s})\varepsilon_0+\tfrac{\varepsilon}{2}\varepsilon_0)\\
&\geq\ sN_0(\beta(\tfrac{r}{s})-\varepsilon)\ >\ 0.
\end{aligned}
\tag{5.34}
$$

We now construct a branching process in random environment $(\psi_n)_{n\in\mathbb{N}_0}$ which is dominated by $\big(\eta_{nsN_0}(nrN_0)\big)_{n\in\mathbb{N}_0}$. After starting with one particle at 0, we count all the particles that are at time sN_0 at position rN_0 and denote this number by ψ_1. The remaining particles are removed from the system and no longer considered. After that, we count the number of particles at time $2sN_0$ at position $2rN_0$ and denote this number by ψ_2. Repeating this procedure yields the process $(\psi_n)_{n\in\mathbb{N}_0}$ which is supercritical due to (5.34). In fact, (5.34) and Theorem 4.2.3 by Tanny now imply that for sufficiently small ε_0,

$$
\liminf_{n\to\infty}\tfrac{1}{n}\log\eta_{nsN_0}(nrN_0)\geq sN_0(\beta(\tfrac{r}{s})-\varepsilon)
\tag{5.35}
$$

a.s. on $\{\psi_n\not\to 0\}$. Since we assume condition (5.5), Theorem 4.2.1 by Tanny implies

$$
P_\omega(\psi_n\to 0)<1
\tag{5.36}
$$

for P-a.e. ω. Combining (5.35) and (5.36) now completes the proof of the lemma. $\square$

Lemma 5.5.3 yields the following

Corollary 5.5.4. *Let ε, r, s and N_0 be as in Lemma 5.5.3. Then, there exists $\nu>0$ such that for P-a.e. ω there exists an increasing sequence $(x_l)_{l\in\mathbb{N}_0}=(x_l(\omega))_{l\in\mathbb{N}_0}$ in $\mathbb{N}_0$ such that for all $l\in\mathbb{N}_0$ we have*

$$
P_\omega^{x_l}\Big(\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log\eta_{nsN_0}(nrN_0)\geq\beta(\tfrac{r}{s})-\varepsilon\Big)>\nu.
$$

Proof of Corollary 5.5.4. Due to Lemma 5.5.3, there exists $\nu>0$ such that

$$
\mathsf{P}\Big(\Big\{\omega:\ P_\omega\Big(\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log\eta_{nsN_0}(nrN_0)\geq\beta(\tfrac{r}{s})-\varepsilon\Big)>\nu\Big\}\Big)>0.
$$

Since the sequence

$$
\begin{aligned}
&\Big(P_\omega^{x}\Big(\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log\eta_{nsN_0}(nrN_0)\geq\beta(\tfrac{r}{s})-\varepsilon\Big)\Big)_{x\in\mathbb{N}_0}\\
=\ &\Big(P_{\theta^x\omega}\Big(\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log\eta_{nsN_0}(nrN_0)\geq\beta(\tfrac{r}{s})-\varepsilon\Big)\Big)_{x\in\mathbb{N}_0}
\end{aligned}
$$

is ergodic with respect to P, the ergodic theorem yields

$$
\lim_{n\to\infty}\tfrac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}\Big\{P_{\theta^x\omega}\big(\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log\eta_{nsN_0}(nrN_0)\geq\beta(\tfrac{r}{s})-\varepsilon\big)>\nu\Big\}>0
$$

for P-a.e. ω and this completes the proof of the corollary. $\square$

Let $(x_l)_{l\in\mathbb{N}_0}$ be an increasing sequence of positions as in Corollary 5.5.4. We now show in two steps that a.s. on the event of non-extinction there will eventually be some particle at one of the positions x_l such that the descendants of this particle constitute a process with the desired growth.

(i) As a first step, we show that a.s. on the event of survival $(Z_n)_{n\in\mathbb{N}_0}$ grows as desired along some subsequence $(j+nsN_0)_{n\in\mathbb{N}_0}$ for some $j\in\{0,\dots,sN_0-1\}$. To obtain this, as in the proof of Theorem 5.3.5, let Γ again denote the set of all existing particles and for $\sigma\in\Gamma$ let $\eta_n^\sigma(y)$ denote the number of descendants of σ among the particles which belong to $\eta_n(y)$. With the sets $(G_l)_{l\in\mathbb{N}_0}$ as in (5.27) and the sequence $(x_l)_{l\in\mathbb{N}_0}$ as in Corollary 5.5.4, we define:

$$\begin{aligned} A_{x_l} &:= \Big\{\exists\,\sigma\in G_{x_l} : \liminf_{n\to\infty} \tfrac{1}{nsN_0}\log \eta^\sigma_{|\sigma|+nsN_0}(x_l+nrN_0) \geq \beta(\tfrac{r}{s})-\varepsilon\Big\} \\ B_{x_l} &:= \Big\{|G_{x_l}|\geq l\Big\} \end{aligned}$$

Due to Corollary 5.5.4 and since the descendants of all particles belonging to G_{x_l} evolve independently, we get

$$P_\omega\left(A^c_{x_l}\cap B_{x_l}\right)\leq (1-\nu)^l \quad \text{for } \mathsf{P}\text{-a.e. } \omega,$$

and therefore, we conclude with the Borel-Cantelli lemma that

$$P_\omega\left(\limsup_{l\to\infty}\left(A^c_{x_l}\cap B_{x_l}\right)\right)=0 \quad \text{for } \mathsf{P}\text{-a.e. } \omega. \tag{5.37}$$

According to Theorem 4.2.3 by Tanny, we have a.s. exponential growth of the process $\big(|G_l|\big)_{l\in\mathbb{N}_0}$ on the event of survival and therefore it holds that we have a.s.

$$\liminf_{l\to\infty} B_{x_l}=\big\{Z_n\not\to 0\big\}.$$

Together with (5.37), this yields

$$P_\omega\left(\limsup_{l\to\infty} A^c_{x_l}\ \middle|\ Z_n\not\to 0\right)=0 \quad \text{for } \mathsf{P}\text{-a.e. } \omega.$$

Thus a.s. on $\{Z_n\not\to 0\}$, there is $l\in\mathbb{N}_0$ and $\sigma\in G_{x_l}$ such that

$$\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log\eta^\sigma_{|\sigma|+nsN_0}(x_l+nrN_0)\geq\beta(\tfrac{r}{s})-\varepsilon$$

and hence we have for P-a.e. ω

$$\begin{aligned} & P_\omega\left(\bigcup_{\sigma\in\Gamma}\Big\{\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log Z_{|\sigma|+nsN_0}\geq\beta(\tfrac{r}{s})-\varepsilon\Big\}\ \middle|\ Z_n\not\to 0\right) \\ = \ & P_\omega\left(\bigcup_{j\in\mathbb{N}_0}\Big\{\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log Z_{j+nsN_0}\geq\beta(\tfrac{r}{s})-\varepsilon\Big\}\ \middle|\ Z_n\not\to 0\right) \end{aligned}$$

$$= \quad P_\omega\left(\bigcup_{j=1}^{sN_0}\left\{\liminf_{n\to\infty}\tfrac{1}{nsN_0}\log Z_{j+nsN_0} \geq \beta(\tfrac{r}{s})-\varepsilon\right\}\;\middle|\; Z_n \not\to 0\right) \;=\; 1. \tag{5.38}$$

(ii) The last step of this part of the proof is to show that the growth along some subsequence $(j+nsN_0)_{n\in\mathbb{N}_0}$ already implies sufficiently strong growth of $(Z_n)_{n\in\mathbb{N}_0}$.
Due to the ellipticity condition (5.5), we have (recalling $\delta_0 = \delta^2(1-\delta)$)

$$P_\omega^x\big(\eta_i(x)\geq 1\big) \geq \delta_0^i \quad \text{for all } i, x\in\mathbb{N}_0.$$

A large deviation bound for the binomial distribution therefore implies

$$P_\omega\left(Z_{n+i}\leq Z_n\cdot\tfrac{\delta_0^i}{2}\;\middle|\; Z_n = m\right) \leq \exp(-m\cdot\lambda_0) \quad \forall\, m\in\mathbb{N} \tag{5.39}$$

for $i\in\{1,...,sN_0\}$ and some constant $\lambda_0=\lambda_0(N_0)>0$. We now define:

$$\begin{aligned} C_{j,n} &:= \bigcup_{i=1}^{sN_0}\left\{Z_{j+nsN_0+i}\leq \tfrac{\delta_0^{sN_0}}{2}\exp\big(nsN_0\cdot(\beta(\tfrac{r}{s})-\varepsilon)\big)\right\} \\ D_{j,n} &:= \left\{\tfrac{1}{nsN_0}\log Z_{j+nsN_0}\geq\beta(\tfrac{r}{s})-\varepsilon\right\} \end{aligned}$$

Then, due to (5.39) for P-a.e. ω, we have for all $j\in\{1,\dots,sN_0\}$

$$\begin{aligned} & P_\omega\left(C_{j,n}\cap D_{j,n}\right) \\ \leq\;& sN_0\cdot\exp\big(-\lambda_0\exp(n\cdot\lambda_1)\big) \end{aligned} \tag{5.40}$$

where $\lambda_1 := sN_0\cdot(\beta(\frac{r}{s})-\varepsilon)$.
Since the upper bound in (5.40) is summable in $n\in\mathbb{N}_0$, we can apply the Borel-Cantelli lemma and conclude that for P-a.e. ω we have for all $j\in\{1,\dots,sN_0\}$

$$\begin{aligned} & P_\omega\left(\limsup_{n\to\infty} C_{j,n}\;\middle|\;\liminf_{n\to\infty} D_{j,n}\right) \\ \leq\;& P_\omega\left(\liminf_{n\to\infty} D_{j,n}\right)^{-1}\cdot\; P_\omega\left(\limsup_{n\to\infty}(C_{j,n}\cap D_{j,n})\right) \;=\; 0 \end{aligned}$$

Thus, for P-a.e. ω, we have for all $j\in\{1,\dots,sN_0\}$

$$P_\omega\left(\liminf_{n\to\infty}\tfrac{1}{n}\log Z_n\leq\beta(\tfrac{r}{s})-2\varepsilon\;\middle|\;\liminf_{n\to\infty} D_{j,n}\right)=0$$

and this implies

$$P_\omega\left(\liminf_{n\to\infty}\tfrac{1}{n}\log Z_n\leq\beta(\tfrac{r}{s})-2\varepsilon\;\middle|\;\bigcup_{j=1}^{sN_0}\liminf_{n\to\infty} D_{j,n}\right)=0. \tag{5.41}$$

Using (5.38) and (5.41), we obtain

$$P_\omega\left(\liminf_{n\to\infty}\tfrac{1}{n}\log Z_n\leq\beta(\tfrac{r}{s})-2\varepsilon\;\middle|\; Z_n\not\to 0\right)$$

$$\begin{aligned} &\leq \; P_\omega\left(Z_n \not\to 0\right)^{-1} \cdot P_\omega\left(\left\{\liminf_{n\to\infty} \tfrac{1}{n}\log Z_n \leq \beta(\tfrac{r}{s}) - 2\varepsilon\right\} \cap \bigcup_{j=1}^{sN_0} \liminf_{n\to\infty} D_{j,n}\right) \\ &= \; 0 \end{aligned}$$

which yields

$$P_\omega\left(\liminf_{n\to\infty} \tfrac{1}{n}\log Z_n > \beta(\tfrac{r}{s}) - 2\varepsilon \;\middle|\; Z_n \not\to 0\right) = 1 \quad \text{for } \mathsf{P}\text{-a.e. } \omega. \tag{5.42}$$

Since r and s can be chosen such that $\beta(\frac{r}{s})$ is arbitrarily close to $\max_{x\in[0,1]}\beta(x)$, (5.42) implies (5.33) as $\varepsilon \to 0$ and the proof is complete.

■

Proof of Theorem 5.3.7. If $M \leq 1$, then

$$\log\left(\frac{m_0 h}{1 - m_0(1-h)}\right) \leq 0 \quad \mathsf{P}\text{-a.s.}$$

and therefore Theorem 5.3.2 implies *(i)*.

We continue with proving *(ii)* and assume that $M > 1$. If $m_0 = M$ P-a.s., then

$$\log\left(\frac{m_0 h}{1 - m_0(1-h)}\right) > 0 \quad \mathsf{P}\text{-a.s.}$$

and thus $\varphi(h) > 0$ for all $h \in (h_{LS}, 1]$. This case is included in *(c)*.
In the following, we assume that m_0 is not deterministic. We notice that φ is finite and continuously differentiable for $h \in (h_{LS}, 1]$ since

$$\frac{\partial}{\partial h}\log\left(\frac{m_0 h}{1 - m_0(1-h)}\right) = \frac{1}{h} - \frac{m_0}{1 - m_0(1-h)}$$

is a.s. uniformly bounded for $h \in [h_{LS} + \varepsilon, 1]$ with $\varepsilon > 0$. Thus, we have

$$\frac{\partial}{\partial h}\varphi(h) = \mathsf{E}\left[\frac{1}{h} - \frac{m_0}{1 - m_0(1-h)}\right]. \tag{5.43}$$

Now assume that there exists $h^* \in (h_{LS}, 1]$ with $\varphi(h^*) = 0$. Then,

$$\mathsf{E}\left[\log\left(\frac{m_0}{1 - m_0(1-h^*)}\right)\right] = \log\left(\frac{1}{h^*}\right). \tag{5.44}$$

Due to the strict concavity of $y \longmapsto \log y$, we have

$$\log\left(\mathsf{E}\left[\frac{m_0}{1 - m_0(1-h^*)}\right]\right) > \log\left(\frac{1}{h^*}\right) \tag{5.45}$$

by Jensen's inequality and (5.44). Thus, we obtain that φ is strictly decreasing in $h = h^*$ by (5.43) and (5.45).

Now assume $\varphi(h_{LS}) = 0$. As above, Jensen's inequality yields (5.45) for h_{LS} instead of h^*. Since the mapping

$$h \longmapsto \frac{m_0}{1 - m_0(1-h)}$$

is decreasing in $h > 1 - \frac{1}{m_0}$, we have

$$\lim_{\varepsilon \downarrow 0} \mathsf{E}\left[\frac{m_0}{1 - m_0(1 - h_{LS} + \varepsilon)}\right] = \mathsf{E}\left[\frac{m_0}{1 - m_0(1 - h_{LS})}\right] > \frac{1}{h_{LS}}$$

by the monotone convergence theorem. Thus, φ is strictly decreasing and therefore negative in $h \in (h_{LS}, h_{LS} + \varepsilon)$ for some sufficiently small $\varepsilon > 0$.

Now, we obtain *(a)* – *(c)* by the continuity of φ and the fact that φ is strictly decreasing in every zero in $[h_{LS}, 1]$. ■

5.6 Examples

1. A basic and natural example to illustrate our results is the following. Let $\mu^{(+)}$ and $\mu^{(-)}$ be two different non-trivial offspring distributions. We define

$$m^{(+)} := \sum_{k=0}^{\infty} k\, \mu^{(+)}(k) \quad \text{and} \quad m^{(-)} := \sum_{k=0}^{\infty} k\, \mu^{(-)}(k)$$

and suppose

$$0 < m^{(-)} < m^{(+)} \le \infty.$$

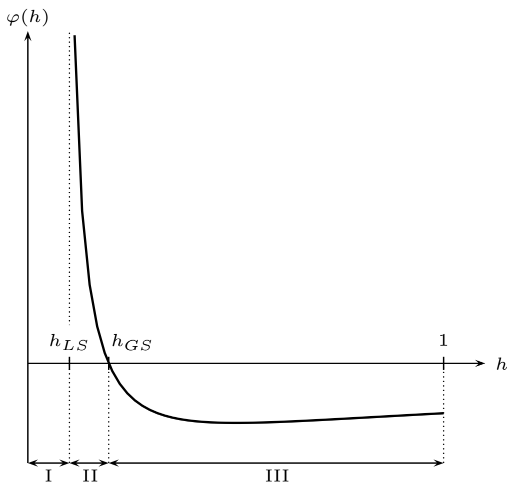

Figure 5.1: *There are three regimes: I: LS, II: GS but no LS, III: no GS*

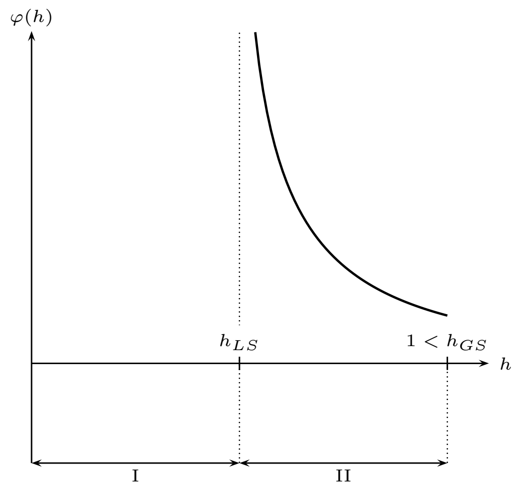

Figure 5.2: *There are two regimes: I: LS, II: GS but no LS*

Furthermore, let

$$\mathsf{P}\left(\mu_0 = \mu^{(+)}\right) = 1 - \mathsf{P}\left(\mu_0 = \mu^{(-)}\right) = q \in (0,1).$$

This setting obviously satisfies condition (5.1). For Figures 5.1 and 5.2, we have chosen

$$\begin{aligned} q &= \frac{3}{4}, \qquad & m^{(+)} &= \frac{10}{9}, \qquad & m^{(-)} &= \frac{2}{5}, \\ q &= \frac{1}{2}, \qquad & m^{(+)} &= \ 2, \qquad & m^{(-)} &= \frac{2}{3}, \end{aligned}$$

respectively.

2. As already announced above, we now provide an example for a setting in which $h_{GS} = h_{LS} < 1$. Let the law P^{m_0} of the mean offspring m_0 be given by

$$\frac{d\mathsf{P}^{m_0}}{d\lambda}(x) := 1.6 \cdot \mathbb{1}_{[0.5,1]}(x) + 0.2 \cdot \mathbb{1}_{(1,2]}(x)$$

where λ denotes the Lebesgue measure. Obviously, $h_{LS} = 0.5$ and a simple computation yields

$$\varphi(h_{LS}) = 0.2 \cdot \Big(2 \cdot \log(2)\Big) + 1.6 \cdot \Big(2 \cdot \log(2) - 1.5 \cdot \log(3)\Big) < 0.$$

Bibliography

[An07] Andreoletti, P. (2007). Almost sure estimates for the concentration neighborhood of Sinai's walk. *Stochastic Process. Appl.*, **117**, 1473–1490.

[BCGH93] Baillon, B. and Clement, Ph. and Greven, A and den Hollander, F. (1993). A variational approach to branching random walk in random environment. *Ann. Probab.*, **21**, 290–317.

[BCGH94] Baillon, B. and Clement, Ph. and Greven, A and den Hollander, F. (1994). On a variational problem for an infinite particle system in a random medium. *J. reine angew. Math.*, **454**, 181–217.

[BGK09] Bartsch, C. and Gantert, N. and Kochler, M. (2009). Survival and growth of a branching random walk in random environment. *Markov Proc. Relat. Fields*, **15**, 525–548.

[CP03a] Campanino, M. and Pétritis, D. (2003). Random walks on randomly oriented lattices. *Markov Proc. Relat. Fields*, **3**, 491–512.

[Ch62] Chernov, A. (1962). Replication of a multicomponent chain, by the "lightning mechanism". *Biophysics*, **12**, 336–341.

[CP03b] Comets, F. and Popov, S. (2003). Limit law for transition probabilities and moderate deviations for Sinai's random walk in random environment. *Probab. Theory Related Fields*, **126**, 571–609.

[CP07] Comets, F. and Popov, S. (2007). Shape and local growth for multidimensional branching random walks in random environment. *ALEA*, **3**, 273–299.

[DGPS07] Dembo, A. and Gantert, N. and Peres, Y. and Shi, Z. (2007). Valleys and the maximum local time for random walk in random environment. *Probab. Theory Related Fields*, **137**, 443–473.

[GMPV08] Gantert, N. and Müller, S. and Popov, S. and Vachkovskaia, M. (2010). Survival of branching random walks in random environment. *Journal of Theoretical Probability*, **23**, 1002–1014.

[GKP12] Gantert, N. and Kochler, M. and Pène, F. (2012). RWRE on randomly oriented lattices. *In preparation.*

[GPS10] Gantert, N. and Peres, Y. and Shi, Z. (2010). The infinite valley for a recurrent random walk in random environment. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, **46**, 525–536.

[Go84] Golosov, A. O. (1984). Localization of random walks in one-dimensional random environments. *Comm. Math. Phys.*, **92**, 491–506.

[Go86] Golosov, A. O. (1986). On limiting distributions for a random walk in a critical one-dimensional random environment. *Russian Math. Surveys*, **41**, 199–200.

[GH91] Greven, A. and den Hollander, F. (1991). Population growth in random media. I. Variational formula and phase diagram. *Journal of Statistical Physics*, **65**, Nos. 5/6, 1123–1146.

[GH92] Greven, A. and den Hollander, F. (1992). Branching random walk in random environment: Phase transition for local and global growth rates. *Prob. Theory and Related Fields*, **91**, 195–249.

[GH94] Greven, A. and den Hollander, F. (1994). On a variational problem for an infinite particle system in a random medium. Part II: The local growth rate. *Prob. Theory Related Fields.*, **100**, 301–328.

[Gut05] Gut, A. (2005). Probability: a graduate course. *Springer Texts in Statistics*, Springer, New York.

[HS06] den Hollander, F. and Steiff, J. (2006) Random walk in random scenery: A survey of some recent results. *IMS Lecture Notes-Monograph Series*, **48**, 53–65.

[HS98] Hu, Y. and Shi, Z. (1998). The Local time of Simple Random Walk in Random Environment. *Journal of Theoretical Probability*, **11**, 765–793.

[JP75] Jain, N. and Pruitt, W. (1975). The other law of the iterated logarithm. *Ann. Probab.*, **3**, 1046–1049.

[Ke86] Kesten, H. (1986). The limit distribution of Sinaĭ's random walk in random environment. *Physica A. Statistical and Theoretical Physics*, **138**, 299–309.

[KS79] Kesten, H. and Spitzer, F. (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete*, **50**, 5–25.

[Kl06] Klenke, A. (2006). Probability theory. Springer, London.

[KMT76] Komlós, J. and Major, P. and Tusnády, G. (1976). An approximation of partial sums of independent RV's and the sample DF. II. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **34**, 33–58.

[MP10] Mörters, P. and Peres, Y. (2010). Brownian motion. *Cambridge University Press.*

[Ré05] Révész, P. (2005). Random walk in random and non-random environments (*second edition*). World Scientific, Singapore.

[Sh98] Shi, Z. (1998). A local time curiosity in random environment. *Stochastic Process. Appl.*, **76**, 231–250.

[SZ07] Shi, Z. and Zindy, O. (2007). A weakness in strong localization for Sinai's walk. *Ann. Probab.*, **35**, 1118–1140.

[Si82] Sinaĭ, Y. G. (1982). The limit behavior of a one-dimensional random walk in a random environment. *Theory Probab. Appl.*, **27**, 256–268.

[Sm68] Smith, W. (1968). Necessary conditions for almost sure extinction of a branching process with random environment. *Ann. Math. Statist.*, **39**, 2136–2140.

[SW69] Smith, W. and Wilkinson, W. (1969). On branching processes in random environments. *Ann. Math. Statist.*, **40**, 814–827.

[So75] Solomon, F. (1975). Random walks in a random environment. *Ann. Probab.*, **3**, 1–31.

[Ta77] Tanny, D. (1977). Limit theorems for branching processes in a random environment. *Ann. Probab.*, **5**, 100–116.

[Te72] Temkin, D. (1972). One-dimensional random walks in a two-component chain. *Soviet Math.Dokl.*, **13**, 1172–1176.

[Ze04] Zeitouni, O. (2004). Random walks in random environment. *Lecture Notes in Math.*, **1837**, 189–312. Springer, Berlin.

[Zi08] Zindy, O. (2008). Upper limits of Sinai's walk in random scenery. *Stochastic Process. Appl.*, **118**, 981–1003.