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Cutoff and cookies – interacting walks in random environment

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Contents

Introduction

This thesis consists of two parts which differ in the considered model. In the first part, we study *random walks in random environment* (RWRE) and in the second part *cookie branching random walks* (CBRW). The focus of the first part will be on the mixing properties of transient RWRE and we prove whether there is a cutoff or not. In the second part of this thesis we give an explicit criterion for the recurrence/transience of CBRW.

RWRE as well as CBRW are models of random motion in random media whose probabilistic investigation started in the seventies. The randomness of the media models irregularities (due to impurities, fluctuations etc.) of the environment in which the motion takes place. During the last forty years, there has been a lot of research in this area, however, the understanding of multidimensional random media is still not satisfactory.

In the first part of this thesis, we consider the (one-dimensional nearest neighbour) RWRE in discrete time on $\mathbb{Z}$. This model was first studied by Chernov in [Che62] and Temkin in [Tem72] as a toy model for the replication of DNA sequences. In contrast to a random walk in a deterministic environment, the transition probabilities at each position are not fixed but random themselves.

A RWRE has two parts of randomness: The first part is choosing an environment according to an environment distribution $\mathbf{P}$ and the second part is considering a Markov chain $(X_n)_{n\in\mathbb{N}_0}$ in this chosen environment. The measure averaging over both parts of randomness is called the annealed measure of the RWRE. Given one fixed environment the law of the Markov chain $(X_n)_{n\in\mathbb{N}_0}$ is called quenched law. Therefore, it is possible to get two types of statements. We call the first type – statements with respect to the annealed measure – annealed and such statements can be interpreted as statements for $(X_n)_{n\in\mathbb{N}_0}$ averaged over all possible environments. The second type of statement is called quenched and here properties of the RWRE are considered which hold for almost every environment. We want to emphasize that $(X_n)_{n\in\mathbb{N}_0}$ is a Markov chain only with respect to the quenched law but not with respect to the annealed measure. Thus, it is not surprising that sometimes one needs different techniques to prove quenched or annealed statements and we will see that one cannot always go from one to the other.

In this thesis, we assume the environment distribution $\mathbf{P}$ to be a product measure such that at every position x of $\mathbb{Z}$ a value ω_x between 0 and 1 is chosen independently of all the other positions according to the same distribution. Given one realization of this $(0,1)$-valued random variables, we can define a nearest neighbour random walk on $\mathbb{Z}$: Given that the random walk is at position x, it moves with probability ω_x to the right and with probability $(1-\omega_x)$ to the left.

In the following, we first give an overview on the historical development and introduce the quantities of the environment which determine the behaviour of the RWRE. For simplicity's sake, we usually omit the assumptions of the results for which we refer to Section 1.2 and the quoted papers.

The first seminal work on one-dimensional RWRE was published 1975 by Solomon (cf. [Sol75]). In this work, he gave a criterion for recurrence and transience respectively. In analogy to the nearest neighbour random walk in a deterministic environment, one might expect the critical parameter to depend on the mean drift $\mathbf{E}[\omega_0-(1-\omega_0)]$, which turns out not to be the case. Solomon showed that for $\mathbf{P}$-almost every environment a RWRE is recurrent if and only if $\mathbf{E}\ln\rho_0 = 0$, where $\rho_0 := \frac{1-\omega_0}{\omega_0}$ is the ratio of the probabilities to move to the left and to the right.

If the RWRE is transient, it is a natural question to ask at which speed the walk escapes to infinity. Also in [Sol75], Solomon showed that the asymptotic linear speed $\lim\limits_{n\to\infty}\frac{X_n}{n}$ can be zero as well as positive. A transient RWRE is called *ballistic* if it has a positive linear speed and *sub-ballistic* otherwise. For a RWRE which is transient to the right the critical parameter is $\mathbf{E}\rho_0$ and the RWRE is ballistic if and only if $\mathbf{E}\rho_0 < 1$.

Even a very simple environment distribution $\mathbf{P}$ which just chooses between two values α and $1-\alpha$ with probabilities

$$\mathbf{P}(\omega_x = \alpha) = p, \quad \text{and} \quad \mathbf{P}(\omega_x = 1-\alpha) = 1-p, \qquad 0 < p,\ \alpha < 1 \tag{1}$$

shows all these different regimes. In this example, we can easily compute the quantities $\mathbf{E}\ln\rho_0$ and $\mathbf{E}\rho_0$. For the phase diagram see Figure 1.

In the case of a recurrent RWRE, there is a remarkable slowdown. In 1982, Sinai proved (cf. [Sin82]) that with respect to the annealed measure a recurrent RWRE normalized by $(\ln n)^2$ converges in distribution. In this paper, Sinai introduced the potential associated with the environment (cf. Section 1.5) which is a very important tool for analysing one-dimensional RWRE and was often used afterwards.

Shortly after Solomon's paper, Kesten, Kozlov and Spitzer proved annealed limit laws (cf. [KKS75]) for the transient regime. They assumed that there exists a $\kappa > 0$ which solves the following equation

$$\mathbf{E}\rho_0^\kappa = 1. \tag{2}$$

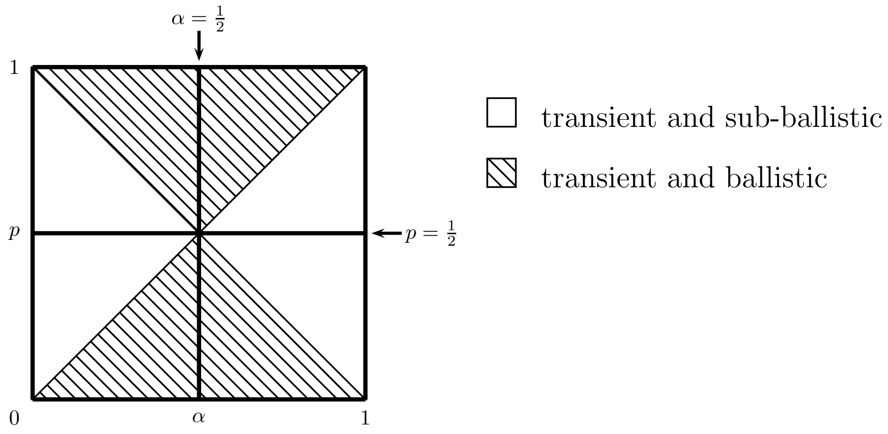

Figure 1: *Phase diagram for the model according to* (1). *We note that the RWRE is recurrent only for the cases $\alpha = \frac{1}{2}$ or $p = \frac{1}{2}$ and that the case $\alpha = \frac{1}{2}$ corresponds to the case of a simple random walk. Further, the RWRE is transient to the right for the top right and bottom left quarter and transient to the left for the top left and bottom right quarter.* [1]

In [KKS75], Kesten, Kozlov and Spitzer provided sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ for all $\kappa > 0$ such that $\frac{X_n - a_n}{b_n}$ converges in distribution and they identified the corresponding annealed limit distributions. Their analysis revealed that all environment distributions $\mathbf{P}$ which belong to the same κ show the same limiting behaviour. The only regime for which we have an annealed central limit theorem is the case $\kappa > 2$. Their proofs rely on an embedded branching process first introduced in [Koz73]. Further, we note that Enriquez, Sabot and Zindy in 2008 (cf. [ESZ09]) and together with Tournier in 2010 (cf. [ESTZ10a]) refined the annealed limit laws of [KKS75] for $\kappa < 2$.

The first quenched limit results for $(X_n)_{n\in\mathbb{N}_0}$ in the transient regime were obtained independently by Goldsheid in [Gol07] and Peterson in [Pet08]. They proved a quenched central limit theorem for $\kappa > 2$ but in contrast to the annealed result the centering depends on the environment. The first step to identify a quenched limit distribution for $\kappa < 1$ and $1 < \kappa < 2$ was made by Peterson and Zeitouni in [Pet09] and [PZ09], respectively. They proved that for almost every environment and any choice of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ there exist no non-trivial distributional limit of $\frac{X_n - a_n}{b_n}$ (only along subsequences). An important part of their proofs is a close analysis of the associated potential.

Very recently Peterson and Samorodnitsky in [PS10] and [PS11] identified a quenched limit in a weaker sense. They considered the quenched distribution of X_n (suitably centered and scaled) as a random probability measure and identified a distributional limit on the space of random probability measures on $\mathbb{R}$ which can be expressed using Poisson point processes. To prove this, they first showed a limit result for the quenched distribution of the (suitably centered and scaled) hitting times T_n

[1] This figure is adapted from Figure 6.1 in [Hug96].

of position n. Enriquez, Sabot, Tournier and Zindy in [ESTZ10b] and Dolgopyat and Goldsheid in [DG10] independently of each other and independently of Peterson and Samorodnitsky gave similar limit results for the quenched distribution of T_n.

In 2000, Comets, Gantert and Zeitouni (cf. [CGZ00]) proved quenched and annealed large deviation results for ergodic environments which, of course, include i.i.d. environments. Also by analysing the potential very carefully, Fribergh, Gantert and Popov (cf. [FGP10]) recently proved quenched and annealed moderate deviations for the transient sub-ballistic regime ($\kappa \leq 1$). From their results, one easily can derive that for $\kappa < 1$ at time n the distance to the origin is almost surely roughly of order n^{κ} .

In contrast to the multidimensional case, a transient nearest neighbour RWRE on $\mathbb{Z}$ has only one way to escape to infinity and has to hit all positions on its way. Further, we note that T_n can be decomposed into a sum of n independent (with respect to the quenched law) increments $T_n = \sum_{i=1}^{n}(T_i - T_{i-1})$ and therefore is a lot easier to analyse. Very often one first proves a statement for the sequence of hitting times $(T_n)_{n\in\mathbb{N}}$ and then transfers it into a statement for the walk.

In this thesis, we investigate the regime in which the RWRE is transient to the right. As mentioned above, we also gain a good understanding (of the behaviour we are later interested in) while first analysing the asymptotics of $(T_n)_{n\in\mathbb{N}}$. The constant κ defined by (2) plays an important role and we note that the annealed expectation of $(T_n)^s$ exists only for $s < \kappa$.

We first investigate the quenched expectation and the quenched variance of T_n as random variables with respect to the law of the environment for the cases in which the analogous annealed quantities are infinite. The first main result is Theorem 1.4.5, where we show that for $\kappa < 1$ the quenched expectation is roughly of order $n^{\frac{1}{\kappa}}$ and for $\kappa < 2$ we prove that the quenched variance is roughly of order $n^{\frac{2}{\kappa}}$. A good understanding of the fluctuations of the potential will be the key tool for these proofs. Afterwards, we use this result to analyse the mixing behaviour of transient RWRE on $\{0, ..., n\}$. To avoid problems with the periodicity of the model, we consider the lazy RWRE. The transition probabilities of the lazy RWRE are the same as for the RWRE, but in every step we first in addition toss a coin to decide if we move or if we stay at the current position.

From the convergence theorem for aperiodic and irreducible Markov chains we know that the quenched law of the RWRE on $\{0, ..., n\}$ converges to its associated stationary distribution in the long run. The mixing properties of a Markov chain give information about how long we have to wait until the law of the Markov chain is very close to the stationary distribution. The behaviour we are interested in is the *cutoff phenomenon* (cf. Definition 1.3.3). If a sequence of Markov chains exhibits a cutoff, asymptotically the transition to stationarity is very sharp in the following sense: For a long time, the law of the Markov chain does not approach its stationary distribution at all and then very rapidly – around the so called *cutoff time* – the distance to the stationary distribution drops to zero. Therefore, the existence of a cutoff shows us for example how long we should run a simulation to end with a Markov chain whose law is approximately the associated stationary distribution. On the one hand, if we stop before the cutoff time, we end with a Markov chain whose distribution is still close to the starting distribution. On the other hand, it is useless to run a simulation a lot longer than the cutoff time. For a historical development and an overview of existing results see Section 3.1.

In Theorem 1.4.1 and Theorem 1.4.2, we show that there is a phase transition for the (quenched) cutoff behaviour of lazy transient RWRE. For $\kappa < 1$, we show that the transition to stationarity is not sharp enough as required for a cutoff and for $\kappa > 1$ we prove that the lazy RWRE exhibits a cutoff. The following observation will be crucial for the proofs. With the help of the potential, we divide the environment into different blocks and we will identify "deep blocks" in which the RWRE spends most of its time before it hits position n. We remark that, in contrast to many other results, our assumptions do not exclude all lattice distributions. For example, the non-lattice assumption excludes all environment distributions of the form (1) which are mostly covered by our results. Further, we determine the order of the time around which the transition to stationarity takes place (cf. Theorem 1.4.3).

The second part of this thesis is based on the paper "Cookie branching random walks" by the author et al. (cf. [BKKMP11]). Our model is related to the *excited random walk* model introduced by Benjamini and Wilson in 2003 (cf. [BW03]). This model has attracted a lot of attention during the last years and informally can be described in the following way. The behaviour of the random walker depends on whether he finds himself in an already visited site, or not. This model is also frequently called *cookie random walk* (CRW). The idea of this interpretation is that initially all sites contain one cookie, and when the site is visited, the cookie from the site is eaten and this changes the behaviour of the random walker in that step.

The paper [BW03] introduced the CRW on $\mathbb{Z}^d$. When the random walker visits a vertex of $\mathbb{Z}^d$ for the first time he eats the cookie and this gives him an arbitrarily small (but fixed) drift in one direction. On subsequent visits to that vertex the walker chooses one of the neighbours uniformly at random. In [BW03], they proved that the CRW on $\mathbb{Z}^d$ is transient if and only if $d \geq 2$ and therefore they showed that the small drift on the first visit is already enough to turn the simple random walk on $\mathbb{Z}^2$ from a recurrent into a transient walk. Further, they showed that there is a positive linear speed for dimensions $d \geq 4$ which was later extended to dimensions $d = 2$ and $d = 3$ (cf. [MPRV12] or [Kozm03], [Kozm05]). In [Zer05], Zerner introduced an extension of this model, called *multi-excited random walk*. In this model, there can be several cookies at every vertex. As long as there are cookies at a vertex, the random walker eats one cookie on every visit which changes his behaviour in that step. If there are initially two cookies at every vertex of $\mathbb{Z}$, Zerner showed that both recurrence and transience are possible depending on the strength of the drift the random walker gets by eating a cookie.

After 2003 the CRW has been studied for different initial cookie distributions in many subsequent papers. For the one-dimensional case, where, as usual, more complete results are available, we refer to [BS08], [KZ08] and for the multi-dimensional case and trees respectively, compare [BS09], [BR07], [HH10], [MPRV12].

A further ingredient for the CBRW model are *branching random walks* (BRW). A BRW on a state space $\mathcal{X}$ consists of two parts of randomness: an offspring distribution μ and a transition mechanism P of a random walk on $\mathcal{X}$. The evolution of the BRW can be described in the following way. At time 0, there is one particle at some position of $\mathcal{X}$. This particle now first produces offspring according to μ and then dies. Afterwards, the newly created particles perform independently of each other one step of the random walk described by P. In the following steps this procedure is repeated independently of each other by all alive particles. We call a BRW transient if almost surely only finitely many particles visit the starting point and recurrent otherwise. For a general overview on BRW we refer to [Shi11].

In the second part of this thesis we study a BRW on $\mathbb{Z}$ with initially one cookie at every integer. In contrast to a CRW, here the transition and branching parameters depend on whether the particle finds a cookie or not. The cookie of a vertex is eaten when the vertex is visited by at least one particle and the behaviour of all particles which visit this vertex at that time is changed in the next step. We thus call our model *cookie branching random walk* (CBRW). We note that a CBRW can also be interpreted as a random walk in a random environment but here the walk is interacting with the environment.

During the last years, there also has been research on BRW in random environment, where the environment is modeled as in the first part of this work (cf. [CP07a], [CP07b], [Mue08], [GMPV10]). To our knowledge, the situation when the behaviour of the BRW is changed in the already visited sited was previously not considered.

However, it is interesting to note that there is a model that lies in some sense in between the excited random walk and the CBRW. It is usually called *frog model* (cf. [AMPR01]), and can be described in the following way: the particles do not branch in already visited sites, and when one or several particles visit a new site, *exactly one of them* is allowed to branch. Another interpretation is that initially every site contains a number of sleeping particles and an active particle is placed somewhere; when an active particle enters a site which contains sleeping particles, those are activated too.

The main results of the second part are Theorems 5.3.1 - 5.3.3, where we give an explicit and complete characterisation for the recurrence/transience of CBRW. It turns out that the critical parameter is the mean number of offspring of an embedded branching process. The most interesting case is the case in which the branching random walk without cookies is transient. Here we show that adding a cookie can change the behaviour of the walk.

The work is structured as follows: The first part (Chapters 1 - 4) treats transient RWRE. In the first chapter we formally introduce the model. Then we give some of the known results and preliminary considerations. In Section 1.4, we present our results. In the last section, we construct a set of "typical" environments on which calculations are simplified and which will be very helpful for the following chapters.

In the second chapter, we prove our first main result (Theorem 1.4.5) on the almost sure behaviour of the quenched expectation and quenched variance of T_n. Further, we prove two lemmata for the quenched variance of the crossing time of "deep blocks".

In the third chapter, we first give an introduction to the cutoff phenomenon in general. Then we prove that a transient lazy RWRE exhibits a cutoff for $\kappa > 1$ and that there is no cutoff for $\kappa < 1$ (Theorem 1.4.1 and Theorem 1.4.2). Further, we compare our results with the case of a deterministic environment. In the last section of this chapter, we analyse the mixing time of the lazy RWRE.

In the forth chapter, we derive statements about the asymptotic behaviour of the spectral gap using our results from Chapter 3.

In the second part (Chapter 5), we analyse the CBRW. We first introduce the model and give recurrence/transience criteria for BRW and CRW. Afterwards, in Section 5.3, we state our criterion for the recurrence/transience of CBRW. In the last section, we prove our main results Theorems 5.3.1 - 5.3.3.

Chapter 1

Transient RWRE

1.1 Model

Let $\omega := (\omega_k)_{k\in\mathbb{Z}}$ be a family of i.i.d. random variables taking values in $(0,1)$. We denote the distribution of ω by $\mathbf{P}$ and the corresponding expectation by $\mathbf{E}$. Further, we define the sequence $\widetilde{\omega} := (\widetilde{\omega}_k)_{k\in\mathbb{N}}$ by

$$\widetilde{\omega}_k := \begin{cases} 1 & \text{for } k = 0, \\ \omega_k & \text{for } k > 0 \end{cases} \tag{1.1.1}$$

and for $n \in \mathbb{N}$ the sequence $\omega^n := (\omega_k^n)_{k\in\{0,..,n\}}$ by

$$\omega_k^n := \begin{cases} 1 & \text{for } k = 0, \\ \omega_k & \text{for } k = 1, ..., n-1, \\ 0 & \text{for } k = n. \end{cases}$$

After choosing an environment ω at random according to the law $\mathbf{P}$, we define the random walk in random environment (RWRE) as the nearest neighbour random walk $(X_k)_{k\in\mathbb{N}_0}$ on $\mathbb{Z}$ with transition probabilities given by ω: With respect to P_ω^z $(z \in \mathbb{Z})$, $(X_k)_{k\in\mathbb{N}_0}$ is the (time homogeneous) Markov chain on $\mathbb{Z}$ with $P_\omega^z(X_0 = z) = 1$ and

$$\begin{aligned} P_\omega^z\left[X_{k+1} = i+1 \mid X_k = i\right] &= \omega_i, \\ P_\omega^z\left[X_{k+1} = i-1 \mid X_k = i\right] &= 1 - \omega_i. \end{aligned} \tag{1.1.2}$$

for $k \in \mathbb{N}_0$, $i \in \mathbb{Z}$. Analogously, we define $P_{\widetilde{\omega}}^z$ as the distribution of a RWRE on $\mathbb{N}_0$ with reflection in 0 and $P_{\omega^n}^z$ as the distribution of a RWRE on $\{0, ..., n\}$ with reflection in 0 and n.

As usual, P_ω^z is called the quenched law of $(X_k)_{k\in\mathbb{N}_0}$ starting from $X_0 = z$ and we denote by E_ω^z the corresponding quenched expectation. Let $\mathbb{Z}^{\mathbb{N}_0}$ be the space of the paths of the RWRE and let $\mathcal{F}$ be the associated σ-algebra generated by all cylinder sets. By $\mathbb{P}^z := \mathbf{P} \times P_\omega^z$ we denote the measure on $\left((0,1)^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{N}}, \left(\mathcal{B}_{(0,1)}\right)^{\mathbb{Z}} \otimes \mathcal{F}\right)$ defined by the relation

$$\mathbb{P}^z(B \times F) \;=\; \int_B P_\omega^z(F)\mathbf{P}(d\omega), \quad B \in \left(\mathcal{B}_{(0,1)}\right)^{\mathbb{Z}}, F \in \mathcal{F},$$

where $\mathcal{B}_{(0,1)}$ is the Borel-σ-algebra on $(0,1)$. The expectation under $\mathbb{P}^z$ is denoted by $\mathbb{E}^z$. We will refer to $\mathbb{P}^z$ and $\mathbb{E}^z$ as the annealed law and the annealed expectation respectively. If $z = 0$, we simply write P_ω, E_ω, $\mathbb{P}$ and $\mathbb{E}$.

1.2 Classical results and basic notation

In this section, we state classical results in particular about the limit behaviour of transient RWRE. Further, we introduce some basic notation.

First, for $i \in \mathbb{Z}$ we define

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \quad W_i := \sum_{j=-\infty}^{i} \prod_{k=j}^{i} \rho_k \tag{1.2.1}$$

and for $i \in \mathbb{N}$

$$W_i^0 := \sum_{j=1}^{i} \prod_{k=j}^{i} \rho_k. \tag{1.2.2}$$

In the following, we assume that the expectations $\mathbf{E} \ln \rho_0 < 0$, $\mathbf{E}\rho_0$ and $\mathbf{E}\left(\rho_0\right)^{-1}$ exist whenever they appear.

1.2.1 Recurrence/transience and linear speed

In the first seminal work on RWRE, Solomon proved (cf. [Sol75]) a criterion for recurrence/transience:

Theorem 1.2.1 (cf. Theorem 1.7 in [Sol75])**.** *Let $\boldsymbol{P}$ be a product measure such that $(\omega_x)_{x\in\mathbb{Z}}$ is i.i.d. with $0 \le \omega_0 < 1$ or $0 < \omega_0 \le 1$ $\boldsymbol{P}$-a.s.*

(a) If $\boldsymbol{E} \ln \rho_0 < 0$, then $\lim_{n\to\infty} X_n = \infty \quad \mathbb{P}$-a.s.

(b) If $\boldsymbol{E} \ln \rho_0 > 0$, then $\lim_{n\to\infty} X_n = -\infty \quad \mathbb{P}$-a.s.

(c) If $\boldsymbol{E} \ln \rho_0 = 0$, then $(X_n)_{n\in\mathbb{N}_0}$ is P_ω-a.s. recurrent for $\boldsymbol{P}$-almost every environment ω; in fact $\liminf_{n\to\infty} X_n = -\infty$ and $\limsup_{n\to\infty} X_n = \infty \quad \mathbb{P}$-a.s.

Further, let

$$T_n := \min\{l \in \mathbb{N}_0 \;:\; X_l = n\}$$

be the first hitting time of position n. One can recursively compute an explicit formula for the quenched expectation of T_n as a function of the environment (cf. (2.1.14) in [Zei04]), and if we assume $\mathbf{E}\ln\rho_0 < 0$, we have (cf. [Sol75])

$$E^i_\omega T_{i+1} \;= 1 + 2W_i \;<\; \infty \quad \mathbf{P}\text{-}a.s. \tag{1.2.3}$$

and therefore $E^i_{\tilde\omega} T_{i+1} = 1 + 2W^0_i$ for $i \in \mathbb{N}$. Note that the fact that ρ_k is an i.i.d. sequence yields

$$\mathbb{E}T_1 \;= 1 + 2\mathbf{E}W_0 = 1 + \sum_{k=1}^{\infty} (\mathbf{E}\rho_0)^k$$

and consequently

$$\mathbb{E}T_1 \;< \infty \text{ iff } \mathbf{E}\rho_0 < 1. \tag{1.2.4}$$

In [Sol75], Solomon analysed the limiting behaviour of the hitting times and determined the speed of the RWRE:

Theorem 1.2.2 (cf. Theorem 1.16 in [Sol75])**.** *Let $\boldsymbol{P}$ be a product measure such that $(\omega_x)_{x\in\mathbb{Z}}$ is i.i.d. with $0 \le \omega_0 < 1$ or $0 < \omega_0 \le 1$ $\boldsymbol{P}$-a.s and let $\boldsymbol{E}\ln\rho_0 \neq 0$.*

(a) If $\boldsymbol{E}\rho_0 < 1$, then $\lim_{n\to\infty} \frac{T_n}{n} = \frac{1+\boldsymbol{E}\rho_0}{1-\boldsymbol{E}\rho_0}$ $\mathbb{P}$-a.s.

(b) If $\boldsymbol{E}(\rho_0)^{-1} < 1$, then $\lim_{n\to\infty} \frac{T_{-n}}{n} = \frac{1+\boldsymbol{E}(\rho_0)^{-1}}{1-\boldsymbol{E}(\rho_0)^{-1}}$ $\mathbb{P}$-a.s.

(c) If $(\boldsymbol{E}\rho_0)^{-1} \le 1 \le \boldsymbol{E}(\rho_0)^{-1}$, then $\lim_{n\to\infty} \frac{T_n}{n} = \infty = \lim_{n\to\infty} \frac{T_{-n}}{n}$ $\mathbb{P}$-a.s.

If we assume that

$$\lim_{n\to\infty} \frac{T_n}{n} = \alpha \in (0,\infty] \quad \mathbb{P}\text{-a.s.},$$

we can easily determine the limit of $(X_n)_{n\in\mathbb{N}_0}$ itself (cf. Lemma 2.1.17 in [Zei04]). Let k_n be the unique random integer such that

$$T_{k_n} \;\le\; n \;<\; T_{k_n+1}.$$

We note that we therefore have

$$X_n \;<\; k_n + 1 \;\text{ and }\; X_n \;\ge\; k_n - (n - T_{k_n})$$

and hence
$$\frac{k_n}{n} - \left(1 - \frac{T_{k_n}}{n}\right) \leq \frac{X_n}{n} \leq \frac{k_n + 1}{n}.$$
Further, the definition of k_n yields
$$\frac{k_n}{k_n+1}\frac{k_n+1}{T_{k_n+1}} < \frac{k_n}{n} \leq \frac{k_n}{T_{k_n}},$$
and, using that $\lim\limits_{n\to\infty} \frac{T_n}{n} = \alpha$ $\mathbb{P}$-a.s., we get
$$\frac{1}{\alpha} \leq \liminf_{n\to\infty} \frac{X_n}{n} \leq \limsup_{n\to\infty} \frac{X_n}{n} \leq \frac{1}{\alpha} \quad \mathbb{P}\text{-a.s.}$$

Therefore, we can conclude:

Theorem 1.2.3 (cf. Theorem 1.16 in [Sol75]). *Let $\boldsymbol{P}$ be a product measure such that $(\omega_x)_{x\in\mathbb{Z}}$ is i.i.d. with $0 \leq \omega_0 < 1$ or $0 < \omega_0 \leq 1$ $\boldsymbol{P}$-a.s and let $\boldsymbol{E}\ln\rho_0 \neq 0$.*

(a) If $\boldsymbol{E}\rho_0 < 1$, then $\lim\limits_{n\to\infty} \frac{X_n}{n} = \frac{1-\boldsymbol{E}\rho_0}{1+\boldsymbol{E}\rho_0}$ $\mathbb{P}$*-a.s.*

(b) If $\boldsymbol{E}(\rho_0)^{-1} < 1$, then $\lim\limits_{n\to\infty} \frac{X_n}{n} = -\frac{1-\boldsymbol{E}(\rho_0)^{-1}}{1+\boldsymbol{E}(\rho_0)^{-1}}$ $\mathbb{P}$*-a.s.*

(c) If $(\boldsymbol{E}\rho_0)^{-1} \leq 1 \leq \boldsymbol{E}(\rho_0)^{-1}$, then $\lim\limits_{n\to\infty} \frac{X_n}{n} = 0$ $\mathbb{P}$*-a.s.*

1.2.2 Annealed and quenched limit laws

In the following, we will compare the annealed and quenched limit laws of RWRE which are transient to the right ($\mathbf{E}\ln\rho_0 < 0$). Most of the results in this section require the following assumptions:

(a) Let $\mathbf{P}$ be an i.i.d. product measure such that
$$-\infty \leq \mathbf{E}\ln\rho_0 < 0.$$

(b) There exists $0 < \kappa < \infty$ for which we have
$$\mathbf{E}\rho_0^\kappa = 1, \quad \mathbf{E}\rho_0^\kappa \log^+ \rho_0 < \infty.$$

(c) The distribution of $\ln\rho_0$ is non-lattice.

In [KKS75], Kesten, Kozlov and Spitzer provided annealed limit laws:

Theorem 1.2.4 (cf. Theorem 1 in [KKS75]). *Let assumptions (a)-(c) hold and further let $v_P := \lim\limits_{n\to\infty} \frac{X_n}{n}$. Then the following limit laws hold for $(X_n)_{n\in\mathbb{N}_0}$ with respect to the annealed law:*

(a) *If* $\kappa < 1$*, then* $\dfrac{X_n}{n^\kappa}$ *converges to a stable law with index* κ*.*

(b) *If* $\kappa = 1$*, then there exists a sequence* $\delta(n) \sim c(\ln n)^{-1} n$ *such that* $\dfrac{X_n - \delta(n)}{n(\ln n)^{-2}}$ *converges to a stable law with index* 1*.*

(c) *If* $1 < \kappa < 2$*, then* $\dfrac{X_n - nv_P}{n^{\frac{1}{\kappa}}}$ *converges to a stable law with index* κ*.*

(d) *If* $\kappa = 2$*, then* $\dfrac{X_n - nv_P}{\sqrt{n \ln n}}$ *converges to a normal distribution.*

(e) *If* $\kappa > 2$*, then* $\dfrac{X_n - nv_P}{\sqrt{n}}$ *converges to a normal distribution.*

The approach Kesten, Kozlov and Spitzer used in [KKS75] was first to prove analogous statements for the hitting times $(T_n)_{n\in\mathbb{N}}$ using an associated branching process (with immigration) and then to transfer these results to $(X_n)_{n\in\mathbb{N}_0}$. We further notice that the boundary cases $\kappa = 1$ and $\kappa = 2$ are the most difficult cases. Enriquez, Sabot and Zindy in [ESZ09] and together with Tournier in [ESTZ10a] refined the results for the non-Gaussian regime ($\kappa < 2$) and they gave an explicit probabilistic representation for the appearing stable laws.

The question about a quenched analogue of this result was recently considered in several papers and we will see that for $\kappa < 2$ the annealed stable behaviour comes from the fluctuations in the environment and not from the walk itself.

Theorem 1.2.5 (cf. Theorem 1.2, 1.3 in [PZ09])**.** *Let assumptions (a)-(c) hold and let* $\kappa < 1$*.*

(a) *For* $\boldsymbol{P}$*-almost every environment* ω *there exist random subsequences* $t_m = t_m(\omega)$ *and* $u_m = u_m(\omega)$ *such that for any* $\delta > 0$*,*

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m} - u_m}{(\ln t_m)^2} \;\in\; [-\delta, \delta]\right) \;=\; 1.$$

(b) *For* $\boldsymbol{P}$*-almost every environment* ω *there exist a random subsequence* $n_{k_m} = n_{k_m}(\omega)$ *of* $n_k = 2^{2^k}$ *and a random subsequence* $t_m = t_m(\omega)$ *such that*

$$\lim_{m\to\infty} \frac{\ln t_m}{\ln n_{k_m}} \;=\; \frac{1}{\kappa},$$

and

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m}}{n_{k_m}} \;\leq\; x\right) \;=\; \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{1}{2}, & \text{if } 0 < x < \infty. \end{cases}$$

Since we will later use the same property of the environment which causes the strong localisation behaviour in (a), we will shortly present the strategy of the proof. For the case $\kappa < 1$, Peterson and Zeitouni identified a sequence of small blocks of the environment for which the expected crossing time is very large compared to the expected time to
reach these blocks. Further, all of these blocks are "typically" smaller than $(\ln n)^2$ and thus, for a suitable choice of time points, the RWRE is (with high probability) located within these blocks. We remark that we will refine the arguments in such a way that we are able to prove a version of Theorem 1.2.5 (a) under weaker assumptions (cf. Theorem 1.4.4).

Theorem 1.2.6 (cf. Theorem 1.1, Theorem 1.2 in [Pet09])**.** *Let* $1 < \kappa < 2$ *and* $v_P := \lim_{n\to\infty} \frac{X_n}{n}$. *Then* $\boldsymbol{P}$*-a.s. there exist random subsequences* $n_{k_m} = n_{k_m}(\omega)$ *and* $n_{l_m} = n_{l_m}(\omega)$ *of* $n_k = 2^{2^k}$ *and non-deterministic random variables* $\nu_{k_m,\omega}$ *and* $\nu_{l_m,\omega}$ *such that*

(a)

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_{k_m}} - n_{k_m}}{v_P\sqrt{\nu_{k_m,\omega}}}\right) = \Phi(x) \qquad \forall\, x \in \mathbb{R},$$

(b)

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_{l_m}} - n_{l_m}}{v_P\sqrt{\nu_{l_m,\omega}}}\right) = \Psi(x) \qquad \forall\, x \in \mathbb{R},$$

where Φ *and* Ψ *denote the distribution function of a standard normal distribution and an exponential distribution with parameter* 1 *respectively and* $t_k := t_k(\omega) = \lfloor E_\omega T_{n_k} \rfloor$.

We notice that this theorem precludes the existence of a quenched analogue of Theorem 1.2.4 for $1 < \kappa < 2$. The key to find these random subsequences is again to look at small blocks of the environment whose expected crossing times are large. Since these blocks are far apart and (with high probability) a transient RWRE does not backtrack to far, these blocks are "almost" independent. To obtain a Gaussian limit, Peterson identified a subsequence $(n_{k_m})_{m\in\mathbb{N}}$ for which none of these blocks in the interval $[0, n_{k_m}]$ dominates and then proved the Lindeberg-Feller condition for triangular arrays. For part (b) of the theorem, Peterson showed that it is also possible to find a subsequence $(n_{l_m})_{m\in\mathbb{N}}$ such that there is one single block in the interval $[0, n_{l_m}]$ for which the variance of the crossing time of this block is approximately of the same size as the variance of $T_{n_{l_m}}$. Further, he showed that the crossing time of these dominating blocks is approximately exponentially distributed.

Recently, Peterson and Samorodnitsky in [PS10], Enriquez, Sabot, Tournier and Zindy in [ESTZ10b] and Dolgopyat and Goldsheid in [DG10] independently of each other identified a limit for the quenched distribution of suitably centered and rescaled hitting times in a weaker sense. They considered

$$\mu_{n,\omega} := P_\omega\left(\frac{T_n - E_\omega T_n}{n^{\frac{1}{\kappa}}} \in \cdot\right)$$

as a random variable on the space of probability measures on $\mathbb{R}$ equipped with the topology of convergence in distribution ([PS10],[DG10]) and the topology of convergence in Wasserstein distance ([ESTZ10b]), respectively. Recall that the Wasserstein distance $W^1(\mu,\nu)$ between two probability measures μ and ν on $\mathbb{R}$ is given by

$$W^1(\mu,\nu) := \inf_{\substack{(X,Y):\\ X\sim\mu, Y\sim\nu}} E\,|X-Y|\,.$$

In [ESTZ10b], Enriquez, Sabot, Tournier and Zindy showed that for large n the quenched law of $\mu_{n,\omega}$ is approximately a law of a weighted sum of centered exponential random variables. For a random variable X let $\mathcal{L}(X)$ denote the law of X.

Theorem 1.2.7 (cf. Theorem 1 in [ESTZ10b]). *Let assumptions (a)-(c) hold and let $\kappa < 2$. Then we have*

$$W^1\left(\mu_{n,\omega},\ \mathcal{L}\left(\frac{1}{n^{\frac{1}{\kappa}}}\sum_{k=0}^{n_0-1} E_\omega\left(T_{\nu_{k+1}} - T_{\nu_k}\right)\overline{\boldsymbol{e}}_k\right)\right) \stackrel{n\to\infty}{\longrightarrow} 0 \quad \textit{in } \boldsymbol{P}\textit{-probability}$$

with $\overline{\boldsymbol{e}}_k := \boldsymbol{e}_k - 1$, where $(\boldsymbol{e}_k)_{k\in\mathbb{N}}$ are i.i.d. exponential random variables of parameter 1 and independent of ω. For the definition of $(\nu_k)_{k\in\mathbb{N}_0}$ and n_0 see (1.5.1) *and* (1.5.4)*, respectively.*

In [PS10], [ESTZ10a] and [DG10], the authors identified a distributional limit of $\mu_{n,\omega}$ in terms of a Poisson point process. Since we would have to introduce further notation to state the precise results, we refer to the papers for more details. We note that these results cover the non-Gaussian regime ($0 < \kappa < 2$) and also show that no quenched analogue of Theorem 1.2.4 is possible. Further, Peterson and Samorodnitsky also identified a distributional limit of the quenched distributions of X_n (cf. [PS10] and [PS11]).

In contrast to the non-Gaussian regime, there is a quenched analogue of Theorem 1.2.4 for $\kappa > 2$:

Theorem 1.2.8 (cf. Theorem 5 in [Gol07] and Theorem 3.4.2 in [Pet08]). *Let assumptions (a) and (b) hold and let the environment be uniformly elliptic (i.e. $\exists\ \delta > 0$ s.t. $\omega_0 \in [\delta, 1-\delta]$ $\boldsymbol{P}$-a.s.). Further, let $\boldsymbol{P}$ be α-mixing with $\alpha(n) = \exp(-n\ln n)^{1+\eta}$ for some $\eta > 0$. Then we have*

$$\lim_{n\to\infty} P_\omega\left(\frac{X_n - b_n(\omega)}{\sigma\sqrt{n}}\right) = \Phi(x),$$

for a suitable centering $b_n(\omega)$ depending on the environment and a suitable constant $\sigma > 0$.

We note that the results in [Gol07] and [Pet08] also hold for certain other ergodic environments and that in [Pet08] a quenched functional CLT is shown. The approach they use is again to show a quenched CLT for the hitting times and then to transfer the result to $(X_n)_{n\in\mathbb{N}_0}$. A first quenched CLT for the hitting times (but not for the walk) was obtained by Alili (cf. Theorem 5.1 in [Ali99]).

1.3 Assumptions and preliminary considerations

Throughout this work, we make the following assumptions on the environment distribution $\mathbf{P}$:

Assumption 1. $\mathbf{E}\ln\rho_0 < 0$.
Assumption 2. There exists a unique $\kappa > 0$ such that

$$\mathbf{E}[\rho_0^\kappa] = 1 \text{ and } \mathbf{E}[\rho_0^\kappa \ln^+ \rho_0] < \infty.$$

From now on, we always assume Assumptions 1 and 2. We sometimes need a further technical assumption which we mention if it is needed:

Assumption 3. Let D be the support of the distribution of $\ln\rho_0$ with respect to $\mathbf{P}$. Then $D \cup \{0\}$ is non-lattice.

Remark 1.3.1.
1. Assumption 1 implies transience to the right (cf. Theorem 1.2.1).
2. The constant in Assumption 2 has a significant influence on the behaviour of the RWRE. If it exists, its value separates the ballistic ($\kappa > 1$) from the sub-ballistic ($\kappa \leq 1$) regime. By the law of large numbers (cf. Theorem 1.2.2 and Theorem 1.2.3), we have

$$\lim_{n\to\infty}\frac{X_n}{n} \;=\; \lim_{n\to\infty}\frac{n}{T_n} \;=\; \frac{1}{\mathbb{E}T_1} \;=\; v_{\mathrm{P}} \quad \mathbb{P}\text{-}a.s.$$

and $v_{\mathrm{P}} > 0$ if and only if $\kappa > 1$ (cf. (1.2.4)). We will also refer to the case $v_{\mathrm{P}} > 0$ as the case with positive linear speed. Further, we notice that due to Lemma 2.4 in [DPZ96] we have for all $0 < s < \kappa$ and $n \in \mathbb{N}$

$$\mathbb{E}\,(T_n)^s \;\leq\; c_s n^s \;<\; \infty$$

for a suitable constant $c_s > 0$ but

$$\mathbb{E}\,(T_1)^\kappa \;=\; \infty.$$

3. We note that Assumptions 1 and 2 exclude all deterministic environments.
4. Notice that Assumption 3 is weaker than the condition that the distribution of $\ln\rho_0$ is non-lattice, which is used for example in [KKS75], [Gol07], [PZ09], and [Pet09] to

show annealed and quenched limit theorems. In particular, our setting includes many environment distributions which consist of just two possible choices for the transition probabilities. These distributions are all excluded in the above papers because for such choices of the transition probabilities the distribution of $\ln \rho_0$ is always lattice.

We proceed to the definitions of the mixing time and the cutoff:
For each $n \in \mathbb{N}$ let $(U_k^n)_{k\in\mathbb{N}_0}$ be an aperiodic and irreducible Markov chain on a finite state space Ω_n and let $(\pi_n)_{n\in\mathbb{N}}$ denote the sequence of associated stationary distributions. Further, we assume

$$|\Omega_n| \overset{n\to\infty}{\longrightarrow} \infty.$$

Definition 1.3.2. For the sequence $(U_k^n)_{k\in\mathbb{N}_0}$ the mixing time $t_{\text{mix}}(n)$ is defined by

$$t_{\text{mix}}(n) := \min\left\{ l \in \mathbb{N} : d_n(l) \leq \frac{1}{4} \right\},$$

where

$$d_n(l) := \max_{x\in\Omega_n} \left\| \mathbb{P}^x(U_l^n \in \cdot) - \pi_n(\cdot) \right\|_{TV} \tag{1.3.1}$$

and $|| \cdot ||_{TV}$ denotes distance in total variation.

We note that due to the convergence theorem for a sequence of aperiodic and irreducible Markov chains we have that $t_{\text{mix}}(n)$ is finite for every fixed n because

$$d_n(l) \overset{l\to\infty}{\longrightarrow} 0.$$

Nevertheless, in most cases $t_{\text{mix}}(n)$ tends to infinity with growing state space. In this work, we are interested in the growths rate of $t_{\text{mix}}(n)$ as a function of n. Further, we notice that $d_n(k)$ can be interpreted as the worst case distance to stationarity after k steps.

Next, we define the cutoff phenomenon for a sequence of aperiodic and irreducible Markov chains. This effect describes a sharp transition of the total variation distance of the distribution of the Markov chain and its stationary distribution from 1 to 0 in a small window around the mixing time.

Definition 1.3.3. The sequence $(U^n)_{n\in\mathbb{N}}$ exhibits a *cutoff* with cutoff times $(t_n)_{n\in\mathbb{N}}$ and window size $(f_n)_{n\in\mathbb{N}}$ if

(1) $f_n = o(t_n)$,

(2) $\lim_{c\to\infty} \liminf_{n\to\infty} d_n(t_n - cf_n) \;=\; 1$ and

(3) $\lim_{c\to\infty} \limsup_{n\to\infty} d_n(t_n + cf_n) \;=\; 0.$

We refer to Chapter 3 for a more detailed introduction to the cutoff phenomenon and an overview on existing results.

To avoid problems with the periodicity of the model, we study the lazy RWRE $(Y_k)_{k\in\mathbb{N}_0}$, which is – as $(X_n)_{n\in\mathbb{N}_0}$ under P_{ω^n} – a nearest neighbour random walk on $\{0,...,n\}$ and moves according to the following rules: First, we toss a fair coin to decide if the random walk moves or stays at its position. If we decide to move, the random walk moves to the left and right with the same probabilities as the RWRE given by ω^n described in (1.1.2): With respect to $P^z_{\omega^n}$, $(Y_k)_{k\in\mathbb{N}_0}$ is the Markov chain on $\{0,...,n\}$ with $P^z_{\omega^n}(Y_0 = z) = 1$ and with the following transition probabilities: For $i \in \{0,...,n\}$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} P^z_{\omega^n}\left[Y_{k+1} = i+1 \mid Y_k = i\right] &= \frac{\omega^n_i}{2}, \\ P^z_{\omega^n}\left[Y_{k+1} = i \mid Y_k = i\right] &= \frac{1}{2}, \\ P^z_{\omega^n}\left[Y_{k+1} = i-1 \mid Y_k = i\right] &= \frac{1-\omega^n_i}{2}. \end{aligned} \tag{1.3.2}$$

Further, for $n \in \mathbb{N}$ let

$$(Z_k)_{k\in\mathbb{N}} = \left(\mathbb{1}_{\{Y_{k-1}\neq Y_k\}}\right)_{k\in\mathbb{N}} \tag{1.3.3}$$

be the sequence of random variables indicating if the random walk $(Y_k)_{k\in\mathbb{N}_0}$ moves or stays at its position. Therefore, $(Z_k)_{k\in\mathbb{N}}$ is under P^z_ω a sequence of i.i.d. Bernoulli random variables with success probability $\frac{1}{2}$. Note that the lazy RWRE is transient or has positive linear speed if and only if the underlying RWRE is transient or has positive linear speed. Furthermore, let

$$T^Y_n := \min\{k \in \mathbb{N} \;:\; Y_k = n\}$$

be the first time that the lazy RWRE hits position n.

The next Lemma relates the quenched expectation and quenched variance of T^Y_n with the corresponding quantities of T_n.

Lemma 1.3.4. *We have*

(a) $E_\omega T^Y_n = 2E_\omega T_n$,

(b) $\mathrm{Var}_\omega T^Y_n = 4\mathrm{Var}_\omega T_n + 2E_\omega T_n$.

Proof. For $k \in \mathbb{N}$ we define

$$\tau_k := \inf\left\{ l \in \mathbb{N} \;:\; \sum_{k=1}^{l} Z_k = k \right\}$$

and note that τ_k is negative binomially distributed with parameters k and $\frac{1}{2}$ (waiting for the kth success). Thus, we have

$$E_\omega\left[T_n^Y\middle|T_n = k\right] \;=\; E_\omega \tau_k \;=\; 2k$$

and we get

$$E_\omega T_n^Y \;=\; E_\omega\left[E_\omega\left[T_n^Y\middle|T_n\right]\right] \;=\; 2E_\omega T_n.$$

To obtain *(b)*, we first consider

$$E_\omega\left[\left(T_n^Y\right)^2\middle|T_n = k\right] \;=\; E_\omega\left[\,(\tau_k)^2\,\right] \;=\; 2k + 4k^2$$

and we therefore get together with *(a)*

$$\begin{aligned}\mathrm{Var}_\omega T_n^Y \;&=\; E_\omega\left[E_\omega\left[\left(T_n^Y\right)^2\middle|T_n\right]\right] - \left(E_\omega T_n^Y\right)^2\\ &=\; 4E_\omega(T_n)^2 + 2E_\omega T_n - 4\left(E_\omega T_n\right)^2\\ &=\; 4\mathrm{Var}_\omega T_n + 2E_\omega T_n.\end{aligned}$$

■

Further, we note that analogously we can show that

$$E_{\omega^n} T_n^Y \;=\; 2E_{\omega^n} T_n \quad\text{and}\quad \mathrm{Var}_{\omega^n} T_n^Y \;=\; 4\mathrm{Var}_{\omega^n} T_n \;+ 2E_{\omega^n} T_n. \tag{1.3.4}$$

1.4 Results

In the following, we investigate for which $\kappa > 0$ a sequence of lazy RWRE on $(\{0, ..., n\})_{n\in\mathbb{N}}$ exhibits a cutoff. We show that although the lazy RWRE is transient to the right for all $\kappa > 0$, we only observe a sharp transition of the distance in total variation to its stationary distribution in the case of positive linear speed ($\kappa > 1$). Let $t^\omega_{\text{mix}}(n)$ denote the mixing time of the lazy RWRE with respect to P_{ω^n}.

Theorem 1.4.1. *Let Assumptions 1 and 2 hold and assume $\kappa > 1$. Then for $\boldsymbol{P}$-almost every environment ω a sequence of lazy RWRE $(Y_k^n)_{k\in\mathbb{N}_0}$ on $(\{0, ..., n\})_{n\in\mathbb{N}}$ exhibits a cutoff with cutoff times*

$$t_\omega(n) :=\; 2E_{\omega^n}(T_n)$$

and window size

$$f_\omega(n) :=\; \sqrt{\mathrm{Var}_{\omega^n}(T_n)}.$$

Note that although $\mathbb{E}(T_n^2) = \infty$ for $\kappa \leq 2$, we have $\mathrm{Var}_{\omega^n}(T_n) < \infty$ for $\mathbf{P}$-almost every environment ω (cf. Theorem 1.4.5) and all $\kappa > 0$. We remark that for $n \in \mathbb{N}$ $d_n(k)$ is only defined for $k \in \mathbb{N}$ and monotone decreasing in k. For simplicity's sake, we omit the ceiling function in the results.

In the case $\kappa < 1$ we show that there is no cutoff:

Theorem 1.4.2. *Let Assumptions 1-3 hold and assume $\kappa < 1$. Then for $\boldsymbol{P}$-almost every environment ω a sequence of lazy RWRE $(Y_k^n)_{k\in\mathbb{N}_0}$ on $(\{0,...,n\})_{n\in\mathbb{N}}$ does not exhibit a cutoff.*

To prove that for $\kappa < 1$ there is no cutoff under Assumptions 1-3, we show that for $\mathbf{P}$-almost every environment ω the window within which the total variation distance drops from 1 to 0 has the same order as the mixing time, and therefore the transition cannot be sharp in the sense of a cutoff.

Furthermore, we determine the order of the mixing time:

Theorem 1.4.3. *Let Assumptions 1 and 2 hold. Then for $\boldsymbol{P}$-almost every environment ω we have*

(a) $\lim_{n\to\infty} \frac{\ln t^{\omega}_{\mathrm{mix}}(n)}{\ln n} = \frac{1}{\kappa}$ *for $0 < \kappa \leq 1$ and*

(b) $\lim_{n\to\infty} \frac{t^{\omega}_{\mathrm{mix}}(n)}{n} = 2\mathbb{E}T_1$ *for $\kappa > 1$.*

With the help of the construction in the proof of Theorem 1.4.2 we can prove the "strong localisation" theorem of Peterson and Zeitouni (cf. Theorem 1.2 in [PZ09]) under the weaker Assumptions 1-3. In [PZ09], they additionally assume that the distribution of $\ln \rho_0$ is non-lattice with respect to $\mathbf{P}$.

Theorem 1.4.4. *Let Assumptions 1-3 hold and assume $\kappa < 1$. Then for $\boldsymbol{P}$-almost every environment ω there exist environment dependent sequences $t_m = t_m(\omega)$ and $u_m = u_m(\omega)$ such that for any $\delta > 0$ we have*

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m} - u_m}{(\ln t_m)^2} \in [-\delta, \delta]\right) = 1.$$

The theorem shows that the RWRE at time t_m is with high probability in an interval of length $(\ln t_m)^2$. Note that the localisation theorem is related to (but cannot be deduced from) Theorem 1.1 in [GS02] where Gantert and Shi prove that for $0 < \kappa \leq 1$ there exists an environment dependent sequence of times $(t_m)_{m\in\mathbb{N}}$ at which the local time of the RWRE is a positive fraction of n. They used the same assumptions as [PZ09] for their proof.

To decide if there is a cutoff or not, we have to control the distance in total variation very precisely. In Chapter 3, we show that we can give sharp bounds based on estimates of the quenched expectation and quenched variance of T_n. In Chapter 2 we prove the following **P**-a.s. behaviour of these quantities, which we then use to prove Theorems 1.4.1 - 1.4.3.

Theorem 1.4.5. *Let Assumptions 1 and 2 hold. Then we have*

$$(a)\ \lim_{n\to\infty} \frac{\ln E_\omega(T_n)}{\ln n} = \max\left\{\frac{1}{\kappa}, 1\right\} \qquad \boldsymbol{P}-a.s.,$$

$$(b)\ \lim_{n\to\infty} \frac{\ln \mathrm{Var}_\omega(T_n)}{\ln n} = \max\left\{\frac{2}{\kappa}, 1\right\} \qquad \boldsymbol{P}-a.s.$$

Note that because we consider an i.i.d. environment, the shift Θ on the product space is ergodic with respect to **P**. Therefore, Birkhoff's ergodic theorem yields the following stronger statements for the cases in which the annealed expectation and the annealed variance, respectively, exist:

For $\kappa > 1$ we have

$$\lim_{n\to\infty} \frac{E_\omega T_n}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} E_{\Theta^{-j}\omega} T_1 = \mathbb{E}T_1 \quad \mathbf{P}-a.s. \tag{1.4.1}$$

and for $\kappa > 2$ we get

$$\lim_{n\to\infty} \frac{\mathrm{Var}_\omega T_n}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} Var_{\Theta^{-j}\omega}(T_1) = \mathbf{E}(\mathrm{Var}_\omega T_1) \quad \mathbf{P}-a.s. \tag{1.4.2}$$

Using the monotonicity of the function $d_n(k)$ in k (cf. Exercise 4.4 in [LPW09]), we can deduce a version of Theorem 1.4.1 with a deterministic cutoff window size from Theorem 1.4.5 and equation (1.4.2):

Corollary 1.4.6. *Let Assumptions 1 and 2 hold and assume $\kappa > 1$. Then for $\boldsymbol{P}$-almost every environment ω and all $0 < \delta < 1 - \frac{1}{\kappa}$ a sequence of lazy RWRE $(Y_k^n)_{k\in\mathbb{N}_0}$ on $(\{0, ..., n\})_{n\in\mathbb{N}}$ exhibits a cutoff with cutoff times*

$$t_\omega(n) := 2E_{\omega^n}(T_n)$$

and (deterministic) window size

$$f_\omega^\kappa(n) := \begin{cases} n^{\frac{1}{\kappa}+\delta} & for\ \ 1 < \kappa \le 2 \\ \sqrt{n} & for\ \ \kappa > 2. \end{cases}$$

For $1 < \kappa < 2$, we do not believe that the cutoff window size can be replaced by $n^{\frac{1}{\kappa}}$ for $\mathbf{P}$-almost every environment. If we additionally assume that the distribution of $\ln \rho_0$ is non-lattice, we have due to Theorem 1.3 in [Pet09]

$$\lim_{n\to\infty} \mathbf{P}\left(n^{-\frac{2}{\kappa}}\mathrm{Var}_\omega T_n \;\leq\; x\right) \;=\; L_{\frac{\kappa}{2},b}(x),$$

where $L_{\frac{\kappa}{2},b}(x)$ denotes the distribution function of a stable distribution for some constant $b > 0$. To get a cutoff window size of $n^{\frac{1}{\kappa}}$, we would need a $\mathbf{P}$-almost sure convergence of $n^{-\frac{2}{\kappa}}\mathrm{Var}_\omega T_n$ which we do not believe to be true.

Further, to state a version of Theorem 1.4.1 with deterministic cutoff times, one has to analyse the order of the deviation of $E_\omega T_n$ from $n\mathbb{E}T_1$, which is not precisely known so far. Due to equation (1.4.1), we know that it is of order $o(n)$. Therefore, it is possible to state a version of the cutoff with (deterministic) cutoff times $2n\mathbb{E}T_1$ and window size chosen to be the maximum of $n^{\frac{1}{\kappa}}$ and the order of the just mentioned deviations. But at least for the case $\kappa > 2$ this enlarges the window size because in this case the proofs of the CLT (cf. Theorem 5.1 in [Ali99], Theorem 3.1.1 in [Pet08] and Theorem 3 in [Gol07]) have shown that the deviation is bigger than $\sqrt{n}$ which is also the reason why there is no CLT with deterministic centering.

There is a close connection between the mixing time and the spectral gap of a Markov chain (cf. Theorem 4.1.1). Let $\widetilde{P}^\omega_n$ denote the transition matrix of the lazy RWRE with respect to P_{ω^n} and let $\mathrm{gap}(\widetilde{P}^\omega_n)$ denote its spectral gap. In Chapter 4 we show:

Theorem 1.4.7. *Let Assumptions 1 and 2 hold.*

(a) Let $\kappa < 1$ and additionally assume Assumption 3. Then we have for $\boldsymbol{P}$-almost every environment ω

$$\liminf_{n\to\infty} \frac{\ln \mathrm{gap}\left(\widetilde{P}^\omega_n\right)}{\ln n} \;=\; -\frac{1}{\kappa}.$$

(b) Let $\kappa > 1$. Then we have for $\boldsymbol{P}$-almost every environment ω

$$\lim_{n\to\infty} \mathrm{gap}\left(\widetilde{P}^\omega_n\right)\cdot n \;=\; \infty.$$

(c) Let $1 < \kappa < 2$ and additionally assume that the distribution of $\ln \rho_0$ is non-lattice with respect to $\boldsymbol{P}$. Then we have for $\boldsymbol{P}$-almost every environment ω

$$\limsup_{n\to\infty} \frac{\ln \mathrm{gap}\left(\widetilde{P}^\omega_n\right)}{\ln n} \;\leq\; 1 - \frac{2}{\kappa}.$$

In particular, Theorem 1.4.7 (c) shows that we have

$$\lim_{n\to\infty} \mathrm{gap}\left(\widetilde{P}^\omega_n\right) \;=\; 0$$

for $1 < \kappa < 2$. This is in sharp contrast to the case of a deterministic environment, where the spectral gap of a random walk with drift to the right is bounded away from 0 uniformly in n (cf. Remark 4.1.4).

1.5 Estimates on the environment

In this section, we have a closer look at the environment. For this purpose, we analyse the potential associated with the environment, which was introduced by Sinai in [Sin82]. A good understanding of the potential will be the key to analyse the quenched expectation and quenched variance of T_n in the next chapter. The potential, denoted by $(V(x))_{x\in\mathbb{Z}}$, is a function of the environment ω defined in the following way:

$$V(x) := \begin{cases} -\sum_{i=x}^{-1} \ln \rho_i & \text{if } x \leq -1, \\ 0 & \text{if } x = 0, \\ \sum_{i=0}^{x-1} \ln \rho_i & \text{if } x \geq 1. \end{cases}$$

Due to Assumption 1 ($\mathbf{E} \ln \rho_0 < 0$), the potential is a random walk with negative drift. But there are some "blocks" of the environment where the potential is increasing. We will see that the height of these increases depends on κ and that the RWRE needs most of the time before it hits position n to cross the highest increase of the potential in the interval $[0, n]$. We note that an increase of the potential in x means that the probability to move to the left (i.e. $x-1$) is higher than to move to the right (i.e. $x+1$). In contrast to that, at positions where the potential decreases it is the other way around. For the influence of the parameter κ on the shape of the potential see Figure 1.3 and Figure 1.4 on page 34.

In the following, we use the partition of the potential in blocks introduced in [PZ09] (cf. Figure 1.1). We define "ladder locations" $(\nu_i(\omega))_{i\in\mathbb{Z}}$ of the environment ω by

$$\nu_i(\omega) := \begin{cases} \sup \{n < \nu_{i+1}(\omega) : V(n) < V(k) \;\; \forall\, k < n\} & \text{for } i \leq -1, \\ 0 & \text{for } i = 0, \\ \inf \{n > \nu_{i-1}(\omega) : V(n) < V(\nu_{i-1}(\omega))\} & \text{for } i \geq 1. \end{cases} \tag{1.5.1}$$

If no confusion is possible, we often drop the dependence of ω and simply write $(\nu_i)_{i\in\mathbb{Z}}$. The portion of the environment $[\nu_i, \nu_{i+1})$ is called the ith block. Note that the block from ν_{-1} to -1 is different from all the other blocks.

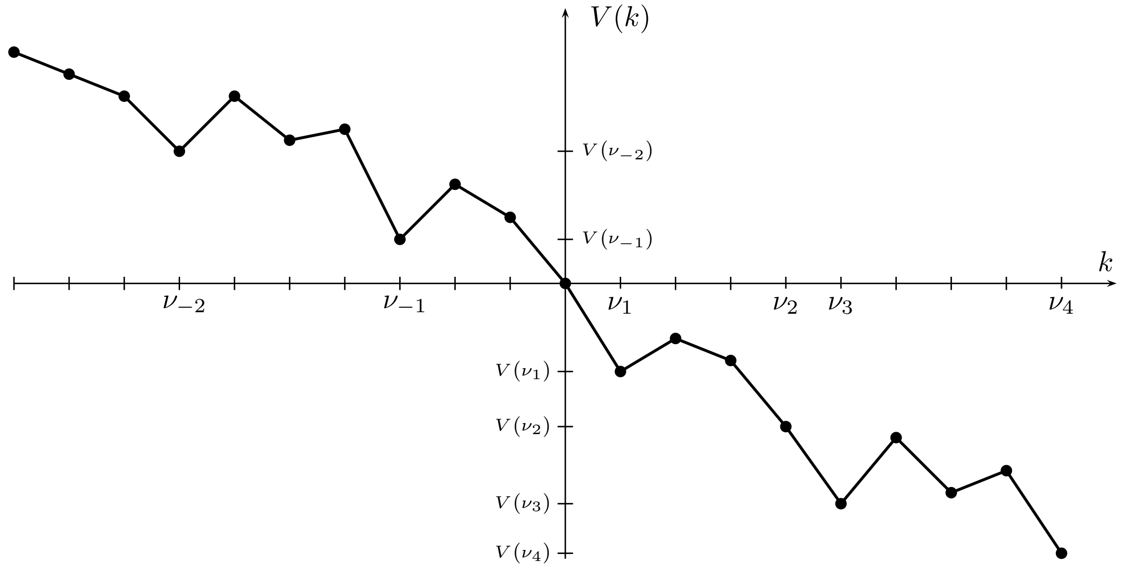

Figure 1.1: *On the definition of the ladder locations.*

Next, we define another measure $\mathbf{Q}$ on the environments by

$$\mathbf{Q}(\cdot) := \mathbf{P}(\cdot|\mathcal{R}), \quad \text{where} \quad \mathcal{R} := \left\{ \omega \in \Omega \ : \ \sum_{i=-k}^{-1} \ln \rho_i < 0, \forall\, k \in \mathbb{N} \right\}. \tag{1.5.2}$$

We notice that $\mathbf{P}(\mathcal{R}) > 0$ due to Assumption 1 and that the sequence $(\nu_i - \nu_{i-1})_{i\in\mathbb{Z}}$ is i.i.d. with respect to $\mathbf{Q}$. Further, we define for $i \in \mathbb{Z}$

$$H_i := \max_{\nu_i \le j < \nu_{i+1}} (V(j) - V(\nu_i)) \ = \max_{\nu_i \le j < k < \nu_{i+1}} (V(k) - V(j)) \tag{1.5.3}$$

as the height of the ith block (cf. Figure 1.2) and for $n \in \mathbb{N}$ let

$$n_0 := \max\{l \in \mathbb{N}_0 \ : \ \nu_l \le n\} \tag{1.5.4}$$

denote the number of the block to which n belongs.

Later, we estimate the height of blocks using results on the asymptotic of the maximum of random walks with negative drift. First, let us assume that the distribution of $\ln \rho_0$ is non-lattice. Then, additionally assuming Assumptions 1 and 2, Theorem 1 in [Igl72] yields constants $0 < K_1 < K_2$ such that for $h > 0$ we have

$$K_1 \exp(-\kappa \cdot h) \ \le \ \mathbf{P}(S \ge h) \ \le \ K_2 \exp(-\kappa \cdot h), \tag{1.5.5}$$

where $S := \max_{j \ge 0} V(j)$ denotes the maximum of the potential on $\mathbb{N}_0$.

Next, let the distribution of $\ln \rho_0$ be concentrated on $x + y\mathbb{Z}$ for $x \in \mathbb{R}, y \in \mathbb{R}^{>0}$.

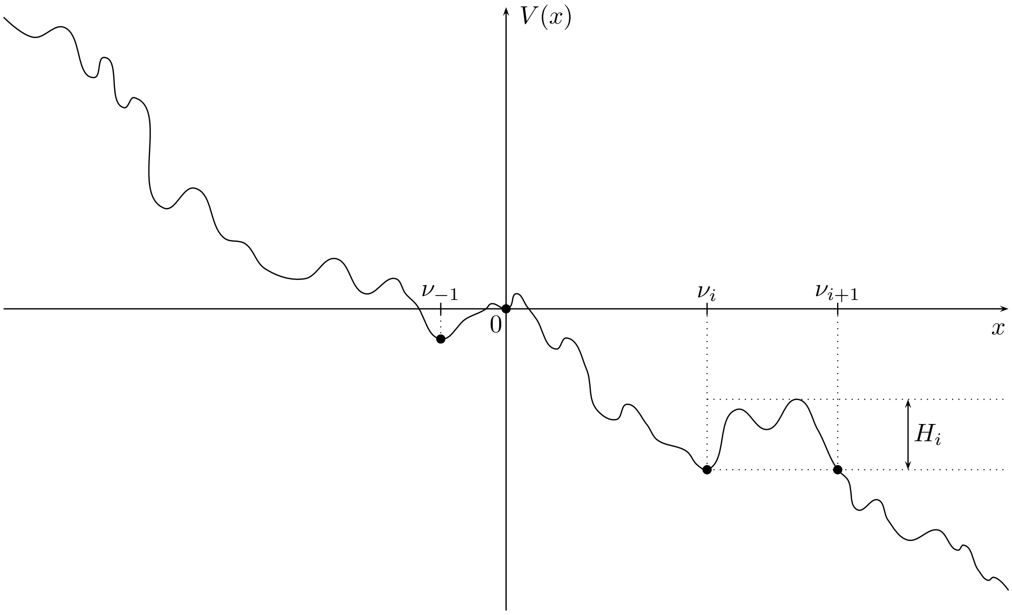

Figure 1.2: *On the definition of H_i. Note that this realisation cannot be seen under* $\boldsymbol{Q}$ *because here we have $V(\nu_{-1}) < 0$.*

Therefore, the potential is a Markov chain with i.i.d. increments of a lattice distribution. Again assuming Assumptions 1 and 2, we get for the case in which the potential is aperiodic using E19.4 in [Spi76] with $r = \exp(\kappa)$

$$K_1' \exp(-\kappa \cdot (y \cdot n + x)) \;\leq\; \mathbf{P}\left(S \geq y \cdot n + x\right) \;\leq\; K_2' \exp(-\kappa \cdot (y \cdot n + x)), \quad (1.5.6)$$

for $n \in \mathbb{N}$ and constants $0 < K_1' < K_2'$. If the potential $V(\cdot)$ is a periodic Markov chain with period $d \in \mathbb{N}$, we still have that $(V(nd+k))_{n\in\mathbb{N}_0}$ is aperiodic for every $k \in \{0, ..., d-1\}$ and therefore we get the same asymptotic as in (1.5.6) using the minimum and the maximum, respectively, of the appearing constants.

Now combining (1.5.5) and (1.5.6), we get that under Assumptions 1 and 2 there exist constants $0 < \widetilde{C}_1 < C_1$ such that we have for all $h > 0$

$$\widetilde{C}_1 \exp(-\kappa \cdot h) \;\leq\; \mathbf{P}\left(S \geq h\right) \;\leq\; C_1 \exp(-\kappa \cdot h). \quad (1.5.7)$$

In the next lemmata, we identify "typical" and "good" subsets of the environment, which simplify calculations in the following.

First, we show that it is very unlikely for the potential to stay at a certain level for a long time because of the negative drift. We define

$$B_1(n) := \left\{ \nexists\, k \in \mathbb{N},\ i, j \in \{-n, ..., n\} \;:\; j - i \geq k(\ln n)^2,\ V(j) > V(i) - k \ln n \right\} \quad (1.5.8)$$

as the set of environments for which on the interval $[-n,n]$ the potential decreases at least by $k\ln n$ every $\lceil (k\ln n)^2\rceil$ steps. In particular, we have for environments $\omega\in B_1(n)$ that all blocks in the interval $[-n,n]$ are smaller than $(\ln n)^2$.

Lemma 1.5.1. *We have*

$$\boldsymbol{P}\Big(B_1(n)^{\mathrm{c}}\Big) \;=\; O\left(n^{-2}\right).$$

Proof. First, we note that for $k\in\mathbb{N}$ we have

$$\begin{aligned}
&\Big\{\exists\; j\geq k(\ln n)^2 \;:\; V(j)>-k\ln n\Big\}\\
\subseteq &\left\{V\big(\lfloor k(\ln n)^2\rfloor\big) > -\left(k+\frac{4}{\kappa}\right)\ln n\right\}\cup\left\{\max_{j\geq k(\ln n)^2}\big(V(j)-V\big(\lfloor k(\ln n)^2\rfloor\big)\big)>\frac{4}{\kappa}\ln n\right\}
\end{aligned}\tag{1.5.9}$$

because either $V\big(\lfloor k(\ln n)^2\big)\rfloor$ is bigger than $-\left(k+\frac{4}{\kappa}\right)\ln n$ or there has to be an increase of the potential of more than $\frac{4}{\kappa}\ln n$ afterwards. Using (1.5.7), we get for arbitrary $l\in\mathbb{N}$

$$\mathbf{P}\left(\max_{j\geq l}\Big(V(j)-V(l)\Big)>\frac{4}{\kappa}\ln n\right) \;=\; O\left(n^{-4}\right).$$

This together with (1.5.9) yields

$$\begin{aligned}
&\mathbf{P}\Big(B_1(n)^{\mathrm{c}}\Big)\\
&=\mathbf{P}\Big(\exists\; k\in\mathbb{N},\; i,j\in\{-n,...,n\}\;:\; j-i\geq k(\ln n)^2,\; V(j)>V(i)-k\ln n\Big)\\
&\leq \sum_{k=0}^{\left\lfloor\frac{2n}{(\ln n)^2}\right\rfloor} 2n\mathbf{P}\Big(\exists\; j\geq k(\ln n)^2 \;:\; V(j)>-k\ln n\Big)\\
&\leq \sum_{k=0}^{\left\lfloor\frac{2n}{(\ln n)^2}\right\rfloor} 2n\Bigg(\mathbf{P}\left(V\Big(\lfloor k(\ln n)^2\rfloor\Big)>-\left(k+\frac{4}{\kappa}\right)\ln n\right)\\
&\qquad\qquad +\mathbf{P}\left(\max_{j\geq k(\ln n)^2}\Big(V(j)-V\Big(\lfloor k(\ln n)^2\rfloor\Big)\Big)>\frac{4}{\kappa}\ln n\right)\Bigg)\\
&\leq \sum_{k=0}^{\left\lfloor\frac{2n}{(\ln n)^2}\right\rfloor} 2n\mathbf{P}\left(\Big|V\Big(\lfloor k(\ln n)^2\rfloor\Big)-\lfloor k(\ln n)^2\rfloor\mathbf{E}\ln\rho_0\Big| \;>\; \frac{|\mathbf{E}\ln\rho_0|}{2}\lfloor k(\ln n)^2\rfloor\right)\\
&\quad +O\left(n^{-2}\right).
\end{aligned}\tag{1.5.10}$$

Since the potential is a sum of i.i.d. random variables with finite exponential moments in a neighbourhood of zero due to Assumption 2 and negative expectation due to

Assumption 1, we can apply Cramér's Theorem (cf. Theorem 2.2.3 in [DZ98]) to obtain an upper bound for (1.5.10), that is

$$\mathbf{P}\Big(B_1(n)^{\mathrm{c}}\Big) \;\leq 4n^2 \exp\big(-c(\ln n)^2\big) + O\left(n^{-2}\right)$$

for a constant $c > 0$, and this finishes the proof. ■

Further, we show that the first n appearing blocks on the right side of 0 and on the left side of 0 are together not to wide.

Lemma 1.5.2. *We have*

$$\boldsymbol{P}\Big(B_2^{\mathrm{c}}(n)\Big) = O\left(n^{-2}\right),$$

where

$$B_2(n) := \;\{-2\bar{\nu}n \leq \nu_{-n}, \nu_n \leq 2\bar{\nu}n\} \quad \textit{with } \bar{\nu} := \;\boldsymbol{E}\nu_1.$$

Proof. First, we show that $(\nu_i - \nu_{i-1})$ has exponential tails for all $i \in \mathbb{Z}^{\neq 0}$. Again using Cramér's Theorem for the sequence $(\ln \rho_k)_{k\in\mathbb{Z}}$, we get for large $x > 0$ and $i \in \mathbb{Z}^{\neq 0}$

$$\begin{aligned}
\mathbf{P}\big(\nu_i - \nu_{i-1} \;>\; x\big) \;&=\; \mathbf{P}\big(\nu_1 \;>\; x\big)\\
&\leq\; \mathbf{P}\left(V(\lfloor x \rfloor) \;\geq\; 0\right)\\
&\leq\; \mathbf{P}\Big(\big|V(\lfloor x \rfloor) - \lfloor x \rfloor \mathbf{E}\ln\rho_0\big| \;\geq\; |\mathbf{E}\ln\rho_0|\lfloor x \rfloor\Big)\\
&\leq\; \exp\big(-c\cdot x\big)
\end{aligned}$$

for a constant $c > 0$. Thus, we have

$$\mathbf{E}\exp\left(\widetilde{c}\cdot\nu_1\right) \;<\; \infty \quad \forall\, \widetilde{c} < c$$

and therefore we also can apply Cramér's Theorem for the sequence $(\nu_i - \nu_{i-1})_{i\in\mathbb{Z}^{\neq 0}}$ to obtain

$$\mathbf{P}\left(\sum_{i=1}^{n}(\nu_i - \nu_{i-1}) > 2\bar{\nu}n\right) + \mathbf{P}\left(\sum_{i=-n+1}^{-1}(\nu_i - \nu_{i-1}) > \frac{3}{2}\bar{\nu}n\right) \;=\; O\left(n^{-2}\right). \tag{1.5.11}$$

Furthermore, we have due to Lemma 1.5.1

$$\mathbf{P}\Big(\nu_{-1} < -(\ln n)^2\Big) \;\leq\; \mathbf{P}\Big(B_1(n)^{\mathrm{c}}\Big) \;=\; O\left(n^{-2}\right),$$

and this together with (1.5.11) finishes the proof. ■

Next, we are interested in the height of the highest block in the interval $[-n, n]$. We define

$$B_3(n) := \;\left\{\max_{-n\leq i\leq n}\max_{k\geq i}\left(V(k) - V(i)\right) \;\leq\; \frac{1}{\kappa}\Big(\ln n + 2\ln\ln n\Big)\right\}$$

and notice:

Lemma 1.5.3. *For $\boldsymbol{P}$-almost every environment ω there exists a $N(\omega)$, such that $\omega \in B_3(n)$ for all $n \geq N(\omega)$.*

For a proof see for instance Lemma 3.4 in [FGP10].

Furthermore, we are interested in a lower bound for the height of the highest block in $[-n, n]$. We define

$$B_4(n) := \left\{ \max_{-n \leq i \leq n} \max_{k \geq i} (V(k) - V(i)) \; > \frac{1}{\kappa}(\ln n - 4 \ln \ln n) \right\}$$

and note:

Lemma 1.5.4. *We have*

$$\boldsymbol{P}\Big(B_4(n)^{\mathrm{c}}\Big) \;=\; O\left(n^{-2}\right).$$

For a proof see for instance Lemma 3.5 in [FGP10].

As a next step, we want to analyse the frequency of the appearance of blocks with certain heights. We therefore define for $0 < a < 1$ (cf. (1.5.4) for a definition of n_0)

$$D_n(a) := \left\{ \left| \left\{ 0 \leq i \leq n_0 \;:\; H_i \geq \frac{a}{\kappa}\Big(\ln n + 2 \ln \ln n \Big) \right\} \right| \;<\; n^{1-a} \right\}.$$

In the next lemma, we show that asymptotically we do not have "too many" high blocks:

Lemma 1.5.5. *For all $m \in \mathbb{N}$ we have*

$$\boldsymbol{P}\Big(D(n, m)^c\Big) \;=\; O\left(n^{-2}\right),$$

where

$$D(n, m) := \bigcap_{l=1}^{m-1} D_n\left(\frac{l}{m}\right). \tag{1.5.12}$$

Proof. Because m is fixed, it is enough to show

$$\mathbf{P}\left(D_n\left(\frac{l}{m}\right)^{\mathrm{c}} \right) \;=\; O\left(n^{-2}\right)$$

for arbitrary $l, m \in \mathbb{N}$, $l < m$.

First, we note that the number of blocks with a height of more than $\frac{l}{\kappa m}\Big(\ln n + 2 \ln \ln n \Big)$ can stochastically be dominated by a binomial random variable B_n^l with parameters n and success probability

$$\mathbf{P}\left(S \geq \frac{l}{\kappa m}\Big(\ln n + 2 \ln \ln n \Big) \right),$$

where $S = \max_{j\geq 0} V(j)$ denotes the maximum of the potential on $\mathbb{N}_0$. Due to (1.5.7) we have

$$\mathbf{P}\left(S \geq \frac{l}{\kappa m}\Big(\ln n + 2\ln\ln n\Big)\right) \;=\; C_1 \cdot \left(n \cdot ln(n)^2\right)^{-\frac{l}{m}}. \tag{1.5.13}$$

Therefore, we get

$$\mathbf{P}\left(D_n\left(\frac{l}{\kappa m}\right)^{\mathrm{c}}\right) \;\leq\; \mathbf{P}\left(B_n^l \geq n^{1-\frac{l}{\kappa m}}\right).$$

Now, using the exponential Markov inequality, this together with (1.5.13) yields:

$$\begin{aligned}\mathbf{P}\left(D_n\left(\frac{l}{\kappa m}\right)^{\mathrm{c}}\right) \;&\leq\; \exp\left(-n^{1-\frac{l}{m}}\right)\cdot \mathbf{E}\exp(B_n^l)\\ &=\; \exp\left(-n^{1-\frac{l}{m}}\right)\cdot\left(1 + C_1(e-1)\left(n(\ln n)^2\right)^{-\frac{l}{m}}\right)^n\\ &\leq\; \exp\left(C_1(e-1)(\ln n)^{-\frac{2l}{m}} n^{1-\frac{l}{m}} - n^{1-\frac{l}{m}}\right)\\ &= O\left(n^{-2}\right),\end{aligned} \tag{1.5.14}$$

where we used that we have $1+x \leq \exp(x)$ for $x \geq 0$ to obtain the second last line. ∎

Further, we define for $0 < a < 1$

$$E_n(a) := \Bigg\{\nexists\; n \leq k < n + (\ln n)^2 \;:\; \max_{n\leq l<k}(V(k) - V(l)) > \frac{a}{\kappa}\ln n,\; \max_{k<i<j\leq k+(\ln n)^2}(V(j) - V(i)) > \frac{1-\frac{3a}{4}}{\kappa}\ln n\Bigg\}$$

as the set of environments which do not have two "large" increases of the potential in a "small" interval after n.

Lemma 1.5.6. *For all $0 < a < 1$ we have*

$$\boldsymbol{P}\Big(E_n(a)^{\mathrm{c}}\Big) \;=\; O\left(n^{-\left(1+\frac{a}{5}\right)}\right).$$

Proof. For environments $\omega \in E_n(a)^{\mathrm{c}}$ we have two "large" increases of the potential in the interval $[n, n + 2(\ln n)^2]$. The first increase is bigger than $\frac{a}{\kappa}\ln n$ and the second bigger than $\frac{1-\frac{3a}{4}}{\kappa}\ln n$. We therefore have

$$\mathbf{P}\Big(E_n(a)^{\mathrm{c}}\Big) \;\leq (\ln n)^4 \cdot \mathbf{P}\left(S \geq \frac{a}{\kappa}\ln n\right)\cdot \mathbf{P}\left(S \geq \frac{1-\frac{3a}{4}}{\kappa}\ln n\right) \;\leq\; (\ln n)^4 C_1^2 n^{-(1+\frac{a}{4})},$$

where we used (1.5.7) to obtain the last inequality. ∎

Now, we want to use Lemma 1.5.6 to show that for n large enough in the interval $[0, n]$ we do not find two "big" increases of the potential in an interval of size $2(\ln n)^2$. We define

$$E(n,a) := \Bigg\{ \nexists\, 0 \le k \le n \ : \max_{k-(\ln n)^2 \le l < k} (V(k) - V(l)) > \frac{a}{\kappa} \ln n, \\ \max_{k < i < j \le k + (\ln n)^2} (V(j) - V(i)) > \frac{1 - \frac{3a}{4}}{\kappa} \ln n \Bigg\}$$

and we observe:

Lemma 1.5.7. *For all $0 < a < 1$ and $\boldsymbol{P}$-almost every environment ω there exists $N(\omega)$ such that we have $\omega \in E(n,a)$ for all $n \ge N(\omega)$.*

Proof. Let

$$N_0(\omega) := \min \{ j \ge 0 \ : \ \omega \in E_i(a) \ \ \forall\, i > j \}$$

and let

$$M(\omega) := \max \Bigg\{ \max_{k < r < s \le k + (\ln N_0(\omega))^2} (V(s) - V(r)) \ : \\ k \in [0, N_0(\omega)] \text{ with } \max_{k-(\ln N_0(\omega))^2 < l < k} (V(k) - V(l)) > \frac{a}{\kappa} \ln N_0(\omega) \Bigg\}$$

be the maximal increase of the potential in an interval of size $(\ln N_0(\omega))^2$ after an increase of more than $\frac{a}{\kappa} \ln N_0(\omega)$ in the interval $[0, N_0(\omega)]$.

Due to Lemma 1.5.6 and the Borel-Cantelli lemma, we have that $N_0(\omega)$ is finite for $\mathbf{P}$-almost every environment ω. Now, we take $N(\omega)$ large enough such that $N(\omega) \ge N_0(\omega)$ and

$$\frac{1 - \frac{3a}{4}}{\kappa} \ln N(\omega) \ \ge M(\omega).$$

Then, for $n \ge N(\omega)$ let K be the size of the maximal increase of the potential in an interval of size $(\ln n)^2$ after an increase of more than $\frac{a}{\kappa} \ln n$ in the interval $[0, n]$. Then, we have

$$K \ \le \ M(\omega) \ \le \ \frac{1 - \frac{3a}{4}}{\kappa} \ln n$$

if the increase is in the interval $[0, N_0(\omega)]$ or if the increase is in the interval $(N_0(\omega), n]$ we have

$$K \ \le \ \frac{1 - \frac{3a}{4}}{\kappa} \ln n$$

by the definition of $N_0(\omega)$. ■

Next, we prove three technical statements, which will help us to compare the quenched expectations and the quenched variances with respect to P_ω and P_{ω^n} in the next chapters.

Lemma 1.5.8. *We have for* $\boldsymbol{P}$*-almost every environment* ω

(a) $C^-(\omega) := \sum_{j=-\infty}^{-1} \exp\big(-V(j)\big) \ < \ \infty,$

(b) $C^+(\omega) := \sum_{j=0}^{\infty} \exp\big(V(j)\big) \ < \ \infty,$

(c) $D^-(\omega) := \sum_{j=-\infty}^{-1} \exp\big(-V(j+1)\big)\big(W_j + W_j^2\big) \ < \ \infty.$

Before we prove the lemma, we note that due to (1.2.3) we have

$$\begin{aligned} E_\omega T_n - E_{\omega^n} T_n \ &= \ 2\sum_{j=0}^{n-1}\Big(W_j - W_j^0\Big) \\ &= \ 2\sum_{j=0}^{n-1}\sum_{i=-\infty}^{0} \exp\big(V(j+1) - V(i)\big) \\ &= \ 2\sum_{j=0}^{n-1} \exp\big(V(j+1)\big) \sum_{i=-\infty}^{0} \exp\big(-V(i)\big) \\ &\le \ 2\Big(C^-(\omega) + 1\Big)C^+(\omega), \end{aligned}$$

which, using Lemma 1.5.8, yields

$$\lim_{n\to\infty} \frac{E_\omega T_n}{E_{\omega^n} T_n} \ = \ 1 \tag{1.5.15}$$

for $\mathbf{P}$-almost every environment ω.

Proof of Lemma 1.5.8. First, we note that for $\mathbf{P}$-almost every environment ω by the SLLN there existis $N_1(\omega)$ large enough such that

$$\begin{aligned} V(j) \ &\ge \ \frac{-\mathbf{E}\ln\rho_0}{2} j \ \text{ for all } \ j \le -N_1(\omega), \\ V(j) \ &\le \ \frac{\mathbf{E}\ln\rho_0}{2} j \ \text{ for all } \ j \ge N_1(\omega). \end{aligned} \tag{1.5.16}$$

Due to Assumption 1 ($\mathbf{E}\ln\rho_0 < 0$), we therefore get that for $\mathbf{P}$-almost every environment ω we have

$$\max\{C^-(\omega), C^+(\omega)\} \ < \ \infty$$

and this finishes the proof of (a) and (b).

Next, we note that Lemma 1.5.1 and 1.5.3 together with the Borel-Cantelli lemma yield

$$N_2(\omega) := \inf\{k \in \mathbb{N} \ : \ \omega \in B_1(n) \cap B_3(n) \ \forall\, n \geq k\} \ < \ \infty \ \ \mathbf{P}\text{-a.s.}$$

We get for $j \leq -N_2(\omega)$

$$\begin{aligned} W_j \ &= \ \sum_{i=2j+1}^{j} \exp\big(V(j+1) - V(i)\big) + \sum_{i=-\infty}^{2j} \exp\big(V(j+1) - V(i)\big) \\ &\leq \ (-j)\,(-2j)^{\frac{1}{\kappa}}\,(\ln(-2j))^{\frac{2}{\kappa}} + \sum_{i=-\infty}^{2j} \frac{1}{i^2} \\ &\leq \ (-j)^{2+\frac{1}{\kappa}}, \end{aligned} \tag{1.5.17}$$

where we used for the second last inequality that for $\omega \in B_3(-2j)$ the biggest increase of the potential in the interval $[2j, j]$ is smaller than $\frac{1}{\kappa}\Big(\ln(-2j) + 2\ln\ln(-2j)\Big)$. Further, we used that since $(-i) - (-j) \geq 2(\ln(-i))^2$ for all $i \leq 2j$, we have $V(j) - V(i-1) < -2\ln i$ for $i \leq 2j$ and $\omega \in B_1(-i)$.

Next, we notice that for $-N_2(\omega) < j \leq 0$ we get

$$\begin{aligned} W_j \ &= \ \sum_{i=((-N_2(\omega))\wedge 2j)+1}^{j} \exp\big(V(j+1) - V(i)\big) + \sum_{i=-\infty}^{(-N_2(\omega))\wedge 2j} \exp\big(V(j+1) - V(i)\big) \\ &\leq \ N_2(\omega)^{2+\frac{1}{\kappa}}, \end{aligned} \tag{1.5.18}$$

where we this time used that by the definition of $N_2(\omega)$ we have $\omega \in B_3(N_2(\omega))$ and $\omega \in B_1(-i)$ for all $i \leq (-N_2(\omega)) \wedge 2j$.

Using (1.5.17) and (1.5.18), we get

$$\begin{aligned} D^-(\omega) \ &= \ \sum_{j=-\infty}^{-1} \exp\big(-V(j+1)\big)\big(W_j + W_j^2\big) \\ &\leq 2\sum_{j=-\infty}^{-1} \exp\big(-V(j+1)\big)W_j^2 + 2C^-(\omega) \\ &\leq 2\sum_{j=-\infty}^{-N_2(\omega)} \exp\big(-V(j+1)\big)W_j^2 + 2\sum_{j=-N_2(\omega)+1}^{-1} \exp\big(-V(j+1)\big)W_j^2 + 2C^-(\omega) \\ &\leq 2\sum_{j=-\infty}^{-N_2(\omega)} \exp\big(-V(j+1)\big)(-j)^{4+\frac{2}{\kappa}} + 2C^-(\omega)\Big(N_2(\omega)^{4+\frac{2}{\kappa}} + 1\Big). \end{aligned} \tag{1.5.19}$$

Due to (1.5.16), we have that the sum in (1.5.19) is **P**-a.s. summable. Therefore, together with (a) we can conclude

$$D^{-}(\omega) < \infty$$

for **P**-almost every environment ω. ■

We define

$$C := \left\{\omega \ : \ \max\{C^{-}(\omega), C^{+}(\omega), D^{-}(\omega)\} < \infty\right\},$$

where we have $\mathbf{P}(C) = 1$ due to Lemma 1.5.8.

As the last step in this section, we combine all the results of the previous lemmata and define what we will call a "good" and "typical" environment:

Lemma 1.5.9. *For all $m \in \mathbb{N}$, $0 < a < 1$ and for $\boldsymbol{P}$-almost every environment ω, there exists $N(\omega)$ such that*

$$\omega \in B(n) \cap D(n, m) \cap E(n, a) \quad \forall\, n \geq N(\omega),$$

where

$$B(n) := \ B_1(n) \cap B_2(n) \cap B_3(n) \cap B_4(n) \cap C.$$

Proof. The statement follows directly from the Borel-Cantelli lemma together with Lemma 1.5.1 - 1.5.8. ■

For these "good" and "typical" environments, we have a lower and an upper bound on the height of the highest block in the interval $[0, n]$ and we know that the potential does not stay at a certain level for a long time. In addition to that, we can control the number of high blocks and do not have two large increases in a small interval. For the influence of κ on the shape of the potential compare the following two simulations (Figure 1.3 and Figure 1.4). We see that the potential in both cases follows the line with slope $-\mathbf{E} \ln \rho_0 = -0.14$ but the fluctuations are a lot bigger in the case in which κ is smaller.

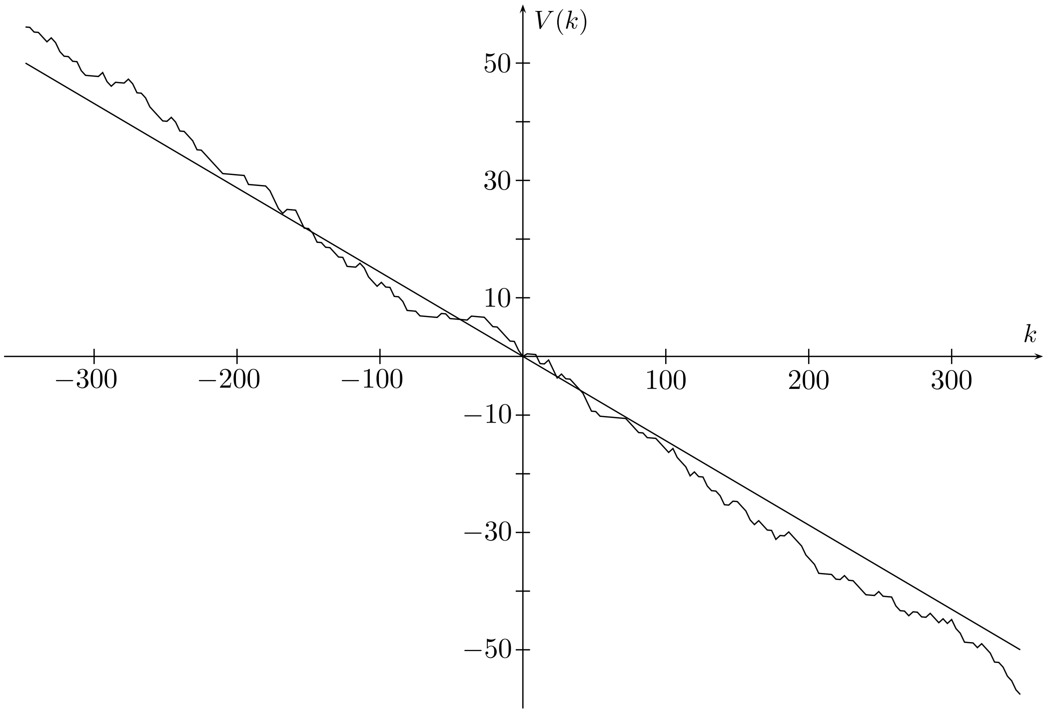

Figure 1.3: *Simulation of the potential of an environment distribution with* $\kappa = 2.39$ *and* $\boldsymbol{E} \ln \rho_0 = -0.14$.

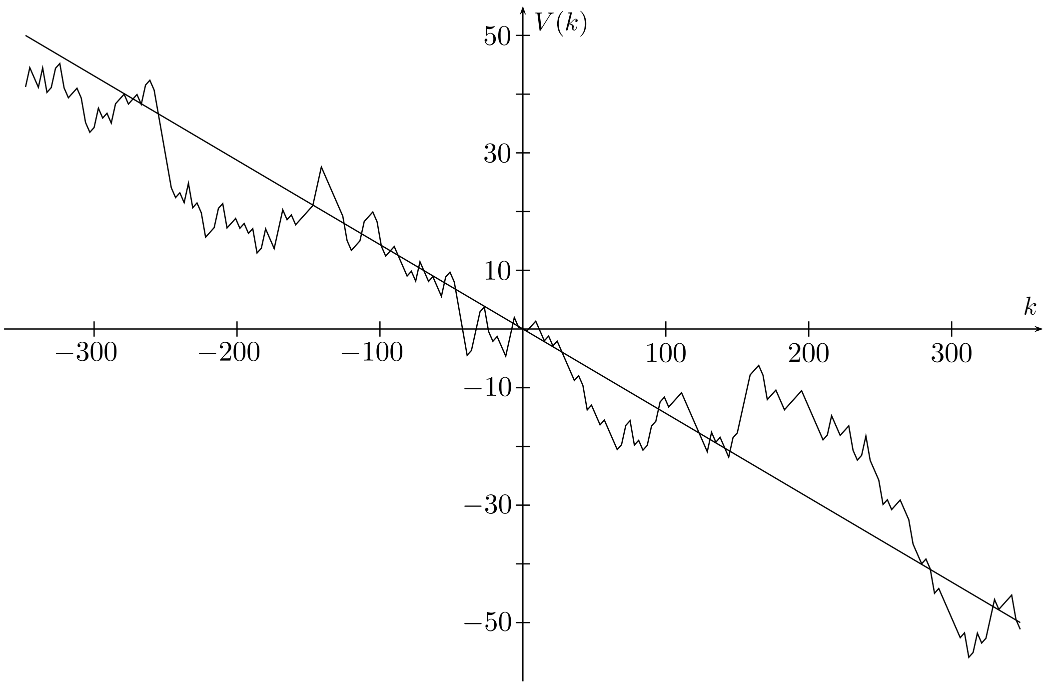

Figure 1.4: *Simulation of the potential of an environment distribution with* $\kappa = 0.19$ *and* $\boldsymbol{E} \ln \rho_0 = -0.14$.

Chapter 2

Quenched expectation and quenched variance

We will see that the distance in total variation of the distribution of the lazy RWRE with respect to P_{ω^n} to its stationary distribution can be bounded using the tails of T_n (cf. Chapter 3). To show a cutoff, we need to control the fluctuations of T_n very precisely. The main result of this chapter is Theorem 1.4.5. For $\kappa < 1$ and for $\mathbf{P}$-almost every environment, we show that the quenched expectation of T_n is of order $n^{\frac{1}{\kappa}}$. Further, we prove that for $\kappa < 2$ the quenched variance of T_n is of order $n^{\frac{2}{\kappa}}$. We notice at this point that the annealed expectation and variance of T_n are infinite for these κ. Furthermore, we analyse the quenched variance of the crossing time of "deep" blocks.

2.1 Asymptotic of the quenched expectation and quenched variance

A formula for the quenched variance as a function of the environment is given in equation (2.1) in [Gol07]. In our notation, we get for $k \in \mathbb{Z}$ (cf. (1.2.1) for the definition of W_i)

$$\mathrm{Var}_\omega\big(T_{k+1} - T_k\big) \;= 4\left(W_k + W_k^2\right) + 8 \sum_{i=-\infty}^{k-1} \exp\big(V(k+1) - V(i+1)\big) \cdot \left(W_i + W_i^2\right). \tag{2.1.1}$$

This yields for $n \in \mathbb{N}$

$$\mathrm{Var}_\omega\big(T_n\big) \;= 4 \cdot \sum_{j=0}^{n-1} \left(W_j + W_j^2\right) + 8 \cdot \sum_{j=0}^{n-1} \sum_{i=-\infty}^{j-1} \exp\big(V(j+1) - V(i+1)\big) \cdot \left(W_i + W_i^2\right). \tag{2.1.2}$$

By equations (2.1.1) and (2.1.2) we see that the quenched variance of T_n is not – as the expectation – a function of the sequence $(W_j)_{j\in\mathbb{N}_0}$. But we will see that the W_j-terms in (2.1.2) still determine the order of $\mathrm{Var}_\omega(T_n)$. First, we give an upper bound for W_j depending on the biggest increase of the potential in a small neighbourhood to the left of j.

Lemma 2.1.1. *For $\omega \in B_1(n)\cap C$ and n large enough such that $C^-(\omega) < \ln n$ we have for $j = 1, ..., n$ (cf. (1.5.4) for the definition of j_0)*

$$W_j^0 \;<\; W_j \;\le\; (\ln n)^2\Big(1 + 2\exp\big(V(j+1) - V(\nu_{j_0})\big)\Big).$$

For $\nu_{i+1} < n$ and n large enough, we note that Lemma 2.1.1 yields for $\omega \in B_1(n)\cap C$ (cf. (1.5.3) for the definition of H_i)

$$\max_{j\in[\nu_i,\nu_{i+1})} W_j \;\le\; (\ln n)^2 + 2(\ln n)^2\exp(H_i). \tag{2.1.3}$$

Therefore, we get together with (1.2.3) and using that for $\omega \in B_1(n)$ all blocks in the interval $[-n, n]$ are smaller than $(\ln n)^2$

$$E_\omega^{\nu_i} T_{\nu_{i+1}} \;\le\; 2(\ln n)^4 + 2(\ln n)^4\exp(H_i).$$

Proof of Lemma 2.1.1. We note that the first inequality of the statement follows directly by the definition. Further, for all $j \in \mathbb{N}_0$ we have by the definition of the ladder locations $(\nu_k)_{k\in\mathbb{N}_0}$ (cf. (1.5.1))

$$\max_{0\le k\le j}\Big(V(j+1) - V(k)\Big) \;=\; V(j+1) - V(\nu_{j_0}).$$

For $\omega \in B_1(n)\cap C$ we therefore get for $j \ge \nu_1$ and n large enough (cf. Lemma 1.5.8 for the definition of $C^-(\omega)$)

$$\begin{aligned}
W_j \;&=\; \sum_{k=-\infty}^{j} \exp\big(V(j+1) - V(k)\big)\\
&\le\; \sum_{k=-\infty}^{-1} \exp\big(V(j+1) - V(k)\big) + \sum_{l=2}^{\left\lceil \frac{j}{\lceil (\ln n)^2\rceil}\right\rceil}\ \sum_{k=j-l\lceil(\ln n)^2\rceil+2}^{j-(l-1)\lceil(\ln n)^2\rceil+1} \exp\big(V(j+1) - V(k)\big)\\
&\quad+ \sum_{k=j-\lceil(\ln n)^2\rceil+2}^{j} \exp\big(V(j+1) - V(k)\big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \exp\big(V(j+1)\big)C^-(\omega) + \lceil(\ln n)^2\rceil \sum_{l=1}^{\left\lceil \frac{j}{\lceil(\ln n)^2\rceil}\right\rceil -1} \exp\big(-l\ln n\big) \\
&\quad + \sum_{k=j-\lceil(\ln n)^2\rceil+2}^{j} \exp\big(V(j+1)-V(k)\big) \\
&\leq 2(\ln n)^2 \exp\big(V(j+1)-V(\nu_{j_0})\big),
\end{aligned}
$$

where we used the definition of the set $B_1(n)$ for the second last inequality and that $\exp\big(V(j)\big) < \exp\big(V(j+1)-V(\nu_{j_0})\big)$ for $j \geq \nu_1$ to obtain the last inequality. Further, for $j = 1, ..., \nu_1 - 1$ and n large enough we have

$$
\max_{j\in\{1,\ldots,\nu_1-1\}} W_j \;\leq\; (\ln n)^2.
$$

■

For environments ω which belong to the "typical" and "good" sets constructed in Section 1.5, we next prove with the help of Lemma 2.1.1 that for $\kappa < 1$ the quenched expectation $E_\omega T_n$ is roughly of size $n^{\frac{1}{\kappa}}$ and that the quenched variance $\mathrm{Var}_\omega(T_n)$ is roughly of size $n^{\frac{2}{\kappa}}$ for $\kappa < 2$.

Proof of Theorem 1.4.5. For environments $\omega \in B(n)$ we have due to (1.2.3) and (2.1.3) (cf. (1.5.4) for the definition of n_0)

$$
\begin{aligned}
E_\omega T_n \;&=\; n + 2\sum_{i=0}^{n-1} W_i \;\leq\; n + 2\sum_{j=0}^{n_0} \big((\nu_{j+1}\wedge n) - \nu_j\big)(\ln n)^2\Big(1 + 2\exp(H_j)\Big) \\
&\leq\; 3n(\ln n)^2 + 4(\ln n)^4 \sum_{l=0}^{n_0} \exp(H_l) \qquad (2.1.4)
\end{aligned}
$$

because for $\omega \in B_1(n)$ every block in the interval $[0,n]$ is smaller than $(\ln n)^2$.

Next, we recall that for environments $\omega \in D(n,m)$ (cf. (1.5.12) for the definition) we have for all $i \in \{1, ..., m-1\}$ at most $n^{1-\frac{i}{m}}$ blocks with a height of more than $\frac{i}{\kappa m}(\ln n + 2\ln\ln n)$. We therefore get for environments $\omega \in B(n) \cap\ D(n,m)$

$$
\begin{aligned}
\sum_{l=0}^{n_0} \exp\big(H_l\big) &\leq \sum_{i=0}^{m-1}\sum_{l=0}^{n_0} \exp\big(H_l\big)\mathbb{1}_{\left\{\frac{i}{\kappa m}(\ln n+2\ln\ln n)\leq H_l \leq \frac{i+1}{\kappa m}(\ln n+2\ln\ln n)\right\}} \\
&\leq \sum_{i=0}^{m-1} \exp\left(\frac{i+1}{\kappa m}(\ln n + 2\ln\ln n)\right) n^{1-\frac{i}{m}} \\
&\leq (\ln n)^{\frac{2}{\kappa}} \sum_{i=0}^{m-1} n^{\frac{(i+1)+\kappa(m-i)}{\kappa m}}, \qquad (2.1.5)
\end{aligned}
$$

where we additionally used that for environments in $B(n)$ the highest block in $[0,n]$ is smaller than $\frac{1}{\kappa}(\ln n + 2\ln\ln n)$ due to the definition of $B_3(n)$.

Let $\delta > 0$ be arbitrary and m large enough, such that

$$\frac{1}{m} \;<\; \frac{\delta}{2}. \tag{2.1.6}$$

Further, we note that the function $(i+1)+\kappa(m-i)$ is increasing in i for $\kappa \leq 1$ and decreasing for $\kappa > 1$. Therefore, (2.1.4) and (2.1.5) yield for environments $\omega \in B(n) \cap\ D(n,m)$ and n large enough

$$\begin{aligned}
E_\omega T_n \;&\leq\; 3n(\ln n)^2 + 4(\ln n)^{4+\frac{2}{\kappa}} \sum_{i=0}^{m-1} n^{\frac{(i+1)+\kappa(m-i)}{\kappa m}} \\
&\leq\; \begin{cases} 3n(\ln n)^2 + 4m(\ln n)^{4+\frac{2}{\kappa}} n^{\frac{m+\kappa}{\kappa m}} & \text{if } \ \kappa \leq 1 \\ 3n(\ln n)^2 + 4m(\ln n)^{4+\frac{2}{\kappa}} n^{\frac{1+\kappa m}{\kappa m}} & \text{if } \ \kappa > 1 \end{cases} \\
&\leq\; \begin{cases} n^{\frac{1}{\kappa}+\delta} & \text{if } \ \kappa \leq 1 \\ n^{1+\delta} & \text{if } \ \kappa > 1. \end{cases}
\end{aligned}$$

Since $\delta > 0$ was arbitrary, Lemma 1.5.9 yields

$$\limsup_{n\to\infty} \frac{\ln E_\omega(T_n)}{\ln n} \;\leq\; \max\left\{\frac{1}{\kappa}, 1\right\} \tag{2.1.7}$$

for $\mathbf{P}$-almost every environment ω.

To obtain the lower bound, we note that $T_n \geq n$ and therefore in the case $\kappa \geq 1$ there is nothing to show. For the case $\kappa < 1$ we can use that for environments $\omega \in B_4(n)$ we know that in the interval $[0,n]$ there exists a block with a height of more than $\frac{1}{\kappa}(\ln n - 4\ln\ln n)$ and we therefore have

$$E_\omega T_n \;=\; n + 2\sum_{i=0}^{n-1} W_i \;\geq\; (\ln n)^{-\frac{4}{\kappa}} n^{\frac{1}{\kappa}}.$$

Thus, Lemma 1.5.9 together with (2.1.7) finishes the proof of part (a).

Next, we analyse $\mathrm{Var}_\omega T_n$. For $j = 0, ..., n$ we define

$$H_j^r(n) := \max_{j\leq k\leq j+\lceil(\ln n)^2\rceil} (V(k+1) - V(j+1))$$

as the biggest increase of the potential in a neighbourhood of size $(\ln n)^2$ to the right of position j. We get for the quenched variance of T_n by changing the order of the summation in equation (2.1.2) for environments $\omega \in B(n)$ and n large enough (cf. Lemma

1.5.8 for the definition of $C^+(\omega)$ and $D^-(\omega)$)

$$\begin{aligned}
\mathrm{Var}_\omega(T_n) &= 4\sum_{j=0}^{n-1}\left(W_j+W_j^2\right)+8\sum_{j=0}^{n-1}\sum_{i=-\infty}^{-1}\exp\left(V(j+1)-V(i+1)\right)\left(W_i+W_i^2\right)\\
&\quad +8\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\exp\left(V(j+1)-V(i+1)\right)\left(W_i+W_i^2\right)\\
&= 4\sum_{j=0}^{n-1}\left(W_j+W_j^2\right)+8D^-(\omega)\sum_{j=0}^{n-1}\exp(V(j+1))\\
&\quad +8\sum_{i=0}^{n-2}\left(\sum_{j=i+1}^{i+\lceil(\ln n)^2\rceil-1}\exp\left(V(j+1)-V(i+1)\right)\right.\\
&\qquad\qquad \left. +\sum_{j=i+\lceil(\ln n)^2\rceil}^{n-1}\exp\left(V(j+1)-V(i+1)\right)\right)\left(W_i+W_i^2\right)\\
&\le 4\sum_{j=0}^{n-1}\left(W_j+W_j^2\right)+8D^-(\omega)C^+(\omega)\\
&\quad +8\sum_{i=0}^{n-2}\left((\ln n)^2\exp\left(H_i^r\right)+(\ln n)^2\sum_{k=1}^{\infty}n^{-k}\right)\left(W_i+W_i^2\right)\\
&\le 12(\ln n)^2\sum_{i=0}^{n-1}\left(1+\exp\left(H_i^r\right)\right)\left(W_i+W_i^2\right).
\end{aligned}\tag{2.1.8}$$

We notice that for all $i=0,...,n$ we have

$$\begin{aligned}
(V(i+1)-V(\nu_{i_0}))+H_i^r(n) &= \max_{i\le k<i+\lceil(\ln n)^2\rceil}(V(k+1)-V(\nu_{i_0}))\\
&\le \max_{i_0\le k\le i_0+\lceil(\ln n)^2\rceil}H_k.
\end{aligned}$$

Therefore, Lemma 2.1.1 yields for $\omega\in B(n)$ and n large enough as an upper bound for (2.1.8)

$$\begin{aligned}
\mathrm{Var}_\omega(T_n) &\le 24(\ln n)^6\sum_{i=0}^{n-1}\left(1+\exp\left(H_i^r\right)\right)\left(1+2\exp\left(V(i+1)-V(\nu_{i_0})\right)\right)^2\\
&\le 24(\ln n)^6\sum_{i=0}^{n-1}\left(1+\exp\left(H_i^r\right)\right)\left(2+8\exp\left(2\left(V(i+1)-V(\nu_{i_0})\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq 192(\ln n)^6 \sum_{i=0}^{n-1} \Bigg(\exp\left(H_i^r + 2\left(V(i+1) - V(\nu_{i_0})\right)\right) + \exp\left(H_i^r\right) \\
&\qquad\qquad + \exp\left(2\left(V(i+1) - V(\nu_{i_0})\right)\right)\Bigg) + 48(\ln n)^6 n \\
&\leq 576(\ln n)^{10} \sum_{l=0}^{n_0} \exp\left(2H_l\right) + 48(\ln n)^6 n. \qquad (2.1.9)
\end{aligned}
$$

Analogously to (2.1.5), we get for environments $\omega \in B(n) \cap\ D(n,m)$

$$
\begin{aligned}
\sum_{l=0}^{n_0} \exp\left(2H_l\right) &\leq \sum_{i=0}^{m-1} \sum_{l=0}^{n_0} \exp\left(2H_l\right) \mathbb{1}_{\left\{\frac{i}{\kappa m}(\ln n + 2\ln\ln n) \leq H_l \leq \frac{i+1}{\kappa m}(\ln n + 2\ln\ln n)\right\}} \\
&\leq \sum_{i=0}^{m-1} \exp\left(2\frac{i+1}{\kappa m}(\ln n + 2\ln\ln n)\right) n^{1-\frac{i}{m}} \\
&\leq (\ln n)^{\frac{4}{\kappa}} \sum_{i=0}^{m-1} n^{\frac{2(i+1)+\kappa(m-i)}{\kappa m}}. \qquad (2.1.10)
\end{aligned}
$$

Let $\delta > 0$ be arbitrary and m chosen as in (2.1.6). Note that the function $2(i+1)+\kappa(m-i)$ is increasing in i for $\kappa \leq 2$ and decreasing for $\kappa > 2$. We therefore get as an upper bound for $\mathrm{Var}_\omega T_n$, using equations (2.1.9) and (2.1.10) for environments $\omega \in B(n) \cap D(n,m)$, and n large enough

$$
\begin{aligned}
\mathrm{Var}_\omega\left(T_n\right) &\leq 576(\ln n)^{10+\frac{4}{\kappa}} \sum_{i=0}^{m-1} n^{\frac{2(i+1)+\kappa(m-i)}{\kappa m}} + 48(\ln n)^6 n \\
&\leq \begin{cases} 576m(\ln n)^{10+\frac{4}{\kappa}} n^{\frac{2m+\kappa}{\kappa m}} + 48(\ln n)^6 n & \text{if } \kappa \leq 2 \\ 576m(\ln n)^{10+\frac{4}{\kappa}} n^{\frac{2+\kappa m}{\kappa m}} + 48(\ln n)^6 n & \text{if } \kappa > 2 \end{cases} \\
&\leq \begin{cases} n^{\frac{2}{\kappa}+\delta} & \text{if } \kappa \leq 2 \\ n^{1+\delta} & \text{if } \kappa > 2. \end{cases} \qquad (2.1.11)
\end{aligned}
$$

Since $\delta > 0$ was arbitrary, (2.1.11) together with Lemma 1.5.9 yields

$$
\limsup_{n\to\infty} \frac{\ln \mathrm{Var}_\omega(T_n)}{\ln n} \leq \max\left\{\frac{2}{\kappa}, 1\right\} \qquad (2.1.12)
$$

for **P**-almost every environment ω.

Now we turn to the lower bound. We first consider the case $\kappa < 2$. For environments $\omega \in B_4(n)$ we have at least one block with a height of more than $\frac{1}{\kappa}(\ln n - 4\ln\ln n)$. Together with (2.1.2) this yields for environments $\omega \in B_4(n)$

$$
\mathrm{Var}_\omega(T_n) \geq \max_{0\leq i\leq n-1} W_i^2 \geq \exp\left(\frac{2}{\kappa}(\ln n - 4\ln\ln n)\right) = (\ln n)^{-\frac{8}{\kappa}} n^{\frac{2}{\kappa}}. \qquad (2.1.13)
$$

For $\kappa \geq 2$ we have

$$\mathrm{Var}_\omega(T_n) \;\geq\; \sum_{i=0}^{n-1} W_i \;\geq\; \sum_{i=0}^{n-1} \exp\big(V(i+1) - V(i)\big) \;=\; \sum_{i=0}^{n-1} \rho_i$$

and applying the SLLN we get

$$\mathrm{Var}_\omega(T_n) \;\geq\; \frac{1}{2} n \mathbf{E}\rho_0 \quad \mathbf{P} - a.s. \tag{2.1.14}$$

for all n large enough. Note that due to Jensen's inequality we have $\mathbf{E}\rho_0 \leq (\mathbf{E}\rho_0^\kappa)^{\frac{1}{\kappa}} = 1$ for $\kappa > 1$.

Lemma 1.5.9 combined with (2.1.13) and (2.1.14) shows that for $\mathbf{P}$-almost every environment ω we have

$$\liminf_{n\to\infty} \frac{\ln \mathrm{Var}_\omega(T_n)}{\ln n} \;\geq\; \max\left\{\frac{2}{\kappa}, 1\right\},$$

which together with (2.1.12) finishes the proof of part (b). ■

2.2 Lower bound for the quenched variance of the crossing time of high blocks

In this section, we analyse the crossing time of high blocks. Later, we will use the results of this section in particular for the deepest block in the interval $[0, n]$. We prove that the quenched expectation and the quenched variance of the crossing time of high blocks are of the same order. This will be important for the proof that there is no cutoff in the case of $\kappa < 1$.

First, we define a modified RWRE which we force not to backtrack too far. Let $(\widetilde{X}_k^{(n)})_{k\in\mathbb{N}_0}$ be the random walk which has the same transition probabilities as $(X_k)_{k\in\mathbb{N}_0}$ with the following additional condition: After reaching a new block ν_k for the first time, the process forms from that time on a random walk in the environment $\widetilde{\omega}^k$, which is defined by

$$\widetilde{\omega}_i^k := \begin{cases} 1 & \text{for } i = \nu_{(k-\lceil(\ln n)^2\rceil)\vee 0}, \\ \omega_i & \text{else.} \end{cases}$$

Now, the transition probabilities stay the same until the process reaches the next new block to the right. From that time on, the process forms a random walk in the environment $\widetilde{\omega}^{k+1}$ and so on. Due to this definition, the process $(\widetilde{X}_k^{(n)})_{k\in\mathbb{N}_0}$ cannot backtrack more than $\lceil(\ln n)^2\rceil$ blocks after reaching a new block for the first time.

Note that there exists a coupling of the processes $(X_k)_{k\in\mathbb{N}_0}$ and $(\widetilde{X}_k^{(n)})_{k\in\mathbb{N}_0}$ such that we have

$$\widetilde{X}_k^{(n)} \;\geq\; X_k$$

for all $k \in \mathbb{N}_0$ with equality holding until the process $(X_k)_{k\in\mathbb{N}_0}$ backtracks more than $\lceil \ln(n)^2 \rceil$ blocks for the first time. For $n \in \mathbb{N}$ we define

$$\widetilde{T}_n^{(r)} : \, \inf\Big\{k \;:\; \widetilde{X}_k^{(r)} = n\Big\} \tag{2.2.1}$$

as the first time the restricted process which backtracks not more than $\lceil (\ln r)^2 \rceil$ blocks hits position n. We further define

$$A(n) := \Big\{ \; T_n \;=\; \widetilde{T}_n^{(n)} \; \Big\} \tag{2.2.2}$$

as the event that the random walk with reflection $(\widetilde{X}_k^{(n)})_{k\in\mathbb{N}_0}$ reaches position n at the same time as the random walk $(X_k)_{k\in\mathbb{N}_0}$. The next lemma shows that for analysing the distance in total variation of the distribution of the RWRE and its stationary distribution it is sufficient to consider the distribution of $(\widetilde{X}_k^{(n)})_{k\in\mathbb{N}_0}$.

Lemma 2.2.1 (cf. Lemma 4.5 in [PZ09])**.** *For all* $k \in \{1, ..., n\}$ *we have for* $\boldsymbol{P}$*-almost every environment* ω *(cf.* (1.1.1) *for the definition of* $\tilde{\omega}$*)*

$$\lim_{n\to\infty} P_{\tilde{\omega}}^k\Big(A(n)^c\Big) \;=\; 0.$$

Proof. We get for all $k \in \{1, ..., n\}$ and all $\varepsilon > 0$ using the Markov inequality

$$\mathbf{P}\left(P_{\tilde{\omega}}^k\left(A(n)^c\right) > \varepsilon\right) \;\leq\; \frac{1}{\varepsilon}\mathbf{P}\left(P_{\tilde{\omega}}^k\left(A(n)^c\right)\right). \tag{2.2.3}$$

The event $A(n)^{\mathrm{c}}$ only occurs if the random walk $(X_t)_{t\in\mathbb{N}_0}$ does at least $(\ln n)^2$ steps to the left. We therefore get

$$\begin{aligned}
\mathbf{P}\Big(P_{\tilde{\omega}}^k\left(A(n)^{\mathrm{c}}\right)\Big) &\leq \sum_{i=(\lceil(\ln n)^2\rceil)\vee k}^{n-1} \mathbf{P}\Big(P_{\tilde{\omega}}^i\left(T_{\lfloor i-(\ln n)^2\rfloor} < \infty\right)\Big) \\
&\leq\; \big(n-(\ln n)^2\big)\cdot\mathbf{P}\left(P_{\omega}^{\lceil(\ln n)^2\rceil}\left(T_0 < \infty\right)\right),
\end{aligned}$$

where we used for the last inequality that the sequence $(\rho_k)_{k\in\mathbb{Z}}$ is i.i.d. with respect to $\mathbf{P}$. Due to Lemma 3.3 in [GS02], there exists a constant $c > 0$ such that

$$\mathbf{P}\left(P_{\omega}^{\lceil(\ln n)^2\rceil}\left(T_0 < \infty\right)\right) \;\leq\; \exp\big(-c(\ln n)^2\big).$$

We thus showed that the probabilities in (2.2.3) are summable and an application of the Borel-Cantelli lemma finishes the proof. ∎

Lemma 2.2.2. *For environments*

$$\omega \in F(\nu_{n+1}) := B(\nu_{n+1}) \cap E\left(\nu_{n+1}, \frac{2}{3}\right) \cap C, \tag{2.2.4}$$

n large enough such that $C^-(\omega) < \ln n$ and blocks with

$$H_n \;>\; \frac{3}{4\kappa}\ln(\nu_{n+1}) \tag{2.2.5}$$

we have

$$\left(E_\omega^{\nu_n} T_{\nu_{n+1}}\right)^2 \;\leq\; 4Var_\omega\big(T_{\nu_{n+1}} - T_{\nu_n}\big).$$

Further, we have $\left(E_{\widetilde{\omega}}^{\nu_n} T_{\nu_{n+1}}\right)^2 \leq 4Var_{\widetilde{\omega}}\big(T_{\nu_{n+1}} - T_{\nu_n}\big)$ (cf. (1.1.1) for the definition of $\widetilde{\omega}$) and $\left(E_\omega^{\nu_n} \widetilde{T}_{\nu_{n+1}}^{(n)}\right)^2 \leq 4Var_\omega\left(\widetilde{T}_{\nu_{n+1}}^{(n)} - \widetilde{T}_{\nu_n}^{(n)}\right)$ (cf. (2.2.1) for the definition of $\widetilde{T}_{\nu_n}^{(n)}$).

Proof. First, we note that due to (1.2.3) we have

$$\begin{aligned}
&\left(E_\omega^{\nu_n} T_{\nu_{n+1}}\right)^2 \\
&= \left((\nu_{n+1} - \nu_n) + 2\sum_{j=\nu_n}^{\nu_{n+1}-1} W_j\right)^2 \\
&= (\nu_{n+1} - \nu_n)^2 + 4(\nu_{n+1} - \nu_n)\sum_{j=\nu_n}^{\nu_{n+1}-1} W_j + 4\sum_{j=\nu_n}^{\nu_{n+1}-1} W_j^2 + 8\sum_{j=\nu_n}^{\nu_{n+1}-2}\sum_{l=j+1}^{\nu_{n+1}-1} W_j W_l.
\end{aligned}$$

Together with (2.1.1) this yields

$$\begin{aligned}
&\left(E_\omega^{\nu_n} T_{\nu_{n+1}}\right)^2 - Var_\omega(T_{\nu_{n+1}} - T_{\nu_n}) \\
&= \big(\nu_{n+1} - \nu_n\big)^2 + 4\big(\nu_{n+1} - \nu_n - 1\big)\sum_{j=\nu_n}^{\nu_{n+1}-1} W_j && (2.2.6)\\
&\quad + 8\sum_{j=\nu_n}^{\nu_{n+1}-2}\sum_{l=j+1}^{\nu_{n+1}-1} W_j\Big(W_l - \exp\big(V(l+1) - V(j+1)\big)\big(1 + W_j\big)\Big) && (2.2.7)\\
&\quad - 8\sum_{i=-\infty}^{\nu_n-1}\sum_{j=\nu_n}^{\nu_{n+1}-1} \exp\big(V(j+1) - V(i+1)\big)\left(W_i + W_i^2\right). && (2.2.8)
\end{aligned}$$

Note that (2.2.8) is negative, since all terms of the sums are positive and therefore can be neglected on our way to find an upper bound.

Next, recall that for $\omega \in B_1(\nu_{n+1})$ blocks in the interval $[-\nu_{n+1}, \nu_{n+1}]$ are smaller than

$(\ln \nu_{n+1})^2$, and thus we obtain using Lemma 2.1.1 for $\omega \in F(\nu_{n+1})$ the following upper bound for (2.2.6)

$$\begin{aligned}
\big(\nu_{n+1}-\nu_n\big)^2 + 4\big(\nu_{n+1}-\nu_n-1\big)\sum_{j=\nu_n}^{\nu_{n+1}-1} W_j \ &\leq \ \big(\ln \nu_{n+1}\big)^4 + 4\big(\ln \nu_{n+1}\big)^6\left(1+2\exp\big(H_n\big)\right) \\
&\leq \ \exp\left(2H_n\right) \\
&\leq Var_\omega\big(T_{\nu_{n+1}} - T_{\nu_n}\big). \qquad (2.2.9)
\end{aligned}$$

Furthermore, we observe that for $j < l$ we have

$$\begin{aligned}
& W_l - \exp\big(V(l+1)-V(j+1)\big)\big(1+W_j\big) \\
= & \sum_{k=-\infty}^{l} \exp\big(V(l+1)-V(k)\big) \\
& - \exp\big(V(l+1)-V(j+1)\big)\left(1+\sum_{k=-\infty}^{j}\exp\big(V(j+1)-V(k)\big)\right) \\
= & \begin{cases} \displaystyle\sum_{k=j+2}^{l} \exp\big(V(l+1)-V(k)\big) & \text{if } j < l-1 \\ 0 & \text{if } j = l-1. \end{cases}
\end{aligned}$$

This simplifies (2.2.7) for environments $\omega \in F(\nu_{n+1})$ to

$$\begin{aligned}
& 8\sum_{j=\nu_n}^{\nu_{n+1}-2}\sum_{l=j+1}^{\nu_{n+1}-1} W_j\Big(W_l - \exp\big(V(l+1)-V(j+1)\big)\big(1+W_j\big)\Big) \\
= \ & 8\sum_{j=\nu_n}^{\nu_{n+1}-3}\sum_{l=j+2}^{\nu_{n+1}-1} W_j \sum_{k=j+2}^{l}\exp\big(V(l+1)-V(k)\big) \\
\leq \ & 8\ln(\nu_{n+1})^4\exp\big(H_n\big)\sum_{j=\nu_n}^{\nu_{n+1}-3} W_j \mathbb{1}_{\left\{W_j \leq \frac{\exp(H_n)}{8\ln(\nu_{n+1})^6}\right\}} \\
& + 8\sum_{j=\nu_n}^{\nu_{n+1}-3} W_j \mathbb{1}_{\left\{W_j > \frac{\exp(H_n)}{8\ln(\nu_{n+1})^6}\right\}} \sum_{l=j+2}^{\nu_{n+1}-1}\sum_{k=j+2}^{l}\exp\big(V(l+1)-V(k)\big) \\
\leq \ & \exp\big(2H_n\big) + 8\sum_{j=\nu_n}^{\nu_{n+1}-3} W_j \mathbb{1}_{\left\{W_j > \frac{\exp\left(H_n\right)}{8\ln(\nu_{n+1})^6}\right\}} \sum_{l=j+2}^{\nu_{n+1}-1}\sum_{k=j+2}^{l}\exp\big(V(l+1)-V(k)\big).
\end{aligned}$$

(2.2.10)

To get an upper bound for (2.2.7), we therefore have to control the last two sums in (2.2.10). We note that for an environment $\omega \in F(\nu_{n+1})$ and $\nu_n < j < \nu_{n+1}$ with $W_j > \frac{\exp(H_n)}{8\ln(\nu_{n+1})^6}$ we have

$$\exp\big(V(j+1) - V(\nu_n)\big) \;\geq\; \frac{\exp\big(H_n\big)}{17\ln(\nu_{n+1})^8}$$

because otherwise

$$\begin{aligned} W_j \;&=\; \sum_{k=-\infty}^{j} \exp\big(V(j+1) - V(k)\big) \\ &\leq \sum_{k=-\infty}^{\nu_n - \lceil(\ln\nu_n)^2\rceil} \exp\big(V(j+1) - V(k)\big) + \lceil(\ln\nu_n)^2\rceil \\ &\quad + (j+1-\nu_n + \lceil(\ln\nu_n)^2\rceil)\exp\big(V(j+1) - V(\nu_n)\big) \\ &\leq 2(\ln\nu_n)^2 + 2(\ln\nu_n)^2 \frac{\exp\big(H_n\big)}{17\ln(\nu_{n+1})^8} \\ &\leq \frac{\exp\big(H_n\big)}{8\ln(\nu_{n+1})^6}. \end{aligned}$$

Thus, we get for $\omega \in F(\nu_{n+1})$ using assumption (2.2.5)

$$V(j+1) - V(\nu_n) \;\geq\; \frac{2}{3\kappa}\ln(\nu_{n+1}). \tag{2.2.11}$$

This yields the following upper bound for the summands in (2.2.10) for environments $\omega \in F(\nu_{n+1})$

$$\begin{aligned} &8W_j\mathbb{1}_{\left\{W_j > \frac{\exp(H_n)}{8\ln(\nu_{n+1})^6}\right\}} \sum_{l=j+2}^{\nu_{n+1}-1}\sum_{k=j+2}^{l} \exp\big(V(l+1) - V(k)\big) \\ &\leq 8W_j\mathbb{1}_{\left\{W_j > \frac{\exp(H_n)}{8\ln(\nu_{n+1})^6}\right\}} \ln(\nu_{n+1})^4\,(\nu_{n+1})^{\frac{1}{2\kappa}} \\ &\leq W_j^2\mathbb{1}_{\left\{W_j > \frac{\exp(H_n)}{8\ln(\nu_{n+1})^6}\right\}}, \end{aligned} \tag{2.2.12}$$

where we used that for $\omega \in E\left(\nu_{n+1}, \frac{2}{3}\right)$ the potential does not increase more than $\frac{1}{2\kappa}\ln(\nu_{n+1})$ on the interval $\{j, ..., \nu_{n+1}\}$ due to (2.2.11).

For environments $\omega \in F(\nu_{n+1})$, (2.2.10) and (2.2.12) together now imply the following upper bound for (2.2.7)

$$\begin{aligned}
&8\sum_{j=\nu_n}^{\nu_{n+1}-2}\sum_{l=j+1}^{\nu_{n+1}-1} W_j\Big(W_l - \exp\big(V(l+1)-V(j+1)\big)\big(1+W_j\big)\Big)\\
&\le \exp\big(2H_n\big) + \sum_{j=\nu_n}^{\nu_{n+1}-3} W_j^2\\
&\le 2\mathrm{Var}_\omega\big(T_{\nu_{n+1}} - T_{\nu_n}\big). \qquad (2.2.13)
\end{aligned}$$

Therefore, (2.2.9) and (2.2.13) finally yield

$$\Big(E_\omega^{\nu_n}(T_{\nu_{n+1}})\Big)^2 - \mathrm{Var}_\omega\big(T_{\nu_{n+1}} - T_{\nu_n}\big) \le 3\mathrm{Var}_\omega\big(T_{\nu_{n+1}} - T_{\nu_n}\big).$$

Note that the proof of the statement is the same for $\widetilde{\omega}$ instead of ω and for $\widetilde{T}^{(n)}_{\nu_{n+1}}$ instead of $T_{\nu_{n+1}}$. The considered quenched expectation in these cases is even smaller. In line (2.2.8), the number of summands is different in these cases but since they are all negative we can use the same upper bound 0 for all cases. ■

Lemma 2.2.3. *For $\boldsymbol{P}$-almost every environment ω we have*

$$\limsup_{n\to\infty} \frac{E_{\widetilde{\omega}}(T_n) - E_{\widetilde{\omega}}\left(T_{n-\lceil 2(\ln n)^2\rceil}\right)}{\sqrt{\mathrm{Var}_{\widetilde{\omega}}\left(T_n\right)}} \le 2.$$

Proof. First, we note that for environments $\omega \in E\left(n, \frac{2}{3}\right)$ we have at most one increase of the potential of more than $\frac{2}{3\kappa}\ln n$ in the interval $[n - \lceil 2(\ln n)^2\rceil, n]$. Therefore, we get for $\omega \in B(n) \cap E\left(n, \frac{2}{3}\right) \cap C$ and n large enough

$$\begin{aligned}
\frac{E_{\widetilde{\omega}}^{a_n}(T_n)}{\sqrt{\mathrm{Var}_{\widetilde{\omega}}\left(T_n\right)}} &\le \frac{1}{\sqrt{\mathrm{Var}_{\widetilde{\omega}}\left(T_n\right)}} \sum_{k=n-\lceil 2(\ln n)^2\rceil}^{n-1} E_{\widetilde{\omega}}^k T_{k+1}\left(\mathbb{1}_{\left\{H_{k_0}\le \frac{2}{3\kappa}\ln n\right\}} + \mathbb{1}_{\left\{H_{k_0}> \frac{2}{3\kappa}\ln n\right\}}\right)\\
&\le \frac{1}{\sqrt{\mathrm{Var}_{\widetilde{\omega}}\left(T_n\right)}}\left(\lceil 2(\ln n)^2\rceil(\ln n)^2\left(n^{\frac{2}{3\kappa}}+1\right) + 2\sqrt{\mathrm{Var}_{\widetilde{\omega}}\left(T_n\right)}\right),
\end{aligned}$$

where we additionally used Lemma 2.1.1 and 2.2.2 to obtain the last line. Lemma 1.5.9 and Theorem 1.4.5 (note that Theorem 1.4.5 is also true for $\widetilde{\omega}$) now finish the proof. ■

2.3 Uniform Integrability

Lemma 2.3.1. *Assume Assumptions 1 and 2. For any $\varepsilon < \frac{1}{3}$, there exists an $\eta > 0$ such that for*

$$A_n := \left\{ \exists\, i, j \in \mathbb{N},\ 1 \le i \le n \ :\ H_i \ > \ \frac{1-\varepsilon}{\kappa} \ln n,\ E_\omega^{\nu_i} \left(\widetilde{T}_{\nu_{i+1}}^{(n)} \right)^j \ > \ 3j! 2^j \left(E_\omega^{\nu_i} \widetilde{T}_{\nu_{i+1}}^{(n)} \right)^j \right\}$$

we have (cf. (1.5.2) *for the definition of* $\boldsymbol{Q}$*)*

$$\boldsymbol{Q}(A_n) \ = \ o\left(n^{-\eta}\right).$$

Proof. The proof of this Lemma is a refinement of the proof of Lemma 5.9 in [PZ09], where they additionally use that the distribution of $\ln \rho_0$ is non-lattice. We first consider a modification of the environment. For $x \in \mathbb{Z}$ we define $\omega^{(x)}$ by

$$\omega_i^{(x)} := \begin{cases} 1 & \text{if } i = x, \\ \omega_i & \text{if } i \neq x \end{cases}$$

as the environment which is the same as ω with an added reflection at x. Therefore, $(X_n)_{n \in \mathbb{N}_0}$ forms under $P_{\omega^{(x)}}^x$ a random walk in the environment ω with a reflection at its starting point x.

Let $(X_t^\star)_{t \ge 0}$ be the continuous time version of $(X_n)_{n \in \mathbb{N}_0}$ constructed in such a way that there exists a family $(\zeta_k)_{k \in \mathbb{N}}$ of independent exponentially random variables of parameter 1 indicating the times of the jumps of $(X_t^\star)_{t \ge 0}$. Therefore, we can couple $(X_n)_{n \in \mathbb{N}_0}$ and $(X_t^\star)_{t \ge 0}$ such that for all $k \in \mathbb{N}$ we have

$$X_{\Upsilon_k}^\star \ = \ X_k, \quad \text{where } \Upsilon_k := \sum_{i=1}^k \zeta_i.$$

Additionally, we choose the family $(\zeta_k)_{k \in \mathbb{N}}$ to be independent of the environment and define $T_k^\star$ as the hitting time of position k of $(X_t^\star)_{t \ge 0}$. Note that

$$T_k^\star \ = \ \sum_{i=1}^{T_k} \zeta_i,$$

and thus, using Wald's identity, we have for all $k > x$

$$E_{\omega^{(x)}}^x T_k^\star \ = \ E_{\omega^{(x)}}^x T_k. \tag{2.3.1}$$

Further, we note that for $n \in \mathbb{N}$ the sum of n independent exponentially distributed random variables is Erlang distributed with parameters 1 and n, and due to Lemma A.1 we have for all $n \in \mathbb{N}$

$$P_{\omega^{(x)}}^x \left(\sum_{i=1}^n \zeta_i \ \ge \ n \right) \ \ge \ \frac{1}{3}.$$

Therefore, we get for all $k > x$ and all $j \in \mathbb{N}$

$$\begin{aligned} P^x_{\omega^{(x)}}\left(T^\star_k \ > \ T_k\right) \ &= \ \sum_{l=k-x}^{\infty} P^x_{\omega^{(x)}}\left(T^\star_k \ > \ T_k \ \middle| \ T_k = l\right) \cdot P^x_{\omega^{(x)}}\left(T_k = l - x\right) \\ &= \ \sum_{l=k-x}^{\infty} P^x_{\omega^{(x)}}\left(\sum_{i=1}^{l} \zeta_i \ > \ l\right) \cdot P^x_{\omega^{(x)}}\left(T_k = l\right) \\ &\geq \frac{1}{3}, \end{aligned}$$

and this yields for $t > 0$

$$\begin{aligned} \frac{1}{3} P^x_{\omega^{(x)}}\left(T_k \ > \ t\right) \ &\leq \ P^x_{\omega^{(x)}}\left(T_k \ > \ t\right) \cdot P^x_{\omega^{(x)}}\left(T^\star_k \ > \ T_k\right) \\ &= \ P^x_{\omega^{(x)}}\left(T_k \ > \ t, \ T^\star_k \ > \ T_k\right) \\ &\leq \ P^x_{\omega^{(x)}}\left(T^\star_k \ > \ t\right). \end{aligned} \tag{2.3.2}$$

Using Kac's moment formula (cf. (6) in [FP99]) and the Markov property, we obtain for $k > x$

$$E^x_{\omega^{(x)}}\left(T^\star_k\right)^j \ \leq \ j!\left(E^x_{\omega^{(x)}} T^\star_k\right)^j$$

because we have $E^x_{\omega^{(x)}} T^\star_k > E^y_{\omega^{(y)}} T^\star_k$ for all $y \in (x, k)$. Together with (2.3.1) and (2.3.2) this yields

$$\begin{aligned} \frac{1}{3} E^x_{\omega^{(x)}}\left(T_k\right)^j \ &= \ \frac{1}{3} \int_0^\infty j t^{j-1} P^x_{\omega^{(x)}}\left(T_k \ > \ t\right) \ dt \ \leq \ \int_0^\infty j t^{j-1} P^x_{\omega^{(x)}}\left(T^\star_k \ > \ t\right) \ dt \\ &= \ E^x_{\omega^{(x)}}\left(T^\star_k\right)^j \ \leq \ j!\left(E^x_{\omega^{(x)}} T^\star_k\right)^j \ = \ j!\left(E^x_{\omega^{(x)}} T_k\right)^j. \end{aligned} \tag{2.3.3}$$

Next, we define

$$\omega^i_\star := \ \omega^{\left(\nu_{i - \lceil (\ln n)^2 \rceil}\right)},$$

and we get using (2.3.3)

$$\begin{aligned} E^{\nu_i}_\omega\left(\widetilde{T}^{(n)}_{\nu_{i+1}}\right)^j \ &\leq \ E^{\nu_{i-\lceil (\ln n)^2 \rceil}}_{\omega^i_\star}\left(T_{\nu_{i+1}}\right)^j \\ &\leq \ 3j!\left(E^{\nu_{i-\lceil (\ln n)^2 \rceil}}_{\omega^i_\star} T_{\nu_{i+1}}\right)^j \\ &\leq \ 3j!\left(E^{\nu_{i-\lceil (\ln n)^2 \rceil}}_{\omega^i_\star} T_{\nu_i} + E^{\nu_i}_\omega \widetilde{T}^{(n)}_{\nu_{i+1}}\right)^j. \end{aligned}$$

Due to the shift invariance of $\mathbf{Q}$ for ladder locations, this yields

$$\mathbf{Q}(A_n) \ \leq \mathbf{Q}\left(\exists \ i \leq n \ : \ H_i > \frac{1-\varepsilon}{\kappa} \ln n, \ E^{\nu_{i-\lceil (\ln n)^2 \rceil}}_{\omega^i_\star} T_{\nu_i} > E^{\nu_i}_\omega \widetilde{T}^{(n)}_{\nu_{i+1}}\right)$$

$$\leq n\mathbf{Q}\left(H_1 > \frac{1-\varepsilon}{\kappa}\ln n,\ E_\omega^{\nu_{-\lceil(\ln n)^2\rceil}}\widetilde{T}_0^{(n)} > n^{\frac{1-\varepsilon}{\kappa}}\right)$$
$$= n\mathbf{Q}\left(H_1 > \frac{1-\varepsilon}{\kappa}\ln n\right)\mathbf{Q}\left(E_\omega^{\nu_{-\lceil(\ln n)^2\rceil}}\widetilde{T}_0^{(n)} > n^{\frac{1-\varepsilon}{\kappa}}\right), \qquad (2.3.4)$$

where we used in second last line that

$$E_{\omega_\star^i}^{\nu_{-\lceil(\ln n)^2\rceil}}T_0 \ \leq\ E_\omega^{\nu_{-\lceil(\ln n)^2\rceil}}\widetilde{T}_0^{(n)}$$

and in the last equality that H_1 is independent of the environment to the left side of 0. Now, for $\omega \in B_1(n)$ we have due to (1.2.3)

$$\begin{aligned} E_\omega^{\nu_{-\lceil(\ln n)^2\rceil}}\widetilde{T}_0^{(n)} &= \sum_{k=-\lceil(\ln n)^2\rceil}^{-1} E_\omega^{\nu_k}\widetilde{T}_{\nu_{k+1}}^{(n)} \\ &\leq \lceil(\ln n)^4\rceil + 2\sum_{k=-\lceil(\ln n)^2\rceil}^{-1}\ \sum_{i=\nu_k}^{\nu_{k+1}-1}\ \sum_{j=\nu_{k-\lceil(\ln n)^2\rceil}}^{j} \exp\big(V(i+1)-V(j)\big) \\ &\leq 3(\ln n)^6 \sum_{k=-\lceil(\ln n)^2\rceil}^{-1} \exp\big(H_k\big). \end{aligned} \qquad (2.3.5)$$

Further, we note that $\mathbf{Q}(H_1 > h) = \mathbf{P}(H_1 > h)$. Then, (2.3.5) together with (1.5.7) yields for arbitrary $\varepsilon > 0$ and large n (cf. (1.5.2) for the definition of $\mathcal{R}$)

$$\begin{aligned} &\mathbf{Q}\left(E_\omega^{\nu_{-\lceil(\ln n)^2\rceil}}\widetilde{T}_0 > n^{\frac{1-\varepsilon}{\kappa}}\right) \\ &\leq \mathbf{Q}\left(3(\ln n)^6\sum_{i=-\lceil(\ln n)^2\rceil}^{-1}\exp(H_i) > n^{\frac{1-\varepsilon}{\kappa}}\right) + \mathbf{Q}\left(B_1(n)^{\mathrm{c}}\right) \\ &\leq \ \big((\ln n)^2+1\big)\,\mathbf{Q}\left(H_1 > \frac{1-\varepsilon}{\kappa}\ln n - 7\ln\ln n\right) + \mathbf{P}\left(\mathcal{R}\right)\mathbf{P}\left(B_1(n)^{\mathrm{c}}\right) \\ &\leq 2C_1(\ln n)^{\frac{7}{\kappa}+2}n^{-(1-\varepsilon)}, \end{aligned}$$

where we used that $\mathbf{P}\big(B_1(n)^{\mathrm{c}}\big) = O(n^{-2})$ by Lemma 1.5.1. Plugging this in (2.3.4), we get again using (1.5.7)

$$\mathbf{Q}\big(A_n\big) \ \leq\ 2C_1(\ln n)^{\frac{7}{\kappa}+2}n^{-1+2\varepsilon} \ =\ o\left(n^{-1+3\varepsilon}\right).$$

This finishes the proof for all $\varepsilon < \frac{1}{3}$. ■

To get an asymptotic behaviour with respect to $\mathbf{P}$, we change the set A_n slightly in such a way that it does not depend on the environment to the left side of 0. We get:

Corollary 2.3.2. *Assume Assumptions 1 and 2. For any $\varepsilon < \frac{1}{3}$, there exists an $\eta > 0$ such that for*

$$\widetilde{A}_n := \left\{ \exists\ i, j \in \mathbb{N}, (\ln n)^2 < i \leq n : H_i > \frac{1-\varepsilon}{\kappa} \ln n, E_\omega^{\nu_i} \left(\widetilde{T}_{\nu_{i+1}}^{(n)} \right)^j > 3j!2^j \left(E_\omega^{\nu_i} \widetilde{T}_{\nu_{i+1}}^{(n)} \right)^j \right\}$$

we have

$$\boldsymbol{P}\left(\widetilde{A}_n \right) \ = \ o\left(n^{-\eta}\right).$$

Lemma 2.3.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a (environment dependent) subsequence of $n_k = 2^{2^k}$ and there exists $\varepsilon < \frac{1}{3}$ such that we have for all $n \in \mathbb{N}$*

$$H_{a_{n-1}} \ > \ \frac{1-\varepsilon}{\kappa} \ln(\nu_{a_n}). \tag{2.3.6}$$

Then the sequence

$$\left(\left(\frac{\widetilde{T}_{\nu_{a_n}}^{(a_n)} - \widetilde{T}_{\nu_{a_n-1}}^{(a_n)}}{E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}} \right)^2 \right)_{n\in\mathbb{N}}$$

is uniformly integrable with respect to P_ω for $\boldsymbol{P}$-almost every environment ω.

Proof. At first, we note that the distribution of $\widetilde{T}_{\nu_{a_n}}^{(a_n)} - \widetilde{T}_{\nu_{a_n-1}}^{(a_n)}$ under P_ω is the same as the distribution of $\widetilde{T}_{\nu_{a_n}}^{(a_n)}$ under $P_\omega^{\nu_{a_n-1}}$.

We have for all blocks a_n and all $c > 0$

$$\begin{aligned}
&\frac{1}{\left(E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right)^2} \int\limits_{\left\{\widetilde{T}_{\nu_{a_n}}^{(a_n)} \ > \ \sqrt{c} E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right\}} \left(\widetilde{T}_{\nu_{a_n}}^{(a_n)}\right)^2 dP_\omega^{\nu_{a_n-1}} \\
&= \frac{c\left(E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right)^2 P_\omega^{\nu_{a_n-1}} \left(\widetilde{T}_{\nu_{a_n}}^{(a_n)} \ > \ \sqrt{c} E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right)}{\left(E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right)^2} \\
&\quad + \frac{1}{\left(E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right)^2} \int\limits_{\sqrt{c} E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}}^{\infty} 2t P_\omega^{\nu_{a_n-1}} \left(\widetilde{T}_{\nu_{a_n}}^{(a_n)} \ > \ t\right) dt \\
&= c P_\omega^{\nu_{a_n-1}} \left(\widetilde{T}_{\nu_{a_n}}^{(a_n)} \ > \ \sqrt{c} E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right) + \int\limits_{\sqrt{c}}^{\infty} 2x P_\omega^{\nu_{a_n-1}} \left(\widetilde{T}_{\nu_{a_n}}^{(a_n)} \ > \ x E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right) dx.
\end{aligned} \tag{2.3.7}$$

Due to (2.3.6), Corollary 2.3.2 and the fact that $(a_n)_{n\in\mathbb{N}}$ is growing at least as fast as 2^{2^n}, the Borel-Cantelli lemma yields that for $\mathbf{P}$-almost every environment ω there exists a $N = N(\omega)$ such that we have

$$E_\omega^{\nu_{a_n-1}} \left(\widetilde{T}_{\nu_{a_n}}^{(a_n)}\right)^j \ \leq \ 3j!2^j \left(E_\omega^{\nu_{a_n-1}} \widetilde{T}_{\nu_{a_n}}^{(a_n)}\right)^j$$

for all $j \in \mathbb{N}$ and all $n \geq N$.

We have $\widetilde{T}^{(a_n)}_{\nu_{a_n}} \geq 0$ by the definition, and therefore we obtain for $n \geq N$ and $\mathbf{P}$-almost every environment ω

$$E^{\nu_{a_n-1}}_{\omega} \exp\left(\frac{\widetilde{T}^{(a_n)}_{\nu_{a_n}}}{4E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}}\right) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{E^{\nu_{a_n-1}}_{\omega}\left(\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right)^j}{4^j\left(E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right)^j} \leq \sum_{j=0}^{\infty} \frac{3}{2^j} = 6.$$

Then, applying Markov inequality yields for $n \geq N$ and $\mathbf{P}$-almost every environment ω

$$P^{\nu_{a_n-1}}_{\omega}\left(\widetilde{T}^{(a_n)}_{\nu_{a_n}} > xE_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right) \leq 6\exp\left(-\frac{x}{4}\right), \qquad x \geq 0.$$

Thus, for all $n \geq N$ and $\mathbf{P}$-almost every environment ω we get as an upper bound for (2.3.7)

$$\begin{aligned}
&\frac{1}{\left(E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right)^2} \int\limits_{\left\{\widetilde{T}^{(a_n)}_{\nu_{a_n}} > \sqrt{c}E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right\}} \left(\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right)^2 dP^{\nu_{a_n-1}}_{\omega} \\
&\leq 6c\exp\left(-\frac{\sqrt{c}}{4}\right) + 12\int_{\sqrt{c}}^{\infty} x\exp\left(-\frac{x}{4}\right)dx \\
&= 6c\exp\left(-\frac{\sqrt{c}}{4}\right) + 12\left(4\sqrt{c}+16\right)\exp\left(-\frac{\sqrt{c}}{4}\right) \\
&= 6\exp\left(-\frac{\sqrt{c}}{4}\right)\left(c+8\sqrt{c}+32\right).
\end{aligned}$$

Therefore, we conclude that we have for $\mathbf{P}$-almost every environment ω

$$\begin{aligned}
&\lim_{c\to\infty}\sup_{n\in\mathbb{N}} \int\limits_{\left\{\widetilde{T}^{(a_n)}_{\nu_{a_n}} > \sqrt{c}E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right\}} \left(\frac{\widetilde{T}^{(a_n)}_{\nu_{a_n}}}{E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}}\right)^2 dP^{\nu_{a_n-1}}_{\omega} \\
&= \max_{n\in\{1,\dots,N-1\}}\lim_{c\to\infty} \int\limits_{\left\{\widetilde{T}^{(a_n)}_{\nu_{a_n}} > \sqrt{c}E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right\}} \left(\frac{\widetilde{T}^{(a_n)}_{\nu_{a_n}}}{E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}}\right)^2 dP^{\nu_{a_n-1}}_{\omega} \\
&\quad + \lim_{c\to\infty}\sup_{n\geq N} \int\limits_{\left\{\widetilde{T}^{(a_n)}_{\nu_{a_n}} > \sqrt{c}E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}\right\}} \left(\frac{\widetilde{T}^{(a_n)}_{\nu_{a_n}}}{E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}}\right)^2 dP^{\nu_{a_n-1}}_{\omega} \\
&= 0,
\end{aligned}$$

which shows that

$$\left(\left(\frac{\widetilde{T}^{(a_n)}_{\nu_{a_n}} - \widetilde{T}^{(a_n)}_{\nu_{a_n-1}}}{E^{\nu_{a_n-1}}_{\omega}\widetilde{T}^{(a_n)}_{\nu_{a_n}}}\right)^2\right)_{n\in\mathbb{N}}$$

is uniformly integrable with respect to P_ω for $\mathbf{P}$-almost every environment ω. ■

Chapter 3

Cutoff

In the first section of this chapter, we give an introduction to the cutoff phenomenon and an overview of existing results. Then, we prove our main results Theorem 1.4.1 and Theorem 1.4.2. We show that a sequence of transient lazy RWRE on $(\{0, ..., n\})_{n\in\mathbb{N}}$ exhibits a cutoff under Assumptions 1 and 2 for $\kappa > 1$ and that there is no cutoff for $\kappa < 1$ if we additionally assume Assumption 3. In Section 3.5, we compare our main results with the case of a deterministic environment. Further, in the last section of this chapter we prove that the mixing time of a transient RWRE is roughly of order $n^{\frac{1}{\kappa}}$ for $\kappa \leq 1$ and $\mathbf{P}$-almost surely of order n for $\kappa > 1$.

3.1 Introduction and overview

For each $n \in \mathbb{N}$ let $(U_k^n)_{k\in\mathbb{N}_0}$ be an aperiodic and irreducible Markov chain on a finite state space Ω_n and let $(\pi_n)_{n\in\mathbb{N}}$ denote the sequence of associated stationary distributions. Further, we assume

$$|\Omega_n| \overset{n\to\infty}{\longrightarrow} \infty$$

and recall the definition of $d_n(l)$ (cf. Definition 1.3.2).

From the convergence theorem for irreducible and aperiodic Markov chains, we know that for all $n \in \mathbb{N}$ we have

$$d_n(l) \overset{l\to\infty}{\longrightarrow} 0.$$

Further, we know that $d_n(l)$ is decreasing in l (cf. Exercise 4.4 in [LPW09]) and we have

$$d_n(0) \overset{n\to\infty}{\longrightarrow} 1.$$

We recall that $t_{\mathrm{mix}}(n)$ is defined as the first time at which the distance to stationarity is smaller than $\frac{1}{4}$. In general, we expect that $t_{\mathrm{mix}}(n)$ is growing in n and tends to infinity as n goes to infinity. But the next easy example shows that this is not always the case. Let us consider the simple random walk on the complete graph with n vertices. The random walk moves from its current position to one of its $n-1$ neighbours with probability $\frac{1}{n}$ and stays at its current position with probability $\frac{1}{n}$. The associated stationary distribution is the uniform distribution on the n vertices and we observe the following "unusual" behaviour. Since the distribution after one time step is already the uniform distribution, we have for all $n \in \mathbb{N}$

$$d_n(1) \;=\; 0 \quad \text{and} \quad t_{\text{mix}}(n) \;=\; 1.$$

The properties of d_n mentioned before the example yield a general shape of the transition to stationarity but we will see that the transition still can look very different from example to example. In the following, we give examples which differ in the size of the window in which the transition to stationarity takes place. The quantity we use to distinguish the different cases is

$$\Sigma := \sup_{0<\varepsilon<\frac{1}{2}} \limsup_{n\to\infty} \frac{t^{\varepsilon}_{\text{mix}}(n)}{t^{1-\varepsilon}_{\text{mix}}(n)} \quad \in [1,\infty], \tag{3.1.1}$$

where for $0<c<1$ we define

$$t^{c}_{\text{mix}}(n) := \;\min\left\{k \in \mathbb{N} \;:\; d_n(k) \le c\right\}.$$

A big value of Σ (especially $\Sigma=\infty$) indicates a slow transition to stationary (cf. Figure 3.1).

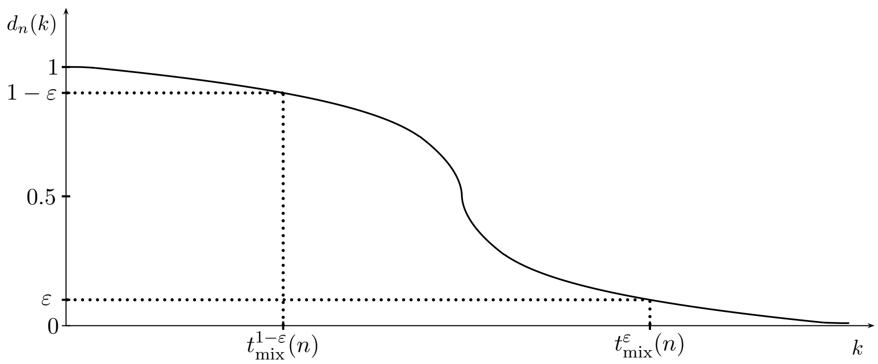

Figure 3.1

1. $\Sigma = \infty$:

This case includes all sequences of chains which show a slow transition to stationarity even if considered on the time-scale of the mixing time $t_{\text{mix}}(n)$. An easy example of a chain for which we have $\Sigma=\infty$ is the simple random walk on a cycle with n vertices. If n is odd, we note that this chain is irreducible and aperiodic and that the stationary distribution is the uniform distribution on the n vertices. In

Theorem 2 of Chapter 3 in [Dia88] it is shown that for $7 \leq n$ and arbitrary k we have

$$\left\|P_n(X_k \in \cdot) - \pi\right\|_{TV} \;\geq\; \frac{1}{2}\exp\left(-\frac{\pi^2}{2}\frac{k}{n^2} - \frac{\pi^4}{11}\frac{k}{n^4}\right).$$

Furthermore, for $7 \leq n^2 \leq k$ and n odd, there exist the following upper bound

$$\left\|P_n(X_k \in \cdot) - \pi\right\|_{TV} \;\leq\; \exp\left(-\frac{\pi^2}{2}\frac{k}{n^2}\right)$$

of the same order. Using these bounds, we get

$$t_{\text{mix}}(n) \;\leq\; \frac{2\ln 4}{\pi^2}n^2$$

for large odd n and

$$t^{\varepsilon}_{\text{mix}}(n) \;\geq\; \frac{-\ln(2\varepsilon)}{\pi^2}n^2$$

for $\varepsilon < \frac{1}{2}$. Therefore, we can conclude

$$\Sigma \;\geq\; \sup_{0<\varepsilon<\frac{1}{2}} \limsup_{n\to\infty} \frac{t^{\varepsilon}_{\text{mix}}(2n+1)}{t_{\text{mix}}(2n+1)} \;\geq\; \sup_{0<\varepsilon<\frac{1}{2}} \frac{-\ln(2\varepsilon)}{2\ln 4} \;=\; \infty.$$

We note that for the lazy simple random walk on $\{0, ..., n\}$ with reflection in 0 and n we also have $\Sigma = \infty$ (cf. Section 3.5).

2. $\Sigma < \infty$:

We say that a sequence of Markov chains for which we have $\Sigma < \infty$ exhibits a **pre-cutoff**. In this case the time within the total variation distance drops from 1 to 0 has at most the same order as the mixing time.

For the cutoff phenomenon (which we are mainly interested in), this condition is not strong enough. For this it is required that the transition from 1 to 0 takes place in a window whose size is negligible compared to the order of the mixing time. More precisely, a sequence of Markov chains exhibits a **cutoff** if we have

$$\lim_{n\to\infty} \frac{t^{\varepsilon}_{\text{mix}}(n)}{t^{1-\varepsilon}_{\text{mix}}(n)} \;=\; 1 \tag{3.1.2}$$

for all $\varepsilon > 0$. Another equivalent characterisation, which provides a good illustration (cf. Figure 3.2) of the cutoff phenomenon, is given in the following lemma:

Lemma 3.1.1 (cf. Lemma 18.1 in [LPW09])**.** *The sequence $((U^n_k)_{k\in\mathbb{N}_0})_{n\in\mathbb{N}}$ exhibits a cutoff if and only if*

$$\lim_{n\to\infty} d_n\left(c\cdot t_{\text{mix}}(n)\right) \;=\; \begin{cases} 1 & \textit{if } c < 1, \\ 0 & \textit{if } c > 1. \end{cases}$$

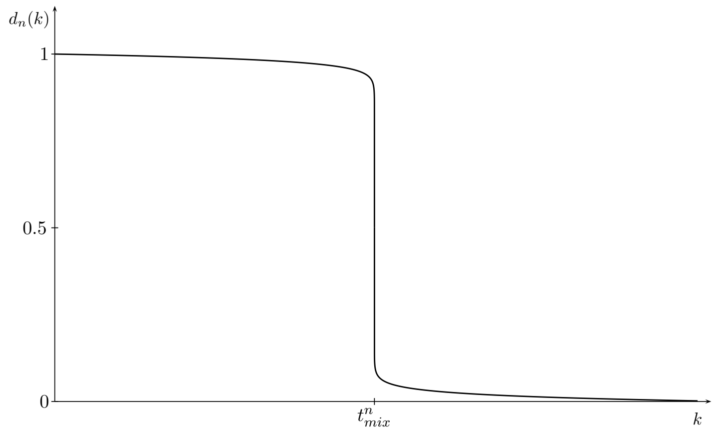

Figure 3.2: *If we rescale the time by the mixing time, Lemma 3.1.1 states for sequences with a cutoff that the function $d_n(\cdot)$ approaches a step function.* [1]

We note that a sequence of simple random walks on the complete graph with n vertices fulfils condition (3.1.2) but of course we are more interested in examples with a growing mixing time in n.

The definition of the cutoff phenomenon in (3.1.2) or Lemma 3.1.1 only considers the question if the transition is sharp, but does not specify the time it takes for the total variation distance to drop from 1 to 0. This leads to the following definition:

Definition 3.1.2 (cf. Definition 1.3.3)**.** The sequence $((U_k^n)_{k\in\mathbb{N}_0})_{n\in\mathbb{N}}$ exhibits a *cutoff* with cutoff times $(t_n)_{n\in\mathbb{N}}$ and window size $(f_n)_{n\in\mathbb{N}}$ if

(1) $f_n = o(t_n)$,

(2) $\lim_{c\to\infty} \liminf_{n\to\infty} d_n(t_n - cf_n) \;=\; 1$ and

(3) $\lim_{c\to\infty} \limsup_{n\to\infty} d_n(t_n + cf_n) \;=\; 0.$

We note that a smaller order of the window size implies a sharper transition to stationarity.

The first example for which a cutoff was proved (although the term *cutoff* was not introduced until [AD86]) is the case of random transpositions on the symmetric group by Diaconis and Shahshahani in [DS81]. They were motivated by the analysis of algorithms to generate random permutations. A very easy algorithm to generate a random permutation of the first n integers consists of the following two steps in every iteration:
First, choose a random integer uniformly between 1 and n and then transpose this

[1] This figure is adapted from Figure 18.1 in [LPW09].

integer with 1. We continue this procedure such that in the jth step we choose a random integer uniformly between j and n and afterwards transpose this integer with the number which is at position j at that time.

It is easily seen that for $n \geq 2$ the distribution of the permutation after $n - 1$ iterations is the uniform distribution on the set of all permutations of the first n integers. Diaconis and Shahshahani compared this algorithm with the algorithm which chooses in every iteration 2 integers independently of each other and uniformly between 1 and n and then transposes these integers. The distribution of the permutation is never exactly uniform. But how long do we have to wait until it is approximately uniform and is this time possibly smaller than $n - 1$?

In [DS81], Diaconis and Shahshahani proved that there is a cutoff in the situation of the second algorithm with cutoff times $\frac{1}{2}n \ln n$ and window size n, and thus they answered both questions. Asymptotically, this shows that we have to wait at least about $\frac{1}{2}n \ln n$ steps to be close to the uniform distribution. Further, we know that the convergence to stationarity of the first algorithm is a lot faster because for large n the distance in total variation to stationarity after $n-1$ iterations is still very close to 1 for the second algorithm while the distribution of the first algorithm is already uniform.

In the following years mainly Aldous and Diaconis examined the transition to stationarity of card shuffling algorithms. A reasonable algorithm should in the long run approach the uniform distribution on all permutations of the deck. The question they tried to answer is: How long do we have to wait to be reasonable close to this distribution?

Aldous identified a cutoff for the case of the riffle-shuffle (and the random walk on the hypercube) in [Ald83]. Then, in [AD86] Aldous and Diaconis proved the cutoff phenomenon for the top-in-at-random card shuffling. In this fundamental work the expression *cutoff* was first used. The paper "Trailing the dovetail shuffle to its lair" published by Bayer and Diaconis in 1992 (cf. [BD92]) is the probably most famous paper in this context. Their result was also on the cutoff of the riffle shuffle and improved the result in [Ald83]. It even attracted attention of a general audience. The New York Times wrote on 09.01.1990 (cf. [Kol90]) as their headline

"In shuffling cards, 7 is winning number."

More precisely, you should riffle a deck of 52 cards (used for example in blackjack) 7 times to end with a totally random order of cards. Further, one can conclude from their result that the order of the cards after riffling 5 times is with high probability as bad as at the start and riffling more than 9 times is useless because the cards are already in a random order (cf. Figure 3.3).

To state their precise result, we first describe the mathematical model they used to model the riffling of n cards (cf. Section 8 in [LPW09]):
Let M be a binomial$(n, 1/2)$ random variable and split the deck into its top M cards and its bottom $n - M$ cards. The two piles are then held over the table and cards are

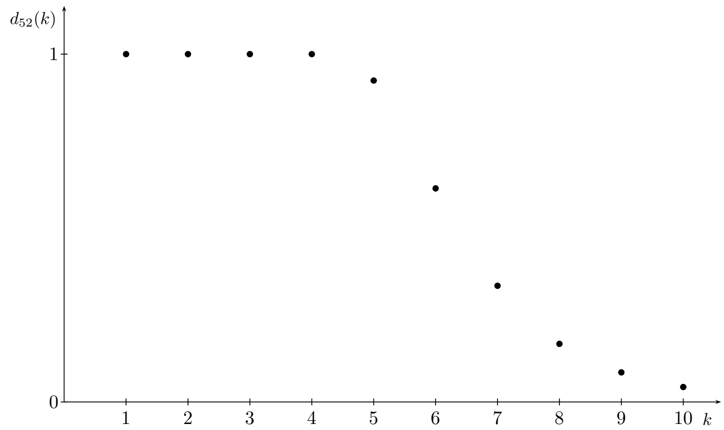

Figure 3.3: *On the distance in total variation of distribution of the riffle shuffle algorithm to its stationary distribution for a deck with* 52 *cards.*

dropped one by one, forming a single pile again, according to the following recipe: If at a particular moment, the left pile contains b cards and the right pile contains c cards, then drop the card on the bottom of the left pile with probability $\frac{b}{b+c}$ and the card on the bottom of the right pile with probability $\frac{c}{b+c}$. Repeat this procedure until all cards have been dropped.

They proved:

Theorem 3.1.3 (cf. Theorem 2 in [BD92])**.** *Let P_n^m denote the distribution of a deck of n cards after m riffle shuffles (starting from the deck in order) and let π_n denote the uniform distribution. Then we have for $m = \frac{3}{2}\log_2(n) + \theta$ and large n*

$$\left\|P_n^m - \pi_n\right\|_{TV} \;=\; 1 - 2\Phi\left(-\frac{2^{-\theta}}{4\sqrt{3}}\right) + O\left(\frac{1}{n^{\frac{1}{4}}}\right),$$

where Φ denotes the distribution function of a standard Gaussian random variable.

We want to emphasize that in this situation there is a very sharp transition which can be seen at the fact that the cutoff window size can be chosen as a constant. The theorem yields that for large n we have

$$d_n\left(\frac{3}{2}\log_2(n) - 5\right) \;\geq 0.99 \quad \text{and} \quad d_n\left(\frac{3}{2}\log_2(n) + 5\right) \;\leq\; 0.01.$$

The mixing properties of models in the statistical mechanics have also been studied a lot recently. In these models, the stationary distribution is the equilibrium distribution, and therefore identifying a cutoff is very helpful for simulations. During the last years, there have been proofs for a cutoff in several spin systems (cf. for example [LLP10], [LS09], and [LS12]).

In general, proving a cutoff turns out to be a challenging task even if one is not interested in the window size. The proofs usually require a very detailed analysis of the family of underlying distributions and there are still nearly no necessary and sufficient conditions to answer the question when to expect a cutoff. But nevertheless there are a lot of examples which are believed to have a cutoff. In 2004 Peres (cf. [Per04]) formulated a very simple condition just involving the mixing time and the spectral gap of the transition matrix (cf. Chapter 4 for more details). He conjectured that a family of Markov chains exhibits a cutoff if and only if the product of the mixing time and the spectral gap of the transition matrix tends to infinity as n goes to infinity. It turned out that this conjecture is not true in general (cf. Aldous' example, Section 6 in [CS08]) but still yields a necessary condition:

Lemma 3.1.4 (cf. Lemma 18.4 in [LPW09]). *For a sequence of irreducible and aperiodic Markov chains with spectral gaps $(gap_n)_{n\in\mathbb{N}}$ and mixing times $(t_{mix}(n))_{n\in\mathbb{N}}$, if $gap_n \cdot t_{mix}(n)$ is bounded above, then there is no pre-cutoff.*

Despite the fact that it turned out that this product-condition does not imply a cutoff in general, Peres still conjectures that there is an equivalence between the existence of a cutoff and the validity of this product-condition for many natural types of chains. Recently, this was proved for birth and death chains (cf. [DLP10]).

In the rest of this chapter, we are interested in the mixing properties (and especially the cutoff phenomenon) of transient RWRE on $\mathbb{Z}$. In the case $\Sigma = \infty$, we have seen that a simple random walk on the cycle does not exhibit a cutoff. If we choose one d-regular ($d \geq 3$) graph out of the set of all d-regular graphs uniformly at random, Lubetzky and Sly proved in [LS10] that with high probability the simple random walk on this d-regular graph exhibits a cutoff. Further, it is known that the simple random walk with drift on $\{0, ..., n\}$ exhibits a cutoff (cf. Section 3.5).

3.2 Stationary distribution

To show a cutoff or to prove that no cutoff is possible, we need to understand very precisely which events have a high probability with respect to the stationary distribution. In this section, we show that the mass of the stationary distribution of the RWRE under P_{ω^n} is asymptotically concentrated on the interval $[n - 2(\ln n)^2, n]$.

At first, we note that for $x = 2, ..., n-1$ we have

$$\begin{aligned} &\omega_{x-1}\Big(\exp\big(-V(x)\big) + \exp\big(-V(x-1)\big)\Big) \\ =\ &\exp\big(-V(x)\big) \ =\ \big(1-\omega_x\big)\Big(\exp\big(-V(x+1)\big) + \exp\big(-V(x)\big)\Big) \end{aligned}$$

and

$$\big(1-\omega_1\big)\Big(\exp\big(-V(2)\big)+\exp\big(-V(1)\big)\Big) \;=\; 1\cdot\exp\big(-V(1)\big) \;=\; \omega_0^n,$$
$$\omega_{n-1}\Big(\exp\big(-V(n)\big)+\exp\big(-V(n-1)\big)\Big) = \exp\big(-V(n)\big)\big(1-\omega_n^n\big).$$

Therefore, the reversible (and hence stationary) probability distribution of the RWRE $(X_n)_{n\in\mathbb{N}_0}$ under P_{ω^n} is defined by

$$\begin{aligned}
\pi_{\omega^n}(0) &:= \frac{\exp\big(-V(1)\big)}{C_n}\,,\\
\pi_{\omega^n}(x) &:= \frac{\exp\big(-V(x+1)\big)+\exp\big(-V(x)\big)}{C_n} \qquad \text{for } x=1,...,n-1,\\
\pi_{\omega^n}(n) &:= \frac{\exp\big(-V(n)\big)}{C_n}\,,
\end{aligned} \tag{3.2.1}$$

where

$$\begin{aligned}
C_n &:= \sum_{x=1}^{n-1}\Big(\exp\big(-V(x+1)\big)+\exp\big(-V(x)\big)\Big)+\exp\big(-V(n)\big)+\exp\big(-V(1)\big) \\
&= 2\sum_{x=1}^{n}\exp\big(-V(x)\big)
\end{aligned} \tag{3.2.2}$$

is the normalizing constant.

Lemma 3.2.1. *We have*

$$\lim_{n\to\infty}\pi_{\omega^n}\Big(\Big[n-2(\ln n)^2,n\Big)\Big) \;=\; 1$$

for $\boldsymbol{P}$-almost every environment ω.

Proof. We get

$$\begin{aligned}
&\pi_{\omega^n}\big([0,n-2(\ln n)^2)\big)\\
&= \frac{\sum_{x=1}^{\lfloor n-2(\ln n)^2\rfloor}\big(\exp\big(-V(x+1)\big)+\exp\big(-V(x)\big)\big)+\exp\big(-V(1)\big)}{2\sum_{x=1}^{n}\exp\big(-V(x)\big)}\\
&\leq \frac{2n\exp\Big(-\min_{0\leq i\leq\lfloor n-2(\ln n)^2\rfloor}V(i)\Big)+\exp(-V(1))}{2\exp\big(-V(n)\big)}\\
&= n\exp\Big(V(n)-\min_{0\leq i\leq\lfloor n-2(\ln n)^2\rfloor}V(i)\Big)+\exp\big(V(n)-V(1)\big).
\end{aligned} \tag{3.2.3}$$

By the definition of $B_1(n)$, we know that for all $i \leq n - 2(\ln n)^2$ we have

$$V(n) - V(i) \;<\; -2\ln(n).$$

For environments $\omega \in B_1(n)$, this therefore yields as an upper bound for (3.2.3)

$$\begin{aligned}\pi_{\omega^n}\Big([0, n-2(\ln n)^2)\Big) &\leq 2n\exp\big(-2\ln n\big) + \exp\big(V(n)-V(1)\big)\\ &= \frac{2}{n} + \exp\big(V(n)-V(1)\big),\end{aligned}$$

which shows that for $\omega \in B_1(n)$ we have

$$\lim_{n\to\infty} \pi_{\omega^n}\Big(\,\big[n-2(\ln n)^2, n\big]\,\Big) \;=\; 1.$$

Lemma 1.5.1 and the Borel-Cantelli lemma now finish the proof. ■

Note here that with respect to P_{ω^n} the lazy random walk $(Y_k)_{k\in\mathbb{N}_0}$ has the same stationary distribution π_{ω^n} as $(X_k)_{k\in\mathbb{N}_0}$.

Remark 3.2.2.
Let π_n^{SRW} denote the stationary distribution of the simple random walk (in deterministic environment) on $\{0, ..., n\}$ with reflection in 0 and n (cf. Section 3.5). In this case, the formula of the stationary distribution (cf. (3.2.1)) simplifies to

$$\begin{aligned}\pi_n^{\mathrm{SRW}}(k) &= \frac{1}{n} \quad \text{for} \quad k \in \{1, ..., n-1\} \quad \text{and}\\ \pi_n^{\mathrm{SRW}}(0) &= \pi_n^{\mathrm{SRW}}(n) \;=\; \frac{1}{2n}.\end{aligned}$$

Further, let us consider the nearest neighbour random walk on $\{0, ..., n\}$ with reflection in 0 and n and transition probabilities $\frac{p}{2}$ to move to the right, $\frac{1-p}{2}$ to move to the left and $\frac{1}{2}$ to stay at the current position. We get for the associated stationary distribution for $k = 0, ..., n$

$$\pi_n^p(k) \;=\; \frac{\frac{p}{1-p} - 1}{\left(\frac{p}{1-p}\right)^{n+1} - 1}\left(\frac{p}{1-p}\right)^k.$$

If $p > \frac{1}{2}$, this yields

$$\lim_{n\to\infty} \pi_n^p\Big([n-f(n), n]\Big) \;=\; 1, \tag{3.2.4}$$

for all functions f for which we have $\lim\limits_{n\to\infty} f(n) = \infty$.

3.3 Cutoff for $\kappa > 1$

In this section, we prove Theorem 1.4.1. In a first step, we show that we can bound the distance in total variation of the lazy RWRE to its stationary distribution using the hitting times T_n.

Lemma 3.3.1. *We have for all $n, k \in \mathbb{N}$*

$$\max_{x\in\{0,\dots,n\}} \left\| P^x_{\omega^n}(Y_k \in \cdot) - \pi_{\omega^n} \right\|_{TV} \le P_{\omega^n}\left(T^Y_n > k\right).$$

Proof. We have for all $k \in \mathbb{N}$ using Corollary 5.3 in [LPW09]:

$$\max_{x\in\{0,\dots,n\}} \left\| P^x_{\omega^n}(Y_k \in \cdot) - \pi_{\omega^n} \right\|_{TV} \le \max_{x,y\in\{0,\dots,n\}} P^{\vec{z}}_{\omega^n}\Big(\min\{s \in \mathbb{N} : Y^x_s = Y^y_s\} > k\Big), \tag{3.3.1}$$

where for all $x, y \in \{0, ..., n\}$ under $P^{\vec{z}}_{\omega^n}$ we consider a coupling $(Y^x_k, Y^y_k)_{k\in\mathbb{N}_0}$ of two lazy RWRE on $\{0, ..., n\}$ with $P^{\vec{z}}_{\omega^n}(Y^x_0 = x, Y^y_0 = y) = 1$ and marginal distribution $P^x_{\omega^n}$ and $P^y_{\omega^n}$, respectively, defined in the following way: Until the two chains meet for the first time, the chains move according to the following two steps: First, we toss a coin to decide which chain moves. After that, the chosen chain performs a step of a RWRE in the environment ω^n described in (1.1.2) and the other chain stays at its position. After they met for the first time, we move them together according to the law of a lazy RWRE.

Note that due to this coupling the two chains cannot cross each other without meeting, and therefore we have

$$\max_{x,y\in\{0,\dots,n\}} P^{\vec{z}}_{\omega^n}\left(\min\{s \in \mathbb{N} : X^x_s = X^y_s\} > t\right) \le P_{\omega^n}\left(T^Y_n > t\right),$$

which together with (3.3.1) yields the statement. ∎

Proof of Theorem 1.4.1. First, we note that due to (2.1.2) and Theorem 1.4.5 we have for $\kappa > 1$ and $\mathbf{P}$-almost every environment ω

$$\sqrt{\mathrm{Var}_{\omega^n}(T_n)} \le \sqrt{\mathrm{Var}_{\omega}(T_n)} = o(n). \tag{3.3.2}$$

Therefore, (1) in Definition 1.3.3 is valid for

$$t_\omega(n) := 2E_{\omega^n}(T_n) \quad \text{and} \quad f_\omega(n) := \sqrt{\mathrm{Var}_{\omega^n}(T_n)}$$

because we have for $\mathbf{P}$-almost every environment ω due to (1.4.1)

$$1 \le \lim_{n\to\infty} \frac{E_{\omega^n}(T_n)}{n} \le \lim_{n\to\infty} \frac{E_{\omega}(T_n)}{n} = \mathbb{E}T_1.$$

Further, we define

$$t_\omega^+(c,n) := \lceil t_\omega(n) + c \cdot f_\omega(n) \rceil.$$

Then, using Lemma 1.3.4, Lemma 3.3.1 and Chebyshev's inequality, we get

$$\begin{aligned} d_n\left(t_\omega^+(c,n)\right) &= \max_{x\in\{0,\dots,n\}} \left\| P_{\omega^n}^x(Y_{t_\omega^+(c,n)} \in \cdot) - \pi_{\omega^n} \right\|_{TV} \\ &\leq P_{\omega^n}\left(T_n^Y > t_\omega^+(c,n)\right) \\ &\leq P_{\omega^n}\left(\left|T_n^Y - E_{\omega^n} T_n^Y\right| > c\sqrt{\mathrm{Var}_{\omega^n}(T_n)}\right) \\ &\leq \frac{1}{c^2} \frac{\mathrm{Var}_{\omega^n}(T_n^Y)}{\mathrm{Var}_{\omega^n}(T_n)}. \end{aligned} \tag{3.3.3}$$

Therefore, Theorem 1.4.5 together with (1.3.4) and (2.1.14) yields for **P**-almost every environment ω

$$\lim_{c\to\infty} \limsup_{n\to\infty} d_n\left(t_\omega^+(c,n)\right) \;\leq\; \lim_{c\to\infty} \frac{4 + O(1)}{c^2} \;=\; 0,$$

and thus we showed (2) in Definition 1.3.3.

As the last step, we are interested in the lower bound of the cutoff. The idea is to show that before the cutoff window the lazy RWRE with start in 0 has with high probability not reached position $\lceil n - 2(\ln n)^2 \rceil$ and therefore is still in the interval $[0, n - 2(\ln n)^2)$. On the other hand, the mass of the stationary distribution π_{ω^n} is for large n concentrated on the interval $[n - 2(\ln n)^2, n]$ due to Lemma 3.2.1. We define

$$t_\omega^-(c,n) := \lfloor t_\omega(n) - c \cdot f_\omega(n) \rfloor \quad \text{and} \quad a_n := n - \left\lceil 2(\ln n)^2 \right\rceil.$$

Then, we get for c and n large enough using (1.3.4)

$$\begin{aligned} P_{\omega^n}\left(Y_{t_\omega^-(c,n)} \;\geq\; a_n\right) &\leq P_{\omega^n}\left(T_{a_n}^Y \leq t_\omega^-(c,n)\right) \\ &\leq P_{\omega^n}\left(T_{a_n}^Y - E_{\omega^n} T_{a_n}^Y \;\leq\; -c\sqrt{\mathrm{Var}_{\omega^n} T_n} + 2E_{\omega^n}^{a_n} T_n\right) \\ &\leq P_{\omega^n}\left(\left|T_{a_n}^Y - E_{\omega^n} T_{a_n}^Y\right| \geq \frac{c}{2}\sqrt{\mathrm{Var}_{\omega^n}(T_n)}\right), \end{aligned} \tag{3.3.4}$$

where to obtain the last inequality we used that we have for **P**-almost every environment ω due to Lemma 2.2.3

$$c - \frac{2E_{\omega^n}^{a_n} T_n}{\sqrt{\mathrm{Var}_{\omega^n}(T_n)}} > \frac{c}{2}$$

for all $n \geq n(\omega)$ and c large enough.

Now, again applying Chebyshev's inequality and (1.3.4), we get as an upper bound for (3.3.4) for c and n large enough

$$P_{\omega^n}\left(Y_{t_\omega^-(c,n)} \;\geq\; a_n\right) \;\leq\; \frac{\mathrm{Var}_{\omega^n}(T_{a_n}^Y)}{\mathrm{Var}_{\omega^n}(T_n)} \cdot \frac{4}{c^2} \;=\; \frac{4\mathrm{Var}_{\omega^n}\left(T_{a_n}\right) + 2E_{\omega^n} T_{a_n}}{\mathrm{Var}_{\omega^n}(T_n)} \cdot \frac{4}{c^2}.$$

Finally, using Lemma 3.2.1, Theorem 1.4.5 and (2.1.14), we conclude that we have for **P**-almost every environment ω

$$\begin{aligned}
&\lim_{c\to\infty}\liminf_{n\to\infty} d_n\left(t_\omega^-(c,n)\right)\\
&= \lim_{c\to\infty}\liminf_{n\to\infty}\max_{x\in\{0,\dots,n\}}\left\|P_{\omega^n}^x\left(Y_{t_\omega^-(c,n)}^n\in\cdot\right)-\pi_{\omega^n}\right\|_{TV}\\
&\geq \lim_{c\to\infty}\liminf_{n\to\infty}\left(\pi_{\omega^n}\left([n-2(\ln n)^2,n]\right)-P_{\omega^n}\left(Y_{t_\omega^-(c,n)}\;\geq\; n-2(\ln n)^2\right)\right)\\
&\geq 1-\lim_{c\to\infty}\limsup_{n\to\infty}\frac{4\mathrm{Var}_{\omega^n}(T_{a_n})+2E_{\omega^n}T_{a_n}}{\mathrm{Var}_{\omega^n}(T_n)}\cdot\frac{4}{c^2}\\
&= 1-\lim_{c\to\infty}\frac{16+O(1)}{c^2}\;=\;1,
\end{aligned}$$

which shows (3) in Definition 1.3.3. ■

In the next proposition, we consider the question if the window size in Theorem 1.4.1 is optimal. Under stronger assumptions, we prove that the exponent of n of the window size $f_\omega(n)=\sqrt{\mathrm{Var}_{\omega^n}T_n}$ is optimal which is $\frac{1}{\kappa}$ for $1<\kappa\leq 2$ and $\frac{1}{2}$ for $\kappa>2$ due to Theorem 1.4.5.

Proposition 3.3.2. *Let Assumptions 1 and 2 hold.*

(a) Let $1<\kappa<2$ and additionally assume that the distribution of $\ln\rho_0$ is non-lattice. Then for $\boldsymbol{P}$-almost every environment ω and all $\delta>0$ the cutoff window size has to be bigger than $n^{\frac{1}{\kappa}-\delta}$.

(b) Let $\kappa>2$. Then for $\boldsymbol{P}$-almost every environment ω the optimal cutoff window size is $\sqrt{n}$.

Proof. First, we assume $1<\kappa<2$. Then, Theorem 1.1 in [Pet09] yields that for **P**-almost every environment ω there exists an environment dependent subsequence $n_{k_m}=n_{k_m}(\omega)$ of $n_k=2^{2^k}$ such that

$$\frac{T_{n_{k_m}}-E_\omega T_{n_{k_m}}}{\sqrt{\mathrm{Var}_\omega\left(T_{n_{k_m}}^{(d_{k_m})}-T_{n_{k_m}-1}^{(d_{k_m})}\right)}}\xrightarrow{d}\mathcal{N}(0,1)\tag{3.3.5}$$

with $d_{k_m}=n_{k_m}-n_{k_m-1}$. Further, $\xrightarrow{d}$ means convergence in distribution and see (2.2.1) for the definition of the hitting times $T_n^{(r)}$ of the restricted process. Note that the proof of this theorem needs the assumption that the distribution of $\ln\rho_0$ is non-lattice.

Recall that the random variables $(Z_k)_{k\in\mathbb{N}}$ indicate if the lazy RWRE moves in step k or stays at its position. We get using Lemma 3.3.1

$$
\begin{aligned}
&\liminf_{n\to\infty} d_n\left(\left\lceil 2E_\omega T_n - 2n^{\frac{1}{\kappa}-\delta}\right\rceil\right)\\
\le\ &\liminf_{n\to\infty} P_{\omega^n}\left(T_n^Y > \left\lceil 2E_\omega T_n - 2n^{\frac{1}{\kappa}-\delta}\right\rceil\right)\\
\le\ &\liminf_{n\to\infty} P_\omega\left(T_n > E_\omega T_n - n^{\frac{1}{\kappa}-\delta} - \sqrt{\left\lceil 2E_\omega T_n - 2n^{\frac{1}{\kappa}-\delta}\right\rceil}\right)\\
&+\liminf_{n\to\infty} P_\omega\left(\frac{\sum_{i=1}^{\left\lceil 2E_\omega T_n - 2n^{\frac{1}{\kappa}-\delta}\right\rceil}\left((1-Z_i)-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}\left\lceil 2E_\omega T_n - 2n^{\frac{1}{\kappa}-\delta}\right\rceil}} > 2\right)\\
\le\ &\lim_{m\to\infty} P_\omega\left(\frac{T_{n_{k_m}} - E_\omega T_{n_{k_m}}}{\sqrt{\mathrm{Var}_\omega\left(T^{(d_{k_m})}_{n_{k_m}} - T^{(d_{k_m})}_{n_{k_m}-1}\right)}} > -\frac{n_{k_m}^{\frac{1}{\kappa}-\delta} + \sqrt{\left\lceil 2E_\omega T_n - 2n^{\frac{1}{\kappa}-\delta}\right\rceil}}{\sqrt{\mathrm{Var}_\omega\left(T^{(d_{k_m})}_{n_{k_m}} - T^{(d_{k_m})}_{n_{k_m}-1}\right)}}\right)\\
&+\Big(1-\Phi(2)\Big),
\end{aligned}
\tag{3.3.6}
$$

where to obtain the last inequality we applied the CLT for the i.i.d. sequence $(Z_i)_{i\in\mathbb{N}}$ and used that $E_\omega T_n$ is of order n due to (1.4.1) ($\kappa > 1$). With a similar proof as for Lemma 1.5.4, we can find for all m large enough and for **P**-almost every environment ω a block with a depth of more than $\frac{1}{\kappa}\left(\ln n_{k_m} - 4\ln\ln n_{k_m}\right)$ in the interval $(n_{k_m-1}, n_{k_m}]$. Therefore, we have

$$
\mathrm{Var}_\omega\left(T^{(d_{k_m})}_{n_{k_m}} - T^{(d_{k_m})}_{n_{k_m}-1}\right) \ > \ (n_{k_m})^{\frac{2}{\kappa}-\delta}.
$$

Now, additionally using (3.3.5), we get an upper bound for (3.3.6), that is

$$
\liminf_{m\to\infty} d_{n_{k_m}}\left(\left\lceil 2E_\omega T_n - 2n^{\frac{1}{\kappa}-\delta}\right\rceil\right) \ \le\ \frac{1}{2} + \Phi(-2) \ <\ 1.
$$

Finally, we can conclude that the cutoff window size has to be bigger than $n^{\frac{1}{\kappa}-\delta}$ for all $\delta > 0$.

Next, we assume $\kappa > 2$. A similar proof as in Theorem 5.1 in [Ali99] yields

$$
\frac{T_n^Y - E_\omega T_n^Y}{\sqrt{n}} \ \xrightarrow{d}\ \mathcal{N}(0,\sigma^2),
$$

for a suitable $\sigma > 0$.

Recall that $2E_\omega T_n = E_\omega T_n^Y$ (cf. Lemma 1.3.4) and thus we get using Lemma 3.3.1

$$\begin{aligned}\liminf_{n\to\infty} d_n\left(\lceil 2E_\omega T_n - \sqrt{n}\rceil\right) &\leq \liminf_{n\to\infty} P_\omega\left(T_n^Y > E_\omega T_n^Y - \sqrt{n}\right)\\ &= 1-\Phi\left(-\frac{1}{\sigma}\right) = \Phi\left(\frac{1}{\sigma}\right) < 1.\end{aligned}$$

Therefore, the order of the cutoff window size has to be bigger or equal to $\sqrt{n}$. Corollary 1.4.6 then shows that this is the optimal size. ■

3.4 No cutoff for $0 < \kappa < 1$

In this section, we consider the case $\kappa < 1$. We show that in this case the transition to stationarity is not as sharp as required for the cutoff phenomenon. To show this, we first construct an environment dependent sequence of deep blocks in which the RWRE (with start in 0) spends most of its time before it hits the endpoint of this deep block (cf. Lemma 3.4.4). Afterwards, we show that the existence of this sequence excludes that the lazy RWRE exhibits a cutoff.

The following construction basically refines the arguments of Section 4 in [PZ09], where the authors additionally use that the distribution of $\ln\rho_0$ with respect to $\mathbf{P}$ is non-lattice. For $k\in\mathbb{N}$ we define

$$n_k := 2^{2^k} \quad\text{and}\quad d_k := n_k - n_{k-1}. \tag{3.4.1}$$

Lemma 3.4.1. *Assume Assumptions 1 and 2 and let $\kappa \leq 1$. Then for any $0<\delta<\frac{6}{\kappa}$ we have*

$$\boldsymbol{P}\left(E_\omega \widetilde{T}_{\nu_{n_k}}^{(d_k+1)} \geq n_k^{\frac{1}{\kappa}+\delta}\right) = o\left(n_k^{-\frac{\delta\kappa}{3}}\right).$$

Proof. First, we recall that for $n\in\mathbb{N}$ $(X_k^{(n)})_{k\in\mathbb{N}_0}$ is a restricted RWRE which cannot backtrack more than $\lceil(\ln n)^2\rceil$ blocks after reaching a new block for the first time.

For $\omega\in B_1(2\bar\nu n_k)\cap B_2(n_k)$ (cf. Lemma 1.5.2 for the definition of $\bar\nu$) we have that all blocks in the interval $[-\nu_{n_k},\nu_{n_k}]$ are smaller than $(\ln(2\bar\nu n_k))^2$, and therefore we get using equation (1.2.3)

$$\begin{aligned}E_\omega \widetilde{T}_{\nu_{n_k}}^{(d_k+1)} &= \nu_{n_k} + 2\sum_{j=1}^{n_k}\sum_{i=\nu_{j-1}}^{\nu_j-1}\sum_{l=\nu_{j-1-\lceil(\ln d_{k+1})^2\rceil}+1}^{i}\exp\left(V(i+1)-V(l)\right)\\ &\leq 2\bar\nu n_k + 2\lceil\ln(2\bar\nu n_k)\rceil^6\sum_{j=1}^{n_k}\Big(1+\exp\left(H_j\right)\Big).\end{aligned}$$

This yields for arbitrary $0 < \delta$

$$\begin{aligned}
&\mathbf{P}\left(E_\omega \widetilde{T}^{(d_k+1)}_{\nu_{n_k}} \geq n_k^{\frac{1}{\kappa}+\delta}\right) \\
&\leq \mathbf{P}\left(\left\{E_\omega \widetilde{T}^{(d_k+1)}_{\nu_{n_k}} \geq n_k^{\frac{1}{\kappa}+\delta}\right\} \cap B_1(2\bar{\nu}n_k) \cap B_2(n_k)\right) + \mathbf{P}\Big(B_1(2\bar{\nu}n_k)^{\mathrm{c}}\Big) + \mathbf{P}\Big(B_2(n_k)^{\mathrm{c}}\Big) \\
&\leq \mathbf{P}\left(2\bar{\nu}n_k + 2\lceil \ln(2\bar{\nu}n_k)\rceil^6 \sum_{j=1}^{n_k} \Big(1+\exp(H_j)\Big) \;\geq\; n_k^{\frac{1}{\kappa}+\delta}\right) + O\left(n_k^{-2}\right), \qquad (3.4.2)
\end{aligned}$$

where we used Lemma 1.5.1 and Lemma 1.5.2 to obtain the last inequality. To complete the proof of this Lemma, we therefore have to show that the first term in (3.4.2) is of order $o\left(n_k^{-\frac{\delta\kappa}{3}}\right)$ for all $0 < \delta < \frac{6}{\kappa}$.

First, we observe that for all $0 < \beta < \kappa \leq 1$ we have due to (1.5.7)

$$\mathbf{E}\exp(\beta H_1) \;=\; \int_0^\infty \mathbf{P}\left(H_1 > \frac{1}{\beta}\ln t\right) dt \;\leq\; C_1 \int_0^\infty t^{-\frac{\kappa}{\beta}} dt \;<\; \infty.$$

Therefore, we get for large k and all $0 < \beta < \kappa \leq 1$ using the Markov inequality and that $(H_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence under $\mathbf{P}$

$$\begin{aligned}
&\mathbf{P}\left(2\bar{\nu}n_k + 2\lceil \ln(2\bar{\nu}n_k)\rceil^6 \sum_{j=1}^{n_k} \Big(1+\exp(H_j)\Big) \;\geq\; n_k^{\frac{1}{\kappa}+\delta}\right) \\
&\leq \mathbf{P}\left(\sum_{j=1}^{n_k} \exp(H_j) \geq n_k^{\frac{1}{\kappa}+\frac{\delta}{2}}\right) \\
&\leq n_k^{-\beta\left(\frac{1}{\kappa}+\frac{\delta}{2}\right)} \mathbf{E}\left(\sum_{j=1}^{n_k} \exp(H_j)\right)^\beta \\
&\leq n_k^{-\beta\left(\frac{1}{\kappa}+\frac{\delta}{2}\right)} n_k \mathbf{E}\exp(\beta H_1) \\
&= n_k^{-\frac{\beta\delta}{2}+\left(1-\frac{\beta}{\kappa}\right)} \mathbf{E}\exp(\beta H_1), \qquad (3.4.3)
\end{aligned}$$

where in the first line we used that for large k and all $\delta > 0$ we have

$$\frac{n_k^{\frac{1}{\kappa}+\delta}}{2\bar{\nu}n_k + 2\lceil \ln(2\bar{\nu}n_k)\rceil^6} \;\geq\; n_k^{\frac{1}{\kappa}+\frac{\delta}{2}}.$$

Now, choosing β in (3.4.3) to be arbitrarily close to κ and combining this with (3.4.2) finishes the proof for all $0 < \delta < \frac{6}{\kappa}$. ■

Lemma 3.4.2. *Assume Assumptions 1 and 2 and let $\kappa < 1$. Then we have for $\boldsymbol{P}$-almost every environment ω*

$$\lim_{k\to\infty} \frac{E_\omega^{\nu_{n_{k-1}}} \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k}{E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k} \;=\; 1,$$

where

$$\mu_k := \;\max\left\{ E_\omega^{\nu_{j-1}} \widetilde{T}^{(d_k)}_{\nu_j} \;:\; n_{k-1} < j \leq n_k \right\}.$$

Proof. First, we note that for all environments ω and all k the fraction in the statement is obviously smaller or equal to 1. Thus, we only have to show the lower bound. Let $\varepsilon > 0$ be arbitrary. Then, we get for $0 < \delta < \frac{6}{\kappa}$

$$\begin{aligned}
&\mathbf{P}\left(\frac{E_\omega^{\nu_{n_{k-1}}} \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k}{E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k} \;\leq\; 1 - \varepsilon \right) \\
&= \mathbf{P}\left(E_\omega^{\nu_{n_{k-1}}} \widetilde{T}^{(d_k)}_{\nu_{n_k}} \;\leq\; (1-\varepsilon) E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} + \varepsilon \mu_k \right) \\
&= \mathbf{P}\left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_{k-1}}} \;\geq\; \varepsilon \left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \right) \right) \\
&\leq \mathbf{P}\left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_{k-1}}} \;\geq\; \varepsilon \left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \right),\; E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \;>\; \frac{1}{\varepsilon} n_{k-1}^{\frac{1}{\kappa}+\delta} \right) \\
&\qquad + \mathbf{P}\left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \;\leq\; \frac{1}{\varepsilon} n_{k-1}^{\frac{1}{\kappa}+\delta} \right) \\
&\leq \mathbf{P}\left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_{k-1}}} \;\geq\; n_{k-1}^{\frac{1}{\kappa}+\delta} \right) + \mathbf{P}\left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \;\leq\; \frac{1}{\varepsilon} n_{k-1}^{\frac{1}{\kappa}+\delta} \right),
\end{aligned} \tag{3.4.4}$$

where due to Lemma 3.4.1 the first term on the right side of (3.4.4) is of order $o\left(n_{k-1}^{-\frac{\delta\kappa}{3}} \right)$ and hence summable in k. Next, we analyse the second term and we first note that because of the choice of the sequence $(n_k)_{k\in\mathbb{N}}$ we have for $\delta < \frac{1}{3\kappa}$ and k large enough

$$\frac{1}{\varepsilon} n_{k-1}^{\frac{1}{\kappa}+\delta} \;=\; \frac{1}{\varepsilon}\left(2^{2^{k-1}\left(\frac{1}{\kappa}+\delta\right)} \right) \;\leq\; \left(2^{2^{k-1}\cdot 2\left(\frac{1}{\kappa}-\delta\right)} \right) \;=\; n_k^{\frac{1}{\kappa}-\delta}$$

and hence

$$\mathbf{P}\left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \;\leq\; \frac{1}{\varepsilon} n_{k-1}^{\frac{1}{\kappa}+\delta} \right) \;\leq\; \mathbf{P}\left(E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \;\leq\; n_k^{\frac{1}{\kappa}-\delta} \right). \tag{3.4.5}$$

But

$$E_\omega \widetilde{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \;\leq\; n_k^{\frac{1}{\kappa}-\delta}$$

implies, that the expected crossing time $E_\omega^{\nu_{j-1}} \widetilde{T}_{\nu_j}^{(d_k)}$ has to be smaller than $n_k^{\frac{1}{\kappa}-\delta}$ for at least $n_k - 1$ blocks. Therefore, we get using equation (1.5.7) and the fact that $\exp(H_{j-1}) < E_\omega^{\nu_{j-1}} \widetilde{T}_{\nu_j}^{(d_k)}$

$$\begin{aligned}
\mathbf{P}\left(E_\omega \widetilde{T}_{\nu_{n_k}}^{(d_k)} - \mu_k \ \le \ n_k^{\frac{1}{\kappa}-\delta}\right) &\le \ n_k \left(1 - \mathbf{P}\left(H_1 > \left(\frac{1}{\kappa} - \delta\right) \ln n_k\right)\right)^{n_k - 1} \\
&\le \ n_k \exp\left(-(n_k - 1)\mathbf{P}\left(H_1 > \left(\frac{1}{\kappa} - \delta\right) \ln n_k\right)\right) \\
&\le \ n_k \exp\left(-(n_k - 1)\widetilde{C}_1 n_k^{-1+\delta\kappa}\right). \qquad (3.4.6)
\end{aligned}$$

Finally, (3.4.5) and (3.4.6) yield that for $\delta < \frac{1}{3\kappa}$ the second probability in (3.4.4) is also summable in k for all $\varepsilon > 0$, and thus the Borel-Cantelli lemma gives us the lower bound of the statement for $\mathbf{P}$-almost every environment ω. ■

Lemma 3.4.3. *Assume $\kappa < 1$ and let $\widehat{\boldsymbol{P}}$ be an environment distribution on $\mathbb{Z}$ which fulfils Assumptions 1 and 2 and additionally has the property that the distribution of $\ln \rho_0$ is non-lattice with respect to $\widehat{\boldsymbol{P}}$. Further, we define*

$$\widehat{\boldsymbol{Q}}(\cdot) := \ \widehat{\boldsymbol{P}}(\cdot|\mathcal{R}), \quad \textit{where} \ \ \mathcal{R} = \left\{\omega \in \Omega \ : \ \sum_{i=-k}^{-1} \ln \rho_i < 0, \forall \ k \in \mathbb{N}\right\}.$$

Then we have for any $C > 1$

$$\liminf_{n\to\infty} \widehat{\boldsymbol{Q}}\left(\exists \ k \in \left[1, \frac{n}{2}\right] : \exp(H_k) \ge C \sum_{j \in [1,n]\setminus\{k\}} E_\omega^{\nu_{j-1}} \widetilde{T}_{\nu_j}^{(n)}\right) > \gamma L_{\kappa,b}(1) C^{-\kappa},$$

where $\gamma > 0$ is a constant and $L_{\kappa,b}$ denotes the distribution function of a stable random variable with index κ and characteristic function

$$\widehat{L}_{\kappa,b}(t) = \exp\left(-b|t|^\kappa \left(1 - it|t|^{-1} \tan\left(\frac{1}{2}\pi\kappa\right)\right)\right).$$

For a proof see Lemma 4.1 in [PZ09].

We note that in the proof the assumption that the distribution of $\ln \rho_0$ is non-lattice is needed to use that $n^{-\frac{1}{\kappa}} E_\omega T_{\nu_n}$ converges with respect to $\widehat{\mathbf{Q}}$ in distribution to a stable distribution with characteristic function $\widehat{L}_{\kappa,b}$.

Further, we note that the lower bound in Lemma 3.4.3 only depends on κ which is defined by the equation $\mathbf{E}_{\widehat{\mathbf{P}}} \rho^\kappa = 1$. Therefore, we get the same lower bound for all environment distributions $\widehat{\mathbf{P}}$ which fulfil the previous assumptions for the same κ. Furthermore, we have $\widehat{\mathbf{P}}(\mathcal{R}) > 0$ since $\mathbf{E}_{\widehat{\mathbf{P}}} \ln \rho_0 < 0$ by Assumption 1.

Lemma 3.4.4. *Assume Assumptions 1-3 and let $\kappa < 1$. For $\boldsymbol{P}$-almost every environment ω, there exists a random subsequence $a_m = a_m(\omega)$ such that*

$$\exp\left(H_{a_m}\right) \;\geq\; m^2 E_\omega \widetilde{T}^{(a_m)}_{\nu_{a_{m-1}}}.$$

Proof. First, we consider the following modification of our environment:

With respect to $\mathbf{P}$ (on a possibly enlarged probability space), let $(\Gamma_k)_{k\in\mathbb{Z}}$ be a sequence of i.i.d. Bernoulli random variables with success probability $\frac{1}{2}$ and further let let $(\Gamma_k)_{k\in\mathbb{Z}}$ be independent of the environment ω. We define a new environment $\tau := (\tau_k)_{k\in\mathbb{Z}}$ by

$$\tau_k := \;\omega_k \mathbb{1}_{\{\Gamma_k=0\}} + \frac{1}{2}\mathbb{1}_{\{\Gamma_k=1\}}, \qquad k \in \mathbb{Z}.$$

Recall that the quantities $(\rho_k(\tau))_{k\in\mathbb{Z}}$, $(\nu_k(\tau))_{k\in\mathbb{Z}}$ and $(H_k(\tau))_{k\in\mathbb{Z}}$ depend on the environment, and we have for $k \in \mathbb{Z}$

$$\rho_k(\tau) \;=\; \rho_k(\omega)\mathbb{1}_{\{\Gamma_k=0\}} + \mathbb{1}_{\{\Gamma_k=1\}}.$$

Note that we get an environment with the same distribution as ω by “deleting” the “$\frac{1}{2}$’s” in τ (which are at positions k with $\Gamma_k = 1$) and defining $\omega_0 = \tau_l$ with $l = \inf\{m \in \mathbb{N}_0 : \Gamma_m = 0\}$ as the environment in 0. We define

$$\Delta : \;\; (0,1)^{\mathbb{Z}} \;\longrightarrow\; (0,1)^{\mathbb{Z}}, \; \tau \;\mapsto\; \Delta(\tau)$$

to be the function which realises this “deleting procedure” (cf. Figure 3.4). Further, for a set of environments A we define

$$\Delta(A) := \;\{\Delta(\mu) \;:\; \mu \in A\}\,.$$

We note that we have

$$V(\nu_k(\tau)) = V\Big(\nu_k\big(\Delta(\tau)\big)\Big) \;\text{ and }\; H_k(\tau) = H_k\Big(\Delta(\tau)\Big)$$

and

$$\mathbf{P}^{\Delta(\tau)} \;=\; \mathbf{P}, \tag{3.4.7}$$

where $\mathbf{P}^{\Delta(\tau)}$ denotes the image measure of $\Delta(\tau)$ with respect to $\mathbf{P}$.

Due to the definition, we still have that τ is an i.i.d. environment with

$$\begin{aligned}
\mathbf{E}\left(\rho_0(\tau)\right)^\kappa \;&=\; \mathbf{P}\left(\Gamma_0 = 0\right)\mathbf{E}(\rho_0(\tau))^\kappa + \mathbf{P}\left(\Gamma_0 = 1\right)1^\kappa \;=\; 1,\\
\mathbf{E}\ln\rho_0(\tau) \;&=\; \frac{1}{2}\mathbf{E}\ln\rho_0(\omega) \;<\; 0 \;\text{ and}\\
\mathbf{E}\Big[(\rho_0(\tau))^\kappa \ln^+\rho_0(\tau)\Big] \;&=\; \frac{1}{2}\mathbf{E}\Big[(\rho_0(\omega))^\kappa \ln^+\rho_0(\omega)\Big] \;<\; \infty.
\end{aligned}$$

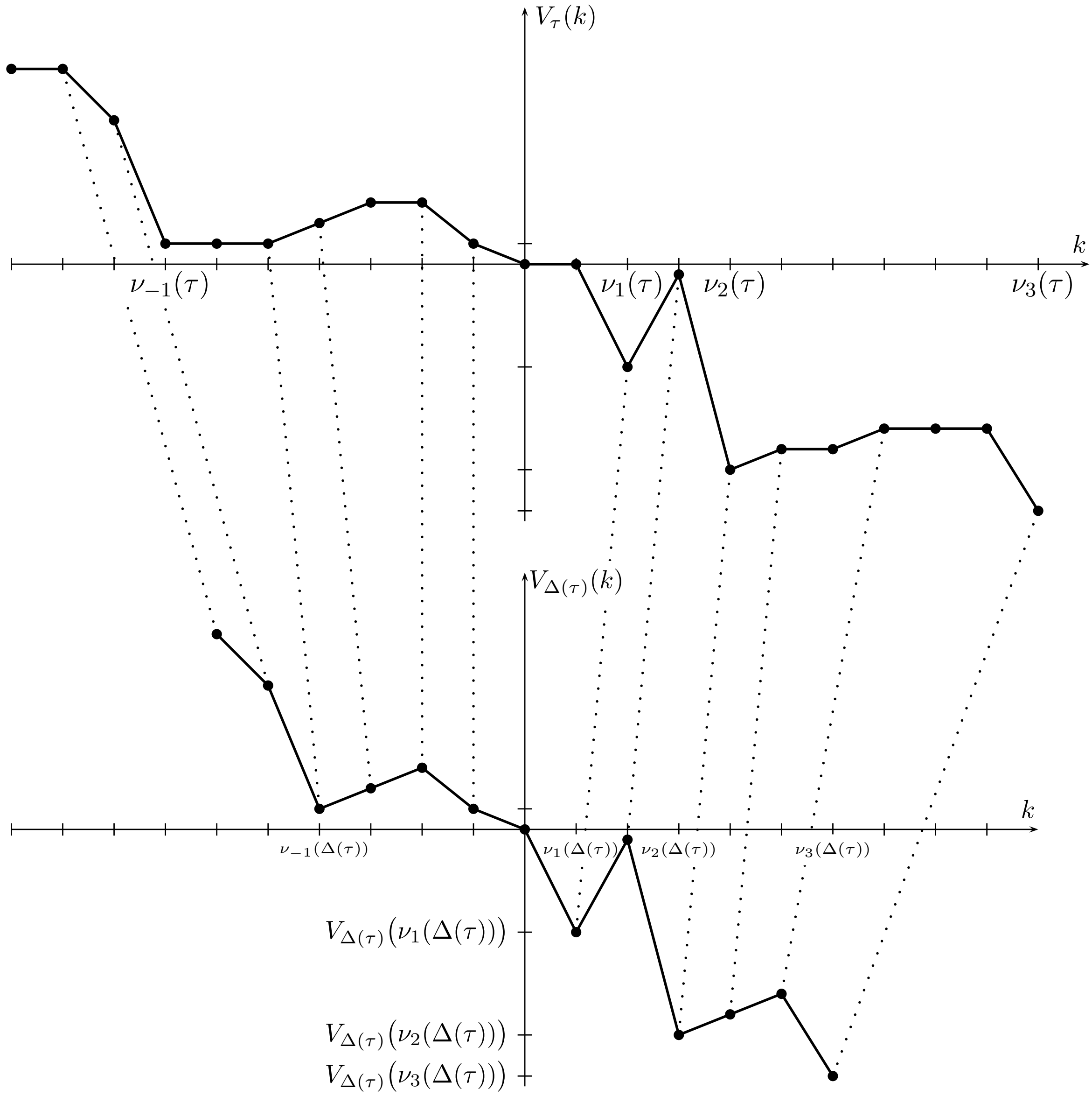

Figure 3.4: *A realisation of the "deleting procedure" Δ. Note that here we have $\Gamma_k = 0$ for $k \in \{-8, -7, -4, -3, -1, 0, 2, 3, 4, 5, 7, 10\}$ and $\Gamma_k = 1$ for $k \in \{-9, -6, -5, -2, 1, 6, 8, 9\}$.*

In particular, τ still fulfils Assumptions 1 and 2 for the same κ as ω. Note that the distribution of $\ln \rho_0(\tau)$ is non-lattice with respect to $\mathbf{P}$ because of Assumption 3.

As the next step, we want to use Lemma 3.4.3 in order to find a sequence of events on which we have one dominating block in the environment τ. Afterwards, with the help of the function Δ we will transfer this property back to the environment ω.

For $k \in \mathbb{N}$ and $C > 1$ we define (cf. (3.4.1) for the definition of d_k)

$$A_{C,k} := \left\{ \mu \in (0,1)^{\mathbb{Z}} \ : \ \exists j \in \left(n_{k-1}, n_{k-1} + \frac{d_k}{2} \right] \right.$$
$$\left. \text{with } \exp\Big(H_j(\mu)\Big) \ \geq \ C \sum_{k \in [n_{k-1}, n_k] \setminus \{j\}} E_{\mu}^{\nu_k(\mu)} \widetilde{T}^{(d_k)}_{\nu_{k+1}(\mu)} \right\}.$$

Note that this event only depends on the increments of the potential between $\nu_{n_{k-1} - \lceil (\ln d_k)^2 \rceil}(\mu)$ and $\nu_{n_k}(\mu)$. Further, we have

$$n_{k-1} - \left\lceil (\ln d_k)^2 \right\rceil > n_{k-2} \ \text{ for } k \geq 4.$$

Therefore, with respect to $\mathbf{P}$ the events $(A_{C,2k})_{k \in \mathbb{N}}$ are all independent of each other. Using (3.4.7), we get

$$\mathbf{P}\left(A_{C,k}\right) \ = \ \mathbf{P}^{\tau}\left(\Delta^{-1}(A_{C,k})\right) \ \geq \ \mathbf{P}^{\tau}\left(A_{C,k}\right) \tag{3.4.8}$$

because due to (1.2.3) and by the definition of W_i (cf. (1.2.1)) we have for all $k \in \mathbb{N}$

$$E_{\omega}^{\nu_k(\Delta(\tau))} \widetilde{T}^{(d_k)}_{\nu_{k+1}(\Delta(\tau))} \ \leq \ E_{\tau}^{\nu_k(\tau)} \widetilde{T}^{(d_k)}_{\nu_{k+1}(\tau)}.$$

Further, since for $k \geq 2$ the events $A_{C,k}$ do not depend on the environment on the left side of 0, they have the same probability under $\mathbf{Q}$ as under $\mathbf{P}$. The shift invariance of ladder locations of $\mathbf{Q}^{\tau}$ together with Lemma 3.4.3 yields for all $C > 1$

$$\begin{aligned}
\liminf_{k \to \infty} \mathbf{P}\left(A_{C,k}\right) \ &= \ \liminf_{k \to \infty} \mathbf{Q}\left(A_{C,k}\right) \\
&\geq \ \liminf_{k \to \infty} \mathbf{Q}^{\tau}\left(A_{C,k}\right) \\
&\geq \ \liminf_{k \to \infty} \mathbf{Q}^{\tau}\left(\exists j \in \left[1, \frac{d_k}{2}\right] : \exp(H_j(\mu)) \geq C \sum_{j \in [1, d_k] \setminus \{j\}} E_{\mu}^{\nu_{j-1}(\mu)} \widetilde{T}^{(d_k)}_{\nu_j(\mu)} \right) \\
&\geq C_{\kappa},
\end{aligned}$$

for a κ depending constant $C_{\kappa} > 0$.

Since the events $(A_{C,2k})_{k \in \mathbb{N}}$ are independent, we can conclude using the Borel-Cantelli lemma that for each $C > 1$ and for $\mathbf{P}$-almost every environment ω infinitely many of the events $A_{C,2k}$ occur. Hence, for $\mathbf{P}$-almost every environment ω we can find a random subsequence k_m of integers such that for each m there exists an $a_m \in \left(n_{k_{m-1}}, n_{k_{m-1}} + \frac{d_{k_m}}{2}\right]$ with

$$\exp(H_{a_m}(\omega)) \ \geq 2m^2 \left(E_{\omega}^{\nu_{n_{k_m - 1}}(\omega)} \widetilde{T}^{(d_{k_m})}_{\nu_{a_{m-1}}(\omega)} + E_{\omega}^{\nu_{a_m}(\omega)} \widetilde{T}^{(d_{k_m})}_{\nu_{n_{k_m}}(\omega)} \right). \tag{3.4.9}$$

Therefore, (3.4.9) and Lemma 3.4.2 yield for m large enough

$$\begin{aligned}\exp\left(H_{a_m}(\omega)\right) &\geq 2m^2\left(E_\omega^{\nu_{n_{k_m-1}}(\omega)}\widetilde{T}^{(d_{k_m})}_{\nu_{a_{m-1}}(\omega)} + E_\omega^{\nu_{a_m}(\omega)}\widetilde{T}^{(d_{k_m})}_{\nu_{n_{k_m}}(\omega)}\right)\\ &= 2m^2\left(E_\omega^{\nu_{n_{k_m-1}}(\omega)}\widetilde{T}^{(d_{k_m})}_{\nu_{n_{k_m}}(\omega)} - \mu_{k_m}\right)\\ &\geq m^2\left(E_\omega\widetilde{T}^{(d_{k_m})}_{\nu_{n_{k_m}}(\omega)} - \mu_{k_m}\right)\\ &\geq m^2 E_\omega\widetilde{T}^{(d_{k_m})}_{\nu_{a_{m-1}}(\omega)}.\end{aligned}$$

For $k \in \mathbb{N}$ we have

$$d_k - \left(n_{k-1} + \frac{d_k}{2}\right) \;=\; \frac{1}{2}2^{2^k} - \frac{3}{2}2^{2^{k-1}} \;=\; 2^{2^{k-1}}\left(2^{2^{k-1}-1} - \frac{3}{2}\right) \;>\; 0,$$

and therefore $a_m < d_{k_m}$. Since allowing less backtracking only decreases the crossing time, we finally get

$$\exp\left(H_{a_m}(\omega)\right) \;\geq\; m^2 E_\omega\widetilde{T}^{(d_{k_m})}_{\nu_{a_{m-1}}(\omega)} \;\geq\; m^2 E_\omega\widetilde{T}^{(a_m)}_{\nu_{a_{m-1}}(\omega)}.$$

■

Now, we are able to prove that a sequence of lazy RWRE on $(\{0,...,n\})_{n\in\mathbb{N}}$ cannot exhibit a cutoff for $\kappa < 1$:

Proof of Theorem 1.4.2. In the following, we use the sequence of deep blocks of Lemma 3.4.4 in order to construct two sequences of the same order as the mixing time with the property that along the first (smaller) one the distance to stationarity is bounded away from 1 and along the second (bigger) one bounded away from 0.

In Lemma 3.4.4, we constructed for **P**-almost every environment ω a sequence $(j_m)_{m\in\mathbb{N}} = (j_m(\omega))_{m\in\mathbb{N}}$ which fulfils

$$\exp\left(H_{j_m-1}\right) \;\geq\; m^2 E_\omega\left(\widetilde{T}^{(j_m)}_{\nu_{j_m-1}}\right) \tag{3.4.10}$$

for all $m \in \mathbb{N}$. Obviously, this can only be the case, if block $j_m - 1$ is the highest block in the interval $[0, \nu_{j_m}]$. Therefore, for environments ω which are additionally in $F(\nu_{j_m})$ (cf. (2.2.4) for the definition of $F(\nu_{j_m})$), we have that condition (2.2.5) holds. Hence, we can use Lemma 2.2.2 to obtain that for $\omega \in F(\nu_{j_m})$ we have

$$\left(E_\omega^{\nu_{j_m-1}}\left(\widetilde{T}^{(j_m)}_{\nu_{j_m}}\right)\right)^2 \;\leq\; 4\mathrm{Var}_\omega\left(\widetilde{T}^{(j_m)}_{\nu_{j_m}} - \widetilde{T}^{(j_m)}_{\nu_{j_m-1}}\right). \tag{3.4.11}$$

Further, we note that the laws of

$$\left(\frac{\widetilde{T}^{(j_m)}_{\nu_{j_m}} - \widetilde{T}^{(j_m)}_{\nu_{j_m-1}}}{E_\omega^{\nu_{j_m-1}}\widetilde{T}^{(j_m)}_{\nu_{j_m}}}\right)_{m\in\mathbb{N}} \;=\; \left(\frac{\widetilde{T}^{(j_m)}_{\nu_{j_m}} - \widetilde{T}^{(j_m)}_{\nu_{j_m-1}}}{E_\omega\left(\widetilde{T}^{(j_m)}_{\nu_{j_m}} - \widetilde{T}^{(j_m)}_{\nu_{j_m-1}}\right)}\right)_{m\in\mathbb{N}}$$

are tight with respect to P_ω due to the Markov inequality. Thus, Prohorov's Theorem yields that there exists a subsequence $\left(\nu_{j_{m_k}}\right)_{k\in\mathbb{N}}$ such that

$$\frac{\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}}}{E_\omega^{\nu_{j_{m_k}-1}} \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}}} \xrightarrow{d} Z \tag{3.4.12}$$

for some positive random variable Z as $k \to \infty$, where $\xrightarrow{d}$ means convergence in distribution.

Note that $(j_{m_k})_{k\in\mathbb{N}}$ is a subsequence of $(n_k)_{k\in\mathbb{N}} = (2^{2^k})_{k\in\mathbb{N}}$ by construction. Therefore and due to (3.4.10), assumption (2.3.6) of Lemma 2.3.3 is valid for $\omega \in F(\nu_{j_{m_k}})$ (using the property of set $B_4(\nu_{j_{m_k}})$), and we get that for $\mathbf{P}$-almost every environment ω the sequence

$$\left(\left(\frac{\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}}}{E_\omega^{\nu_{j_{m_k}-1}} \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}}}\right)^2\right)_{k\in\mathbb{N}}$$

is uniformly integrable with respect to P_ω. Thus, the first two moments of

$$\frac{\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}}}{E_\omega^{\nu_{j_{m_k}-1}} \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}}}$$

converge to the corresponding moments of Z for $\mathbf{P}$-almost every environment ω as $k \to \infty$.

Further, for $\mathbf{P}$-almost every environment ω we have $\omega \in F(n)$ for all n large enough due to Lemma 1.5.9. Hence, using (3.4.11), we get for $\mathbf{P}$-almost every environment ω

$$\mathrm{Var}_\omega(Z) \;=\; \lim_{k\to\infty} \frac{1}{\left(E_\omega^{\nu_{j_{m_k}-1}} \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}}\right)^2} \mathrm{Var}_\omega\left(\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}}\right) \;\geq\; \frac{1}{4}.$$

Therefore, for $\mathbf{P}$-almost every environment ω we have that the distribution of Z with respect to P_ω cannot be the Dirac measure in 1. Since $E_\omega Z = 1$, there exists an interval (a, b) with $a < 1 < b$, such that

$$\lim_{k\to\infty} P_\omega\left(\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}} \;>\; a \cdot E_\omega^{\nu_{j_{m_k}-1}} \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}}\right) \;<\; 1, \tag{3.4.13}$$

$$\lim_{k\to\infty} P_\omega\left(\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}} \;<\; b \cdot E_\omega^{\nu_{j_{m_k}-1}} \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}}\right) \;<\; 1. \tag{3.4.14}$$

Using Lemma 3.3.1, we get (cf. (2.2.2) for the definition of $A(n)$ and cf. (1.3.3) for the definition of $(Z_k)_{k\in\mathbb{N}}$)

$$\begin{aligned} d_n\left(\left\lceil (a+b)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right\rceil\right) &\le P_{\omega^n}\left(T_n^Y > (a+b)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right) \\ &\le P_{\widetilde{\omega}}\left(A(n)^{\mathrm{c}}\right) + P_{\omega^n}\left(\widetilde{T}_n^{(n_0)} \;>\; \left(a+\frac{1}{4}(b-a)\right)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right) \\ &\quad + P_{\widetilde{\omega}}\left(\sum_{i=1}^{\left\lceil (a+b)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right\rceil} (1-Z_i) > \left(\frac{3}{4}b+\frac{1}{4}a\right)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right). \end{aligned}$$

We note that for arbitrary $a>0$ we have $P_{\omega^n}\left(\widetilde{T}_n^{(n_0)} > a\right) \le P_\omega\left(\widetilde{T}_n^{(n_0)} > a\right)$, and therefore, we get

$$\begin{aligned} &\liminf_{n\to\infty} d_n\left(\left\lceil (a+b)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right\rceil\right) \\ &\le \lim_{k\to\infty}\left(P_{\widetilde{\omega}}\left(A\left(\nu_{j_{m_k}}\right)^{\mathrm{c}}\right) + P_\omega\left(\widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})} \;>\; \left(a+\frac{1}{4}(b-a)\right) E_\omega \widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})}\right)\right), \end{aligned}$$

where we additionally used that $\frac{3}{4}b+\frac{1}{4}a > \frac{a+b}{2}$ and thus by Cramér's Theorem

$$P_{\widetilde{\omega}}\left(\sum_{i=1}^{\left\lceil (a+b)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right\rceil} (1-Z_i) > \left(\frac{3}{4}b+\frac{1}{4}a\right)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Further, we note that due to (3.4.10) we have

$$\frac{E_\omega \widetilde{T}_{\nu_{j_{m_k}-1}}^{(j_{m_k})}}{E_\omega \widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})}} \;\le\; \frac{E_\omega \widetilde{T}_{\nu_{j_{m_k}-1}}^{(j_{m_k})}}{W_{j_{m_k}-1}} \;\le \frac{E_\omega \widetilde{T}_{\nu_{j_{m_k}-1}}^{(j_{m_k})}}{\exp(H_{j_{m_k}-1})} \;\le\; \frac{1}{m_k^2},$$

which together with Lemma 2.2.1 and equation (3.4.13) yields

$$\begin{aligned} &\liminf_{n\to\infty} d_n\left((a+b)\cdot E_\omega \widetilde{T}_n^{(n_0)}\right) \\ &\le \lim_{k\to\infty}\Bigg(P_\omega\left(\widetilde{T}_{\nu_{j_{m_k}-1}}^{(j_{m_k})} > \frac{1}{4}(b-a) E_\omega \widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})}\right) \\ &\qquad\qquad + P_\omega\left(\left(\widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})} - \widetilde{T}_{\nu_{j_{m_k}-1}}^{(j_{m_k})}\right) > a\cdot E_\omega \widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})}\right)\Bigg) \\ &\le \lim_{k\to\infty}\left(\frac{4}{b-a}\frac{E_\omega \widetilde{T}_{\nu_{j_{m_k}-1}}^{(j_{m_k})}}{E_\omega \widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})}} + P_\omega\left(\left(\widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})} - \widetilde{T}_{\nu_{j_{m_k}-1}}^{(j_{m_k})}\right) > a\cdot E_\omega \widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})}\right)\right) \\ &\le \lim_{k\to\infty}\left(\frac{4}{b-a}\cdot\frac{1}{m_k} + P_\omega\left(\left(\widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k^2})} - \widetilde{T}_{\nu_{j_{m_k}-1}}^{(j_{m_k})}\right) > a\cdot E_\omega \widetilde{T}_{\nu_{j_{m_k}}}^{(j_{m_k})}\right)\right) \\ &< 1. \end{aligned} \tag{3.4.15}$$

Next, we want to construct a sequence of (bigger) time points of the same order as $(a+b)\cdot E_\omega \widetilde{T}_n^{(n_0)}$ at which the distance to stationarity is strictly bigger than 0. We define the sequence $(a_m)_{m\in\mathbb{N}}$ by

$$a_m := \lfloor \nu_{j_m} + 2(\ln \nu_{j_m})^2 \rfloor,$$

and using (3.4.10) we get for $\omega \in F(a_m)$

$$\exp(H_{j_m-1}) \geq \frac{1}{2}m^2 \left(E_\omega \widetilde{T}^{(j_m)}_{\nu_{j_m-1}} + E^{\nu_{j_m}}_\omega \widetilde{T}_{a_m} \right)$$

because after the "large" increase in block $j_m - 1$ there cannot be a second "large" increase in the interval $[\nu_{j_m}, a_m]$ for environments $\omega \in E\left(a_m, \frac{2}{3}\right) \subset F(a_m)$. Therefore, we can conclude for $\omega \in F(a_m)$

$$\frac{E_\omega \widetilde{T}^{(j_m)}_{\nu_{j_m-1}} + E^{\nu_{j_m}}_\omega \widetilde{T}_{a_m}}{E_\omega \widetilde{T}^{(j_m)}_{a_m}} \leq \frac{E_\omega \widetilde{T}^{(j_m)}_{\nu_{j_m-1}} + E^{\nu_{j_m}}_\omega \widetilde{T}_{a_m}}{\exp(H_{j_m-1})} \leq \frac{2}{m^2} \overset{m\to\infty}{\longrightarrow} 0,$$

which yields

$$\frac{E^{\nu_{j_m-1}}_\omega \widetilde{T}^{(j_m)}_{\nu_{j_m}}}{E_\omega \widetilde{T}^{(j_m)}_{a_m}} \overset{m\to\infty}{\longrightarrow} 1. \tag{3.4.16}$$

Let $\varepsilon > 0$ be small enough such that $\left(\frac{3}{2} - \varepsilon\right) b + \frac{1}{2}a > b + a$. We get

$$\begin{aligned} & P_{\omega^n}\left(Y_{\left\lceil \left(\left(\frac{3}{2}-\varepsilon\right)b+\frac{1}{2}a\right)\cdot E_\omega \widetilde{T}^{(n)}_n \right\rceil} \geq n - 2(\ln n)^2 \right) \\ \leq & \left(P_{\omega^n}\left(T_{\lfloor n-2(\ln n)^2 \rfloor} < (1-\varepsilon)\, b \cdot E_\omega \widetilde{T}^{(n)}_n \right) \right. \\ & \left. + P_{\widetilde{\omega}}\left(\sum_{i=1}^{\left\lceil \left(\left(\frac{3}{2}-\varepsilon\right)b+\frac{1}{2}a\right)\cdot E_\omega \widetilde{T}^{(n)}_n \right\rceil} (1 - Z_i) < \left(\frac{1}{2}b + \frac{1}{2}a \right) \cdot E_\omega \widetilde{T}^{(n)}_n \right) \right) \end{aligned} \tag{3.4.17}$$

Due to the choice of ε, Cramér's Theorem yields

$$P_{\widetilde{\omega}}\left(\sum_{i=1}^{\left\lceil \left(\left(\frac{3}{2}-\varepsilon\right)b+\frac{1}{2}a\right)\cdot E_\omega \widetilde{T}^{(n)}_n \right\rceil} Z_i < \left(\frac{1}{2}b + \frac{1}{2}a \right) \cdot E_\omega \widetilde{T}^{(n)}_n \right) \overset{n\to\infty}{\longrightarrow} 0. \tag{3.4.18}$$

Again due to Lemma 1.5.9, we have that $\omega \in F(n)$ for $\mathbf{P}$-almost every environment ω and all n large enough. Thus, we get for $\mathbf{P}$-almost every environment ω using (3.4.17) and (3.4.18) (cf. (2.2.2) for the definition of the set $A(n)$)

$$\begin{aligned} & \liminf_{n\to\infty} P_{\omega^n}\left(Y_{\left\lceil \left(\left(\frac{3}{2}-\varepsilon\right)b+\frac{1}{2}a\right)\cdot E_\omega \widetilde{T}^{(n)}_n \right\rceil} \geq n - 2(\ln n)^2 \right) \\ \leq & \lim_{k\to\infty} \left(P_{\omega^n}\left(\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} < (1-\varepsilon) b \cdot E_\omega \widetilde{T}^{(j_{m_k})}_{a_{m_k}} \right) + P_{\widetilde{\omega}}\left(A\left(\nu_{j_{m_k}}\right)^{\mathrm{c}} \right) \right). \end{aligned}$$

Further, using that $\left(\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}}\right)$ has the same distribution with respect to P_ω as with respect to P_{ω^n}, we get together with (3.4.16) and Lemma 2.2.1

$$\liminf_{n\to\infty} P_{\omega^n}\left(Y_{\left\lceil\left(\left(\frac{3}{2}-\varepsilon\right)b+\frac{1}{2}a\right)\cdot E_\omega \widetilde{T}^{(n)}_n\right\rceil} \;\geq\; n-2(\ln n)^2\right)$$
$$\leq \lim_{k\to\infty}\left(P_\omega\left(\left(\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}}\right) \;<\; b\cdot E^{\nu_{j_{m_k}-1}}_\omega \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}}\right)\right).$$

Finally, this together with (3.4.14) and Lemma 3.2.1 yields

$$\begin{aligned}
&\limsup_{n\to\infty} d_n\left(\left\lceil\left(\left(\frac{3}{2}-\varepsilon\right)b+\frac{1}{2}a\right)\cdot E_\omega \widetilde{T}^{(n)}_n\right\rceil\right)\\
&\geq \limsup_{n\to\infty}\left(\pi_{\omega^n}\left(\left[n-2(\ln n)^2, n\right]\right) - P_{\omega^n}\left(Y_{\left\lceil\left(\left(\frac{3}{2}-\varepsilon\right)b+\frac{1}{2}a\right)\cdot E_\omega \widetilde{T}^{(n)}_n\right\rceil} \;\geq\; n-2(\ln n)^2\right)\right)\\
&\geq 1-\lim_{k\to\infty} P_\omega\left(\left(\widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}} - \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}-1}}\right) \;<\; b\cdot E^{\nu_{j_{m_k}-1}}_\omega \widetilde{T}^{(j_{m_k})}_{\nu_{j_{m_k}}}\right)\\
&> 0.
\end{aligned} \tag{3.4.19}$$

Since for $a<b$ we have
$$a+b<\left(\frac{3}{2}-\varepsilon\right)b+\frac{1}{2}a,$$
(3.4.15) and (3.4.19) show that the order of the window size has to be bigger or equal to the order of $E_\omega \widetilde{T}^{(n)}_n$. But due to Theorem 1.4.3, this is the order of the mixing time. Consequently, the sequence of lazy RWRE $(Y_n)_{n\in\mathbb{N}_0}$ cannot exhibit a cutoff defined in Definition 1.3.3. ■

The construction of the environment dependent sequence of high blocks is the key for the strong localisation theorem in [PZ09]. We are therefore able to prove Theorem 1.2 in [PZ09] under the weaker Assumptions 1-3:

Proof of Theorem 1.4.4. We can follow the proof of Theorem 1.2 in [PZ09]. We note at this point that Lemma 3.4.4 refines Corollary 4.4 in [PZ09] where we only need Assumption 3 instead of the stronger assumption that the distribution of $\ln\rho_0$ is non-lattice with respect to $\mathbf{P}$. ■

3.5 Simple random walk and biased random walk

In this section, we compare the results of the previous sections with the case in which we have a deterministic environment.

Theorem 3.5.1. *Let $(P_n)_{n\in\mathbb{N}}$ denote the transition matrices of the lazy simple random walk on $\{0,...,n\}$ with reflection in 0 and n. Then the sequence $(P_n)_{n\in\mathbb{N}}$ does not exhibit a pre-cutoff.*

Proof. Let $t_{\text{mix}}^{\text{SRW}}(n)$ denote the mixing time and $\text{gap}(P_n)$ the spectral gap of the lazy simple random walk on $\{0,...,n\}$ with reflection in 0 and n. Due to Proposition 10.14 in [LPW09], we have

$$t_{\text{mix}}^{\text{SRW}}(n) \;\leq\; 2 \max_{0\leq x,y\leq n} E_y T_x + 1 \;=\; 2E_0T_n + 1.$$

Using that the effective resistance between 0 and n is n, the Commute Time Identity (cf. Proposition 10.6 in [LPW09]) yields

$$t_{\text{mix}}^{\text{SRW}}(n) \;\leq\; 2n^2 + 1.$$

Now, in the situation of the lazy simple random walk on $\{0,...,n\}$ we can calculate the eigenvalues of the transition matrix explicitly and get for the spectral gap (cf. Remark 4.1.4)

$$\text{gap}(P_n) \;=\; O\left(n^{-2}\right).$$

Therefore, Lemma 3.1.4 yields that there is no pre-cutoff. ■

Next, we consider the lazy simple random walk on $\{0,...,n\}$ with a drift to the right and reflection in 0 and n.

Theorem 3.5.2 (cf. Theorem 18.2 in [LPW09])**.** *Suppose $p > \frac{1}{2}$. Let $(Y_k)_{k\in\mathbb{N}_0}$ be the lazy nearest neighbour random walk on $\{0,...,n\}$ with reflection in 0 and n and transition probabilities $\frac{p}{2}$ to move to the right, $\frac{1-p}{2}$ to move to the left and $\frac{1}{2}$ to stay at the current position. Then $(Y_k)_{k\in\mathbb{N}_0}$ has a cutoff with cutoff times*

$$t_n := \frac{2n}{2p-1}$$

and window size $\sqrt{n}$.

Proof. Note that we can also use the proof of Theorem 1.4.1 for a deterministic environment $\omega_i^n := p$ for $0 < i < n$ and use (3.2.4) instead of Lemma 3.2.1. In the case of a deterministic environment, the quenched expectation and the quenched variance are deterministic, we have (cf. (1.2.3))

$$\begin{aligned}
E_pT_n \;&=\; n + 2\sum_{i=0}^{n-1}\sum_{j=1}^{i}\left(\frac{1-p}{p}\right)^{i-j+1} \\
&=\; n + 2\sum_{i=0}^{n-1}\left(\frac{p}{2p-1}\left(1-\left(\frac{1-p}{p}\right)^{i+1}\right)-1\right) \\
&=\; n\left(1+2\frac{1-p}{2p-1}\right) + c_1 + o(1) \\
&=\; \frac{n}{2p-1} + c_1 + o(1)
\end{aligned}$$

for a suitable constant $c_1 > 0$ just depending on p. Further, we note that we have for all $i \in \mathbb{N}$

$$W_i^0 \;=\; \sum_{j=1}^{i} \left(\frac{1-p}{p}\right)^{i-j+1} \;\leq\; \frac{1-p}{2p-1} =: c_2,$$

and therefore we get (cf. (2.1.2))

$$\begin{aligned}\mathrm{Var}_p(T_n) &= 4 \cdot \sum_{j=0}^{n-1} \left(W_j^0 + \left(W_j^0\right)^2\right) + 8 \cdot \sum_{j=0}^{n-1} \sum_{i=-\infty}^{j-1} \exp(V(j) - V(i)) \cdot \left(W_i^0 + \left(W_i^0\right)^2\right) \\ &\leq 4n\left(c_2 + c_2^2\right) + 8\left(c_2 + c_2^2\right) \sum_{j=0}^{n-1} \sum_{i=-\infty}^{j-1} \left(\frac{1-p}{p}\right)^{j-i} \\ &\leq c_3 n\end{aligned}$$

for a constant $c_3 > 0$. Obviously, we also have

$$\mathrm{Var}_p(T_n) \;\geq\; c_4 n$$

for $0 < c_4 < c_3$. Therefore, the lazy simple random walk on $\{0, ..., n\}$ with a drift to the right as described in Theorem 3.5.2 exhibits a cutoff with cutoff times $t_n := \frac{2n}{2p-1}$ and window size $\sqrt{n}$. ■

The mixing properties of the transient RWRE consequently lie between those of a simple random walk and those of a random walk with drift to the right (both in deterministic environment). A comparison of the associated potentials helps to understand why. While the potential of a simple random walk is 0 for all x, the potential of a random walk with drift to the right $\left(p > \frac{1}{2}\right)$ is a line with slope $\ln \frac{1-p}{p} < 0$. In contrast to that, the potential of a RWRE with a very large κ is not a line. But it still follows a line with slope $\mathbf{E} \ln \rho_0 < 0$ and has just very small excursions (cf. Lemma 1.5.3). For smaller κ, the excursions are larger and start to dominate the behaviour of the RWRE. Although the potentials look very different, the associated stationary distributions look for all $\kappa > 0$ very similar to the case of a random walk with drift to the right (cf. Lemma 3.2.1 and (3.2.4) and Figures 1.3 and 1.4 on page 34).

3.6 Mixing time

In this section, we analyse the mixing time of a sequence of lazy RWRE on $(\{0, ..., n\})_{n \in \mathbb{N}}$ and prove Theorem 1.4.3.

Proof of Theorem 1.4.3. At first, let us assume $\kappa \leq 1$. For large n, we get using the

Markov inequality and due to Lemma 3.3.1

$$\begin{aligned} d_n\left(\lceil 12E_{\omega^n}(T_n)\rceil\right) &\leq P_{\omega^n}\left(T_n^Y > \lceil 12E_{\omega^n}(T_n)\rceil\right) \\ &\leq P_{\omega^n}\left(T_n > 5E_{\omega^n}(T_n)\right) + P_{\omega^n}\left(\sum_{i=1}^{\lceil 12E_{\omega^n}(T_n)\rceil} Z_i > 7E_{\omega^n}(T_n)\right) \\ &\leq \frac{1}{4} \end{aligned}$$

and therefore

$$t^{\omega}_{\text{mix}}(n) \;\leq\; \lceil 12E_{\omega^n}T_n\rceil \;\leq\; \lceil 12E_{\omega}T_n\rceil.$$

Due to Theorem 1.4.5 (a), this yields for $\kappa \leq 1$

$$\limsup_{n\to\infty} \frac{\ln(t^{\omega}_{\text{mix}}(n))}{\ln n} \;\leq\; \frac{1}{\kappa}. \tag{3.6.1}$$

Further, for the lower bound of the mixing time we get for any constant $c > 0$

$$\begin{aligned} & d_n\left(\left\lfloor cn^{\frac{1}{\kappa}}(\ln n)^{-\frac{4}{\kappa}}\right\rfloor - 2\right) \\ \geq\ & \pi_{\omega^n}\left([n-2(\ln n)^2, n]\right) - P_{\omega^n}\left(Y_{\left\lfloor cn^{\frac{1}{\kappa}}(\ln n)^{-\frac{4}{\kappa}}\right\rfloor - 2} \geq n - 2(\ln n)^2\right) \\ \geq\ & \pi_{\omega^n}\left([n-2(\ln n)^2, n]\right) - P_{\omega^n}\left(T^Y_{\lfloor n-2(\ln n)^2\rfloor} \leq \left\lfloor cn^{\frac{1}{\kappa}}(\ln n)^{-\frac{4}{\kappa}}\right\rfloor - 2\right) \\ \geq\ & \pi_{\omega^n}\left([n-2(\ln n)^2, n]\right) - P_{\omega^n}\left(T_{\lfloor n-2(\ln n)^2\rfloor} < \left\lfloor cn^{\frac{1}{\kappa}}(\ln n)^{-\frac{4}{\kappa}}\right\rfloor - 1\right). \end{aligned} \tag{3.6.2}$$

To reach position $\lfloor n - 2(\ln n)^2\rfloor$, the lazy RWRE has first of all to cross the highest block on the interval $[0, \lfloor n-2(\ln n)^2\rfloor]$. For environments $\omega \in B_4(\lfloor n-2(\ln n)^2\rfloor)$, we have at least one block with a height of more than $\frac{1}{\kappa}\left(\ln\left(\frac{n}{2}\right) - 4\ln\ln n\right)$. Now, we use Proposition 4.2 in [FGP10] which yields an upper bound on the probability that the crossing time of a high block is small:

Lemma 3.6.1. *There exists $\gamma > 0$ such that for any $0 \leq x < y \leq n$ and $h \in [x, y]$ we have*

$$P^x_{\omega^n}(T_y < s) \;\leq\; \gamma(1+s)\frac{\pi_{\omega^n}(h)}{\pi_{\omega^n}(x)}.$$

For a proof see Proposition 4.2 in [FGP10].

Applying this Lemma to equation (3.6.2), we get for $c := \frac{1}{\gamma\cdot 2^{1-\frac{1}{\kappa}}}$ and environments $\omega \in B_4(\lfloor n-2(\ln n)^2\rfloor)$

$$\begin{aligned} d_n\left(\left\lfloor cn^{\frac{1}{\kappa}}(\ln n)^{-\frac{4}{\kappa}}\right\rfloor - 2\right) &\geq \pi_{\omega^n}\left([n-2(\ln n)^2, n]\right) - \gamma\cdot c\cdot 2^{\frac{1}{\kappa}} \\ &= \pi_{\omega^n}\left([n-2(\ln n)^2, n]\right) - \frac{1}{2}. \end{aligned}$$

Finally, Lemma 1.5.9 and Lemma 3.2.1 yield for **P**-almost every environment ω and large n

$$t^{\omega}_{\mathrm{mix}}(n) \;\geq\; \left\lfloor \frac{1}{\gamma \cdot 2^{1-\frac{1}{\kappa}}} n^{\frac{1}{\kappa}} (\ln n)^{-\frac{4}{\kappa}} \right\rfloor - 2.$$

and therefore

$$\liminf_{n\to\infty} \frac{\ln t^{\omega}_{\mathrm{mix}}(n)}{\ln n} \;\geq\; \frac{1}{\kappa}.$$

This together with (3.6.1) shows that for $\kappa \leq 1$ and **P**-almost every environment ω we have

$$\lim_{n\to\infty} \frac{\ln t^{\omega}_{\mathrm{mix}}(n)}{\ln n} \;=\; \frac{1}{\kappa}.$$

In Theorem 1.4.1, we show that a sequence of lazy RWRE exhibits a cutoff for $\kappa > 1$. Therefore, the mixing time has the same order as the cutoff times, and due to equations (1.4.1), (1.5.15) and Lemma 1.5.8 we have

$$\lim_{n\to\infty} \frac{1}{n} \cdot t^{\omega}_{\mathrm{mix}}(n) \;=\; 2\mathbb{E}T_1$$

for **P**-almost every environment ω. ■

Chapter 4

Spectral gap

In this chapter, we use the connection between the mixing time and the spectral gap to derive statements about the asymptotic behaviour of the spectral gap of a lazy RWRE from the results in Chapter 3.

4.1 Lower and upper bounds

Let P be the transition matrix of a reversible and irreducible Markov chain with finite state space Ω and let

$$1 \; = \; \lambda_1(P) \; > \; \lambda_2(P) \; \geq \; \ldots \; \geq \; \lambda_{|\Omega|}(P) \; \geq = -1$$

denote the corresponding eigenvalues in decreasing order. We define

$$\lambda^\star(P) := \; \max\{|\lambda| \; : \; \lambda \text{ is an eigenvalue of } P, \; \lambda \neq 1\}$$

as the largest absolute-value of all non trivial eigenvalues of P and

$$\operatorname{gap}(P) := \; 1 - \lambda^\star(P)$$

as the spectral gap of P. Further let

$$\widetilde{P} := \; \frac{1}{2} \left(P + I_{|\Omega|} \right)$$

be the transition matrix of the corresponding lazy Markov chain on the same state space Ω and $I_{|\Omega|}$ denotes the $|\Omega| \times |\Omega|$ unity matrix.

We note that for all $1 \leq i \leq |\Omega|$ we have

$$\lambda_i \left(\widetilde{P} \right) \; = \; \frac{1 + \lambda_i(P)}{2}$$

and therefore

$$\lambda_i \left(\widetilde{P} \right) \; \geq \; 0.$$

This yields

$$\operatorname{gap}\left(\widetilde{P}\right) \;=\; \frac{1-\lambda_2(P)}{2}.$$

The mixing time of a Markov chain can be bounded from below and above by its spectral gap:

Theorem 4.1.1. *Let P be the transition matrix of a reversible and irreducible Markov chain with finite state space Ω and stationary distribution π. Further, let $\pi_{\min} := \min_{x\in\Omega} \pi(x)$ and t_{mix} denote the mixing time. Then we have*

$$\ln 2\left(\operatorname{gap}(P)^{-1}-1\right) \;\le\; t_{\text{mix}} \;\le\; \ln\left(\frac{4}{\pi_{\min}}\right)\operatorname{gap}(P)^{-1}.$$

For a proof see Theorem 12.3 and 12.4 in [LPW09].

The following Theorem gives a connection between the cutoff phenomenon and the asymptotic of the spectral gap:

Theorem 4.1.2. *Let $((Y_k^n)_{k\in\mathbb{N}_0})_{n\in\mathbb{N}}$ be a sequence of lazy irreducible birth-and-death chains with transition matrices $(\widetilde{P}_n)_{n\in\mathbb{N}}$ and mixing times $(t^n_{\text{mix}})_{n\in\mathbb{N}}$. Then $((Y_k^n)_{k\in\mathbb{N}_0})_{n\in\mathbb{N}}$ exhibits cutoff iff*

$$\lim_{n\to\infty} t^n_{mix}\cdot\operatorname{gap}\left(\widetilde{P}_n\right) \;=\; \infty.$$

Furthermore, the cutoff window size is at most the geometric mean between the mixing time and the inverse of the spectral gap of $\widetilde{P}_n$.

For a proof see Corollary 2 in [DLP10].

The last two theorems give us the following lower bound for the spectral gap if there is no cutoff:

Corollary 4.1.3. *Let $((Y_k^n)_{k\in\mathbb{N}_0})_{n\in\mathbb{N}}$ be a sequence of lazy irreducible birth-and-death chains with transition matrices $(\widetilde{P}_n)_{n\in\mathbb{N}}$ and mixing times $(t^n_{\text{mix}})_{n\in\mathbb{N}}$. If $((Y_k^n)_{k\in\mathbb{N}_0})_{n\in\mathbb{N}}$ does not exhibit a cutoff, then we have*

$$c_1 \;\le\; \liminf_{n\to\infty}\operatorname{gap}\left(\widetilde{P}_n\right)\cdot t^n_{\text{mix}} \;\le\; c_2$$

for constants $0 < c_1 \le c_2 < \infty$.

Proof. If $((Y_k^n)_{k\in\mathbb{N}_0})_{n\in\mathbb{N}}$ does not exhibit a cutoff, we can use Theorem 4.1.2 to obtain

$$\liminf_{n\to\infty}\operatorname{gap}\left(\widetilde{P}_n\right)\cdot t^n_{\text{mix}} \;<\; \infty.$$

Furthermore, the lower bound of Theorem 4.1.1 yields

$$(t^n_{\text{mix}})^{-1} \;=\; O\left(\operatorname{gap}\left(\widetilde{P}_n\right)\right)$$

which shows the lower bound of the corollary. ∎

From now on, let P_n^ω and $\widetilde{P}_n^\omega$ denote the transition matrices of a RWRE $(X_k)_{k\in\mathbb{N}_0}$ and a lazy RWRE $(Y_k)_{k\in\mathbb{N}_0}$, respectively, with respect to P_{ω^n}. With the help of Theorem 4.1.1 and 4.1.2 we can now proof Theorem 1.4.7:

Proof of Theorem 1.4.7. First we assume $0 < \kappa < 1$. Part (a) of the Theorem follows directly by Theorem 1.4.2, Theorem 1.4.3 *(a)* and Corollary 4.1.3.

Next, we assume $\kappa > 1$. Then due to Theorem 1.4.1 the sequence of lazy RWRE $(Y_k^n)_{k\in\mathbb{N}_0}$ exhibits for $\mathbf{P}$-almost every environment ω a cutoff with cutoff times

$$t_\omega(n) = 2E_{\omega^n}T_n.$$

Therefore, Theorem 4.1.2 yields

$$\lim_{n\to\infty} \operatorname{gap}\left(\widetilde{P}_n^\omega\right) \cdot t_{\text{mix}}^\omega(n) \;=\; \infty \quad \mathbf{P} - a.s.$$

and together with (1.4.1) we get

$$\lim_{n\to\infty} \operatorname{gap}\left(\widetilde{P}_n^\omega\right) \cdot n \;=\; \infty \quad \mathbf{P} - a.s.$$

Further, for $1 < \kappa < 2$ we know due to Proposition 3.3.2 that for $\mathbf{P}$-almost every environment ω the cutoff window size has to be bigger than $n^{\frac{1}{\kappa}-\delta}$ for all $\delta > 0$ if we additionally assume that the distribution of $\ln \rho_0$ is non-lattice. Together with Theorem 4.1.2 we can conclude

$$\operatorname{gap}\left(\widetilde{P}_n^\omega\right) \;=\; O\left(n^{-\frac{2}{\kappa}+\delta} \cdot t_{\text{mix}}^\omega(n)\right) \quad \mathbf{P} - a.s.$$

for all $\delta > 0$. Finally, (1.4.1) yields that for $1 < \kappa < 2$ and $\mathbf{P}$-almost every environment ω we have

$$\limsup_{n\to\infty} \frac{\ln \operatorname{gap}\left(\widetilde{P}_n^\omega\right)}{\ln n} \;=\; 1 - \frac{2}{\kappa} \;<\; 0,$$

which finishes the proof of part (b). ■

Remark 4.1.4.
For nearest neighbour random walks on $\{0, ..., n\}$ with $(n+1) \times (n+1)$ transition matrix

$$A_n \;=\; \begin{pmatrix} p_s + p_l & p_r & 0 & 0 & \dots & 0 \\ p_l & p_s & p_r & 0 & \dots & 0 \\ 0 & p_l & p_s & p_r & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p_l & p_s + p_r \end{pmatrix}$$

with $p_l + p_s + p_r = 1, p_l, p_s, p_r > 0$, we can calculate the eigenvalues of A_n explicitly (for example cf. Theorem 3.3 in [Par09]). We get that the eigenvalues of A_n are given by $\lambda = 1$ and

$$\lambda_k = 2\sqrt{p_l \cdot p_r} \cos\left(\frac{\pi \cdot k}{n+1}\right) + p_s, \quad \text{for } 1 \leq k \leq n.$$

Therefore, we get for the second largest eigenvalue of a lazy symmetric random walk $\left(p_s = \frac{1}{2}, p_l = p_r = \frac{1}{4}\right)$ using Taylor expansion around 0

$$\lambda_2(A_n^{\text{symm}}) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{n+1}\right) = 1 - \frac{\pi^2}{2(n+1)^2} + O\left(n^{-4}\right),$$

which yields that

$$\text{gap}(A_n^{\text{symm}}) = O\left(n^{-2}\right).$$

For lazy random walks with positive drift $\left(p_s = \frac{1}{2}, p_r > p_l\right)$, we get that for all $n \in \mathbb{N}$

$$\lambda_2(A_n^{\text{drift}}) \leq \frac{1}{2} + 2\sqrt{p_l \cdot p_r} < 1.$$

Therefore, the spectral gap is bounded by $\frac{1}{2} - 2\sqrt{p_l \cdot p_r} > 0$ uniformly in n.

Chapter 5

Cookie branching random walks

In this chapter, we consider cookie branching random walks on $\mathbb{Z}$. We give an explicit and complete criterion for the recurrence/transience behaviour for the case in which there is one cookie at every integer.

5.1 Model and basic notation

In order to define the *cookie branching random walk* (CBRW), we first have to choose the initial configuration of the cookies. We restrict ourselves to the case in which we have one cookie at every non-negative integer and no cookies at the negative integers. Thus, denoting by $c_n(x)$ the number of cookies at position $x \in \mathbb{Z}$ at time $n \in \mathbb{N}_0$, this situation is described through

$$c_0(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

As it turns out, the above configuration of cookies is a natural choice for an initial configuration in order to point out the essential differences in the evolution of the process. In particular, further results for the initial configuration ($c_0(x) = 1$ for all $x \in \mathbb{Z}$) can be derived easily (cf. Section 5.6). At time 0 the CBRW starts with one initial particle at the origin. To specify the evolution of the population of particles, we need the following ingredients:

- *the cookie offspring distribution* $\mu_c = \Big(\mu_c(k)\Big)_{k\in\mathbb{N}_0}$ with mean $m_c := \sum_{k=1}^{\infty} k\mu_c(k)$;
- *the cookie transition probabilities* $p_c \in (0,1)$, $q_c := 1 - p_c$;
- *the no-cookie offspring distribution* $\mu_0 = \Big(\mu_0(k)\Big)_{k\in\mathbb{N}_0}$ with mean $m_0 := \sum_{k=1}^{\infty} k\mu_0(k)$;
- *the no-cookie transition probabilities* $p_0 \in (0,1)$, $q_0 := 1 - p_0$.

We say a particle produces offspring according to a offspring distribution $\mu = (\mu(k))_{k\in\mathbb{N}_0}$ if the probability of having k offspring is $\mu(k)$. With these above quantities fixed, the population of particles evolves at every discrete time unit $n \in \mathbb{N}_0$ according to the following rules:

(1) First, every existing particle produces offspring independently of the other particles. Each particle either reproduces according to the offspring distribution μ_c if there is a cookie at its position or according to μ_0 otherwise. After that the parent particle dies.

(2) Second, after the branching the newly produced offspring particles move independently of each other either one step to the right or one step to the left. Again the movement depends on whether the particles are at a position with or without a cookie. If there is a cookie, each particle moves to the right (left) with probability p_c (q_c). Otherwise, if there is no cookie, the transition probabilities are given by p_0 and q_0.

(3) Finally, each cookie which is located at a position where at least one particle has produced offspring is removed. Note that different particles share the same cookie if they are at a position with a cookie at the same time. Moreover, due to the chosen initial configuration of the cookies only the leftmost cookie can be consumed at every time step.

We now introduce some essential notations and assumptions. Since we do not want the process to die out, we assume that

$$\mu_c(0) \;=\; \mu_0(0) \;=\; 0$$

holds. Further to avoid additional technical difficulties, we suppose that we have

$$M := \;\sup\{k \in \mathbb{N}_0 : \mu_c(k) + \mu_0(k) > 0\} \;<\; \infty. \tag{5.1.1}$$

In fact, we believe that the results remain true if we replace (5.1.1) by the assumption that the cookie and the no-cookie offspring variance is finite. In the following we want to distinguish different particles of the CBRW by using the usual Ulam-Harris labelling. Therefore, we enumerate the offspring of every particle and introduce the set

$$\mathbb{V} := \bigcup_{n\in\mathbb{N}_0} \mathbb{N}^n$$

as the set of all particles which may be produced at some time in the whole process. Thereby $\mathbb{N}^0$ consists of the root $\emptyset$ which denotes the initial particle. In this setting,

$$\nu \;=\; (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{V}$$

labels the particle which is the ν_n-th offspring of the particle $(\nu_1, \nu_2, \ldots \nu_{n-1})$. By iteration we can trace back the ancestral line of ν to the initial particle $\emptyset$. Further, we denote the generation (length) of the particle $\nu \in \mathbb{V}$ by $|\nu|$, and for two particles $\nu, \eta \in \mathbb{V}$ we write

$$\nu \succ \eta \quad \text{(respectively, } \nu \succeq \eta)$$

if ν is a descendant of the particle η (respectively, if ν is a descendant of η or η itself). We use the same notation

$$\nu \succeq U \quad \text{(respectively, } \nu \succ U)$$

for some set $U \subseteq \mathbb{V}$ if there is a particle $\eta \in U$ with $\nu \succeq \eta$ (respectively, $\nu \succ \eta$). With the above notations, we can consider the actually produced particles in the CBRW. For $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}$ let

$$Z_n(x) \subset \mathbb{N}^n \subset \mathbb{V}$$

denote the random set of particles which are at position x at time n. Thus

$$Z_n := \bigcup_{x \in \mathbb{Z}} Z_n(x)$$

is the set of all particles which exist at time n and using this we can define

$$\mathcal{Z} := \bigcup_{n \in \mathbb{N}_0} Z_n$$

as the set of all particles ever produced. Then, for every particle $\nu \in \mathcal{Z}$ we write X_ν for its random position in $\mathbb{Z}$ and the collection of all positions of all particles

$$(X_\nu)_{\nu \in \mathcal{Z}}$$

is what we call CBRW. Further, we denote the position of the leftmost cookie by

$$l(n) := \min\{x \in \mathbb{N} : c_n(x) = 1\}.$$

Now, we are able to define the set of particles $Z^{(l)}n$ which is crucial for our considerations:

$$Z^{(l)}n := Z_n(l(n))$$

The particles that belong to $Z^{(l)}n$ are located at the position of the leftmost cookie and thus they are the only particles which produce offspring according to μ_c. We call the process $\left(Z^{(l)}n\right)_{n \in \mathbb{N}_0}$ *leading process* (and use the abbreviation LP) since it contains the rightmost particles if $Z^{(l)}n \neq \emptyset$. One key observation for the understanding of the CBRW is that the particles in the LP constitute a Galton-Watson process (GWP) as long as there are particles in the LP. The associated mean offspring is given by $p_c m_c$ and thus we call the LP supercritical (respectively, subcritical, or critical) when $p_c m_c$ is greater than 1 (respectively, smaller than 1, or equal to 1).

As it is usually done in the context of branching random walks, we now define three different regimes:

Definition 5.1.1. A CBRW is called

(1) *strongly recurrent* if a.s. infinitely many particles visit the origin, i.e. $\mathsf{P}\left(|Z_n(0)| \xrightarrow[n\to\infty]{} 0\right) = 0$,

(2) *weakly recurrent* if $\mathsf{P}\left(|Z_n(0)| \xrightarrow[n\to\infty]{} 0\right) \in (0,1)$,

(3) *transient* if $\mathsf{P}\left(|Z_n(0)| \xrightarrow[n\to\infty]{} 0\right) = 1$.

We mention that these regimes may have different names in the literature; for instance, strong local survival, local survival, and local extinction of [GMPV10] correspond to strong recurrence, recurrence, and transience of this chapter. The transient regime may be subdivided into *transient to the left* (resp. *transient to the right*) if the negative (resp. positive) integers are visited infinitely many times.

5.2 Branching random walk and cookie random walk

Criteria for the recurrence/transience behaviour of *branching random walks* (BRW) and *cookie random walks* (CRW) are well-known in the literature.

In our setting the BRW of interest is the process related to the behaviour of the particles without cookies. In the following we call this process BRW without cookies. It is a BRW in the usual sense started with one particle at 0, with offspring distribution μ_0 and transition probabilities p_0, q_0 to the nearest neighbours. In this case we have the following proposition that goes back to classical work of Biggins [Big76], Hammersley [Ham74], and Kingman [Kin75]; for a proof we refer to Theorem 18.3 in [Per99] and Theorem 3.2 in [GM06].

Proposition 5.2.1. *The BRW without cookies is*

(1) transient to the right iff

$$p_0 > \frac{1}{2} \quad \text{and} \quad m_0 \leq \frac{1}{2\sqrt{p_0 q_0}},$$

(2) transient to the left iff

$$p_0 < \frac{1}{2} \quad \text{and} \quad m_0 \leq \frac{1}{2\sqrt{p_0 q_0}},$$

(3) and strongly recurrent in the remaining cases.

On the other hand, if we consider a CBRW with

$$\mu_0(1) \;=\; \mu_c(1) \;=\; 1, \quad p_0 \;=\; \frac{1}{2} \quad \text{and} \;\; c_0(x) = 1 \; \forall \; x \in \mathbb{Z}$$

our model corresponds to the *exited random walk model* on $\mathbb{Z}$ introduced by Benjamini and Wilson in 2003 (cf. [BW03]). In [BW03], they analysed the excited random walk on $\mathbb{Z}^d$ which evolves in the following way: If the random walker visits a vertex for the first time, he steps right with probability $(1+\varepsilon)/(2d)$, left with probability $(1-\varepsilon)/(2d)$, and in other directions with probability $1/(2d)$. On subsequent visits to a vertex, the walker chooses one neighbour uniformly at random. This model is also frequently called *cookie random walk* (CRW) and the interpretation of the cookies is the same as in the CBRW model. Initially, there is one cookie at every vertex, and when the random walker visits a vertex for the first time he eats the cookie and this gives him a drift in one direction. Afterwards, the cookie is removed and on subsequent visits to that vertex the random walker chooses one of the neighbours uniformly at random. The following theorem says that this modification of the walk is already enough to turn the simple random walk on $\mathbb{Z}^2$ from a recurrent into a transient walk.

Theorem 5.2.2 (cf. Theorem 4 in [BW03])**.** *Suppose $p_c \in (1/2, 1]$. Then the CRW is transient iff $d \geq 2$.*

Note that by a paper of Davis (cf. [Dav99]) it was already known that an excited random walk on $\mathbb{Z}$ is recurrent.

Further, it is interesting to notice that there is both recurrence and transience possible if we consider a CRW on $\mathbb{Z}$ with initially two cookies at every integer of $\mathbb{Z}$.

Theorem 5.2.3 (cf. Theorem 12 in [Zer05])**.** *Suppose $c_0(x) = 2 \; \forall \; x \in \mathbb{Z}$ and let $p_0 = \frac{1}{2}$. Then the CRW is recurrent iff $p_c \in [1/4, 3/4]$.*

5.3 Results

First, we define

$$\varphi_\ell := \begin{cases} \frac{1}{2p_0 m_0}\left(1 - \sqrt{1 - 4p_0 q_0 m_0^2}\right), & \text{if } m_0 > 1, \\ \min\left\{1, \frac{q_0}{p_0}\right\}, & \text{if } m_0 = 1 \end{cases}$$

(the meaning of the quantity φ_ℓ is explained in Section 5.4 below).

Next, we formulate the main results of this part of the thesis. Theorems 5.3.1 - 5.3.3 give a complete classification of the process with respect to weak/strong recurrence in the sense of Definition 5.1.1.

Theorem 5.3.1. *Suppose that the BRW without cookies is transient to the right.*

(a) If the LP is supercritical, i.e. $p_c m_c > 1$ holds, then

(i) the CBRW is strongly recurrent iff $p_c m_c \varphi_\ell \geq 1$,

(ii) and the CBRW is transient iff $p_c m_c \varphi_\ell < 1$.

(b) If the LP is subcritical or critical, i.e. $p_c m_c \leq 1$ holds, then the CBRW is transient.

Theorem 5.3.2. *Suppose that the BRW without cookies is strongly recurrent.Then the CBRW is strongly recurrent, no matter whether the LP is subcritical, critical or supercritical.*

Theorem 5.3.3. *Suppose that the BRW without cookies is transient to the left.*

(a) If the LP is supercritical, i.e. $p_c m_c > 1$ holds, then the CBRW is weakly recurrent.

(b) If the LP is critical or subcritical, i.e. $p_c m_c \leq 1$ holds, then the CBRW is transient.

5.4 Preliminaries

In this section, we define an embedded *Galton-Watson process* (GWP) which will be very helpful for the proofs in the next section. Further, we collect some known facts about GWP.

Analogously to the notation which we use for the CBRW, let $\big(Y_\nu\big)_{\nu\in\mathcal{Y}}$ denote the BRW without cookies. Thereby $\mathcal{Y}$ denotes the set of all ever produced particles and (for every $\nu \in \mathcal{Y}$) Y_ν denotes the random position of the particle ν. We define

$$\begin{aligned}
\Lambda_0^+ = \Lambda_0^- &:= 1, \\
\Lambda_n^+ &:= \sum_{\nu\in\mathcal{Y}} \mathbb{1}_{\{Y_\nu = n,\ Y_\eta < n\ \forall \eta \prec \nu\}}, \\
\Lambda_n^- &:= \sum_{\nu\in\mathcal{Y}} \mathbb{1}_{\{Y_\nu = -n,\ Y_\eta > -n\ \forall \eta \prec \nu\}}
\end{aligned} \tag{5.4.1}$$

for $n \in \mathbb{N}$. Here Λ_n^+ (respectively, Λ_n^-) denotes the random number of particles which are the first in their ancestral line to reach the position n (respectively, $-n$). In addition, we define

$$\begin{aligned}
\varphi_r &:= \mathsf{E}[\Lambda_1^+], \\
\varphi_\ell &:= \mathsf{E}[\Lambda_1^-].
\end{aligned} \tag{5.4.2}$$

Note that we have

$$\mathsf{P}\big(\Lambda_1^+ < \infty\big) \;=\; \mathsf{P}\big(\Lambda_1^- < \infty\big) \;=\; 1$$

if the BRW without cookies $\big(Y_\nu\big)_{\nu\in\mathcal{Y}}$ is transient. In this case the processes $\big(\Lambda_n^+\big)_{n\in\mathbb{N}_0}$ and $\big(\Lambda_n^-\big)_{n\in\mathbb{N}_0}$ are both GWPs. An important observation is that φ_r and φ_ℓ can be expressed using the first visit generating function of the underlying random walk. Thus, denote by X_n the nearest neighbour random walk defined by

$$\begin{aligned}\mathsf{P}(X_{n+1} = x+1 \mid X_n = x) \;&=\; p_0\\ \mathsf{P}(X_{n+1} = x-1 \mid X_n = x) \;&=\; q_0.\end{aligned}$$

The *first visit generating function* is defined by

$$F(x,y|z) \;=\; \sum_{n=0}^{\infty} \mathsf{P}(X_n = y, X_k \neq y \;\forall k < n \mid X_0 = x)z^n.$$

A (short) thought reveals that $\varphi_r = F(0,1|m_0)$ and $\varphi_\ell = F(0,-1|m_0)$ and standard calculations yield the following formulas; for both arguments one might also consult Chapter 5 in [Woe09].

Proposition 5.4.1. *If the BRW without cookies is transient, we have*

$$\varphi_r \;=\; \frac{1}{2q_0m_0}\left(1-\sqrt{1-4p_0q_0m_0^2}\right), \tag{5.4.3}$$

and

$$\varphi_\ell \;=\; \frac{1}{2p_0m_0}\left(1-\sqrt{1-4p_0q_0m_0^2}\right). \tag{5.4.4}$$

Remark 5.4.2.
A natural special case is the situation where $\mu_0(1) = 1$. In this model particles can only branch at positions with a cookie. In sites without cookies the process reduces to an asymmetric random walk $\big(Y_n\big)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ with transition probabilities p_0 and q_0. Here φ_r and φ_ℓ simplify to the probabilities of an asymmetric random walk to ever reach $+1$ or -1, respectively, i.e.

$$\varphi_r \;=\; \mathsf{P}\big(\exists n\in\mathbb{N}:\; Y_n = +1\big) \;=\; \min\left\{1, \tfrac{p_0}{q_0}\right\}$$

and

$$\varphi_\ell \;=\; \mathsf{P}\big(\exists n\in\mathbb{N}:\; Y_n = -1\big) \;=\; \min\left\{1, \tfrac{q_0}{p_0}\right\}.$$

Remark 5.4.3.
The quantities φ_r and φ_ℓ play an important role for our argument. As long as the particles do not reach a cookie, the particles in the CBRW behave as in the BRW without cookies. If we assume for example that the BRW without cookies is transient to the right, then $\varphi_\ell < 1$ and the probability of a particle to produce an offspring which moves n steps to the left before hitting a cookie again decays like $F(n,0|m_0) = (\varphi_\ell)^n$.

Next, we collect some known facts about Galton-Watson processes that will be needed in the sequel. An important tool for the proofs is to identify GWPs which are embedded in the CBRW. For the rest of this chapter the processes

$$\big(GW_n^{\text{super}}\big)_{n\in\mathbb{N}_0},\ \big(GW_n^{\text{sub}}\big)_{n\in\mathbb{N}_0} \text{ and } \big(GW_n^{\text{cr}}\big)_{n\in\mathbb{N}_0}$$

shall denote a supercritical, subcritical or critical GWP started with $z \in \mathbb{N}$ particles with respect to the probability measure P_z. Furthermore, let $T^{\text{super}}, T^{\text{sub}}$ and T^{cr} denote the time of extinction corresponding to the above GWPs, i.e.

$$T^{\text{super}} := \inf\{n \geq 0 :\ GW_n^{\text{super}} = 0\}$$

and analogously for the subcritical and critical case.

Proposition 5.4.4. *For a subcritical GWP $\big(GW_n^{\text{sub}}\big)_{n\in\mathbb{N}_0}$ with strictly positive and finite offspring variance there is a constant $c > 0$ such that*

$$\lim_{n\to\infty} \frac{\mathsf{P}_1\left(GW_n^{\text{sub}} > 0\right)}{\mathsf{E}_1\left[GW_1^{\text{sub}}\right]^n} \ = \ c.$$

For a proof see for instance Theorem 2.6.1 in [Jag75].

Proposition 5.4.5. *For a critical GWP $\big(GW_n^{\text{cr}}\big)_{n\in\mathbb{N}_0}$ with strictly positive and finite offspring variance there is a constant $c > 0$ such that*

$$\lim_{n\to\infty} n\, \mathsf{P}_1\left(GW_n^{\text{cr}} > 0\right) \ = \ c.$$

For a proof see for instance Theorem I.9.1 in [AN72]. Using the inequality $1 - x \leq \exp(-x)$ we obtain the following consequence of Proposition 5.4.5.

Proposition 5.4.6. *For the extinction time T^{cr} of a critical GWP with strictly positive and finite offspring variance there exists a constant $C > 0$ such that*

$$\mathsf{P}_z\big(T^{\text{cr}} \leq n\big) \ \leq \ \exp\left(-C\frac{z}{n}\right)$$

for all $n \in \mathbb{N}$ and for all $z \in \mathbb{N}$.

Proposition 5.4.7. *For the extinction time T^{cr} of a critical GWP with strictly positive and finite offspring variance there exists a constant $C > 0$ such that*

$$\mathsf{P}_z\big(T^{\text{cr}} = n\big) \ \leq \ C\frac{z}{n^2}$$

for all $n \in \mathbb{N}$ and for all $z \in \mathbb{N}$.

Proof. Due to Corollary I.9.1 in [AN72] (with $s = 0$), there is a constant $c > 0$ such that

$$\lim_{n\to\infty} n^2 \mathsf{P}_1\big(T^{\text{cr}} = n+1\big) \ = \ c.$$

Therefore, we get for $n \in \mathbb{N}$

$$\mathsf{P}_z\big(T^{\text{cr}} = n\big) \ \leq \ z\, \mathsf{P}_1\big(T^{\text{cr}} = n\big) \ = \ z\frac{1}{(n-1)^2}\big(c + o(1)\big) \ \leq \ C\frac{z}{n^2}$$

for a suitable constant $C > 0$. □

5.5 Proofs of the main results

5.5.1 Proof of Theorem 5.3.1

Proof of part (a).
In this part of the proof we suppose $p_c m_c > 1$, i.e. the LP is supercritical. For $n \in \mathbb{N}$ we define inductively the n-th extinction time and the n-th rebirth time of the LP by

$$\begin{aligned} \tau_n &:= \inf\big\{ i > \sigma_{n-1} : |Z^{(l)} i| = 0 \big\}, \\ \sigma_n &:= \inf\big\{ i > \tau_n : |Z^{(l)} i| \geq 1 \big\} \end{aligned}$$

with $\sigma_0 := 0$ and $\inf \emptyset := \infty$. Since $p_0 > 1/2$ and the LP is supercritical we have that

$$\mathsf{P}(\sigma_n < \infty \mid \tau_n < \infty) \;=\; 1$$

and

$$\mathsf{P}(\tau_{n+1} = \infty \mid \tau_n < \infty) \;\geq\; \mathsf{P}(\tau_1 = \infty) > 0$$

for all $n \geq 0$. Hence, we a.s. have

$$\sigma^* := \inf\{ n \in \mathbb{N}_0 : |Z^{(l)} i| \geq 1 \;\forall\, i \geq n \} \;<\; \infty. \tag{5.5.1}$$

It is a well-known fact that conditioned on survival a supercritical GWP with finite second moment normalized by its mean converges to a strictly positive random variable (e.g. see Theorem I.6.2 in [AN72]). Considering the LP seperately on the sets $\{\sigma^* = k\}$ for $k \in \mathbb{N}_0$ yields

$$\lim_{n\to\infty} \frac{|Z^{(l)} n|}{(p_c m_c)^n} \;=\; W \;>\; 0 \tag{5.5.2}$$

for a strictly positive random variable W.

(i) Now, we suppose $p_c m_c \varphi_\ell \geq 1$. For $n \in \mathbb{N}_0$, let us introduce

$$L_n := \big\{ \nu \in Z_{n+1}(l(n) - 1) : \nu \succ Z^{(l)} n \big\}.$$

The set L_n contains all particles that are produced in the LP at time n and then leave the LP to the left. Thus they are located at the position $l(n) - 1$ at time $n + 1$. We define for $n \in \mathbb{N}_0$ the sets

$$A_n := \big\{ \exists\, \nu \succeq L_n : X_\nu = 0 \big\}.$$

In order to show strong recurrence of the CBRW it is now sufficient to proof that

$$\mathsf{P}\Big(\limsup_{n\to\infty} A_n \Big) \;=\; 1. \tag{5.5.3}$$

As a first step to achieve this, we consider the events

$$B_n := \big\{ |L_n| \geq \big(p_c m_c\big)^n n^{-1},\; n \geq \sigma^* \big\}$$

for $n \in \mathbb{N}_0$ and show that

$$\mathsf{P}\left(\liminf_{n\to\infty} B_n\right) \;=\; 1. \tag{5.5.4}$$

This provides a lower bound for the growth of $|L_n|$ for large n. To see that (5.5.4) holds, we define

$$C_n := \;\left\{|Z^{(l)}n| \geq (p_c m_c)^n n^{-1/2}\right\}$$

and notice that due to (5.5.2) we have

$$\mathsf{P}\left(\liminf_{n\to\infty} C_n\right) \;=\; 1. \tag{5.5.5}$$

We observe that, given the event C_n, the random variable $|L_n|$ can be bounded from below by a random sum of $\lceil (p_c m_c)^n n^{-1/2}\rceil$ i.i.d. Bernoulli random variables with success probability q_c. For a sum $S(n)$ of n i.i.d. Bernoulli random variables with success probability q_c a large deviation bound yields

$$\mathsf{P}\big(S(n) \leq \tfrac{q_c}{2}\cdot n\big) \;\leq\; \exp\big(-I_1\cdot n\big) \tag{5.5.6}$$

for all $n \in \mathbb{N}_0$ and some $I_1 > 0$ (see for instance Theorem 2.2.3 and Exercise 2.2.23 in [DZ98]). Therefore by using (5.5.6), we get

$$\begin{aligned}
&\mathsf{P}\left(\left\{|L_n| < (p_c m_c)^n\cdot n^{-1}\right\}\cap C_n\right)\\
&\leq \mathsf{P}\left(|L_n| < (p_c m_c)^n\cdot n^{-1} \;\middle|\; C_n\right)\\
&\leq \mathsf{P}\left(S\big(\lceil (p_c m_c)^n\cdot n^{-1/2}\rceil\big) < \frac{(p_c m_c)^n\cdot n^{-1}}{\lceil (p_c m_c)^n\cdot n^{-1/2}\rceil}\cdot\lceil (p_c m_c)^n\cdot n^{-1/2}\rceil\right)\\
&\leq\; \exp\left(-I_1\cdot (p_c m_c)^n\cdot n^{-1/2}\right)
\end{aligned} \tag{5.5.7}$$

for large n. Since the sum over the upper bound in (5.5.7) converges, we can conclude (using the Borel-Cantelli lemma) that we have

$$\mathsf{P}\left(\limsup_{n\to\infty}\left(\left\{|L_n| < (p_c m_c)^n n^{-1}\right\}\cap C_n\right)\right) \;=\; 0. \tag{5.5.8}$$

Since $\sigma^* < \infty$ a.s., (5.5.8) together with (5.5.5) yields (5.5.4).

Next, we observe that on $\{n \geq \sigma^*\}$ the amount of offspring of every particle in L_n which ever moves $1, 2, \ldots$ steps to the left for the first time in their genealogy constitutes an embedded GWP in the CBRW. Its mean is given by φ_ℓ, where $\varphi_\ell < 1$ holds since the BRW without cookie is transient to the right (cf. Remark 5.4.3). Additionally, Lemma B.1 yields that the second moment of this GWP is finite. Using Proposition 5.4.4, we therefore get

$$\begin{aligned}
\mathsf{P}\big(A_n \mid |L_n| \geq (p_c m_c)^n n^{-1},\ n \geq \sigma^* \big) &\geq 1 - (1 - c(\varphi_\ell)^n)^{(p_c m_c)^n n^{-1}} \\
&\geq 1 - \exp\big(-c(\varphi_\ell)^n (p_c m_c)^n n^{-1}\big) \\
&\geq 1 - \exp\big(-\tfrac{c}{n}\big) \\
&\geq \tfrac{C}{n} \qquad (5.5.9)
\end{aligned}$$

for some $c, C > 0$. Here we used that the position of a particle $\nu \in L_n$ is bounded by n (in fact by $n-1$). Notice also that we have $p_c m_c \varphi_\ell \geq 1$ by assumption. Since $\mathbb{1}_{B_n}$ is measurable with respect to the σ-algebra generated by $|L_n|$ and σ^*, we have for $i, j \in \mathbb{N}$ with $i < j$

$$\begin{aligned}
\mathsf{P}\left(\bigcap_{n=i}^{j} \big(A_n^{\mathrm{c}} \cap B_n\big) \right) &= \mathsf{E}\left[\mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{A_n^{\mathrm{c}} \cap B_n} \;\middle|\; |L_i|, \ldots, |L_j|,\ \sigma^* \right]\right] \\
&= \mathsf{E}\left[\Big(\prod_{n=i}^{j} \mathbb{1}_{B_n} \Big) \mathbb{1}_{\{i \geq \sigma^*\}} \mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{A_n^{\mathrm{c}}} \;\middle|\; |L_i|, \ldots, |L_j|,\ \sigma^* \right]\right].
\end{aligned}$$

Now, we observe that on $\{i \geq \sigma^*\}$ the random variables $\big(\mathbb{1}_{A_n^{\mathrm{c}}}\big)_{i \leq n \leq j}$ are conditionally independent given $|L_i|, \ldots, |L_j|$ and σ^*. This holds because on $\{i \geq \sigma^*\}$ all the particles in $\bigcup_{n=i}^{j} L_n$ start independent BRWs which cannot reach the cookies anymore. For the same reason on $\{i \geq \sigma^*\}$ each of the random variables $(\mathbb{1}_{A_n^{\mathrm{c}}})_{i \leq n}$ is conditionally independent of $(|L_k|)_{k \neq n}$ given $|L_n|$ and σ^*. Using these two facts we obtain

$$\begin{aligned}
&\mathsf{E}\left[\left(\prod_{n=i}^{j} \mathbb{1}_{B_n} \right) \mathbb{1}_{\{i \geq \sigma^*\}} \cdot \mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{A_n^{\mathrm{c}}} \;\middle|\; |L_i|, \ldots, |L_j|,\ \sigma^* \right]\right] \\
&= \mathsf{E}\left[\prod_{n=i}^{j} \left(\mathbb{1}_{B_n} \mathbb{1}_{\{i \geq \sigma^*\}} \mathsf{E}\left[\mathbb{1}_{A_n^{\mathrm{c}}} \;\middle|\; |L_i|, \ldots, |L_j|,\ \sigma^* \right] \right) \right] \\
&= \mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{B_n} \mathsf{E}\left[\mathbb{1}_{A_n^{\mathrm{c}}} \;\middle|\; |L_n|,\ \sigma^* \right] \right]. \qquad (5.5.10)
\end{aligned}$$

With the help of (5.5.9) and (5.5.10) we can now conclude that we have

$$\begin{aligned}
\mathsf{P}\left(\bigcap_{n=i}^{j} \big(A_n^{\mathrm{c}} \cap B_n\big) \right) &= \mathsf{E}\left[\prod_{n=i}^{j} \mathbb{1}_{B_n} \mathsf{E}\left[\mathbb{1}_{A_n^{\mathrm{c}}} \;\middle|\; |L_n|,\ \sigma^* \right] \right] \\
&\leq \prod_{n=i}^{j} \big(1 - \tfrac{C}{n}\big) \xrightarrow[j \to \infty]{} 0. \qquad (5.5.11)
\end{aligned}$$

Therefore, for all $i \in \mathbb{N}$ we have

$$\mathsf{P}\left(\bigcap_{n=i}^{\infty} \big(A_n^{\mathrm{c}} \cap B_n\big) \right) = 0,$$

which implies

$$\mathsf{P}\left(\liminf_{n\to\infty}\left(A_n^c \cap B_n\right)\right) \;=\; 0. \tag{5.5.12}$$

Since (5.5.4) holds, (5.5.12) yields

$$\mathsf{P}\left(\liminf_{n\to\infty} A_n^c\right) \;=\; 0.$$

Thus, we have established (5.5.3) and so (i) of Theorem 5.3.1(a) is proven.

(ii) Now suppose $p_c m_c \varphi_\ell < 1$. Again we consider the event

$$A_n \;=\; \Big\{\exists\, \nu \succeq L_n : X_\nu = 0\Big\}$$

but in contrast to *(i)* we here show in a first step that we have

$$\mathsf{P}\left(\limsup_{n\to\infty} A_n\right) \;=\; 0.$$

Similar to *(i)*, for $\varepsilon > 0$ we define the event

$$B_n' := \;\big\{|L_n| \le \big(p_c m_c + \varepsilon\big)^n,\; l(n) - 1 \ge n\cdot(1-\varepsilon)\big\}$$

and as above we show that

$$\mathsf{P}\left(\liminf_{n\to\infty} B_n'\right) \;=\; 1. \tag{5.5.13}$$

Analogously to the definition of C_n, we define $C_n' := \big\{|Z^{(l)}n| \le (p_c m_c + \frac{\varepsilon}{2})^n\big\}$ and observe that due to (5.5.2) we have

$$\mathsf{P}\left(\liminf_{n\to\infty} C_n'\right) \;=\; 1. \tag{5.5.14}$$

Then we can make use of a very similar argument as above based on a large deviation bound for the sum of $M \cdot \lfloor (p_c m_c + \frac{\varepsilon}{2})^n \rfloor$ i.i.d. Bernoulli random variables with success probability q_c (cf. (5.5.7)). This time we just have to take into account that each particle produces at most M offspring. Therefore, we have

$$\mathsf{P}\left(\big\{|L_n| > (p_c m_c + \varepsilon)^n\big\} \cap C_n'\right) \;\le\; \exp\Big(-I_2 \cdot \big(M \cdot \lfloor (p_c m_c + \tfrac{\varepsilon}{2})^n \rfloor\big)\Big)$$

for large n and some $I_2 > 0$. Anyway, we can conclude that we have

$$\mathsf{P}\left(\limsup_{n\to\infty}\left(\big\{|L_n| > \big(p_c m_c + \varepsilon\big)^n\big\} \cap C_n'\right)\right) \;=\; 0.$$

Together with (5.5.14) this implies

$$\mathsf{P}\left(\liminf_{n\to\infty} \big\{|L_n| \le \big(p_c m_c + \varepsilon\big)^n\big\}\right) \;=\; 1. \tag{5.5.15}$$

Moreover due to the fact that $\sigma^* < \infty$ (cf. (5.5.1)), we a.s. have

$$\lim_{n\to\infty} \frac{l(n)}{n} = 1$$

and this together with (5.5.15) yields (5.5.13).

Due to (5.5.13) it is enough to consider the behaviour of the process on the sets $\big(B'_n\big)_{n\in\mathbb{N}_0}$. Using Proposition 5.4.4 for the same embedded GWP with mean $\varphi_\ell < 1$ as in equation (5.5.9), we can estimate the probability that there exists a particle in L_n which has a descendant returning to the origin in the following way:

$$\begin{aligned}
\mathsf{P}\big(A_n \cap B'_n\big) &\le \mathsf{P}\big(A_n \big| B'_n\big) \\
&= 1 - \mathsf{P}\big(\nexists\, \nu \succeq L_n : X_\nu = 0 \,\big| B'_n\big) \\
&\le 1 - \Big(1 - \big(\varphi_\ell\big)^{n\cdot(1-\varepsilon)} \cdot \big(c + o(1)\big)\Big)^{(p_c m_c + \varepsilon)^n} && (5.5.16)
\end{aligned}$$

Note that for $x \in [0, \frac{1}{2}]$ we have $1 - x \ge \exp(-2x)$. Since $\varphi_\ell < 1$ (cf. Remark 5.4.3), we have $\big(\varphi_\ell\big)^{n\cdot(1-\varepsilon)} \cdot \big(c + o(1)\big) \in (0, \frac{1}{2}]$ for large $n \in \mathbb{N}$ and therefore we get

$$\begin{aligned}
&\Big(1 - \big(\varphi_\ell\big)^{n\cdot(1-\varepsilon)} \cdot \big(c + o(1)\big)\Big)^{(p_c m_c + \varepsilon)^n} \\
\ge\ & \exp\Big(-2 \cdot \big(\varphi_\ell\big)^{n\cdot(1-\varepsilon)} \cdot \big(c + o(1)\big) \cdot (p_c m_c + \varepsilon)^n\Big)
\end{aligned}$$

for large n. This together with (5.5.16) and the estimate $1 - \exp(-x) \le x$ yields

$$\begin{aligned}
\mathsf{P}\big(A_n \cap B'_n\big) &\le 1 - \exp\Big(-2 \cdot \big(\varphi_\ell\big)^{n\cdot(1-\varepsilon)} \cdot \big(c + o(1)\big) \cdot (p_c m_c + \varepsilon)^n\Big) \\
&\le 2 \cdot \big(\varphi_\ell\big)^{n\cdot(1-\varepsilon)} \cdot \big(c + o(1)\big) \cdot (p_c m_c + \varepsilon)^n && (5.5.17)
\end{aligned}$$

for large $n \in \mathbb{N}$. Since $p_c m_c \cdot \varphi_\ell < 1$ holds, we can choose $\varepsilon > 0$ small enough such that we still have

$$\big(p_c m_c + \varepsilon\big) \cdot (\varphi_\ell)^{1-\varepsilon} \ < \ 1.$$

Thus for such a choice of ε, the Borel-Cantelli lemma implies

$$\mathsf{P}\left(\limsup_{n\to\infty} (A_n \cap B'_n)\right) = 0.$$

This together with (5.5.13) shows

$$\mathsf{P}\left(\limsup_{n\to\infty} A_n\right) \ = \ \mathsf{P}\left(\limsup_{n\to\infty} \big\{\exists\, \nu \succeq L_n : X_v = 0\big\}\right) \ = \ 0. \qquad (5.5.18)$$

To show that (5.5.18) suffices for the transience of the CBRW, we define

$$N := \ \sup\{n \in \mathbb{N}_0 : \exists\, \nu \succeq L_n : \ X_\nu = 0\}$$

and observe that due to (5.5.18) we a.s. have $N < \infty$. Further, for every particle ν (except for the initial particle) which is located at the origin there is a largest $n \in \mathbb{N}_0$ for which ν has an ancestor in L_n. Thus we have

$$\sum_{\nu \in \mathcal{Z}} \mathbb{1}_{\{X_\nu = 0\}} \;\leq\; \sum_{\substack{\nu \in \mathcal{Z} \\ |\nu| \leq \sigma^*}} \mathbb{1}_{\{X_\nu = 0\}} + \sum_{n=\sigma^*}^{N \vee \sigma^*} \sum_{\nu \in L_n} \left|\{\eta \succeq \nu :\; X_\eta = 0\}\right|.$$

Since we have $\sigma^*, N < \infty$ a.s. and since for every $\nu \in L_n$ with $n \geq \sigma^*$ the offspring of ν behave as a BRW without cookies, which is transient to right by assumption, we a.s. have

$$\sum_{\nu \in \mathcal{Z}} \mathbb{1}_{\{X_\nu = 0\}} \;<\; \infty.$$

Therefore, we can finally conclude that a.s. only finitely many particles visit the origin, i.e. the CBRW is transient. This completes part *(a)* of the proof. ∎

Proof of part (b)

In this part of the proof we suppose that the LP is subcritical or critical, i.e. that $p_c m_c \leq 1$. We start with Lemma 5.5.1, which states that except for finitely many times the particles at a single position $x \in \mathbb{Z}$ produce an amount of offspring which is close to the expected amount as long as there are many particles at this position. To do so, we first split the set of particles $Z_n(x)$ into the following two sets

$$\begin{aligned} Z^+_{n+1}(x) &:= \;\{\nu \in Z_{n+1}(x) : \nu \succ Z_n(x-1)\}, \\ Z^-_{n+1}(x) &:= \;\{\nu \in Z_{n+1}(x) : \nu \succ Z_n(x+1)\} \end{aligned}$$

containing the particles which have moved to the right or to the left from time n to time $n+1$. For $\varepsilon > 0$, which we specify later (cf. (5.5.36) and (5.5.53)), we introduce the following sets:

$$\begin{aligned} D^+_n(x) &:= \;\{x < l(n),\, |Z_n(x)| \geq n\} \cap \left(\left\{ \frac{|Z^+_{n+1}(x+1)|}{|Z_n(x)|} < (p_0 m_0 - \varepsilon) \right\} \right. \\ &\qquad \left. \cup \left\{ (p_0 m_0 + \varepsilon) < \frac{|Z^+_{n+1}(x+1)|}{|Z_n(x)|} \right\} \right), \\ D^-_n(x) &:= \;\{x < l(n),\, |Z_n(x)| \geq n\} \cap \left(\left\{ \frac{|Z^-_{n+1}(x-1)|}{|Z_n(x)|} < (q_0 m_0 - \varepsilon) \right\} \right. \\ &\qquad \left. \cup \left\{ (q_0 m_0 + \varepsilon) < \frac{|Z^-_{n+1}(x-1)|}{|Z_n(x)|} \right\} \right), \\ E^+_n &:= \;\{Z^{(l)} n \geq n\} \cap \left(\left\{ \frac{|Z^{(l)} n + 1|}{|Z^{(l)} n|} < (p_c m_c - \varepsilon) \right\} \right. \\ &\qquad \left. \cup \left\{ (p_c m_c + \varepsilon) < \frac{|Z^{(l)} n + 1|}{|Z^{(l)} n|} \right\} \right), \end{aligned}$$

$$
\begin{aligned}
E_n^- &:= \{Z^{(l)}n \geq n\} \cap \left(\left\{ \frac{|Z_{n+1}^-(l(n)-1)|}{|Z^{(l)}n|} < (q_c m_c - \varepsilon) \right\} \right. \\
&\qquad \left. \cup \left\{ (q_c m_c + \varepsilon) < \frac{|Z_{n+1}^-(l(n)-1)|}{|Z^{(l)}n|} \right\} \right), \\
F_n &:= E_n^+ \cup E_n^- \cup \bigcup_{x \in \mathbb{Z}} \Big(D_n^+(x) \cup D_n^-(x) \Big). \qquad (5.5.19)
\end{aligned}
$$

Lemma 5.5.1. *We have*

$$
\mathsf{P}\left(\limsup_{n\to\infty} F_n \right) = 0. \qquad (5.5.20)
$$

Proof of Lemma 5.5.1. A large deviation estimate (note that the number of offspring of a single particle is bounded by M) for the random sum $|Z_{n+1}^+(x+1)|$ of $|Z_n(x)|$ i.i.d. random variables with mean $p_0 m_0$ yields

$$
\mathsf{P}\Big(|Z_{n+1}^+(x+1)| > (p_0 m_0 + \varepsilon)|Z_n(x)| \Big| \sigma(|Z_n(x)|) \Big) \leq \exp\big(-|Z_n(x)|C_1 \big) \qquad (5.5.21)
$$

for some constant $C_1 > 0$ and

$$
\mathsf{P}\Big(|Z_{n+1}^+(x+1)| < (p_0 m_0 - \varepsilon)|Z_n(x)| \Big| \sigma(|Z_n(x)|) \Big) \leq \exp\big(-|Z_n(x)|C_2 \big) \qquad (5.5.22)
$$

for some constant $C_2 > 0$. From (5.5.21) and (5.5.22) we can conclude

$$
\mathsf{P}\Big(D_n^+(x) \Big) \leq \exp(-nC_1) + \exp(-nC_2). \qquad (5.5.23)
$$

The same argument leads to analogue estimates for the sets $D_n^-(x)$, E_n^+ and E_n^- with constants $C_i > 0$ for $i = 3, \ldots, 8$. Since at time $n \in \mathbb{N}_0$ particles can only be located at the $n+1$ positions $-n, -n+2, \ldots, n-2, n$, we get

$$
\mathsf{P}\left(E_n^+ \cup E_n^- \cup \bigcup_{x \in \mathbb{Z}} \Big(D_n^+(x) \cup D_n^-(x) \Big) \right) \leq 2(2 + 2(n+1)) \exp(-nC)
$$

for $C := \min_{i=1,\ldots,8} C_i > 0$. Therefore, the Borel-Cantelli lemma implies (5.5.20). $\square$

In the considered case the CBRW behaves very differently depending on whether we have $p_0 m_0 \leq 1$ or $p_0 m_0 > 1$:

(i) For $p_0 m_0 \leq 1$ the offspring of a single particle which move to the right in every step behave as a critical or subcritical GWP as long as the particles do not reach the cookies. Therefore, we can expect that the amount of particles which reach a cookie at the same time is not very large. More precisely, we will show in Proposition 5.5.2 that the amount of particles in the LP does not grow exponentially.

(ii) For $p_0 m_0 > 1$ the amount of offspring which moves to the right in every time step in the corresponding BRW without cookies constitutes a supercritical GWP.

Therefore, the number of particles at the rightmost occupied position in the BRW without cookies a.s. grows with exponential rate $p_0m_0 > 1$. In this case the following proposition shows that the amount of particles in the LP is essentially bounded by the growth rate of the rightmost occupied position of the corresponding BRW without cookies.

Proposition 5.5.2. *For every $\alpha > \max\{1, p_0m_0\} =: m^*$ we have*

$$\mathsf{P}\left(\liminf_{n\to\infty}\{|Z^{(l)}n| < \alpha^n\}\right) \;=\; 1. \tag{5.5.24}$$

Proof of Proposition 5.5.2. For the proof we start with the following lemma which states that a large LP at time n leads to a long survival of the LP afterwards (except for finitely many times). For $\beta > 0$ we define

$$G_n := \; G_n(\beta) := \; \left\{|Z^{(l)}n| \geq n,\, \tau(n) \leq \beta \log |Z^{(l)}n|\right\}, \tag{5.5.25}$$

where

$$\tau(n) := \; \inf\{\ell \geq n : \; |Z^{(l)}\ell| = 0\} \tag{5.5.26}$$

denotes the time of the next extinction of the LP beginning from time n.

Lemma 5.5.3. *There exists some $\beta > 0$ such we have*

$$\mathsf{P}\left(\limsup_{n\to\infty} G_n\right) = 0. \tag{5.5.27}$$

Proof of Lemma 5.5.3. Let us first look at a subcritical GWP $\left(GW_n^{\text{sub}}\right)_{n\in\mathbb{N}_0}$ with reproduction mean $p_cm_c < 1$ and strictly positive, finite offspring variance and its extinction time T^{sub}. Assuming that we have an initial population of $z \in \mathbb{N}$ particles, we get using Proposition 5.4.4

$$\begin{aligned}
\mathsf{P}_z(T^{\text{sub}} \leq n) \;&=\; \left(1 - \mathsf{P}(GW_n^{\text{sub}} > 0)\right)^z \\
&\leq\; \left(1 - c(p_cm_c)^n\right)^z \\
&\leq\; \exp\left(-c(p_cm_c)^n z\right) \\
&\leq\; \exp\left(-C\frac{z}{n}\right),
\end{aligned}$$

for suitable constants $c, C > 0$. Together with Proposition 5.4.6 we conclude that in the two cases of a subcritical and a critical LP there exists $C > 0$ such that we have for $n \in \mathbb{N}$

$$\mathsf{P}\left(|Z^{(l)}n| \geq n,\, \tau(n) \leq \beta \log |Z^{(l)}n|\right) \;\leq\; \exp\left(-C\frac{n}{\beta\log(n)}\right).$$

Therefore, the Borel-Cantelli lemma implies (5.5.27). □

In the following we want to investigate the behaviour of the CBRW on the set

$$H_{n_0} := \bigcap_{n \geq n_0} \left(F_n^c \cap G_n^c\right) \tag{5.5.28}$$

for fixed $n_0 \in \mathbb{N}_0$. On this set we have upper and lower bounds for

$$\frac{|Z_{n+1}^+(x+1)|}{|Z_n(x)|} \quad \text{and} \quad \frac{|Z_{n+1}^-(x-1)|}{|Z_n(x)|}$$

for positions $x \in \mathbb{Z}$ containing at least n particles at time $n \geq n_0$ (cf. (5.5.19)). Additionally, we have a lower bound for the time for which a LP with at least n particles at time $n \geq n_0$ will stay alive afterwards (cf. (5.5.25)). Note that we have

$$\mathsf{P}\left(\liminf_{n\to\infty} \left(F_n^c \cap G_n^c\right)\right) \;=\; \lim_{n\to\infty} \mathsf{P}(H_n) \;=\; 1 \tag{5.5.29}$$

due to Lemma 5.5.1 and Lemma 5.5.3.

For the next lemma we need some additional notation. We define

$$\sigma_0 := \;\inf\{n > n_0 : \,|Z^{(l)}n-1| = 0,\, |Z^{(l)}n| \neq 0,\, l(n) \leq n - 2n_0 - 1\},$$

which is the time of the first rebirth of the LP after time n_0 for which we have

$$l(\sigma_0) - (\sigma_0 - n_0) \;\leq\; -2n_0 - 1 + n_0 \;=\; -n_0 - 1.$$

This implies

$$Z_{n_0}\big(l(\sigma_0) - (\sigma_0 - n_0)\big) \;=\; 0, \tag{5.5.30}$$

which is an important fact which we make use of in the following calculations (cf. Figure 5.1). Since the LP is critical or subcritical and the BRW without cookies is transient to the right, we a.s. have $\sigma_0 < \infty$. We now define the random times

$$\begin{aligned} \tau_n &:= \;\inf\{\ell > \sigma_n : \;|Z^{(l)}\ell| = 0\} - \sigma_n,\; \text{ for } n \geq 0, \\ \sigma_n &:= \;\inf\{\ell > \sigma_{n-1} + \tau_{n-1} : \;|Z^{(l)}\ell| \neq 0\},\; \text{ for } n \geq 1, \end{aligned}$$

which denote the time period of survival and the time of extinction of the LP, inductively. Due to the assumptions of the CBRW all of these random times are a.s. finite. Using (5.5.30) we see that we have

$$Z_{n_0}\big(l(\sigma_j) - (\sigma_j - n_0 + k)\big) = 0 \tag{5.5.31}$$

for all $j, k \in \mathbb{N}_0$.

As the next step of the proof, we state the following upper bounds for the size of the restarted LP at time σ_{j+1} on the set H_{n_0}:

Lemma 5.5.4. *For arbitrary $\gamma > 0$ and large n_0 we have*

$$\begin{aligned}
|Z^{(l)}\sigma_{j+1}| \;&\leq\; (m^* + 3\gamma)^{\sigma_{j+1}} && (5.5.32)\\
&\quad \textit{on } H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\} \cap \{|Z^{(l)}\sigma_j| \leq (m^* + \gamma)^{\sigma_j}\},\\
|Z^{(l)}\sigma_{j+1}| \;&\leq\; |Z^{(l)}\sigma_j| \cdot (m^* + 4\gamma)^{\tau_j} && (5.5.33)\\
&\quad \textit{on } H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\} \cap \{|Z^{(l)}\sigma_j| > (m^* + \gamma)^{\sigma_j}\},\\
|Z^{(l)}\sigma_{j+1}| \;&\leq\; (m^* + 2\gamma)^{\sigma_{j+1}} && (5.5.34)\\
&\quad \textit{on } H_{n_0} \cap \{\sigma_{j+1} > \sigma_j + \tau_j + 2\},
\end{aligned}$$

where $m^* = \max\{1, p_0 m_0\}$.

Proof of Lemma 5.5.4. First we choose $0 < \delta < \gamma$ in such a way that

$$1 + \delta \;\leq\; \frac{m^* + 2\gamma}{m^* + \gamma}, \qquad 1 + \delta \;\leq\; \left(\frac{m^* + 3\gamma}{m^* + 2\gamma}\right)^{\beta \log(m^* + \gamma)}, \tag{5.5.35}$$

where $\beta > 0$ satisfies Lemma 5.5.3. Then we choose $\varepsilon > 0$ for the definitions of the sets $\big(F_n\big)_{n\in\mathbb{N}_0}$ (cf. (5.5.19)) sufficiently small such that

$$p_c m_c + \varepsilon \;\leq\; 1 + \delta, \qquad \frac{p_0 m_0 + \varepsilon}{p_0 m_0 - \varepsilon} \;\leq\; 1 + \delta, \qquad p_0 m_0 + \varepsilon \;\leq\; m^* + \gamma. \tag{5.5.36}$$

For the upcoming estimates we use the following properties of the set H_{n_0}. For $n > n_0$ we have

$$\begin{aligned}
1.)\quad |Z_{n-1}(x-1)| \;&\leq\; n - 1 && (5.5.37)\\
&\quad \text{on } H_{n_0} \cap \{|Z_n(x)| = 0\},
\end{aligned}$$

which means that there cannot be very many particles at position $x - 1$ one time step before n if we know that the position x stays empty at time n. Similarly, the knowledge of $|Z_n(x)|$ gives us upper estimates for $(|Z_{n-k}(x-k)|)_{k\in\mathbb{N}}$. If we are in the case in which the cookies are always to the right of the considered positions, we have for $n > n_0$

$$\begin{aligned}
2.)\quad |Z_{n-1}(x-1)| \;&\leq\; z \cdot (p_0 m_0 - \varepsilon)^{-1} + n - 1\\
&\quad \text{on } H_{n_0} \cap \{|Z_n(x)| = z,\ l(n-1) > (x-1)\},\\
|Z_{n-k}(x-k)| \;&\leq\; z \cdot (p_0 m_0 - \varepsilon)^{-k} + (n-1) \cdot (p_0 m_0 - \varepsilon)^{-k+1} && (5.5.38)\\
&\quad \text{on } H_{n_0} \cap \{|Z_n(x)| = z,\ l(n-1) > (x-1)\}
\end{aligned}$$

for $n-k \geq n_0$. The first estimate is easily obtained using a proof by contradiction and an iteration of it yields the second inequality.
We obtain similar estimates for the size of the LP before the next extinction at time $\tau(n)$ (for the definition of $\tau(n)$ see (5.5.26)):

$$\begin{aligned} 3.)\quad |Z^{(l)}n+1| &\leq z\cdot(p_c m_c+\varepsilon)+M\cdot n \\ &\quad \text{on } H_{n_0}\cap\{Z^{(l)}n=z\}, \\ |Z^{(l)}n+2| &\leq z\cdot(p_c m_c+\varepsilon)^2+2\cdot M\cdot(n+1)\cdot\max\{1,p_c m_c+\varepsilon\} \\ &\quad \text{on } H_{n_0}\cap\{|Z^{(l)}n|=z\}\cap\{\tau(n)\geq n+2\}, \\ |Z^{(l)}n+k| &\leq z\cdot(p_c m_c+\varepsilon)^k+k\cdot M\cdot(n+k-1)\cdot(1+\delta)^{k-1} \\ &\quad \text{on } H_{n_0}\cap\{|Z^{(l)}n|=z\}\cap\{\tau(n)\geq n+k\}. \end{aligned} \tag{5.5.39}$$

Now, we introduce two processes $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$, which help us – together with the estimates (5.5.37), (5.5.38), and (5.5.39) – to control the number of particles that restart the LP at time σ_{j+1} (cf. Figure 5.1 and 5.2). For $j\in\mathbb{N}_0$ and $n\in\mathbb{N}$ we define

$$\Phi_n := \Phi_n^{(j)} := Z_n(l(\sigma_{j+1})-\sigma_{j+1}+n)$$

and

$$\Psi_n := \Psi_n^{(j)} := Z_n(l(\sigma_{j+1})-\sigma_{j+1}+2+n).$$

For sake of a better presentation we drop the superscript j and write just Φ_n and Ψ_n if there is no room for confusion. We observe that we have

$$|\Phi_{n+1}| = |\Psi_n| = 0 \tag{5.5.40}$$

for all $n\leq n_0$ due to (5.5.31). Furthermore, by definition we have

$$\Phi_{\sigma_{j+1}} = Z^{(l)}\sigma_{j+1} \tag{5.5.41}$$

and

$$|\Psi_{\sigma_{j+1}}| = |\Psi_{\sigma_{j+1}-1}| = |\Psi_{\sigma_{j+1}-2}| = 0. \tag{5.5.42}$$

Again, we split the set of particles Φ_n into the particles which have moved one step to the right from time $n-1$ to time n and the particles which have moved to the left:

$$\begin{aligned} \Phi_n^+ &:= Z_n^+(l(\sigma_{j+1})-\sigma_{j+1}+n), \\ \Phi_n^- &:= Z_n^-(l(\sigma_{j+1})-\sigma_{j+1}+n) \end{aligned}$$

To obtain an upper bound for $\Phi_{\sigma_{j+1}}=Z^{(l)}\sigma_{j+1}$, we use the following recursive structure. We have

$$|\Phi_n^-| \leq M|\Psi_{n-1}|$$

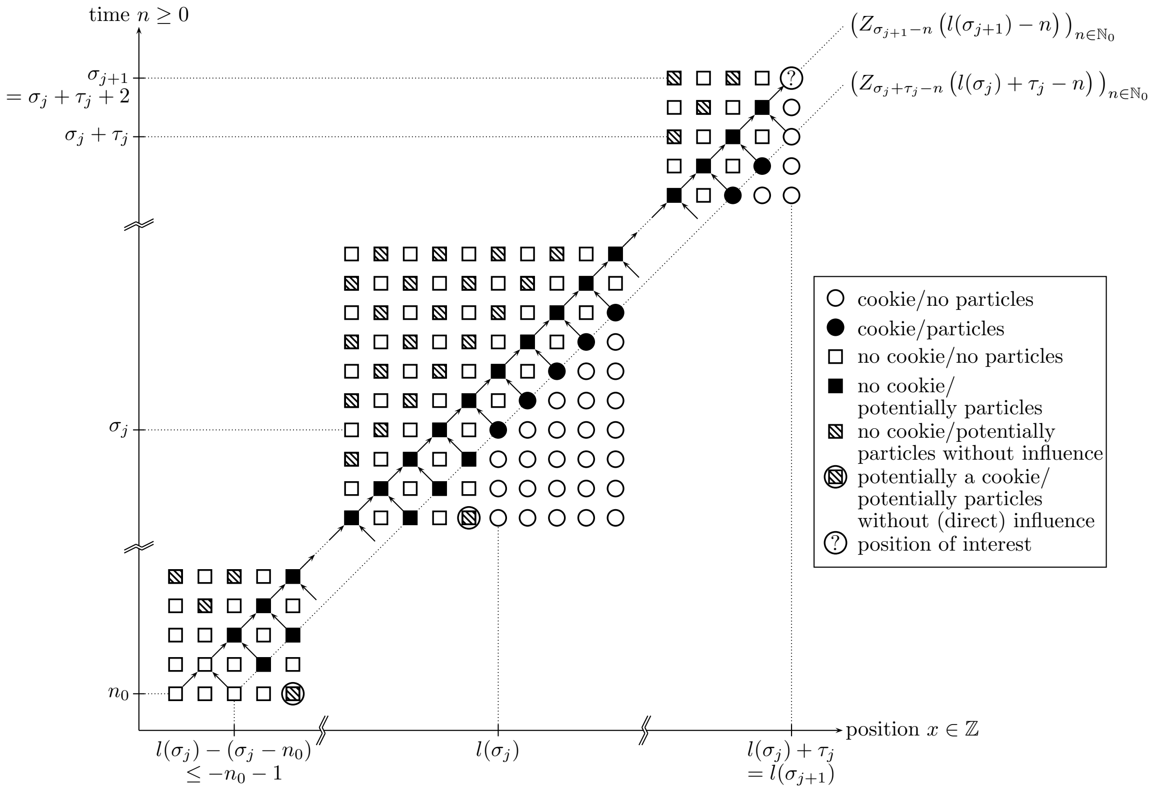

Figure 5.1: *The LP is restarted at time σ_{j+1} two time steps after the last extinction at time $\sigma_j + \tau_j$. The two diagonals represent the processes $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$.*

for $n \in \mathbb{N}$ due to assumption (5.1.1). Moreover, on H_{n_0} we have

$$|\Phi_n^+| \;\leq\; |\Phi_{n-1}|(p_0m_0 + \varepsilon) + M\sigma_{j+1}$$

for $n_0 + 2 \leq n \leq \sigma_{j+1}$ (since the particles reproduce and move without cookies) and these two facts yield

$$|\Phi_n| \;=\; |\Phi_n^+| + |\Phi_n^-| \;\leq\; |\Phi_{n-1}|(p_0m_0 + \varepsilon) + M\sigma_{j+1} + M|\Psi_{n-1}| \tag{5.5.43}$$

for $n_0 + 2 \leq n \leq \sigma_{j+1}$. Using (5.5.40), (5.5.42), and $\sigma_{j+1} - n_0 - 1$ iterations of the recursion in (5.5.43), we obtain the following upper bound for the particles which start the LP at time σ_{j+1} on H_{n_0}:

$$\begin{aligned}
|\Phi_{\sigma_{j+1}}| &\leq M \sum_{k=3}^{\sigma_{j+1}-n_0-1} |\Psi_{\sigma_{j+1}-k}|(p_0m_0+\varepsilon)^{k-1} + M\sigma_{j+1} \sum_{k=1}^{\sigma_{j+1}-n_0-1} (p_0m_0+\varepsilon)^{k-1} \\
&\leq M \sum_{k=1}^{\sigma_{j+1}-n_0-3} |\Psi_{\sigma_{j+1}-k-2}|(p_0m_0+\varepsilon)^{k+1} + M\sigma_{j+1}^2(m^*+\gamma)^{\sigma_{j+1}}. \qquad (5.5.44)
\end{aligned}$$

Note that this bound just depends on σ_{j+1} and the process $(\Psi_n)_{n\in\mathbb{N}}$. For this reason we now take a closer look at $(\Psi_n)_{n\in\mathbb{N}}$ and distinguish between the following two cases:

- In the first case we assume that the LP restarts right after it has died out and we therefore have $\sigma_{j+1} = \sigma_j + \tau_j + 2$. In this case the process $(\Psi_n)_{n\in\mathbb{N}}$ coincides with the LP between time σ_j and $\sigma_j + \tau_j$ (cf. Figure 5.1).
- In the second case we assume that we have $\sigma_{j+1} > \sigma_j + \tau_j + 2$. From this we know that there are no particles in the LP at time $\sigma_{j+1} - 2$ and thus the process $(\Psi_n)_{n\in\mathbb{N}}$ is always left of the cookies (cf. Figure 5.2).

In both cases the crucial observation is that the amount of particles in $(\Psi_n)_{n\in\mathbb{N}}$ does not exceed a certain level since none of its offspring reaches the leftmost cookie at time $\sigma_{j+1} - 2$.

- At first, we consider the case $H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\}$.

We apply the estimations (5.5.38) and (5.5.39) to give upper bounds for

$$|\Psi_{\sigma_{j+1}-k}| \;=\; |\Psi_{\sigma_j+\tau_j+2-k}|$$

for $1 \le k \le \sigma_{j+1} - n_0$. We know by definition of σ_j that we have

$$l(\sigma_j - 1) \;=\; l(\sigma_j) > l(\sigma_j) - 1.$$

Thus, we can apply (5.5.38) and conclude that on the set H_{n_0} for $1 \le k \le \sigma_j - n_0$ we have

$$|\Psi_{\sigma_j-k}| \;= |Z_{\sigma_j-k}(l(\sigma_j) - k)| \;\le\; |Z^{(l)}\sigma_j|(p_0 m_0 - \varepsilon)^{-k} + \sigma_j (p_0 m_0 - \varepsilon)^{-k+1}$$

and by using (5.5.39) for $0 \le k \le \tau_j - 1$ we get

$$|\Psi_{\sigma_j+k}| \;= |Z^{(l)}\sigma_j + k| \;\le\; |Z^{(l)}\sigma_j|(p_c m_c + \varepsilon)^k + kM(\sigma_j + k - 1)(1+\delta)^{k-1}.$$

Applying these two estimates to (5.5.44) yields

$$\begin{aligned}
|\Phi_{\sigma_{j+1}}| \le\; & M \sum_{k=1}^{\tau_j} |\Psi_{\sigma_j+(\tau_j-k)}|(p_0 m_0+\varepsilon)^{k+1} + M \sum_{k=\tau_j+1}^{\sigma_j+\tau_j-n_0-1} |\Psi_{\sigma_j-(k-\tau_j)}|(p_0 m_0+\varepsilon)^{k+1} \\
& + M\sigma_{j+1}^2 (m^*+\gamma)^{\sigma_{j+1}} \\
\le\; & M \sum_{k=1}^{\tau_j} \Big(|Z^{(l)}\sigma_j|(p_c m_c + \varepsilon)^{\tau_j - k} \\
& \qquad + (\tau_j - k) M (\sigma_j + \tau_j - k - 1)(1+\delta)^{\tau_j-k-1} \Big) (p_0 m_0 + \varepsilon)^{k+1} \\
& + M \sum_{k=\tau_j+1}^{\sigma_j+\tau_j-n_0-1} \Big(|Z^{(l)}\sigma_j|(p_0 m_0 - \varepsilon)^{-k+\tau_j} \\
& \qquad + \sigma_j (p_0 m_0 - \varepsilon)^{-k+\tau_j+1} \Big) (p_0 m_0+\varepsilon)^{k+1} + M\sigma_{j+1}^2 (m^*+\gamma)^{\sigma_{j+1}}
\end{aligned}$$

$$
\begin{aligned}
&\le \tau_j^2 M^2(|Z^{(l)}\sigma_j| + \sigma_{j+1})(1+\delta)^{\tau_j-1}(m^*+\gamma)^{\tau_j+1} \\
&\quad + \sigma_{j+1} M\big(|Z^{(l)}\sigma_j| + \sigma_j(p_0 m_0 - \varepsilon)\big)\left(\frac{p_0 m_0+\varepsilon}{p_0 m_0-\varepsilon}\right)^{\sigma_j}(p_0 m_0+\varepsilon)^{\tau_j+1} \\
&\quad + M\sigma_{j+1}^2(m^*+\gamma)^{\sigma_{j+1}} \\
&\le 2M^2\sigma_{j+1}^2(|Z^{(l)}\sigma_j| + \sigma_{j+1})(1+\delta)^{\sigma_j+\tau_j-1}(m^*+\gamma)^{\tau_j+2} + M\sigma_{j+1}^2(m^*+\gamma)^{\sigma_{j+1}}.
\end{aligned}
\tag{5.5.45}
$$

Here we used (5.5.36) in the last two steps.

If we first investigate $|Z^{(l)}\sigma_{j+1}|$ on the subset

$$\{|Z^{(l)}\sigma_j| \le (m^*+\gamma)^{\sigma_j}\} \cap H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\},$$

on which we have a limited amount of particles in $Z^{(l)}\sigma_j$, we get by using (5.5.45)

$$
\begin{aligned}
|Z^{(l)}\sigma_{j+1}| &= |\Phi_{\sigma_{j+1}}| \\
&\le 2M^2\sigma_{j+1}^2\big((m^*+\gamma)^{\sigma_j} + \sigma_{j+1}\big)(1+\delta)^{\sigma_j+\tau_j-1}(m^*+\gamma)^{\tau_j+2} \\
&\quad + M\sigma_{j+1}^2(m^*+\gamma)^{\sigma_{j+1}} \\
&\le 3M^2(\sigma_{j+1}+1)^3(1+\delta)^{\sigma_{j+1}}(m^*+\gamma)^{\sigma_{j+1}} \\
&\le (m^*+3\gamma)^{\sigma_{j+1}}
\end{aligned}
$$

for n_0 and thus $\sigma_{j+1} \ge n_0$ large enough due to (5.5.35). This shows (5.5.32) in Lemma 5.5.4.

On the other hand, if we consider the remaining subset

$$\{|Z^{(l)}\sigma_j| > (m^*+\gamma)^{\sigma_j}\} \cap H_{n_0} \cap \{\sigma_{j+1} = \sigma_j + \tau_j + 2\},$$

(5.5.45) yields

$$
\begin{aligned}
|Z^{(l)}\sigma_j|^{-1}|Z^{(l)}\sigma_{j+1}| &= |Z^{(l)}\sigma_j|^{-1}|\Phi_{\sigma_{j+1}}| \\
&\le 2M^2\sigma_{j+1}^2(1+\sigma_{j+1})(1+\delta)^{\sigma_j+\tau_j-1}(m^*+\gamma)^{\tau_j+1} \\
&\quad + M\sigma_{j+1}^2(m^*+\gamma)^{\tau_j+2} \\
&\le 3M^2(\sigma_j+\tau_j+3)^3(1+\delta)^{\sigma_j}(m^*+2\gamma)^{\tau_j+2} \\
&\le 3M^2(\sigma_j+\tau_j+3)^3(1+\delta)^{\frac{1}{\beta\log(m^*+\gamma)}\tau_j}(m^*+2\gamma)^{\tau_j+2} \\
&\le (m^*+4\gamma)^{\tau_j}
\end{aligned}
$$

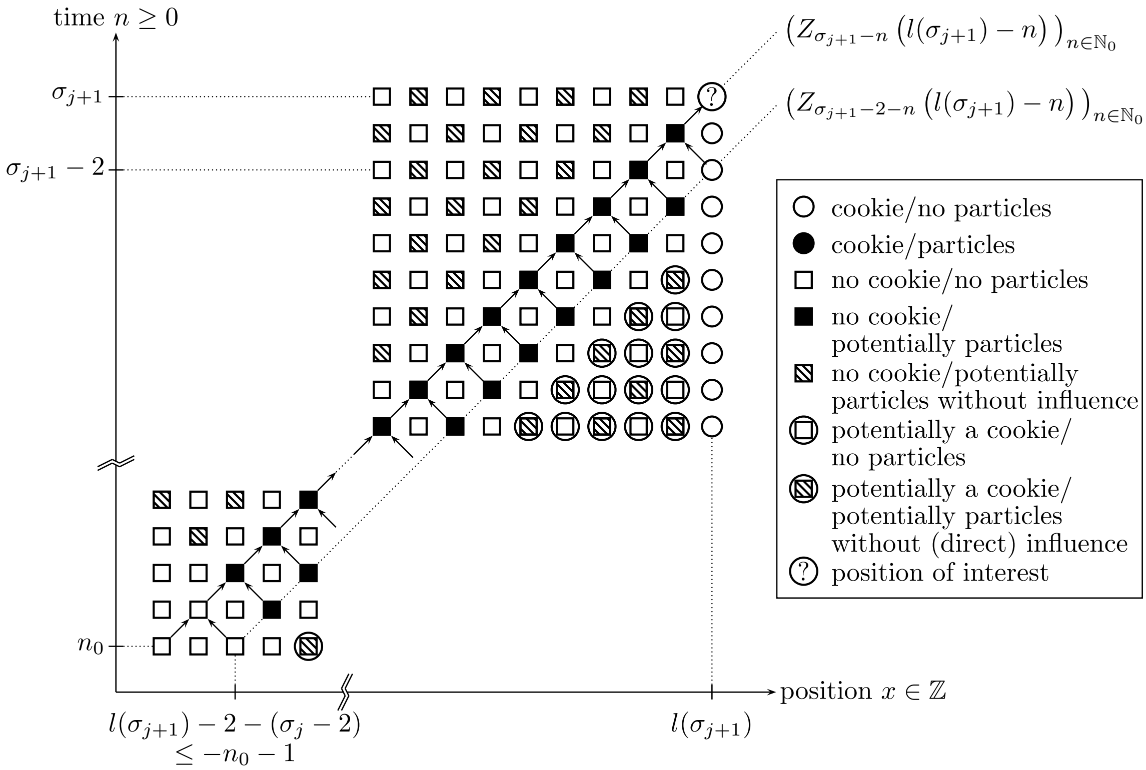

Figure 5.2: *The LP is restarted at time σ_{j+1} more than two time steps after the last extinction at time $\sigma_j+\tau_j$. The two diagonals represent the processes $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$.*

for n_0 and thus $\sigma_j \geq n_0$ large enough. Thereby we used (5.5.35) and the fact that we have $\{\tau_j > \beta \log\big((m^*+\gamma)^{\sigma_j}\big)\}$ on the considered set (cf. Lemma 5.5.3). This shows (5.5.33) in Lemma 5.5.4.

• We now turn to the set $H_{n_0} \cap \{\sigma_{j+1} > \sigma_j + \tau_j + 2\}$. First, we observe that on this set, due to (5.5.37), we have

$$|\Psi_{\sigma_{j+1}-2-1}| \;=|Z_{\sigma_{j+1}-2-1}(l(\sigma_{j+1})-1)| \;\leq\; \sigma_{j+1}-2-1 \;\leq \sigma_{j+1} \tag{5.5.46}$$

since $|\Psi_{\sigma_{j+1}-2}| \;=|Z_{\sigma_{j+1}-2}(l(\sigma_{j+1}))| = 0$ holds. Further, we observe that the particles which belong to $(\Psi_n)_{n\in\mathbb{N}}$ are always to the left of the cookies. In particular, we have

$$l(\sigma_{j+1}-2-1) \;=\; l(\sigma_{j+1}) \;>\; l(\sigma_{j+1})-1.$$

Therefore, we can apply (5.5.38) and conclude, by using (5.5.46),

$$\begin{aligned}|\Psi_{\sigma_{j+1}-2-k}| \;&= |Z_{\sigma_{j+1}-2-k}(l(\sigma_{j+1})-k)| \\ &\leq \sigma_{j+1}(p_0m_0-\varepsilon)^{-k} + (\sigma_{j+1}-2-1)(p_0m_0-\varepsilon)^{-k+1} \\ &\leq 2\sigma_{j+1}(p_0m_0-\varepsilon)^{-k} \end{aligned}\tag{5.5.47}$$

for $2 \le k \le \sigma_{j+1} - 2 - n_0$.

With the help of (5.5.44) and (5.5.47) we get on the set $H_{n_0} \cap \{\sigma_{j+1} > \sigma_j + \tau_j + 2\}$

$$\begin{aligned}
|\Phi_{\sigma_{j+1}}| &\le M \sum_{k=1}^{\sigma_{j+1}-n_0-3} |\Psi_{\sigma_{j+1}-2-k}|(p_0 m_0 + \varepsilon)^{k+1} + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\le M \sum_{k=1}^{\sigma_{j+1}-n_0-3} 2\sigma_{j+1}(p_0 m_0 - \varepsilon)^{-k}(p_0 m_0 + \varepsilon)^{k+1} + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\le 2M\sigma_{j+1}^2 \left(\frac{p_0 m_0 + \varepsilon}{p_0 m_0 - \varepsilon}\right)^{\sigma_{j+1}-n_0-3} (p_0 m_0 + \varepsilon) + M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\le 3M\sigma_{j+1}^2 (m^* + \gamma)^{\sigma_{j+1}} \\
&\le (m^* + 2\gamma)^{\sigma_{j+1}}
\end{aligned}$$

for n_0 and thus $\sigma_{j+1} \ge n_0$ large enough. Here we used (5.5.35) and (5.5.36) in the last two steps. This shows (5.5.34) in Lemma 5.5.4. $\square$

We now return to the proof of Proposition 5.5.2. First, we choose $\gamma \in \mathbb{R}$ with

$$0 \ < \ 6\gamma \ < \ \alpha - m^*$$

and n_0 large enough such that the estimations (5.5.32), (5.5.33) and (5.5.34) from Lemma 5.5.4 hold. Using these estimations, we can conclude that on H_{n_0} we a.s. have

$$\eta := \ \inf\{n \ge n_0 : \ |Z^{(l)}\sigma_n| < (m^* + 5\gamma)^{\sigma_n}\} \ < \ \infty. \tag{5.5.48}$$

To see this, we just have to see what happens on the event

$$H_{n_0} \cap \bigcap_{j=1}^{k} \Big(\big\{|Z^{(l)}\sigma_j| > (m^* + \gamma)^{\sigma_j}\big\} \cap \big\{\sigma_{j+1} = \sigma_j + \tau_j + 2\big\}\Big).$$

On this set we can use (5.5.33) k times in a row and we get

$$|Z^{(l)}\sigma_k| \ \le \ |Z^{(l)}\sigma_0| \prod_{j=1}^{k} (m^* + 4\gamma)^{\tau_j} \ \le \ |Z^{(l)}\sigma_0|(m^* + 4\gamma)^{\sigma_k},$$

from which we conclude that (5.5.48) indeed holds on H_{n_0}.

Again by using the three estimations (5.5.32), (5.5.33), and (5.5.34) of Lemma 5.5.4, we can see inductively that on the set H_{n_0} we have

$$|Z^{(l)}\sigma_n| \ \le \ (m^* + 5\gamma)^{\sigma_n}$$

for all $n \geq \eta$. Additionally, if we assume

$$|Z^{(l)}\sigma_n + i - 1| \leq (m^* + 5\gamma)^{\sigma_n + i - 1},$$

we see inductively by using (5.5.39) that on the set H_{n_0} we have for all $n \geq \eta$ and for all $1 \leq i \leq \tau_n - 1$

$$\begin{aligned} |Z^{(l)}\sigma_n + i| &\leq |Z^{(l)}\sigma_n + i - 1|(p_c m_c + \varepsilon) + (\sigma_n + i - 1)M \\ &\leq (m^* + 5\gamma)^{\sigma_n + i - 1}(p_c m_c + \varepsilon) + (\sigma_n + i - 1)M \\ &\leq (m^* + 5\gamma)^{\sigma_n + i - 1}(m^* + \gamma) + (\sigma_n + i - 1)M \\ &\leq (m^* + 6\gamma)^{\sigma_n + i} < \alpha^{\sigma_n + i} \end{aligned}$$

for n_0 (and thus also $\sigma_n \geq n_0$) large enough. Since by definition of $(\sigma_n)_{n\in\mathbb{N}_0}$ and $(\tau_n)_{n\in\mathbb{N}_0}$ the LP is empty at the remaining times, we conclude that we have

$$\mathsf{P}\left(\liminf_{n\to\infty}\left(H_n \cap \{|Z^{(l)}n| < \alpha^n\}\right)\right) = 1. \tag{5.5.49}$$

Finally, Lemma 5.5.1 and Lemma 5.5.3 imply that

$$\mathsf{P}\left(\liminf_{n\to\infty} H_n\right) = 1,$$

and this together with (5.5.49) yields (5.5.24). □

After having investigated the growth of the LP, we are now interested in the speed at which the cookies are consumed:

Proposition 5.5.5. *(a) There exists $\lambda > 0$ such that we a.s. have*

$$\liminf_{n\to\infty}\frac{l(n)}{n} > \lambda. \tag{5.5.50}$$

(b) In fact, for $p_0 m_0 > 1$ we a.s. have

$$\lim_{n\to\infty}\frac{l(n)}{n} = 1. \tag{5.5.51}$$

Proof of Proposition 5.5.5. (a) We compare the CBRW with the following process $(W_n)_{n\in\mathbb{N}_0}$, that behaves similarly to an excited random walk. It is determined by the initial configuration $W_0 := 0$ and the transition probabilities

$$\mathsf{P}\big(W_{n+1} = W_n + 1 \mid (W_j)_{1\leq j\leq n}\big) = \begin{cases} 0 & \text{on } \left\{\max\limits_{j=0,1,\dots,n-1} W_j < W_n\right\} \\ p_0 & \text{on } \left\{\max\limits_{j=0,1,\dots,n} W_j > W_n\right\} \end{cases}$$

and

$$\mathsf{P}\big(W_{n+1} = W_n - 1 \mid (W_j)_{1\le j\le n}\big) \;=\; \begin{cases} 1 & \text{on } \left\{\max\limits_{j=0,1,\dots,n-1} W_j < W_n\right\} \\ q_0 & \text{on } \left\{\max\limits_{j=0,1,\dots,n} W_j > W_n\right\} \end{cases}$$

for $n \in \mathbb{N}_0$. The process $(W_n)_{n\in\mathbb{N}_0}$ moves to the left with probability 1 every time it reaches a position $x \in \mathbb{N}_0$ for the first time and otherwise it behaves as an asymmetric random walk on $\mathbb{Z}$ with transition probabilities p_0 and q_0. For the random times

$$T_x := \inf\{n \in \mathbb{N}_0 : W_n = x\} \quad (\text{for } x \in \mathbb{N}_0),$$

we notice that $\big(T_{x+1} - T_x\big)_{x\in\mathbb{N}_0}$ is a sequence of i.i.d. random variables with

$$\mathsf{E}[T_1 - T_0] \;=\; \mathsf{E}[T_1] \;=\; 1 + \frac{2p_0}{2p_0 - 1}.$$

Therefore, the strong law of large numbers implies that we a.s. have

$$\lim_{x\to\infty} \frac{T_x}{x} \;=\; \lim_{x\to\infty} \frac{1}{x}\sum_{i=0}^{x-1}(T_{i+1} - T_i) \;=\; \mathsf{E}[T_1 - T_0] \;=\; 1 + \frac{2p_0}{2p_0 - 1} \;<\; \infty.$$

Since we can couple the CBRW and the process $(W_n)_{n\in\mathbb{N}_0}$ in a natural way such that we have

$$\max_{\nu\in Z_n} X_\nu \;\ge\; W_n$$

for all $n \in \mathbb{N}_0$, we can conclude that (5.5.50) holds for $0 < \lambda < \left(1 + \frac{2p_0}{2p_0-1}\right)^{-1}$.

(b) We start this part of the proof with the following lemma:

Lemma 5.5.6. *For a CBRW with $m_0 > 1$, there exists $\gamma > 1$ such that we a.s. have*

$$\lim_{n\to\infty} \frac{|Z_n|}{\gamma^n} = \infty. \tag{5.5.52}$$

Proof of Lemma 5.5.6. Let us treat the case where $m_c > 1$ first. Let $\big(V_{n,k}\big)_{n,k\in\mathbb{N}}$ be i.i.d. random variables with

$$1 - \mathsf{P}(V_{1,1} = 1) \;=\; \mathsf{P}(V_{1,1} = 2) \;=\; \min\left\{\sum_{i=2}^{\infty}\mu_0(i), \sum_{i=2}^{\infty}\mu_c(i)\right\},$$

and we define the corresponding GWP $\big(\widetilde{Z}_n\big)_{n\in\mathbb{N}_0}$ by $\widetilde{Z}_0 := 1$,

$$\widetilde{Z}_{n+1} := \sum_{i=1}^{\widetilde{Z}_n} V_{n+1,i}.$$

Observe, that $\mathsf{E}[V_{1,1}] > 1$. A standard coupling argument reveals that

$$\widetilde{Z}_n \;\leq\; |Z_n|.$$

Now, the claim follows since $\widetilde{Z}_n$ grows exponentially, e.g. Theorem I.10.3 on page 30 in [AN72].

The other case is similar: Consider now the i.i.d. random variables $\big(V_{n,k}\big)_{n,k\in\mathbb{N}}$ with

$$1 - \mathsf{P}(V_{1,1} = 1) \;=\; \mathsf{P}(V_{1,1} = 2) \;=\; \min\{q_0, q_c\} \sum_{i=2}^{\infty} \mu_0(i),$$

and define as above the corresponding GWP $\big(\widetilde{Z}_n\big)_{n\in\mathbb{N}_0}$. For the coupling we observe that the probability of every particle in the CBRW to produce a particle which moves to the left is bounded from below by $\min\{q_0, q_c\}$. Such a particle cannot be at a position with a cookie and therefore its offspring distribution is given by $\big(\mu_0(i)\big)_{i\in\mathbb{N}_0}$. Eventually, the corresponding coupling yields

$$\widetilde{Z}_n \;\leq\; |Z_{2n}|$$

and the claim follows as above. □

We now return to the proof of Proposition 5.5.5(b). Let us choose $\varepsilon > 0$ such that

$$p_0 m_0 - \varepsilon > 1. \tag{5.5.53}$$

We use this ε for the definition of the sets $\big(F_n\big)_{n\in\mathbb{N}_0}$ and $\big(H_n\big)_{n\in\mathbb{N}_0}$, see (5.5.19) and (5.5.28). Due to Lemma 5.5.6 we can choose $\gamma > 1$ such that we a.s. have

$$\lim_{n\to\infty} \frac{|Z_n|}{\gamma^{2n}} \;=\; \infty \quad \text{and} \quad \gamma^2 \;<\; p_0 m_0 - \varepsilon. \tag{5.5.54}$$

In addition, we choose n_0 sufficiently large such that we have for all $n \geq n_0$

$$\gamma^n \;>\; n, \quad \gamma^n(q_c m_c - \varepsilon) \;>\; (n+1), \quad \gamma^{\beta \log(\gamma^n)}(q_c m_c - \varepsilon) \;\geq\; 1 \tag{5.5.55}$$

for some $\beta > 0$ which fulfills Lemma 5.5.3. In the following we again investigate the behaviour of the CBRW on the set H_{n_0} on which the process does not show certain unlikely behaviour after time n_0 (cf. (5.5.19) and (5.5.25)). We show that already the offspring of one position with "many" particles cause the leftmost cookie to move to the right with speed 1. For this, we introduce the random time

$$\eta := \; \inf\{n \geq n_0 : \; \exists x \in \mathbb{Z} \text{ such that } |Z_n(x)| \geq \gamma^n\}.$$

At time η we have sufficiently many particles at the random position

$$x_0 := \; \sup\{x \in \mathbb{Z} : \; Z_\eta(x) \geq \gamma^\eta\}.$$

Due to (5.5.54) we a.s. have $\eta < \infty$ since at time n only $n+1$ positions can be occupied. Additionally, we introduce the random time

$$\sigma_0 := \inf\{n \geq \eta :\ l(n) = x_0 + n - \eta\}$$

at which offspring of the particles belonging to $Z_\eta(x_0)$ can potentially reach the LP for the first time after time η. Since $p_c m_c \leq 1$, the LP dies out infinitely often and therefore we a.s. have $\sigma_0 < \infty$. Then, we define inductively the random times

$$\begin{aligned} \tau_j &:= \inf\{n \geq \sigma_j :\ |Z^{(l)}n| = 0\} - \sigma_j, && \text{for } j \geq 0,\\ \sigma_j &:= \inf\{n \geq \sigma_{j-1} + \tau_{j-1} :\ |Z^{(l)}n| \neq 0\}, && \text{for } j \geq 1, \end{aligned}$$

denoting the time period of survival and the time of the restart of the LP after time σ_0. Due to (5.5.55) we have

$$|Z_\eta(x_0)| \ \geq\ \gamma^\eta \geq \eta \tag{5.5.56}$$

which allows us to use the lower bound for $|Z^+_{\eta+1}(x_0+1)|$ on H_{n_0}. By using (5.5.54) and (5.5.56) we get on the set $H_{n_0} \cap \{l(\eta) > x_0\}$

$$|Z_{\eta+1}(x_0+1)| \ \geq\ |Z^+_{\eta+1}(x_0+1)| \ \geq\ \gamma^\eta(p_0 m_0 - \varepsilon) \ \geq\ \gamma^{\eta+1}.$$

Iterating the last step, we see that on the set $H_{n_0} \cap \{l(\eta+k) > x_0 + k\}$ we have

$$|Z_{\eta+k}(x_0+k)| \ \geq\ \gamma^{\eta+k}$$

and therefore we conclude that

$$|Z^{(l)}\sigma_0| \ =\ |Z_{\eta+\sigma_0-\eta}(x_0+\sigma_0-\eta)| \ \geq\ \gamma^{\eta+\sigma_0-\eta} \ =\ \gamma^{\sigma_0}$$

holds on H_{n_0}. In the following we see that already the offspring particles of $Z^{(l)}\sigma_0$ which move to the left at time σ_0 and afterwards move to the right in every step lead to a very large LP at the next restart at time σ_1. To see this, we first notice that (5.5.55) implies on the set H_{n_0}

$$|Z_{\sigma_0+1}(l(\sigma_0)-1)| \ \geq\ |Z^-_{\sigma_0+1}(l(\sigma_0)-1)| \ \geq\ \gamma^{\sigma_0}(q_c m_c - \varepsilon) \ \geq\ (\sigma_0+1)$$

since we have $|Z_{\sigma_0}(l(\sigma_0))| \geq \gamma^{\sigma_0} > \sigma_0$. An iteration of this together with (5.5.54) and (5.5.55) yield for $k \in \mathbb{N}$

$$\begin{aligned} |Z_{\sigma_0+1+k}(l(\sigma_0)-1+k)| &\geq |Z^+_{\sigma_0+1+k}(l(\sigma_0)-1+k)|\\ &\geq \gamma^{\sigma_0}(q_c m_c - \varepsilon)(p_0 m_0 - \varepsilon)^k\\ &\geq \gamma^{\sigma_0+2k}(q_c m_c - \varepsilon) \geq\ \sigma_0 + 2k + 1 \geq \sigma_0 + k + 1 \end{aligned}$$

on the set $H_{n_0} \cap \{\tau_0 \geq k-1\}$. In particular, this implies

$$\begin{aligned} |Z^{(l)}\sigma_0 + \tau_0 + 2| &= |Z_{\sigma_0+\tau_0+2}(l(\sigma_0) + \tau_0)| \\ &\geq \gamma^{\sigma_0+2(\tau_0+1)}(q_c m_c - \varepsilon) \\ &\geq \gamma^{\sigma_0+\tau_0+2}\gamma^{\beta\log(\gamma^{\sigma_0})}(q_c m_c - \varepsilon) \;\geq\; \gamma^{\sigma_0+\tau_0+2} > \; 0 \end{aligned}$$

on the set H_{n_0}. Here we used that, due to Lemma 5.5.3, we have $\tau_0 \geq \beta\log(\gamma^{\sigma_0})$ and recalled (5.5.55) for the last inequality. Further, we conclude that we have $\sigma_1 = \sigma_0 + \tau_0 + 2$ on H_{n_0}, which implies that the LP is restarted two time steps after it has died out at time $\sigma_0 + \tau_0$. Iterating this argument finally implies

$$|Z^{(l)}\sigma_{j+1}| \;\geq\; \gamma^{\sigma_{j+1}} \quad \text{and} \quad \sigma_{j+1} \;=\; \sigma_j + \tau_j + 2 \tag{5.5.57}$$

for all $j \in \mathbb{N}_0$ on the set H_{n_0}. For

$$\beta^* := \; \beta\log(\gamma) \; > \; 0$$

we further conclude from (5.5.57) and Lemma 5.5.3 by induction that on H_{n_0} we have for $j \in \mathbb{N}_0$

$$\tau_j \;\geq\; \beta\sigma_j\log(\gamma) \;\geq\; \beta^*(1+\beta^*)^j\sigma_0 \tag{5.5.58}$$

and thus

$$\sigma_{j+1} \;=\; \sigma_j + \tau_j + 2 \;\geq\; (1+\beta^*)^j\sigma_0 + \beta^*(1+\beta^*)^j\sigma_0 \;=\; (1+\beta^*)^{j+1}\sigma_0. \tag{5.5.59}$$

Hence, on the set H_{n_0} we have for $n \geq \sigma_0$

$$\begin{aligned} \frac{l(n)}{n} &\geq \frac{l(\sigma_0) + n - \sigma_0 - 2|\{j \geq 0 : \, \sigma_j + \tau_j \leq n\}|}{n} \\ &\geq \frac{l(\sigma_0) + n - \sigma_0 - 2\frac{\log(n)-\log(\sigma_0)}{\log(1+\beta^*)}}{n} \;\xrightarrow[n\to\infty]{}\; 1. \end{aligned}$$

Here we used (5.5.57) in the first step and in the second step we used the fact that due to (5.5.58) and (5.5.59) we have

$$\sigma_j + \tau_j \;\geq\; (1+\beta^*)^{j+1}\sigma_0$$

for $j \in \mathbb{N}_0$. This yields that on the set H_{n_0} we have

$$\lim_{n\to\infty} \frac{l(n)}{n} \;=\; 1.$$

Since by (5.5.29) we have

$$\lim_{n\to\infty} \mathsf{P}(H_n) \;=\; 1,$$

we finally established (5.5.51). $\square$

With Proposition 5.5.2 and Proposition 5.5.5 we are now prepared to prove Theorem 5.3.1(b). Similarly to the proof of Theorem 5.3.1(a), we introduce the event

$$A_n := \{\exists \nu \succeq L_n : X_\nu = 0, X_\eta < l(|\eta|) \,\forall\, L_n \prec \eta \prec \nu\}$$

with $L_n = \{\nu \in Z_{n+1}(l(n)-1) : \nu \succ Z^{(l)}n\}$ for $n \in \mathbb{N}$. On A_n, there exists a particle ν which returns to the origin after time n and additionally the last ancestor of ν which has been at a position containing a cookie was the ancestor at time n. For $\lambda_0, \gamma > 0$, which we will choose later (cf. (5.5.61) and (5.5.63)), we get the following estimate with $m^* = \max\{1, p_0 m_0\}$:

$$\begin{aligned}
&\mathsf{P}\Big(A_n \mid \{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \leq (m^*+\gamma)^n\}\Big) \\
&= 1 - \mathsf{P}\Big(A_n^c \mid \{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \leq (m^*+\gamma)^n\}\Big) \\
&\leq 1 - \mathsf{P}\Big(\Lambda^-_{\lceil n\lambda_0 - 1\rceil} = 0\Big)^{M(m^*+\gamma)^n}.
\end{aligned}$$

Here we used the fact that the number of offspring of every particle belonging to L_n which return to the origin is bounded by the amount of offspring in $\Lambda^-_{l(n)-1}$. Additionally, we have

$$|L_n| \;\leq\; M|Z^{(l)}n|$$

due to assumption (5.1.1). Since the GWP $\big(\Lambda_n^-\big)_{n\in\mathbb{N}_0}$ with mean φ_ℓ is subcritical we can use Proposition 5.4.4 to obtain for some constants $c, C > 0$

$$\begin{aligned}
\mathsf{P}\Big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \leq (m^*+\gamma)^n\}\Big) &\leq 1 - \big(1 - c(\varphi_\ell)^{\lceil n\lambda_0 - 1\rceil}\big)^{M(m^*+\gamma)^n} \\
&\leq 1 - \exp\big(-2c(\varphi_\ell)^{\lceil n\lambda_0-1\rceil} M(m^*+\gamma)^n\big) \\
&\leq 2c(\varphi_\ell)^{n\lambda_0 - 1} M(m^*+\gamma)^n \\
&= C(\varphi_\ell)^{n\lambda_0}(m^*+\gamma)^n \qquad (5.5.60)
\end{aligned}$$

for large n. In the above display we used the inequalities $1-x \geq \exp(-2x)$ for $x \in [0, \frac{1}{2}]$ (note that we have $\varphi_\ell < 1$) and $1 - \exp(-x) \leq x$ for all $x \in \mathbb{R}$.

(i) Let us first assume that we have $m^* = \max\{1, p_0 m_0\} = 1$:

We choose

$$\lambda_0 \;=\; \frac{1}{2} \cdot \lambda$$

for some $\lambda > 0$ which fulfills Proposition 5.5.5(a). We have

$$\varphi_\ell \;\leq\; 2q_0 m_0 \;<\; 1$$

and therefore can choose $\gamma > 0$ such that

$$(\varphi_\ell)^{\lambda_0}(m^*+\gamma) \;\leq\; (2q_0m_0)^{\lambda_0}(1+\gamma) \;\leq\; (1-\gamma). \qquad (5.5.61)$$

By applying (5.5.61) to (5.5.60), we get

$$\mathsf{P}\Big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \leq (m^* + \gamma)^n\}\Big) \;\leq\; o(1)(1-\gamma)^n.$$

Therefore, the Borel-Cantelli lemma implies

$$\mathsf{P}\left(\limsup_{n\to\infty}\Big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \leq (m^* + \gamma)^n\}\Big)\right) \;=\; 0. \tag{5.5.62}$$

Thereby, Proposition 5.5.2 and Proposition 5.5.5 together with the choices of λ_0 and γ yield

$$\mathsf{P}\left(\liminf_{n\to\infty}\Big(\{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \leq (m^* + \gamma)^n\}\Big)\right) \;=\; 1.$$

Finally, we can conclude from (5.5.62) that we have

$$\mathsf{P}\left(\limsup_{n\to\infty} A_n\right) \;=\; 0,$$

which implies the transience of the CBRW in this case.

(ii) We now assume that we have $m^* = p_0 m_0 > 1$:

Due the assumption of the transience of the BRW without cookies, we have

$$\varphi_\ell p_0 m_0 \;\leq\; 2q_0 m_0 \cdot p_0 m_0 \;\leq\; \frac{1}{2}.$$

Therefore, we can choose $0 < \gamma < 1$ such that

$$(\varphi_\ell)^{1-\gamma}(p_0 m_0 + \gamma) \;\leq\; \frac{3}{4}. \tag{5.5.63}$$

For $\lambda_0 := 1 - \gamma$, (5.5.60) and (5.5.63) imply

$$\mathsf{P}\Big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \;\leq\; (m^* + \gamma)^n\}\Big) \;\leq\; C\left(\frac{3}{4}\right)^n.$$

Again by applying the Borel-Cantelli lemma, we get

$$\mathsf{P}\left(\limsup_{n\to\infty}\big(A_n \cap \{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \leq (m^* + \gamma)^n\}\big)\right) \;=\; 0.$$

Additionally, Proposition 5.5.2 and Proposition 5.5.5 together with the choices of λ_0 and γ yield

$$\mathsf{P}\left(\liminf_{n\to\infty}\big(\{l(n) \geq n\lambda_0\} \cap \{Z^{(l)}n \leq (m^* + \gamma)^n\}\big)\right) \;=\; 1.$$

Therefore, we conclude that we have

$$\mathsf{P}\left(\limsup_{n\to\infty} A_n\right) \;=\; 0,$$

which implies the transience of the CBRW in the case $p_0 m_0 > 1$. ■

5.5.2 Proof of Theorem 5.3.2

For this theorem we only have to make sure that the cookies cannot displace the cloud of particles too far to the right. But similar to the case of a cookie random walk (cf. [BS08]) one single cookie at every position $x \in \mathbb{N}_0$ is not enough for such a behaviour. We divide the proof of the theorem into two cases. At first we consider the case $m_0 = 1$, i.e. particles can only branch at positions with a cookie, and in the second part we consider the case $m_0 > 1$.

(a) Let us first assume $m_0 = 1$. In this case the BRW without cookies reduces to a nearest neighbour random walk on $\mathbb{Z}$ and is therefore strongly recurrent iff $p_0 = \frac{1}{2}$ holds. For $x \geq 0$ let us denote the random time at which the cookie at position x is eaten by

$$T_x := \inf\{n \in \mathbb{N}_0 : \exists \nu \in Z_n \text{ with } X_\nu = x\}.$$

For $\nu \in \mathcal{Z}$ let $(\nu, e_1) := (\nu, 1)$ denote the first offspring particle of ν and similarly we define $(\nu, e_n) := (\nu, 1, \dots, 1)$ where we add n ones to the vector ν. Since we assume that each particle has at least one offspring, we have

$$\mathsf{P}\big((\nu, e_n) \in \mathcal{Z} \big| \nu \in \mathcal{Z}\big) = 1 \quad \forall n \in \mathbb{N}.$$

By assumption we have $p_0 = \frac{1}{2}$. Therefore, for every $\nu \in \mathcal{Z}$ the process $\big(X_{(\nu,e_n)}\big)_{n\in\mathbb{N}_0}$ behaves like a symmetric random walk started at the random position X_ν as long as the particles do not reach a cookie. Since q_c is the probability for the particle (ν, e_1) to move to the left and $\frac{2}{x+1}$ is the probability for a symmetric random walk started in $x-1$ to reach 0 before $x+1$, we can conclude that

$$\begin{aligned}
&\mathsf{P}\big(\exists \eta \succeq \nu : \; X_\eta = 0, \; X_\tau \leq x \; \forall \nu \preceq \tau \preceq \eta \big| \nu \in \mathcal{Z}, \; X_\nu = x\big) \\
&\geq \mathsf{P}\big(\exists n \in \mathbb{N} : \; X_{(\nu,e_n)} = 0, X_{(\nu,e_m)} \leq x \; \forall m \leq n \big| \nu \in \mathcal{Z}, \; X_\nu = x\big) \\
&= q_c \cdot \frac{2}{x+1}.
\end{aligned}$$

With the help of the inequality $1 - x \leq \exp(-x)$ this yields for $N \in \mathbb{N}_0$

$$\begin{aligned}
&\mathsf{P}\big(\nexists \; \nu \in Z_n, \; n \geq N : \; X_\nu = 0\big) \\
&\leq \mathsf{P}\big(\nexists \; \nu \in Z_n, \; n \geq T_N : \; X_\nu = 0\big) \\
&\leq \prod_{x=N}^{\infty} \mathsf{P}\big(\nexists \; \nu \succeq Z^{(l)} T_x : \; \exists \eta \succeq \nu \text{ such that } \; X_\eta = 0, \; X_\tau \leq x \; \forall \nu \preceq \tau \preceq \eta\big)
\end{aligned}$$

$$\leq \prod_{x=N}^{\infty} \left(1 - q_c \cdot \tfrac{2}{x+1}\right)$$
$$\leq \exp\left(-\sum_{x=N}^{\infty} q_c \cdot \tfrac{2}{x+1}\right) \;=\; 0.$$

This implies that arbitrarily late a particle returns to the origin with probability 1, which yields the strong recurrence of the CBRW.

(b) Now we suppose that we have $m_0 > 1$. From Proposition 5.2.1 we know that we have

$$\log(m_0) > -\frac{1}{2}\log\left(4 \cdot p_0 \cdot q_0\right) = I(0)$$

where $I(\cdot)$ denotes the rate function of the nearest neighbour random walk on $\mathbb{Z}$ with transition probabilities p_0 and q_0. Since the rate function is continuous on $(-1, 1)$, there even exists $0 < \varepsilon < 1$ such that

$$\log(m_0) > I(-\varepsilon).$$

Let $\big(S_n\big)_{n\in\mathbb{N}_0}$ denote such a nearest neighbour random walk started in 0 and with transition probabilities p_0 and q_0. We have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathsf{P}\big(S_n \leq -n\varepsilon\big) \;=\; \begin{Bmatrix} -I(-\varepsilon) & \text{for } 2p_0 - 1 > -\varepsilon \\ 0 & \text{for } 2p_0 - 1 \leq -\varepsilon \end{Bmatrix} \;\geq\; -I(-\varepsilon).$$

Therefore, for every $\delta > 0$ there exists k_0 such that

$$\mathsf{P}\big(S_{k_0} \leq -k_0\varepsilon\big) \;\geq\; \exp\big(-k_0(I(-\varepsilon) + \delta)\big).$$

This yields for the BRW without cookies $\big(Y_\nu\big)_{\nu\in\mathcal{Y}}$ that

$$\mathsf{E}\Big[\sum_{\nu\in\mathcal{Y}_{k_0}} \mathbb{1}_{\{Y_\nu \leq -k_0\varepsilon\}}\Big] \;\geq\; (m_0)^{k_0} \exp\big(-k_0(I(-\varepsilon) + \delta)\big) \;>\; 1 \tag{5.5.64}$$

for $\delta > 0$ small enough. Therefore, we can conclude that the embedded GWP of those particles which move at least $k_0\varepsilon$ to the left between time 0 and k_0, between k_0 and $2k_0$ and so on is supercritical and therefore survives with strictly positive probability p_{sur}. Let us now turn back to the CBRW. For every existing particle the probability

$$\mathsf{P}\big(\exists \eta \in \mathcal{Z}:\; \eta \succeq \nu,\; |\eta| - |\nu| = k_0,\; X_\eta = X_\nu - k_0 \mid \nu \in \mathcal{Z}\big) \;\geq\; \min(q_c, q_0) q_0^{k_0}$$

to produce an offspring which moves k_0 times to the left in the next k_0 time steps is bounded away from 0. From this we conclude that the probability

$$\mathsf{P}\big(\exists \eta \in \mathcal{Z}:\; \eta \succeq \nu,\; X_\tau \leq l(|\nu|)\; \forall \nu \preceq \tau \preceq \eta,\; X_\eta \leq 0 \mid \nu \in \mathcal{Z}\big) \;\geq q_c q_0^{k_0} p_{\text{sur}} \;=: c \;>\; 0 \tag{5.5.65}$$

for every existing particle in the CBRW to produce an offspring which moves to the negative semi-axis before it hits the cookies again is also bounded away from 0. Thereby the lower bound is a lower estimate for the probability for each existing particle in the CBRW to produce an offspring which moves k_0 times to the left in the next k_0 time steps and then starts a surviving embedded GWP which moves at least $k_0\varepsilon$ to the left between time 0 and k_0, between k_0 and $2k_0$ and so on. Since the particles we consider for this embedded GWP cannot hit the cookies in between, this GWP has the same probability for survival p_{sur} as in the case of the BRW without cookies (cf. (5.5.64)). Using (5.5.65), we can conclude the strong recurrence of the CBRW since the particles on the negative semi-axis behave as the strongly recurrent BRW without cookies before they can reach a cookie again. ■

5.5.3 Proof of Theorem 5.3.3

Proof of part (a).
Here, we suppose that the LP is supercritical, i.e. $p_c m_c > 1$. On the one hand the probability that all particles which are produced in the first step move to the left and their offspring then escape to $-\infty$ without returning to 0 is strictly positive since every offspring particle starts an independent BRW without cookies at position -1 as long as the offspring does not return to the origin. Thereby the probability for the BRW started at -1 never to return to the origin is strictly positive since the BRW without cookies is transient to the left by assumption.

On the other hand the LP which is started at 0 is a supercritical GWP and therefore survives with positive probability. If it survives, a.s. infinitely many particles leave the LP (to the left) at time $n \geq 1$. Afterwards each of those particles starts a BRW without cookies at position $n-1 \geq 0$ since the offspring cannot reach a cookie again. Each of those BRWs without cookies will a.s. produce at least one offspring which visits the origin since the BRW without cookies is transient to the left by assumption. ■

Proof of part (b).
Here, we suppose that the LP is critical or subcritical, i.e. $p_c m_c \leq 1$. In the following we want to consider the following three quantities. The first one is the amount of particles in the LP. The second one is the number of particles which are descendants of the non-LP particles of generation n and which are the first in their ancestral line to reach the position $l(n)$. Thus these particles can potentially change the position of the leftmost cookie. The third group is the number of offspring particles of the generation n which will not reach the position of the leftmost cookie anymore. More precisely, for all $n \in \mathbb{N}_0$ we define (observe that $X_\nu = l(n)$ implies that $|\nu| = n$):

$$\zeta_1(n) := |Z^{(l)}n|,$$
$$\zeta_2(n) := \sum_{\nu \succeq Z_n \setminus Z^{(l)}n} \mathbb{1}_{\{X_\nu = l(n),\, X_\eta < l(n)\, \forall \eta \prec \nu\}},$$

$$\zeta_3(n) := \sum_{\nu \in Z_n \setminus Z^{(l)} n} \mathbb{1}_{\{X_\eta < l(n), \forall \eta \succeq \nu\}}.$$

Note that for the definition of $\zeta_2(n)$ we count the amount of descendants of the non-LP particles at time n which will reach the position $l(n)$ in the future. Thus the type-2 particles belong to a generation larger than n.
In the following we want to allow arbitrary starting configurations from the set

$$\mathcal{S} := \left\{ (a,b) \in \mathbb{N}_0^{\mathbb{Z}} \times \mathbb{N}_0 : \sum_{k \in \mathbb{Z}} a(k) < \infty, \max\{k \in \mathbb{Z} : a(k) > 0\} \leq b \right\}.$$

Here a contains the information about the number of particles at each position $k \in \mathbb{Z}$ and b is the position of the leftmost cookie. In particular, every configuration of the CBRW which can be reached within finite time is contained in the set $\mathcal{S}$. For each $(a,b) \in \mathcal{S}$ we consider the probability measure $\mathsf{P}_{(a,b)}$ under which the CBRW starts in the configuration (a,b) and then evolves in the usual way.

The main idea of the proof is the following. We show that there is a critical level for the total amount of the type-1 and type-2 particles. Once this level is exceeded the total amount has the tendency to fall back below this level. There are two reasons which cause this behaviour. On one hand, the expected amount of type-2 particles which stay type-2 particles for another time step decreases every time the leftmost cookie is consumed by a type-1 particle. On the other hand, if there are many type-1 particles, the LP survives for a long time with high probability and meanwhile the remaining particles have time to escape to the left.

For the proof we need to know the relation between the type-1 and type-2 particles. Thereby we have to distinguish between two different situations. In the first one, there are type-1 particles at time n and therefore the leftmost cookie is consumed. In the second case there are no type-1 particles and therefore the position of the leftmost cookie does not change. Let us first assume that there are type-1 particles at time n. Then, on the set $\{\zeta_1(n) \neq 0\}$ we a.s. have

$$\begin{aligned}
\mathsf{E}_{(a,b)}\left[\zeta_1(n+1) \mid \zeta_1(n), \zeta_2(n)\right] &= \zeta_1(n) p_c m_c, \\
\mathsf{E}_{(a,b)}\left[\zeta_2(n+1) \mid \zeta_1(n), \zeta_2(n)\right] &= \zeta_1(n) q_c m_c (\varphi_r)^2 + \zeta_2(n) \varphi_r.
\end{aligned} \tag{5.5.66}$$

Here the last equality holds since each type-1 particle produces an expected amount of $q_c m_c$ particles which leave the LP to the left. To decide how many of these particles are type-2 particles at time $n+1$ we have to count the number of their offspring which will reach position $l(n+1) = l(n) + 1$ in the future. For each of these particles the distribution of this random number coincides with the distribution of Λ_2^+ whose expectation is given by $(\varphi_r)^2$. Additionally, since one cookie is consumed the amount of type-2 particles, which are still type-2 particles at time $n+1$, decreases in expectation by φ_r. Observe that due to the transience to the left of the BRW without cookies, the process $\left(\Lambda_n^+\right)_{n\in\mathbb{N}_0}$ is a GWP with mean $\varphi_r < 1$ (cf. Remark 5.4.3).

Let us now assume that the LP is empty. Then, on $\{\zeta_1(n) = 0\}$ we a.s. have

$$\mathsf{E}_{(a,b)}\big[\zeta_1(n+1) + \zeta_2(n+1) \mid \zeta_1(n), \zeta_2(n)\big] \;=\; \zeta_2(n), \tag{5.5.67}$$

since the position of the leftmost cookie does not change, i.e. $l(n{+}1) = l(n)$. Therefore, each type-2 particle of time n either still is a type-2 particle at time $n+1$ or becomes a type-1 particle.

(i) First, we deal with the subcritical case, i.e. $p_c m_c < 1$. For fixed $h \in \mathbb{N}$ (which will be specified later, cf. (5.5.69)) we define the following random times

$$\eta_{n+1} := \begin{cases} (\eta_n + h) \wedge \inf\{i > \eta_n : \zeta_1(i) = 0\}, & \text{if } \zeta_1(\eta_n) > 0, \\ (\eta_n + h) \wedge \inf\{i > \eta_n : \zeta_1(i) > 0\}, & \text{if } \zeta_1(\eta_n) = 0, \end{cases}$$

for $n \in \mathbb{N}_0$ and $\eta_0 := 0$. Note that we have $\eta_{n+1} - \eta_n \le h$. For $n \in \mathbb{N}_0$ we define

$$\xi_1(n) := \zeta_1(\eta_n), \quad \xi_2(n) := \zeta_2(\eta_n)$$

as the amount of type-1 and type-2 particles along the sequence $(\eta_n)_{n\in\mathbb{N}_0}$ and the associated filtration $\mathcal{F}_n := \sigma\big(\xi_1(i),\ \xi_2(i),\ \eta_i :\ i \le n\big)$. We want to adapt Theorem 2.2.1 of [FMM95] and start with the following lemma:

Lemma 5.5.7. *For suitable (large) $h, u \in \mathbb{N}$ we have*

$$\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \mid \mathcal{F}_n] \;\le\; \xi_1(n) + \xi_2(n) \tag{5.5.68}$$

a.s. on $\{\xi_1(n) + \xi_2(n) \ge u\}$ for all $(a,b) \in \mathcal{S}$.

Proof of Lemma 5.5.7. Let us fix $(a,b) \in \mathcal{S}$. We choose $h \in \mathbb{N}$ large enough such that

$$\big(p_c m_c\big)^h + q_c m_c \sum_{i=0}^{h-1} \big(p_c m_c\big)^i (\varphi_r)^{h-i+1} \;<\; \frac{1}{2} \tag{5.5.69}$$

and

$$(\varphi_r)^h \;<\; \frac{1}{2}. \tag{5.5.70}$$

Such a choice is possible since $p_c m_c < 1$ and $\varphi_r < 1$. Then, we fix $c = c(h)$ such that

$$0 \;<\; c \;\le\; \frac{1}{M^h}\,(1 - \varphi_r) \tag{5.5.71}$$

holds. Recall that the particles in the LP constitute a subcritical GWP. Let $\big(GW_n^{\text{sub}}\big)_{n\in\mathbb{N}_0}$ denote such a GWP (with the same offspring distribution). Then, for every $\delta > 0$ there is $u = u(\delta, h, c) \in \mathbb{N}$ such that

$$\mathsf{P}_{\lfloor uc/(c+1)\rfloor}\big(GW_h^{\text{sub}} \ge 1\big) \ge 1 - \delta \tag{5.5.72}$$

since the probability for each existing particle to have at least one offspring which moves to the right is strictly positive.

We now verify (5.5.68) separately on the following three sets:

$$\begin{aligned}
A_1 &:= \{\xi_1(n) + \xi_2(n) \geq u\} \cap \{\xi_1(n) = 0\}, \\
A_2 &:= \{\xi_1(n) + \xi_2(n) \geq u\} \cap \{0 < \xi_1(n) \leq c\xi_2(n)\}, \\
A_3 &:= \{\xi_1(n) + \xi_2(n) \geq u\} \cap \{\xi_1(n) > c\xi_2(n)\}.
\end{aligned}$$

Note that $A_1 \cup A_2 \cup A_3 = \{\xi_1(n) + \xi_2(n) \geq u\}$.

• On the set A_1 there is no particle in the LP between time η_n and time η_{n+1} by definition. Thus, the position of the leftmost cookie does not change during this period. Hence we a.s. have

$$\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \,|\, \mathcal{F}_n]\mathbb{1}_{A_1} \;=\; \xi_2(n)\mathbb{1}_{A_1}$$

due to (5.5.67).

• On the set A_2 there is at least one particle in the LP and thus the leftmost cookie is consumed at time η_n. Using $\eta_{n+1} - \eta_n \leq h$ and the fact that the total number of offspring of each particle is bounded by M, we a.s. obtain on the set A_2

$$\begin{aligned}
\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \,|\, \mathcal{F}_n] &\leq \big(\xi_1(n)M^h + \varphi_r\xi_2(n)\big) \\
&\leq \xi_2(n)\big(cM^h + \varphi_r\big) \\
&\leq \xi_2(n).
\end{aligned}$$

Here we used (5.5.71) in the last step.

• Next, recall that

$$L_n = \big\{\nu \in Z_{n+1}(l(n) - 1) : \nu \succ Z^{(l)}n\big\}$$

denotes the number of particles which leave the leading process to the left at time n. Using (5.5.66) we a.s. get on the set A_3

$$\begin{aligned}
&\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1) \,|\, \mathcal{F}_n] \\
&= \mathsf{E}_{(a,b)}\big[\big(\xi_1(n+1) + \xi_2(n+1)\big)\mathbb{1}_{\{\eta_{n+1} - \eta_n < h\}} \mid \mathcal{F}_n\big] \\
&\quad + \mathsf{E}_{(a,b)}\big[\big(\xi_1(n+1) + \xi_2(n+1)\big)\mathbb{1}_{\{\eta_{n+1} - \eta_n = h\}} \mid \mathcal{F}_n\big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \big(M^{h-1}\xi_1(n) + \varphi_r\xi_2(n)\big)\, \mathsf{E}_{(a,b)}\left[\mathbb{1}_{\{\eta_{n+1}-\eta_n<h\}} \mid \mathcal{F}_n\right] \\
&\quad + (\varphi_r)^h\xi_2(n)\mathsf{E}_{(a,b)}\left[\mathbb{1}_{\{\eta_{n+1}-\eta_n=h\}} \mid \mathcal{F}_n\right] \\
&\quad + \mathsf{E}_{(a,b)}\left[|Z^{(l)}\eta_n + h|\mathbb{1}_{\{\eta_{n+1}-\eta_n=h\}} \,\middle|\, \mathcal{F}_n\right] \\
&\quad + \sum_{i=0}^{h-1}\mathsf{E}_{(a,b)}\left[\sum_{\nu\succeq L_{\eta_n+i}} \mathbb{1}_{\{X_\nu=l(\eta_n)+h,\, X_\eta<l(\eta_n)+h\,\forall\eta\prec\nu\}}\mathbb{1}_{\{\eta_{n+1}-\eta_n=h\}} \,\middle|\, \mathcal{F}_n\right].
\end{aligned}
$$

Here in the second step we used that on the set $\{\eta_{n+1}-\eta_n<h\}$ (in expectation) the proportion at most φ_r of the type-2 particles does not escape to the left since at least one cookie is consumed. On the set $\{\eta_{n+1}-\eta_n=h\}$ we consider three summands. The first corresponds to the type-2 particles at time η_n that are still type-2 particles at time η_{n+1}. The second corresponds to the particles that are still in the LP at time η_{n+1} and the third to the particles which have left the LP in the meantime. Using (5.5.66) and the fact that we have at least $\lfloor uc/(c+1)\rfloor$ type-1 particles on the set A_3, we continue the calculation and obtain that on the set A_3 we a.s. have

$$
\begin{aligned}
\mathsf{E}_{(a,b)}[\xi_1(n+1)+\xi_2(n+1)\,|\,\mathcal{F}_n] \;&\leq \Bigg[\big(M^{h-1}\xi_1(n)+\varphi_r\xi_2(n)\big)\,\mathsf{P}_{\lfloor uc/(c+1)\rfloor}\left(GW_h^{\text{sub}}=0\right) \\
&\qquad + (\varphi_r)^h\xi_2(n) + \big(p_cm_c\big)^h\xi_1(n) \\
&\qquad + \sum_{i=0}^{h-1}\xi_1(n)(p_cm_c)^i(q_cm_c)(\varphi_r)^{h-i+1}\Bigg] \\
&\leq \left(M^{h-1}\delta+\frac{1}{2}\right)\xi_1(n) + \left(\varphi_r\delta+\frac{1}{2}\right)\xi_2(n) \\
&\leq \xi_1(n)+\xi_2(n)
\end{aligned}
$$

for $\delta=\delta(M,h,\varphi_r)$ sufficiently small. Here we used (5.5.69), (5.5.70), and (5.5.72) for the latter estimates. □

(ii) We now turn to the case when we have a critical leading process, i.e., $p_cm_c=1$. Again for some $c>0$, which we specify later (cf. (5.5.74)), we inductively define the following random times

$$
\eta_{n+1} := \begin{cases} \eta_n+1, & \text{if } \zeta_2(\eta_n)\geq c\zeta_1(\eta_n), \\ \inf\{n>\eta_n:\ \zeta_1(n)=0\}, & \text{if } \zeta_2(\eta_n)<c\zeta_1(\eta_n), \end{cases}
$$

for $n\in\mathbb{N}_0$ and $\eta_0:=0$. Similarly to above, we define for $n\in\mathbb{N}_0$

$$
\xi_1(n):=\zeta_1(\eta_n),\quad \xi_2(n):=\zeta_2(\eta_n)
$$

and the associated filtration

$$\mathcal{F}_n := \sigma\big(\xi_1(i),\ \xi_2(i), \eta_i : i \leq n\big).$$

Analogously to Lemma 5.5.7, we continue with the following

Lemma 5.5.8. *For suitable (large) $u \in \mathbb{N}$ we have*

$$\mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1)|\mathcal{F}_n] \ \leq\ \xi_1(n) + \xi_2(n) \tag{5.5.73}$$

a.s. on $\{\xi_1(n) + \xi_2(n) \geq u\}$ for all $(a,b) \in \mathcal{S}$.

Proof of Lemma 5.5.8. Let us fix $(a,b) \in \mathcal{S}$. Again for some $u = u(c) \in \mathbb{N}$, which we specify later (cf. (5.5.85)), we introduce the following sets

$$\begin{aligned} A_1 &:= \ \{\xi_1(n) + \xi_2(n) \geq u\} \cap \{\xi_2(n) \geq c\xi_1(n)\}, \\ A_2 &:= \ \{\xi_1(n) + \xi_2(n) \geq u\} \cap \{\xi_2(n) < c\xi_1(n)\}. \end{aligned}$$

and show (5.5.73) on the sets A_1 and A_2 separately.

• On the set A_1 we a.s. have

$$\begin{aligned} \mathsf{E}_{(a,b)}[\xi_1(n+1) + \xi_2(n+1)|\mathcal{F}_n] &\leq \mathbb{1}_{\{\xi_1(n)=0\}}\xi_2(n) + \mathbb{1}_{\{\xi_1(n)>0\}}\left(\varphi_r\xi_2(n) + M\xi_1(n)\right) \\ &\leq \mathbb{1}_{\{\xi_1(n)=0\}}\xi_2(n) + \mathbb{1}_{\{\xi_1(n)>0\}}\left(\varphi_r\xi_2(n) + Mc^{-1}\xi_2(n)\right) \\ &\leq \left[\mathbb{1}_{\{\xi_1(n)=0\}} + \mathbb{1}_{\{\xi_1(n)>0\}}\left(\varphi_r + Mc^{-1}\right)\right]\xi_2(n) \\ &\leq \xi_2(n) \end{aligned}$$

for any

$$0 \ <\ c\ \leq\ M\left(1-\varphi_r\right)^{-1}. \tag{5.5.74}$$

Thereby we used that on the set A_1 we have $\eta_{n+1} = \eta_n + 1$. If $\xi_1(n) = 0$ holds, then no cookie is eaten at time η_n and therefore we have $\xi_2(n+1) = \xi_2(n)$. If $\xi_1(n) > 0$ holds, the leftmost cookie is consumed and therefore in expectation the amount of the type-2 particles is reduced by the factor φ_r.

• Next, to investigate the behaviour on the set A_2, consider first the case $(\xi_1(n), \xi_2(n)) = (v, 0)$ with $v \in \mathbb{N}$. From this we can easily derive the general case later on since each time a cookie is consumed the number of type-2 particles is reduced by the factor $\varphi_r < 1$. Therefore, the type-2 particles do not essentially contribute to the growth of the process. We have:

$$\begin{aligned}
&\mathsf{E}_{(a,b)}[\xi_1(n+1)+\xi_2(n+1) \mid \mathcal{F}_n]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \\
&= \mathsf{E}_{(a,b)}[\xi_2(n+1)|\mathcal{F}_n]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \\
&= \Bigg(\mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{\left\{\eta_{n+1}-\eta_n\le v^{1/3}\right\}} \mid \mathcal{F}_n\right] \\
&\quad + \sum_{j>v^{1/3}} \mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{\{\eta_{n+1}-\eta_n=j\}} \mid \mathcal{F}_n\right]\Bigg)\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \qquad (5.5.75)
\end{aligned}$$

We now consider the first summand in (5.5.75). For this we define

$$\begin{aligned}
E^0 &:= \left\{\max_{\ell=1,\dots,\lfloor v^{1/3}\rfloor} \zeta_1(\eta_n+\ell) \le v^{2/3}\right\}, \\
E^k &:= \left\{\max_{\ell=1,\dots,\lfloor v^{1/3}\rfloor} \zeta_1(\eta_n+\ell) \in \left(2^{k-1}v^{2/3}, 2^k v^{2/3}\right]\right\} \quad \text{for } k\ge 1,
\end{aligned}$$

in order to control the maximum number of particles in the LP. Using these definitions, we write

$$\begin{aligned}
\mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{\left\{\eta_{n+1}-\eta_n\le v^{1/3}\right\}} \;\middle|\; \mathcal{F}_n\right] &= \sum_{k=0}^{\infty} \mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{E^k\cap\left\{\eta_{n+1}-\eta_n\le v^{1/3}\right\}} \;\middle|\; \mathcal{F}_n\right] \\
&\le v^{1/3}Mv^{2/3}\mathsf{P}_{(a,b)}\left(\eta_{n+1}-\eta_n\le v^{1/3} \;\middle|\; \mathcal{F}_n\right) \\
&\quad + \sum_{k=1}^{\infty} v^{1/3}M2^k v^{2/3}\mathsf{P}_{(a,b)}\big(\mathcal{B}_k(n,v) \mid \mathcal{F}_n\big), \qquad (5.5.76)
\end{aligned}$$

where we used the notation

$$\mathcal{B}_k(n,v) := \left\{\exists \ell\in\{\eta_n+1,\dots,\eta_{n+1}\} : \zeta_1(\ell) > 2^{k-1}v^{2/3}, \eta_{n+1}-\eta_n\le v^{1/3}\right\}.$$

Note here that each particle that leaves the LP starts a new BRW without cookies (as long as the offspring particles do not reach a cookie again) which is transient to the left by assumption. Thus for each of those particles the expected number of descendants which reach the position $l(\eta_{n+1})$ (and therefore are type-2 particles at time η_{n+1}) is less than one since they have to move at least two steps to the right. Now we observe that on the set $\{(\xi_1(n),\xi_2(n))=(v,0)\}$ we a.s. have

$$\mathsf{P}_{(a,b)}\left(\eta_{n+1}-\eta_n\le v^{1/3} \;\middle|\; \mathcal{F}_n\right) = \mathsf{P}_v\left(T^{\mathrm{cr}}\le v^{1/3}\right) \qquad (5.5.77)$$

and

$$\mathsf{P}_{(a,b)}\big(\mathcal{B}_k(n,v) \;\big|\; \mathcal{F}_n\big) \le v^{1/3}\mathsf{P}_{\lceil 2^{k-1}v^{2/3}\rceil}\left(T^{\mathrm{cr}}\le v^{1/3}\right) \qquad (5.5.78)$$

where T^{cr} denotes the extinction time of a critical GWP whose offspring distribution is given by the number of particles produced by a single particle in the LP which stay in the LP. (Note that this coincides with the number of type-1 offspring of a type-1 particle.) Now we apply (5.5.77), (5.5.78) and Proposition 5.4.6 to (5.5.76) and a.s. obtain

$$\begin{aligned}
&\mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{\{\eta_{n+1}-\eta_n\le v^{1/3}\}}\middle|\,\mathcal{F}_n\right]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}\\
\le\;&\left[Mv\exp\left(-C\frac{v}{v^{1/3}}\right)+\sum_{k=1}^{\infty}M2^k v^{4/3}\exp\left(-C\frac{2^{k-1}v^{2/3}}{v^{1/3}}\right)\right]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}\\
=\;&o(v)\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}
\end{aligned}\tag{5.5.79}$$

where $C>0$ is the constant of Proposition 5.4.6.

Now we deal with the second summand in (5.5.75). For some $\delta\in(0,\frac{1}{3})$ and $j\in\mathbb{N}$ we introduce the events

$$\begin{aligned}
F_j^0 &:= \left\{\max_{\ell=1,\dots,\lfloor j^\delta\rfloor}\zeta_1\big(\eta_n+j-\lfloor j^\delta\rfloor+\ell\big)\le j^{2\delta}\right\},\\
F_j^k &:= \left\{\max_{\ell=1,\dots,\lfloor j^\delta\rfloor}\zeta_1\big(\eta_n+j-\lfloor j^\delta\rfloor+\ell\big)\in\big(2^{k-1}j^{2\delta},2^k j^{2\delta}\big]\right\}\quad\text{for } k\ge 1,
\end{aligned}$$

and

$$\begin{aligned}
G_j^0 &:= \left\{\max_{\ell=1,\dots,j}\zeta_1(\eta_n+\ell)\le j^{1+\delta}\right\},\\
G_j^k &:= \left\{\max_{\ell=1,\dots,j}\zeta_1(\eta_n+\ell)\in\big(2^{k-1}j^{1+\delta},2^k j^{1+\delta}\big]\right\}\quad\text{for } k\ge 1.
\end{aligned}$$

On the events G_j^k we control the maximum number of particles in the LP up to time j, whereas on F_j^k we control the maximum number during the $\lfloor j^\delta\rfloor$ time steps before j. We observe that on the event $F_j^k\cap G_j^\ell$ not more than $M\cdot 2^\ell j^{2+\delta}$ particles leave the LP up to time $\eta_n+j-\lfloor j^\delta\rfloor$ (because of G_j^ℓ). Each of those particles starts a BRW without cookies and in average it contributes not more than $(\varphi_r)^{\lfloor j^\delta\rfloor+1}\le(\varphi_r)^{j^\delta}$ to the number of type-2 particles at time η_n+j. Similarly, on $F_j^k\cap G_j^\ell$ not more than $M2^k j^{3\delta}$ particles leave the LP from time $\eta_n+j-\lfloor j^\delta\rfloor+1$ to time η_n+j (because of F_j^k). Further, it holds that each particle that leaves the LP starts a new BRW without cookies and for each of those particles the expected number of descendants which reach the position $l(\eta_{n+1})$ is less than one since they have to move at least two steps to the right. Thus, we have

$$\begin{aligned}
&\mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{F_j^k\cap G_j^\ell\cap\{\eta_{n+1}-\eta_n=j\}}\mid\mathcal{F}_n\right]\\
\le\;&\left(M2^\ell j^{2+\delta}(\varphi_r)^{j^\delta}+M2^k j^{3\delta}\right)\mathsf{P}_{(a,b)}\left(F_j^k\cap G_j^\ell\cap\{\eta_{n+1}-\eta_n=j\}\middle|\,\mathcal{F}_n\right).
\end{aligned}\tag{5.5.80}$$

Now suppose that $\ell \geq k$ and $(k,\ell) \neq (0,0)$. Then due to Proposition 5.4.6 we have

$$\begin{aligned}
&\mathsf{P}_{(a,b)}\left(F_j^k \cap G_j^\ell \cap \{\eta_{n+1} - \eta_n = j\} \,\middle|\, \mathcal{F}_n\right) \\
&\leq \mathsf{P}_{(a,b)}\left(\exists i \in \{1,\ldots,j\} : \zeta_1(\eta_n + i) > 2^{\ell-1} j^{1+\delta},\, \zeta_1(\eta_n + j) = 0 \,\middle|\, \mathcal{F}_n\right) \\
&\leq j \mathsf{P}_{\lceil 2^{\ell-1} j^{1+\delta} \rceil}\left(T^{\mathrm{cr}} \leq j\right) \\
&\leq j \exp\left(-\tfrac{1}{2} C 2^{(\ell+k)/2} j^\delta\right). \qquad (5.5.81)
\end{aligned}$$

If otherwise $k \geq \ell$ and $(k,\ell) \neq (0,0)$, then again due to Proposition 5.4.6 we have

$$\begin{aligned}
&\mathsf{P}_{(a,b)}\left(F_j^k \cap G_j^\ell \cap \{\eta_{n+1} - \eta_n = j\} \,\middle|\, \mathcal{F}_n\right) \\
&\leq \mathsf{P}_{(a,b)}\left(\exists i \in \{j - \lfloor j^\delta \rfloor + 1,\ldots,j\} : \zeta_1(\eta_n + i) > 2^{k-1} j^{2\delta},\, \zeta_1(\eta_n + j) = 0 \,\middle|\, \mathcal{F}_n\right) \\
&\leq j \mathsf{P}_{\lceil 2^{k-1} j^{2\delta} \rceil}\left(T^{\mathrm{cr}} \leq j^\delta\right) \\
&\leq j \exp\left(-\tfrac{1}{2} C 2^{(\ell+k)/2} j^\delta\right). \qquad (5.5.82)
\end{aligned}$$

With the help of (5.5.81) and (5.5.82) together with (5.5.80) we a.s. obtain

$$\begin{aligned}
&\mathsf{E}_{(a,b)}\left[\xi_2(n+1) \mathbb{1}_{\{\eta_{n+1} - \eta_n = j\}} \mid \mathcal{F}_n\right] \\
&= \sum_{k,\ell=0}^{\infty} \mathsf{E}_{(a,b)}\left[\xi_2(n+1) \mathbb{1}_{F_j^k \cap G_j^\ell \cap \{\eta_{n+1} - \eta_n = j\}} \,\middle|\, \mathcal{F}_n\right] \\
&\leq \left(M j^{2+\delta} (\varphi_r)^{j^\delta} + M j^{3\delta}\right) \mathsf{P}_{(a,b)}\left(\eta_{n+1} - \eta_n = j \mid \mathcal{F}_n\right) \\
&\quad + \sum_{(k,\ell) \neq (0,0)} \left(M 2^\ell j^{2+\delta} (\varphi_r)^{j^\delta} + M 2^k j^{3\delta}\right) j \exp\left(-\tfrac{1}{2} C 2^{(\ell+k)/2} j^\delta\right) \\
&\leq O\left(j^{3\delta}\right) \mathsf{P}_{(a,b)}\left(\eta_{n+1} - \eta_n = j \mid \mathcal{F}_n\right) + \sum_{i=1}^{\infty} O\left(j^{1+3\delta}\right) (i+1) 2^i \exp\left(-\tfrac{1}{2} C 2^{i/2} j^\delta\right). \qquad (5.5.83)
\end{aligned}$$

By Proposition 5.4.7, on the set $\{(\xi_1(n), \xi_2(n)) = (v, 0)\}$ we a.s. have

$$\mathsf{P}_{(a,b)}\left(\eta_{n+1} - \eta_n = j \mid \mathcal{F}_n\right) \;\leq\; C \frac{v}{j^2},$$

and therefore (5.5.83) yields

$$\begin{aligned}
&\mathsf{E}_{(a,b)}\left[\xi_2(n+1)\mathbb{1}_{\{\eta_{n+1}-\eta_n=j\}} \mid \mathcal{F}_n\right]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}\\
&= \left[O(j^{3\delta-2})v + \sum_{i=1}^{\infty} O(j^{1+3\delta})(i+1)2^i \exp\left(-\tfrac{1}{2}C2^{i/2}j^{\delta}\right)\right]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}\\
&\le \Bigg[O(j^{3\delta-2})v + O(j^{1+3\delta})\exp\left(-\tfrac{1}{4}C2^{\frac{1}{2}}j^{\delta}\right)\\
&\qquad \sum_{i=1}^{\infty}(i+1)2^i\exp\left(-\tfrac{1}{4}C2^{i/2}1\right)\Bigg]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}\\
&= O(j^{3\delta-2})v\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}.
\end{aligned} \tag{5.5.84}$$

Using the estimates (5.5.79) and (5.5.84) for the two summands in (5.5.75), we conclude

$$\begin{aligned}
&\mathsf{E}_{(a,b)}[\xi_1(n+1)+\xi_2(n+1)|\mathcal{F}_n]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}\\
&\le \Big[o(v) + v\sum_{j>v^{1/3}} O(j^{3\delta-2})\Big]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}\\
&= vo(v)\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}},
\end{aligned}$$

and therefore there exists $v_0 \in \mathbb{N}$ such that

$$\mathsf{E}_{(a,b)}[\xi_1(n+1)+\xi_2(n+1) \mid \mathcal{F}_n]\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}} \ \le\ v\mathbb{1}_{\{(\xi_1(n),\xi_2(n))=(v,0)\}}$$

for $v \ge v_0$.

For the general case, in which we can also have type-2 particles at time η_n, we notice that for

$$u \ \ge\ (1+c)v_0 \tag{5.5.85}$$

we have

$$\mathsf{E}_{(a,b)}[\xi_1(n+1)+\xi_2(n+1) \mid \mathcal{F}_n]\mathbb{1}_{A_2} \ \le\ \big[\xi_1(n)+\xi_2(n)\big]\mathbb{1}_{A_2}$$

since on A_2 the type-2 particles which exist at time η_n evolve independently of the LP until time η_{n+1}. $\square$

Now we fix $u \in \mathbb{N}$ such that Lemma 5.5.7 and Lemma 5.5.8 hold. Further, we define

$$\tau := \ \inf\{n \in \mathbb{N}_0 : \ \xi_1(n)+\xi_2(n) \le u\}.$$

Due to Lemma 5.5.7 and, respectively, Lemma 5.5.8, we see that in the subcritical (i.e. $p_c m_c < 1$) as well as in the critical (i.e. $p_c m_c = 1$) case

$$\big(\xi_1(n \wedge \tau) + \xi_2(n \wedge \tau)\big)_{n\in\mathbb{N}_0}$$

is a non-negative supermartingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ and $\mathsf{P}_{(a,b)}$ for arbitrary $(a,b) \in \mathcal{S}$. Thus, it converges $\mathsf{P}_{(a,b)}$-a.s. to a finite random variable $\mathcal{X}(a,b)$. Since we have $\xi_1(n\wedge\tau)+\xi_2(n\wedge\tau) \in \mathbb{N}_0$ for all $n \in \mathbb{N}_0$ and since the probability for this process to eventually stay at a constant level $v > u$ for all times is equal to 0, we conclude that

$$\mathcal{X}(a,b) \;\le\; u$$

holds $\mathsf{P}_{(a,b)}$-a.s. Therefore, for all $(a,b) \in \mathcal{S}$ we have

$$\mathsf{P}_{(a,b)}\left(\exists n \in \mathbb{N}_0 :\, \xi_1(n) + \xi_2(n) \le u\right) \;=\; 1,$$

and hence

$$\mathsf{P}_{(a,b)}\left(\exists n \in \mathbb{N}_0 :\, \zeta_1(n) + \zeta_2(n) \le u\right) \;=\; 1. \tag{5.5.86}$$

We now introduce the following random times

$$\begin{aligned} \sigma_i &:= \inf\{n > \tau_i :\; l(n) = l(\tau_i) + 2\}, && \text{for } i \ge 0,\\ \tau_i &:= \inf\{n \ge \sigma_{i-1} :\; \zeta_1(n) + \zeta_2(n) \le u\}, && \text{for } i \ge 1, \end{aligned}$$

with $\tau_0 := 0$. Thereby σ_i denotes the first time at which two more cookies have been eaten since τ_i. Moreover, we observe that

$$\big(Y(n)\big)_{n\in\mathbb{N}_0} := \;\Big(\big(Z_n(x)\big)_{x\in\mathbb{Z}}, l(n)\Big)_{n\in\mathbb{N}_0}$$

is a Markov chain with values in $\mathcal{S}$, which can only reach finitely (thus countably) many states within finite time. Therefore, (5.5.86) yields for $i \in \mathbb{N}_0$

$$\mathsf{P}_{(e_0,0)}\big(\tau_{i+1} < \infty \mid \sigma_i < \infty\big) \;=\; 1 \tag{5.5.87}$$

where $(e_0, 0)$ denotes the usual starting configuration with one particle and the leftmost cookie at position 0. Finally, we have

$$\mathsf{P}_{(e_0,0)}\big(\sigma_i = \infty \mid \tau_i < \infty\big) \;\ge\; \big(q_c \mathsf{P}(\Lambda_1^+ = 0)\big)^{Mu} \;=: \gamma \in (0,1). \tag{5.5.88}$$

This inequality holds since at the first time after τ_i, at which any particle reaches the leftmost cookie again, there are not more than u type-1 particles. Each of those type-1 particles cannot produce more than M particles in the next step. Afterwards, the probability for any direct offspring of the type-1 particles to move to the left and then produce offspring which escape to $-\infty$ is given by $q_c\mathsf{P}(\Lambda_1^+ = 0)$. All the remaining type-2 particles escape to the left with probability $\mathsf{P}(\Lambda_1^+ = 0)$ since one more cookie has been eaten. In this case, only one more cookie is consumed after the random time τ_i implying $\sigma_i = \infty$.

Using (5.5.87) and (5.5.88) we can conclude

$$\begin{aligned}
&\mathsf{P}_{(e_0,0)}\big(\sigma_i < \infty\ \forall\ i \in \mathbb{N}\big)\\
&\leq \mathsf{P}_{(e_0,0)}\big(\sigma_k < \infty\big)\\
&= \mathsf{P}_{(e_0,0)}\big(\sigma_0 < \infty\big)\prod_{i=1}^{k}\mathsf{P}_{(e_0,0)}\big(\sigma_i < \infty \mid \tau_i < \infty\big)\mathsf{P}_{(e_0,0)}\big(\tau_i < \infty \mid \sigma_{i-1} < \infty\big)\\
&\leq \big(1-\gamma\big)^k \xrightarrow[k\to\infty]{} 0.
\end{aligned}$$

In particular this implies that a.s. only finitely many cookies are consumed and this yields that the CBRW is transient. ■

5.6 Final remarks

In this section, let us consider a CBRW with one cookie at every position $x \in \mathbb{Z}$, i.e., $c_0(x) := 1 \quad \forall x \in \mathbb{Z}$. In this case the leftmost cookie on the positive semi-axis

$$l(n) = \min\{x \geq 0 :\ c_n(x) = 1\}$$

and the rightmost cookie on the negative semi-axis

$$r(n) := \max\{x \leq 0 :\ c_n(x) = 1\}$$

are of interest. With these two definitions we can introduce the right LP

$$\mathcal{L}^+(n) := Z_n(l(n))$$

and the left LP

$$\mathcal{L}^-(n) := Z_n(r(n)).$$

Using Theorems 5.3.1, 5.3.2, and 5.3.3 and the symmetry of the CBRW with regard to the origin, one can derive the following results:

Theorem 5.6.1. *Suppose that the BRW without cookies is transient to the right.*

(a) If the right LP is supercritical, i.e. $p_c m_c > 1$ holds, then

(i) the CBRW is strongly recurrent iff $p_c m_c \varphi_\ell \geq 1$,

(ii) the CBRW is weakly recurrent iff $p_c m_c \varphi_\ell < 1$ and $q_c m_c > 1$,

(iii) the CBRW is transient iff $p_c m_c \varphi_\ell < 1$ and $q_c m_c \leq 1$.

(b) If the right LP is subcritical or critical, i.e. $p_c m_c \leq 1$ *holds, then*

- *(i) the CBRW is weakly recurrent iff the left LP is supercritical, i.e.* $q_c m_c > 1$,
- *(ii) the CBRW is transient iff the left LP is subcritical or critical, i.e.* $q_c m_c \leq 1$.

Theorem 5.6.2. *Suppose that the BRW without cookies is strongly recurrent. Then the CBRW is strongly recurrent, no matter which kinds of right and left LP we have.*

Theorem 5.6.3. *Suppose that the BRW without cookies is transient to the left. Due to the symmetry of the process we get the same result as in Theorem 5.6.1 if we just replace right LP by left LP,* p_c *by* q_c *and* φ_ℓ *by* φ_r.

Appendix A

Erlang distribution

In this chapter, we show that the probability that an Erlang distribution with parameters 1 and n is bigger than its expectation n is bounded from below by $\frac{1}{3}$ for all $n \in \mathbb{N}$.

Lemma A.1. *For $n \in \mathbb{N}$ let X_n be an Erlang distributed random variable with parameters 1 and n. Then we have for all $n \in \mathbb{N}$*

$$P(X_n \;\geq\; n) \;\geq\; \frac{1}{3}.$$

Proof. Let $n \geq 2$ and recall that the Lebesgue density of an Erlang distribution with parameters 1 and n is given by

$$f_{(1,n)}(x) := \frac{x^{n-1}}{(n-1)!} \exp(-x) \mathbb{1}_{[0,\infty)}(x).$$

We get as the derivative of $f_{(1,n)}$ for $x > 0$

$$f'_{(1,n)}(x) \;=\; \frac{x^{n-2}}{(n-2)!} \exp(-x) \left(1 - \frac{x}{n-1}\right),$$

and we note that $f_{(1,n)}$ has its maximum at position $n-1$ because we have $f_{(1,n)}(x) > 0$ for all x and $f_{(1,n)}(0) = 0$, $\lim_{x\to\infty} f_{(1,n)}(x) = 0$.

Next, we show that for $y \in (0, n-1]$ we have

$$f_{(1,n)}(n-1-y) \;<\; f_{(1,n)}(n-1+y), \tag{A.1}$$

which is equivalent to

$$\left(1 + \frac{y}{n-1}\right) \exp\left(-\frac{y}{n-1}\right) - \left(1 - \frac{y}{n-1}\right) \exp\left(\frac{y}{n-1}\right) \;>\; 0$$

for $y \in (0, n-1]$. Now, substituting $\frac{y}{n-1}$ by z shows that it is enough to prove that the function

$$g(z) := \;(1+z)\exp(-z) - (1-z)\exp(z)$$

is strictly positive for all $z \in (0,1]$. We get

$$\begin{aligned} g'(z) &= \exp(-z) - (1+z)\exp(-z) + \exp(z) - (1-z)\exp(z) \\ &= z(\exp(z) - \exp(-z)). \end{aligned}$$

Therefore, we have $g'(z) > 0$ for $z > 0$ and since $g(0) = 0$ this yields $g(z) > 0$ for all $x > 0$. Thus, we can conclude (A.1).

Using (A.1), we get

$$P\Big(X_n \in \big[0, n-1\big)\Big) \leq P\Big(X_n \in \big[n-1, 2(n-1)\big)\Big)$$

and therefore

$$P\Big(X \in \big[0, n-1\big)\Big) \leq \frac{1}{2}. \tag{A.2}$$

We note that due to Stirling's approximation we have for all $n \in \mathbb{N}$

$$f_{(1,n)}(n-1) = \frac{(n-1)^{n-1}}{(n-1)!}\exp(-(n-1)) \leq \frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{1}{12n-11}\right) \leq \frac{1}{\sqrt{2\pi n}}.$$

This together with (A.2) yields for $n \geq 6$

$$\begin{aligned} P(X_n \geq n) &= 1 - P\Big(X \in \big[0, n-1\big)\Big) - P\Big(X \in \big[n-1, n\big)\Big) \\ &\geq \frac{1}{2} - \frac{1}{\sqrt{2\pi n}} \geq \frac{1}{3}. \end{aligned} \tag{A.3}$$

The distribution function of a Erlang distributed random variable with parameters 1 and n is given by

$$F_{(1,n)}(x) = \left(1 - \exp(-x)\sum_{i=0}^{n-1}\frac{x^i}{i!}\right)\mathbb{1}_{[0,\infty)}(x),$$

and therefore we get

$$F_{(1,1)}(1) = 0.37,\ F_{(1,2)}(2) = 0.41,\ F_{(1,3)}(3) = 0.42,\ F_{(1,4)}(5) = 0.43,\ F_{(1,5)}(5) = 0.44,$$

which together with (A.3) finishes the proof. ■

Appendix B

Second moment of Λ_1^-

Lemma B.1. *We have* $\mathsf{E}\Big[\big(\Lambda_1^-\big)^2\Big] < \infty.$

Proof. For the second moment of Λ_1^- (cf. (5.4.1) for the definition) we get by distinguishing the particles by their last common ancestor:

$$\begin{aligned}
\mathsf{E}\Big[\big(\Lambda_1^-\big)^2\Big] &= \mathsf{E}\Bigg[\bigg(\sum_{\tau\in\mathbb{V}} \mathbb{1}_{\{\tau\in\mathcal{Y}\}}\mathbb{1}_{\{Y_\tau=-1,\ Y_\eta>-1\ \forall\eta\prec\tau\}}\bigg)\cdot\bigg(\sum_{\nu\in\mathbb{V}} \mathbb{1}_{\{\nu\in\mathcal{Y}\}}\mathbb{1}_{\{Y_\nu=-1,\ Y_\sigma>-1\ \forall\sigma\prec\nu\}}\bigg)\Bigg] \\
&= \sum_{\omega\in\mathbb{V}} \mathsf{E}\Big[\mathbb{1}_{\{\omega\in\mathcal{Y}\}}\cdot\mathbb{1}_{\{Y_\omega=-1,\ Y_\eta>-1\ \forall\eta\prec\omega\}}\Big] \\
&\quad + \sum_{\omega\in\mathbb{V}}\sum_{\ell=1}^{M}\sum_{\tau\in\mathbb{V}}\sum_{\substack{n=1\\ n\neq\ell}}^{M}\sum_{\nu\in\mathbb{V}} \mathsf{E}\Big[\mathbb{1}_{\{(\omega,\ell,\tau)\in\mathcal{Y}\}}\cdot\mathbb{1}_{\{Y_{(\omega,\ell,\tau)}=-1,\ Y_\eta>-1\ \forall\eta\prec(\omega,\ell,\tau)\}} \\
&\qquad\qquad \cdot\mathbb{1}_{\{(\omega,n,\nu)\in\mathcal{Y}\}}\cdot\mathbb{1}_{\{Y_{(\omega,n,\nu)}=-1,\ Y_\eta>-1\ \forall\eta\prec(\omega,n,\nu)\}}\Big]
\end{aligned} \tag{B.1}$$

Here the first sum consists of the products of identical factors and the second sum of mutually different factors. Since the branching and the movement of the particle are independent, we can analyse the dependence of the factors in the last expectation separately. We a.s. have for $n\neq\ell$ on $\{(\omega,\ell,\tau)\in\mathcal{Y}\}$:

$$\begin{aligned}
\mathsf{E}\Big[\mathbb{1}_{\{(\omega,n,\nu)\in\mathcal{Y}\}}\Big|\ \sigma\left(\mathbb{1}_{\{(\omega,\ell,\tau)\in\mathcal{Y}\}}\right)\Big] &= \mathsf{P}\Big((\omega,n)\in\mathcal{Y}\Big|(\omega,\ell)\in\mathcal{Y}\Big)\cdot\mathsf{P}\Big(\nu\in\mathcal{Y}\Big) \\
&\leq \mathsf{P}\Big(\nu\in\mathcal{Y}\Big)
\end{aligned} \tag{B.2}$$

Further for $n \neq \ell$, we a.s. have on $\{Y_\eta > -1 \text{ for } \eta \preceq \omega\}$:

$$\begin{aligned}
&\mathsf{E}\Big[\mathbb{1}_{\{Y_{(\omega,n,\nu)}=-1,\ Y_\eta>-1\ \forall \eta\prec(\omega,n,\nu)\}}\Big|\ \sigma\Big(\big(Y_\eta\big)_{\eta\preceq(\omega,\ell,\tau)}\Big)\Big]\\
&=\sum_{k=0}^{\infty}\mathsf{E}\Big[\mathbb{1}_{\{Y_\omega=k,\,Y_{(\omega,n)}=k+1\}}\cdot\mathbb{1}_{\{Y_{(\omega,n,\nu)}-Y_{(\omega,n)}=-1-(k+1),\ Y_\eta-Y_{(\omega,n)}>-1-(k+1)\ \forall\eta\prec(\omega,n,\nu)\}}\Big|\,Y_\omega\Big]\\
&\quad+\sum_{k=0}^{\infty}\mathsf{E}\Big[\mathbb{1}_{\{Y_\omega=k,\,Y_{(\omega,n)}=k-1\}}\cdot\mathbb{1}_{\{Y_{(\omega,n,\nu)}-Y_{(\omega,n)}=-1-(k-1),\ Y_\eta-Y_{(\omega,n)}>-1-(k-1)\ \forall\eta\prec(\omega,n,\nu)\}}\Big|\,Y_\omega\Big]\\
&=\sum_{k=0}^{\infty}\mathbb{1}_{\{Y_\omega=k\}}\cdot p_0\cdot\mathsf{P}\Big(Y_\nu=-k-2,\ Y_\eta>-k-2\ \forall\eta\prec\nu\Big)\\
&\quad+\sum_{k=0}^{\infty}\mathbb{1}_{\{Y_\omega=k\}}\cdot q_0\cdot\mathsf{P}\Big(Y_\nu=-k,\ Y_\eta>-k\ \forall\eta\prec\nu\Big) \qquad \text{(B.3)}
\end{aligned}$$

Note that

$$\mathsf{P}\Big(Y_\nu=0,\ Y_\eta>0\ \forall\eta\prec\nu\Big)=\begin{cases}1 & \text{for } \nu=\emptyset\\ 0 & \text{for } \nu\neq\emptyset\end{cases}.$$

Moreover, we observe for $k\in\mathbb{N}_0$

$$\sum_{\nu\in\mathbb{V}}\mathsf{P}\Big(\nu\in\mathcal{Y}\Big)\cdot\mathsf{P}\Big(Y_\nu=-k,\ Y_\eta>-k\ \forall\eta\prec\nu\Big)\ =\mathsf{E}\left[\Lambda_k^-\right]\ =\ \mathsf{E}\left[\Lambda_1^-\right]^k\ \leq\ 1. \qquad \text{(B.4)}$$

By applying (B.2), (B.3) and (B.4) to (B.1) we get:

$$\begin{aligned}
\mathsf{E}\left[\left(\Lambda_1^-\right)^2\right]\ &\leq\mathsf{E}\big[\Lambda_1^-\big]+\sum_{\omega\in\mathbb{V}}\sum_{\ell=1}^{M}\sum_{\tau\in\mathbb{V}}M\cdot\mathsf{E}\Bigg[\mathbb{1}_{\{(\omega,\ell,\tau)\in\mathcal{Y}\}}\cdot\mathbb{1}_{\{Y_{(\omega,\ell,\tau)}=-1,\ Y_\eta>-1\ \forall\eta\prec(\omega,\ell,\tau)\}}\\
&\qquad\cdot\sum_{k=0}^{\infty}\mathbb{1}_{\{Y_{(\omega,n)}=k\}}\cdot\big(p_0\cdot\mathsf{E}\left[\Lambda_{k+2}^-\right]+q_0\cdot\mathsf{E}\left[\Lambda_k^-\right]\big)\Bigg]\\
&\leq\mathsf{E}\left[\Lambda_1^-\right]\ +M\cdot\sum_{\omega\in\mathbb{V}}\sum_{\ell=1}^{M}\sum_{\tau\in\mathbb{V}}\mathsf{E}\Big[\mathbb{1}_{\{(\omega,\ell,\tau)\in\mathcal{Y}\}}\cdot\mathbb{1}_{\{Y_{(\omega,\ell,\tau)}=-1,\ Y_\eta>-1\ \forall\eta\prec(\omega,\ell,\tau)\}}\Big]\\
&=\mathsf{E}\left[\Lambda_1^-\right]+M\cdot\mathsf{E}\left[\Lambda_1^-\right]<\infty
\end{aligned}$$

■

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