Allocation of Risk Capital to Insurance Portfolios

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1 Introduction

Capital allocation is of special interest to (re-)insurers. As insurance policies are purchased to protect against adverse financial contingencies, insolvency risk plays a special role in the insurance industry. Risk capital is held to assure policyholders that claims can be paid even if larger than expected. Insolvency of an insurer concerns the whole company - bankruptcy is not defined on subportfolios of an insurer. Nevertheless, for certain types of decisions as e.g. pricing and underwriting, but also for financial decisions concerning the investment side of the insurer, it is useful to think of the risk capital as being allocated to different business units or subportfolios.

For an interesting conceptional overview to this topic we refer to Cummins [5]: " Filling in the details to enable insurers to move from the concepts to practical applications in capital allocation provides a promising avenue for future research."

This paper is a contribution along these lines. We report the outcome of an empirical study resting on theoretical investigations on capital allocation to different positions of an insurance portfolio based on various risk measures.

In Section 2, we introduce a minimum set of definitions and define five risk measures and five allocation methods, leading to 25 (not necessarily disjoint) allocation principles.

In Section 3 we show that some of these allocation methods generate equivalent outcomes. Although the covariance allocation principle does not fit into the allocation principles as defined in Section 2, we include it in our considerations, since it plays an important role in practice. This section also contains the main theoretical result of this paper: we investigate the equivalence of the covariance principle to the other allocation principles introduced before.

Section 4 is devoted to the practical analysis of the results and shows the behaviour of the considered allocation principles when they are applied to a small portfolio. With these

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considerations on the sample portfolio we can highlight the different inherent characteristics of the allocation principles. This underlines the importance of the choice of such a principle for the practical application.

2 Definitions and Methods

The claims for the next period (normally one year) of the different positions of an insurance portfolio are given by random variables $S_1, \ldots, S_n$. Under the assumption that cost, commission and interest on risk capital (fluctuation loading) are paid in advance, the premiums are understood to be equal to $E[S_1], \ldots, E[S_n]$. We receive the future income for each claim by $X_i = E[S_i] - S_i$, $i = 1, \ldots, n$. Consequently, the future incomes of all subportfolios have expectation 0.

Now an exogenously quantified risk capital has to be allocated in a "fair" way (fair means according to the risk involved) to the different positions of the portfolio. The applied method consists of two components:

(i) a risk measure and

(ii) an allocation method.

In this context a **risk measure** ρ is a mapping

$$\rho : L \to \mathbb{R},$$

where L is the set of real-valued random variables. (All random quantities in this paper are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.) A risk measure ρ can be seen as the mathematical version of a subjective risk concept, i.e. for two random variables X and Y with X more risky than Y we should have $\rho(X) > \rho(Y)$. Already here we could think of $\rho(X)$ as the risk capital allocated to the risk X.

One axiomatic approach to select risk measures by properties should be mentioned here: The concept of coherent risk measures was introduced by Artzner et al. [2]. They are playing an increasingly important role in academic and practitioners' debates on risk measurement and risk management. Albrecht [1] from the point of view of an insurer investigates this concept as well as other existing concepts in the literature.

Definition 2.1 *A risk measure ρ is called* **coherent** *if it satisfies the following properties:*

(i) **Translation invariance:** $\rho(X + \alpha) = \rho(X) - \alpha$ *for all* $X \in L$ *and* $\alpha \in \mathbb{R}$.

(ii) **Subadditivity:** $\rho(X + Y) \leq \rho(X) + \rho(Y)$ *for all* $X, Y \in L$.

(iii) **Positive homogeneity:** $\rho(\lambda X) = \lambda \rho(X)$ *for all* $X \in L$ *and all* $\lambda \geq 0$.

(iv) **Monotonicity:** *$\rho(X) \leq \rho(Y)$ for all $X, Y \in L$ with*[1] *$X \geq Y$ $\mathbb{P}$-a.s.*

We only work with centred random variables X_i, i.e. we restrict the space L to the space L^0, which consists of all real-valued centred random variables. In L^0 translation invariance is obsolete. Moreover, monotonicity is reduced to $X = Y$ P-a.s. $\Rightarrow \rho(Y) = \rho(X)$, which is for all risk measures considered in this paper always satisfied. So in our context the check up on subadditivity and positive homogeneity is already sufficient to guarantee coherence of a risk measure.

Risk measures should measure rare events, e.g. large claims (earthquake or storm losses) causing high fluctuations in the portfolio. Consequently, some risk measures are based on quantiles. For a random variable X and $\alpha \in (0,1)$ we define the **upper $\boldsymbol{\alpha}$-quantil**

$$Q^\alpha(X) = \sup\{x \in \mathbb{R} \mid \mathbb{P}[X \leq x] \leq \alpha\}\,.$$

For $x \in \mathbb{R}$ we define as usual $x_+ = \max(x, 0)$ and $x_- = (-x)_+$.

Definition 2.2 *For $X \in L$ we define the following risk measures, where we assume that they exist finitely.*

(i) **Variance:** $\rho_{\text{var}}(X) = \text{var}[X]\,.$

(ii) **Standard deviation:** $\rho_{\text{sd}}(X) = \sigma[X] = \sqrt{\text{var}[X]}\,.$

(iii) **Semi-variance:** $\rho_{\text{svar}}(X) = E[((X - EX)_-)^2]\,.$

(iv) **Value-at-Risk (VaR)** $\rho_{\text{VaR}(\alpha)}(X) = -Q^\alpha(X)\,.$

(v) **Expected shortfall (ES):**

$$\rho_{\text{ES}(\alpha)}(X) = -\frac{1}{\alpha}\Big(E[X\mathbf{1}_{\{X \leq Q^\alpha(X)\}}] + Q^\alpha(X)\big(\alpha - \mathbb{P}[X \leq Q^\alpha(X)]\big)\Big).$$

In the following we present some basic properties of these risk measures.

ρ_{var} is the classical risk measure used for instance in statistics, but also in insurance pricing as well as in the context of portfolio optimization. The difference between ρ_{sd} and ρ_{var} is only a square root. However, ρ_{sd} is coherent, but ρ_{var} is not.

The simple but fundamental idea which leads to the semi-variance (as to other downward risk measures) is that only events below a certain target value are considered as risky. We have chosen for ρ_{svar} the expectation as target value, which is in L^0 equal to 0; i.e. $\rho_{\text{svar}}(X) = E(X_-)^2$.

Although ρ_{svar} and ρ_{var} are not coherent, we include both in our study, since they are classical risk measures in insurance. Applying the downside risk restriction to the standard deviation leads to the semi-standard deviation, which is again coherent (see Fischer [9]).

[1] The applicable condition is $X \geq Y$ in the case of result distributions and $X \leq Y$ on loss distributions.

The most common risk measure in the banking sector is $\rho_{\mathrm{VaR}(\alpha)}$, becoming even more important with Basel II. It is also related to the insurance notions of ruin probability and probable maximum loss, thus it measures only extreme events.

$\rho_{\mathrm{ES}(\alpha)}$ is the most prominent coherent risk measure. For a random variable with continuous distribution function the second sum in the definition of $\rho_{\mathrm{ES}(\alpha)}$ is 0 and $\mathbb{P}[X \leq Q^\alpha(X)] = \alpha$. In this case $\rho_{\mathrm{ES}(\alpha)}$ is the negative of the conditional expectation $E[X|X \leq Q^\alpha(X)]$. Tasche [16, 17] presents many important properties of $\rho_{\mathrm{ES}(\alpha)}$ in great detail.

Table 1 shows, which of the five risk measures satisfy subadditivity or positive homogeneity. This remains true even for all random variables in L. For centred random variables this means that the corresponding risk measures are coherent.

Table 1: Properties of risk measures

Property	**variance** ρ_{var}	**standard deviation** ρ_{sd}	**semi-variance** ρ_{svar}	**Value at Risk** $\rho_{\mathrm{VaR}(\alpha)}$	**Expected Shortfall** $\rho_{\mathrm{ES}(\alpha)}$
Subadditivity		$\surd$		$\surd^{1)}$	$\surd$
Positive homogeneity		$\surd$		$\surd$	$\surd$

1) for jointly elliptically distributed random variables X, Y (see Embrechts et al. [8]).

From a naive point of view portfolio positions, where a small risk capital is allocated, may be considered as less risky than those where a high risk capital is allocated. However, this obviously depends on the size of the portfolio position. The size can for example be determined by the premium volumes which can be seen as exogenous parameters. The idea is now to scale each portfolio position by its size and hence allocate risk capital to each portfolio position per unit premium.

The random vector $X = (X_1, \ldots, X_n)'$, which represents the future incomes of the model points, can be scaled by the size of the model points $\lambda = (\lambda_1, \ldots, \lambda_n)'$, i.e. $X = (X_1(\lambda_1), \ldots, X_n(\lambda_n))'$, and $Z(\lambda) = X_1(\lambda_1) + \cdots + X_n(\lambda_n)$. Then $\partial\rho(Z(\lambda))/\partial\lambda_i$ stands for the partial derivative of the risk measure of the whole portfolio with respect to λ_i. They have been calculated for various examples of risk measures by Kalkbrener [10], Rodriguez [13] and Tasche [15]).

We now turn to the second component for the allocation of risk capital; i.e. an appropriate capital allocation method. We denote by R the set of risk measures for centred random variables. A **capital allocation method** Φ is a mapping

$$\Phi : R \times (L^0)^n \to \mathbb{R}^n, \quad (\rho, X_1, \ldots, X_n) \mapsto \begin{pmatrix} \Phi_1(\rho, X_1, \ldots, X_n) \\ \vdots \\ \Phi_n(\rho, X_1, \ldots, X_n) \end{pmatrix},$$

with $\sum_{i=1}^n \Phi_i = 1$ and $\Phi_i \geq 0$ for $i = 1, \ldots, n$. For further use we abbreviate $N = \{1, \ldots, n\}$

With Φ_i we identify the allocation coefficients; i.e. the proportions of risk capital K to be allocated to model point i for $i \in N$.

Definition 2.3 *Assume* $(X_1, \ldots, X_n)' = (X_1(\lambda_1), \ldots, X_n(\lambda_n))'$ *is a random vector, which stands for the future incomes corresponding to the model points,* $\lambda = (\lambda_i, \ldots, \lambda_n)'$ *represents the size of the model points. Define* $Z : \mathbb{R}^n \to L^0$ *with* $Z(\lambda) = \sum_{i=1}^n X_i(\lambda_i)$ *and* ρ *a risk measure whose partial derivatives with respect to* λ_i *exist for all* $i \in N$.

We define the **allocation coefficients** *for* $i \in N$ *for the different capital allocation methods by*

(i) **proportional**

$$\Phi_i^{p,\rho} = \frac{\rho(X_i)}{\sum\limits_{j \in N} \rho(X_j)} ,$$

(ii) **Merton & Perold [11]**

$$\Phi_i^{MP,\rho} = \frac{\rho(Z) - \rho(Z - X_i)}{\sum\limits_{j \in N} \Big[\rho(Z) - \rho(Z - X_j)\Big]} ,$$

(iii) **Myers & Read [12]**

$$\Phi_i^{MR,\rho} = \frac{\partial \rho\big(Z(\lambda)\big)}{\partial \lambda_i} \Big/ \sum_{j=1}^{n} \frac{\partial \rho\big(Z(\lambda)\big)}{\partial \lambda_j} ,$$

(iv) **Shapley [14]**

$$\Phi_i^{S,\rho} = \frac{1}{\rho(Z)} \sum_{S \subseteq N} \frac{(|S|-1)!(n-|S|)!}{n!} \left(\rho\Big(\sum_{j \in S} X_j\Big) - \rho\Big(\sum_{j \in S \setminus \{i\}} X_j\Big) \right) \text{ and}$$

(v) **Aumann & Shapley [4]**

$$\Phi_i^{AS,\rho} = \frac{\lambda_i}{\rho(Z(\lambda))} \int_0^1 \frac{\partial \rho(Z(t\lambda))}{\partial \lambda_i} \, dt .$$

For a positively homogeneous risk measure ρ *this simplifies to*

$$\Phi_i^{AS,\rho} = \frac{1}{\rho(Z(\lambda))} \frac{\partial \rho\big(Z(\lambda)\big)}{\partial \lambda_i} .$$

The allocation methods according to Merton & Perold [11] and to Myers & Read [12] are based on the option pricing model of a firm. This means that risk is measured by the price of an insolvency put option. The insolvency put option refers to the loss of policyholders given that the insurer defaults. For more details we refer to the original literature. Both the allocation methods according to Shapley [14] and to Aumann & Shapley [4] are based on the theory of co-operative games. Denault [6] gives a different view of the connection between the allocation principle according to Shapley and coherent risk measures.

3 Equivalences of allocation principles

In the previous section we have formally introduced **allocation principles** based on combinations of **risk measures** and **allocation methods**. An important allocation principle in practice, which does not fit into this framework, is the so-called covariance principle, which we nevertheless want to include in our analysis.

Definition 3.1 *Suppose $(X_1, \ldots, X_n)'$ is a random vector, which stands for the future income of the model points and $Z = \sum_{i=1}^{n} X_i$.*

Then we define the **allocation coefficients of the covariance principle** *by*

$$\Phi_i^{CP} = \frac{\mathrm{cov}[X_i, Z]}{\sum\limits_{i=1}^{n} \mathrm{cov}[X_i, Z]} = \frac{\mathrm{cov}[X_i, Z]}{\mathrm{var}[Z]} \quad \text{for } i = 1, \ldots, n.$$

Since the covariance principle can not be considered in the framework of compositions of a risk measures and allocation methods it is not consistent with the approach taken formerly in this paper. But it is an ad hoc approach widely used in the insurance industry. Therefore, it is of basic interest to investigate, which allocation principles are equivalent to the covariance principle.

In the following we give a short overview about some statements on the equivalences of allocation principles and the covariance principle.

Theorem 3.2 *Suppose $X = (X_1(\lambda_1), \ldots, X_n(\lambda_n))'$ is a scalable random vector and $Z : \mathbb{R}^n \to L$ is a mapping with $Z(\lambda) = \sum_{i=1}^{n} X_i(\lambda_i) = \sum_{i=1}^{n} \lambda_i \tilde{X}_i = \lambda' \tilde{X}$ (linear scaling to be assumed).*

The following allocation principles of X are equivalent to the covariance principle, i.e. the allocation coefficients of the following allocation methods coincide with the allocation coefficients of the covariance principle (we assume that the risk measures exist finitely):

(i) proportional allocation with ρ_{var}, if the random variables X_i are pairwise uncorrelated;

(ii) proportional allocation with ρ_{sd}, if the random variables X_i are pairwise completely correlated

(iii) allocation according to Merton & Perold with ρ_{var}, if the random variables X_i are pairwise uncorrelated

(iv) allocation according to Myers & Read with ρ_{var} and ρ_{sd};

(v) allocation according to Shapley with ρ_{var};

(vi) allocation according to Aumann & Shapley with ρ_{var} and ρ_{sd};

Proof The following results hold for all $i \in N$.
(i): If all X_i are pairwise uncorrelated, then

$$\Phi_i^{CP} = \frac{\text{cov}[X_i, Z]}{\sum\limits_{j \in N} \text{cov}[X_j, Z]} = \frac{\text{var}[X_i]}{\sum\limits_{j \in N} \text{var}[X_j]} = \Phi_i^{p,\text{var}}.$$

(ii): If all X_i are pairwise completely correlated, then

$$\Phi_i^{CP} = \frac{\text{cov}[X_i, Z]}{\sum\limits_{j \in N} \text{cov}[X_i, Z]} = \frac{\sum\limits_{j \in N} \text{cov}[X_i, X_j]}{\sum\limits_{j,k \in N} \text{cov}[X_j, X_k]} = \frac{\sum\limits_{j \in N} \pm\, \sigma[X_i]\sigma[X_j]}{\sum\limits_{j,k \in N} \pm\, \sigma[X_j]\sigma[X_k]} = \frac{\sigma[X_i]}{\sum\limits_{j \in N} \sigma[X_j]} = \Phi_i^{p,sd}.$$

(iii): If all X_i are pairwise uncorrelated, then

$$\Phi_i^{CP} = \frac{\text{cov}[X_i, Z]}{\sum\limits_{j \in N} \text{cov}[X_j, Z]} = \frac{\text{var}[X_i]}{\sum\limits_{j \in N} \text{var}[X_j]} = \frac{\text{var}[Z] - \text{var}[Z - X_i]}{\sum\limits_{j \in N} \Big(\text{var}[Z] - \text{var}[Z - X_j]\Big)} = \Phi_i^{MP,\text{var}}.$$

(iv): Since the derivative of the variance in direction of the model point X_i is $\dfrac{\partial \text{var}[Z(\lambda)]}{\partial \lambda_i} = 2\,\text{cov}[X_i, Z]$, we have

$$\Phi_i^{MR,\text{var}} = \frac{\partial \text{var}[Z(\lambda)]}{\partial \lambda_i} \Bigg/ \sum_{j=1}^{n} \frac{\partial \text{var}[Z(\lambda)]}{\partial \lambda_j} = \frac{\text{cov}[X_i, Z]}{\sum\limits_{j=1}^{n} \text{cov}[X_j, Z]} = \Phi_i^{CP}.$$

Analogously, since the derivative of the standard deviation in direction of the model point X_i is $\dfrac{\partial \sigma[Z(\lambda)]}{\partial \lambda_i} = \dfrac{\text{cov}[X_i, Z]}{\sqrt{\text{var}[Z]}}$, the same arguments hold as before.

(v): The numerator of $\Phi_i^{S,\mathrm{var}}$ is

$$\begin{aligned}
&\sum_{S\subseteq N}\frac{(|S|-1)!(n-|S|)!}{n!}\left(\mathrm{var}\Big[\sum_{j\in S}X_j\Big]-\mathrm{var}\Big[\sum_{j\in S\setminus\{i\}}X_j\Big]\right)\\
&=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{(|S|-1)!(n-|S|)!}{n!}\left(2\,\mathrm{cov}\Big[X_i,\sum_{j\in S\setminus\{i\}}X_j\Big]+\mathrm{var}[X_i]\right)\\
&=\sum_{s=1}^{n}\sum_{\substack{|S|=s\\ i\in S}}\frac{(s-1)!(n-s)!}{n!}\left(2\sum_{j\in S\setminus\{i\}}\mathrm{cov}\Big[X_i,X_j\Big]+\mathrm{var}[X_i]\right)\\
&=\sum_{s=1}^{n}\left(\frac{(s-1)!(n-s)!}{n!}\binom{n-1}{s-1}\left(\binom{n-2}{s-2}\Big/\binom{n-1}{s-1}\right)2\sum_{j\in N\setminus\{i\}}\mathrm{cov}[X_i,X_j]\right)\\
&\quad+\sum_{s=1}^{n}\frac{(s-1)!(n-s)!}{n!}\binom{n-1}{s-1}\mathrm{var}[X_i]\\
&=\sum_{s=1}^{n}\left(\frac{2(s-1)}{n(n-1)}\sum_{j\in N\setminus\{i\}}\mathrm{cov}[X_i,X_j]+\frac{1}{n}\mathrm{var}[X_i]\right)\\
&=\mathrm{cov}[X_i,Z].
\end{aligned}$$

(vi):

$$\int_0^1\frac{\partial\mathrm{var}[t\lambda'\tilde{X}]}{\partial\lambda_i}\,dt=\int_0^1 t^2\frac{\partial\mathrm{var}[\lambda'\tilde{X}]}{\partial\lambda_i}\,dt=2\int_0^1 t^2\,\mathrm{cov}[X_i,Z]\,dt=\frac{2}{3}\mathrm{cov}[X_i,Z].$$

Because of the homogeneity of ρ_{sd} and the linear scaling the allocation according to Myers & Read is identical to the allocation according to Aumann & Shapley (see Denault [6] Lemma 1). The proof now follows from (v). □

Remark (a) It should be stressed that the equivalences in (iv) and (vi) only hold because of the linear scaling of the random variables concerning portfolio size.
(b) Theorem 3.2 shows that the proportional allocation with variance and standard deviation are only compatible with the covariance principle for extreme dependency concepts.

A further statement concerning the relationship of the covariance principle towards the proportional allocation and the allocation according to Merton & Perold is given in the following theorem.

Theorem 3.3 *Suppose $(X_1,\dots,X_n)'$ is a random vector. Let $i\in N$. Then one of the cases (i)-(iii) holds.*

(i) $\Phi_i^{p,\mathrm{var}}<\Phi_i^{CP}<\Phi_i^{MP,\mathrm{var}}$

(ii) $\Phi_i^{p,\mathrm{var}} > \Phi_i^{CP} > \Phi_i^{MP,\mathrm{var}}$

(iii) $\Phi_i^{p,\mathrm{var}} = \Phi_i^{CP} = \Phi_i^{MP,\mathrm{var}}$.

If the random variable X_i is uncorrelated to $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$, then either (ii) or (iii) hold.

Proof Assume $Z = \sum_{i=1}^n X_i$. Immediately by the definition,

$$\begin{aligned}\Phi_i^{MP,\mathrm{var}} &= \Phi_i^{p,\mathrm{var}} + \frac{2\,y}{\left(2\,\mathrm{var}[Z] - \sum\limits_{j=1}^{n}\mathrm{var}[X_j]\right)\sum\limits_{j=1}^{n}\mathrm{var}[X_j]} \\ &= \Phi_i^{CP} + \frac{y}{\left(2\,\mathrm{var}[Z] - \sum\limits_{j=1}^{n}\mathrm{var}[X_j]\right)\mathrm{var}[Z]},\end{aligned} \tag{1}$$

where

$$y = \mathrm{cov}[X_i, Z]\sum_{j=1}^{n}\mathrm{var}[X_j] - \mathrm{var}[X_i]\mathrm{var}[Z]. \tag{2}$$

Since the denominator of both ratios in (1) is always positive, it follows from (1) and (2) that

$$\begin{aligned}&\Phi_i^{p,\mathrm{var}} > \Phi_i^{CP} \Leftrightarrow \Phi_i^{MP,\mathrm{var}} < \Phi_i^{CP} \Leftrightarrow \Phi_i^{MP,\mathrm{var}} < \Phi_i^{p,\mathrm{var}} \quad \text{and} \\ &\Phi_i^{p,\mathrm{var}} < \Phi_i^{CP} \Leftrightarrow \Phi_i^{MP,\mathrm{var}} > \Phi_i^{CP} \Leftrightarrow \Phi_i^{MP,\mathrm{var}} > \Phi_i^{p,\mathrm{var}}.\end{aligned}$$

From this (i) and (ii) follow immedately; for $y = 0$ (iii) is obvious. □

4 The example portfolio

4.1 Definition of the model

We consider a portfolio of seven different claim distributions represented by the random variables S_i, $i = 1, \ldots, 7$, which will be specified below. By the transformation $X_i = E[S_i] - S_i$ they are converted into future income distributions with expectations 0. The distributions, which we use for our model, are the Poisson, Pareto and lognormal distribution. They are defined as follows.

Definition 4.1 *(i) A random variable N is* **Poisson** *distributed with parameter $\tau \in (0, \infty)$, if it is concentrated on $\mathbb{N}_0$ and*

$$\mathbb{P}[N = n] = \frac{\tau^n}{n!}e^{-\tau}, \quad n \in \mathbb{N}_0$$

It has $E[N] = \text{var}[N] = \tau$.
(ii) A random variable X is **Pareto** *distributed with form parameter* $\gamma \in (0,\infty)$, *scale parameter* $b \in (0,\infty)$ *and shift parameter* $s \in \mathbb{R}$, *if it has density*

$$f_X(x) = \frac{\gamma}{b}\Big(\frac{x-s}{b}\Big)^{-\gamma-1}\mathbf{1}_{[b+s,\infty)}(x)\,.$$

It has $E[X] = b^2\frac{\gamma}{\gamma-1}$ *and* $\text{var}[X] = b^2\Big(\frac{\gamma}{\gamma-2} - \Big(\frac{\gamma}{\gamma-1}\Big)^2\Big)$.
(iii) A random variable X is **lognormally** *distributed with shift parameter* μ, *if it has density*

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2x^2}}\exp\Big(\frac{-(\ln x-\mu)^2}{2\sigma^2}\Big)\mathbf{1}_{(0,\infty)}(x)\,.$$

It has $E[X] = \exp(\mu+\sigma^2/2)$ *and standard deviation* $\sqrt{\text{var}[X]} = \sqrt{\exp(2\mu+\sigma^2)(\exp(\sigma^2)-1)}$.

The different models for the X_i, $i = 1,\ldots,7$ have been chosen by their importance in practice based on long experience. Storms, earthquakes, major engineering and fire claims happen rarely, but when they happen, they can be very large. Consequently, they are modeled by a compound Poisson model with Pareto claim sizes. On the other hand, basic losses in general liability, engineering and fire are not quite as rare and not quite as large; hence a lognormal distribution fits such data very well.

In our examples, for practical reasons, some additional transformations have been introduced into the models. First, the Pareto distributions F_i are truncated at ξ_i for $i = 1,\ldots,4$. The truncated distribution functions are then renormalized by $1 - F_i(\xi_i)$. Furthermore, one of the Pareto distribution is shifted; the amount is given by a shift parameter. Finally, a scale parameter is additionally introduced into the lognormal model.

The different parameters for the collective Poisson/Pareto models are given in Table 2, the parameters for the lognormal distributions in Table 3. These are realistic fits to business data.

The major losses modeled via Poisson/Pareto can be assumed to be independent. Within the basic losses one observes a certain dependency; this is modeled by a Gaussian copula with a pairwise rank correlation coefficient of 0.14.

Table 2: Parameters for the Poisson/Pareto model

Name	**abbre-viation**	**Poisson**	**Pareto**			
		τ	γ	**b**	ξ	**s**
Storm	S	2.43	0.65	1	250	-1
Earthquake	EQ	0.15	0.42	2	634	0
Engineering (major losses)	E(ML)	0.22	0.98	3	200	0
Fire (major losses)	F(ML)	1.57	1.3	4	200	0

Table 3: Parameters for the lognormal model

Name	abbre-viation	mean	standard deviation	scale
General Liability (basic losses)	GL(BL)	0.98	0.120	350
Engineering (basic losses)	E(BL)	0.98	0.105	60
Fire (basic losses)	F(BL)	0.90	0.085	350

4.2 The simulation study

4.2.1 The simulation method

The simulations and calculations were performed on a Intel Pentium III, 1GHz, 256MB RAM machine invoking routines from the simulation software program @Risk$^{\text{TM}}$ in connection with Microsoft Excel$^{\text{TM}}$.

For the computation of the five risk measures defined in Definition 2.2 we simulated $k = 30\,000$ realisations of the random vector $X = (X_1, \dots, X_7)$ from the above marginal models with the Gaussian dependency structure of the three basic losses.

Let X here stand for any of the components of $(X_1, \dots, X_7)$. The estimators for the risk measures for observations x_i, $i = 1, \dots, k$, of the random variable X were then calculated componentwise for the different model points; $\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$.

(i) $\hat{\rho}_{\text{var}}(X) = \frac{1}{k-1}\sum_{i=1}^{k}(x_i - \bar{x}_i)^2$

(ii) $\hat{\rho}_{\text{sd}}(X) = \sqrt{\hat{\rho}_{\text{var}}(X)}$

(iii) $\hat{\rho}_{\text{svar}}(X) = \frac{1}{k-1}\sum_{i=1}^{k}(x_i - \bar{x}_i)^2_-$

For the sample $(X_1, \dots, X_k)$ we denote by $X_{(1)} < \dots < X_{(k)}$ its order statistics. By $\lfloor x \rfloor$ we denote the integer part of $x \in \mathbb{R}$.

(iv) $\hat{\rho}_{\text{VaR}(\alpha)}(X) = X_{(\lfloor k\alpha \rfloor)}$

(v) $\hat{\rho}_{\text{ES}(\alpha)}(X) = \frac{1}{\lfloor k\alpha \rfloor}\sum_{i=1}^{\lfloor k\alpha \rfloor} X_{(i)}$

The estimators of $\rho_{\text{VaR}(\alpha)}$ and $\rho_{\text{ES}(\alpha)}$ given in (iv) and (v) make only sense if the quantile can be computed by the empirical method; i.e. if it lies within the range of the data. However, with a sample size of 30 000 this did not cause problems for the chosen quantiles $\alpha \in \{0.1, 0.05, 0.01\}$. Alternatively to the empirical method, extreme value methods are available to compute extreme quantiles (see Embrechts et al. [7]). We also want to remark that it is rather wasteful to simulate 30 000 data points and use only α percent for the computation of $\rho_{\mathrm{VaR}(\alpha)}$ and $\rho_{\mathrm{ES}(\alpha)}$. Here special simulation methods like importance sampling can be applied (see e.g. Asmussen [3]). Such considerations, however, are left for further investigations.

With these computed risk measures we calculate now the coefficients of the allocation methods as given in Definition 2.3. The results are summarized in Tables 4-6.

The allocation according to Myers & Read and to Aumann & Shapley could not be computed; the problem are the derivatives. The differences $\rho(\lambda_1 X_1 + \cdots + \lambda_n X_n) - \rho((\lambda_1 + \Delta)X_1 + \cdots + \lambda_n X_n)$ for the calculation of the derivatives are not stable for small Δ. Similarly, numerical instabilities were to be expected for these cases. Here numerical methods other than the brute force methods we used are called for. Our analysis can only be considered as a first step to the practical implementation of sophisticated allocation methods. The self-imposed restriction to simulation approaches with 30 000 iterations lead to operability even for large portfolios consisting of more than 100 segments.

To investigate the quality of the estimators of the allocation coefficients given in Tables 4-6 also the mean square errors (MSE) of the estimated allocation coefficients based on 50 simulation runs were computed. The results can be found in Urban [18]

4.2.2 Interpretation of Results

The allocation coefficients for the different combinations of risk measures given in Definition 2.2 and allocation methods as defined in Definition 2.3 are given in Tables 4-6. We have also visualized these results in Figures 1-3.

A first look at Figures 1-3 immediately draws attention to an outlier in the EQ model point, which comes from the allocation according to Merton & Perold in combination with $\rho_{\mathrm{ES}(0.01))}$. With its parameters the EQ-distribution has the heaviest tail and hence for small α the expected shortfall $\mathrm{ES}(\alpha)$ is much bigger than for any other distribution of our model. For this method it is remarkable that $\rho_{\mathrm{ES}(0.01)}$ concentrates more than half of the risk capital to EQ, whereas $\rho_{\mathrm{VaR}(0.01)}$ distributes the risk capital - even for large risks - not in such an extreme way. In the context of $\rho_{\mathrm{ES}(\alpha)}$ and $\rho_{\mathrm{VaR}(\alpha)}$ the allocation coefficients strongly depend on the quantile α. The impact of different α in particular for heavy tailed random variables can clearly be seen by the coefficients of EQ. This means that ES can only be used if the tails of the distributions are well known and modelled, which is usually not the case for all parts of an insurance portfolio.

Another observation is the small range of the allocation coefficients for E(BL). This is due to the concentration of the distribution around 0 and so the risk of E(BL) is small, independent of the risk measure or the allocation method.

In Theorem 3.2(i) it is shown that for uncorrelated random variables the proportional allocation method with the variance as risk measure is equivalent to the covariance principle. In this model, only three random variables are correlated; the other random variables are uncorrelated. Thus the allocation coefficients of these two methods should be quite

Table 4: Allocation coefficients for the proportional method

distri- bution	allocation coefficients[1)] in % for var	 sd	 svar	 $VaR_{0.01}$	 $VaR_{0.05}$	 $VaR_{0.1}$	 $ES_{0.01}$	 $ES_{0.05}$	 $ES_{0.1}$
S	28.6	22.5	32.0	26.8	32.3	26.5	22.4	28.0	28.5
EQ	21.9	19.6	28.0	23.7	3.9	-1.2	32.1	19.5	14.2
GL(BL)	24.9	21.0	18.0	14.6	24.2	31.3	12.1	16.7	19.8
E(BL)	0.6	3.2	0.4	2.2	3.6	4.7	1.8	2.5	2.9
E(ML)	2.0	5.9	2.4	6.8	3.6	2.2	8.9	6.4	5.4
F(BL)	12.5	14.9	8.8	10.1	16.9	22.1	8.3	11.6	13.8
F(ML)	9.6	13.0	10.3	15.7	15.5	14.5	14.4	15.4	15.3

1) allocation coefficients are rounded to one decimal.

Table 5: Allocation coefficients according to Merton & Perold

distri- bution	allocation coefficients[1)] in % for var	 sd	 svar	 $VaR_{0.01}$	 $VaR_{0.05}$	 $VaR_{0.1}$	 $ES_{0.01}$	 $ES_{0.05}$	 $ES_{0.1}$
S	25.2	25.5	28.9	38.2	40.7	29.2	19.3	36.5	35.9
EQ	19.3	19.1	26.7	33.4	13.6	7.0	64.1	32.8	24.1
GL(BL)	27.3	27.8	20.6	13.2	19.2	31.1	7.4	13.4	17.9
E(BL)	2.1	2.0	1.6	1.0	1.6	2.2	0.5	1.0	1.4
E(ML)	1.7	1.6	1.8	1.3	2.3	2.3	0.7	1.5	1.9
F(BL)	16.0	15.8	12.0	7.1	11.5	17.1	4.3	7.7	10.2
F(ML)	8.5	8.2	8.4	6.0	11.0	11.1	3.7	7.1	8.8

1) allocation coefficients are rounded to one decimal.

Table 6: Allocation coefficients according to Shapley

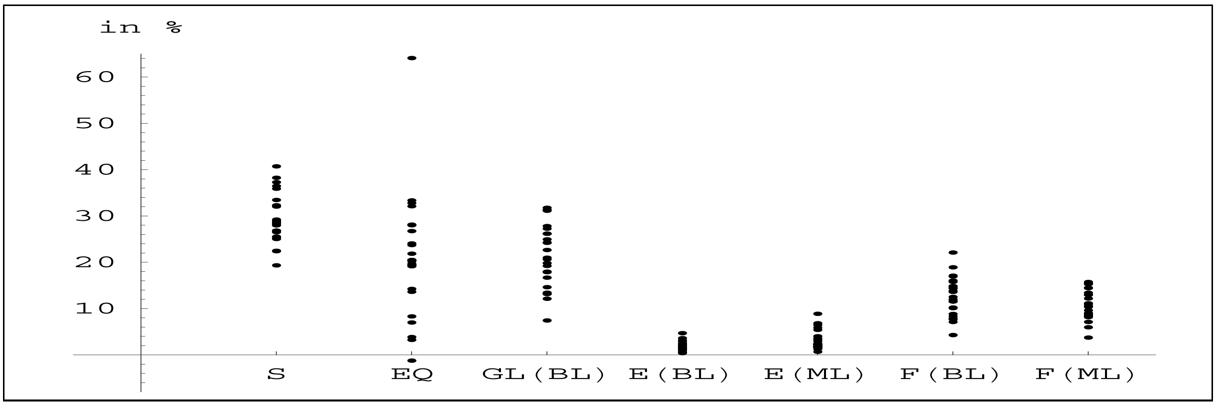
distri- bution	allocation coefficients[1)] in % for var[3)]	 sd	 svar[2)]	 $VaR_{0.01}$	 $VaR_{0.05}$	 $VaR_{0.1}$	 $ES^{2)}_{0.01}$	 $ES^{2)}_{0.05}$	 $ES^{2)}_{0.1}$
S	26.8	25.0	-	33.5	37.3	28.3	-	-	-
EQ	20.5	20.2	-	28.1	8.3	3.2	-	-	-
GL (BL)	26.2	24.3	-	13.2	22.7	31.8	-	-	-
E (BL)	1.4	1.9	-	1.2	1.8	2.5	-	-	-
E (ML)	1.8	3.3	-	4.0	2.9	2.3	-	-	-
F (BL)	14.4	14.7	-	7.9	13.6	18.9	-	-	-
F (ML)	9.0	10.6	-	12.2	13.4	12.9	-	-	-

1) allocation coefficients are rounded to one decimal;

2) could not be calculated due to memory restrictions;

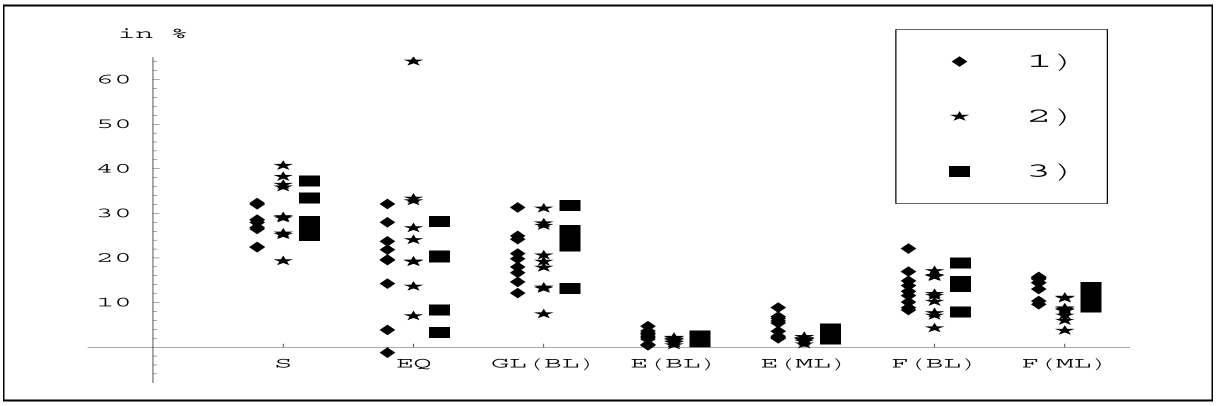
3) allocation method according to Shapley with variance as risk measure is equivalent to the covariance principle; see Theorem 3.2.

Figure 1: Fluctuation margin of the allocation coefficients[1)] in %



1) all estimated allocation coefficients for the seven distributions are drawn

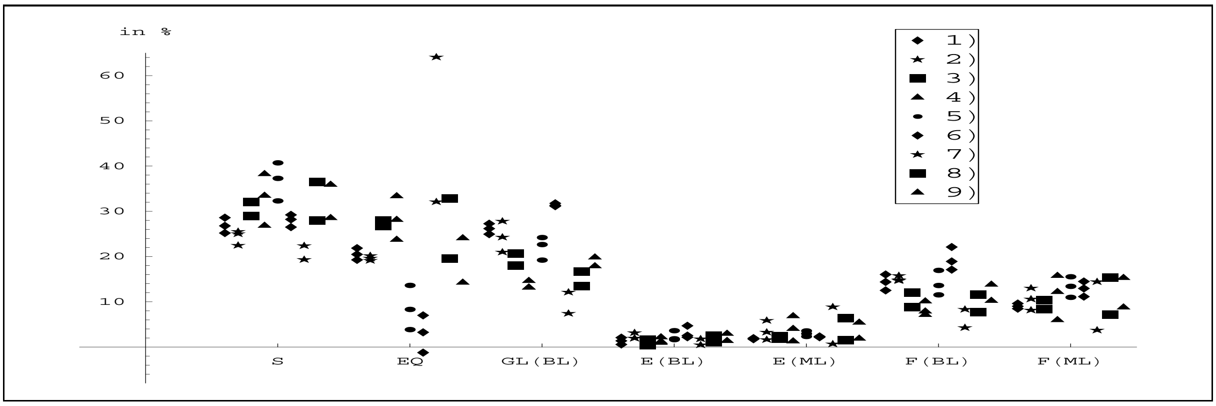
Figure 2: Allocation coefficients[4)] in % grouped by the allocation methods



1) proportional method 2) Merton & Perold 3) Shapley;

4) all estimated allocation coefficients for the seven distributions are drawn.

Figure 3: Allocation coefficients[10)] in % grouped by the risk measures



1) variance 2) standard deviation 3) semi-variance 4) $\mathrm{VaR}_{0.01}$ 5) $\mathrm{VaR}_{0.05}$ 6) $\mathrm{VaR}_{0.1}$ 7) $\mathrm{ES}_{0.01}$ 8) $\mathrm{ES}_{0.05}$ 9) $\mathrm{ES}_{0.1}$ 10) all estimated allocation coefficients for the risk measures in 1)-9) are drawn

similar. This is indeed the case.

On the other hand, we want to emphasize that the allocation method according to Merton & Perold takes care of the dependency of the random variables, but the proportional allocation method does not. Therefore for a "good" risk measure like the standard deviation we would rather think that

$$\Phi_i^{p,sd} \leq \Phi_i^{MP,sd} \quad \forall i \in J\,,$$

if the random variables $(X_i)_{i\in J}$ are pairwise positively correlated and uncorrelated otherwise. But in Tables 4 and 5 we have

$$\Phi_{E(BL)}^{MP,sd} < \Phi_{E(BL)}^{p,sd} \quad \text{and} \quad \Phi_S^{MP,sd} > \Phi_S^{p,sd},$$

contradicting our expectations.

Looking for advise about which allocation method in combination with which risk measure to use, seems to leave us at loss at first. A drawback to note in Figure 1 is that the allocation coefficients are not concentrated around a special value instead they cover the whole range uniformly. It is also not possible to distinguish the distributions by the range of their allocation coefficients. For example E(BL) (lognormal) has the smallest range of all distributions but GL(BL) (lognormal) has the second largest range.

However, a more thorough analysis of our results seems to indicate an interesting direction. Given the large ranges of the allocation coefficients in Figure 1, we distinguished all allocation coefficients by allocation method in Figure 2 and by risk measure in Figure 3.

Figure 2 indicates that the coefficients even for the same allocation method do not concentrate around a special value but again cover the whole range uniformly. Figure 3, however, leads to new insight: the range of the allocation coefficients for a concrete combination of a distribution and a risk measure is in the majority of cases small compared to the range of all coefficients of the distribution.

This leads to the conjecture, that the allocation coefficients are influenced more strongly by the chosen risk measure than by the allocation method. If this conjecture is true, then insurance companies should choose their risk measure very accurately, but could choose a simple allocation principle.

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