Lower bounds for finite wavelet and Gabor systems

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Abstract

Given $\psi \in L^2(\mathbb{R})$ and a finite sequence $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma} \subseteq \mathbb{R}^+ \times \mathbb{R}$ consisting of distinct points, the corresponding wavelet system is the set of functions $\{\frac{1}{a_\gamma^{1/2}}\psi(\frac{x}{a_\gamma} - \lambda_\gamma)\}_{\gamma\in\Gamma}$. We prove that for a dense set of functions $\psi \in L^2(\mathbb{R})$, the wavelet system corresponding to any choice of $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma}$ is linearly independent, and we derive explicite estimates for the corresponding lower (frame) bounds. In particular, this puts restrictions on the choice of a scaling function in the theory for multiresolution analysis. We also obtain estimates for the lower bound for Gabor systems $\{e^{2\pi i a_\gamma x} g(x - \lambda_\gamma)\}_{\gamma\in\Gamma}$ for functions g in a dense subset of $L^2(\mathbb{R})$.

1 Introduction

Given $\psi \in L^2(\mathbb{R})$ and a sequence $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma} \subseteq \mathbb{R}^+ \times \mathbb{R}$, define the *wavelet family* $\{\psi_\gamma\}_{\gamma\in\Gamma}$ by

$$\psi_\gamma(x) = \frac{1}{a_\gamma^{1/2}}\psi(\frac{x}{a_\gamma} - \lambda_\gamma). \tag{1}$$

We find conditions implying that a finite set $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is linearly independent, meaning that $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a basis for its span in $L^2(\mathbb{R})$. It turns out that the set of $\psi \in L^2(\mathbb{R})$ for which $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is linearly independent for all choices of finite

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sequences $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma}$ is dense in $L^2(\mathbb{R})$. We estimate the corresponding lower frame bound, i.e., we find a number $A > 0$ such that

$$\sum_{\gamma\in\Gamma} | < f, \psi_\gamma > |^2 \geq A||f||^2, \quad \forall f \in \text{span}\{\psi_\gamma\}_{\gamma\in\Gamma}. \tag{2}$$

A similar analysis is performed for a finite Gabor system $\{g_\gamma\}_{\gamma\in\Gamma}$, which, for a given function $g \in L^2(\mathbb{R})$ and a sequence $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma} \subseteq \mathbb{R}^2$ is defined by

$$g_\gamma(x) = e^{2\pi i a_\gamma x} g(x - \lambda_\gamma). \tag{3}$$

The motivation behind the results comes from wavelet theory, where frames for $L^2(\mathbb{R})$ play a prominent role. Recall that a set of vectors $\{f_\gamma\}_{\gamma\in\Gamma}$ belonging to a separable Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if

$$\exists A, B > 0 : \ A||f||^2 \leq \sum_{\gamma\in\Gamma} | < f, f_\gamma > |^2 \leq B||f||^2, \ \forall f \in \mathcal{H}. \tag{4}$$

In particular, every finite set of vectors $\{f_\gamma\}_{\gamma\in\Gamma}$ is a frame for span$\{f_\gamma\}_{\gamma\in\Gamma}$. The numbers A, B appearing in (4) are called *(frame) bounds.* They are clearly not unique.

In signal processing, a special role is played by wavelet frames and Gabor frames. Given a function $\psi \in L^2(\mathbb{R})$ and parameters $a > 1, b > 0$, the corresponding wavelet system is the set of functions $\{\frac{1}{a^{m/2}}\psi(\frac{x}{a^m} - bn)\}_{m,n\in\mathbb{Z}}$; we refer to it as the *regular wavelet system* in contrast to the more general system (1). It is known [2] that $\{\frac{1}{a^{m/2}}\psi(\frac{x}{a^m} - bn)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if

$$\begin{aligned} A &:= \frac{1}{b}\inf_{|\gamma|\in[1,a]} \left[\sum_{n\in\mathbb{Z}} |\hat{\psi}(a^n\gamma)|^2 - \sum_{k\neq 0}\sum_{n\in\mathbb{Z}} |\hat{\psi}(a^n\gamma)\hat{\psi}(a^n\gamma + k/b)| \right] > 0 \\ B &:= \frac{1}{b}\sup_{|\gamma|\in[1,a]} \sum_{k,n\in\mathbb{Z}} |\hat{\psi}(a^n\gamma)\hat{\psi}(a^n\gamma + k/b)| < \infty. \end{aligned} \tag{5}$$

Here, the Fourier Transform of $f \in L^1(\mathbb{R})$ is defined by

$$\hat{f}(y) := \int_{\mathbb{R}} f(x) e^{-i2\pi xy} dx$$

As usual, the Fourier transform is extended to a unitary mapping of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

Similar, given $g \in L^2(\mathbb{R})$ and parameters $a, b > 0$, the *regular Gabor system* is the set of functions $\{e^{2\pi iamx}g(x-nb)\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$; it is a frame for $L^2(\mathbb{R})$ with bounds A, B if

$$\begin{aligned} A &:= \frac{1}{b}\inf_{x\in[0,a]}\left[\sum_{n\in\mathbb{Z}}|g(x-na)|^2 - \sum_{k\neq n}|\sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-\frac{k}{b})}|\right] > 0 \\ B &:= \frac{1}{b}\sup_{x\in[0,a]}\sum_{k\in\mathbb{Z}}|\sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-\frac{k}{b})}| < \infty. \end{aligned} \tag{6}$$

For more information about wavelets and Gabor frames we refer to the monographs [1], [6]. Usually, the speed of convergence in algorithms involving frames depends on the ratio $\frac{B}{A}$, making good estimates for the frame bounds an important issue, cf. [7]. However, practical calculations always have to be performed with finite subfamilies; therefore, the question is rather how to find frame bounds for those sets. The upper bound is trivial: every upper bound for $\{\frac{1}{a^{m/2}}\psi(\frac{x}{a^m}-bn)\}_{m,n\in\mathbb{Z}}$ (resp. $\{e^{2\pi iamx}g(x-nb)\}_{m,n\in\mathbb{Z}}$) is also an upper bound for any finite subfamily. However, it is nontrivial to find lower bounds for subfamilies. Lower bounds for finite subfamilies are also needed in other contexts, eg. in approximation problems, cf. [3].

Our estimates are motivated by those problems. Since our calculations work without extra complications for the irregular wavelet systems and Gabor systems defined above, we decided to present the general version. It actually works without assuming that $\{\psi_\gamma\}_{\gamma\in\Gamma}$ (resp. $\{g_\gamma\}_{\gamma\in\Gamma}$) is a subfamily of a frame for $L^2(\mathbb{R})$.

We end this introduction with some definitions and basic results that will be needed throughout the paper. First, and most important, it is clear that a finite family $\{f_\gamma\}$ in a Hilbert space is linearly independent if and only if

$$\exists A > 0: \quad ||\sum_\gamma a_\gamma f_\gamma||^2 \geq A\sum_\gamma |a_\gamma|^2, \tag{7}$$

for all choices of scalar coefficients a_γ. Furthermore, the set of values for A that can be used in (7) coincides with the set of lower frame bounds for $\{f_\gamma\}$. All estimates in this paper will be established using (7).

Definition 1.1 A sequence $\{a_m\}$ of real numbers is *separated by* $\delta > 0$, if $|a_m - a_n| \geq \delta$ for $a_m \neq a_n$. If $\{a_m\}$ is a sequence of positive real numbers, it is called *logarithmically separated by* $a > 1$, if the sequence $\{\log a_m\}$ is separated by $\log a$.

It is clear that a finite sequence $\{a_m\}$ of distinct (positive) real numbers will always be (logarithmically) separated. Furthermore, for any non-degenerate subinterval I of $\mathbb{R}$, $\{e^{2\pi i a_m x}\}$ is linearly independent in $L^2(I)$.

2 Finite wavelet systems

The following Theorem gives a sufficient condition under which a finite wavelet family will be linearly independent. Our assumptions may appear quite complicated. However, in Example 2.3 we show how this Theorem covers the Mexican hat wavelet; another consequence will be Theorem 2.4, which covers all functions ψ whose Fourier Transform is continuous and has compact support. For convenience, we assume that $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma}$ forms an "irregular lattice", i.e., we formulate the results for parameters $\{(a_m, \lambda_n)\}_{m=1,n=1}^{M,N}$. Let

$$\psi_{mn}(x) := \frac{1}{a_m^{1/2}}\psi(\frac{x}{a_m} - \lambda_n),\ m = 1, .., M,\ n = 1, .., N.$$

The case of arbitrary parameters $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma}$ can be treated simply by extending $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma}$ to an irregular lattice; after that, the extimates can be used directly.

Theorem 2.1 *Suppose that* $\{a_m\}_{m=1}^M \subseteq \mathbb{R}^+$ *is logarithmically separated by* $a > 1$, *and that* $\sup_{j,m=1,\ldots,M} a_j/a_m \leq K$ *for some* $K > 1$; *let* $\{\lambda_n\}_{n=1}^N \subseteq \mathbb{R}$ *be separated. Let* $\psi \in L^2(\mathbb{R})$ *and suppose there is a positive number* c *and a non-degenerate interval* $I \subset [c, \infty)$ *such that for any positive number* r *there are numbers* $d_1(r) > 0$, $d_2(r) \geq 0$ *and* $s(r) > -c$ *such that*

$$|\hat{\psi}(x)| \geq d_1(r) \quad \forall\, x \in I + s(r), \tag{8}$$

$$|\hat{\psi}(x)| \leq d_2(r) \quad \forall\, x > (c + s(r))a, \tag{9}$$

$$\frac{d_2(r)}{d_1(r)} \leq r. \tag{10}$$

Let A be a lower frame bound for $\{e^{-2\pi i\lambda_n x}\}_{n=1}^N$ in $L^2(I)$ and B be an upper bound for it in $L^2(KI)$. Denote by B' an upper bound for $\{e^{-2\pi i\lambda_n x}\hat{\psi}(x)\}_{n=1}^N$ in $L^2(\mathbb{R})$. Let $A_1,\dots,A_M$ be a sequence of numbers satisfying

$$0 < A_1 \le d_1^2(r)A \quad \textit{for some } r > 0,$$

$$0 < A_k \le \frac{d_1^2(r_k)AA_{k-1}}{16(\sqrt{r_k}+1)^2B'} \quad \textit{for some} \quad r_k \in \left]0, \frac{AA_{k-1}}{16BB'(k-1)}\right] \qquad (k \ge 2).$$

Then $\{\psi_{mn}\}_{m=1,n=1}^{M,N}$ is linearly independent in $L^2(\mathbb{R})$ with lower frame bound A_M.

Proof: Since the Fourier transform is an isometry of $L^2(\mathbb{R})$, it suffices to show that $\{\widehat{\psi_{mn}}(x)\}_{m=1,n=1}^{M,N} = \{\sqrt{a_m}e^{-2\pi i a_m\lambda_n x}\hat{\psi}(a_m x)\}_{m=1,n=1}^{M,N}$ is linearly independent with lower bound A_M. W.l.o.g. we suppose $a_1 > \ldots > a_M$.
Let $\{c_{mn}\}_{m=1,n=1}^{M,N}$ be a sequence of complex scalars. It suffices to show that

$$\left\|\sum_{m=1}^{k}\sum_{n=1}^{N} c_{mn}\sqrt{a_m}e^{-2\pi i a_m\lambda_n(\cdot)}\hat{\psi}(a_m\cdot)\right\|_{L^2(\mathbb{R})} \ge \sqrt{A_k\sum_{m=1}^{k}\sum_{n=1}^{N}|c_{mn}|^2} \qquad (11)$$

holds for all $k \in \{1,\dots,M\}$. We do this by induction on k:
For $k=1$ we have, with $r>0$ arbitrary,

$$\left\|\sum_{n=1}^{N} c_{1n}\sqrt{a_1}e^{-2\pi i a_1\lambda_n(\cdot)}\hat{\psi}(a_1\cdot)\right\|_{L^2(\mathbb{R})} \ge \left\|\sum_{n=1}^{N} c_{1n}\sqrt{a_1}e^{-2\pi i a_1\lambda_n(\cdot)}\hat{\psi}(a_1\cdot)\right\|_{L^2(\frac{I+s(r)}{a_1})} =$$

$$\left\|\sum_{n=1}^{N} c_{1n}e^{-2\pi i\lambda_n(\cdot)}\hat{\psi}\right\|_{L^2(I+s(r))} \overset{(8)}{\ge} d_1(r)\sqrt{A\sum_{n=1}^{N}|c_{1n}|^2},$$

proving (11) for $k=1$. Now suppose that $k \ge 2$ and that (11) is valid for $k-1$. We distinguish between two cases:

Case 1:

$$\frac{1}{2}\sqrt{A_{k-1}\sum_{m=1}^{k-1}\sum_{n=1}^{N}|c_{mn}|^2} \ge \sqrt{B'\sum_{n=1}^{N}|c_{kn}|^2} \qquad (12)$$

We then have

$$\left\|\sum_{m=1}^{k}\sum_{n=1}^{N} c_{mn}\sqrt{a_m}e^{-2\pi i a_m\lambda_n(\cdot)}\hat{\psi}(a_m\cdot)\right\|_{L^2(\mathbb{R})} \geq$$

$$\left\|\sum_{m=1}^{k-1}\sum_{n=1}^{N} c_{mn}\sqrt{a_m}e^{-2\pi i a_m\lambda_n(\cdot)}\hat{\psi}(a_m\cdot)\right\|_{L^2(\mathbb{R})} - \left\|\sum_{n=1}^{N} c_{kn}\sqrt{a_k}e^{-2\pi i a_k\lambda_n(\cdot)}\hat{\psi}(a_k\cdot)\right\|_{L^2(\mathbb{R})} \geq$$

$$\sqrt{A_{k-1}\sum_{m=1}^{k-1}\sum_{n=1}^{N}|c_{mn}|^2} - \sqrt{B'\sum_{n=1}^{N}|c_{kn}|^2} \overset{(12)}{\geq}$$

$$\frac{1}{2}\sqrt{A_{k-1}\sum_{m=1}^{k-1}\sum_{n=1}^{N}|c_{mn}|^2} \overset{(12)}{\geq}$$

$$\frac{1}{4}\sqrt{A_{k-1}\sum_{m=1}^{k-1}\sum_{n=1}^{N}|c_{mn}|^2} + \frac{1}{2}\sqrt{B'\sum_{n=1}^{N}|c_{kn}|^2} \geq$$

$$\frac{1}{4}\sqrt{A_{k-1}}\sqrt{\sum_{m=1}^{k}\sum_{n=1}^{N}|c_{mn}|^2} \geq \sqrt{A_k\sum_{m=1}^{k}\sum_{n=1}^{N}|c_{mn}|^2}.$$

Case 2:

$$\frac{1}{2}\sqrt{A_{k-1}\sum_{m=1}^{k-1}\sum_{n=1}^{N}|c_{mn}|^2} \leq \sqrt{B'\sum_{n=1}^{N}|c_{kn}|^2} \tag{13}$$

We then have for any positive number r:

$$\left\|\sum_{m=1}^{k}\sum_{n=1}^{N} c_{mn}\sqrt{a_m}e^{-2\pi i a_m\lambda_n(\cdot)}\hat{\psi}(a_m\cdot)\right\|_{L^2(\mathbb{R})} \geq$$

$$\left\|\sum_{m=1}^{k}\sum_{n=1}^{N} c_{mn}\sqrt{a_m}e^{-2\pi i a_m\lambda_n(\cdot)}\hat{\psi}(a_m\cdot)\right\|_{L^2(\frac{I+s(r)}{a_k})} \geq$$

$$\left\|\sum_{n=1}^{N} c_{kn}\sqrt{a_k}e^{-2\pi i a_k\lambda_n(\cdot)}\hat{\psi}(a_k\cdot)\right\|_{L^2(\frac{I+s(r)}{a_k})} - \sum_{m=1}^{k-1}\left\|\sum_{n=1}^{N} c_{mn}\sqrt{a_m}e^{-2\pi i a_m\lambda_n(\cdot)}\hat{\psi}(a_m\cdot)\right\|_{L^2(\frac{I+s(r)}{a_k})} \overset{(8),(9)}{\geq}$$

$$d_1(r)\left\|\sum_{n=1}^{N} c_{kn}\sqrt{a_k}e^{-2\pi i a_k\lambda_n(\cdot)}\right\|_{L^2(\frac{I+s(r)}{a_k})} - d_2(r)\sum_{m=1}^{k-1}\left\|\sum_{n=1}^{N} c_{mn}\sqrt{a_m}e^{-2\pi i a_m\lambda_n(\cdot)}\right\|_{L^2(\frac{I+s(r)}{a_k})} =$$

$$d_1(r)\left\|\sum_{n=1}^{N} c_{kn}e^{-2\pi i\lambda_n(\cdot)}\right\|_{L^2(I+s(r))} - d_2(r)\sum_{m=1}^{k-1}\left\|\sum_{n=1}^{N} c_{mn}e^{-2\pi i\lambda_n(\cdot)}\right\|_{L^2(\frac{a_m}{a_k}(I+s(r)))} \geq$$

$$d_1(r)\sqrt{A\sum_{n=1}^{N}|c_{kn}|^2} - d_2(r)\sum_{m=1}^{k-1}\sqrt{B\sum_{n=1}^{N}|c_{mn}|^2} \overset{(10)}{\geq}$$

$$d_1(r)\left(\sqrt{A\sum_{n=1}^{N}|c_{kn}|^2} - r\sqrt{B(k-1)\sum_{m=1}^{k-1}\sum_{n=1}^{N}|c_{mn}|^2}\right) \overset{(13)}{\geq}$$

$$d_1(r)\left(\sqrt{A} - 2r\sqrt{\frac{BB'(k-1)}{A_{k-1}}}\right)\sqrt{\sum_{n=1}^{N}|c_{kn}|^2}. \tag{14}$$

Now choose $r \in]0, \frac{AA_{k-1}}{16BB'(k-1)}]$. Then $r \leq 1$ and

$$\frac{1}{\sqrt{r}+1}\cdot\frac{1}{\sqrt{r}} \geq \frac{1}{2}\sqrt{\frac{16BB'(k-1)}{AA_{k-1}}} = 2\sqrt{\frac{BB'(k-1)}{AA_{k-1}}},$$

hence

$$\sqrt{A}\left(1-\frac{1}{\sqrt{r}+1}\right)\cdot\frac{1}{r} = \sqrt{A}\cdot\frac{1}{\sqrt{r}+1}\cdot\frac{1}{\sqrt{r}} \geq 2\sqrt{\frac{BB'(k-1)}{A_{k-1}}},$$

and thus

$$\sqrt{A} - 2r\sqrt{\frac{BB'(k-1)}{A_{k-1}}} \geq \frac{\sqrt{A}}{\sqrt{r}+1}.$$

Inserting this in (14), we obtain for $r \in]0, \frac{AA_{k-1}}{16BB'(k-1)}]$:

$$\left\|\sum_{m=1}^{k}\sum_{n=1}^{N} c_{mn}\sqrt{a_m}e^{-2\pi i a_m\lambda_n(\cdot)}\hat{\psi}(a_m\cdot)\right\|_{L^2(\mathbb{R})} \geq d_1(r)\frac{\sqrt{A}}{\sqrt{r}+1}\sqrt{\sum_{n=1}^{N}|c_{kn}|^2} \overset{(13)}{\geq}$$

$$\frac{d_1(r)\sqrt{A}}{\sqrt{r}+1}\left(\frac{1}{2}\sqrt{\sum_{n=1}^{N}|c_{kn}|^2}+\frac{1}{4}\sqrt{\frac{A_{k-1}}{B'}\sum_{m=1}^{k-1}\sum_{n=1}^{N}|c_{mn}|^2}\right)\geq$$

$$\frac{d_1(r)\sqrt{AA_{k-1}}}{4(\sqrt{r}+1)\sqrt{B'}}\sqrt{\sum_{m=1}^{k}\sum_{n=1}^{N}|c_{mn}|^2}\geq\sqrt{A_k\sum_{m=1}^{k}\sum_{n=1}^{N}|c_{mn}|^2},$$

and thus (11) for k, completing the induction step. The proof is over. □

Remark 2.2 Lower bounds A for $\{e^{-2\pi i\lambda_n x}\}_{n=1}^N$ in $L^2(I)$ have been found in [4]. If we choose a separation constant $\delta \leq \frac{1}{|I|}$, then we can use

$$A = 1.6\cdot 10^{-14}\cdot\frac{|I|}{2\pi}\cdot\left(\frac{\delta|I|}{2}\right)^{2N+1}((N+1)!)^{-8}.$$

This is clearly a bad estimate, and the paper [4] describes several cases where better bounds can be obtained. However, it is still desirable to obtain better bounds in the general case.
Explicit values for the constants B and B' are easy given. If $\{\psi_{mn}\}_{m=1,n=1}^{M,N}$ is a subfamily of a frame $\{\psi_{mn}\}_{m,n\in\mathbb{Z}}$ with upper bound C, then we can use $B' = C$, i.e., a bound independent of N. For a regular wavelet system, see the assumption (5). In "worst case" $B' = N\|\psi\|^2$ can be used. Also, we can take $B = NK|I|$.

Example 2.3 The preceeding Theorem yields lower bounds for the *Mexican hat wavelet*, defined by

$$\psi(x) := -\frac{d^2}{dx^2}e^{-x^2/2} \;=\; (1-x^2)e^{-x^2/2}.$$

Thus, its Fourier Transform is given by

$$\hat{\psi}(x) = 4\pi^2x^2e^{-2\pi^2x^2}.$$

Put $I := [1,2]$, $c := 1$, and suppose that the sequences $a_1,\ldots,a_M$ and $\lambda_1,\ldots,\lambda_N$ as well as the constants $a > 1$, K, A, B and B' are as in Theorem 2.1. For $0 < r \leq 1$, define

$$\begin{aligned} s(r) &:= \max\left\{1,\ \frac{16}{a^2-1},\ \sqrt{\frac{\log(a^2/r)}{\pi^2(a^2-1)}}\right\}, \\ d_1(r) &:= 4\pi^2(2+s(r))^2e^{-2\pi^2(2+s(r))^2} = \hat{\psi}(2+s(r)), \\ d_2(r) &:= 4\pi^2((1+s(r))a)^2e^{-2\pi^2(1+s(r))^2a^2} = \hat{\psi}((1+s(r))a). \end{aligned}$$

Since $\hat{\psi}(x)$ is stricly decreasing on $[\frac{1}{\sqrt{2\pi}}, \infty)$, it follows that (8) and (9) are fulfilled for $r \leq 1$. Furthermore, we have

$$\frac{d_2(r)}{d_1(r)} \leq a^2\, e^{-2\pi^2(s(r)^2a^2-(2+s(r))^2)}. \tag{15}$$

Since $s(r) \geq \frac{16}{a^2-1}$ and $s(r) \geq 1$, we have

$$\frac{4}{s(r)^2} + \frac{4}{s(r)} \leq \frac{8}{s(r)} \leq \frac{a^2-1}{2},$$

and hence

$$s(r)^2a^2 - (2+s(r))^2 \geq s(r)^2a^2 - \frac{1+a^2}{2}s(r)^2 = \frac{a^2-1}{2}s(r)^2.$$

Thus (15) and $s(r) \geq \left(\frac{\log(a^2/r)}{\pi^2(a^2-1)}\right)^{1/2}$ give (10). Putting $A_1 := d_1^2(1)$ and observing that $\frac{AA_{k-1}}{16BB'(k-1)} \leq 1$ for $k \geq 2$, an application of Theorem 2.1 gives a lower bound A_M for the wavelet system associated with the Mexican hat wavelet. In particular, the wavelet system is linearly independent.

The following Theorem is a consequence of Theorem 2.1. However, its assumptions are not as complicated and usually easy to check.

Theorem 2.4 *Let $a_1, \ldots, a_M$ be a finite sequence of positive numbers, logarithmically separated by some $a > 1$, and let $\lambda_1, \ldots, \lambda_N$ be a finite sequence of separated real numbers. Let $\psi \in L^2(\mathbb{R})$ and suppose that* $\operatorname{supp}\hat{\psi} \subset (-\infty, p]$ *for some $p > 0$ and that there is a non-degenerate interval $I \subset [\frac{p}{a}, p]$ and a positive number d such that*

$$|\hat{\psi}(x)| \geq d \quad \forall\, x \in I.$$

Denote a lower bound for $\{e^{-2\pi i\lambda_n x}\}_{n=1}^N$ in $L^2(I)$ by A, and an upper bound for $\{e^{-2\pi i\lambda_n x}\hat{\psi}(x)\}_{n=1}^N$ in $L^2(\mathbb{R})$ by B'. Then $\{\psi_{mn}\}_{m=1,n=1}^{M,N}$ is linearly independent with lower bound

$$A_M = d^2 A \left(\frac{d^2 A}{16B'}\right)^{M-1}.$$

Proof: Define $c := \frac{p}{a}$ and for $r > 0$ set

$$s(r) := 0, \quad d_1(r) := d, \quad d_2(r) := 0.$$

Then it is easy to see that (8), (9) and (10) are fulfilled. Since $d_1(r)$ is independent of r, we can choose r_k arbitrarily close to 0 in Theorem 2.1 and obtain in the limiting case $A_k = \frac{d^2 A A_{k-1}}{16 B'}$. The claim follows. □

Remark 2.5 (a) A similar result to the above holds if, for some $q < 0$, we have $\operatorname{supp} \hat{\psi} \subset [q, +\infty)$ and an interval $I \subset [q, \frac{q}{a}]$ such that

$$|\hat{\psi}(x)| \geq d > 0 \quad \forall\, x \in I.$$

(b) Note that the condition on the existence of I and d is in particular fulfilled if $p > 0$ is the right endpoint of $\operatorname{supp} \hat{\psi}$ and if $\hat{\psi}$ is continuous.

Note that the set of all functions $\psi \in L^2(\mathbb{R})$ whose Fourier Transform is continuous and has compact support, forms a dense subspace of $L^2(\mathbb{R})$. We now prove that for those functions, a finite wavelet system will automatically be linearly independent:

Corollary 2.6 *Let $0 \neq \psi \in L^2(\mathbb{R})$ such that $\hat{\psi}$ is continuous and has compact support. Then for any finite set $\Gamma \subset \mathbb{R}^+ \times \mathbb{R}$ consisting of distinct points, the corresponding wavelet system $\{a_\gamma^{-1/2}\, \psi(\frac{x}{a_\gamma} - \lambda_\gamma)\}_{(a_\gamma, \lambda_\gamma) \in \Gamma}$ is linearly independent.*

Proof: By the remark preceeding Theorem 2.1 we can assume that $\Gamma = \{(a_m, \lambda_n)\}_{m=1,n=1}^{M,N}$, where $a_1, \ldots, a_M$ is logarithmically separated by some $a > 1$ and $\lambda_1, \ldots, \lambda_N$ is separated. Let p be the right and q be the left endpoint of $\hat{\psi}$ (i.e., $p := \sup \operatorname{supp} \hat{\psi},\ q := \inf \operatorname{supp} \hat{\psi}$). Then $q < 0$ or $p > 0$. Suppose $p > 0$. Since $\hat{\psi}$ is continuous, there is an interval $I \subset [p/a, p]$, such that $\inf_{x \in I} |\hat{\psi}(x)| > 0$. Then the assumptions of Theorem 2.4 are satisfied. It follows that the corresponding wavelet system is linearly independent. If $q < 0$, the result follows from Remark 2.5 (a). □

Let us relate Corollary 2.6 to the theory of multiresolution analysis, cf. [5]. Recall that multiresolution analysis is based on a function ϕ satisfying a scaling equation; in the case of a function ϕ with compact support, this equation means that for a certain $N \in \mathbb{N}$ and coefficients $\{c_n\}_{n=-N}^{N}$,

$$\phi(x) = \sum_{n=-N}^{N} c_n \phi(2x - n).$$

That is, the wavelet system

$$\{\frac{1}{2^{j/2}}\phi(\frac{x}{2^j} - n)\}_{j=-1,0;\ |n|\leq N}$$

is linearly dependent! In view of the above results this is a very special property.

3 Finite Gabor systems

Let $g \in L^2(\mathbb{R})\backslash\{0\}$ and consider a finite set of points $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma} \subseteq \mathbb{R}^2$. A conjecture by Heil, Ramanathan and Topiwala [8] states that the Gabor family $\{e^{2\pi i a_\gamma x} g(x - \lambda_\gamma)\}_{\gamma\in\Gamma}$ is linearly independent. They proved the result for a dense class of functions $g \in L^2(\mathbb{R})$. Recently, the conjecture has been proved for arbitrary $g \in L^2(\mathbb{R})$ by Linnell [9] for sampling points on a regular lattice, i.e., for $\{(a_\gamma, \lambda_\gamma)\}_{\gamma\in\Gamma} = \{(am, bn)\}_{m,n=1}^{N}$ for $a, b > 0$, but the general case is still open. Below we give estimates for the lower frame bounds for certain finite Gabor systems. Also, our result proves the linear independence of $\{e^{2\pi i a_\gamma x} g(x - \lambda_\gamma)\}_{\gamma\in\Gamma}$ under our conditions.

Theorem 3.1 *Let $a_1, \ldots, a_N$ and $\lambda_1, \ldots, \lambda_M$ be two finite separated sequences of real numbers, the latter separated by $\varepsilon > 0$. Let $g \in L^2(\mathbb{R})$. Suppose there is a non-degenerate interval $I \subset [-\varepsilon, 0]$, such that for any positive number r there are numbers $d_1(r) > 0$, $d_2(r) \geq 0$ and $s(r) \in \mathbb{R}$ such that*

$$|g(x)| \geq d_1(r) \quad \forall\, x \in I + s(r), \tag{16}$$

$$|g(x)| \leq d_2(r) \quad \forall\, x > s(r), \tag{17}$$

$$\frac{d_2(r)}{d_1(r)} \leq r. \tag{18}$$

Furthermore, suppose that A and B are lower and upper frame bounds for $\{e^{2\pi i a_n x}\}_{n=1}^N$ in $L^2(I)$. Let B' be an upper bound for $\{e^{2\pi i a_n x} g(x)\}_{n=1}^N$ in $L^2(\mathbb{R})$. Let $A_1, \ldots, A_M$ be a sequence of numbers satisfying

$$0 < A_1 \le d_1^2(r)A \quad \textit{for some } r > 0,$$

$$0 < A_k \le \frac{d_1^2(r_k)AA_{k-1}}{16(\sqrt{r_k}+1)^2 B'} \;\; \textit{for some} \;\; r_k \in \left]0, \frac{AA_{k-1}}{16BB'(k-1)}\right] \qquad (k \ge 2).$$

Then $\{e^{2\pi i a_n x} g(x-\lambda_m)\}_{m=1,n=1}^{M,N}$ is linearly independent with lower bound A_M.

Proof: The proof parallels that of Theorem 2.1. Therefore we shall only sketch the proof: W.l.o.g. we suppose $\lambda_1 < \ldots < \lambda_M$.
Let $\{c_{mn}\}_{m=1,n=1}^{M,N}$ be a sequence of complex scalars. It suffices to show that

$$\left\| \sum_{m=1}^{k} \sum_{n=1}^{N} c_{mn} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_m) \right\|_{L^2(\mathbb{R})} \ge \sqrt{A_k \sum_{m=1}^{k} \sum_{n=1}^{N} |c_{mn}|^2} \tag{19}$$

holds for all $k \in \{1, \ldots, M\}$. We do this by induction on k:
For $k = 1$ we have, with $r > 0$ arbitrary,

$$\left\| \sum_{n=1}^{N} c_{1n} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_1) \right\|_{L^2(\mathbb{R})} \ge \left\| \sum_{n=1}^{N} c_{1n} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_1) \right\|_{L^2(I+s(r)+\lambda_1)} =$$

$$\left\| \sum_{n=1}^{N} c_{1n} e^{2\pi i a_n (\cdot + \lambda_1)} g \right\|_{L^2(I+s(r))} \overset{(16)}{\ge} d_1(r) \sqrt{A \sum_{n=1}^{N} |c_{1n}|^2}.$$

Now suppose that $k \ge 2$ and that (19) holds for $k-1$. As in the proof of Theorem 2.1, we distinguish between whether (12) or (13) holds. If (12) holds, we have

$$\left\| \sum_{m=1}^{k} \sum_{n=1}^{N} c_{mn} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_m) \right\|_{L^2(\mathbb{R})} \ge$$

$$\left\| \sum_{m=1}^{k-1} \sum_{n=1}^{N} c_{mn} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_m) \right\|_{L^2(\mathbb{R})} - \left\| \sum_{n=1}^{N} c_{kn} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_k) \right\|_{L^2(\mathbb{R})} \ge$$

$$\sqrt{A_k \sum_{n=1}^{N} \sum_{m=1}^{k} |c_{mn}|^2},$$

as in Case 1 in the proof of Theorem 2.1.
If (13) holds, then we have for any positive number r:

$$\left\| \sum_{m=1}^{k} \sum_{n=1}^{N} c_{mn} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_m) \right\|_{L^2(\mathbb{R})} \geq \left\| \sum_{m=1}^{k} \sum_{n=1}^{N} c_{mn} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_m) \right\|_{L^2(I+s(r)+\lambda_k)} \geq$$

$$\left\| \sum_{n=1}^{N} c_{kn} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_k) \right\|_{L^2(I+s(r)+\lambda_k)} - \sum_{m=1}^{k-1} \left\| \sum_{n=1}^{N} c_{mn} e^{2\pi i a_n (\cdot)} g(\cdot - \lambda_m) \right\|_{L^2(I+s(r)+\lambda_k)} \overset{(16),(17)}{\geq}$$

$$d_1(r) \sqrt{A \sum_{n=1}^{N} |c_{kn}|^2} - d_2(r) \sum_{m=1}^{k-1} \sqrt{B \sum_{n=1}^{N} |c_{mn}|^2}.$$

The rest of the proof is as in Theorem 2.1. □

Example 3.2 Much the same as we deduced lower bounds for the Mexican hat wavelet in Example 2.2, Theorem 3.1 can be used to obtain explicit lower bounds for Gabor systems if g is a Gaussian, i.e. if $g(x) = e^{-\alpha x^2}$, where $\alpha > 0$. Here we set $I := [-\varepsilon, -\varepsilon/2]$, and for $r \leq e^{-3\alpha\varepsilon^2/4}$ put

$$s(r) := \frac{1}{\alpha\varepsilon} \log \frac{1}{r} + \frac{\varepsilon}{4}, \quad d_1(r) := e^{-\alpha(s(r)-\varepsilon/2)^2}, \quad d_2(r) := e^{-\alpha s(r)^2}.$$

Then (16), (17) and (18) are fulfilled for $r \leq e^{-3\alpha\varepsilon^2/4}$, and we can apply Theorem 3.1.

The following Theorem presents explicit lower bounds for certain finite Gabor systems under conditions which are easy to check.

Theorem 3.3 *Let $a_1, \ldots, a_N$ and $\lambda_1, \ldots, \lambda_M$ be two finite separated sequences of real numbers, the latter separated by $\varepsilon > 0$. Let $g \in L^2(\mathbb{R})$ be such that* $\operatorname{supp} g \subset (-\infty, c]$ *for some $c \in \mathbb{R}$, and suppose there is a non-degenerate interval $I \subset [c - \varepsilon, c]$ and a positive number d such that*

$$|g(x)| \geq d \quad \forall\, x \in I.$$

Denote a lower bound for $\{e^{2\pi i a_n x}\}_{n=1}^N$ *in* $L^2(I)$ *by* A*, and an upper bound for* $\{e^{2\pi i a_n x} g(x)\}_{n=1}^N$ *in* $L^2(\mathbb{R})$ *by* B'*. Then* $\{e^{2\pi i a_n x} g(x-\lambda_m)\}_{m=1,n=1}^{M,N}$ *is linearly independent with lower bound*

$$A_M = d^2 A \left(\frac{d^2 A}{16 B'} \right)^{M-1}.$$

Proof: For $r > 0$ define

$$s(r) := c, \quad d_1(r) := d, \quad d_2(r) := 0.$$

Then it is easy to see that conditions (16), (17) and (18) are fulfilled with $I - c$ instead of I. Since $d_1(r)$ is independent of r, we can choose r_k arbitrarily close to 0 and obtain from Theorem 3.1 in the limiting case $A_k = \frac{d^2 A A_{k-1}}{16B'}$. □

Remark 3.4 (a) Note that the condition on g is in particularly fulfilled if $\operatorname{supp} g \subset (-\infty, c]$ for some $c \in \mathbb{R}$ and if $0 \neq g$ is continuous.

(b) Similar remarks as in Remark 2.2, in Remark 2.5 or in the one preceeding Theorem 2.1 also hold for the Gabor systems of Theorems 3.1 and 3.3. In particular, if $\{e^{2\pi i a_n x} g(x-\lambda_m)\}_{m=1,n=1}^{M,N}$ is a subfamily of a family $\{e^{2\pi i a_n x} g(x-\lambda_m)\}_{m,n\in\mathbb{Z}}$ satisfying the upper frame condition with bound C, then B' can be replaced by C, i.e., the bound is independent of M, N (see the condition (6)).

(c) Since the Fourier Transform of the function $e^{2\pi i a_n x} g(x-\lambda_m)$ is given by $e^{2\pi i a_n \lambda_m} e^{-2\pi i \lambda_m y} \hat{g}(y-a_n)$ and since $|e^{2\pi i a_n \lambda_m}| = 1$, $\{e^{2\pi i a_n x} g(x-\lambda_m)\}$ is linearly independent if and only if $\{e^{-2\pi i \lambda_m y} \hat{g}(y-a_n)\}$ is, and the lower bounds are the same. Thus it is clear that an analogue statement to Theorem 3.3 holds if suitable conditions are posed on $\hat{g}$ instead on g.

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