A Limit Theorem for Copulas

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Abstract

We characterize convergence of a sequence of d-dimensional random vectors by convergence of the one-dimensional margins and of the copula. The result is applied to the approximation of portfolios modelled by t-copulas with large degrees of freedom, and to the convergence of certain dependence measures of bivariate distributions.

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1 Introduction

Copula functions are widely applied in statistics and econometrics, especially in finance. For example, Bluhm et al [2] and Li [9] apply copula functions for credit risk modelling, and Rosenberg [11] studies the pricing of exchange rate derivatives using copulas. Besides this, copulas in the context of risk management are emphasised by Embrechts et al [7]. In many applications, the asymptotic behaviour of copulas is of interest for approximation and convergence issues. For example, in order to characterize the limiting behaviour of multivariate extremes, Deheuvels [3, Théorème 2.3, Lemma 4.1] has shown that if $X = (X^{(1)}, \ldots, X^{(d)})$ is a random vector with continuous margins, then a sequence of random vectors converges weakly to X if and only if the one-dimensional margins of the sequence converge weakly to the margins $X^{(j)}$, and if additionally the copulas converge pointwise (and hence uniformly) to the copula of X on $[0,1]^d$. See also Deheuvels [5, p. 261], [6, Lemma 2]. In the present paper, we shall generalize Deheuvel's result to the case where X is not assumed to have continuous margins. Since in that case the copula of X does not need to be unique, convergence of the copulas on $[0,1]^d$ cannot be expected. However, we shall show that the copulas converge uniformly on the product of the ranges of the one-dimensional distribution functions of X. As we recently found out, such a result was already anticipated by Deheuvels in [4, Théorème 4]. However, a proof was given only for the case when X has continuous margins. Also, due to the increasing importance of copulas in applications and the fact that some of the literature [3] – [6] may be difficult to access it seems justified to give a full proof of this result in the general case.

2 Main result

An axiomatic definition of copulas is to be found in Joe [8] and Nelsen [10]. According to this a function $C : [0,1]^d \to [0,1]$ is a *(d-dimensional) copula* if C is a d-dimensional distribution function on $[0,1]^d$ having *uniform margins*, i.e. $C(1, \ldots, 1, u^{(j)}, 1, \ldots, 1) = u^{(j)}$ for $u^{(j)} \in [0,1]$.

Let $X = (X^{(1)}, \ldots, X^{(d)})$ be a d–dimensional random vector with distribution function F and marginal distribution functions $F^{(1)}, \ldots, F^{(d)}$. Then a copula C is *associated with* X if it satisfies

$$F(x^{(1)}, \ldots, x^{(d)}) = C(F^{(1)}(x^{(1)}), \ldots, F^{(d)}(x^{(d)})) \quad \forall\, x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d.$$

By Sklar's Theorem, an associated copula always exists and is unique on $\overline{\mathrm{Ran} F^{(1)}} \times \ldots \times \overline{\mathrm{Ran} F^{(d)}}$. On $\mathrm{Ran} F^{(1)} \times \ldots \times \mathrm{Ran} F^{(d)}$ it is given by

$$C(u^{(1)}, \ldots, u^{(d)}) = F((F^{(1)})^{\leftarrow}(u^{(1)}), \ldots, (F^{(d)})^{\leftarrow}(u^{(d)})),$$

where $(F^{(j)})^{\leftarrow}(u^{(j)}) := \inf\{y \in \mathbb{R} : F^{(j)}(y) \geq u^{(j)}\}$ denotes the left inverse of the increasing function $F^{(j)}$, $j \in \{1, \ldots, d\}$.

Now we can proof the limit result for copulas:

Theorem 2.1. *Let N be an ordered index set with limit point n_∞. Let $(X_n)_{n\in N}$ and X be d-dimensional random vectors, where $X_n = (X_n^{(1)}, \ldots, X_n^{(d)})$ and $X = (X^{(1)}, \ldots, X^{(d)})$. Then X_n converges weakly to X as $n \to n_\infty$, if and only if the margins $X_n^{(j)}$ converge weakly to $X^{(j)}$ as $n \to n_\infty$, for $j = 1, \ldots, d$, and if the copulas C_n of X_n converge pointwise to the copula C of X on $\operatorname{Ran} F^{(1)} \times \ldots \times \operatorname{Ran} F^{(d)}$ as $n \to n_\infty$, where $F^{(j)}$ denotes the distribution function of $X^{(j)}$. In that case, the convergence is uniform on $\overline{\operatorname{Ran} F^{(1)}} \times \ldots \times \overline{\operatorname{Ran} F^{(d)}}$.*

Proof. Denote the distribution function of X and X_n by F and F_n, respectively, and the distribution function of $X^{(j)}$ and $X_n^{(j)}$ by $F^{(j)}$ and $F_n^{(j)}$, respectively. Note that any copula D is Lipschitz continuous, more precisely it holds

$$|D(u) - D(v)| \leq \sum_{j=1}^{d} |u^{(j)} - v^{(j)}| \quad \forall\, u = (u^{(1)}, \ldots, u^{(d)}),\ v = (v^{(1)}, \ldots, v^{(d)}) \in [0,1]^d, \quad (1)$$

see Nelsen [10, Theorem 2.10.7]. Suppose that $X_n \xrightarrow{w} X$ as $n \to n_\infty$, where $\xrightarrow{w}$ denotes weak convergence. Then $X_n^{(j)} \xrightarrow{w} X^{(j)}$ as $n \to n_\infty$ by the continuous mapping theorem. For the convergence of the copulas, define $\mathcal{M}^{(j)}$ to be the set of all $u^{(j)} \in [0,1]$ such that there exist $x_{u,j} \in \mathbb{R}$ such that $u^{(j)} = F^{(j)}(x_{u,j})$ and such that $F^{(j)}$ is continuous in $x_{u,j}$. Let $(u^{(1)}, \ldots, u^{(d)}) \in \mathcal{M}^{(1)} \times \ldots \times \mathcal{M}^{(d)}$, and let $x_{u,j}$ be points as appearing in the definition of $\mathcal{M}^{(j)}$. Then (1) gives

$$\begin{aligned}
&|C_n(u^{(1)}, \ldots, u^{(d)}) - C(u^{(1)}, \ldots, u^{(d)})| \\
&\quad = |C_n(F^{(1)}(x_{u,1}), \ldots, F^{(d)}(x_{u,d})) - C(F^{(1)}(x_{u,1}), \ldots, F^{(d)}(x_{u,d}))| \\
&\quad \leq |C_n(F^{(1)}(x_{u,1}), \ldots, F^{(d)}(x_{u,d})) - C_n(F_n^{(1)}(x_{u,1}), \ldots, F_n^{(d)}(x_{u,d}))| \\
&\qquad + |C_n(F_n^{(1)}(x_{u,1}), \ldots, F_n^{(d)}(x_{u,d})) - C(F^{(1)}(x_{u,1}), \ldots, F^{(d)}(x_{u,d}))| \\
&\quad \leq |F^{(1)}(x_{u,1}) - F_n^{(1)}(x_{u,1})| + \ldots + |F^{(d)}(x_{u,d}) - F_n^{(d)}(x_{u,d)})| \\
&\qquad + |F_n(x_{u,1}, \ldots, x_{u,d}) - F(x_{u,1}, \ldots, x_{u,d})|.
\end{aligned}$$

Since the $x_{u,j}$ are continuity points of $F^{(j)}$, it follows that $F_n^{(j)}(x_{u,j})$ converges to $F^{(j)}(x_{u,j})$ as $n \to n_\infty$, and that $P(X \in \partial\{(y^{(1)}, \ldots, y^{(d)}) \in \mathbb{R}^d : y^{(j)} \leq x_{u,j}, j = 1, \ldots, d\}) = 0$. By assumption, this implies convergence of $F_n(x_{u,1}, \ldots, x_{u,d})$ to $F(x_{u,1}, \ldots, x_{u,d})$. Thus, C_n converges pointwise to C on $\mathcal{M}^{(1)} \times \ldots \times \mathcal{M}^{(d)}$, as $n \to n_\infty$. To show uniform convergence, let $\varepsilon > 0$, choose an integer $m \geq 3d/\varepsilon$, and for $k = (k^{(1)}, \ldots, k^{(d)}) \in \{0, \ldots, m-1\}^d$ set

$$A_k := \left\{ u = (u^{(1)}, \ldots, u^{(d)}) \in [0,1]^d : \frac{k^{(j)}}{m} \leq u^{(j)} \leq \frac{k^{(j)}+1}{m}, j = 1, \ldots, d \right\}.$$

Denote by K the set of all $k \in \{0, \dots, m-1\}^d$ such that $A_k \cap (\mathcal{M}^{(1)} \times \ldots \times \mathcal{M}^{(d)})$ is nonempty. Choose $u_k \in A_k \cap (\mathcal{M}^{(1)} \times \ldots \times \mathcal{M}^{(d)})$ for each $k \in K$. Then there exists $n_0 \in N$, such that

$$|C_n(u_k) - C(u_k)| \leq \frac{\varepsilon}{3} \quad \forall\, k \in K,\ n \geq n_0.$$

Then for any $k \in K$ and $u \in A_k$, (1) gives for $n \geq n_0$,

$$\begin{aligned} |C_n(u) - C(u)| &\leq |C_n(u) - C_n(u_k)| + |C_n(u_k) - C(u_k)| + |C(u_k) - C(u)| \\ &\leq \frac{d}{m} + \frac{\varepsilon}{3} + \frac{d}{m} \leq \varepsilon. \end{aligned}$$

Since $\mathcal{M}^{(1)} \times \ldots \times \mathcal{M}^{(d)}$ is dense in $\overline{\operatorname{Ran} F^{(1)}} \times \ldots \times \overline{\operatorname{Ran} F^{(d)}}$, this implies uniform convergence of C_n to C on $\overline{\operatorname{Ran} F^{(1)}} \times \ldots \times \overline{\operatorname{Ran} F^{(d)}}$, as $n \to n_\infty$.

For the converse, suppose that $X_n^{(j)} \xrightarrow{w} X^{(j)}$ for all $j = 1, \dots, d$, and that C_n converges pointwise to C on $\mathcal{M}^{(1)} \times \ldots \times \mathcal{M}^{(d)}$, as $n \to n_\infty$. Let $\mathcal{Q}$ be the set of all $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$ such that $F^{(j)}$ is continuous in $x^{(j)}$ for all $j = 1, \dots, d$. Then (1) gives for any $x \in \mathcal{Q}$,

$$\begin{aligned} &|F_n(x^{(1)}, \dots, x^{(d)}) - F(x^{(1)}, \dots, x^{(d)})| \\ &\quad = |C_n(F_n^{(1)}(x^{(1)}), \dots, F_n^{(d)}(x^{(d)})) - C(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))| \\ &\quad \leq |C_n(F_n^{(1)}(x^{(1)}), \dots, F_n^{(d)}(x^{(d)})) - C_n(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))| \\ &\qquad + |C_n(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)})) - C(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))| \\ &\quad \leq |F_n^{(1)}(x^{(1)}) - F^{(1)}(x^{(1)})| + \ldots + |F_n^{(d)}(x^{(d)}) - F^{(d)}(x^{(d)})| \\ &\qquad + |C_n(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)})) - C(F^{(1)}(x^{(1)}), \dots, F^{(d)}(x^{(d)}))|, \end{aligned}$$

and the latter converges to 0 as $n \to n_\infty$. Thus F_n converges to F in any $x \in \mathcal{Q}$, which then implies weak convergence of X_n to X (e.g. by an obvious modification of the proof of Theorem 29.1 in Billingsley [1]). □

It should be noted that in the case where margins of the limiting vector are supposed to be continuous and strictly increasing, a simpler proof can be given. In fact, then weak convergence of X_n to X implies uniform convergence of $(F_n^{(j)})^{\leftarrow}$ to $(F^{(j)})^{\leftarrow}$ and of F_n to F, so that the copulas converge uniformly, too. In the general case, however, more care has to be taken. Also, convergence of the copulas on the whole unit cube $[0,1]^d$ cannot be expected, as is shown by the following example:

Example 2.2. Let X and Y be two random vectors in $\mathbb{R}^d$ with different copulas. Set $X_n := X/n$ if n is odd and $X_n := Y/n$ if n is even. Then X_n converges weakly to $\mathbf{0}$ as $n \to \infty$, while the copula C_n of X_n is equal to the copula of X or Y, depending whether X is odd or even. Thus C_n cannot converge on $[0,1]^d$. However, it converges on $\times_{j=1}^d \{0,1\}$, which is the product of the ranges of the marginal distribution functions.

3 Applications

In this section we give two applications of Theorem 2.1. The first application is concerned with t-copulas with increasing degrees of freedom.

3.1 Credit Risk and t-Copula

In credit risk theory, modelling portfolios by t-copulas presents a common approach away from multivariate normal models, see e.g. Bluhm et. al. [2], Chapter 2.6. Let Σ be a positive definite $(d \times d)$-matrix with entries 1 on the diagonal and let $n \in \mathbb{N}$. Then the *Gaussian Copula* C_Σ^{Ga} is defined to be the copula of an $N(0, \Sigma)$ distributed vector Y, and the *t-Copula* $C_{n,\Sigma}^t$ is the copula of a multivariate t-distributed vector $X_{n,\Sigma} = \sqrt{n/S}\,Y$, where S is χ_n^2-distributed and independent of Y. Since $X_{n,\Sigma}$ converges weakly to Y as $n \to \infty$, Theorem 2.1 implies that the t-copulas $C_{n,\Sigma}^t$ converge uniformly to the Gaussian copula C_Σ^{Ga} as the degree of freedom n tends to ∞. Then if $(Z_n)_{n\in\mathbb{N}}$ is a sequence of random vectors with t-copula $C_{n,\Sigma}^t$ and if the margins of (Z_n) converge to some random variables with distribution function $F^{(j)}$, $j = 1, \ldots, d$, then (Z_n) converges as $n \to \infty$ to a random variable Z with distribution function $C_\Sigma^{Ga}(F^{(1)}(x^{(1)}), \ldots, F^{(d)}(x^{(d)}))$. In particular, a portfolio which is modelled by a t-Copula with large degrees of freedom can be approximated by a model using a Gaussian copula and the same margins.

3.2 Kendall's Tau, Spearman's Rho, and Tail Dependence

The next application discusses the convergence of three dependence measures of bivariate distributions, namely Kendall's tau, Spearman's rho and tail dependence.

Let $(X^{(1)}, X^{(2)})$, $(Y^{(1)}, Y^{(2)})$ and $(Z^{(1)}, Z^{(2)})$ be three independent and identically distributed random vectors with continuous margins and copula C. Then *Kendall's tau*, τ, and *Spearman's rho*, ρ, are given by

$$\begin{aligned} \tau &:= P((X^{(1)} - Y^{(1)})(X^{(2)} - Y^{(2)}) > 0) - P((X^{(1)} - Y^{(1)})(X^{(2)} - Y^{(2)}) < 0), \\ \rho &:= 3(P((X^{(1)} - Y^{(1)})(X^{(2)} - Z^{(2)}) > 0) - P((X^{(1)} - Y^{(1)})(X^{(2)} - Z^{(2)}) < 0). \end{aligned}$$

From this follows readily that bivariate weak convergence implies convergence of Kendall's tau and Spearman's rho. Another proof of this follows immediately from Theorem 2.1, since τ and ρ can be expressed in terms of the copula C via

$$\tau = 4\int_0^1 \int_0^1 C(u^{(1)}, u^{(2)})\, dC(u^{(1)}, u^{(2)}) - 1, \quad \rho = 12\int_0^1 \int_0^1 C(u^{(1)}, u^{(2)})\, du^{(1)} du^{(2)} - 3,$$

see e.g. Nelsen [10], Theorems 5.1.3 and 5.1.6. Convergence of the lower and upper tail dependence parameter, λ_L and λ_U, however does not follow from bivariate convergence.

For example, the lower tail dependence parameter is given (if it exists) by

$$\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u} = \lim_{u \to 0} P(X^{(2)} \leq (F^{(2)})^{\leftarrow}(u) | X^{(1)} \leq (F^{(1)})^{\leftarrow}(u)),$$

see Joe [8], p. 33. Then if the vector $(X_n^{(1)}, X_n^{(2)})$ has the copula

$$C_n(u^{(1)}, u^{(2)}) := \begin{cases} \min\{u^{(1)}, u^{(2)}\}, & \text{for } \max\{u^{(1)}, u^{(2)}\} \geq 1/n, \\ n\, u^{(1)}\, u^{(2)}, & \text{for } \max\{u^{(1)}, u^{(2)}\} < 1/n, \end{cases}$$

then C_n converges uniformly to the copula $C(u^{(1)}, u^{(2)}) = \min(u^{(1)}, u^{(2)})$. However, the lower tail dependence parameter of C_n is 0, while that of C is 1. So uniform convergence of C_n is not enough to ensure convergence of λ_L. A sufficient condition ensuring convergence of λ_L would be that there is some $\varepsilon > 0$ such that $(C_n(u,u) - C(u,u))/u$ converges uniformly in $u \in (0, \varepsilon]$ to 0 as $n \to \infty$, provided the lower tail dependence parameter of C_n and C exist.

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