

Pricing contingent claims in incomplete markets when the holder can choose among different payoffs*

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Abstract

We suggest a valuation principle to price general claims giving the holder the right to choose (in a predefined way) among several random payoffs in an incomplete financial market. Examples are so-called “chooser options” and American options with finitely many possible exertion times but also some life insurance contracts. Our premium is defined by the minimal amount the writer must receive at time zero such that for **all** possible decision functions of the holder, the writer’s utility is at least as big as the utility he would have if he did not offer this contingent claim. The valuation principle is consistent with no-arbitrage and can be interpreted as a generalization of Schweizer’s indifference principle [17]. We show that in a complete financial market or, in general, if the writer has an exponential utility function, our premium is the supremum over all “utility-indifference premiums” related to all fixed random payoffs we get by fixing the decision function of the holder. For every other utility function our premium can be even larger than this supremum.

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1 Introduction

We are interested in options where the holder purchases the right to choose (in a predefined way) among several random payoffs offered by the seller. Such options could be a chooser option having the feature that, after a specified period of time, the holder can choose whether the option is a call or a put (cf. for example Hull [8]), or an American option that can be exercised at any time up to the expiration date (and so the discounted payoff depends on the stopping time). Another example is an installment option, i.e. a European option in which the premium is paid in a series of installments and the holder has the right to terminate payments at any payment date, but then the option matures automatically (cf. Karsenty and Sikorav [14]). In an insurance context, it could be a pension scheme

where the policy-holder reaching a special age can swap his right to a pension for a single payment. Such choices offered by an insurance contract are called “embedded options”. Many examples for such “options” are given in Held [7].

In all these cases the insurer (writer of the option) has the problem that he does not know at time 0 which random payoff the insured (holder of the option) is going to choose.

In the spirit of Schweizer [17], we consider a general model combining financial market risk and traditional actuarial risk. **Hereafter, we just call the insurer/writer “she” and the insured/holder “he”.**

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, and let the \mathbb{R}^d -valued semimartingale $S = (S_t)_{t \in [0, T]}$ model the *discounted* price processes of the d risky assets available for trade. Θ is a suitable space of admissible trading strategies to be specified later.

Definition 1.1. *Let u be her utility function. It is a mapping from the set of random variables into \mathbb{R} that is monotone in the sense that $X \leq Y$ P -a.s. implies $u(X) \leq u(Y)$.*

The classical actuarial variance principle would correspond to $u(X) = E_P(X) - \lambda \text{Var}(X)$, $\lambda > 0$, but it is known *not* to be monotone. The monotonicity is necessary for our valuation principle to be consistent with no-arbitrage.

For pricing random payoffs in incomplete markets Schweizer [17] introduces - in the most general form - an indifference principle in the framework of financial markets. The idea is as follows: she can decide whether she insures a risk B , an \mathcal{F}_T -measurable random variable, for a premium h or not. The utility-indifference premium is defined as the premium which makes her indifferent with regard to this decision. She also takes into consideration that she can perhaps (partly) hedge the risk.

Definition 1.2. *h is called a “utility-indifference premium” if it satisfies*

$$\sup_{\vartheta \in \Theta} u \left(c + h - B + \int_0^T \vartheta_t dS_t \right) = \sup_{\vartheta \in \Theta} u \left(c + \int_0^T \vartheta_t dS_t \right), \quad (1.1)$$

where c is her initial capital.

For the variance and the standard deviation principle, closed-form valuations for many practically relevant products combining financial and actuarial risk, as for example unit-linked life insurance contracts or so-called financial stop-loss reinsurance contracts, are

given by Møller [15], using general results of Schweizer [17]. For the exponential utility function, more recently, Becherer [1], see Theorem 2.4.1, derives a recursive computation formula for the premium considering a model which consists of a complete financial market and additional independent actuarial risk observed at discrete points of time.

The aim of this paper is to generalize this concept to situations where the random payment is not fixed at the beginning, but during the policy term the holder can choose in a contractually predefined way between several scenarios.

In the first instance, we consider a model with only one predefined decision time at which the holder can choose among a finite number of payoffs (section 2). In section 3, we deal with American style contingent claims where the holder can stop the contract before maturity T . More technical lemmas are left to the appendix.

2 Choice among finite number of payoffs

Let $\mathcal{B} = \{B_1, \dots, B_k\}$ be a set of contingent claims, i.e. each B_i is an \mathcal{F}_T measurable positive random variable. *He* can choose among these k different payoffs at the predefined stopping time τ (using the information \mathcal{F}_τ). This means that there is a set of permissible decision rules

$$\mathcal{D} = \{\delta : \Omega \rightarrow \{1, \dots, k\}, \mathcal{F}_\tau - \text{measurable}\}.$$

The final payment depending on δ is then

$$B^\delta = \sum_{i=1}^k \mathbf{1}(\delta = i) B_i. \tag{2.2}$$

We call $(B^\delta)_{\delta \in \mathcal{D}}$ a *general claim*.

Example 2.1 (Chooser option). *At a fixed time $T^* \in (0, T)$ the holder of a chooser option can decide whether the option is a call or a put (here, with the same strike price K), i.e. $B_1 = (S_T^{(1)} - K)^+$, $B_2 = (K - S_T^{(1)})^+$, and $\tau \equiv T^* \in (0, T)$.*

Our generalization of the utility-indifference principle is as follows:

We suggest to determine the premium as the minimal amount she must receive at time 0 such that for **all** possible decision functions δ (he could hypothetically have) her utility is at least as big as the utility she would have if she did not offer this contingent claim.

This means, she will be on the safe side even if she knows nothing about his preferences/decision function. Such a premium is reasonable in the following sense: if the premium was smaller she would offer him a decision possibility that decreases her utility. For a single decision function δ the random payoff B^δ is uniquely determined. Therefore, in the case of *no* financial market the premium described above would simply be the supremum over all utility-indifference premiums - according to (1.1) - related to all possible B^δ (Theorem 2.9 case (i)).

But the existence of a financial market makes things more complicated: she can (partially) hedge the risk carried by the claim. The crucial point about this is that she does not know his decision function δ , and therefore she must choose her trading strategy independently of it. Only from time $\tau(\omega)$ on she can choose a strategy depending on her information $\delta(\omega)$. We formalize this in the following way:

Definition 2.2. We call h a “still fair premium” if

$$\sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + h - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right) = \sup_{\vartheta \in \Theta} u \left(c + \int_0^T \vartheta_t dS_t \right), \quad (2.3)$$

where

$$\vartheta_t^\delta(\omega) := \begin{cases} \vartheta_t(\omega) & : t \leq \tau(\omega), \\ {}^i\vartheta_t(\omega) & : t > \tau(\omega) \quad \text{and} \quad \delta(\omega) = i. \end{cases} \quad (2.4)$$

As τ is a stopping time, it can be shown by standard arguments that ϑ^δ is predictable if ϑ and ${}^i\vartheta$ are predictable.

Remark 2.3. Definition 2.2 takes into account that from time $\tau(\omega)$ on she knows his *effective* decision $\delta(\omega)$. This is a priori not the same as if she received at time $\tau(\omega)$ his whole *decision function* $\delta : \Omega \rightarrow \{1, \dots, k\}$. In the latter case one would replace the lhs of (2.3) by

$$\sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - B^\delta + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right),$$

at which she could choose an optimal $\tilde{\vartheta}$ depending on δ . But, it turns out that these two concepts are equivalent under all expected utility functions $u(\cdot) = E_P(U(\cdot))$, see Lemma A.2.

The strategy space Θ has to satisfy the

Assumption 2.4. *All elements of Θ are $(\mathcal{F}_t)_{t \in [0, T]}$ -predictable and S -integrable, i.e. $\Theta \subset L(S)$. Θ is linear, and for all $\delta \in \mathcal{D}$ the following implication is valid:*

If $\vartheta, {}^i\vartheta \in \Theta$, $i = 1, \dots, k$, then the compound strategy ϑ^δ is also an element of Θ .

The latter is essential: a strategy ϑ is admissible if and only if its restriction to $(0, \tau]$ and its restriction to $(\tau, T]$ are both admissible. Therefore, it allows us to verify the admissibility of a strategy separately on $(0, \tau]$ and $(\tau, T]$. So, it is a quite natural assumption. But unfortunately, it is not as harmless as it looks like. For example, the set of all predictable trading strategies such that the discounted gain process $\int_0^t \vartheta_u dS_u$ is bounded from below (but not necessarily from above) does obviously not satisfy Assumption 2.4, as the insurer's credit limit for the second period is determined through her trading gains in the first period.

As $L(S)$ is linear an example satisfying Assumption 2.4 is

$$\Theta_1 = \left\{ \vartheta \in L(S) \left| \int_0^t \vartheta_u dS_u \text{ is bounded uniformly in } t \text{ and } \omega \right. \right\}. \quad (2.5)$$

Θ_1 is rather small, but in Delbaen et al. [3] resp. Kabanov and Stricker [10] it is shown that under some restrictions (in particular the standing assumption that S is locally bounded) for exponential utility the maximization problem (1.1) with $\Theta = \Theta_1$ has the same value as for much bigger Θ . Another choice satisfying Assumption 2.4 is

$$\Theta_2 = \left\{ \vartheta \in L(S) \left| \int_0^t \vartheta_u dS_u \text{ is a martingale resp. a special set of martingale measures} \right. \right\}.$$

Remark 2.5. A complementary approach would consist in defining the premium - similarly to Davis and Zariphopoulou [2] - from *his* point of view: the maximum premium he is prepared to pay for the claim. As *he* determines the decision, for him δ is not uncertain but he can maximize over it. The “utility-indifference premium” h' would then be determined by

$$\sup_{\vartheta \in \Theta} \sup_{\delta \in \mathcal{D}} u \left(c' - h' + B^\delta + \int_0^T \vartheta_t dS_t \right) = \sup_{\vartheta \in \Theta} u \left(c' + \int_0^T \vartheta_t dS_t \right),$$

where c' is *his* initial capital. But, in actuarial mathematics the zero utility principle is traditionally considered from the insurer's viewpoint. That makes economically more sense as "utility-indifference" assumes *perfect competition* (and therefore homogeneous preferences) that is more likely between insurance companies than between the insureds. Moreover, we are interested in hedging strategies for the *insurer* against the risk carried by the claim. With (2.3) one can obtain a minimax strategy.

First we show that h solving (2.3) is consistent with no-arbitrage. We take over the concept of Definition 4.2 in Karatzas and Kou [12]:

Definition 2.6. *Suppose that h' is the price of the general claim $(B^\delta)_{\delta \in \mathcal{D}}$ defined in (2.2).*

We say that there is an arbitrage opportunity, if there exists either

(i) *some compound strategy $\widehat{\vartheta}^\bullet$ - according to (2.4) - that satisfies*

$$x + \int_0^T \widehat{\vartheta}_t^\delta dS_t \geq B^\delta \quad P\text{-a.s.} \quad \forall \delta \in \mathcal{D}$$

for some $x < h'$, or

(ii) *some $\widehat{\delta} \in \mathcal{D}$ and some $\widehat{\vartheta} \in \Theta$ such that*

$$-x + \int_0^T \widehat{\vartheta}_t dS_t + B^{\widehat{\delta}} \geq 0 \quad P\text{-a.s.}$$

for some $x > h'$.

Theorem 2.7. *Let (2.3) have unique solution h . Then, for $h' = h$ neither she (case (i)) nor he (case (ii)) has an arbitrage opportunity.*

Proof. (i) Suppose that (i) is satisfied for $h' = h$. Due to the linearity of Θ , the mapping $\vartheta^\delta \mapsto \vartheta^\delta + \widehat{\vartheta}^\delta$ is a bijection of the set of permissible compound trading strategies into itself. This (first equality) and the monotonicity of u (first inequality) yield:

$$\begin{aligned} & \sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + x - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right) \\ &= \sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + x - B^\delta + \int_0^T \widehat{\vartheta}_t^\delta dS_t + \int_0^T \vartheta_t^\delta dS_t \right) \\ &\geq \sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + \int_0^T \vartheta_t^\delta dS_t \right) \\ &= \sup_{\vartheta \in \Theta} u \left(c + \int_0^T \vartheta_t dS_t \right). \end{aligned} \tag{2.6}$$

The last equality holds due to Assumption 2.4. On the other hand, we have by monotonicity of u and the uniqueness of h :

$$\sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + x - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right) < \sup_{\vartheta \in \Theta} u \left(c + \int_0^T \vartheta_t dS_t \right), \quad (2.7)$$

i.e. a contradiction to (2.6).

(ii) For similar reasons, (ii) cannot be satisfied for $h' = h$. \square

Theorem 2.8. *Assume that for every $\delta \in \mathcal{D}$ there exists a unique utility-indifference premium h_δ for the claim B^δ , i.e. h_δ solves equation (1.1) with $B = B^\delta$. If h solves (2.3), then $h \geq \sup_{\delta \in \mathcal{D}} h_\delta$.*

Proof. Let us first show that for all $\delta_0 \in \mathcal{D}$:

$$\begin{aligned} & \sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + h - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right) \\ & \leq \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - B^\delta + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \\ & \leq \inf_{\delta \in \mathcal{D}} \sup_{\vartheta \in \Theta} u \left(c + h - B^\delta + \int_0^T \vartheta_t dS_t \right) \\ & \leq \sup_{\vartheta \in \Theta} u \left(c + h - B^{\delta_0} + \int_0^T \vartheta_t dS_t \right). \end{aligned} \quad (2.8)$$

The first inequality is valid as in the second term, for every $\delta \in \mathcal{D}$, we can choose $\tilde{\vartheta} = \sum_{i=1}^k \mathbf{1}(\delta = i, t > \tau) {}^i\vartheta_t \in \Theta$, by Assumption 2.4. The second inequality holds, since again by Assumption 2.4,

$$\vartheta, \tilde{\vartheta} \in \Theta \implies \mathbf{1}(t \leq \tau)\vartheta + \mathbf{1}(t > \tau)\tilde{\vartheta} \in \Theta.$$

On the other hand, we have by the definitions of h and h_{δ_0}

$$\begin{aligned} & \sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + h - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right) \\ & = \sup_{\vartheta \in \Theta} u \left(c + \int_0^T \vartheta_t dS_t \right) \\ & = \sup_{\vartheta \in \Theta} u \left(c + h_{\delta_0} - B^{\delta_0} + \int_0^T \vartheta_t dS_t \right) \end{aligned} \quad (2.9)$$

for all $\delta_0 \in \mathcal{D}$. Putting (2.8) and (2.9) together, we get

$$\begin{aligned} & \sup_{\vartheta \in \Theta} u \left(c + h - B^{\delta_0} + \int_0^T \vartheta_t dS_t \right) \\ & \geq \sup_{\vartheta \in \Theta} u \left(c + h_{\delta_0} - B^{\delta_0} + \int_0^T \vartheta_t dS_t \right) \quad \forall \delta_0 \in \mathcal{D}, \end{aligned}$$

and, due to monotonicity of u and uniqueness of h_{δ_0} , this implies $h \geq h_{\delta_0}$ for all $\delta_0 \in \mathcal{D}$ and therefore the assertion. \square

Theorem 2.9. *Assume that for every $\delta \in \mathcal{D}$ there exists a unique utility-indifference premium h_δ for the claim B^δ , i.e. h_δ solves equation (1.1) with $B = B^\delta$, and let h be a unique solution of (2.3). If one of the following conditions holds:*

- (i) *there is no financial market, i.e. $\Theta = \{0\}$,*
- (ii) *the financial market is complete, i.e. there is a unique equivalent martingale measure Q , $\Theta = \Theta_2$, and every $B_i \in L^1(\Omega, \mathcal{F}, Q)$, or*
- (iii) *u is the expected exponential utility function, i.e.*

$$u(X) = E_P [-\exp(-\alpha X)], \quad (2.10)$$

for some risk aversion parameter $\alpha > 0$, and $|B_i - B_j| \in L^\infty(\Omega, \mathcal{F}, P)$

then we have $h = \sup_{\delta \in \mathcal{D}} h_\delta$.

Proof. (i) obvious.

(ii) Define

$$\delta_{\max} = \arg \max_{i=1, \dots, k} \{E_Q [B_i | \mathcal{F}_\tau]\}, \quad (2.11)$$

using arbitrary versions of the conditional expectations. It is evident that δ_{\max} is \mathcal{F}_τ -measurable. Theorem 2.7 implies that $h_{\delta_{\max}}$ according to (1.1) is the unique no-arbitrage price for the attainable claim $B^{\delta_{\max}}$, i.e. $h_{\delta_{\max}} = E_Q (B^{\delta_{\max}})$. Due to completeness (cf. e.g. Jacka [9]) and the optional stopping theorem there exists a permissible strategy $\hat{\vartheta}$ such that

$$E_Q [B^{\delta_{\max}} | \mathcal{F}_\tau] = h_{\delta_{\max}} + \int_0^\tau \hat{\vartheta}_t dS_t \quad P\text{-a.s.}$$

Furthermore, there are permissible strategies $\widehat{\vartheta}$ such that

$$B_i = E_Q [B_i | \mathcal{F}_\tau] + \int_\tau^T \widehat{\vartheta}_t^i dS_t \quad P\text{-a.s.}, \quad i = 1, \dots, k.$$

Therefore, starting with initial capital $h_{\delta_{\max}}$ and choosing the compound strategy $\widehat{\vartheta}^\delta$, according to (2.4), we can superhedge all claims $(B^\delta)_{\delta \in \mathcal{D}}$:

$$h_{\delta_{\max}} + \int_0^T \widehat{\vartheta}_t^\delta dS_t = B^\delta + E_Q [B^{\delta_{\max}} | \mathcal{F}_\tau] - E_Q [B^\delta | \mathcal{F}_\tau] \geq B^\delta \quad P\text{-a.s.}$$

So, $h_{\delta_{\max}}$ is the unique no-arbitrage price of the general claim $(B^\delta)_{\delta \in \mathcal{D}}$. Due to Theorem 2.7, this implies $h = h_{\delta_{\max}}$.

(iii) We set

$$\delta_{\max} = \arg \max_{i=1, \dots, k} \left\{ \text{ess inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(B_i - \int_\tau^T \vartheta_t dS_t \right) \right) | \mathcal{F}_\tau \right] \right\}, \quad (2.12)$$

using arbitrary versions of the essential infimums. By Assumption 2.4, the first supremum in (2.13) below can be split into two parts. Then, as $h_{\delta_{\max}}$ exists and $|B_i - B_j| \in L^\infty(\Omega, \mathcal{F}, P)$, Lemma A.1 can be applied. The last equality is the assertion of Lemma A.2:

$$\begin{aligned} & \sup_{\vartheta \in \Theta} u \left(c + h - B^{\delta_{\max}} + \int_0^T \vartheta_t dS_t \right) \\ &= \sup_{\vartheta \in \Theta} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - B^{\delta_{\max}} + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \\ &= \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - B^\delta + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \\ &= \sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + h - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right) \end{aligned} \quad (2.13)$$

Altogether, we obtain $h = h_{\delta_{\max}}$. □

Remark 2.10. We have some kind of minimax-principle:

$$\begin{aligned} & \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} u \left(c + h - B^\delta + \int_0^T \vartheta_t dS_t \right) \\ &\leq \sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + h - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right) \\ &\leq \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - B^\delta + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \\ &\leq \inf_{\delta \in \mathcal{D}} \sup_{\vartheta \in \Theta} u \left(c + h - B^\delta + \int_0^T \vartheta_t dS_t \right) \end{aligned}$$

By Theorem 2.9, in the case of exponential utility, the last two inequalities are equalities. The first inequality, however, can still be strict.

Proposition 2.11. *Let $U \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ be a utility function with $U' > 0$, $U'' < 0$, and $u(\cdot) = E_P[U(\cdot)]$. If U is not of the form $U(c) = a_0 - a_1 \exp\{-\alpha c\}$, $a_0 \in \mathbb{R}$, $a_1, \alpha > 0$, then there exists a general claim $(B^\delta)_{\delta \in \mathcal{D}}$ such that $h > \sup_{\delta \in \mathcal{D}} h_\delta$.*

Remark 2.12. (a) Proposition 2.11 has a nice economical interpretation: for each utility function U with varying risk aversion $r(c) := -U''(c)/U'(c)$, one can construct a counterexample with $h > \sup_{\delta \in \mathcal{D}}$. In this case there need not exist a least favorable decision function δ_{\max} , as in (2.12), which does not depend on her wealth $c + h + \int_0^\tau \vartheta_t dS_t$ at time τ and therefore on her strategy ϑ **until** time τ . We will illustrate this in Example 2.13.

(b) This is an interesting analogy to the assertion in Gerber [5], p. 77, concerning premium calculation principles which is also caused by the (non)constancy of the risk aversion of the utility function: “A principle of zero utility is iterative, if and only if it is an exponential principle or the net premium principle.”

Proof. To construct a counterexample it is sufficient to look at a simple discrete two-period binomial model. She has initial capital c . There are a riskless bond identical 1, a tradeable risky asset with $S_0 = 1$ and for some $s^u \in \mathbb{R}$

$$S_2 = S_1 = \begin{cases} s^u & : \text{ with probability } 1/2 \\ 0 & : \text{ with probability } 1/2 \end{cases}$$

(so trading in the second period can be ignored), and another random variable Y , *stochastically independent* of S , with

$$Y = \begin{cases} y_0 & : \text{ with probability } 1/2 \\ 0 & : \text{ with probability } 1/2 \end{cases}$$

for some $y_0 \in \mathbb{R}$. At time $\tau \equiv 1$, having the information S_1 , the holder can choose between *two* payoffs at time 2, a constant payoff $B_1 \equiv x_0 \in \mathbb{R}$ and $B_2 = Y$. As S_1 can take two different values, there are four possible decision functions $\mathcal{D} = \{\delta^{11}, \delta^{12}, \delta^{21}, \delta^{22}\}$ (where

δ^{ij} means that he decides for B_i if $S_1 = 3$ and for B_j if $S_1 = 0$). We have four free parameters, namely c , s^u , y_0 , and x_0 to construct a counterexample.

For each $(x, c) \in \mathbb{R}_+ \times \mathbb{R}$ let $y = y(x, c)$ be the unique solution of

$$U(c - x) = \frac{1}{2} [U(c) + U(c - y)], \quad y \in [x, 2x]. \quad (2.14)$$

It is given by

$$y(x, c) = c - U^{-1}(2U(c - x) - U(c)),$$

and for the partial derivative with respect to c we have

$$y_c(x, c) = \frac{U'(c) + U'(c - y(x, c)) - 2U'(c - x)}{U'(c - y(x, c))}. \quad (2.15)$$

Taking in equation (2.14) the first and second partial derivative with respect to x , respectively, and then setting $x = 0$, one obtains (note $y(0, c) = 0$)

$$y_x(0, c) = 2, \quad \text{and} \quad y_{xx}(0, c) = \frac{2U''(c)}{U'(c)} = -2r(c), \quad \forall c \in \mathbb{R}.$$

By the Taylor expansion

$$y(x, c) = 2x - r(c)x^2 + x^2 \int_0^1 \lambda [y_{xx}((1 - \lambda)x, c) - y_{xx}(0, c)] d\lambda,$$

and due to $U \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$, we obtain

$$y_c(x, c) = (-r'(c) + o(1))x^2, \quad x \rightarrow 0, \quad (2.16)$$

where the convergence holds uniformly on compacta in c . As U is neither linear nor of exponential type, we have $r' \not\equiv 0$. Thus, there exist some $c_0 \in \mathbb{R}$ and $\varepsilon > 0$ s.t. w.l.o.g.

$$r'(c) < 0, \quad \forall c \in [c_0 - \varepsilon, c_0 + \varepsilon] \quad (2.17)$$

and therefore, due to (2.16) and the continuity of r' , there exists $x_0 > 0$ arbitrary small s.t.

$$y_c(x_0, c) > 0 \quad \forall c \in [c_0 - \varepsilon, c_0 + \varepsilon]. \quad (2.18)$$

We want to lead this to a contradiction to $h = \sup_{\delta \in \mathcal{D}} h_\delta$ or, equivalently, to

$$\sup_{\vartheta \in \mathbb{R}} \inf_{\delta^{ij}} E_P \left[U \left(c_0 + \vartheta(S_1 - S_0) - B^{\delta^{ij}} \right) \right] = \inf_{\delta^{ij}} \sup_{\vartheta \in \mathbb{R}} E_P \left[U \left(c_0 + \vartheta(S_1 - S_0) - B^{\delta^{ij}} \right) \right] \quad (2.19)$$

First of all, choose $c = c_0$ and $s^u > 2$, but close to 2 (i.e. $E_P[S_1 - S_0] \gtrsim 0$) such that for fixed $\delta \in \{\delta^{11}, \delta^{12}, \delta^{21}, \delta^{22}\}$ the maximization problem possesses a maximizer ϑ^{ij} (which is then, due to the strict concavity of U , unique). ϑ^{ij} should be positive but not too big, more precisely,

$$c_0 - \varepsilon < c_0 - \vartheta^{ij} < c_0 + \vartheta^{ij}(s^u - 1) < c_0 + \varepsilon \quad i, j = 1, 2.$$

This is possible as $U''(c_0) < 0$, $E_P[S_1 - S_0] > 0$, and x_0 resp. $y_0 \in (x_0, 2x_0)$ are arbitrarily small. Then, choose $y_0 \in (x_0, 2x_0)$ such that

$$E_P[U(c_0 + \vartheta^{12}(S_1 - S_0) - B^{12})] = E_P[U(c_0 + \vartheta^{11}(S_1 - S_0) - B^{11})] =: V$$

This implies that $y_0 \geq y(x_0, c_0 - \vartheta^{11})$ but $y_0 \leq y(x_0, c_0 - \vartheta^{12})$. As $y(x_0, \cdot)$ is increasing we obtain $(0 <) \vartheta^{12} \leq \vartheta^{11}$ and $y_0 < \min\{y(x_0, c_0 + \vartheta^{11}(s^u - 1)), y(x_0, c_0 + \vartheta^{12}(s^u - 1))\}$. We arrive at

$$\begin{aligned} V &< \min \{E_P[U(c_0 + \vartheta^{12}(S_1 - S_0) - B^{22})], E_P[U(c_0 + \vartheta^{11}(S_1 - S_0) - B^{21})]\} \\ &\leq \min \{E_P[U(c_0 + \vartheta^{22}(S_1 - S_0) - B^{22})], E_P[U(c_0 + \vartheta^{21}(S_1 - S_0) - B^{21})]\}, \end{aligned}$$

i.e. δ^{11} and δ^{12} are indeed the least favorable decision functions.

To disprove (2.19) it is enough to show that $\vartheta^{12} \neq \vartheta^{11}$ (as maximizers are unique). Assume that $\vartheta^{12} = \vartheta^{11} =: \vartheta^{\text{opt}}$. This implies $y_0 = y(x_0, c_0 - \vartheta^{\text{opt}})$. On the one hand we have

$$E_P[U(c_0 + \vartheta(S_1 - S_0) - B^{12})] - E_P[U(c_0 + \vartheta(S_1 - S_0) - B^{11})] = o(\vartheta - \vartheta^{\text{opt}}), \quad \vartheta \rightarrow \vartheta^{\text{opt}},$$

as both expectations take their maximum in ϑ^{opt} . On the other hand we have by (2.15)

$$\begin{aligned} &E_P[U(c_0 + \vartheta(S_1 - S_0) - B^{12})] - E_P[U(c_0 + \vartheta(S_1 - S_0) - B^{11})] \\ &= \frac{1}{4}U(c_0 - \vartheta) + \frac{1}{4}U(c_0 - \vartheta - y(x_0, c_0 - \vartheta^{\text{opt}})) - \frac{1}{2}U(c_0 - \vartheta - x_0) \\ &= \left[\frac{1}{4}U'(c_0 - \vartheta^{\text{opt}}) + \frac{1}{4}U'(c_0 - \vartheta^{\text{opt}} - y(x_0, c_0 - \vartheta^{\text{opt}})) \right. \\ &\quad \left. - \frac{1}{2}U'(c_0 - \vartheta^{\text{opt}} - x_0) + o(1) \right] (\vartheta^{\text{opt}} - \vartheta) \\ &= \frac{1}{4} \left[\underbrace{y_c(x_0, c_0 - \vartheta^{\text{opt}})U'(c_0 - \vartheta^{\text{opt}} - y(x_0, c_0 - \vartheta^{\text{opt}}))}_{>0} + o(1) \right] (\vartheta^{\text{opt}} - \vartheta), \quad \vartheta \rightarrow \vartheta^{\text{opt}}. \end{aligned}$$

But this is a contradiction. □

Example 2.13. We want to illustrate the different situations for exponential utility in contrast to other utility functions. Therefore we take the example above with $U = \log$. She has initial capital $c_0 = 4$ and $s^u = 3$, $x_0 = 1$. The optimal amount ϑ of assets she has to buy at time 0 depends on δ but she must find a joint strategy ϑ . By choosing y_0 in such a way that the optimal utilities for δ^{11} and δ^{12} are the same and smaller than the utilities for δ^{21} , δ^{22} . That is the case for $y \approx 1.7518$ (cf. Figure 1). Then $h_{\delta^{11}} = h_{\delta^{12}} = 1$. But as $\vartheta^{11} \neq \vartheta^{12}$ there exists no **joint** ϑ which brings in at least that utility. Therefore $h > \sup_{\delta \in \mathcal{D}} h_\delta = 1$. In Figure 2 the same situation is plotted, but with exponential utility. Here, there is always a least favorable decision function δ_{\max} (as defined in (2.12), in the example it is δ^{22}). It brings in the smallest expected utility for all strategies ϑ .

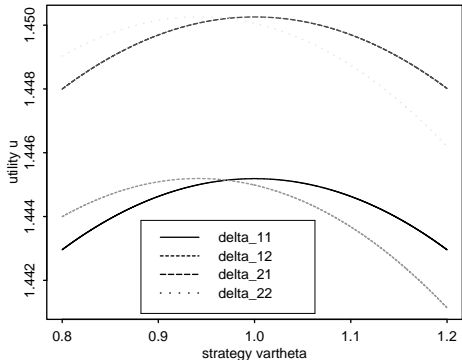


Figure 1: The writer’s expected logarithmic utility as a function of her strategy, plotted for the four different random payoffs B^δ .

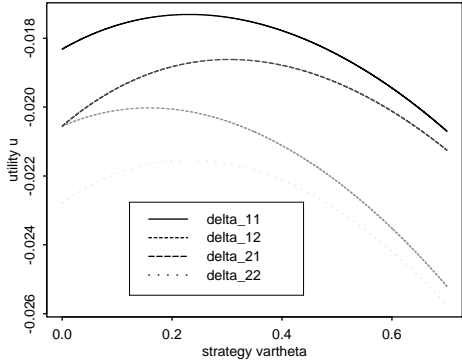


Figure 2: Same situation as in Figure 1, but with expected exponential utility ($\alpha = 1$). $B^{\delta^{22}}$ is least favorable for *every* strategy $\vartheta \in \mathbb{R}$ (cf. (1.26)).

3 American style contingent claims

An American contingent claim is a financial instrument modeled by an $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted process $(f_t)_{t \in [0, T]}$. If he exercises the claim at time t he gets a payoff with *discounted* value f_t . We assume a *deterministic* riskless interest rate $(r_t)_{t \in [0, T]}$. Then, it makes no difference whether he is paid off at time t or at time T with interest. So, in our model we can assume that the paying off takes place at time T (but of course the amount is known at the exercise time t). Here, the claim can only be exercised at $0 = t_0 < t_1 < \dots < t_k = T$.

Definition 3.1. *Let \mathcal{S} be the set of stopping times resp. $(\mathcal{F}_t)_{t \in [0, T]}$ with values in $\{t_0, \dots, t_k\}$.*

In the framework of *complete financial markets* American contingent claims have - due to a superhedging opportunity (for a proof in continuous time cf. Karatzas [11]) - a unique no-arbitrage price. In the presence of constraints on portfolios, Karatzas and Kou [12] give *intervals* of no-arbitrage prices.

Example 3.2 (Surrender option in unit-linked life insurance contract). Consider a pure endowment life insurance contract that is linked to an equity index $(S_t^{(1)})_{t \in [0, T]}$. At time T the amount $\max\{S_T^{(1)}, K\}$ is paid contingent on survival of the policy-holder. Let T_1 be the remaining lifetime of the insured at time 0. Then $f_T = \mathbf{1}(T_1 > T) \max\{S_T^{(1)}, K\} \times e^{-\int_0^T r_s ds}$. But, the policy holder has the right to terminate the contract at t_1, t_2, \dots, t_{k-1} . Then he gets a payoff depending on $S_{t_i}^{(1)}$ and t_i that is predefined in the contract. So, $f_t = \mathbf{1}(T_1 > t)g(S_t^{(1)}, t)e^{-\int_0^t r_s ds}$. Notice that, if the policy-holder dies before the payoff time τ , his following decision would be irrelevant as the payoff is then always 0.

Grosen and Jørgensen [6] describe the practical importance of this example and price such contracts in the context of complete financial markets, not considering mortality risk (as in Example 3.2).

Analogous to Definition 2.3 we define :

Definition 3.3. *We call h a “still fair premium” if*

$$\sup_{(\vartheta, \vartheta, \dots, \vartheta) \in \Theta^{k+1}} \inf_{\tau \in \mathcal{S}} u \left(c + h - f_\tau + \int_0^\tau \vartheta_t^\tau dS_t \right) = \sup_{\vartheta \in \Theta} u \left(c + \int_0^T \vartheta_t dS_t \right), \quad (3.20)$$

where

$$\vartheta_t^\tau(\omega) := \begin{cases} \vartheta_t(\omega) & : t \leq \tau(\omega), \\ {}^i\vartheta_t(\omega) & : t > \tau(\omega) \quad \text{and} \quad \tau(\omega) = t_i, \end{cases} \quad (3.21)$$

and $\tau \in \mathcal{S}$ is his stopping time.

The interpretation is as follows : until the exercise time $\tau(\omega)$ she does only know that $\tau(\omega) > t$ and therefore she has to choose a strategy ϑ independently of τ that comes into effect till $\tau(\omega)$. From $\tau(\omega)$ on she can choose a strategy depending on $\tau(\omega)$ (the information she has). Define for $0 = t_0 < t_1 < \dots < t_k = T$ recursively :

$$\tau_{\max}(t_k) = t_k, \quad (3.22)$$

$$\tau_{\max}(t_{i-1}) := \begin{cases} t_{i-1} & : \omega \in A_{i-1}, \\ \tau_{\max}(t_i) & : \text{otherwise,} \end{cases} \quad (3.23)$$

where

$$\begin{aligned} A_{i-1} &= \left\{ e^{\alpha f_{t_{i-1}}} \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(-\alpha \int_{t_{i-1}}^T \vartheta_t dS_t \right) \middle| \mathcal{F}_{t_{i-1}} \right] \right. \\ &\quad \left. \geq \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] \right\}. \end{aligned}$$

Theorem 3.4. *Assume that for every $\tau \in \mathcal{S}$ there exists a unique utility-indifference premium h^τ for the claim f_τ , i.e. h^τ solves equation (1.1) with $B = f_\tau$, and let h solve (3.20). Then we have*

- (a) $h \geq \sup_{\tau \in \mathcal{S}} h^\tau$ and
- (b) if in addition one of the conditions (i), (ii), (iii) of Theorem 2.9 with f_{t_i} instead of B_i is satisfied, we have $h = \sup_{\tau \in \mathcal{S}} h^\tau$.

Proof. Part (a) is analog to the proof of Theorem 2.8. In part (b), case (i) is evident and (ii) is standard (cf. e.g. Elliott and Kopp [4]). So, we restrict ourself to case (iii).

Recall the proof of Theorem 2.9(iii): there was only one decision time τ and - roughly speaking - it was based on the fact that *at time* τ the claim $B^{\delta_{\max}}$ is less favorable for her than each other B^δ , in the sense of (1.26). Now, the payoff $f_{\tau_{\max}(t_0)}$ is least favorable, but necessarily only at t_0 . But as until τ a joint strategy ϑ comes into effect, we cannot argue

as simple as in the proof of Theorem 2.9. But we know that at t_1 $f_{\tau_{\max}(t_1)}$ is less favorable than all f_τ with $\tau \geq t_1$, etc. (cf. Lemma A.3). So, we can argue successively till $t_k = T$:

For every $\varepsilon > 0$, we take strategies $\vartheta^{(1)}, \dots, \vartheta^{(k)}$ that are “ ε -optimal” for $f_{\tau_{\max}(t_1)}, \dots, f_{\tau_{\max}(t_k)}$ and let them come into effect (depending on $\tau \in \mathcal{S}$) on the stochastic intervals

$$(0, \tau \wedge t_1], (\tau \wedge t_1, \tau \wedge t_2], \dots, (\tau \wedge t_{k-1}, \tau],$$

where $(a, a] := \emptyset$. Each time the approximation error is smaller than ε , uniformly in $\tau \in \mathcal{S}$ (cf. Lemma A.6). With that and by applying Lemma A.3, we obtain

$$\begin{aligned} & \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - f_{\tau_{\max}(t_0)} + \int_0^T \tilde{\vartheta}_t dS_t \right) \\ \stackrel{L. A.3}{\leq} & \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - [\mathbf{1}(\tau = t_0)f_\tau + \mathbf{1}(\tau > t_0)f_{\tau_{\max}(t_1)}] + \int_0^T \tilde{\vartheta}_t dS_t \right) \\ \stackrel{L. A.6}{\leq} & \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - [\mathbf{1}(\tau = t_0)f_\tau + \mathbf{1}(\tau > t_0)f_{\tau_{\max}(t_1)}] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} dS_t \right. \\ & \left. + \int_{\tau \wedge t_1}^T \tilde{\vartheta}_t dS_t \right) + \varepsilon \\ \stackrel{L. A.3}{\leq} & \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - [\mathbf{1}(\tau \leq t_1)f_\tau + \mathbf{1}(\tau > t_1)f_{\tau_{\max}(t_2)}] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} dS_t \right. \\ & \left. + \int_{\tau \wedge t_1}^T \tilde{\vartheta}_t dS_t \right) + \varepsilon \\ \stackrel{L. A.6}{\leq} & \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - [\mathbf{1}(\tau \leq t_1)f_\tau + \mathbf{1}(\tau > t_1)f_{\tau_{\max}(t_2)}] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} dS_t \right. \\ & \left. + \int_{\tau \wedge t_1}^{\tau \wedge t_2} \vartheta_t^{(2)} dS_t + \int_{\tau \wedge t_2}^T \tilde{\vartheta}_t dS_t \right) + 2\varepsilon \\ & \vdots \\ \leq & \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - f_\tau + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} dS_t + \int_{\tau \wedge t_1}^{\tau \wedge t_2} \vartheta_t^{(2)} dS_t \right. \\ & \left. + \dots + \int_{\tau \wedge t_{k-1}}^{\tau} \vartheta_t^{(k)} dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) + k\varepsilon. \end{aligned}$$

By setting

$$\hat{\vartheta}_t = \vartheta_t^{(i)} \quad \text{if } t_{i-1} < t \leq t_i,$$

and taking the infimum over all $\tau \in \mathcal{S}$, we get

$$\begin{aligned}
& \sup_{\vartheta \in \Theta} u \left(c + h - f_{\tau_{\max}(t_0)} + \int_0^T \vartheta_t dS_t \right) \\
& \leq \inf_{\tau \in \mathcal{S}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - f_\tau + \int_0^\tau \hat{\vartheta}_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) + k\varepsilon \\
& \leq \sup_{\vartheta \in \Theta} \inf_{\tau \in \mathcal{S}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - f_\tau + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) + k\varepsilon.
\end{aligned}$$

As ε can be chosen arbitrary small, this implies

$$\begin{aligned}
& \sup_{\vartheta \in \Theta} u \left(c + h - f_{\tau_{\max}(t_0)} + \int_0^T \vartheta_t dS_t \right) \\
& \leq \sup_{\vartheta \in \Theta} \inf_{\tau \in \mathcal{S}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - f_\tau + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right). \tag{3.24}
\end{aligned}$$

Putting (3.24) and Lemma A.4 together yields

$$\begin{aligned}
& \sup_{\vartheta \in \Theta} u \left(c + h - f_{\tau_{\max}(t_0)} + \int_0^T \vartheta_t dS_t \right) \\
& \leq \sup_{\vartheta \in \Theta} \sup_{\tilde{\vartheta} \in \Theta} \inf_{\tau \in \mathcal{S}} u \left(c + h - f_\tau + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \\
& \leq \sup_{(\vartheta, \tilde{\vartheta}, \dots, \tilde{\vartheta}^{k-1}) \in \Theta^{k+1}} \inf_{\tau \in \mathcal{S}} u \left(c + h - f_\tau + \int_0^T \vartheta_t^\tau dS_t \right),
\end{aligned}$$

and therefore, due to monotonicity of u and uniqueness of $h^{\tau_{\max}(t_0)}$, $h \leq h^{\tau_{\max}(t_0)}$. This completes the proof of Theorem 3.4. \square

Remark 3.5. (a) If the denominator in (3.25) below does not vanish with positive probability, we can recursively define the “still fair conditional time t_i premium” X_{t_i} of the *American contingent claim* :

$$\begin{aligned}
& X_{t_k} = f_{t_k}, \\
& X_{t_{i-1}} = \max \left\{ f_{t_{i-1}}, \frac{1}{\alpha} \ln \frac{\operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(X_{t_i} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \mid \mathcal{F}_{t_{i-1}} \right]}{\operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(-\alpha \int_{t_{i-1}}^T \vartheta_t dS_t \right) \mid \mathcal{F}_{t_{i-1}} \right]} \right\}, \tag{3.25}
\end{aligned}$$

for $i = 1, \dots, k$. If the financial market is complete $(X_{t_i})_{i=0, \dots, k}$ coincides with the Snell envelope of the discrete process $(f_{t_i})_{i=0, \dots, k}$.

(b) The denominator in (3.25) can vanish even if we assume the existence of an equivalent (local) martingale measure, let S be locally bounded, and take the rather small set of strategies Θ_1 , defined in (2.5). This shows the example in Lemma 3.8 of Schachermayer [16]. But, if we have in addition an equivalent local martingale measure with finite relative entropy it follows from Theorem 1 in Delbaen et al. [3] that the denominator in (3.25) is P -a.s. positive.

4 Conclusion

In this paper, we have studied the following questions: can the possibility for the holder to choose have a value in itself and how can the writer of a claim hedge simultaneously against different risks related to different decision functions of the holder.

It turns out that in the case of exponential utility the utility-indifference premium which covers the claim related to the least favorable decision function is sufficient for all other decision functions the holder could hypothetically take.

An important application is a unit-linked life insurance contract that can be terminated by the policy-holder (cf. Example 3.2). Working with exponential utility, it would be reasonable to define the payoff, the holder gets if he terminates the contract, as the current conditional premium for the final payoff. Then, the optimal hedging strategies for all possible stopping times coincide until the termination time.

A Appendix

The appendix consists of some more technical lemmas needed for the proofs of Theorem 2.9 and Theorem 3.4.

Lemma A.1. *Let Z be an \mathcal{F}_τ -measurable random variable and u the expected exponential utility function (2.10). If $\sup_{\vartheta \in \Theta} u \left(Z - B_i + \int_\tau^T \vartheta_t dS_t \right) > -\infty$ for $i = 1, \dots, k$ then we have for all $\delta \in \mathcal{D}$*

$$\sup_{\vartheta \in \Theta} u \left(Z - B^{\delta_{\max}} + \int_\tau^T \vartheta_t dS_t \right) \leq \sup_{\vartheta \in \Theta} u \left(Z - B^\delta + \int_\tau^T \vartheta_t dS_t \right), \quad (1.26)$$

where δ_{\max} is defined in (2.12).

Proof. Let us first show that infimum and integral can be interchanged, i.e.

$$\begin{aligned} & \inf_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(-Z + B^\delta - \int_\tau^T \vartheta_t dS_t \right) \right) \right] \\ &= E_P \left[e^{-\alpha Z} \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left(\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_\tau \right) \right]. \end{aligned} \quad (1.27)$$

General properties of the essential infimum (cf. e.g. Karatzas and Shreve [13]) guarantee that there exists a sequence $(\vartheta^n)_{n \in \mathbb{N}}$ of admissible strategies such that

$$\inf_{n \in \mathbb{N}} E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t^n dS_t \right) \right) \middle| \mathcal{F}_\tau \right] = \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_\tau \right].$$

For two strategies $\vartheta^{(1)}, \vartheta^{(2)} \in \Theta$ define

$$\vartheta_t^{(3)} = \begin{cases} \mathbf{1}(t > \tau) \vartheta_t^{(1)} & : & E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_s^{(1)} dS_s \right) \right) \middle| \mathcal{F}_\tau \right], \\ & & \leq E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_s^{(2)} dS_s \right) \right) \middle| \mathcal{F}_\tau \right], \\ \mathbf{1}(t > \tau) \vartheta_t^{(2)} & : & \text{otherwise.} \end{cases}$$

We have $\vartheta^{(3)} \in \Theta$,

$$\begin{aligned} & E_P \left[\exp \left(\alpha \left(B - \int_\tau^T \vartheta_t^{(3)} dS_t \right) \right) \middle| \mathcal{F}_\tau \right] \\ &= \min \left\{ E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t^{(1)} dS_t \right) \right) \middle| \mathcal{F}_\tau \right], E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t^{(2)} dS_t \right) \right) \middle| \mathcal{F}_\tau \right] \right\} \end{aligned}$$

and therefore inf-stability. Hence, there exists a sequence $(\vartheta^n)_{n \in \mathbb{N}} \in \Theta$ such that

$$\begin{aligned} & e^{-\alpha Z} E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t^n dS_t \right) \right) \middle| \mathcal{F}_\tau \right] \\ & \searrow e^{-\alpha Z} \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_\tau \right], \quad n \nearrow \infty, \end{aligned} \quad (1.28)$$

and the left-hand side is dominated by the integrable random variable

$e^{-\alpha Z} E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t^1 dS_t \right) \right) \middle| \mathcal{F}_\tau \right]$. So, (1.27) holds due to the dominated convergence theorem. The result follows immediately from the the definition of u and the fact that for all $\delta \in \mathcal{D}$

$$\operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(B^{\delta_{\max}} - \int_\tau^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_\tau \right] \geq \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(B^\delta - \int_\tau^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_\tau \right].$$

□

Lemma A.2. Let $U : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone utility function and $u(\cdot) = E_P[U(\cdot)]$. If $\sup_{(\vartheta, \vartheta^1, \dots, \vartheta^k) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + h - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right) > -\infty$ then

$$\begin{aligned} & \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - B^\delta + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \\ &= \sup_{(\vartheta, \vartheta^1, \dots, \vartheta^k) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u \left(c + h - B^\delta + \int_0^T \vartheta_t^\delta dS_t \right), \end{aligned}$$

where ϑ^δ is defined in (2.4).

Proof. Let us show that

$$\begin{aligned} & \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} \sup_{\tilde{\vartheta} \in \Theta} E_P \left[U \left(c + h - B^\delta + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \right] \\ &= \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} E_P \left[\operatorname{ess\,sup}_{\tilde{\vartheta} \in \Theta} E_P \left[U \left(c + h - B^\delta + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \mid \mathcal{F}_\tau \right] \right] \\ &= \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} E_P \left[\operatorname{ess\,sup}_{\tilde{\vartheta} \in \Theta} \sum_{i=1}^k \mathbf{1}(\delta = i) E_P \left[U \left(c + h - B_i + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \mid \mathcal{F}_\tau \right] \right] \\ &= \sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} E_P \left[\sum_{i=1}^k \mathbf{1}(\delta = i) \operatorname{ess\,sup}_{\tilde{\vartheta} \in \Theta} E_P \left[U \left(c + h - B_i + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \mid \mathcal{F}_\tau \right] \right] \\ &= \sup_{\vartheta \in \Theta} E_P \left[\min_{i=1, \dots, k} \operatorname{ess\,sup}_{\tilde{\vartheta} \in \Theta} E_P \left[U \left(c + h - B_i + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \mid \mathcal{F}_\tau \right] \right] \\ &= \sup_{\vartheta \in \Theta} \sup_{(\vartheta^1, \dots, \vartheta^k) \in \Theta^k} E_P \left[\min_{i=1, \dots, k} E_P \left[U \left(c + h - B_i + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \vartheta_t^i dS_t \right) \mid \mathcal{F}_\tau \right] \right] \\ &= \sup_{(\vartheta, \vartheta^1, \dots, \vartheta^k) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} E_P \left[\sum_{i=1}^k \mathbf{1}(\delta = i) U \left(c + h - B_i + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \vartheta_t^i dS_t \right) \right] \quad (1.29) \end{aligned}$$

The first equality holds by a similar argument leading to (1.27), the third by Assumption 2.4. For the fourth equality, we use the fact that the infimum is attained in

$$\delta = \arg \min_{i=1, \dots, k} \left\{ \operatorname{ess\,sup}_{\tilde{\vartheta} \in \Theta} E_P \left[U \left(c + h - B_i + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \mid \mathcal{F}_\tau \right] \right\}.$$

The crucial fifth equality in (1.29) holds by the dominated convergence theorem and the argument leading to (1.28). In addition, we use the obvious fact that for sequences of random variables $(Z_n^i)_{n \in \mathbb{N}}$ if $Z_n^i \uparrow Z^i$ as $n \uparrow \infty$ for $i = 1, \dots, k$ then also $\min\{Z_n^1, \dots, Z_n^k\} \uparrow \min\{Z^1, \dots, Z^k\}$ as $n \uparrow \infty$. \square

Lemma A.3. Let $\tilde{\tau} \in \mathcal{S}$, $A \in \mathcal{F}_{\tilde{\tau}}$, Z be an $\mathcal{F}_{\tilde{\tau}}$ -measurable random variable and u the expected exponential utility function (2.10). If $\sup_{\vartheta \in \Theta} u \left(Z - f_i + \int_{\tilde{\tau}}^T \vartheta_t dS_t \right) > -\infty$ for $i = 0, \dots, k$ then we have for all $\tau \in \mathcal{S}$ with $\tau \geq \tilde{\tau}$:

$$\sup_{\vartheta \in \Theta} u \left(Z - f_{\tau_{\max}(\tilde{\tau})} \mathbf{1}_A - f_{\tau} \mathbf{1}_{\Omega \setminus A} + \int_{\tilde{\tau}}^T \vartheta_t dS_t \right) \leq \sup_{\vartheta \in \Theta} u \left(Z - f_{\tau} + \int_{\tilde{\tau}}^T \vartheta_t dS_t \right),$$

where τ_{\max} is defined in (3.22)/(3.23).

Proof. Using (1.27) it remains to show that for all $\tau \geq \tilde{\tau}$

$$\begin{aligned} & \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(\tilde{\tau})} - \int_{\tilde{\tau}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{\tilde{\tau}} \right] \\ & \geq \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau} - \int_{\tilde{\tau}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{\tilde{\tau}} \right]. \end{aligned} \quad (1.30)$$

In addition, it is sufficient to proof (1.30) for deterministic stopping times $\tilde{\tau} = t_i$. This is done by induction (in reverse order of time): for $i = k$ we have $\tau = t_k = \tau_{\max}(t_k)$. $i \rightsquigarrow i - 1$: for all $A \in \mathcal{F}_{t_{i-1}}$ we have per definition of $\tau_{\max}(t_{i-1})$

$$\begin{aligned} & \int_A \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_{i-1})} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] dP \\ & = \int_A \max \left\{ \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{t_{i-1}} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right], \right. \\ & \left. \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] \right\} dP \\ & \geq \int_{A \cap \{\tau = t_{i-1}\}} \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{t_{i-1}} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] dP \\ & + \int_{A \cap \{\tau > t_{i-1}\}} \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] dP. \end{aligned} \quad (1.31)$$

As

$$\begin{aligned} & \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] \\ & = \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(-\alpha \int_{t_{i-1}}^{t_i} \vartheta_t dS_t \right) \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \right) \middle| \mathcal{F}_{t_i} \right] \middle| \mathcal{F}_{t_{i-1}} \right] \end{aligned}$$

we can now apply the induction assumption for $\tau' = \tau \vee t_i$ to the last expression in (1.31)

and obtain

$$\begin{aligned} & \int_A \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_{i-1})} - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] dP \\ & \geq \int_A \operatorname{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp \left(\alpha \left(f_\tau - \int_{t_{i-1}}^T \vartheta_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] dP. \end{aligned}$$

□

Lemma A.4. *Let u be the expected exponential utility function (2.10). If*

$$\sup_{\vartheta \in \Theta} \sup_{\tilde{\vartheta} \in \Theta} \inf_{\tau \in \mathcal{S}} u \left(c + h - f_\tau + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) > -\infty,$$

then

$$\begin{aligned} & \sup_{\vartheta \in \Theta} \inf_{\tau \in \mathcal{S}} \sup_{\tilde{\vartheta} \in \Theta} u \left(c + h - f_\tau + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right) \\ & = \sup_{\vartheta \in \Theta} \sup_{\tilde{\vartheta} \in \Theta} \inf_{\tau \in \mathcal{S}} u \left(c + h - f_\tau + \int_0^\tau \vartheta_t dS_t + \int_\tau^T \tilde{\vartheta}_t dS_t \right). \end{aligned}$$

Proof. We have to show that

$$\begin{aligned} & \inf_{\vartheta \in \Theta} \sup_{\tau \in \mathcal{S}} \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_\tau - \int_0^\tau \vartheta_t dS_t - \int_\tau^T \tilde{\vartheta}_t dS_t \right) \right) \right] \\ & = \inf_{\vartheta \in \Theta} \inf_{\tilde{\vartheta} \in \Theta} \sup_{\tau \in \mathcal{S}} E_P \left[\exp \left(\alpha \left(f_\tau - \int_0^\tau \vartheta_t dS_t - \int_\tau^T \tilde{\vartheta}_t dS_t \right) \right) \right]. \end{aligned}$$

Therefore, it suffices to show that for every fixed $\vartheta \in \Theta$

$$\begin{aligned} & \sup_{\tau \in \mathcal{S}} \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_\tau - \int_0^\tau \vartheta_t dS_t - \int_\tau^T \tilde{\vartheta}_t dS_t \right) \right) \right] \\ & = \inf_{\tilde{\vartheta} \in \Theta} \sup_{\tau \in \mathcal{S}} E_P \left[\exp \left(\alpha \left(f_\tau - \int_0^\tau \vartheta_t dS_t - \int_\tau^T \tilde{\vartheta}_t dS_t \right) \right) \right]. \end{aligned} \quad (1.32)$$

Of course, the right-hand side is at least as big as the left-hand side. For the converse, we use the fact (cf. Lemma A.5) that there exists a sequence of strategies $(\tilde{\vartheta}^n)_{n \in \mathbb{N}} \subset \Theta$ such that for all $i = 0, \dots, k-1$

$$E_P \left[\exp \left(-\alpha \int_{t_i}^T \tilde{\vartheta}_t^n dS_t \right) \middle| \mathcal{F}_{t_i} \right] \searrow \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(-\alpha \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \middle| \mathcal{F}_{t_i} \right]. \quad (1.33)$$

With this special sequence we want to approximate the left-hand side of (1.32) from above:

$$\begin{aligned}
& \sup_{\tau \in \mathcal{S}} E_P \left[\exp \left(\alpha \left(f_\tau - \int_0^\tau \vartheta_t dS_t - \int_\tau^T \tilde{\vartheta}_t^n dS_t \right) \right) \right] \\
& - \sup_{\tau \in \mathcal{S}} \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_\tau - \int_0^\tau \vartheta_t dS_t - \int_\tau^T \tilde{\vartheta}_t dS_t \right) \right) \right] \\
& \leq \sup_{\tau \in \mathcal{S}} \left\{ E_P \left[\exp \left(\alpha \left(f_\tau - \int_0^\tau \vartheta_t dS_t - \int_\tau^T \tilde{\vartheta}_t^n dS_t \right) \right) \right] \right. \\
& \quad \left. - \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_\tau - \int_0^\tau \vartheta_t dS_t - \int_\tau^T \tilde{\vartheta}_t dS_t \right) \right) \right] \right\} \\
& = \sup_{\tau \in \mathcal{S}} \sum_{i=0}^k \left\{ E_P \left[\mathbf{1}(\tau = t_i) \exp \left(\alpha \left(f_{t_i} - \int_0^{t_i} \vartheta_t dS_t - \int_{t_i}^T \tilde{\vartheta}_t^n dS_t \right) \right) \right] \right. \\
& \quad \left. - E_P \left[\mathbf{1}(\tau = t_i) \exp \left(\alpha \left(f_{t_i} - \int_0^{t_i} \vartheta_t dS_t \right) \right) \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left(\exp \left(-\alpha \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \mid \mathcal{F}_{t_i} \right) \right] \right\} \\
& = \sup_{\tau \in \mathcal{S}} \sum_{i=0}^k E_P \left[\mathbf{1}(\tau = t_i) \exp \left(\alpha \left(f_{t_i} - \int_0^{t_i} \vartheta_t dS_t \right) \right) \right. \\
& \quad \left. \times \underbrace{\left\{ E_P \left[\exp \left(-\alpha \int_{t_i}^T \tilde{\vartheta}_t^n dS_t \right) \mid \mathcal{F}_{t_i} \right] - \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(-\alpha \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \mid \mathcal{F}_{t_i} \right] \right\}}_{\geq 0 \text{ } P\text{-a.s.}} \right] \\
& \leq \sum_{i=0}^k E_P \left[\exp \left(\alpha \left(f_{t_i} - \int_0^{t_i} \vartheta_t dS_t \right) \right) \right. \\
& \quad \left. \times \left\{ E_P \left[\exp \left(-\alpha \int_{t_i}^T \tilde{\vartheta}_t^n dS_t \right) \mid \mathcal{F}_{t_i} \right] - \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(-\alpha \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \mid \mathcal{F}_{t_i} \right] \right\} \right].
\end{aligned}$$

Due to (1.33) and the dominated convergence theorem, the last term tends to zero as n tends to infinity. \square

Lemma A.5. *There exists a sequence of strategies $(\vartheta^n)_{n \in \mathbb{N}} \subset \Theta$ such that for all $i = 0, \dots, k-1$*

$$E_P \left[\exp \left(-\alpha \int_{t_i}^T \vartheta_t^n dS_t \right) \mid \mathcal{F}_{t_i} \right] \searrow \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(-\alpha \int_{t_i}^T \vartheta_t dS_t \right) \mid \mathcal{F}_{t_i} \right] \quad P\text{-a.s.} \quad (1.34)$$

Proof. General properties of the essential infimum guarantee that for each $i = 0, \dots, k-1$ there exists a sequence $(\vartheta^{n,i})_{n \in \mathbb{N}} \subset \Theta$ satisfying (1.34). It remains to show that there is a *joint* sequence. We define such a joint sequence $(\hat{\vartheta}^n)_{n \in \mathbb{N}} \subset \Theta$ recursively on the intervals

$(t_{k-1}, T], (t_{k-2}, t_{k-1}], \dots, (0, t_1]$: For $t \in (t_{k-1}, T]$ we set :

$$\widehat{\vartheta}_t^n = \vartheta_t^{n, \widehat{j}}, \quad \text{where } \widehat{j} = \arg \min_{j=0, \dots, k-1} \left\{ E_P \left[\exp \left(-\alpha \int_{t_{k-1}}^T \vartheta_t^{n, j} dS_t \right) \middle| \mathcal{F}_{t_{k-1}} \right] \right\},$$

and for $t \in (t_{i-1}, t_i]$

$$\widehat{\vartheta}_t^n = \vartheta_t^{n, \widehat{j}}, \quad \text{where } \widehat{j} = \arg \min_{j=0, \dots, k-1} \left\{ E_P \left[\exp \left(-\alpha \left(\int_{t_{i-1}}^{t_i} \vartheta_t^{n, j} dS_t + \int_{t_i}^T \widehat{\vartheta}_t^n dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] \right\}.$$

It is obvious that for all $n \in \mathbb{N}$, $i = 0, \dots, k-1$

$$E_P \left[\exp \left(-\alpha \int_{t_i}^T \widehat{\vartheta}_t^n dS_t \right) \middle| \mathcal{F}_{t_i} \right] \leq E_P \left[\exp \left(-\alpha \int_{t_i}^T \vartheta_t^{n, i} dS_t \right) \middle| \mathcal{F}_{t_i} \right].$$

That implies the assertion. \square

Lemma A.6. *Under the conditions of Theorem 3.4(b) case (iii), for every $\varepsilon > 0$ there exist $\vartheta^{(i)} \in \Theta$, $i = 1, \dots, k$ such that for all $\tau \in \mathcal{S}$, $i = 1, \dots, k$*

$$\begin{aligned} & \sup_{\widetilde{\vartheta} \in \Theta} u \left(c + h - [\mathbf{1}(\tau \leq t_{i-1}) f_\tau + \mathbf{1}(\tau > t_{i-1}) f_{\tau_{\max}(t_i)}] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} dS_t + \dots \right. \\ & \quad \left. + \int_{\tau \wedge t_{i-2}}^{\tau \wedge t_{i-1}} \vartheta_t^{(i-1)} dS_t + \int_{\tau \wedge t_{i-1}}^T \widetilde{\vartheta}_t dS_t \right) \\ & \leq \sup_{\widetilde{\vartheta} \in \Theta} u \left(c + h - [\mathbf{1}(\tau \leq t_{i-1}) f_\tau + \mathbf{1}(\tau > t_{i-1}) f_{\tau_{\max}(t_i)}] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} dS_t + \dots \right. \\ & \quad \left. + \int_{\tau \wedge t_{i-2}}^{\tau \wedge t_{i-1}} \vartheta_t^{(i-1)} dS_t + \int_{\tau \wedge t_{i-1}}^{\tau \wedge t_i} \vartheta_t^{(i)} dS_t + \int_{\tau \wedge t_i}^T \widetilde{\vartheta}_t dS_t \right) + \varepsilon. \end{aligned} \quad (1.35)$$

Proof. In the first expression, the supremum over all strategies $(\widetilde{\vartheta})_{t \in (\tau \wedge t_{i-1}, T]}$ can be split into two suprema : one over all $(\widetilde{\vartheta})_{t \in (\tau \wedge t_{i-1}, \tau \wedge t_i]}$ and the other over all $(\widetilde{\vartheta})_{t \in (\tau \wedge t_i, T]}$. So, it remains to show that $\vartheta^{(i)}$ in (1.35) can be chosen independent of $\tau \in \mathcal{S}$. By putting in the definition of u and interchanging infimum and expectation, one can see that the set $\{\tau \leq t_{i-1}\}$ does not have any influence on the difference between the two suprema in (1.35) as $(\tau \wedge t_{i-1}, \tau \wedge t_i) = \emptyset$ on the set $\{\tau \leq t_{i-1}\}$. Therefore, we only consider the set

$\{\tau > t_{i-1}\}$:

$$\begin{aligned}
& \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp \left\{ \alpha \left(f_{\tau_{\max}(t_i)} - \dots - \int_{t_{i-1}}^{\tau \wedge t_i} \vartheta_t^{(i)} dS_t - \int_{\tau \wedge t_i}^T \tilde{\vartheta}_t dS_t \right) \right\} \right] \\
& - \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \dots - \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)} dS_t - \int_{t_{i-1}}^T \tilde{\vartheta}_t dS_t \right) \right) \right] \\
& \leq \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \dots - \int_{t_{i-1}}^{t_i} \vartheta_t^{(i)} dS_t - \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \right) \right] \\
& - \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \dots - \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)} dS_t - \int_{t_{i-1}}^T \tilde{\vartheta}_t dS_t \right) \right) \right] \\
& = E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp \left(-\alpha \left(\int_0^{t_1} \vartheta_t^{(1)} dS_t + \dots + \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)} dS_t \right) \right) \right] \\
& \times \left\{ \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^{t_i} \vartheta_t^{(i)} dS_t - \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] \right. \\
& \left. - \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \tilde{\vartheta}_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] \right\} \\
& \leq E_P \left[\exp \left(-\alpha \left(\int_0^{t_1} \vartheta_t^{(1)} dS_t + \dots + \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)} dS_t \right) \right) \right] \\
& \times \left\{ \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^{t_i} \vartheta_t^{(i)} dS_t - \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] \right. \\
& \left. - \operatorname{ess\,inf}_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \tilde{\vartheta}_t dS_t \right) \right) \middle| \mathcal{F}_{t_{i-1}} \right] \right\} \\
& = \inf_{\tilde{\vartheta} \in \Theta} E_P \left[e^{\alpha(f_{\tau_{\max}(t_i)} - \int_0^{t_1} \vartheta_t^{(1)} dS_t - \dots - \int_{t_{i-1}}^{t_i} \vartheta_t^{(i)} dS_t - \int_{t_i}^T \tilde{\vartheta}_t dS_t)} \right] \\
& - \inf_{\tilde{\vartheta} \in \Theta} \inf_{\tilde{\vartheta} \in \Theta} E_P \left[\exp \left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_0^{t_1} \vartheta_t^{(1)} dS_t \right. \right. \right. \\
& \quad \left. \left. - \dots - \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)} dS_t - \int_{t_{i-1}}^{t_i} \vartheta_t dS_t - \int_{t_i}^T \tilde{\vartheta}_t dS_t \right) \right) \right].
\end{aligned}$$

The last term does not depend on τ any more, and by suitable choice of $\vartheta^{(i)}$ it can be made arbitrary small. \square

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