The number of walks and degree powers in directed graphs

Hanjo Täubig

TUM-I123
The number of walks and degree powers in directed graphs

Hanjo Täubig

Abstract

Fiol and Garriga proved that in undirected graphs the number $w_k$ of walks of length $k$ does not exceed the sum of the $k$-th powers of the vertex degrees, i.e., $w_k \leq \sum_{x \in V} d(x)^k$. Here, we propose a generalization of this inequality for directed graphs using the geometric mean of the sums of the $k$-th powers of in- and out-degrees, namely, $w_k^2 \leq (\sum_{x \in V} d_{\text{in}}(x)^k)(\sum_{x \in V} d_{\text{out}}(y)^k)$. Further, we show that this inequality can be generalized for the case of nonnegative matrices, i.e., the sum of entries of the $k$-th matrix power is bounded from above by the geometric mean of the sums of the $k$-th powers of the row sums and column sums.

1 Introduction

Throughout the paper we assume that $\mathbb{N}$ denotes the set of nonnegative integers. Let $G = (V, E)$ be a directed graph having $n$ vertices, $m$ edges and adjacency matrix $A$. We investigate (the number of) walks, i.e., sequences of vertices, where each pair of consecutive vertices $v_i$ and $v_{i+1}$ is connected by a directed edge $(v_i, v_{i+1}) \in E$. Nodes and edges can be used repeatedly in the same walk. The length $k$ of a walk is counted in terms of edges.

For $k \in \mathbb{N}$ and $x, y \in V$, we denote by $w_k(x, y)$ the number of walks of length $k$ that start at vertex $x$ and end at vertex $y$. Since the graph is directed this number can be different from the number of walks of length $k$ that start at vertex $y$ and end at vertex $x$. By $s_k(x) = \sum_{y \in V} w_k(x, y)$ and $e_k(x) = \sum_{y \in V} w_k(y, x)$ we denote the number of all walks of length $k$ that start or end at node $x$, resp. Consequently, $w_k = \sum_{x \in V} s_k(x) = \sum_{x \in V} e_k(x)$ denotes the total number of walks of length $k$. The set of all walks of length $k$ is denoted by $W_k$, i.e., $w_k = |W_k|$. $d_{\text{in}}(x)$ and $d_{\text{out}}(x)$ denote the in-degree and the out-degree of vertex $x$.

It is a well known fact that the $(i, j)$-entry of $A^k$ is the number of walks of length $k$ that start at vertex $i$ and end at vertex $j$ (for all $k \geq 0$). Fundamental observations about the number of walks are due to their decomposition into two or more segments:

**Observation 1.** For arbitrary graphs $G = (V, E)$ and all vertices $x, z \in V$ holds

$$w_{k+\ell}(x, z) = \sum_{y \in V} w_k(x, y) \cdot w\ell(y, z)$$

and

$$w_{k+p+\ell} = \sum_{(x \rightarrow y) \in W_p} w_k(x) \cdot w\ell(y)$$

In particular, this implies:

$$w_{k+1} = \sum_{x \in V} d_{\text{in}}(x) \cdot s_k(x) = \sum_{x \in V} d_{\text{out}}(x) \cdot e_k(x)$$

$$w_{k+\ell} = \sum_{x \in V} e_k(x) \cdot s_{\ell}(x) = \sum_{x \in V} e_k(x) \cdot s_k(x)$$

*Institut für Informatik, Technische Universität München, D-85748 Garching, Germany, taeubig@in.tum.de*
2 Walks and degree powers

The following inequality for undirected graphs was conjectured by Marc Noy and proven by Fiol and Garriga [FG09]:

**Theorem 2.** In any undirected graph, the number \( w_k \) of walks of length \( k \) does not exceed the sum of the \( k \)-th powers of the vertex degrees, i.e.,

\[
    w_k \leq \sum_{x \in V} d_x^k.
\]

In the following, we discuss possible generalizations of this theorem to directed graphs. The conceivable inequality \( w_k \leq \sum_{x \in V} d_{\text{in}}(x)^k \) is invalid. For instance, it is violated by the graph shown in Figure 1. Because of the reversely directed counterpart of this graph, the same applies to the inequality \( w_k \leq \sum_{x \in V} d_{\text{out}}(x)^k \). Also, trying to generalize the inequality by using direct products of \( d_{\text{in}}(x) \) and \( d_{\text{out}}(x) \) is not successful, since, e.g., \( w_k \leq \sum_{x \in V} d_{\text{in}}(x) \cdot d_{\text{out}}(x)^k \) is violated for \( k = 1 \) by the graph consisting of only one directed edge.

**Observation 3.** The following inequalities are invalid generalizations of Theorem 2:

\[
    w_k \not\leq \sum_{x \in V} d_{\text{in}}(x)^k \\
    w_k \not\leq \sum_{x \in V} d_{\text{out}}(x)^k \\
    w_k \not\leq \sum_{x \in V} \sqrt{d_{\text{in}}(x) \cdot d_{\text{out}}(x)}^k
\]

While the power sum for \( d_{\text{in}}(x) \) or \( d_{\text{out}}(x) \) alone is not suitable for bounding \( w_k \), we will show that a combination (namely, the geometric mean) of both sums is sufficient. To this end, we first show that for the consideration of power sums with exponent \( q \) over the set of walks of length \( p \) the total cannot decrease if we shorten the walk length while at the same time the exponent is increased by the same difference.

**Lemma 4.** For every directed graph \( G = (V, E) \) and for all nonnegative integers \( p, q \in \mathbb{N} \) holds

\[
    \left( \sum_{(x^p \rightarrow y) \in W_p} d_{\text{in}}(x)^q \right) \left( \sum_{(x^p \rightarrow y) \in W_p} d_{\text{out}}(y)^q \right) \leq \left( \sum_{(x^{p-1} \rightarrow y) \in W_{p-1}} d_{\text{in}}(x)^{q+1} \right) \left( \sum_{(x^{p-1} \rightarrow y) \in W_{p-1}} d_{\text{out}}(y)^{q+1} \right)
\]

**Proof.** The proof starts with decomposing and counting walks of length \( p \) from \( x \) to \( y \), denoted by \( (x \xrightarrow{p} y) \), into walks of length \( p - 1 \) which is prepended or followed by a single edge, i.e., \( (x \xrightarrow{p-1} w \rightarrow y) \) and \( (x \rightarrow z \xrightarrow{p-1} y) \), resp.
Theorem 5. For every directed graph $G = (V, E)$ and for all nonnegative integers $p \in \mathbb{N}$ holds

$$w_p^2 \leq \left( \sum_{x \in V} d_{in}(x)^p \right) \left( \sum_{y \in V} d_{out}(y)^p \right).$$
Lemma 7. For every nonnegative matrix with row sums \(r_i\) and column sums \(c_j\) \((i \in [n])\) holds:

\[
\sum_{x \in [n]} \sum_{y \in [n]} a_{xy}^p r_x^q \leq \left( \sum_{x \in [n]} a_{x}^{p-1} c_x^{q+1} \right) \left( \sum_{y \in [n]} a_{y}^{p-1} c_y^{q+1} \right)
\]
Proof.

\[
\left( \sum_{x \in [n]} \sum_{y \in [n]} a_{xy}^{[p]} c_x \right) \left( \sum_{x \in [n]} \sum_{y \in [n]} a_{xy}^{[p]} r_y \right) = \left( \sum_{x,y \in [n]} a_{xy}^{[p-1]} a_{wy}^{[p]} c_x \right) \left( \sum_{x,y \in [n]} a_{xz} a_{yz}^{[p-1]} r_y \right) \\
= \left( \sum_{x \in [n]} \sum_{y \in [n]} a_{xy}^{[p-1]} c_x^{r_y} w \right) \left( \sum_{z \in [n]} \sum_{y \in [n]} a_{zy}^{[p-1]} r_y c_z \right) \\
= \sum_{(x,w,z,y) \in [n]^4} a_{xy}^{[p-1]} a_{wy}^{[p-1]} c_x^{r_y} w a_{zy}^{[p-1]} r_y c_z \\
+ \sum_{(x,w) \in [n]^2} a_{xy}^{[p-1]} a_{wy}^{[p-1]} c_x^{r_y} w a_{zy}^{[p-1]} r_y c_z \\
= \sum_{(x,y) \in [n]^2} a_{xy}^{[p-1]} a_{wy}^{[p-1]} c_x^{r_y} w a_{zy}^{[p-1]} r_y c_z \\
+ \sum_{(x,w) \in [n]^2} a_{xy}^{[p-1]} a_{wy}^{[p-1]} c_x^{r_y} w a_{zy}^{[p-1]} r_y c_z \\
\leq \left( \sum_{x \in [n]} \sum_{y \in [n]} a_{xy}^{[p-1]} c_x^{r_y} \right) \left( \sum_{z \in [n]} \sum_{y \in [n]} a_{zy}^{[p-1]} r_y \right).
\]

Theorem 8. For every nonnegative \( n \times n \)-matrix \( A = (a_{ij}) \) and \( p \in \mathbb{N} \) holds

\[
(\text{sum}(A^p))^2 \leq \left( \sum_{x \in [n]} c_x^p \right) \left( \sum_{y \in [n]} r_y^p \right)
\]

Proof. The proof works by repeatedly applying Lemma 7 to the squared entry sum of matrix \( A^p \):

\[
(\text{sum}(A^p))^2 = \left( \sum_{x \in [n]} \sum_{y \in [n]} a_{xy}^{[p]} c_x \right) \left( \sum_{x \in [n]} \sum_{y \in [n]} a_{xy}^{[p]} r_y \right) \\
\leq \left( \sum_{x \in [n]} \sum_{w \in [n]} a_{xw}^{[p-1]} c_x \right) \left( \sum_{z \in [n]} \sum_{y \in [n]} a_{zy}^{[p-1]} r_y \right) \\
\leq \left( \sum_{x \in [n]} \sum_{w \in [n]} a_{xw}^{[0]} c_x \right) \left( \sum_{y \in [n]} \sum_{z \in [n]} a_{zy}^{[0]} r_y \right) = \left( \sum_{x \in [n]} c_x^p \right) \left( \sum_{y \in [n]} r_y^p \right).
\]

The last equality follows from the fact that \( a_{ij}^{[0]} \) is 1 for \( i = j \) and 0 otherwise, since \( A^0 \) is the identity matrix.
The last theorem implies an even more general form of Theorem 5 for walks:

**Corollary 9.** For every directed graph $G = (V, E)$ and for all nonnegative integers $p \in \mathbb{N}$ holds:

$$w_{pk}^2 \leq \left( \sum_{x \in V} e_k(x)^p \right) \left( \sum_{y \in V} s_k(y)^p \right)$$

**References**
