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A geometric approach to portfolio optimization in models with transaction costs

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Abstract. We consider a continuous-time stochastic optimization problem with infinite horizon, linear dynamics, and cone constraints which includes as a particular case portfolio selection problems under transaction costs for models of stock and currency markets. Using an appropriate geometric formalism we show that the Bellman function is the unique viscosity solution of a HJB equation.

Key words: Currency market, transaction costs, consumption-investment problem, utility function, HJB equation, viscosity solution

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1 Introduction

1.1 From two-asset to multi-asset models

Consumption-investment problems for models with market friction, taking its origin from the paper by Magill and Constantinides [21] and put on a firm theoretical background by Davis and Norman [10] constitute now one of the most actively growing branches of mathematical finance. It happens that the theory of viscosity solutions developed in the early eighties by Crandall, Ishii, and Lions (see their

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famous guide [9] and the books by Fleming and Soner [16] and Bardi and Capuzzo-Dolcetta [5]) can be successfully applied to portfolio selection under transaction costs: Zariphopoulou [25,26], Davis, Panas, and Zariphopoulou [11], Tourin and Zariphopoulou [23], Shreve and Soner [22], Fleming and Soner [16], Cadenillas [8], and many others. A close look at the existing literature shows that nowadays the "hot" subjects of current research in the field are multi-asset models of portfolio optimization for markets with imperfections as well as models driven by Lévy price processes, see, e.g., Akian, Menaldi, and Sulem [1], Akian, Sulem, and Taksar [2], Benth, Karlsen, and Reikvam [6,7], Emmer and Klüppelberg [14], and Framstadt, Øksendal, and Sulem [15]. Apparently, such models are more realistic compared to the two-asset models studied at the early stage of the theory. Unfortunately, available research papers on multidimensional models with market friction require not only a good command of advanced mathematics but also a certain patience to follow manipulations with rather cumbersome formulas. Moreover, known results concern, basically, HARA (and logarithmic) utility functions and do not cover (at least, directly) the interesting case of currency market models.

This note is aimed to show that convex analysis provides a natural language to treat multi-asset investment-consumption models with transaction costs. As we shall see, convex analysis allows us to replace tedious computations by appealing to elementary geometric properties. We extend the geometric approach to markets with transaction costs suggested for the hedging problem in Kabanov [17,18] (see also Kabanov and Last [19], Kabanov and Stricker [20] for further development) to models with consumption. Our presentation is, in fact, a study of a stochastic utility optimization problem with infinite horizon, linear dynamics, and polyhedral cone constraints.

Having in mind that our message is addressed to readers interested, principally, in financial applications, we are looking for a compromise between generality and "accessibility" of results and restrict ourselves to the framework which is adequate, e.g., to cover a currency market model. The main results are assertions that the Bellman function is a viscosity solution of an HJB equation and that this equation has a unique solution. Their proofs rely only on the basic definitions: a few facts from the theory of viscosity solutions are used. We derive the HBJ equation following traditional lines, on the basis of the dynamic programming principle establishing the latter in an elementary and a self-contained way. We isolate the concept of the Lyapunov function for the non-linear operator involved in the HJB equation. We show in particular that the existence of such a function together with the monotonicity of the dual of the utility function (with respect to the partial ordering induced by the dual to the solvency cone) are the only properties needed to guarantee the uniqueness result.

Our research is influenced by Akian et al. [1] and Shreve and Soner [22] which are the starting point of the present study. Needed prerequisites from convex analysis can be found in any textbook (e.g., in Aubin [3]).

1.2 Notations

In our language "cone" means always (nontrivial) "closed convex cone". For a cone $K \subseteq \mathbf{R}^d$ we denote by $\operatorname{int} K$ its interior and by K^* the dual positive cone, i.e.

$$K^* := \{y: \ yx \geq 0 \ \ \forall x \in K\}.$$

Here yx is the scalar product but in more complicate formulae we prefer to write $\langle y, x\rangle$; $|x|$ stands for the Euclidean norm; $\mathcal{O}_r(x) := \{z: \ |z-x| < r\}$ and $\bar{\mathcal{O}}_r(x) := \{z: \ |z-x| \leq r\}$ are, respectively, the open and closed balls in $\mathbf{R}^d$ of radius r with center at x.

Usually, $\mathcal{O}$ will be a non-empty open subset of $\mathbf{R}^d$. We denote by $C^p(\mathcal{O})$ the set of all real functions on $\mathcal{O}$ with continuous derivatives up to the order p.

For a concave function $U: \mathbf{R}^d \to \mathbf{R} \cup \{-\infty\}$, we denote by U^* the *Fenchel dual* of $-U(-.)$, i.e.

$$U^*(p) := \sup_x [U(x) - px].$$

Finally, Σ_G is the *support function* of a set $G \subseteq \mathbf{R}^d$; it is given by the relation $\Sigma_G(p) := \sup_{x \in G} px$. We apologize for using of the capital letter Σ here, reserving the standard notation σ for the diffusion coefficient.

For a function $B: \mathbf{R}_+ \to \mathbf{R}^d$, we denote by $||B||_t$ the *total variation* of B on the interval $[0,t]$. In this definition we use the convention that $B_{0-} = 0$ and $||B||_0 = |\Delta B_0|$. If B is of finite variation, then $\dot{B} := dB/d||B||$.

2 The model

2.1 The dynamics

Let $Y = (Y_t)$ be an $\mathbf{R}^d$-valued semimartingale on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with the trivial initial σ-algebra. Let K and $\mathcal{C}$ be *proper* cones in $\mathbf{R}^d$ such that $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. Define the set $\mathcal{A}$ of controls $\pi = (B, C)$ as the set of adapted càdlàg processes of bounded variation such that, up to an evanescent set,

$$\dot{B} \in -K, \qquad \dot{C} \in \mathcal{C}. \tag{1}$$

Let $\mathcal{A}_a$ be the set of controls with absolutely continuous C and $\Delta C_0 = 0$. For the elements of $\mathcal{A}_a$ we have $c := dC/dt \in \mathcal{C}$.

The controlled process $V = V^{x,\pi}$ is the solution of the linear system

$$dV^i_t = V^i_{t-} dY^i_t + dB^i_t - dC^i_t, \quad V^i_{0-} = x^i, \quad i = 1, ..., d. \tag{2}$$

For $x \in \operatorname{int} K$ we consider the subsets $\mathcal{A}^x$ and $\mathcal{A}^x_a$ of "admissible" controls for which the processes $V^{x,\pi}$ never leave the set $\operatorname{int} K \cup \{0\}$ and has zero as an absorbing point. Thus, if $V_{s-}(\omega) \in \partial K$, then $\Delta B_s(\omega) = -V_{s-}(\omega)$.

The important hypothesis that the cone K is proper, i.e. $K \cap (-K) = \{0\}$, or equivalently, $\operatorname{int} K^* \neq \emptyset$, corresponds to a model with *efficient friction*, the notion isolated in Kabanov [17] and discussed in detail in Delbaen, Kabanov, and Valkeila

[13]. In a financial context K (usually containing $\mathbf{R}^d_+$) is interpreted as the solvency region and $C = (C_t)$ as the consumption process; the process $B = (B_t)$ describes accumulated fund transfers.

Let $G := (-K) \cap \partial \bar{\mathcal{O}}_1(0)$. It is a compact closed set, and $-K = \operatorname{cone} G$.

In this paper we shall work using the following assumption:

$\mathbf{H_1}$. The process Y is a continuous process with independent increments, mean $EY_t = \alpha t$, and variance $DY_t = At$.

To facilitate references we formulate also a more specific hypothesis (frequent in the literature) where the matrix A is diagonal with $a^{ii} = (\sigma^i)^2$.

$\mathbf{H_2}$. The components of Y are of the form $dY^i_t = \alpha^i dt + \sigma^i dw^i_t$ where w is a standard Wiener process in $\mathbf{R}^d$.

In our proof of the dynamic programming principle (needed to derive the HJB equation) we shall assume that the stochastic basis is a canonical one, that is the set of continuous functions with the Wiener measure.

2.2 *Goal functionals*

Let $U : \mathcal{C} \to \mathbf{R}_+$ be a concave function such that $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. With every $\pi = (B, C) \in \mathcal{A}^x_a$ we associate the "utility process"

$$J^\pi_t := \int_0^t e^{-\beta s} U(c_s) ds \,, \quad t \geq 0 \,,$$

where $\beta > 0$. We consider the problem with infinite horizon and the *goal functional* EJ^π_∞ and define the *Bellman function* W by

$$W(x) := \sup_{\pi \in \mathcal{A}^x_a} EJ^\pi_\infty \,, \quad x \in \operatorname{int} K \,. \tag{3}$$

If π_i, $i = 1, 2$, are admissible strategies for the initial points x_i, then the strategy $\lambda\pi_1 + (1-\lambda)\pi_2$ is an admissible strategy for the initial point $\lambda x_1 + (1-\lambda)x_2$ for any $\lambda \in [0, 1]$, and the corresponding absorbing time is the maximum of the absorbing times for both π_i. It follows that the function W is *concave* on $\operatorname{int} K$. Since $A^{x_1} \subseteq A^{x_2}$ when $x_2 - x_1 \in K$, the function W is *increasing* with respect to partial ordering $\geq_K$ generated by the cone K. It is convenient to put W equal to zero on the boundary of K and extend it to the whole space $\mathbf{R}^d$ as a concave function just by putting $W := -\infty$ outside K.

Remark. In this paper we consider a model with mixed "regular-singular" controls. In fact, the assumption that the consumption process has an intensity $c = (c_t)$ and the agent's utility depends on this intensity is far from being realistic. Modern models allow an intertemporal substitution and the consumption by "gulps", i.e. they deal with "singular" controls of the class $\mathcal{A}^x$ and suitably modified goal functionals, see, e.g., Bank and Riedel [4] and references wherein. We shall not discuss this issue here.

2.3 Examples

Now we present, in a chronological order, several consumption–investment problems under transaction costs covered by the above setting.

Example 1 (One bond, one stock)
The price dynamics is given by

$$\begin{aligned} dS_t^1 &= 0, \\ dS_t^2 &= S_t^2(\alpha dt + \sigma dw_t), \end{aligned}$$

where w is a Wiener process, $\sigma > 0$. The first relation means that the first asset ("bond","money", or "bank account") is chosen as the *numéraire*. The price of the risky asset follows a geometric Brownian motion. The portfolio values evolve as

$$\begin{aligned} dV_t^1 &= dL_t^{21} - (1+\lambda^{12})dL_t^{12} - c_t^1 dt, \\ dV_t^2 &= V_t^2(\alpha dt + \sigma dw_t) + dL_t^{12} - (1+\lambda^{21})dL_t^{21} - c_t^2 dt, \end{aligned}$$

where L^{12} and L^{21} are adapted right-continuous increasing processes, the solvency cone K is defined in an obvious way via the transaction costs coefficients (K is proper if $\lambda^{12} + \lambda^{21} > 0$). Here the consumption process C is constrained to be absolutely continuous; typically, it is also assumed that $c^2 = 0$.

The optimization problem is of the form

$$E\int_0^\infty e^{-\beta t} u(c_t^1)dt \to \max \tag{4}$$

where $u : \mathbf{R}_+ \to \mathbf{R}_+$ is a utility function. The maximum is taken over the set of strategies for which the value process evolves in the solvency cone K.

The problem was solved for the power utility $u(c) = c^\gamma/\gamma$, $\gamma \in]0,1[$, (and also for logarithmic utility) by Davis and Norman [10]. This was a remarkable achievement: a special problem with transaction costs allows for an explicit solution. Not surprisingly, many authors contributed to a further study of this model; see the comments in the survey paper by Cadenillas [8].

Between a number of related works it is worth to mention the extensive memoir by Shreve and Soner [22], where a complete solution also for $\gamma < 0$ was given and the properties of the Bellman function were studied in great details using the techniques of viscosity solutions.

Example 2 (One bond, many stocks)
This more realistic case is considered in Akian et al. [1] (see also Akian et al. [2] for the optimization of the long-term growth rate). The price dynamics is given by

$$\begin{aligned} dS_t^1 &= 0, \\ dS_t^i &= S_t^i(\alpha^i dt + \sigma^i dw_t^i), \end{aligned}$$

where w^i are independent Wiener processes, $\sigma^i > 0$, $i = 2, \dots, d$.

It is assumed that all operations are monetary ones, i.e. the investor may sell or buy stocks paying proportional transaction costs. The portfolio dynamics is described by the system

$$\begin{aligned} dV_t^1 &= \sum_{j=1}^d [dL_t^{j1} - (1+\lambda^{1j}) dL_t^{1j}] - c_t^1 dt, \\ dV_t^i &= V_t^i(\alpha^i dt + \sigma^i dw_t^i) + dL_t^{1i} - (1+\lambda^{i1}) dL_t^{i1}, \quad i \geq 2. \end{aligned}$$

The solvency region in Akian et al. [1] is described as follows:

$$K = \{x \in \mathbf{R}^d : \ \mathcal{W}(x) \geq 0\}, \tag{5}$$

where

$$\mathcal{W}(x) := x^1 + \sum_{i=2}^d \min\left\{(1+\lambda^{1i})x^i, (1+\lambda^{i1})^{-1}x^i\right\}$$

represents the net wealth, that is, the amount of money in the bank account after performing transactions that bring the holdings in the risky assets to zero.

As in the previous model, an admissible strategy (defined by $2(d-1)+1$ finite increasing processes L^{i1}, L^{1i}, and C^1, where the last one is absolutely continuous) is such that the corresponding value process does not exit the solvency region. It is assumed that $\lambda^{1i} + \lambda^{i1} > 0$ for all $i = 1, \dots, d$, and, hence, the solvency cone is proper.

The principal theoretical result of Akian et al. [1] is a theorem which asserts that the Bellman function of the problem with the power utility function (for $\gamma \in]0,1[$) is the unique viscosity solution of a HJB equation with zero boundary condition. For the reader's convenience we write this equation in the form of the aforementioned paper:

$$\max\left\{AW + u^*(W_{x_1}), \max_{2\leq i\leq d} L_i W, \max_{2\leq i\leq d} M_i W\right\} = 0, \tag{6}$$

where

$$\begin{aligned} AW &= \frac{1}{2}\sum_{i=2}^d \sigma_i^2 x_i^2 W_{x_i x_i} + \sum_{i=1}^d \alpha_i x_i W_{x_i} - \delta W, \\ L_i W &= -(1+\lambda^{1i})W_{x_1} + W_{x_i}, \\ M_i W &= (1+\lambda^{i1})^{-1} W_{x_1} - W_{x_i}, \\ u^*(p) &= \left(\frac{1}{\gamma} - 1\right) p^{\gamma/(\gamma-1)} \end{aligned}$$

(for esthetic reasons we replaced in the above formulae superscripts by subscripts).

The next model of a general currency market motivated our study.

Example 3 (Currency market)
The model has exactly the form described at the beginning of this section with the cone K (assumed to be proper) having the following specific structure:

$$K := \left\{x : \ \exists\, m \in \mathbf{M}_+^d \text{ such that } x^i \geq \sum_{j=1}^d [(1+\lambda^{ij})m^{ij} - m^{ji}], \ i = 1, \dots, d\right\}.$$

Here $\mathbf{M}_+^d$ is the set of matrices with non-negative entries and zero diagonal. The transaction costs coefficients are given by the matrix $\Lambda \in \mathbf{M}_+^d$. This reflects the fact that any asset can be, in principle, exchanged directly for any other asset. It is easily seen that the solvency region is a polyhedral cone admitting the representation

$$K = \text{cone}\left\{(1+\lambda^{ij})e_i - e_j,\ 1 \le i,j \le d\right\},$$

where e_i is the ith unit vector of the canonical base in $\mathbf{R}^d$, and

$$K^* = \{w \in \mathbf{R}^d : (1+\lambda^{ij})w^i - w^j \ge 0,\ 1 \le i,j \le d\}.$$

Of course, Example 1 is a particular case. In Example 2 the whole matrix Λ of transaction costs coefficients is not specified: only its first row and column are given. It is assumed that the direct transfers of the wealth between the “risky” assets are prohibited. However, the model can be imbedded into the setting of Example 3 if we complete the matrix of coefficients to make these direct transfers more expensive by choosing the other coefficients to meet the condition

$$1+\lambda^{ij} \ge (1+\lambda^{i1})(1+\lambda^{1j}), \quad \forall\, i,j \ge 2.$$

In the literature one can also find a stock market model with direct exchanges organized in such a way that transactions charge only the money account, see Kabanov and Stricker [20]. This model, having a polyhedral solvency cone, falls into the scope of our general framework.

2.4 The Hamilton-Jacobi-Bellman equation

Assume that $\mathbf{H}_1$ holds. Put

$$F(X,p,W,x) := \max\{F_0(X,p,W,x) + U^*(p), \Sigma_G(p)\},$$

where X belongs to $\mathcal{S}_d$, the set of $d \times d$ symmetric matrices, $p, x \in \mathbf{R}^d$, $W \in \mathbf{R}$,

$$F_0(X,p,W,x) := \frac{1}{2}\text{tr}\, A(x)X + \alpha(x)p - \beta W$$

where $A^{ij}(x) := a^{ij}x^i x^j$, $\alpha^i(x) := \alpha^i x^i$, $1 \le i,j \le d$.

If ϕ is a smooth function, we put

$$\mathcal{L}\phi(x) := F(\phi''(x), \phi'(x), \phi(x), x).$$

In a similar way, $\mathcal{L}_0$ corresponds to the function F_0.

We show, under mild hypotheses, that W is the unique viscosity solution of the Dirichlet problem for the HJB equation

$$F(W''(x), W'(x), W(x), x) = 0, \qquad x \in \text{int}\, K, \tag{7}$$

$$W(x) = 0, \qquad x \in \partial K, \tag{8}$$

with the boundary condition understood in the usual classical sense.

2.5 Viscosity solutions

Since, in general, W may have no derivatives at some points $x \in \operatorname{int} K$, the notation (7) needs to be interpreted. The idea of viscosity solutions is to plug into F the derivatives and Hessians of quadratic functions touching W from above and below. Formal definitions (adapted to the case we are interested in) are as follows.

Let f and g be functions defined in a neighborhood of zero. We shall write $f(.) \lessapprox g(.)$ if $f(h) \le g(h) + o(|h|^2)$ as $|h| \to 0$, the notations $f(.) \gtrapprox g(.)$ and $f(.) \approx g(.)$ have the obvious meaning.

For $p \in \mathbf{R}^d$ and $X \in \mathcal{S}_d$ we consider the quadratic function

$$Q_{p,X}(z) := pz + (1/2)\langle Xz, z\rangle, \quad z \in \mathbf{R}^d,$$

and define the *super*- and *subjets* of a function v at point x:

$$J^+v(x) := \{(p,X) : \ v(x+.) \lessapprox v(x) + Q_{p,X}(.)\},$$
$$J^-v(x) := \{(p,X) : \ v(x+.) \gtrapprox v(x) + Q_{p,X}(.)\}.$$

In other words, $J^+v(x)$ (resp. $J^-v(x)$) is the family of all quadratic functions (parameterized by their coefficients) locally dominating the function v (resp. dominated by this function) at point x up to the second order.

A function $v \in C(K)$ is called *viscosity supersolution* of (7) if

$$F(X, p, v(x), x) \le 0 \qquad \forall\, (p, X) \in J^-v(x), \ x \in \operatorname{int} K.$$

A function $v \in C(K)$ is called *viscosity subsolution* of (7) if

$$F(X, p, v(x), x) \ge 0 \qquad \forall\, (p, X) \in J^+v(x), \ x \in \operatorname{int} K.$$

A function $v \in C(K)$ is a *viscosity solution* of (7) if it is a viscosity super- and subsolution of (7).

A function $v \in C(K)$ is called *classical supersolution* of (7) if $v \in C^2(\operatorname{int} K)$ and $\mathcal{L}v \le 0$ on $\operatorname{int} K$. We add the adjective *strict* when $\mathcal{L}v < 0$ on $\operatorname{int} K$.

The next well-known criterion gives a flexibility to manipulate with the above concepts. It allows us to use smooth local majorants/minorants of a function, which is the supposed viscosity solution, as test functions (to be inserted with their derivatives into the operator). For the reader's convenience we recall it with a proof.

Lemma 1 *Let $v \in C(K)$. Then the following conditions are equivalent:*
(a) *the function v is a viscosity supersolution of (7);*
(b) *for any ball $\mathcal{O}_r(x) \subseteq K$ and any $f \in C^2(\mathcal{O}_r(x))$, such that $v(x) = f(x)$ and $v \ge f$ on $\mathcal{O}_r(x)$, the inequality $\mathcal{L}f(x) \le 0$ holds.*

Proof. $(a) \Rightarrow (b)$ Obvious: the pair $(f'(x), f''(x))$ is in $J^-v(x)$.

$(b) \Rightarrow (a)$ Take (p, X) in $J^-v(x)$. To conclude, we construct a smooth function f with $f'(x) = p$, $f''(x) = X$ satisfying the requirements of (b).

By definition,

$$v(x+h) - v(x) - Q_{p,X}(h) \ge |h|^2 \varphi(|h|),$$

where $\varphi(u) \to 0$ as $u \downarrow 0$. We consider on $]0, r[$ the non-decreasing continuous function

$$\delta(u) := \sup_{\{h:\ |h| \le u\}} |h|^{-2}(v(x+h) - v(x) - Q_{p,X}(h))^- \le \sup_{\{y:\ 0 \le y \le u\}} (\varphi(y))^-.$$

Obviously, δ is continuous, non-decreasing and $\delta(u) \to 0$ as $u \downarrow 0$. The function

$$\Delta(u) := \frac{2}{3} \int_u^{2u} \int_\eta^{2\eta} \delta(\xi) d\xi d\eta$$

vanishes at zero with its two right derivatives; $u^2\delta(u) \le \Delta(u) \le u^2\delta(2u)$. Thus,

$$v(x+h) - v(x) - Q_{p,X}(h) \ge -|h|^2\delta(|h|) \ge -\Delta(|h|)$$

and $f(y) := v(x) + Q_{p,X}(y-x) - \Delta(|y-x|)$ is a needed function. □

For subsolutions we have a similar result with the inverse inequalities.

2.6 Ishii's lemma

The only result we need from the theory of viscosity solutions (or, better to say, from convex analysis) is the following simplified version of Ishii's lemma, see Crandall et al. [9] or Fleming and Soner [16].

Lemma 2 *Let v and $\tilde{v}$ be two continuous functions on $\mathcal{O}$. Consider the function $\Delta(x,y) := v(x) - \tilde{v}(y) - \frac{1}{2}n|x-y|^2$ with $n > 0$. Suppose that Δ attains a local maximum at $(\widehat{x}, \widehat{y})$. Then there are symmetric matrices X and Y such that*

$$(n(\widehat{x} - \widehat{y}), X) \in \bar{J}^+ v(\widehat{x}), \qquad (n(\widehat{x} - \widehat{y}), Y) \in \bar{J}^- \tilde{v}(\widehat{y}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{9}$$

In this statement I is the identity matrix and $\bar{J}^+ v(x)$ and $\bar{J}^- v(x)$ are values of the set-valued mappings whose graphs are closures of graphs of $J^+ v$ and $J^- v$, respectively.

Of course, if v is smooth, the claim follows directly from the necessary conditions of a local maximum (with $X = v''(\widehat{x})$, $Y = \tilde{v}''(\widehat{y})$ and the constant 1 instead of 3 in inequality (9)).

3 Uniqueness of the solution and Lyapunov functions

3.1 Uniqueness theorem

The following concept plays the crucial role in the proof of the purely analytic result on the uniqueness of the viscosity solution which we establish by a classical method of doubling variables, making use of Ishii's lemma.

Definition We say that a non-negative function $\ell \in C(K) \cap C^2(\operatorname{int} K)$ is the *Lyapunov function* if $\ell' \in \operatorname{int} K^*$ and $\mathcal{L}_0 \ell \le 0$ on $\operatorname{int} K$ and $\ell(x) \to \infty$ as $|x| \to \infty$.

Theorem 1 *Suppose that there exists a Lyapunov function ℓ. Assume also that $U^*(p) \le U^*(q)$ if $p - q \in K^*$. Then the Dirichlet problem (7), (8) has at most one viscosity solution in the class of continuous functions satisfying the growth condition*

$$W(x)/\ell(x) \to 0, \quad |x| \to \infty. \tag{10}$$

Proof. Let W and $\tilde{W}$ be viscosity solutions of (7) coinciding on ∂K. Suppose that $W(z) > \tilde{W}(z)$ for some $z \in K$. Take $\varepsilon > 0$ such that $W(z) - \tilde{W}(z) - 2\varepsilon\ell(z) > 0$. Put

$$\Delta_n(x, y) := W(x) - \tilde{W}(y) - \frac{1}{2} n|x - y|^2 - \varepsilon[\ell(x) + \ell(y)].$$

Since $\Delta_n(x, y) \to -\infty$ as $|x| + |y| \to \infty$, there exists $(x_n, y_n) \in K \times K$ such that

$$\Delta_n(x_n, y_n) = \bar{\Delta}_n := \sup_{(x,y) \in K \times K} \Delta_n(x, y) \ge \bar{\Delta} := \sup_{x \in K} \Delta_0(x, x) > 0.$$

All (x_n, y_n) belong to the bounded set $\{(x, y) : \ \Delta_0(x, y) \ge 0\}$. It follows that the sequence $n|x_n - y_n|^2$ is bounded. We continue to argue (without introducing new notations) with a subsequence along which (x_n, y_n) converge to some limit $(\widehat{x}, \widehat{x})$. Necessarily, $n|x_n - y_n|^2 \to 0$ (otherwise we would have $\Delta_0(\widehat{x}, \widehat{x}) > \bar{\Delta}$). It is easily seen that $\bar{\Delta}_n \to \Delta_0(\widehat{x}, \widehat{x}) = \bar{\Delta}$. Thus, $\widehat{x}$ is an interior point of K and so are x_n and y_n for sufficiently large n.

By Ishii's lemma applied to the functions $v := W - \varepsilon\ell$ and $\tilde{v} := \tilde{W} + \varepsilon\ell$ at the point (x_n, y_n) there exist matrices X and Y satisfying (9) such that

$$(n(x_n - y_n), X) \in \bar{J}^+ v(x_n), \qquad (n(x_n - y_n), Y) \in \bar{J}^- \tilde{v}(y_n).$$

Using the notations $p_n := n(x_n - y_n) + \varepsilon\ell'(x_n)$, $q_n := n(x_n - y_n) - \varepsilon\ell'(y_n)$, $X_n := X + \varepsilon\ell''(x_n)$, $Y_n := Y - \varepsilon\ell''(y_n)$, we may rewrite the last relations in the following equivalent form:

$$(p_n, X_n) \in \bar{J}^+ W(x_n), \qquad (q_n, Y_n) \in \bar{J}^- \tilde{W}(y_n). \tag{11}$$

Since W and $\tilde{W}$ are viscosity sub- and supersolutions,

$$F(X_n, p_n, W(x_n), x_n) \ge 0 \ge F(Y_n, q_n, \tilde{W}(y_n), y_n).$$

The second inequality implies that $mq_n \le 0$ for each $m \in G$. By our assumption $\ell'(x) \in \operatorname{int} K^*$ for $x \in \operatorname{int} K$ and, therefore,

$$mp_n = mq_n + \varepsilon m(\ell'(x_n) + \ell'(y_n)) < 0.$$

Since G compact, $\Sigma_G(p_n) < 0$. It follows that

$$F_0(X_n, p_n, W(x_n), x_n) + U^*(p_n) \ge 0 \ge F_0(Y_n, q_n, \tilde{W}(y_n), y_n) + U^*(q_n).$$

Taking into account that $U^*(p_n) \le U^*(q_n)$, we obtain the inequality

$$b_n := F_0(X_n, p_n, W(x_n), x_n) - F_0(Y_n, q_n, \tilde{W}(y_n), y_n) \ge 0.$$

Clearly,

$$b_n = \frac{1}{2}\sum_{i,j=1}^{d}(a^{ij}x_n^i x_n^j X_{ij} - a^{ij}y_n^i y_n^j Y_{ij}) + n\sum_{i=1}^{d}\alpha^i(x_n^i - y_n^i)^2$$
$$+\frac{1}{2}\beta n|x_n - y_n|^2 - \beta\Delta_n(x_n, y_n) + \varepsilon(\mathcal{L}_0\ell(x_n) + \mathcal{L}_0\ell(y_n)).$$

By virtue of (9) the first sum is dominated by $\mathrm{const} \times n|x_n - y_n|^2$; a similar bound for the second sum is obvious; the last term is negative. It follows that $\limsup b_n \le -\beta\bar{\Delta} < 0$ and we get a contradiction arising from the assumption $W(z) > \tilde{W}(z)$. $\square$

Remarks (a) The definition of the Lyapunov function does not depend on U (it is a property of the operator with $U^* = 0$) and we have the uniqueness for all utility function U for which U^* is decreasing with respect to the partial ordering induced on K^*. But to apply the theorem we should know that W is not growing faster than a certain Lyapunov function.

(b) Notice that, if U is defined on $\mathcal{C}$ and increasing with respect to the partial ordering $\ge_{\mathcal{C}}$, then U^* is decreasing with respect to $\ge_{\mathcal{C}^*}$, hence, $\ge_{K^*}$, see Deelstra, Pham, and Touzi [12].

3.2 Existence of Lyapunov functions and classical supersolutions

Results on the uniqueness of a solution to the HJB equation are all based on work with specific Lyapunov functions. The following general considerations explain how the latter can be constructed.

Let $u \in C(\mathbf{R}_+)$ be an increasing strictly concave function, smooth on $\mathbf{R}_+ \setminus \{0\}$ with $u(0) = 0$ and $u(\infty) = \infty$. Introduce the function $R := -u'^2/(u''u)$. Assume that $\bar{R} := \sup_{z>0} R(z) < \infty$.

For $p \in K^*$ we define the function $f(x) = f_p(x) := u(px)$ on K. If $y \in G$ and $x \ne 0$, then $yf'(x) = (py)u'(px) \le 0$. The inequality is strict when $p \in \operatorname{int} K^*$.

Recall that $A(x)$ is the matrix with $A^{ij}(x) = A^{ij}x^i x^j$ and the vector $\alpha(x)$ has components $\alpha^i x^i$. Suppose that $\langle A(x)p, p\rangle \neq 0$. Putting $z := px$ for brevity, we obtain by obvious transformations,

$$\mathcal{L}_0 f(x) = \frac{1}{2}\left[\langle A(x)p,p\rangle u''(z) + 2\langle \alpha(x),p\rangle u'(z) + \frac{\langle \alpha(x),p\rangle^2}{\langle A(x)p,p\rangle}\frac{u'^2(z)}{u''(z)}\right]$$
$$+\frac{1}{2}\frac{\langle \alpha(x),p\rangle^2}{\langle A(x)p,p\rangle}R(z)u(z) - \beta u(z). \tag{12}$$

Since we have [...] ≤ 0 in the above formula, its left-hand side is non-positive if $\beta \geq \eta(p)\bar{R}$ where

$$\eta(p) := \frac{1}{2}\sup_{x\in G}\frac{\langle \alpha(x),p\rangle^2}{\langle A(x)p,p\rangle}.$$

This simple observation leads to the following existence result for Lyapunov functions:

Proposition 3 *Let $p \in \operatorname{int} K^*$. Suppose that $\langle A(x)p,p\rangle \neq 0$ for all $x \in \operatorname{int} K$. If $\beta \geq \eta(p)\bar{R}$, then f_p is a Lyapunov function.*

It is easily seen that the same conclusion holds if $\langle \alpha(x), p\rangle$ vanishes on the set $\{x \in \operatorname{int} K : \langle A(x)p,p\rangle = 0\}$.

Let $\bar{\eta} := \sup_{p\in K^*}\eta(p)$. Continuity considerations show that this quantity is finite if $\langle A(x)p,p\rangle \neq 0$ for all nontrivial $x \in K$ and $p \in K^*$. Obviously, if $\beta \geq \bar{\eta}\bar{R}$, then f_p is a Lyapunov function for $p \in \operatorname{int} K^*$.

The representation (12) is useful also in the search of classical supersolutions for the operator $\mathcal{L}$. Since $\mathcal{L}(f) = \mathcal{L}_0(f) + U^*(f')$, it is natural to choose u related to U. For a particular case, where $\mathcal{C} = \mathbf{R}^d_+$ and $U(c) = u(e_1 c)$, with u satisfying the postulated properties (except, maybe, unboundedness) and, where also the inequality

$$u^*(au'(z)) \leq g(a)u(z) \tag{13}$$

holds, we get (using the homogeneity of $\mathcal{L}_0$) the following result.

Proposition 4 *Assume that $\langle A(x)p,p\rangle \neq 0$ for all $x \in \operatorname{int} K$ and $p \in K^*$. Suppose that (13) holds for every $a, z > 0$ with $g(a) = o(a)$ as $a \to \infty$. If $\beta > \bar{\eta}\bar{R}$, then there exists a_0 such that for every $a \geq a_0$ the function af_p is a classical supersolution of (7), whatever $p \in K^*$ with $p^1 \neq 0$ is. Moreover, if $p \in \operatorname{int} K^*$, then af_p is a strict supersolution on any compact subset of* $\operatorname{int} K$.

For the power utility function $u(z) = z^\gamma/\gamma$, $\gamma \in]0,1[$, we have

$$R(z) = \gamma/(1-\gamma) = \bar{R}$$

and $u^*(au'(z)) = (1-\gamma)a^{\gamma/(\gamma-1)}u(z)$.

If Y satisfies $\mathbf{H}_2$ with $\sigma^1 = 0$, $\alpha^1 = 0$ (i.e. the first asset is the *numéraire*) and $\sigma^i \neq 0$ for $i \neq 1$, then, by the Cauchy–Schwarz inequality applied to $\langle \alpha(x), p\rangle$,

$$\eta(p) \leq \frac{1}{2}\sum_{i=2}^{d}\left(\frac{\alpha^i}{\sigma^i}\right)^2.$$

The inequality

$$\beta > \frac{\gamma}{1-\gamma}\frac{1}{2}\sum_{i=2}^{d}\left(\frac{\alpha^i}{\sigma^i}\right)^2 \tag{14}$$

(implying the relation $\beta > \bar{\eta}\bar{R}$) is a standing assumption in many studies on the consumption-investment problem under transaction costs, see Akian et al. [1] and Davis and Norman [10].

As we shall see in the next section, the existence of supersolutions has important implications for the Bellman function.

4 Supersolutions and properties of the Bellman function

4.1 When is W finite on K?

We first present sufficient conditions for W to be finite.

Let Φ be the set of continuous functions $f : K \to \mathbf{R}_+$ increasing with respect to the partial ordering $\geq_K$ and such that for every $x \in \operatorname{int} K$ and $\pi \in \mathcal{A}_a^x$ the non-negative process $X^f = X^{f,x,\pi}$ with representation

$$X_t^f := e^{-\beta t} f(V_t) + J_t^\pi, \tag{15}$$

where $V = V^{x,\pi}$, is a supermartingale.

The set Φ of f with this property is convex, stable under the operation $\wedge$ (recall that the minimum of two supermartingales is a supermartingale). Any continuous function which is a monotone limit (increasing or decreasing) of functions in Φ also belongs to Φ.

Lemma 5 (a) *If $f \in \Phi$, then $W \leq f$;*

(b) *if for any $y \in \partial K$ there exists some $f \in \Phi$ such that $f(y) = 0$, then W is continuous on K.*

Proof. (a) Using the positivity of f, the supermartingale property X^f, and, finally, the monotonicity of f we have that

$$EJ_t^\pi \leq EX_t^f \leq f(V_0) \leq f(x).$$

(b) The concave function W is locally Lipschitz continuous on its domain $\operatorname{int} K$. The continuity at the point $y \in \partial K$ follows because $0 \leq W \leq f$. □

Lemma 6 *Let $f : K \to \mathbf{R}_+$ be a function in $C(K) \cap C^2(\operatorname{int} K)$ increasing with respect to the partial ordering $\geq_K$. If f is a classical supersolution of (7), then $f \in \Phi$, i.e. X^f is a supermartingale.*

Proof. In order to be able to apply Itô's formula in a comfortable way we introduce the process $\tilde{V} = V^{\sigma-}$, where σ is the first hitting time of zero by the process V. This process coincides with V on $[0, \sigma[$ but, in contrast to the latter, either always remains in $\operatorname{int} K$ (due to the stopping at σ if $V_{\sigma-} \in \operatorname{int} K$) or exits to the boundary in a continuous way and stops there. Let $\tilde{X}^f$ be defined by (15) with V replaced by $\tilde{V}$. Since

$$X^f = \tilde{X}^f + e^{-\beta\sigma}(f(V_{\sigma-} + \Delta B_\sigma) - f(V_{\sigma-}))I_{[\sigma,\infty[},$$

by the assumed monotonicity of f it is sufficient to verify that $\tilde{X}^f$ is a supermartingale.

Applying Itô's formula to $e^{-\beta t} f(\tilde{V}_t)$ we obtain on $[0,\sigma[$ the representation

$$\tilde{X}_t^f = f(x) + \int_0^t e^{-\beta s}[\mathcal{L}_0 f(V_s) - c_s f'(V_s) + U(c_s)]ds + R_t + m_t, \quad (16)$$

where m is a process such that m^{σ_n} are continuous martingales, σ_n are stopping times increasing to σ, and

$$R_t := \int_0^t e^{-\beta s} f'(\tilde{V}_{s-})dB_s^c + \sum_{s\le t} e^{-\beta s}[f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})]. \quad (17)$$

By definition of a supersolution, for any $x \in \operatorname{int} K$,

$$\mathcal{L}_0 f(x) \le -U^*(f'(x)) \le cf'(x) - U(c) \quad \forall\, c \in K.$$

Thus, the integral in (16) is a decreasing process. By the finite increments formula

$$f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-}) = f'(\tilde{V}_{s-} + \theta_s \Delta B_s)\Delta B_s, \quad (18)$$

where θ_s takes values in $[0,1]$ and, therefore,

$$R_t = \int_0^t e^{-\beta s} f'(\tilde{V}_{s-} + \theta_s \Delta B_s)\dot{B}_s d||B||_s. \quad (19)$$

This makes clear that the process R is also decreasing because $\dot{B} \in -K$ and for a classical supersolution of (7) we have $f'(x)c \ge 0$ whenever $c \in K$. Since $\tilde{X}^f \ge 0$, it follows easily that each process $\tilde{X}^f_{t\wedge\sigma_n}$ is a supermartingale and, hence, by Fatou's lemma, $\tilde{X}^f$ is a supermartingale as well. □

Lemma 6 implies that the existence of a smooth non-negative supersolution f of (7) ensures the finiteness of W on K. Sometimes, e.g., in the case of power utility, it is possible to find such a function in a rather explicit form, see Sect. 7.

4.2 Strict local supersolutions

The next more technical result, which is also based on an analysis of (16), will be used to show that W is a subsolution of the HJB equation.

We fix a ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$. Define τ^π as the exit time of $V^{\pi,x}$ from $\bar{\mathcal{O}}_r(x)$ and consider the set $\mathcal{A}_a^{x,r} \subseteq \mathcal{A}_a^x$ consisting of strategies for which the stopped processes $(V^{\pi,x}_{t\wedge\tau^\pi})$ evolve in $\bar{\mathcal{O}}_r(x)$.

Lemma 7 *Let $f \in C^2(\bar{\mathcal{O}}_r(x))$ be such that $\mathcal{L}f \le -\varepsilon < 0$ on $\bar{\mathcal{O}}_r(x)$. Then there exists a function $\eta :]0,\infty[\to \mathbf{R}_+$, strictly positive on a certain interval $]0,t_0[$ and such that on this interval*

$$\sup_{\pi \in \mathcal{A}_a^{x,r}} EX^{f,x,\pi}_{t\wedge\tau^\pi} \le f(x) - \eta_t.$$

Proof. Since the strategy π is fixed in the arguments below, we omit it in the notations of this proof. As in the proof of Lemma 6, we apply Itô's formula (but now directly to the process $V_{t\wedge\tau}$, because we suppose that f is smooth in a ball larger than $\mathcal{O}_r(x)$ and which the process never leaves). Using the bound $\mathcal{L}_0(V_s) \le -\varepsilon - U^*(V_s)$ for $s \le \tau$ as well as (19), we infer the inequality

$$X^{f,x}_{t\wedge\tau} \le f(x) - e^{-\beta t} N_t + m_t,$$

where m is a martingale and

$$N_t := \varepsilon(t\wedge\tau) + \int_0^{t\wedge\tau} H(c_s, f'(V_s))ds + \int_0^{t\wedge\tau} (-f'(V_{s_-} + \theta_s \Delta B_s)\dot{B}_s)d||B||_s$$

with $H(c,y) := U^*(y) + cy - U(c) \ge 0$. It remains to verify that EN_t dominate, on a certain interval $]0,t_0]$, a strictly positive function (independent on π).

Let us introduce the compact set $\Gamma := f'(\bar{\mathcal{O}}_r(x)) \subseteq K^*$. For $y \in \Gamma$ and $c \in \mathcal{C}$ we have that $(c/|c|)y \ge \varepsilon$ and $U(c)/|c| \to 0$ as $c \to \infty$. Thus, there are constants κ ("large") and κ_1 ("small") such that the following inequality holds:

$$\inf_{y\in\Gamma} H(c,y) \ge \kappa_1 |c|, \qquad |c| \ge \kappa, \quad c \in \mathcal{C}.$$

It follows that one can check the domination property for $E\tilde{N}_t$ with the simpler processes

$$\tilde{N}_t := t\wedge\tau + \int_0^{t\wedge\tau} I_{\{|c_s|\ge\kappa\}}|c_s|ds + ||B||_{t\wedge\tau}. \tag{20}$$

Take $\delta > 1$. By the stochastic Cauchy formula the solution of the linear equation (2) can be written as

$$V^i_t = \mathcal{E}_t(Y^i)x + \mathcal{E}_t(Y^i)\int_0^t \mathcal{E}_s^{-1}(Y^i)d(B^i_s - C^i_s), \qquad i = 1,\dots,d,$$

with the Girsanov exponential

$$\mathcal{E}_t(Y^i) := e^{Y^i_t - (1/2)\langle Y^i\rangle_t}.$$

It follows that there exists a number $t_0 > 0$ and a set Γ with $P(\Gamma) > 0$ on which

$$|V^{x,\pi} - x| \le r/2 + \delta(||B|| + ||C||) \quad \text{on } [0,t_0]$$

whatever the control $\pi = (B,C)$ is. Of course, diminishing t_0, we may assume without loss of generality that $\kappa t_0 \le r/(4\delta)$. For any $t \le t_0$ we have on the set $\Gamma \cap \{\tau \le t\}$ the inequality $||B||_\tau + ||C||_\tau \ge r/(2\delta)$. Thus, the expectation of $E\tilde{N}_t$ on $[0,t_0]$ dominates the piecewise linear function $(t\wedge(r/(4\delta))P(\Gamma)$. □

5 Dynamic programming principle

The following property of the Bellman function is usually referred to as the (weak) "dynamic programming principle":

Theorem 2 *Assume that $W(x) < \infty$ for $x \in \operatorname{int} K$. Then for any finite stopping time τ*

$$W(x) = \sup_{\pi \in \mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right). \tag{21}$$

We work on the canonical filtered space of continuous functions equipped with the Wiener measure. The generic point $\omega = \omega_.$ of this space is a continuous function on $\mathbf{R}_+$, zero at the origin. Let $\mathcal{F}_t^\circ := \sigma\{\omega_s,\ s \le t\}$ and $\mathcal{F}_t := \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$. We add the superscript P to denote σ-algebras augmented by all P-null sets from Ω. Recall that $\mathcal{F}_t^{\circ,P}$ coincides with $\mathcal{F}_t^P$ (this assertion follows easily from the predictable representation theorem).

A particular structure of Ω allows us to consider such operators as the stopping $\omega_. \mapsto \omega_.^s$, $s \ge 0$, where $\omega_.^s = \omega_{s\wedge .}$ and the translation $\omega_. \mapsto \omega_{s+.} - \omega_s$. Taking Doob's theorem into account, one can describe $\mathcal{F}_s^\circ$-measurable random variables as those of the form $g(w_.) = g(w_.^s)$ where g is a measurable function on Ω.

We define also the "concatenation" operator as the measurable mapping

$$g : \mathbf{R}_+ \times \Omega \times \Omega \to \Omega$$

with $g_t(s, \omega_., \tilde{\omega}_.) = \omega_t I_{[0,s[}(t) + (\tilde{\omega}_{t-s} + \omega_s) I_{[s,\infty[}(t)$.

Notice that

$$g_t(s, \omega_.^s, \omega_{.+s} - \omega_s) = \omega_t.$$

Thus, $\pi(\omega) = \pi(g(s, \omega_.^s, \omega_{.+s} - \omega_s))$.

Let π be a fixed strategy from $\mathcal{A}_a^x$ and let $\vartheta = \vartheta^{x,\pi}$ be a hitting time of zero for the process $V^{x,\pi}$.

We need the following general fact on conditional distributions.

Let ξ and η be two random variables taking values in Polish spaces X and Y equipped with their Borel σ-algebras $\mathcal{X}$ and $\mathcal{Y}$. Then ξ admits a regular conditional distribution given $\eta = y$ which we shall denote by $p_{\xi|\eta}(\Gamma, y)$ and

$$E(f(\xi,\eta)|\eta) = \int f(x,y) p_{\xi|y}(dx, y)\Big|_{y=\eta} \qquad (a.s.)$$

for any measurable function $f(x,y) \ge 0$.

We shall apply the above relation to the random variables $\xi = (\omega_{.+\tau} - \omega_\tau)$ and $\eta = (\tau, \omega^\tau)$. In this case, according to the Dynkin–Hunt theorem, the conditional distribution $p_{\xi|\eta}(\Gamma, y)$ admits a version which is independent of y and coincides with the Wiener measure P.

At last, for fixed s and w^s, the shifted control $\pi(g(s, \omega_.^s, \tilde{\omega}.), s + dr)$ is admissible for the initial condition $V_{s-}^{x,\pi}(\omega)$. Here we denote by $\tilde{\omega}.$ a generic point of the canonical space.

Lemma 8 *Let $\mathcal{T}_f$ be the set of all finite stopping time. Then*

$$W(x) \le \sup_{\pi \in \mathcal{A}_a^x} \inf_{\tau \in \mathcal{T}_f} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right). \tag{22}$$

Proof. For arbitrary $\pi \in \mathcal{A}_a^x$ and $\mathcal{T}_f$ we have that

$$EJ_\infty^{\pi,x} = EJ_\tau^{\pi,x} + Ee^{-\beta\tau}\int_0^\infty e^{-\beta r} u(c_{r+\tau})dr.$$

According to the above discussion we can rewrite the second term of the right-hand side as

$$Ee^{-\beta\tau}\int\left(\int_0^\infty e^{-\beta r} u(c_{r+\tau}(g(\tau,\omega^\tau,\tilde\omega)))dr\right)P(d\tilde\omega)$$

and dominate it by $Ee^{-\beta\tau}W(V_{\tau-}^{x,\pi})$. This leads to the desired inequality. □

The proof of the opposite inequality is based on different ideas.

Lemma 9 *Assume that $W(x) < \infty$ for $x \in \operatorname{int} K$. Then for any finite stopping time τ*

$$W(x) \ge \sup_{\pi \in \mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right). \tag{23}$$

Proof. Fix $\varepsilon > 0$. Being concave, the function W is continuous on $\operatorname{int} K$. For every $x \in \operatorname{int} K$ we can find an open ball $\mathcal{O}_r(x) = x + \mathcal{O}_r(0)$ with $r = r(\varepsilon, x) < \varepsilon$ containing the intersection of $\operatorname{int} K$ and the open set $\{y : \ |W(y) - W(x)| < \varepsilon\}$. Moreover, we can find a smaller ball $\mathcal{O}_{\tilde r}(x)$ contained in the set $y(x) + K$ with $y(x) \in \mathcal{O}_r(x)$. Indeed, take a ball $x_0 + \mathcal{O}_\delta(0) \subseteq K$. Since K is a cone,

$$x + \mathcal{O}_{\lambda\delta}(0) \subseteq x - \lambda x_0 + K$$

for every $\lambda > 0$. Clearly, the requirement is met for $y(x) = x - \lambda x_0$ and $\tilde r = \lambda\delta$ when $\lambda|x_0| < \varepsilon$ and $\lambda\delta < r$. The family of sets $\mathcal{O}_{\tilde r(x)}(x)$, $x \in \operatorname{int} K$, is an open covering of $\operatorname{int} K$. But any open covering of a separable metric space contains a countable subcovering (this is the Lindelöf property; in our case, where $\operatorname{int} K$ is a countable union of compacts, it is obvious). Take a countable subcovering indexed by points x_n. For simplicity, we shall denote its elements by $\mathcal{O}_n$ and $y(x_n)$ by y_n. Put $A_1 := \mathcal{O}_1$, and $A_n = \mathcal{O}_n \setminus \cap_{k<n}\mathcal{O}_k$. The sets A_n are disjoint and their union is $\operatorname{int} K$.

Let $\pi^n = (B^n, C^n) \in \mathcal{A}_a^{y_n}$ be an ε-optimal strategy for the initial point y_n, i.e. such that

$$EJ^{y_n,\pi_n} \ge W(y_n) - \varepsilon.$$

Let $\pi \in \mathcal{A}_a^x$ be an arbitrary strategy. We consider the strategy $\tilde\pi \in \mathcal{A}_a^x$ defined by the relation

$$\tilde\pi = \pi I_{[0,\tau[} + \sum_{n=1}^\infty [(y_n - V_{\tau-}^{x,\pi}, 0) + \bar\pi^n] I_{[\tau,\infty[} I_{A_n} I_{\{\tau<\vartheta\}}$$

where $\bar\pi^n$ is the translation of the strategy π^n: for a point $\omega_.$ with $\tau(\omega) = s < \infty$

$$\bar\pi_t^n(\omega_.) := \pi_{t-s}^n(\omega_{.+s} - \omega_s).$$

In other words, the measure $d\tilde{\pi}$ coincides with $d\pi$ on $[0, \tau[$ and with the shift of $d\pi^n$ on $]\tau, \infty[$ when $V^{x,\pi}_{\tau-}$ is a subset of A_n; the correction term guarantees that in the latter case the trajectory of the control system corresponding to the control $\tilde{\pi}$ passes at time τ through the point y_n.

Now, using the same considerations as in the previous lemma, we have:

$$\begin{aligned} W(x) \geq EJ^{\tilde{\pi}}_{\infty} &= EJ^{\pi}_{\tau} + \sum_{n=1}^{\infty} EI_{A_n}(V^{x,\pi}_{\tau-}) I_{\{\tau<\vartheta\}} \int_{\tau}^{\infty} e^{-\beta s} u(\bar{c}^n_s) ds \\ &\geq EJ^{\pi}_{\tau} + \sum_{n=1}^{\infty} EI_{A_n}(V^{x,\pi}_{\tau-}) I_{\{\tau<\vartheta\}} e^{-\beta\tau} (W(y_n) - \varepsilon) \\ &\geq EJ^{\pi}_{\tau} + Ee^{-\beta\tau} W(V^{x,\pi}_{\tau-}) - 2\varepsilon. \end{aligned}$$

Since π and ε are arbitrary, the result follows. □

Remarks It is easily seen that the above proof goes well when also the driving noise is a Lévy process (the canonical basis in this case is the Skorohod space of regular functions), see Benth et al. [6] where the claimed property is taken as an assumption. Notice also that the previous lemmas imply the identity

$$W(x) = \sup_{\pi \in \mathcal{A}^x_a} \inf_{\tau \in \mathcal{T}_f} E\left(J^{\pi}_{\tau} + e^{-\beta\tau} W(V^{x,\pi}_{\tau-})\right).$$

Comment The dynamic programming principle (DPP) has a clear intuitive meaning. However, there is a difference between universal "principles" and specific theorems: the latter require precise formulations and proofs. By an abuse of terminology such theorems are often also referred to as DPP.

Our analysis of the literature reveals that it is rather difficult to find a paper with a self-contained and complete proof even if the model of interest is of the simplest class, for instance, with a linear dynamics. Typically, some "formal" arguments are given indicating that "rigorous" proofs can be found elsewhere, preferably in treatises on controlled Markov processes. Tracking further references, one can observe that they often deal with slightly different models, other definitions of optimality, "regular" controls and so on. For instance, in Fleming and Soner [16] and Yong and Zhou [24], the concept of control involves a choice of a stochastic basis. Furthermore, often proofs rely on specific properties of the utility function or use methods from PDEs which can hardly be generalized for driving processes other than diffusions. Since we aim at future generalizations to Lévy processes, we provide a complete proof of DPP for our simple model, which requires only elementary prerequisites. We hope that a complete understanding of this simple situation allows us the generalization to more general driving processes.

6 The Bellman function and the HJB equation

Theorem 10 *Assume that the Bellman function W is in $C(K)$. Then W is a viscosity solution of (7).*

Proof. The claim follows from the two lemmas below. □

Lemma 11 *If (23) holds then W is a viscosity supersolution of (7).*

Proof. Let $x \in \mathcal{O} \subseteq \operatorname{int} K$. Let $\phi \in C^2(\mathcal{O})$ be such that $\phi(x) = W(x)$ and $W \geq \phi$ in $\mathcal{O}$. Fix $m \in K$ and take $\varepsilon > 0$ small enough to ensure that $x - \varepsilon m \in \mathcal{O}$. The function W is increasing with respect to the partial ordering generated by K. Thus, $\phi(x) = W(x) \geq W(x - \varepsilon m) \geq \phi(x - \varepsilon m)$. It follows that $-m\phi'(x) \leq 0$.

Take now π with $B_t = 0$ and $c_t = c \in \mathcal{C}$. Let τ_r be the exit time for the process $V = V^{x,\pi}$ from a small ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$. Applying (23) we have, using Itô's formula (16), that

$$\begin{aligned} 0 &\geq E\left(\int_0^{t\wedge\tau_r} e^{-\beta s}U(c_s)ds + e^{-\beta(t\wedge\tau_r)}W(V_{t\wedge\tau_r})\right) - \phi(x) \\ &\geq E\int_0^{t\wedge\tau_r} e^{-\beta s}[\mathcal{L}_0\phi(V_s) - c\phi'(V_s) + U(c)]ds \\ &\geq \min_{y\in \mathrm{B}_r(x)} [\mathcal{L}_0\phi(y) - c\phi'(y) + U(c)]E\left[\frac{1}{\beta}\left(1 - \exp(-\beta(t\wedge\tau_r))\right)\right]. \end{aligned}$$

Dividing by t and taking successively the limits as t and r converge to zero we get that $\mathcal{L}_0\phi(x) - c\phi'(x) + U(c) \leq 0$. Thus, $\mathcal{L}_0\phi(x) + U^*(\phi'(x)) \leq 0$. □

Lemma 12 *If (22) holds then W is a viscosity subsolution of (7).*

Proof. Let $x \in \mathcal{O} \subseteq \operatorname{int} K$. Let $\phi \in C^2(\mathcal{O})$ be a function such that $\phi(x) = W(x)$ and $W \leq \phi$ on $\mathcal{O}$. Assume that the subsolution inequality for ϕ fails at x. Thus, there exists $\varepsilon > 0$ such that $\mathcal{L}\phi \leq -\varepsilon$ on some ball $\bar{\mathcal{O}}_r(x) \subseteq \mathcal{O}$. Fix $t > 0$ such that $\eta_t > 0$ where η is as in Lemma 7. Take $\pi \in \mathcal{A}_a^x$ such that

$$W(x) \leq E\left(J^\pi_{t\wedge\tau} + e^{-\beta\tau}W(V^{x,\pi}_{t\wedge\tau})\right) + \frac{1}{2}\eta_t, \tag{24}$$

where τ is the exit time of $V^{x,\pi}$ from $\bar{\mathcal{O}}_r(x)$. Since W is increasing with respect to the partial ordering induced by K, we may assume without loss of generality that $\pi \in \mathcal{A}_a^{x,r}$. Using the inequality $W \leq \phi$ and applying Lemma 7 we obtain from (24) that $W(x) = \phi(x) - (1/2)\eta_t$. A contradiction. □

7 Power utility functions

Summarizing the previous discussions we arrive at the following result for the Bellman function W in our basic stochastic control model for the utility function $U(c) = (e_1 c)^\gamma/\gamma$, $\gamma \in]0,1[$.

Theorem 13 *Assume that $\langle A(x)p, p\rangle \neq 0$ for all $x \in \operatorname{int} K$ and $p \in K^*$ and*

$$\beta > \frac{\gamma}{1-\gamma}\bar{\eta} = \frac{\gamma}{1-\gamma}\frac{1}{2}\sup_{x\in G}\sup_{p\in K^*}\frac{\langle\alpha(x),p\rangle^2}{\langle A(x)p,p\rangle}. \tag{25}$$

Then the Dirichlet problem (7), (8) has a unique viscosity solution in the class of functions whose growth rates are strictly less than $\gamma' := \beta/(\beta + \bar{\eta})$, and this solution is the Bellman function W having growth rate γ.

Proof. According to Proposition 3 the function $l(x) = (px)^{\gamma'}$, $p \in \text{int } K^*$, is a Lyapunov function having γ' as its growth rate. Theorem 1 guarantees the uniqueness of the Dirichlet problem in the class of functions whose growth rates are strictly smaller than that of l, i.e. γ'.

By virtue of Proposition 4 for sufficiently large a the function $f(x) = a\,u(px)$ is a classical supersolution of (7) whatever $p \in K^*$ with $p^1 \neq 0$ is. It follows from Lemmas 5 and 6 that W has growth rate γ, vanishes on the boundary of K, and belongs to $C(K)$. The latter property allows us to apply Theorem 10 which claims that the Bellman function is a viscosity solution of (7). $\square$

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