Computing Optimal Descriptions of Stratifications of Actions of Compact Lie Groups

Thomas Bayer

Institut für Informatik
Technische Universität München
Boltzmannstr. 3, 85748 Garching, Germany
bayert@in.tum.de
www: http://www14.in.tum.de/personen/bayert

Abstract. We provide a constructive approach to the stratification of the representation- and the orbit space of linear actions of compact Lie groups contained in $GL_n(\mathbf{R})$ on $\mathbf{R}^n$ and we show that any d-dimensional stratum, respectively, its closure can be described by d sharp, respectively, relaxed polynomial inequalities and that d is also a lower bound for both cases. Strata of the representation space are described as differences of closed sets given by polynomial equations while d-dimensional strata of the orbit space are represented by means of polynomial equations and inequalities. All algorithms have been implemented in SINGULAR V2.0.

Introduction

In 1983 Abud and Sartori [1] pointed out the relation between spontaneous symmetry breaking and stratifications of linear actions of compact Lie groups and presented several applications in particle physics. Spontaneous symmetry breaking can briefly be described as follows. Let G be a compact Lie group which acts linearly on $\mathbf{R}^n$, let $\phi_0 \in \mathbf{R}^n$ be the ground state of a physical system and let $V_\gamma(z)$ be a G-invariant potential which determines ϕ_0 and depends on the parameter γ. Varying γ might change ϕ_0 into ϕ_0' and the stabilizer group $G_{\phi_0'}$ of ϕ_0' may be "smaller" than the stabilizer G_{ϕ_0} (i.e., moving from ϕ_0 to ϕ_0' amounts to a loss of symmetry), which can be seen as a breaking of symmetry. In this way various patterns of spontaneous symmetry breaking occur, which correspond to distinct phases of the model. It is well-known (see for instance [15]) that the orbit space $\mathbf{R}^n/G$ is a semialgebraic set and there exists a disjoint decomposition of $\mathbf{R}^n/G$ in finitely many semialgebraic sets, called strata, whereas any stratum consists of points of the same symmetry type. The knowledge of a description of each stratum in terms of polynomial equations and inequalities is important for numerous applications (e.g., construction of invariant potentials, symmetric bifurcation theory, see [1], [4], [9], [10]).

There are several approaches for constructing the stratification of the orbit space of a compact Lie group[1] starting with Abud and Sartori, see [1], while Gatermann [9] provides a systematic exposition for compact Lie groups.

These algorithms (except [4], [5]) construct a stratification of the orbit space $\mathbf{R}^n/G$ of a compact Lie group G by using the matrix $\text{grad}(z)$ which is defined on $\mathbf{R}^n/G$. We propose a different approach, namely, to compute a stratification of the representation space of G, and only then to construct the stratification of the orbit space (or the images of relevant strata) by means of elimination theory (equations) and refinements of results of Procesi and Schwarz (inequalities), see [15]. Additional, our algorithms describe any d-dimensional stratum and its closure by at most d inequalities, which turns out to be optimal. This approach has several advantages compared to the present approach[2], namely: Primary decomposition is done before the (nonlinear) Hilbert map is applied, no superfluous components in the orbit space are computed, the association of strata and their stabilizers is quite obvious and, finally, it is possible to compute only those strata, which are relevant for the application under consideration. We also show how to compute inequalities which describe a stratum only up to generic equivalence but contain fewer terms. For several applications, like the construction of continuous potentials on the orbit space, this approach may lead to easier computations. For polynomial potentials, inequalities need not be calculated since the Zariski-closure of a stratum suffices.

[1] Explicit algorithms for finite groups, which yield a minimal number of inequalities, are given in [5].

[2] Equations for the (Zariski-closure) of strata are computed out of rank conditions on the matrix $\text{grad}(z)$. The locus where $\text{rank}\,(\text{grad}(z)) \leq d$ contains all d-dimensional strata of the orbit space and must be decomposed in irreducible components in order to obtain equations defining these strata. Some of these components may be superfluous, i.e., $\text{grad}(z)$ is not positive semidefinite for some points.

In addition, we show that each d-dimensional stratum, respectively its closure, can be presented by at most d strict, respectively relaxed, inequalities and that d is also a lower bound.

1 On Invariant Theory of Compact Lie Groups and Orbit Spaces

We present some background on invariants of compact Lie groups and orbit spaces. In both sections we use fundamental facts from semialgebraic geometry like the Tarksi-Seidenberg principle, for which we refer to [7]. For short, an *basic open* (*basic closed*) *semialgebraic subset* of the algebraic set $V \subseteq \mathbf{R}^n$ is of the form $\{v \in V \mid g_i(v) > 0, 1 \leq i \leq r\}$, respectively, $\geq$ instead of $>$, where $g_1, g_2, \ldots, g_r \in \mathbf{R}[x_1, x_2, \ldots, x_n]$. In the sequel we call an inequality of the form $f > 0$, respectively, $f \geq 0$ strict, respectively, relaxed. An *open* (*closed*) *semialgebraic subset* of V is a finite union of basic open (basic closed) semialgebraic subsets of V.

1.1 Invariants of Lie Groups

Let G be a compact Lie group and $\rho : G \to GL_n(\mathbf{R})$ be a faithful representation. In the sequel we identify G and its image $\rho(G) \subset GL_n(\mathbf{R})$. It is well-known that $\mathbf{R}^n$ admits a G-invariant scalar product $(\_,\_)_G$ on $\mathbf{R}^n$ (see for instance [8]). By the Gram-Schmidt orthonormalization process there exists $A \in GL_n(n)$ such that $A \cdot G \cdot A^{-1} \subseteq O_{\mathbf{R}}$, i.e, the representation ρ is equivalent to an orthogonal representation. From now on we assume $G \subseteq O_{\mathbf{R}}$ and that G acts as usual on $\mathbf{R}^n$. In the sequel let $\mathbf{K}$ be on of the fields $\mathbf{R}$ or $\mathbf{C}$. For $X \subseteq \mathbf{K}^n$ we define $\mathcal{I}(X) := \{f \in \mathbf{K}[t_1, t_2, \ldots, t_n] \mid f(x) = 0 \text{ for all } x \in X\}$, the *ideal* of X and for an ideal $I \subseteq \mathbf{K}[x_1, x_2, \ldots, x_n]$ we define $\mathcal{V}(I) := \{x \in \mathbf{K}^n \mid f(x) = 0 \text{ for } f \in I\}$, the *variety* associated to I. A subset $U \subseteq \mathbf{K}^n$ is *closed* in the Zariski topology if and only if $U = \mathcal{V}(I)$ for some ideal $I \subseteq \mathbf{K}[x_1, x_2, \ldots, x_n]$. A polynomial $f \in \mathbf{K}[x_1, x_2, \ldots, x_n]$ is *invariant* w.r.t. G if $f(g^{-1} \cdot \mathbf{x}) = f(\mathbf{x})$ for all $g \in G$. The ring $\mathbf{K}[x_1, x_2, \ldots, x_n]^G$, consisting of all invariant polynomials w.r.t. G, is called the *invariant ring* of G (ρ will be omitted). By Hilbert's Finiteness Theorem, the invariant ring is finitely generated as a $\mathbf{K}-$algebra. Homogeneous generators $\pi_1, \pi_2, \ldots, \pi_m$ of $\mathbf{K}[x_1, x_2, \ldots, x_n]^G$ are called *fundamental invariants* (i.e., each invariant polynomial is a polynomial in $\pi_1, \pi_2, \ldots, \pi_m$). Fundamental invariants define the projection

$$\begin{aligned} \pi : \mathbf{K}^n &\longrightarrow \mathbf{K}^n/G \subseteq \mathbf{K}^m \\ \mathbf{x} &\longmapsto (\pi_1(\mathbf{x}), \pi_2(\mathbf{x}), \ldots, \pi_m(\mathbf{x})) \end{aligned}$$

of $\mathbf{K}^n$ onto an embedding of the orbit space $\mathbf{K}^n/G \subseteq \mathbf{K}^m$, also called the *Hilbert map.* Note that π maps closed sets to closed sets[3] and that each fiber contains precisely one closed orbit (see for instance[13]). For $\mathbf{K} = \mathbf{C}$ the image of $\pi(\mathbf{C}^n) \subseteq \mathbf{C}^m$ equals the variety of the ideal of relations of $\pi_1, \pi_2, \ldots, \pi_m$ (see for instance[13]). Over $\mathbf{R}$ it is well-known that the image of π is a semialgebraic set.

Proposition 1. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group. The orbit space $\mathbf{R}^n/G$ of G is a semialgebraic set semialgebraically homeomorphic to $\pi(\mathbf{R}^n)$.*

Proof. It is well-known that the orbits of G can be separated by fundamental invariants of G (see for instance Theorem 3.4.3. in [14]). By the Tarski-Seidenberg principle (see for instance [7]) the real image of π is a semialgebraic set (it equals the projection of the graph, which is a real algebraic set). □

Note that the orbit space of an algebraic group parameterizes all closed orbits. Hence the orbit space of a compact Lie group G parameterizes all orbits of G since they are closed. Orbits which are not closed cannot be separated by polynomials so group actions having non-closed orbits cannot be stratified by using their invariant rings, see [16].

1.2 Inequalities defining Orbit Spaces

Procesi and Schwarz have constructed polynomial inequalities which have to be added to the equations coming from the Hilbert map of a compact Lie group G, which need not be a subgroup of $O_n(\mathbf{R})$, in order to describe an embedding of the quotient $\mathbf{R}^n/G \subset \mathbf{R}^m$. Essential parts of the proof are the existence of a closed orbit in each fiber of π (see for instance [13]) and the existence of a G-invariant inner

[3] Note that the map π is proper.

product $(\_,\_)$ on $\mathbf{R}^n$, which is used to construct the $m \times m$ matrix $\mathrm{grad}(v) = (d\pi_i(v), d\pi_j(v))_{i,j=1,\ldots,m}$ for $v \in \mathbf{C}^n$ where $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is the Hilbert map. Here we have used the identification[4] of $\mathbf{R}^n$ with its dual $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$. They proved that a point $z \in \mathcal{V}(I)$, where $I \subset \mathbf{R}[z_1, z_2, \ldots, z_m]$ is the ideal of relations among $\pi_1, \pi_2, \ldots, \pi_m$, lies in $\mathbf{R}^n/G$ if and only if the matrix $\mathrm{grad}(z)$ is positive semidefinite. The constraint that $\mathrm{grad}(z)$ must be semidefinite yields inequalities for describing $\mathbf{R}^n/G$. Recall that the type of a real $m \times m$ Matrix M equals (p, q) where p, respectively, q denote the number of positive, respectively, negative eigenvalues counted with multiplicities. Obviously, $\mathrm{rank}\,(M) = p + q$.

Proposition 2. *An $m \times m$ matrix M over $\mathbf{R}$ is positive semidefinite (denoted by $M \geq 0$) iff all symmetric minors of M are non-negative. The matrix M is positive definite (denoted by $M > 0$) iff all principal minors of M are positive.*

Proof. We refer to, e.g., Section IX.72 in [18]. □

In order to define the matrix $\mathrm{grad}(z)$ on the orbit space we have to show that all entries are invariant w.r.t. G. By $d\pi(z)$ we denote the Jacobian matrix of π at z.

Proposition 3. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group. For $\sigma \in G_v$ the Jacobian of the Hilbert map $\pi : \mathbf{R}^n \to \mathbf{R}^n/G$ satisfies $d\pi(v) = d\pi(v) \circ \sigma$. In particular, the functions $v \mapsto \mathrm{grad}(v)_{ij}$ are invariant.*

Proof. Follows from $\pi(v) = \pi(\sigma \cdot v)$, the chain rule, and the fact that σ is linear. □

Therefore the matrix $\mathrm{grad}(v)$ is also defined on $\mathbf{K}^n/G \subseteq \mathbf{K}^m$ and can be extended to the whole of $\mathbf{K}^m$. Procesi and Schwarz provided the following description of the orbit space.

Theorem 1. *(Procesi-Schwarz [15]) Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group and let $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ be such that $\pi_1, \pi_2, \ldots, \pi_m$ generate $\mathbf{R}[x_1, x_2, \ldots, x_n]^G$. The quotient space is given by*

$$\mathbf{R}^m/G = \pi(\mathbf{R}^n) = \{z \in \mathbf{R}^m \mid \mathrm{grad}(z) \geq 0, z \in \mathcal{V}(I)\}$$

where $I \subset \mathbf{R}[y_1, y_2, \ldots, y_m]$ is the ideal of relations of $\pi_1, \pi_2, \ldots, \pi_m$.

Proof. We refer to [15]. □

Inequalities for the orbit space can be obtained from the condition $\mathrm{grad}(z) \geq 0$. This can be checked by means of Proposition 1.2.2, i.e., testing if all $2^n - 1$ symmetric minors of $\mathrm{grad}(z)$ are ≥ 0. In subsequent sections we use the theorem of Procesi and Schwarz and a modification of Decartes rule of signs to provide an optimal description[5] of the orbit space and all of its strata and their closures (defined in the following section), which are useful for several applications.

Example 1. Consider the action of the compact Lie group $G = O_2 \subset GL_2(\mathbf{R})$ on $\mathbf{R}^4$, given by $(g \cdot x, g \cdot y), g \in G, x, y \in \mathbf{R}^2$, and its complexification $G_\mathbf{C}$ (see Section 3.1). We may choose three algebraically independent fundamental invariants $\pi_1 = t_1^2 + t_2^2, \pi_2 = t_1 t_3 + t_2 t_4, \pi_3 = t_3^2 + t_4^2$. The invariant ring of G, respectively, $G_\mathbf{C}$ equals $\mathbf{K}[t_1, t_2, t_3, t_4]^G = \mathbf{K}[\pi_1, \pi_2, \pi_3]$ where $\mathbf{K} = \mathbf{R}$, respectively, $\mathbf{K} = \mathbf{C}$. The Hilbert map is $\pi = (\pi_1, \pi_2, \pi_3) : \mathbf{K}^4 \to \mathbf{K}^3$. Since π_1, π_2, π_3 are algebraically independent, we obtain $\mathbf{C}^4/G_\mathbf{C} = \mathbf{C}^3 = \mathrm{im}(\pi)$. Over the reals, we apply Theorem 1.2.1 and Proposition 2.2.10 to the matrix $\mathrm{grad}(z) = \begin{pmatrix} 4z_1 & 2z_2 & 0 \\ 2z_2 & z_1 + z_3 & 2z_2 \\ 0 & 2z_2 & 4z_3 \end{pmatrix}$ and obtain the description

$$\mathbf{R}^4/G = \mathrm{im}(\pi) = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbf{R}^3 \;\middle|\; \begin{array}{l} z_1 + z_3 \geq 0, z_1^2 - 2z_2^2 + 6z_1 z_3 + z_3^2 \geq 0, \\ z_1^2 z_3 + z_1 z_3^2 - z^2(z_1 + z_3) \geq 0 \end{array} \right\} \subsetneq \mathbf{R}^3.$$

Remark 1. (a) For practical purposes the dependence on a G-invariant scalar product may be problematic.
(b) It is not necessary that $G \subseteq O_\mathbf{R}$ for computing inequalities if a G-invariant inner product is given in an effective form.

[4] Note that $d\pi_j$ is a differential form, so $d\pi_j(z) : \mathbf{R}^n \to \mathbf{R}$ is a linear form.
[5] The description is optimal in the number of inequalities , i.e., we show that this number is an upper and lower bound.

2 On the Stratification of the Representation and Orbit Space

Consider a compact Lie group $G \subset GL_n(\mathbf{R})$, the set of points having the same symmetry type w.r.t. G form a partition of $\mathbf{R}^n$ in finitely many distinct open sets, also called a stratification. We present underlying definitions and properties of of strata and their closures (Zariski- or Euclidean topology). These properties will be used in subsequent sections to compute equations and inequalities for describing strata and their closures.

2.1 On the Stratification of the representation- and orbit space

We provide the definition of strata, respectively, stratifications and associated objects like orbit type, etc. In the sequel $G \subset GL_n(\mathbf{R})$ denotes a compact Lie group and $\mathrm{cl}_Z(X)$, respectively, $\mathrm{cl}_E(X)$ denote the closure of the set X in the Zariski, respectively, Euclidean topology.

Definition 1. *Let $E \subseteq \mathbf{R}^n$ be a semialgebraic set. A stratification of E is a finite partition E_λ of E where each E_λ is a semialgebraically connected locally closed*[6] *equidimensional semialgebraic subset (or a finite set of points) of $\mathbf{R}^n$ such that $E_\lambda \cap \mathrm{cl}_E(E_\beta) \neq \emptyset$ and $\lambda \neq \beta$ implies $E_\lambda \subset E_\beta$ and $\dim E_\lambda < \dim E_\beta$. For $\lambda \in \Lambda$ the set E_λ is called a stratum and $\mathrm{cl}_E(E)_\lambda$ is called a semi-stratum of the stratification, and if $d = dim E_\lambda$ then E_λ is called a $d-$stratum.*

Given $x \in \mathbf{R}^n$, the set $G(x) = \{g \cdot x \mid g \in G\}$ is called the *orbit* of x and the group $G_x = \{g \in G \mid g \cdot x = x\}$ is called the *stabilizer* of x.

Proposition 4. *Let G be an algebraic group (defined over the field $\mathbf{K}$) which acts algebraically (via α) on $\mathbf{K}^n$. For $x \in \mathbf{K}^n$ the stabilizer G_x and the set $X_d = \{x \in X \mid \dim G_x \geq d\}$ are closed.*

Proof. Let $\pi_2 : X \times X \to X$ be the projection onto the second component, $i_x : G \hookrightarrow G \times X, i_x(g) = (g, x)$ be an injection for $x \in X$ and define $\alpha' : G \times X \to X \times X$ by $\alpha'(g, x) = (\alpha(g, x), x)$. All maps are continuous (w.r.t. the Zariski-topology), hence the fibers of $\pi_2 \circ \alpha' \circ i$ are closed. The stabilizer of x is closed since G_x is isomorphic to $\alpha'^{-1}(x, x) = \{(g, x) \mid \alpha(g, x) = x\}$. We also obtain that $X_d = \{x \in X \mid \dim(\pi_2 \circ \alpha' \circ i)^{-1}(x) \geq d\}$ hence the claim follows from upper-continuity of the fiber dimension. □

Definition 2. *For a subgroup $H \subseteq G$ we denote the conjugacy class of H in G by by $[H] = \{gHg^{-1} \mid g \in G\}$. The orbit type of $x \in \mathbf{R}^n$ is $[x] := [G_x]$. For $u, v \in \mathbf{R}^n$ we define $[u] < [v]$ if $G_u \subset H$ for some $H \in [v]$. The associated stratum, respectively, semi-stratum of $[x]$ is $\Sigma_x := \{y \in \mathbf{R}^n \mid [x] = [y]\}$, respectively, $\mathrm{cl}_E(\Sigma_x)$.*

The orbit type is a measure for the symmetry of the points of $\mathbf{R}^n$. We have $[x] > [y]$ if the point x has more symmetries than the point y, i.e., $gG_yg^{-1} \subset G_x$ form some $g \in G$. The notation of strata is justified by the fact that these sets, respectively, their images under the Hilbert map form a stratification of the representation-, respectively, orbit space.

Proposition 5. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group.*

(a) There are only finitely many different orbit types, i.e,. the set $\{[G_x] \mid x \in \mathbf{R}^n\}$ is finite.
(b) The orbit types form a lattice. For $v \in \Sigma_p := \{x_0 \in \mathbf{R}^n \mid \mathrm{rank}\,(d\pi(x)x_0)$ is maximal$\}$ the orbit type $[v]$ is the least element.
(c) For each $v \in \mathbf{R}^n$ there exists a small neighborhood $U \subset \mathbf{R}^n$ of v such that $u \in U$ implies $[u] \leq [v]$.

Proof. (a) see for instance Ch. IV.10 in [8].
(b) Note that $\mathrm{rank}\,(d\pi(x)v)$ is maximal iff $\dim N_v^0$ is maximal (see Section 2.2) hence the stabilizer of v is contained in $[w]$ for all $w \in \mathbf{R}^n$.
(c) We refer, e.g., to [1].

□

The set Σ_p, which is dense in $\mathbf{R}^n$, is called the *principal stratum* of G.

Proposition 6. *Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group.*

[6] The set E_λ is open in its metric closure $\mathrm{cl}_E(E_\lambda)$.

(a) *For a subgroup* $H \subseteq G$ *of* G *the set* $\mathbf{R}^n_H = \{x \in \mathbf{R}^n \mid H \subseteq G_x\}$ *is a vectorspace. In particular, the set* $\{x \in \mathbf{R}^n \mid G_x = H\}$ *is Zariski-open in* $\mathbf{R}^n_H$*.*
(b) *For* $0 \neq x \in \mathbf{R}^n$ *each stratum* Σ_x *is open in its closure (both metric and Zariski) and* $G(x)$ *is a proper subset of* Σ_x*.*

Proof. (a) Let $x, y \in \mathbf{R}^n_H$ and $g \in H$. Obviously, $g \cdot (x + y)$ and $g \cdot \lambda x, \lambda \in \mathbf{R}$, are contained in $\mathbf{R}^n_H$. The set $S = \{x \in \mathbf{R}^n_H \mid G_x \supset H\}$ is of dimension less than $\mathbf{R}^n_H$ and can be written as the union of all strata Σ_y with $[y] > [H]$ intersected with $\mathbf{R}^n_H$. By Proposition 2.1.5, the set S is closed, hence $\mathbf{R}^n_H \setminus S$ is Zariski-open.
(b) The first claim follows from Theorem 2.2.2. For the second claim note that $G(x)$ is compact, hence the set $\{\lambda x \mid \lambda \in \mathbf{R}, \lambda > 0\}$ is not contained in $G(x)$ but in Σ_x.

□

Note that the closure of a stratum of the representation space need not be a finite union of vectorspaces, as it is the case for finite groups, see Example 3.4.4. We conclude this section by giving a description of the orbit space (and its stratification) in terms of equations and relaxed inequalities obtained from Procesi's and Schwarz's Theorem. Here strata are described as differences of closed semialgebraic sets.

Corollary 1. *Let* $G \subset GL_n(\mathbf{R})$ *be a compact Lie group, let* $x \in \mathbf{R}^n$ *and* $y = \pi(x)$*.*

(a) *Let* $\Sigma_x \subseteq \mathbf{R}^n$ *be a stratum. Then* $\mathrm{cl}_E(\hat{\Sigma}_y) = \pi(\mathrm{cl}_E(\Sigma_x)) = \{z \in \mathbf{R}^m \mid \mathrm{grad}(z) \geq 0, z \in \mathcal{V}(J)\}$ *where* $J \subset \mathbf{R}[z_1, z_2, \ldots, z_m]$ *is the ideal of the image of* Σ_x *under* π*.*
(b) *Let* $\mathrm{cl}_E(\Sigma_x) = \Sigma_x \cup B_x$ *be a disjoint union (*B_x *is a finite union of lower-dimensional strata). Then* $\hat{\Sigma}_x = \pi(\Sigma_x) = \pi(\mathrm{cl}_E(\Sigma_x)) - \pi(B_x)$*, i.e.,*

$$\hat{\Sigma}_x = \{z \in \mathbf{R}^m \mid z \in \mathrm{cl}_Z(\pi(\Sigma_x)), z \notin \pi(B_x), \mathrm{grad}(z) \geq 0\}$$

2.2 Properties of Strata

We describe properties of strata and semi-strata on the representation and orbit space. In the representation space closures of strata, respectively, strata can be described by closed sets, respectively, differences of closed sets. For a description of the orbit space Procesi and Schwarz have derived the condition that $\mathrm{grad}(z) \geq 0$ (see Theorem 1.2.1), but they only provide the criterium given in Proposition 1.2.2, which yields $2^d - 1$ inequalities (provided that d equals the dimension of the orbit space). These inequalities may also be used to describe all topological closures of strata on the orbit space and therefore also all strata by forming differences of closed sets (see Corollary 2.1.1). We show that a d-dimensional stratum respectively, its closure can be described by d sharp, respectively, relaxed inequalities and the ideal of its Zariski-closure in $\mathbf{R}^n/G$ and that d is also a lower bound. In particular, we provide effective descriptions relying on equations and inequalities.

The stratification of the representation space of a compact Lie group is completely determined by the matrix $d\pi(x)v$. Since $\mathbf{R}^n$ admits a G-invariant inner product $(\_, \_)_G$ we may define the orthogonal complement N_v to $T_v(G(v))$ and the decomposition $N_= N^0_v \oplus N^1_v$, where $N^0_v = \{w \in N_v \mid w \text{ is } G_v\text{-invariant}\}$ and N^1_v is the orthogonal complement of N^0_v in N_v. Note that G need not be a subgroup of the orthogonal group.

Proposition 7. *Let* $G \subset GL_n(\mathbf{R})$ *be a compact Lie group. We have*

$$\ker d\pi(x)x_0 = T_{x_0}G(x_0) \oplus N^1_{x_0} \text{ and } \mathrm{im} d\pi(x)x_0 \cong N^0_{x_0}.$$

Proof. Note that $v \in T_{x_0}G(x_0)$ implies $v \in \ker d\pi(x_0)$ since π is G-invariant. Let V be the the vectorspace generated by the gradients (considered as elements of $\mathbf{R}^n$) $d\pi_1(x_0), d\pi_2(x_0), \ldots, d\pi_m(x_0)$, i.e., $V = \mathrm{im}\ d\pi(x_0)$. Note that $v \in \ker d\pi(x_0)$ implies $d\pi_i(x_0) \cdot v = 0$ so $v \in N_{x_0}$. By Proposition 2.2.3 we have $d\pi_i(x_0) \circ \sigma = d\pi_i(x_0)$ for $\sigma \in G_v$, hence $V \subseteq N^0_v$. Now $v \in N^0_{x_0} \setminus V$ implies $v \in \ker d\pi(x_0)$. Hence the rank of the matrix $d\pi(x_0)$ augmented by the column v equals the rank of $d\pi(x_0)$ and so $v \in V$. □

Proposition 8. *Let* $G \subset GL_n(\mathbf{R})$ *be a compact Lie group. We have*

$$T_{x_0}\Sigma_{x_0} = T_{x_0}G(x_0) \oplus N^0_{x_0}.$$

In particular, $T_{\pi(x_0)}\hat{\Sigma}_{x_0} \cong N^0_{x_0}$*.*

Proof. One has to show that any curve through x_0 and contained in Σ_{x_0} has a tangent vector at x_0 which is contained in $T_{x_0}G(x_0) \oplus N^0_{x_0}$. This proof can be found in Section V of [1]. □

Corollary 2. *We have* $\dim \Sigma_{x_0} = \dim T_{x_0} + \dim N^0_{x_0} = \dim G - \dim G_{x_0} + \dim N^0_{x_0}$ *and* $\dim \hat{\Sigma}_{\pi(x_0)} = \dim N^0_{x_0}$.

Theorem 2. *Let* $G \subset GL_n(\mathbf{R})$ *be a compact Lie group and* $\pi : \mathbf{R}^n \to \mathbf{R}^n/G \subseteq \mathbf{R}^m$ *be the Hilbert map.*

(a) The union $\Sigma^{(d)}$ *of all strata whose image under* π *is of dimension* d *equals the open semi-algebraic set*

$$\Sigma^{(d)} = \{v \in \mathbf{R}^n \mid \operatorname{rank}(d\pi(v)) = d\}.$$

(b) The union Σ^d *of all strata whose image under* π *is of dimension at most* d *equals the closed semi-algebraic set*

$$\Sigma^d = \{v \in \mathbf{R}^n \mid \operatorname{rank}(d\pi(v)) \leq d\}$$

In addition, $\mathrm{cl}_Z(\Sigma^{(d)}) = \mathrm{cl}_E(\Sigma^{(d)}) = \Sigma^d$.

Proof. (a) Note that a stratum is a smooth semi-algebraic set, so by Proposition 2.2.8 we have $\operatorname{rank}(d\pi(v)) = \dim \operatorname{im}(d\pi(v)) = \dim T_{\pi(v)}\hat{\Sigma}_{\pi(v)} = \dim \hat{\Sigma}_{\pi(v)}$.

(b) The set Σ^d can be defined by the vanishing of all $(d+i) \times (d+i)$ minors of $\frac{\partial \pi}{\partial x}$ where $i \geq 1$. If $d \geq \min\{n, m\}$ then $\Sigma^d = \mathbf{R}^n$. Note that $\Sigma^{(d)} = \Sigma^d \setminus \Sigma^{d-1}$.

□

So far we have only considered semistrata, respectively, strata on the representation space. Unfortunately, we need at most $2^n - 1$ inequalities, obtained from the symmetric minors of $\mathrm{grad}(z)$. A direct description of a d-dimensional stratum by means of equations and (strict) inequalities can be obtained from the constraint that the type of $\mathrm{grad}(z)$ equals $(d, 0)$. We apply Decartes rule of sign to the characteristic polynomial of the matrix $\mathrm{grad}(z)$ in order to obtain an optimal number of inequalities.

A sequence $a_0, a_1, \ldots, a_n$ has a sign change if there exists i, j s.t. $a_i a_{i+j} < 0$ and $a_i a_{i+k} \geq 0$ for $1 \leq k < j$. For a polynomial $f = \sum_{i=0}^n a_i t^i$ we define the number of sign changes $N_+(f)$ respectively alternative sign changes $N_-(f)$ by the total number of sign changes of the sequence $a_0, a_1, \ldots, a_n$ respectively of the sequence $a_0, -a_1, a_2, \ldots, (-1)^i a_i, \ldots, (-1)^n a_n$. By $Z_+(f)$ respectively $Z_-(f)$ we denote the number of positive respectively negative real roots of f.

Proposition 9. *(Descartes rule of sign; see [18]) Let* $f \in \mathbf{R}[t]$ *be a nonzero polynomial. There exist* $\rho_+, \rho_- \in \mathbb{N}$ *s.t.* $N_+(f) = Z_+(f) - 2\rho_+$ *and* $N_-(f) = Z_-(f) - 2\rho_-$. *Moreover, if* f *has only real roots then* $N_+(f) = Z_+(f)$ *and* $N_-(f) = Z_-(f)$.

We state a refinement of a well-known result in matrix analysis (see for instance Ch. 7 in [12]).

Corollary 3. *Let* $M \in \mathrm{Mat}_n(\mathbf{R})$ *be a symmetric matrix of* $\operatorname{rank}(M) = d > 0$ *and* $p(t) = \sum_{i=0}^n a_i t^i$ *be its characteristic polynomial. Then* M *is of type* $(d, 0)$ *iff* $(-1)^i a_{n-i} > 0$ *for* $1 \leq i \leq d$.

Proof. Note that $a_{n-d-1} = \ldots = a_0 = 0$ and all roots of $p(t)$ are real. By Proposition 2.2.9 we have $N_+(p) = Z_+(f)$ as required. □

By relaxing all inequalities obtained from conditions about sign changes of the characteristic polynomial we obtain a criterium for positive semidefiniteness without assumptions about the rank. This yields an upper bound for the description of closures of strata.

Proposition 10. *Let* $M \in \mathrm{Mat}_n(\mathbf{R})$ *be a symmetric matrix and* $p(t) = \sum_{i=0}^n a_i t^i$ *be its characteristic polynomial. Then* M *is positive semidefinite iff* $(-1)^i a_{n-i} \geq 0$ *for* $1 \leq i \leq n$.

Proof. Let M be a symmetric matrix of $\operatorname{rank}(M) = d > 0$ having a negative eigenvalue. Note that $a_{n-d-1} = a_{n-d-2} = \ldots = a_0 = 0$ and $a_n = 1$. By Decartes rule of sign (Proposition 2.2.9) there exists a minimal $i > 0$ s.t. $(-1)^n(-1)^{n-i}a_i < 0$. For n even we obtain $(-1)^{n-i}a_i < 0$ a contradiction to $(-1)^{n-i}a_{n-i} \geq 0$ since $(-1)^i = (-1)^{n-i}$. In case n odd the sign change gives $(-1)(-1)^{n-i}a_{n-i} = (-1)^{n-i+1}a_{n-i} < 0$, a contradiction to $(-1)^i a_{n-i} = (-1)^{n-i+1}a_{n-i} \geq 0$. □

Theorem 3. *Let* $G \subset GL_n(\mathbf{R})$ *be a compact Lie group, let* $\pi_1, \pi_2, \ldots, \pi_m$ *be fundamental invariants of* G *and let* I *be their ideal of relations. Let* $d \leq \dim \mathbf{R}^n/G$ *be an integer and* I_d *be the ideal of all* $d \times d$ *minors of* $\mathrm{grad}(z)$. *By* $p_d(t) = \sum_{i=0}^m (-1)^{m-i}\delta_i t^i$ *we denote the characteristic polynomial of* $\mathrm{grad}\, z$ *modulo* I_d.

(a) *We have*

$$\{z \in \mathcal{V}_{\mathbf{R}}(I) \mid \operatorname{grad}(z) \geq 0, \operatorname{rank}(\operatorname{grad}(z)) = d\} = \{z \in \mathcal{V}_{\mathbf{R}}(I_d) \mid \delta_1(z) > 0, \delta_2(z) > 0, \ldots, \delta_d(z) > 0\}.$$

(b) *Relaxing the strict inequalities in part (a) gives the set* $\{z \in \mathcal{V}_{\mathbf{R}}(I) \mid \operatorname{grad}(z) \geq 0\}$.

(c) *Let* J *be the ideal of the Zariski-closure of a* d*-dimensional stratum* $\hat{\Sigma}_d$ *and* $\delta_i' = \delta_i \mod J$. *Then* $\hat{\Sigma}_d = \{z \in \mathcal{V}_{\mathbf{R}}(J) \mid \delta_1'(z) > 0, \delta_2'(z) > 0, \ldots, \delta_d'(z) > 0\}$. *For the topological closure of* $\hat{\Sigma}_d$ *we obtain* $\operatorname{cl}_E(\hat{\Sigma}_d) = \{z \in \mathcal{V}_{\mathbf{R}}(J) \mid \delta_1'(z) \geq 0, \delta_2'(z) \geq 0, \ldots, \delta_d'(z) \geq 0\}$.

(d) *Let* J *be the ideal of the Zariski-closure of a* d*-dimensional stratum* $\hat{\Sigma}_d$ *and suppose that* $\operatorname{grad}(z)$ *is so arranged that the first* d *principal minors do not vanish identically on* $\hat{\Sigma}_d$. *Then* $\hat{\Sigma}_d$ *is generically equivalent (the symmetric difference has codimension at least* 1*) to* $\{z \in \mathcal{V}_{\mathbf{R}}(J) \mid \Delta_1(z) > 0, \Delta_2(z) > 0, \ldots, \Delta_d(z) > 0\}$ *where* $\Delta_1, \Delta_2, \ldots, \Delta_d$ *are the first* d *principal minors of* $\operatorname{grad}(z)$.

(e) *Suppose that* $\pi_1, \pi_2, \ldots, \pi_d$ *are algebraically independent. The principal stratum of* $\mathbf{R}^n/G$ *is given by* $\hat{\Sigma}_p = \{z \in \mathbf{R}^d \mid \Delta_1(z) > 0, \Delta_2(z) > 0, \ldots, \Delta_d(z) > 0\}$ *where* $\Delta_1, \Delta_2, \ldots, \Delta_d$ *are all principal minors of* $\operatorname{grad}(z)$.

Proof. Part (a),(b), and (c) follow from Proposition 2.2.9. For Part (d) note that $\Delta_i(z) = 0$ defines a hypersurface in $\hat{\Sigma}_d$. Part (e) follows from $I = \{0\}$, i.e, $\mathcal{V}(I) = \mathbf{R}^d$ and from $\operatorname{rank}(\operatorname{grad}(z)) = d$ for all $z \in \hat{\Sigma}_p$. □

For a given d-dimensional stratum respectively its topological closure, the number of d inequalities obtained from the previous theorem is optimal, as shown by the following example.

Example 2. Let $G \subset GL_n(\mathbf{R})$ be the finite group generated by all $n \times n$ diagonal matrices of the form $(1, 1, \ldots, 1, -1, 1, \ldots, 1)$. Fundamental invariants are given by $t_1^2, t_2^2, \ldots, t_n^2$. Hence the orbit space is the positive orthant $z_1 \geq 0, z_2 \geq 0, \ldots z_n \geq 0$ and any d-dimensional stratum respectively its topological closure is given by equations $z_{i_1} = \ldots = z_{i_{n-d}} = 0$ and inequalities $z_{i_{n-d+1}} > 0, \ldots, z_{i_n} > 0$ respectively $\leq$ instead of $>$, where $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$. It is well-known that any such set cannot be described by fewer than d inequalities (see for instance [7]).

We obtain the following geometric statement:

Corollary 4. *Let* $\hat{\Sigma}_d$ *be a* d*-dimensional stratum of a compact Lie group* $G \subset GL_n(\mathbf{R})$. *The semialgebraic set* $\hat{\Sigma}_d$ *is basic open in its Zariski-closure. The topological closure of* $\hat{\Sigma}_d$ *is a basic closed semialgebraic set in its Zariski-closure. Moreover both sets can be described by at most* d *strict respectively relaxed inequalities, which is optimal.*

Remark 2. (a) Bröcker and Scheiderer have proved that any basic open set of dimension d can be described by at most d sharp inequalities (unpublished, see Chapter 6.5 in [7]) and that d is also a lower bound. For basic closed sets of dimension d Scheiderer has proved that $\frac{d(d+1)}{2}$ is an upper and lower bound for the number of (relaxed) inequalities required for a description (see [17]). Since Theorem 2.2.3 states that for the (topological) closure of a d-dimensional stratum d inequalities suffice, closures of strata (in particular orbit spaces) form a class of basic closed sets which are easier to describe. Note that the dimension is still a lower bound. Hence there is no gain in efficiency when using generic descriptions.

(b) Suppose that there exist algebraically independent fundamental invariants $\pi_1, \pi_2, \ldots, \pi_m$ of G. If $|G| < \infty$, any d-dimensional stratum can be described by the first d principal minors of $\operatorname{grad}(z)$ (after a permutation of $\pi_1, \pi_2, \ldots, \pi_m$), see [5]. If G is not finite, this is no longer true, see, e.g., Example 3.4.4 or Example 3 in [1].

(c) The upper bound d holds for all d-dimensional basic closed sets, where inequalities are obtained from positive-semidefiniteness conditions on matrices.

3 Constructing the Stratification

As shown in Section 2.2 the d-dimensional components of the strata can be computed by conditions on the rank of the matrix $d\pi(v)$. In this section we provide an algorithm together with necessary tools for the construction of a stratification of the representation- and the orbit space.

More precisely, given a d-dimensional connected component C of a stratum (obtained from rank conditions), the corresponding stratum is given by the orbit of C. The same holds true for the associated semistrata. In this way we construct the stratification of the orbit space out of the stratification of the representation space by computing the image of π (recall Corollary 2.2.1). It remains to add a set of inequalities obtained from the Theorem of Procesi and Schwarz (Theorem 1.2.1), and its refinement (Corollary 2.2.3 and Theorem 2.2.3). We also present an algorithm for computing the stabilizer of a given vector subspace of $\mathbf{K}^n$, which may be used to distinguish the symmetry type of strata[7] of the same dimension.

All used algorithms but the computation of inequalities rely on algebraically closed ground fields. For this reason we present properties of complexifications of real varieties below.

3.1 On the Complexification of a Group-Action

We briefly mention some relations between a compact Lie group G and its complexification and the real- and complex orbit space. More precisely, given fundamental invariants $\pi_1, \pi_2, \ldots, \pi_m \in \mathbf{R}[x_1, x_2, \ldots, x_n]$ of G, in order to describe the orbit space we have to compute the image of the morphism π by Elimination Theory, i.e., one computes the ideal I of relations among $\pi_1, \pi_2, \ldots, \pi_m$, which requires an algebraically closed ground field. As we have already seen, the orbit space of G may be properly be contained in the real algebraic set $\mathcal{V}(I) \subseteq \mathbf{R}^m$. Therefore we have to take care if the computations performed over an algebraically closed field are valid over $\mathbf{R}$. Several important results are based on Kempf-Ness Theory. We refer, e.g., to [19].

Let $G \subset GL_n(\mathbf{R})$ be a compact Lie group defined by the ideal[8] $I_G \subset \mathbf{R}[s_1, s_2, \ldots, s_m]$. The complexification of G is the zero set of I_G over the complex numbers, denoted by $G_{\mathbf{C}}$. Note that $G_{\mathbf{C}}$ is a complex reductive group with coordinate ring $\mathbf{C}[s_1, s_2, \ldots, s_m]/I_G = \mathbf{R}[s_1, s_2, \ldots, s_m]/I_G \otimes_{\mathbf{R}} \mathbf{C}$ and that G is Zariski-dense in $G_{\mathbf{C}}$. The ideals defining the (real) orbit and the stabilizer of a point $v \in \mathbf{R}^n$ can be computed by Elimination Theory from the ideal I_G and the necessary constructions.

By Hilbert's Finiteness Theorem the invariant ring of G is finitely generated, hence $\mathbf{R}[t_1, t_2, \ldots, t_n]^G = \mathbf{R}[h_1, h_2, \ldots, h_m]$ for some homogeneous invariants $h_1, h_2, \ldots, h_m$. The action of G complexifies to an action of $G_{\mathbf{C}}$ on $\mathbf{C}^n$ and the invariant ring of $G_{\mathbf{C}}$ equals $\mathbf{C}[t_1, t_2, \ldots, t_n]^{G_{\mathbf{C}}} = \mathbf{R}[h_1, h_2, \ldots, h_m] \otimes_{\mathbf{R}} \mathbf{C}$. Hence the Hilbert map $\pi : \mathbf{R}^n \to \mathbf{R}^m$ complexifies to $\pi_{\mathbf{C}} : \mathbf{C}^n \to \mathbf{C}^m$ and $\pi_{\mathbf{C}}(\mathbf{C}^n) = \mathrm{cl}_Z(\pi(\mathbf{R}^n))$ (closure in $\mathbf{C}^m$). Let I be the ideal of relations of $h_1, h_2, \ldots, h_m$. Since $\mathcal{V}(I) = \mathrm{cl}_Z(\pi(\mathbf{R}^m))$ over $\mathbf{R}$, by Procesi and Schwarz (see Theorem 1.2.1) we have $\mathbf{R}^n/G = \{z \in \mathcal{V}(I) \cap \mathbf{R}^m \mid \mathrm{grad}(z) \geq 0\}$ where the latter closure is taken in $\mathbf{R}^m$.

3.2 Stratification of the Representation Space

By using the results stated in Section 2 we are now able to provide an algorithm for computing a stratification $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$ of the representation space of a compact Lie group G. The stratification of the orbit space $\mathbf{R}^m/G$ is obtained by computing the ideals of the images $\pi(\Sigma_1), \pi(\Sigma_2), \ldots, \pi(\Sigma_r)$ and adding appropriate inequalities to each set of equations.

Algorithm 1 RepSpaceStrata*(I_G, ψ)*
In: *Ideal defining a compact Lie group $G \subset GLn\mathbf{R}$, ψ a list of polynomials in $\mathbf{R}[s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_n]$ defining the action of G.*
Out: *list of equations defining the closures $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$ of G and their generic stabilizer .*
begin
$\pi = (\pi_1, \pi_2, \ldots, \pi_r)$; *// algebra generators of* $\mathbf{R}[t_1, t_2, \ldots, t_n]^G$;
$d = \dim \mathbf{R}^m/G$ *// dimension of the orbit space*
for $i = 1$ **to** d **do**
  $J_d = d \times d$ *minors of* $d\pi$; *// all* $d \times d$ *minors of the Jacobian*
  collectedSpaces = primary decomposition of $\sqrt{J_d}$.
  $c := 1$;
  for *each* $V \in$ *collectSpaces*$[i]$ **do**
    orbitV $= \psi(G, V)$; *// orbit of V*
    if *orbitV* $\notin \bigcup_{j=1}^{c-1}$ *Semistrata*$[d][j]$ **then begin**
      Semistrata$[d][c]$ = *Semistrata*$[d][c] \cup$ *orbitV*;

[7] Strata of the same dimension may have different stabilizers of the same dimension but different number of connected components

[8] Compact Lie groups are algebraic groups, see for instance [14].

$stabilizer[d][c] = Stabilizer(I_G, \psi, V)$; // *representative of the orbit-type*
$c = c + 1$;
end
end-for;
end-for;
return($[Semistrata, stabilizer]$);
end RepSpaceStrata.

A set of fundamental invariants for G may be computed by the algorithm given in [6], which works for all reductive groups. Algorithms restricted to compact Lie groups can be found in [9].

We are left with the problem of computing a representative of an orbit type $[v]$, i.e, given the closure $\mathrm{cl}_Z(\Sigma_x)$, find equations for the 'generic' stabilizer G_ξ of $\mathrm{cl}_Z(\Sigma_x)$. By computing a primary decomposition of the ideal of G_ξ we obtain the index $G_\xi/(G_\xi)_0$

Proposition 11. *Let G be an algebraic group defined by the ideal $I_G \subseteq \mathbf{K}[s_1, s_2, \ldots, s_m]$, let $\alpha : G \times \mathbf{K}^n \to \mathbf{K}^n$ be a linear action, let $V \subseteq \mathbf{K}^n$ be an irreducible variety of dimension d, defined by the ideal J_V and let $J_a = \langle t_i - a_i : 1 \le i \le n \rangle \subsetneq \mathbf{K}(a_1, a_2, \ldots, a_d)[t_1, t_2, \ldots, t_n]$. Define the ideals $I = \langle I_G, J_V, J_a, \alpha_i(s,t) - t_i \ : \ 1 \le i \le n \rangle \subset \mathbf{K}(a_1, a_2, \ldots, a_k)[s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n]$ and $J = I \cap \mathbf{K}(a_1, a_2, \ldots, a_k)[s_1, s_2, \ldots, s_m]$ and the (partial) substitution map $\varphi_{\mathbf{b}} : \mathbf{K}(a_1, a_2, \ldots, a_d) \to \mathbf{K}, a_i \longmapsto b_i$ for $(b_1, b_2, \ldots, b_n) \in \mathbf{K}^d$. There exists a non-empty Zariski-open set $U \subseteq V$ such that $u \in U$ implies $\varphi_u(J) = \mathcal{I}(G_u)$.*

Proof. After a finite number of steps we obtain a Gröbner basis of I. In each step we collect the following data: If multiplication by a polynomial f occurs then let P_f be the set of all coefficients of monomials in f which contain some a_i. When computing $f - g$ then add all rational functions in $a_1, a_2, \ldots, a_d$ which are obtained from solving $f - g = 0$ by comparing coefficients. Exclude these sets from $\mathbf{K}^n$. □

Algorithm 2 Stabilizer*(I_G, ψ, I_V)*
In: *ideal I_G of a compact group G, ideal I_V of a component of a stratum.*
Out: *equations of the stabilizer*
Note: *Basering is $\mathbf{K}(a_1, a_2, \ldots, a_k)[s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_n]$.*
begin
$I = \mathrm{GroebnerBasis}(I_V)$;
$c = 0$;
for $i = 1$ **to** n **do**
if $\deg(\mathrm{NormalForm}(t_i, I)) > 0$ *then* **begin**
$c := c + 1$;
$I = \mathrm{GroebnerBasis}(I \cup \{t_i - a_c\})$;
end-if
end-for
$I = I \cup \{\psi_i - t_i : 1 \le i \le n\}$;
$J = \mathrm{GroebnerBasis}(I) \cap \mathbf{K}(a_1, a_2, \ldots, a_k)[s_1, s_2, \ldots, s_k]$;
return(J);
end Stabilizer.

Remark 3. An alternative way to compute the number of connected components of the stabilizer is as follows. Compute the generic orbit $G(\xi)$ of V and determine a primary decomposition and the multiplicity of $G(\xi)$ (see [4]).

3.3 Stratification of the Orbit Space

Given a (semi-)stratification of the representation space, the computation of the stratification of the orbit space is essentially the computation of the matrix $\mathrm{grad}(z)$ and its symmetric minors. If G is not finite then the dimension of the representation space is strictly greater than the dimension of the orbit space.

The algorithm returns a list of strata of the orbit space of G sorted by dimension. Each stratum $\hat{\Sigma}_{d,i}$ is described as a triple $[[f_1, f_2, \ldots, f_r], [g_1, g_2, \ldots, g_{2^d-1}], [h_1, h_2, \ldots, h_s]]$ where $\hat{\Sigma}_{d,i} = \{z \in \mathbf{R}^m \mid f_1(z) = 0, \ldots, f_r(z) = 0, g_1(z) > 0, \ldots, g_{2^d-1}(z) > 0, h_1(z) \neq 0, \ldots, h_s(z) \neq 0\}$.

Algorithm 3 ORBITSPACESTRATA*(π, repStrata)*
In: $\pi = \pi_1, \pi_2, \ldots, \pi_m$ *fundamental invariants of* $G \subseteq O_{\mathbf{R}}$*,list of closures of strata of the representation space. Assume that* $d = \dim \mathbf{R}^n/G$.
Out: *list of strata of the orbit space (given by equations and inequalities)*
begin
$\mathrm{grad}(z) = (d\pi_i, d\pi_j)_{i=1..n}^{j=1..n}$;
$c = 0$;
$p(t) = det(\mathrm{grad}(z) - t \cdot \mathrm{id}_n)$; // *assume* $p(z) = t^{m-d} \sum_{i=0}^{d} (-1)^i \delta_i t^i$*, characteristic polynomial of* $\mathrm{grad}(z)$
for $k = 1$ **to** $|repStrata|$ **do**
  for $i = 1$ **to** $|repStrata[k]|$ **do**
    J = *image of* $repStrata[k][i]$ *under* π. // *by Elimination Theory*
    $ineq = \{\mathrm{NormalForm}(\delta_i, J) > 0 \mid 1 \leq i \leq d\}$
    $strata[d][i] = [semistratum[k][i], I, J]$;
  end-for
end-for
return*(strata);*
end ORBITSPACESTRATA.

Remark 4. A stratification up to generic equivalence can be obtained by replacing the line defining I by the line

I := set of first $d \times d$ principal minors of $\mathrm{grad}(z)$; // $\mathrm{grad}(z)$ arranged s.t. no principal minor vanishes identically on $repStrata[k][i]$.

Example 3. We consider the compact Lie group $G = O_2 \times \mathbf{Z}_2 \subset GL_2(\mathbf{R})$ (O_2 acts on the first two coordinates, $\mathbf{Z}_2$ acts on the third coordinate) defined by the ideal $\langle s_1^2 + s_2^2 - 1, s_3^2 + s_4^2 - 1, s_1 s_3 + s_2 s_4, s_5^2 - 1 \rangle$. The Jacobian of $\pi : \mathbf{R}^3 \to \mathbf{R}^2, (t_1, t_2, t_3) \mapsto (t_1^2 + t_2^2, t_3^2)$ equals $\begin{pmatrix} 2t_1 & 2t_2 & 0 \\ 0 & 0 & 2t_3 \end{pmatrix}$, hence we have (all variables range over $\mathbf{R}$)

$$\begin{aligned} \Sigma_0 &= \{v = (a, b, c) \mid \mathrm{rank}\,(d\pi(v)) = 0\} = \{(0, 0, 0)\} \\ \Sigma_{1,1} \cup \Sigma_{1,2} &= \{v = (a, b, c) \mid \mathrm{rank}\,(d\pi(v)) = 1\} = \{(a, b, 0) \mid a \neq 0 \text{ or } b \neq 0\} \cup \{(0, 0, c) \mid c \neq 0\} \\ \Sigma_2 &= \{v = (a, b, c) \mid \mathrm{rank}\,(d\pi(v)) = 2\} = \{(a, b, c) \mid ac \neq 0 \text{ or } bc \neq 0\} \end{aligned}$$

By using the algorithm STABILIZER we obtain for the associated stabilizers the table

Stratum	Σ_0	$\Sigma_{1,1}$	$\Sigma_{1,2}$	Σ_2
Stabilizer	$O_2 \times \mathbf{Z}_2$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	O_2	$\mathbf{Z}_2$

As an example, the ideal $I \subset \mathbf{R}(a_1, a_2)[s_1, s_2, \ldots, s_5]$ defining the generic stabilizer of $\Sigma_{1,1}$ is given by

$$\begin{aligned} I = \langle & a_1 s_3 + a_2 s_4 - a_2, a_1^3 s_2 + a_1 a_2^2 s_3 + a_1^2 a_2 + a_2^3 s_4 - a_1^2 a_2 - a_2^3, \\ & a_1 s_1 + a_2 s_2 - a_1, s_5^2 - 1, a_1^2 + a_2^2 s_4^2 + a_1 a_2 s_3 - a_2^2 s_4 - a_1^2 \rangle \end{aligned}$$

Substitution of $(a, b) \in \Sigma_{1,1}$ for (a_1, a_2) yields the the ideal of the stabilizer of the point (a, b). Inequalities for describing strata of the orbit space are derived from the matrix $\mathrm{grad}(z) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$:

$$\begin{aligned} \hat{\Sigma}_0 &= \{(0, 0)\} \\ \hat{\Sigma}_{1,1} \cup \hat{\Sigma}_{1,2} &= \{(z_1, 0) \mid z_1 > 0\} \cup \{(0, z_2) \mid z_2 > 0\} \\ \hat{\Sigma}_2 &= \{(z_1, z_2) \mid z_1 > 0, z_1 z_2 > 0\} \end{aligned}$$

3.4 Examples

We present three examples, two from [1], in order to demonstrate our algorithms. Note that in all three examples the ideals of minors of $\mathrm{grad}(z)$ contain primary components, which do contribute to the stratification, while our algorithms avoid the occurrence of superfluous components. All computations have been performed in the computer algebra system `SINGULAR 2.0` [11], wherein all algorithms have been implemented. Fundamental invariants have been computed by means of the algorithm given in [6], but in example 5 we have used invariants given in [1].

Example 4. (See Example 1.2.1) We consider the action of the representation $\mathrm{id} \oplus \mathrm{id}$ on $\mathbf{R}^4$ of $G = O_2 \subset GL_n(2)\mathbf{R}$, where $\mathrm{id} : G \to GL_n(2)\mathbf{R}$. Note that the chosen fundamental invariants are algebraically independent. The representation- and orbit space can be decomposed in three strata of dimension $0, 2, 3$ respectively. The matrix $\mathrm{grad}(z)$ is given by $\mathrm{grad}(z) = \begin{pmatrix} 4z_1 & 2z_2 & 0 \\ 2z_2 & z_1 + z_3 & 2z_2 \\ 0 & 2z_2 & 4z_3 \end{pmatrix}$. Note that the ideal of 3×3 minors of $\mathrm{grad}(z)$ contains the ideal $\langle z_1 + z_3 \rangle$ as a primary component, but $z_1 = -z_3, z_1 \neq 0$ prevents $\mathrm{grad}(z)$ to be positive semidefinite, hence this component does not contribute to the stratification. Strata of the representation space are obtained from rank conditions on $d\pi(x)$.

Dim.	strata on rep. space	strata of orbit space
0	$\Sigma_0 = \{(0,0,0,0)\}$	$\hat{\Sigma}_0 = \{(0,0,0)\}$
2	$\Sigma_2 = \left\{ \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} \middle\vert t_1t_4 - t_2t_3 = 0 \right\} \setminus \Sigma_0$	$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbf{R}^3 \middle\vert \begin{array}{l} z_2^2 - z_1z_3 = 0 \\ z_1 + z_3 > 0, z_1^2 + 4z_1z_3 + z_3^2 > 0 \end{array} \right\}$
3	$\Sigma_3 = \mathbf{R}^4 \setminus (\Sigma_0 \cup \Sigma_2)$	$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbf{R}^3 \middle\vert \begin{array}{l} z_1 + z_3 > 0, z_1^2 - 2z_2^2 + 6z_1z_3 + z_3^2 > 0, \\ z_1^2z_3 + z_1z_3^2 - z^2(z_1 + z_3) > 0 \end{array} \right\}$

Note that the inequality $z_1^2 + 4z_1z_3 + z_3^2 > 0$ for the description of $\hat{\Sigma}_2$ can be omitted. The inequality $z_1 + z_3 > 0$ cannot be substituted by the principal minors z_1, respectively z_3, which do not vanish identically on $\mathrm{cl}_Z(\hat{\Sigma}_2)$, since such a choice excludes points of the form $z = (0, 0, z_3), z_3 > 0$, respectively $z = (z_1, 0, 0), z_1 > 0$. By using the algorithm STABILIZER we obtain for the associated stabilizers the table

Stratum	Σ_0	Σ_2	Σ_3
Stabilizer	O_2	$\mathbf{Z}_2$	$\{\mathrm{id}\}$

Example 5. The action of $\mathrm{id} \oplus \mathrm{id}$ of the group $G = SO_2$ on $\mathbf{R}^4$ (cf. Example 1 in [1]). The polynomials $\pi_1 = t_1^2 + t_2^2 + t_3^2 + t_4^2, \pi_2 = t_1^2 + t_2^2 - t_3^2 - t_4^2, \pi_3 = -2t_1t_4 + 2t_2t_3, \pi_4 = 2t_1t_3 + 2t_2t_4$, as given in [1], form a minimal set of fundamental invariants of $\mathbf{R}[t_1, t_2, \ldots, t_4]^G$. Since $\pi_1, \pi_2, \ldots, \pi_4$ satisfy the relation $\pi_1^2 - \pi_2^2 - \pi_3^2 - \pi_4^2$, the orbit space is embedded in the hypersurface of $\mathbf{R}^4$. There are only two strata of the orbit space. The 4×4 matrix $\mathrm{grad}(z)$ has rank at most 3 and is given by $\mathrm{grad}(z) = \begin{pmatrix} 4z_1 & 4z_2 & 4z_3 & 4z_4 \\ 4z_2 & 4z_1 & 0 & 0 \\ 4z_3 & 0 & 4z_1 & 0 \\ 4z_4 & 0 & 0 & 4z_1 \end{pmatrix}$.
Note that the ideal of 3×3 minors of $\mathrm{grad}(z)$ contains the ideal $\langle z_1 \rangle$ as a primary component, but, as in the previous example, $z_1 = 0, z_i \neq 0$ for some $1 < i \leq 4$ prevents $\mathrm{grad}(z) \geq 0$, hence this component does not contribute to the stratification. We obtain the following description.

Dim.	strata on rep. space	strata of orbit space
0	$\Sigma_0 = \{(0,0,0,0)\}$	$\{(0,0,0,0)\}$
3	$\Sigma_3 = \mathbf{R}^4 \setminus \Sigma_0$	$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbf{R}^4 \middle\vert \begin{array}{l} z_1^2 - z_2^2 - z_3^2 - z_4^2 = 0 \\ z_1 > 0, z_2^2 + z_3^2 + z_4^2 > 0, \\ z_1z_2^2 + z_1z_3^2 + z_1z_4^2 > 0 \end{array} \right\} \subsetneq \mathbf{R}^4.$

Obviously, $\hat{\Sigma}_3$, respectively $\mathbf{R}^4/G$ can be described by the inequality $z_1 > 0$, respectively $z_1 \geq 0$. The stabilizer associated to Σ_0, respectively, Σ_3 is SO_2, respectively, $\{\mathrm{id}\}$.

Example 6. We consider the action of the representation $\mathrm{id} \oplus \det \cdot \mathrm{id}$ on $\mathbf{R}^6$ of $G = O_3 \subset GL_n(3)\mathbf{R}$, where $\mathrm{id} : G \to GL_n(3)\mathbf{R}$ (i.e., $(g, (x, y)) \mapsto (g \cdot x, \det(g) g \cdot y)$. Algebraically independent fundamental invariants are given by $\pi_1 = t_4^2 + t_5^2 + t_6^2, \pi_2 = t_1^2 + t_2^2 + t_3^2, \pi_3 = t_1^2t_5^2 + t_1^2t_6^2 - 2t_1t_2t_4t_5 - 2t_1t_3t_4t_6 + t_2^2t_4^2 + t_2^2t_6^2 - 2t_2t_3t_5t_6 + t_3^2t_4^2 + t_3^2t_5^2$. The representation- and orbit space can be decomposed in six strata of dimension $0, 1, 1, 2, 2, 3$ respectively. The matrix $\mathrm{grad}(z)$ is given by $\begin{pmatrix} 4z_1 & 0 & 4z_3 \\ 0 & 4z_2 & 4z_3 \\ 4z_3 & 4z_3 & 4z_1z_3 + 4z_2z_3 \end{pmatrix}$. As above, the ideal of 3×3 minors of $\mathrm{grad}(z)$ contains the primary component $\langle z_1 + z_2 \rangle$, which must be excluded, because $\mathrm{grad}(z) \not\geq 0$ for $z_1 = -z_2 \neq 0$. A decomposition of the representation- and the 3-dimensional orbit space is given in the table below.

rk	strata of rep.space	strata of orbit space
0	$\Sigma_0 = \{(0,0,0,0,0,0)\}$	$\hat{\Sigma}_0 = \{(0,0,0)\} \in \mathbf{R}^3$
1	$\Sigma_{1,1} = \{(t_1,t_2,t_3,0,0,0)\} \setminus \Sigma_0$	$\hat{\Sigma}_{1,1} = \left\{ \begin{pmatrix} 0 \\ z_2 \\ 0 \end{pmatrix} \in \mathbf{R}^3 \;\middle|\; z_2 > 0 \right\}$
1	$\Sigma_{1,2} = \{(0,0,0,t_4,t_5,t_6)\} \setminus \Sigma_0$	$\hat{\Sigma}_{1,2} = \left\{ \begin{pmatrix} z_1 \\ 0 \\ 0 \end{pmatrix} \in \mathbf{R}^3 \;\middle|\; z_1 > 0 \right\}$
2	$\Sigma_{2,1} = \{\mathbf{t} \mid t_1t_4 + t_2t_5 + t_3t_6 = 0\} \setminus (\Sigma_{1,1} \cup \Sigma_{1,2} \cup \Sigma_0)$	$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_1z_2 \end{pmatrix} \in \mathbf{R}^4 \;\middle|\; \begin{matrix} z_1^2z_3 + z_2^2z_3 + z_3 > 0, \\ z_1z_3 + z_2z_3 + z_1 + z_2 > 0 \end{matrix} \right\}$
2	$\Sigma_{2,2} = \left\{ \mathbf{t} \;\middle|\; \begin{matrix} t_2t_6 - t_3t_5 = 0 \\ t_1t_6 - t_3t_4 = 0 \\ t_1t_5 - t_2t_4 = 0 \end{matrix} \right\} \setminus (\Sigma_{1,1} \cup \Sigma_{1,2} \cup \Sigma_0)$	$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix} \in \mathbf{R}^4 \;\middle|\; z_1 + z_2 > 0, z_1z_2 > 0 \right\}$
3	$\Sigma_p = \mathbf{R}^6 \setminus (\Sigma_{2,1} \cup \Sigma_{2,2} \cup \Sigma_{1,1} \cup \Sigma_{1,2} \cup \Sigma_0))$	$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbf{R}^3 \;\middle|\; \begin{matrix} z_1 > 0, z_1z_2 > 0, \\ z_1^2z_2z_3 + z_1z_2^2z_3 - z_1z_3^2 - z_2z_3^2 > 0 \end{matrix} \right\}$

Without Theorem 2.2.3.(e), the description of the principal stratum is

$$\hat{\Sigma}_p = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbf{R}^3 \;\middle|\; \begin{matrix} z_1^2z_2z_3 + z_1z_2^2z_3 - z_1z_3^2 - z_2z_3^2 > 0 \\ z_1^2z_3 + 2z_1z_2z_3 + z_2^2z_3 + z_1z_2 - 2z_3^2 > 0 \\ z_1z_3 + z_2z_3 + z_1 + z_2 > 0 \end{matrix} \right\}.$$

The algorithm STABILIZER yields equations defining the stabilizer for each stratum, in particular, the dimension and the number of connected components. After some (easy) calculation for the strata $\Sigma_{1,1}$ and $\Sigma_{1,2}$ we obtain the following table.

Stratum	Σ_0	$\Sigma_{1,1}$	$\Sigma_{1,2}$	$\Sigma_{2,1}$	$\Sigma_{2,2}$	Σ_p
Stabilizer	O_3	O_2	$\mathbf{Z}_2 \times SO_2$	$\mathbf{Z}_2$	SO_2	$\{\mathrm{id}\}$

Conclusion

We have presented an alternative approach for the computation of stratifications of compact Lie groups and have pointed out, that the dimension of a stratum, respectively, its closure is an upper and lower bound for the number of inequalities, which are necessary in order to describe it. In particular, the number of inequalities for describing orbit spaces is bounded by their dimension. The advantage of the approach lies in the fact, that several applications (like the construction of polynomial potentials) do not necessarily need inequalities at all, and that primary decomposition is faster on the representation space than on the orbit space. Additionally, if the representation of G is not orthogonal, out approach may be used to compute the Zariski-closures of the strata of the orbit space. From a practical point of view, the dependence on orthogonal representations should be avoided.

Acknowledgments

The author is very grateful to Peter Ullrich for numerous useful discussions, particularly about semialgebraic geometry.

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