Technische Universität München
Department of Electrical Engineering and Information Technology
Institute for Electronic Design Automation

Optimization Methods for Circuit Design

Exercises

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Contents

Exercise 1: Optimality conditions *without* constraints

1. The following functions $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ must be minimized:

$$\min_{\mathbf{x}} f_1(\mathbf{x}) \triangleq f_1(x_1, x_2) \triangleq x_1^2 + x_2^2 + 2x_1 + 3$$
$$\min_{\mathbf{x}} f_2(\mathbf{x}) \triangleq f_2(x_1, x_2) \triangleq -2x_1^2 + x_2^2$$

   a) Calculate the stationary point of $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ with the help of the 1^{st} order optimality condition.

   b) Use the 2^{nd} order optimality condition to check whether the stationary point of $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ is a minimum.

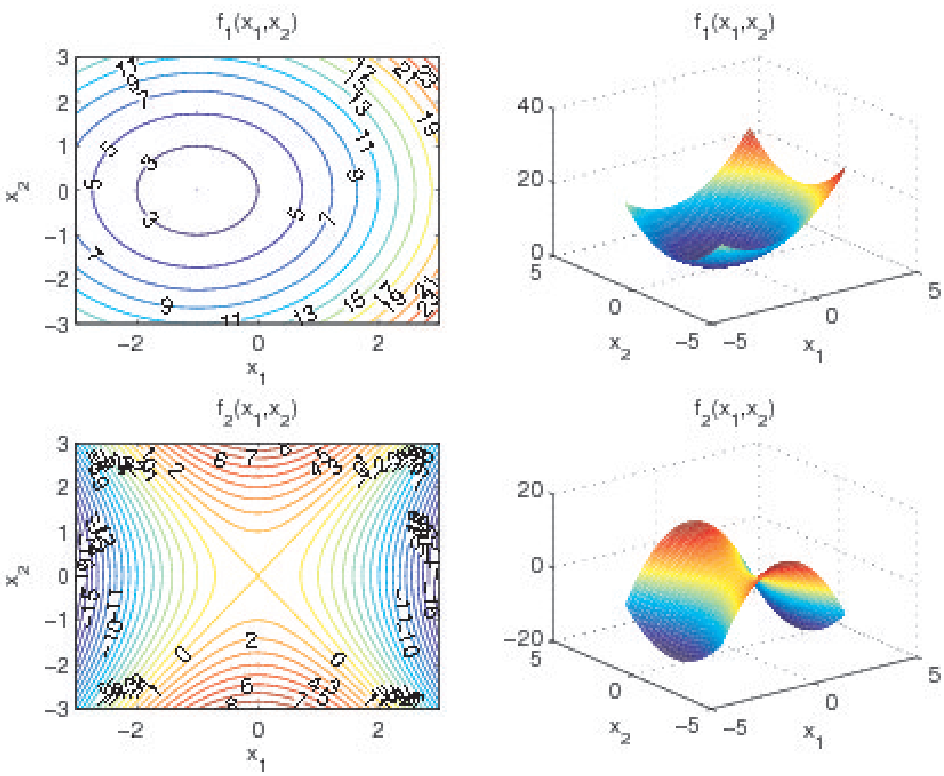


Figure 1.1: function $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$

Exercise 1: Solution

Optimality Conditions:
First Order:

$$\nabla f(\mathbf{x}^*) = \mathbf{0} \tag{I}$$

Second Order:

$$\forall_{\mathbf{r}\neq\mathbf{0}}\ \mathbf{r}^T\,\nabla^2 f(\mathbf{x}^*)\,\mathbf{r}\ \geq\ 0 \quad \Leftrightarrow \quad \nabla^2 f(\mathbf{x}^*) \text{ is positive semidefinite} \tag{II}$$

1)

$f_1(\mathbf{x}) = f_1(x_1, x_2) = x_1^2 + x_2^2 + 2x_1 + 3$

a) find stationary point

$$\nabla f_1(\mathbf{x}) = \text{grad} f_1(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + 2 \\ 2x_2 \end{bmatrix} \stackrel{!}{=} \mathbf{0}$$

$$\mathbf{x}^* = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \text{using} \quad \nabla f_1(\mathbf{x}^*) = \mathbf{0} \quad \text{(first order conditions)}$$

b) check second order condition

$$H = \nabla^2\ f_1(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2} & \frac{\partial^2 f_1}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f_1}{\partial x_2 \partial x_1} & \frac{\partial^2 f_1}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Second Order Condition:

$$\forall_{\mathbf{r}\neq\mathbf{0}}\ \mathbf{r}^T H \mathbf{r} > 0$$

$$\mathbf{r}^T H \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 2r_1^2 + 2r_2^2$$

$$\forall_{\mathbf{r}\neq 0}\ 2\ r_1^2 + 2r_2^2 > 0 \text{ holds} \Rightarrow H \text{ positive definite } \Rightarrow \text{ minimum}$$

$f_2(\mathbf{x}) = -2\ x_1^2 + x_2^2$

a)

$$\nabla f_2(\mathbf{x}) = \begin{bmatrix} -4x_1 \\ 2x_2 \end{bmatrix} \stackrel{!}{=} \mathbf{0} \Rightarrow \mathbf{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b)

$$H = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{r}^T H \mathbf{r} = -4r_1^2 + 2r_2^2$$

$$\forall_{\mathbf{r}\neq\mathbf{0}} - 4r_1^2 + 2r_2^2 > 0 \quad \textbf{not} \text{ fulfilled } \Rightarrow H \text{ not positive definite } \Rightarrow \text{ no minimum}$$

$$\text{e.g., } \mathbf{r} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{r}^T H \mathbf{r} = -4$$

Exercise 2: Optimality conditions *with* constraints

1. The following optimization problem features a linear objective function and two constraints:

$$\min_{\mathbf{x}} f(\mathbf{x}) \triangleq x_1 + x_2 \quad \text{s.t.} \quad c_1(\mathbf{x}) \triangleq -x_1^2 - x_2^2 + 2 \geq 0 \ \wedge \ c_2(\mathbf{x}) \triangleq x_2 \geq 0$$

   a) Calculate all values of $\mathbf{x}$ which meet the 1^{st} order optimality condition.
   b) Check with the help of the 2^{nd} order optimality condition which of these values of $\mathbf{x}$ correspond to a local minimum of the function.

2. The following optimization problem features a quadratic objective function and one constraint:

$$\min_{\mathbf{x}} f(\mathbf{x}) \triangleq \frac{1}{2}(x_1^2 + x_2^2) \text{ s.t. } c(\mathbf{x}) \triangleq -(x_1 + 1) + \beta x_2^2 = 0$$

   a) Calculate all values of $\mathbf{x}$ as a function of β, that meet the 1^{st} order optimality condition.
   b) Check with the help of the 2^{nd} order optimality condition which of these values of $\mathbf{x}$ correspond to a local minimum of the function.

3. The following optimization problem features a linear objective function and four constraints:

$$\begin{aligned}
\min_{\mathbf{x}} f(\mathbf{x}) &\triangleq x_1 + x_2 \\
\text{s.t. } c_1(\mathbf{x}) &\triangleq x_1 \cdot x_2 - 0.5 \geq 0 \ \wedge \\
c_2(\mathbf{x}) &\triangleq -x_1 \cdot x_2 + 2 \geq 0 \ \wedge \\
c_3(\mathbf{x}) &\triangleq x_2 - 0.5 \geq 0 \ \wedge \\
c_4(\mathbf{x}) &\triangleq -x_2 + 2 \geq 0
\end{aligned}$$

   a) Calculate all values of $\mathbf{x}$ which meet the 1^{st} order optimality condition.
   b) Check with the help of the 2^{nd} order optimality condition which of these values of $\mathbf{x}$ correspond to a local minimum of the function.

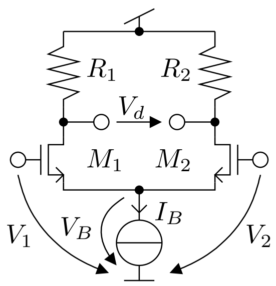
4. We want to construct a warehouse with width w, height h and length l in meters (m). The warehouse must have a capacity of more than 1500 m^3. The construction costs are as follows: walls 4 Euro/m^2, ceiling 6 Euro/m^2 and floor plus ground 12 Euro/m^2. Regulations demand that the width must be exactly double the height of the warehouse.

   a) Formulate the optimization problem to minimize the construction costs, K, of the warehouse.

   b) Calculate the width, length and height of the warehouse that lead to minimal construction costs by applying the 1^{st} order optimality condition.

   c) How do the construction costs approx. change if capacity is reduced by 10%.

5. The simple OTA depicted on the right side is given. Width to length ratios (W_1/L_1), (W_2/L_2) and bias current I_B for maximum static gain $A = v_d/(v_2 - v_1)$ should be found, under the following conditions,



   - $V_{dd} = 1.8V$, $V_1 = V_2 = 0.9V$,
   - M_1 and M_2 are in strong inversion by at least $100mV$,
   - M_1 and M_2 are in saturation by at least $100mV$,
   - $W_1 = W_2$, $L_1 = L_2$,
   - $R_1 = R_2 = 10k\Omega$,
   - $V_B \geq 100mV$.

   Technology data: $\mu_0 C_{ox} = 100\frac{\mu A}{V^2}$, $V_{th0} = 400mV$

   a) Formulate the corresponding optimization problem.

   b) Formulate an equivalent minimization problem for $x_1 = (V_{gs1} - V_{th})/10mV$ and $x_2 = I_B/1\mu A$.

   c) Find the stationary points of the minimization problem using first order optimality conditions.

   d) Determine which of these stationary points are minima.

   e) A change in the design rules requires to increase the minimal inversion voltage to $110mV$. Estimate how maximum gain changes.

Exercise 2: Solution

$$\min\ f(\boldsymbol{x}) \text{ s.t. } \forall_{i\in I}\ c_i \geq 0\ \wedge \forall_{i\in E}\ c_i = 0$$

Lagrange-Function

$$\min\ \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \min\ f(\boldsymbol{x}) - \sum_{i\,\in\,E\cup I} \lambda_i c_i(\boldsymbol{x})$$

First-order conditions (Karush-Kuhn-Tucker)

$$\nabla\mathcal{L}(\mathbf{x}^*) = 0 \tag{I}$$
$$c_i(\mathbf{x}^*) = 0 \qquad i \in E \tag{II}$$
$$c_i(\mathbf{x}^*) \geq 0 \qquad i \in I \tag{III}$$
$$\lambda_i^* \geq 0 \qquad i \in I \tag{IV}$$
$$\lambda_i^* c_i(\mathbf{x}^*) = 0 \qquad i \in I \cup E \tag{V}$$

Second-order conditions

Set of active constraints:

$$\mathcal{A}_+^* = \{j \in \mathcal{A}(\mathbf{x}^*) \mid j \in E \vee \lambda_j^* > 0\} \tag{VI}$$

Set of feasible stationary directions:

$$\mathcal{F}_r = \left\{ \mathbf{r} \left| \begin{array}{rlll} \mathbf{r} & \neq \mathbf{0} & & \\ \nabla c_i(\mathbf{x}^*)^T\mathbf{r} & = 0 & i & \in \mathcal{A}_+^* \\ \nabla c_i(\mathbf{x}^*)^T\mathbf{r} & \geq 0 & i & \in \mathcal{A}(\mathbf{x}^*)\backslash\mathcal{A}_+^* \end{array} \right. \right\} \tag{VII}$$

Minimum condition:

$$\forall_{\mathbf{r}\in\mathcal{F}_r}\ \mathbf{r}^T\nabla^2\mathcal{L}(\mathbf{x}^*)\mathbf{r} \geq 0 \tag{VIII}$$

1)

a)

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = x_1 + x_2 - \lambda_1(-x_1^2 - x_2^2 + 2) - \lambda_2 x_2$$

$$\nabla\mathcal{L}(\mathbf{x}) = \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial x_1} \\ \frac{\partial\mathcal{L}}{\partial x_2} \end{bmatrix} \overset{!}{=} 0$$

$$\lambda_i c_i(\mathbf{x}) = 0 \qquad i \in \{1,2\} \quad \text{compare to (V)}$$
$$\lambda_1(-x_1^2 - x_2^2 + 2) = 0 \tag{2.1}$$
$$\lambda_2 x_2 = 0 \tag{2.2}$$
$$c_i(\mathbf{x}) \geq 0 \qquad i \in \{1,2\} \quad \text{compare to (III)}$$
$$-x_1^2 - x_2^2 + 2 \geq 0$$
$$x_2 \geq 0$$
$$\lambda_1 \geq 0 \qquad \text{compare to (IV)}$$
$$\lambda_2 \geq 0$$

$$\frac{\partial \mathcal{L}}{x_1} = \nabla_{x_1}\mathcal{L} = 1 + 2\lambda_1 x_1 \stackrel{!}{=} 0 \qquad \text{compare to (I)} \tag{2.3}$$

$$\frac{\partial \mathcal{L}}{x_2} = \nabla_{x_2}\mathcal{L} = 1 + 2\lambda_1 x_2 - \lambda_2 \stackrel{!}{=} 0 \tag{2.4}$$

Find active set (case differentiation)

1. $c_1(\mathbf{x})$ inactive $\Rightarrow \lambda_1 = 0$
   Equation (2.3) is always violated $\Rightarrow$ $c_1(\mathbf{x})$ active $\Rightarrow$ $c_1(\boldsymbol{x}) = 0$
$$-x_1^2 - x_2^2 + 2 = 0 \tag{2.5}$$

2. $c_2(\mathbf{x})$ inactive $\Rightarrow \lambda_2 = 0$
$$\left.\begin{aligned} 1 + 2\lambda_1 x_1 &= 0 \\ 1 + 2\lambda_1 x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 = -\frac{1}{2\lambda_1} < 0 \tag{2.6}$$
   $\Rightarrow$ always contradicts $c_2(\mathbf{x}^*) = x_2^* \geq 0 \Rightarrow c_2(\mathbf{x}^*)$ active $\Rightarrow x_2^\star = 0$

$$\stackrel{(2.4)}{\Rightarrow} 1 - \lambda_2 = 0 \Leftrightarrow \lambda_2^* = 1$$

$$\stackrel{(2.5)}{\Rightarrow} -x_1^2 + 2 = 0 \Leftrightarrow x_1^\star = \pm\sqrt{2}$$

$$(2.3) \Leftrightarrow \lambda_1^* = -\frac{1}{2x_1^*} = \frac{1}{2(\pm\sqrt{2})} \stackrel{!}{>} 0 \text{ because of (IV)}$$

$$x_1^* = -\sqrt{2} \qquad \lambda_1^* = \frac{1}{2\sqrt{2}}$$

b)

$$H = \nabla^2\mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = \begin{bmatrix} 2\lambda_1^* & 0 \\ 0 & 2\lambda_1^* \end{bmatrix} \stackrel{\lambda_1^* = \frac{1}{2\sqrt{2}}}{=} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Two ways of reasoning:

1) Determine active set (VI):

$$\lambda_1^* = \frac{1}{2\sqrt{2}} > 0 \qquad \lambda_2^* = 1 > 0$$

$$\Rightarrow A_+^* = \{1, 2\}$$

Determine set of feasible stationary directions (VII):

$$\nabla c_1(\boldsymbol{x}^*)^T \boldsymbol{r} = 0$$
$$\Leftrightarrow \begin{bmatrix} -2x_1^* & -2x_2^* \end{bmatrix} \boldsymbol{r} = 0$$
$$\Leftrightarrow \begin{bmatrix} 2\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0$$
$$\Leftrightarrow 2\sqrt{2}\, r_1 = 0 \;\Leftrightarrow\; r_1 = 0$$

$$\nabla c_2(\boldsymbol{x}^*)^T \boldsymbol{r} = 0$$
$$\Leftrightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0$$
$$\Leftrightarrow r_2 = 0$$

$$\Rightarrow \mathcal{F}_r = \emptyset$$

$\Rightarrow$ No direction of descent $\Rightarrow$ minimum.

2) H is positive semidefinite for all $\boldsymbol{r}$ $\Rightarrow$ Condition (VIII) fulfilled for all possible $\mathcal{F}_r$.

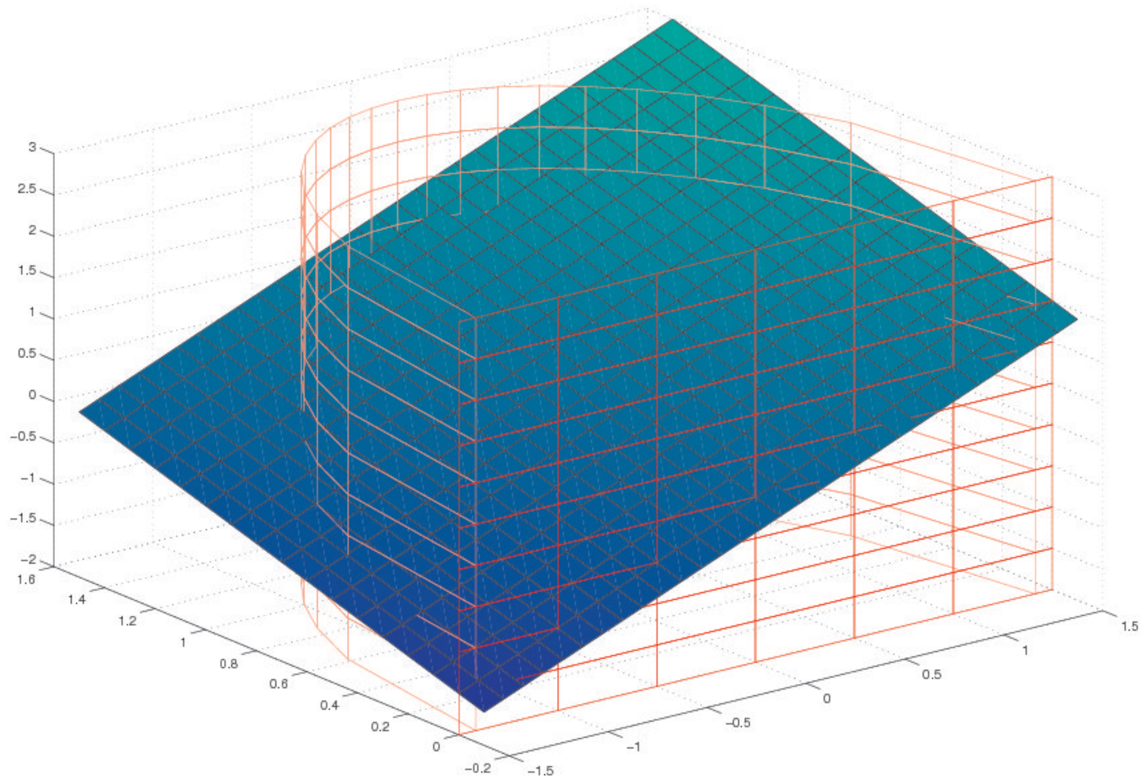


Figure 2.1: Objective function and constraints of task 1

2)

a)

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \frac{1}{2}(x_1^2 + x_2^2) - \lambda[-(x_1 + 1) + \beta x_2^2]$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_1^* + \lambda \overset{!}{=} 0 \Rightarrow x_1^* = -\lambda^*$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = x_2^* - 2\lambda\beta x_2^* \overset{!}{=} 0$$

$$\Leftrightarrow x_2^*(1 - 2\lambda\beta) = 0$$

$$\Leftrightarrow x_2^* = 0 \qquad \vee (1 - 2\lambda\beta) = 0$$

<u>Two cases</u>

1. $x_2^* = 0$
   Because of constraint: $c(\boldsymbol{x}^*) = 0$

$$\Leftrightarrow -(x_1^* + 1) = 0 \Leftrightarrow x_{1,1}^* = -1 \Rightarrow \lambda_1^* = 1$$

2. $1 - 2\lambda^*\beta = 0$

$$\lambda_2^* = \frac{1}{2\beta} \;\Leftrightarrow\; x_{1,2}^* = -\frac{1}{2\beta}$$

$$c(\boldsymbol{x}^*) = 0 \Rightarrow x_{2,2}^* = \pm\sqrt{\frac{2\beta - 1}{2\beta^2}}$$

Three stationary points: $(-1|0)$ $(-\frac{1}{2\beta}|\sqrt{\frac{2\beta-1}{2\beta^2}})$ $(-\frac{1}{2\beta}|-\sqrt{\frac{2\beta-1}{2\beta^2}})$

b)

$$H = \nabla^2\mathcal{L} = \begin{bmatrix} 1 & 0 \\ 0 & 1 - 2\lambda^*\beta \end{bmatrix}$$

$$\nabla c = \begin{bmatrix} -1 \\ 2\beta x_2^\star \end{bmatrix} \qquad \nabla c^T \boldsymbol{r} \Leftrightarrow -r_1 + 2\beta x_2^\star r_2 = 0 \Rightarrow r_1 = 2\beta x_2^\star r_2 \tag{2.7}$$

$$\mathcal{F}_r = \{\boldsymbol{r} | r_2 \neq 0 \wedge r_1 = 2\beta x_2^\star r_2\} \tag{2.8}$$

$$\boldsymbol{r}^T H \boldsymbol{r} = \boldsymbol{r}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 - 2\lambda\beta \end{bmatrix} \boldsymbol{r} = r_1^2 + (1 - 2\lambda\beta) r_2^2$$

$$\forall_{\boldsymbol{r} \in \mathcal{F}_r} \boldsymbol{r}^T H \boldsymbol{r} > 0 \Leftrightarrow \forall_{r_2 \neq 0} \left[4\beta^2 r_2^2 x_2^{\star 2} + (1 - 2\lambda\beta) r_2^2\right] \geq 0$$

$$\Leftrightarrow \forall_{r_2 \neq 0} (4\beta^2 x_2^{\star 2} - 2\lambda\beta + 1) r_2^2 > 0$$

$$\Leftrightarrow (4\beta^2 x_2^{\star 2} - 2\lambda\beta + 1) > 0 \tag{2.9}$$

For the two cases from above

Case 1 $\lambda_1^\star = 1, x_{2,1}^\star = 0 \overset{(2.9)}{\Rightarrow} (-2\beta + 1) > 0 \overset{\beta<\frac{1}{2}}{\Leftrightarrow}$ true $\rightarrow$ minimum

Case 2 $\lambda_2^\star = \frac{1}{2\beta}, x_{2,2}^\star = \pm\sqrt{\frac{2\beta-1}{2\beta^2}} \overset{(2.9)}{\Rightarrow} 2\beta - 1 > 0 \overset{\beta>\frac{1}{2}}{\Leftrightarrow}$ true $\rightarrow$ minimum

3)

a)

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = x_1 + x_2 - \lambda_1(x_1x_2 - 0,5) - \lambda_2(-x_1x_2 + 2) - \lambda_3(x_2 - 0,5) - \lambda_4(-x_2 + 2)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda_1 x_2 + \lambda_2 x_2 \overset{!}{=} 0 \Rightarrow x_2 = \frac{1}{\lambda_1 - \lambda_2} \tag{2.10}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \lambda_1 x_1 + \lambda_2 x_1 - \lambda_3 + \lambda_4 \overset{!}{=} 0 \Rightarrow x_1 = \frac{1 - \lambda_3 + \lambda_4}{\lambda_1 - \lambda_2} \tag{2.11}$$

Determine active set graphically by plotting constraints and gradient:

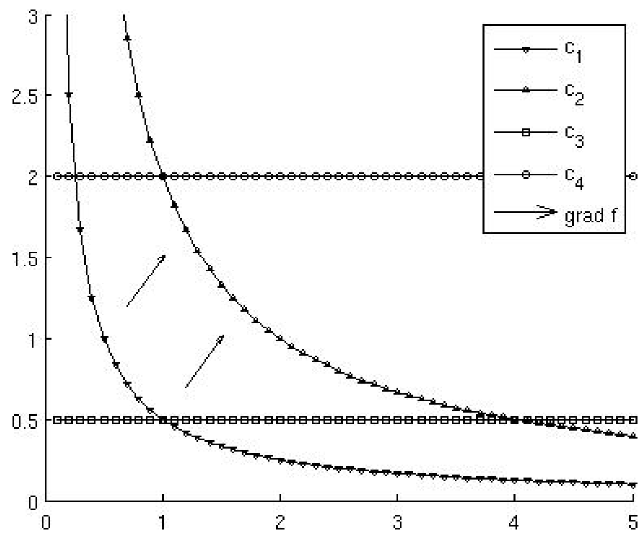


Figure 2.2: Gradient of f and constraints

$$\Rightarrow c_1(\boldsymbol{x}) = 0 \ \wedge \ \lambda_2^* = \lambda_3^* = \lambda_4^* = 0$$

$$\Rightarrow x_1^* x_2^* = 0.5$$

$$(2.10) \Rightarrow \ x_1^* = \frac{1}{\lambda_1} \qquad (2.11) \Rightarrow \ x_2^* = \frac{1}{\lambda_1} \Rightarrow \lambda_1^* = \sqrt{2}$$

$$\Rightarrow x_1^* = x_2^* = \frac{1}{\sqrt{2}}$$

b)

$$H = \nabla^2 \mathcal{L} = \begin{bmatrix} 0 & -\lambda_1^* \\ -\lambda_1^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix}$$

$$\nabla c_1(\boldsymbol{x}^*) = \begin{bmatrix} x_2^* \\ x_1^* \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Determine $\mathcal{F}_r$:

$$\nabla c_1^T \boldsymbol{r} = 0 \Leftrightarrow \frac{1}{\sqrt{2}} r_1 + \frac{1}{\sqrt{2}} r_2 = 0 \Leftrightarrow r_1 + r_2 = 0 \Leftrightarrow r_1 = -r_2$$

$$\mathcal{F}_r = \{\boldsymbol{r} \mid r_1 = -r_2\}$$

Condition (VIII):

$$\boldsymbol{r}^T H \boldsymbol{r} = -2\sqrt{2} r_1 r_2$$

$$\forall_{\boldsymbol{r} \in \mathcal{F}_r} \boldsymbol{r}^T H \boldsymbol{r} \geq 0 \Leftrightarrow \forall_{r_1} 2\sqrt{2} r_1^2 \geq 0 \Leftrightarrow \text{True} \Longrightarrow \text{Minimum}$$

4)

a)

$$\min_{b,h,l} K = 18\, w\, l + 8\, w\, h + 8\, l\, h \text{ s.t. } w\, h\, l \geq 1500 \wedge w = 2\, h (\wedge w > 0 \wedge l > 0 \wedge h > 0)$$

<u>Simplification:</u> Substitute w by $2\,h$:

$$\min_{h,l} K = 36\, h\, l + 16\, h^2 + 8\, l\, h = 44\, h\, l + 16\, h^2 \text{ s.t. } 2\, h^2\, l - 1500 \geq 0$$

b)

$$\mathcal{L}(h, l, \lambda) = 44\, h\, l + 16h^2 - \lambda(2\, h^2\, l - 1500)$$

Assumption: inactive, i.e., $\lambda = 0 \Rightarrow h = 0$, constraint violated $\Rightarrow$ constraint active.

$$\frac{\partial \mathcal{L}}{\partial h} = 44l + 32h - 4\lambda h\, l \overset{!}{=} 0 \tag{2.12}$$

$$\frac{\partial \mathcal{L}}{\partial l} = 44h - 2\lambda h^2 \overset{!}{=} 0 \Leftrightarrow \lambda^* = \frac{22}{h^*} \; (h > 0) \tag{2.13}$$

$$2h^2\, l - 1500 = 0 \Leftrightarrow l^* = \frac{1500}{2h^{*2}} \tag{2.14}$$

Insert (2.13) and (2.14) into (2.12):

$$44 \frac{1500}{2h^{*2}} + 32h^* - 4\frac{22}{h^*} h^* \frac{1500}{2h^{*2}} = 0$$

$$\Leftrightarrow -\frac{33000}{h^{*2}} + 32h^* = 0$$

$$\Leftrightarrow 32h^{*3} = 33000$$

$$\Leftrightarrow h^* = \sqrt[3]{\frac{33000}{32}} \approx 10 \quad b^* \approx 20 \quad l^* \approx 7.5$$

$$\Rightarrow K^* = 4900$$

c)

$$\lambda^* = \frac{22}{h^*} \approx 2.2$$
$$\Delta K = \Delta c \lambda^* \approx -1500 \cdot 0.1 \cdot 2.2 \approx -330$$

5)

a)

$$\max_{\frac{W_1}{L_1}, \frac{W_2}{L_2}, I_B} A \quad \text{s.t.} \quad V_{gs1} - V_{th} \geq 100mV \wedge V_{gs2} - V_{th} \geq 100mV \wedge V_B \geq 100mV$$
$$\wedge V_{ds1} - (V_{gs1} - V_{th}) \geq 100mV \wedge V_{ds2} - (V_{gs2} - V_{th}) \geq 100mV$$
$$\wedge W_1 = W_2 \wedge L_1 = L_2 \wedge R_1 = R_2 = 10k\Omega$$
$$\wedge \text{Kirchhoff-Laws, transistor models}$$

b)

$$(W_1 = W_2 \wedge L_1 = L_2 \wedge R_1 = R_2) \Rightarrow I_{ds1} = I_{ds2} = \frac{I_B}{2} = x_2 \cdot 0.5\mu A$$
$$I_{ds1} = \frac{\mu C_{ox}}{2} \frac{W_1}{L_1} (V_{gs1} - V_{th})^2$$
$$\Leftrightarrow \frac{W_1}{L_1} = \frac{2I_{ds1}}{\mu C_{ox}(V_{gs1} - V_{th})^2} = \frac{x_2}{100 \cdot 0.01^2 x_1^2} = 100 \frac{x_2}{x_1^2}$$
$$g_{m1} = \frac{2I_{ds1}}{V_{gs1} - V_{th}} = 100\mu S \frac{x_2}{x_1}$$
$$V_B = V_1 - V_{gs1}$$
$$V_{ds1} = V_{dd} - \frac{R_1 I_B}{2} - V_B = V_{dd} - \frac{R_1 I_B}{2} - V_1 + V_{gs1}$$
$$A = g_{m1} R_{out} = 100\mu S \frac{x_2}{x_1} \cdot 10k\Omega = \frac{x_2}{x_1}$$
$$V_{gs1} - V_{th} \geq 100mV \Leftrightarrow x_1 \geq 10$$
$$V_B \geq 100mV \Leftrightarrow V_{gs1} - V_{th} \leq V_1 - V_{th} - 100mV \Leftrightarrow V_{gs1} - V_{th} \leq 400mV$$
$$\Leftrightarrow x_1 \leq 40$$
$$V_{ds1} - (V_{gs1} - V_{th}) \geq 100mV \Leftrightarrow V_{dd} - \frac{R_1 I_B}{2} - V_1 + \cancel{V_{gs1}} - \cancel{V_{gs1}} + V_{th} \geq 100mV$$
$$\Leftrightarrow 5k\Omega 1\mu A x_2 \leq 1.2V \Leftrightarrow x_2 \leq 240$$

$$\min_{x_1, x_2} -\frac{x_2}{x_1} \quad \text{s.t.} \quad x_1 \geq 10 \wedge x_1 \leq 40 \wedge x_2 \leq 240$$

c)

$$\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3) = -\frac{x_2}{x_1} - \lambda_1(x_1-10) - \lambda_2(40-x_1) - \lambda_3(240-x_2)$$
$$\frac{\partial\mathcal{L}}{\partial x_1} = \frac{x_2}{x_1^2} - \lambda_1 + \lambda_2 \stackrel{!}{=} 0$$
$$\frac{\partial\mathcal{L}}{\partial x_2} = -\frac{1}{x_1} + \lambda_3 \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \lambda_3 = \frac{1}{x_1}$$
$$\Rightarrow \lambda_3 > 0 \Rightarrow c_3 \text{active} \Rightarrow x_2 = 240$$

1. Assume c_1 inactive, i.e., $\lambda_1 = 0$
   a) Assume c_2 inactive, i.e., $\lambda_2 = 0 \Rightarrow x_2 = 0 \Rightarrow$ contradiction
   b) Assume c_2 active, i.e., $x_1 = 40 \Rightarrow \lambda_3 = \frac{1}{40}, \lambda_2 = -\frac{3}{20} < 0 \Rightarrow$ contradiction
2. Assume c_1 active, i.e., $x_1 = 10$
   a) Assume c_2 inactive, i.e., $\lambda_2 = 0 \Rightarrow \lambda_1 = \frac{240}{100}, \lambda_3 = \frac{1}{10}$ solution
   b) Assume c_2 active, i.e., $x_1 = 40$ contradiction

Solution:

$$A = 24 \qquad\qquad \frac{W_1}{L_1} = \frac{W_2}{L_2} = 240$$

d)

$$H = \begin{bmatrix} -2\frac{x_2}{x_1^3} & \frac{1}{x_1^2} \\ \frac{1}{x_1^2} & 0 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -48 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\nabla c_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \nabla c_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \nabla c_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$\mathcal{A} = \mathcal{A}_+^\star = \{1,3\}$$
$$\nabla c_1 \cdot \boldsymbol{r} = r_1 \qquad \nabla c_3 \cdot \boldsymbol{r} = -r_2$$
$$\mathcal{F}_r = \{\boldsymbol{r} | \boldsymbol{r} \neq 0 \wedge r_1 = 0 \wedge -r_2 = 0\} = \emptyset$$
$$\Rightarrow \text{minimum}$$

e)

$$\Delta A = \lambda_1^\star \Delta c_1$$
$$c_1(\boldsymbol{x}) = x_1 - 10 \Rightarrow \tilde{c}_1(\boldsymbol{x}) = x_1 - 11$$
$$\Delta c_1 = -1$$
$$\Delta A = \frac{24}{10}\cdot(-1) = -2.4$$
$$\text{Real } \tilde{x}_1^\star = 11 \quad \Delta A = \frac{240}{11} - 24 = -2.2$$

Exercise 3: Worst-case analysis

1. The following RC circuit has a time constant of $\tau = R \cdot C$ to measure its performance. The nominal values of the device parameters are $R_0 = 1$ and $C_0 = 1$. The specified performance is $0.5 \leq \tau \leq 2$. The device parameters vary uncorrelated with a standard deviation of $\sigma_C = 0.2$ and $\sigma_R = 0.2$ around the nominal device parameter values due to process variations.

   With the help of worst-case analysis, we try to find out if the worst-case performance value does meet the specification.

   a) Apply a classical worst-case analysis and calculate the worst-case performance value. The tolerance box is defined by a maximum deviation of 3σ of the device parameters. Compare the worst-case performance value calculated by classical worst-case analysis with the real worst-case parameter value in the tolerance region.

   b) Apply a realistic worst-case analysis and calculate the worst-case performance value for the given specification bounds for $\beta_w = 3$. Compare the worst-case performance value calculated by realistic worst-case analysis with the real worst-case parameter value in the tolerance region and with classical worst-case analysis.

   c) Apply a general worst-case analysis and and calculate the worst-case performance value for the given specification bounds for $\beta_w = 3$.

2. The robustness of the simple OTA from Exercise 2.5 shall be investigated. The process variations of mobility μ and treshhold voltage V_{th} of a transistor are modeled as follows.

$$\mu = \mu_0(1 + 0.05x_1) \qquad V_{th} = V_{th0}(1 + 0.05x_2)$$
$$\boldsymbol{x} = [x_1\ x_2]^T \sim \mathcal{N}(\mathbf{0}, I)$$

   Random variables x_1 and x_2 are global statistical parameters of the circuit, i.e., they have the same value for all transistors. The static gain A of the circuit should be greater than 20.

   a) Show, that $A = A_{nom}(1 + 0.05x_1)(1 - 0.2x_2)$ holds, where A_{nom} is the gain in the nominal case.

   b) Determine worst-case parameter set and performance value for static gain using a classical worst-case analysis and a tolerance box of 3σ.

   c) Determine the worst-case parameter set and performance value for static gain using a realistic worst-case analysis and $\beta_w = 3$.

   d) Determine the worst-case parameter set and performance value for static gain using a general worst-case analysis and $\beta_w = 3$.

Exercise 3: Solution

1)

a) Classical worst case analysis

$$T_B = \left\{ \begin{bmatrix} R \\ C \end{bmatrix} \middle| \begin{bmatrix} R_0 - 3\sigma_R \\ C_0 - 3\sigma_C \end{bmatrix} \leq \begin{bmatrix} R \\ C \end{bmatrix} \leq \begin{bmatrix} R_0 + 3\sigma_R \\ C_0 + 3\sigma_C \end{bmatrix} \right\}$$
$$= \left\{ \begin{bmatrix} R \\ C \end{bmatrix} \middle| 0.4 \leq R \leq 1.6 \ \wedge 0.4 \leq C \leq 1.6 \right\}$$

Linearization of τ in the nominal point:

$$\begin{bmatrix} R_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\nabla\tau\left(\begin{bmatrix} R \\ C \end{bmatrix}\right) = \begin{bmatrix} C \\ R \end{bmatrix}$$
$$\overline{\tau} = \tau(R_0, C_0) + \nabla\tau(R_0, C_0)^T \begin{bmatrix} R - R_0 \\ C - C_0 \end{bmatrix}$$
$$\overline{\tau} = 1 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} R - 1 \\ C - 1 \end{bmatrix} = R + C - 1$$

Lower Specification: $\tau \geq 0.5 \ \Rightarrow \ \tau - 0.5 \geq 0 \Rightarrow \max\tau$

$$\left.\frac{\partial\overline{\tau}}{\partial R}\right|_{R_0,C_0} = 1 > 0 \ \Leftrightarrow \ R_{w1} = R_L = 0.4$$
$$\left.\frac{\partial\overline{\tau}}{\partial C}\right|_{R_0,C_0} = 1 > 0 \ \Leftrightarrow \ C_{w1} = C_L = 0.4$$
$$\Rightarrow \quad \overline{\tau}(R, C) = -0.2$$
$$\text{for comparison } \tau(R, C) = 0.16$$

Upper Specification: $\tau \leq 2 \ \Rightarrow \ 2 - \tau \geq 0 \Rightarrow \min\tau$

$$\left.\frac{\partial\overline{\tau}}{\partial R}\right|_{R_0,C_0} = 1 > 0 \ \Leftrightarrow \ R_{w2} = R_U = 1.6$$
$$\left.\frac{\partial\overline{\tau}}{\partial C}\right|_{R_0,C_0} = 1 > 0 \ \Leftrightarrow \ C_{w2} = C_U = 1.6$$
$$\Rightarrow \quad \overline{\tau}(R, C) = 2.2$$
$$\text{for comparison } \tau(R, C) = 2.56$$

b) Realistic worst case analysis

$$T_E = \left\{ \boldsymbol{x} \middle| (\boldsymbol{x} - \boldsymbol{x}_0)^T C^{-1} (\boldsymbol{x} - \boldsymbol{x}_0) \leq \beta_w^2 \right\}$$
$$\text{covariance } K(R, C) = \rho_{RC}\sigma_R\sigma_C$$

C = covariance matrix (explained later):

$$C = \begin{bmatrix} \sigma_R^2 & K(R, C) \\ K(R, C) & \sigma_C^2 \end{bmatrix} = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix}$$

$$\sigma_{\overline{\tau}} = \sqrt{\boldsymbol{g}^T C \boldsymbol{g}} = \sqrt{\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \sqrt{0.08} = 0.283$$

Lower Specification: $\tau \geq 0.5 \;\Rightarrow\; \tau - 0.5 \geq 0 \Rightarrow \max \tau$

$$\begin{aligned} \boldsymbol{x}_{WL} - \boldsymbol{x_0} &= -\frac{\beta_w}{\sigma_{\overline{\tau}}} C \boldsymbol{g} \\ \begin{bmatrix} R_{WL} \\ C_{WL} \end{bmatrix} &= \begin{bmatrix} R_0 \\ C_0 \end{bmatrix} - \frac{\beta_w}{\sigma_{\overline{\tau}}} C \boldsymbol{g} \\ \begin{bmatrix} R_{WL} \\ C_{WL} \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{3}{0.283} \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.58 \\ 0.58 \end{bmatrix} \\ \overline{\tau}_{WL} &= \tau_0 - \beta_w \sigma_{\overline{\tau}} = 1 - 3 \cdot 0.283 = 0.151 \\ \text{for comparison } \tau(R_{WL}, C_{WL}) &= 0.336 \end{aligned}$$

Upper Specification: $\tau \leq 2 \;\Rightarrow\; 2 - \tau \geq 0 \Rightarrow \min \tau$

$$\begin{aligned} \boldsymbol{x}_{WU} - \boldsymbol{x_0} &= \frac{\beta_w}{\sigma_{\overline{\tau}}} C \boldsymbol{g} \\ \begin{bmatrix} R_{WU} \\ C_{WU} \end{bmatrix} &= \begin{bmatrix} R_0 \\ C_0 \end{bmatrix} + \frac{\beta_w}{\sigma_{\overline{\tau}}} C \boldsymbol{g} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{0.283} \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1.42 \\ 1.42 \end{bmatrix} \\ \overline{\tau}_{WU} &= \tau_0 + \beta_w \sigma_{\overline{\tau}} = 1 + 3 \cdot 0.283 = 1.85 \\ \text{for comparison } \tau(R_{WU}, C_{WU}) &= 2.02 \end{aligned}$$

c) General worst case analysis

Upper Specification: $\tau \leq 2 \;\Rightarrow\; 2-\tau \geq 0 \Rightarrow \min\tau \Rightarrow$ WC-Analysis $\max\tau \Rightarrow \min -\tau$

$$\min -\tau \text{ s.t. } (\boldsymbol{x}-\boldsymbol{x}_0)^T C^{-1}(\boldsymbol{x}-\boldsymbol{x}_0) \leq \beta_w^2$$
$$\Leftrightarrow \quad \min -R\,C \text{ s.t. } \begin{bmatrix} R-1 \\ C-1 \end{bmatrix}^T \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix}^{-1} \begin{bmatrix} R-1 \\ C-1 \end{bmatrix} \leq 3^2$$
$$\begin{bmatrix} R-1 \\ C-1 \end{bmatrix}^T \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} R-1 \\ C-1 \end{bmatrix} \leq 9$$

$$\Rightarrow \quad \min -R\,C \text{ s.t. } 9-25(R-1)^2-25(C-1)^2 \geq 0$$

$$\mathcal{L}(R,C,\lambda) = -R\,C - \lambda[9-25(R-1)^2-25(C-1)^2]$$
$$\frac{\partial\mathcal{L}}{\partial R} = -C + 25\lambda\,2\,(R-1) \overset{!}{=} 0 \tag{3.1}$$
$$\frac{\partial\mathcal{L}}{\partial C} = -R + 25\lambda\,2\,(C-1) \overset{!}{=} 0 \tag{3.2}$$

<u>Determine active set:</u>
Assume constraint inactive $\Rightarrow \lambda = 0 \overset{(3.1),(3.2)}{\Rightarrow} R = 0, C = 0$

$$9-25-25 \geq 0 \Rightarrow \text{ constraint violated } \Rightarrow \text{ constraint active}$$

$$9-25(R-1)^2-25(C-1)^2 = 0 \tag{3.3}$$

$$\text{(3.1) - (3.2):} \quad -C+50\lambda(R-1)+R-50\lambda(C-1) = 0$$
$$\Leftrightarrow \quad (R-C)+50\lambda(R-C) = 0$$
$$\Leftrightarrow \quad \underbrace{(50\lambda+1)}_{\lambda\geq 0\Rightarrow(\ldots)>0}(R-C) = 0$$
$$\Rightarrow R = C$$

$$\text{into (3.3)} \quad 25(C-1)^2+25(C-1)^2 = 9$$
$$\Leftrightarrow \quad (C-1)^2 = \frac{9}{50}$$
$$\Leftrightarrow \quad C = \pm\sqrt{\frac{9}{50}}+1$$

$$\begin{bmatrix} C_{WU,1} \\ R_{WU,1} \end{bmatrix} = \begin{bmatrix} 1.42 \\ 1.42 \end{bmatrix}; \quad \begin{bmatrix} C_{WU,2} \\ R_{WU,2} \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.58 \end{bmatrix}$$

$$\text{(3.1): } \lambda_{WU,1} = \frac{C_{WU,1}}{50(R_{WU,1}-1)} = 0.07 \overset{!}{\geq} 0 \;\Rightarrow\; \text{o.k.}$$
$$\lambda_{WU,2} = -0.07 \overset{!}{\geq} 0 \;\Rightarrow\; \text{violated}$$

$\Rightarrow$ Solution: $R_{WU,1}$, $C_{WU,1}$, $\lambda_{WU,1}$
Similar for $\tau \geq 0.5$

Comparison of methods

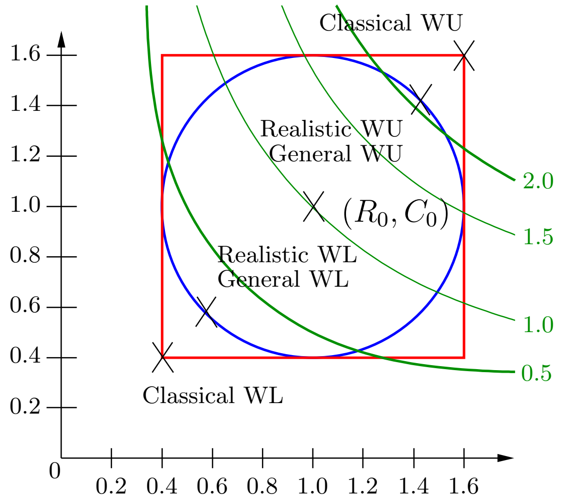
Spec: $\tau \geq 0.5$:

Method	R_{WL}	C_{WL}	$\overline{\tau_{WL}}$	τ_{WL}
Classical WC	0.4	0.4	-0.2	0.16
Realistic WC	0.58	0.58	0.15	0.336
General WC	0.58	0.58	N/A	0.336

Spec: $\tau \leq 2$:

Method	R_{WU}	C_{WU}	$\overline{\tau_{WU}}$	τ_{WU}
Classical WC	1.6	1.6	2.2	2.56
Realistic WC	1.42	1.42	1.85	2.02
General WC	1.42	1.42	N/A	2.02

Realistic and general case exceptionally equal, because of symmetry $R = C$ and $\sigma_R = \sigma_C$.



2)

a)

$$
\begin{aligned}
A &= g_{m1} R_{out} \\
g_{m1} &= \mu C_{ox} \frac{W}{L} (V_{gs1} - V_{th1}) \\
&= (1 + 0.05 x_1) \mu_0 C_{ox} \frac{W}{L} (V_{gs1} - (1 + 0.05 x_2) V_{th}) \\
&= (1 + 0.05 x_1) \mu_0 C_{ox} \frac{W}{L} (V_{gs1} - V_{th} - 0.05 x_2 V_{th}) \\
&= (1 + 0.05 x_1) \mu_0 C_{ox} \frac{W}{L} (V_{ov1} - 0.05 x_2 V_{th}) \\
&= (1 + 0.05 x_1) \underbrace{\mu_0 C_{ox} \frac{W}{L} V_{ov1}}_{=:g_{m1,nom}} \left(1 - 0.05 x_2 \frac{V_{th}}{V_{ov}}\right) \\
&= g_{m1,nom} (1 + 0.05 x_1)(1 - 0.2 x_2) \\
A &= \underbrace{R_1 g_{m1,nom}}_{A_{nom}} (1 + 0.05 x_1)(1 - 0.2 x_2) \\
&= A_{nom} (1 + 0.05 x_1)(1 - 0.2 x_2)
\end{aligned}
$$

b)

$$T_B = \left\{ \boldsymbol{x} | \begin{bmatrix} -3 \\ -3 \end{bmatrix} \leq \boldsymbol{x} \leq \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$$
$$= \{ \boldsymbol{x} | -3 \leq x_1 \leq 3 \wedge -3 \leq x_2 \leq 3 \}$$

linearization of A in the nominal point:

$$\overline{A}(\boldsymbol{x}) = A_0 + \boldsymbol{g}^T (\boldsymbol{x} - \boldsymbol{x_0})$$
$$\boldsymbol{x}_0 = \boldsymbol{0}$$
$$A_0 = A(\boldsymbol{x}_0) = A_{nom} = 24$$
$$\boldsymbol{g} = \nabla A(\boldsymbol{x}_0) = A_{nom} \left. \begin{bmatrix} 0.05(1 - 0.2x_2) \\ -0.2(1 + 0.05x_1) \end{bmatrix} \right|_{\boldsymbol{x}_0} = \begin{bmatrix} 1.2 \\ -4.8 \end{bmatrix}$$
$$\overline{A}(\boldsymbol{x}) = 24 + [1.2 \ \ -4.8] \boldsymbol{x} = 24 + 1.2x_1 - 4.8x_2$$

lower specification: $A \geq 20 \Rightarrow \max A$

$$\frac{\partial A}{\partial x_1} = 1.2 > 0 \quad \Rightarrow x_{1,WL} = -3$$
$$\frac{\partial A}{\partial x_2} = -4.8 < 0 \quad \Rightarrow x_{2,WL} = 3$$
$$\overline{A}(\boldsymbol{x}_{WL}) = 6 \qquad \text{for comparison} \qquad A(\boldsymbol{x}_{WL}) = 8.16$$

c)

$$T_E = \{ \boldsymbol{x} | (\boldsymbol{x} - \boldsymbol{x_0})^T C^{-1} (\boldsymbol{x} - \boldsymbol{x_0}) \leq \beta_w^2 \}$$
$$\boldsymbol{x}_0 = \boldsymbol{0}$$
$$C = \boldsymbol{I} \quad (x_1, \, x_2 \text{ uncorrelated})$$
$$T_E = \{ \boldsymbol{x} | \boldsymbol{x}^T \boldsymbol{x} \leq 3^2 \} = \{ \boldsymbol{x} | x_1^2 + x_2^2 \leq 9 \}$$
$$\sigma_{\overline{A}} = \sqrt{\boldsymbol{g}^T C \boldsymbol{g}} = \sqrt{\begin{bmatrix} 1.2 \\ -4.8 \end{bmatrix}^T \begin{bmatrix} 1.2 \\ -4.8 \end{bmatrix}} = 4.95$$

lower specification: $A \geq 20 \Rightarrow \max A$

$$\boldsymbol{x}_{WL} - \boldsymbol{x}_0 = -\frac{\beta_w}{\sigma_{\overline{A}}} C \boldsymbol{g}$$
$$\boldsymbol{x}_{WL} = -\frac{3}{4.95} \begin{bmatrix} 1.2 \\ -4.8 \end{bmatrix} = \begin{bmatrix} -0.73 \\ 2.9 \end{bmatrix}$$
$$\overline{A}(\boldsymbol{x}_{WL}) = A_0 - \beta_w \sigma_{\overline{A}} = 9.16 \qquad \text{for comparison} \qquad A(\boldsymbol{x}_{WL}) = 9.71$$

d)

lower specification: $A \geq 20 \Rightarrow \max A$

$$\begin{aligned} &\min_{\boldsymbol{x}} A(\boldsymbol{x}) \text{ s.t. } \boldsymbol{x} \in T_E \\ \Leftrightarrow &\min_{\boldsymbol{x}} A_{nom}(1+0.05x_1)(1-0.2x_2) \text{ s.t. } x_1^2+x_2^2 \leq 9 \\ \mathcal{L}(\boldsymbol{x},\lambda) &= A_{nom}(1+0.05x_1)(1-0.2x_2) - \lambda(9-x_1^2-x_2^2) \\ \nabla\mathcal{L}(\boldsymbol{x}) &= \begin{bmatrix} 0.05A_{nom}(1-0.2x_2)+2\lambda x_1 \\ -0.2A_{nom}(1+0.05x_1)+2\lambda x_2 \end{bmatrix} \overset{!}{=} \mathbf{0} \end{aligned} \tag{3.4}$$

assume constraint inactive:

$$\begin{aligned} &\lambda = 0 \\ \Rightarrow &\begin{bmatrix} 0.05A_{nom}(1-0.2x_2) \\ -0.2A_{nom}(1+0.05x_1) \end{bmatrix} \overset{!}{=} \mathbf{0} \\ \Rightarrow &\ x_1 = -20, x_2 = 5 \Rightarrow 20^2+5^2 \leq 9 \text{ contradiction} \end{aligned}$$

constraint active:

$$x_1^2 + x_2^2 = 9 \tag{3.5}$$

x_2(3.4a) - x_1 (3.4b):

$$\begin{aligned} &0.05A_{nom}x_2(1-0.2x_2) + \cancel{2\lambda x_1x_2} + 0.2A_{nom}(1+0.05x_1)x_1 - \cancel{2\lambda x_1x_2} = 0 \\ \Leftrightarrow &1.2x_2 - 0.24x_2^2 + 4.8x_1 + 0.24x_1^2 = 0 \end{aligned} \tag{3.6}$$

(3.5) in (3.6):

$$\begin{aligned} &1.2\sqrt{9-x_1^2} - 0.24(9-x_1^2) + 4.8x_1 + 0.24x_1^2 = 0 \\ \Leftrightarrow &\ 0.4x_1^2 + 4x_1 - 1.8 = -\sqrt{9-x_1^2} \\ \Leftrightarrow &\ \left[0.4x_1^2+4x_1-1.8\right]^2 = 9 - x_1^2 \\ \Leftrightarrow &\ 0.16x_1^4 + (1.6+1.6)x_1^3 + (16-0.72-0.72+1)x_1^2 + (-7.2-7.2)x_1 + (3.24-9) = 0 \\ \Leftrightarrow &\ 0.16x_1^4 + 3.2x_1^3 + 15.56x_1^2 - 14.4x_1 - 5.76 = 0 \end{aligned}$$

find roots of 4th order polynomial (e.g., using solution procedure or numerically):

$$\begin{aligned} x_{1,1} &= 1.04 \qquad x_{1,2} = -0.31 \\ x_{2,1} &= \pm\sqrt{9-x_{1,WL,1}^2} = \pm 2.81 \qquad x_{2,2} = \pm 2.98 \\ \lambda_1 &= \frac{0.2A_{nom}(1+0.05x_{1,1)}}{2x_{2,1}} = \pm 0.90 \overset{!}{>} 0 \ \Rightarrow \ \text{only "+" solution valid} \\ \lambda_2 &= \pm 0.79 \overset{!}{>} 0 \ \Rightarrow \ \text{only "+" solution valid} \end{aligned}$$

check (3.4a)

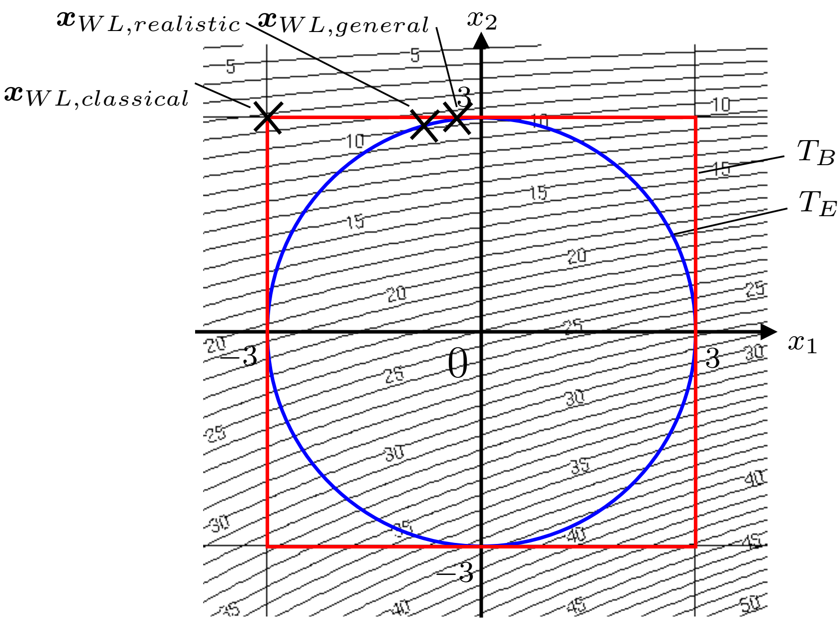
$$0.05A_{nom}(1 - 0.2x_{2,1}) + 2\lambda_1 x_{1,1} = 2.17 \neq 0$$
$$0.05A_{nom}(1 - 0.2x_{2,2}) + 2\lambda_1 x_{1,2} = 0$$

solution

$$\boldsymbol{x}_{WL} = \begin{bmatrix} -0.31 \\ 2.98 \end{bmatrix}$$
$$A_{WL} = 9.54$$

Comparison

Method	$x_{1,WL}$	$x_{2,WL}$	$\overline{A}_{WL}$	A_{WL}
Classical	-3	3	19.8	8.16
Realistic	-0.73	2.9	9.16	9.71
General	-0.31	2.98	N/A	9.54



Exercise 4: Transformation of statistical distributions

1. Calculation of probability density functions (pdf):

   a) The random variable y has a normal distribution. The random variable z depends exponentionally on y:

   $$\text{pdf}_y(y) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{1}{2}(\frac{y-y_0}{\sigma})^2}$$
   $$z = e^y$$

   Calculate the probability density function $\text{pdf}_z(z)$.

   b) The random variable y is standard normal distributed $\mathcal{N}(0,1)$. The random variable z depends quadratically on y:

   $$\text{pdf}_y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$
   $$z = y^2$$

   Calculate the probability density function $\text{pdf}_z(z)$.

2. Generation of normally distributed sample elements: We want to generate a vector of Gaussian correlated random variables $\mathbf{x_s} \sim \mathcal{N}(\mathbf{x}_{s,0}, \mathbf{C})$ from a vector $\mathbf{z}$ of independent uniformly distributed random variables $z_k \sim U(0,1), k = 1...n_{xs}$.

   a) How can you generate the vector $\mathbf{z}$ of independent uniformly distributed random variables?

   b) Find a transformation which maps the distribution of $\mathbf{z}$ on a vector of random variables $\mathbf{y}$ with a standard normal distribution $\mathbf{y} \sim \mathcal{N}(\mathbf{0}, \mathbf{E})$.

   c) Find the transformation which maps $\mathbf{y}$ on the desired distribution of random variables $\mathbf{x_s}$.

Exercise 4: Solution

1)

a)

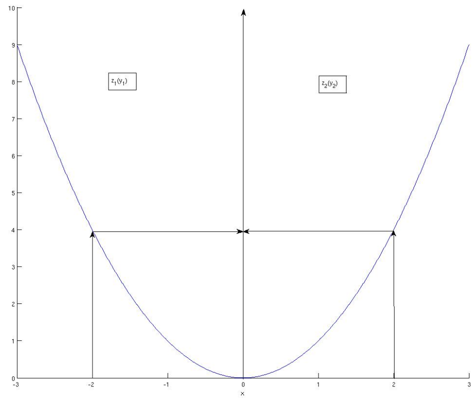
$z = e^y$ is bijective, invertible and continuously differentiable

$$\begin{aligned} \mathrm{pdf}_z(z) &= \mathrm{pdf}_y(y(z)) \left| \frac{\partial y}{\partial z} \right| \qquad \text{where} \quad y = \ln(z) \quad \text{and} \quad \frac{\partial y}{\partial z} = \frac{1}{z} \\ &= \frac{1}{\sqrt{2\pi}\sigma z} e^{-\frac{1}{2}\left(\frac{\ln(z)-y_0}{\sigma}\right)^2} \end{aligned}$$

pdf_z: lognormal distribution

b)

$z = y^2$ is not bijective



distribution function method

$$\begin{aligned} \mathrm{cdf}_Z(z) &= P(Z \leq z) = P(-\sqrt{z} \leq Y \leq \sqrt{z}) \\ &= \mathrm{cdf}_Y(\sqrt{z}) - \mathrm{cdf}_Y(-\sqrt{z}) \\ &= 2\mathrm{cdf}_Y(\sqrt{z}) - 1 \quad \text{symmetry of normal distribution} \\ \overset{\frac{d}{dz}}{\Rightarrow} \quad \mathrm{pdf}_Z(z) &= 2\mathrm{pdf}_Y(\sqrt{z}) \frac{1}{2\sqrt{z}} \\ &= \frac{1}{\sqrt{2\pi}\sqrt{z}} e^{-\frac{1}{2}z} \end{aligned}$$

density function method

$$\begin{aligned}
\mathrm{pdf}_Z(z)dz &= \mathrm{pdf}_Y(\sqrt{z})dy + \mathrm{pdf}_Y(-\sqrt{z})dy \\
\Rightarrow \mathrm{pdf}_Z(z) &= \mathrm{pdf}_Y(\sqrt{z}) \left|\frac{dy}{dz}\right|_{y=\sqrt{z}} + \mathrm{pdf}_Y(-\sqrt{z}) \left|\frac{dy}{dz}\right|_{y=-\sqrt{z}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z} \left|\frac{1}{2\sqrt{z}}\right| + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z} \left|\frac{-1}{2\sqrt{z}}\right| \\
&= \frac{1}{\sqrt{2\pi}\sqrt{z}} e^{-\frac{1}{2}z}
\end{aligned}$$

2)

a)

arithmetic random number generators, e.g. $z_k^{(i+1)} = (az_k^{(i)} + b) \mod m$
linear feedback shift registers (LFSR)

b)

$\boldsymbol{y} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I})$ uncorrelated $\Rightarrow$ independent transformation of each variable $z_k = \mathcal{U}(0,1) \rightarrow y_k = \mathcal{N}(0,1)$

From lecture:

$$y_k = \mathrm{cdf}_{y_k}^{-1}(z_k)$$

Look up in table for $z_k = \mathrm{cdf}(y_k)$ and $y_k = \mathcal{N}(0,1)$.
For example:

$$\begin{aligned}
y_k = 0.51 &\rightarrow z_k = 0.6950 \\
z_k = 0.9406 &\rightarrow y_k = 1.56
\end{aligned}$$

c)

Wanted: $\boldsymbol{x}_s(\boldsymbol{y}) : \ \boldsymbol{y} \rightarrow \boldsymbol{x}_s$

$$\mathrm{pdf}_y(\boldsymbol{y}) = \frac{1}{\sqrt{2\pi}^{n_{x_s}}} e^{-\frac{1}{2}\beta^2(\boldsymbol{y},\boldsymbol{0},\boldsymbol{I})}$$

$$\mathrm{pdf}_{x_s}(\boldsymbol{x}_s) = \frac{1}{\sqrt{2\pi}^{n_{x_s}} \sqrt{\det(\boldsymbol{C})}} e^{-\frac{1}{2}\beta^2(\boldsymbol{x}_s,\boldsymbol{x}_{s,0},\boldsymbol{C})}$$

$$\text{from lecture:} \qquad \mathrm{pdf}_{x_s}(\boldsymbol{x}_s) = \mathrm{pdf}_y(\boldsymbol{y}) \left| \det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_s^T}\right) \right|$$

$$\frac{1}{\cancel{\sqrt{2\pi}^{n_{x_s}}} \sqrt{\det(\boldsymbol{C})}} e^{-\frac{1}{2}\beta^2(\boldsymbol{x}_s,\boldsymbol{x}_{s,0},\boldsymbol{C})} = \frac{1}{\cancel{\sqrt{2\pi}^{n_{x_s}}}} e^{-\frac{1}{2}\beta^2(\boldsymbol{y},\boldsymbol{0},\boldsymbol{I})} \left| \det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_s^T}\right) \right|$$

$$\Rightarrow \qquad \beta^2(\boldsymbol{x}_s, \boldsymbol{x}_{s,0}, \boldsymbol{C}) = \beta^2(\boldsymbol{y}, \boldsymbol{0}, \boldsymbol{I}) \tag{4.1}$$

$$\frac{1}{\sqrt{\det(\boldsymbol{C})}} = \left| \det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_s^T}\right) \right| \tag{4.2}$$

$$(4.1) \Leftrightarrow (\boldsymbol{x}_s - \boldsymbol{x}_{s,0})^T \boldsymbol{C}^{-1} (\boldsymbol{x}_s - \boldsymbol{x}_{s,0}) = \boldsymbol{y}^T \boldsymbol{y}$$

Possible decompositions of $\boldsymbol{C}$:

- Cholesky-Decomposition: $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{A}^T$
- Eigen-Decomposition: $\boldsymbol{C} = \boldsymbol{U} \underbrace{\boldsymbol{D}}_{\substack{\text{diagonal}\\ \text{matrix}}} \boldsymbol{U}^T \quad \Leftrightarrow \quad \boldsymbol{C} = \underbrace{\boldsymbol{U}\boldsymbol{D}^{\frac{1}{2}}}_{\boldsymbol{A}} \underbrace{\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{U}^T}_{\boldsymbol{A^T}}$

$$\Rightarrow \ \boldsymbol{y}^T\boldsymbol{y} = (\boldsymbol{x}_s - \boldsymbol{x}_{s,0})^T \boldsymbol{A}^{-T} \boldsymbol{A}^{-1} (\boldsymbol{x}_s - \boldsymbol{x}_{s,0})$$

$$\Rightarrow \ \boldsymbol{y} = \boldsymbol{A}^{-1}(\boldsymbol{x}_s - \boldsymbol{x}_{s,0})$$

$$\Rightarrow \ \boldsymbol{x}_s = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{x}_{s,0}$$

Check if equation (4.2) is fulfilled:

$$\begin{aligned} \left| \det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_s^T}\right) \right| &= \left|\det(\boldsymbol{A}^{-1})\right| \\ &= \left| \frac{1}{\det(\boldsymbol{A})} \right| = \frac{1}{\sqrt{(\det(\boldsymbol{A}))^2}} \\ &= \frac{1}{\sqrt{\det(\boldsymbol{A}) \cdot \det(\boldsymbol{A}^T)}} = \frac{1}{\sqrt{\det(\boldsymbol{A}\boldsymbol{A}^T)}} \\ &= \frac{1}{\sqrt{\det(\boldsymbol{C})}} \Rightarrow \ \text{o.k.} \end{aligned}$$

Exercise 5: Expected value and estimated value

1. A random number generator is used to generate samples with 10 elements each (see table below). The random numbers are to be normal distributed but mean μ and standard deviation σ are unknown.

   a) Estimate mean value for samples 1 to 16. Draw a histogram for $\hat{\mu} \in [-6; 6]$ and bins of size 0.5.

   b) Estimate standard deviation for samples 1 to 16. Draw a histogram for $\hat{\sigma} \in [0; 6]$ and bins of size 0.5.

#										
1	-2.11	3.61	0.12	0.41	5.02	6.44	-2.55	0.80	9.81	0.81
2	-5.02	-2.82	-1.83	-3.66	-3.67	4.85	4.54	-1.19	-3.64	-3.98
3	-6.39	7.18	-5.04	2.03	7.52	-0.58	-2.54	-1.52	1.29	6.98
4	7.28	3.29	2.27	6.10	3.12	5.51	5.37	4.62	-2.55	6.20
5	4.53	6.94	6.40	7.67	-1.98	2.34	7.49	6.80	1.51	-4.46
6	5.90	0.37	11.00	-6.55	4.48	1.76	5.70	-4.58	-0.98	0.52
7	-2.27	5.88	4.03	7.55	1.65	0.42	-3.87	2.87	1.35	-0.01
8	4.96	2.15	-1.18	2.07	3.65	-3.52	8.54	3.47	0.25	2.88
9	-2.65	7.05	2.10	-1.44	5.78	-0.07	-0.71	4.57	2.53	3.46
10	-2.17	3.89	4.09	3.14	5.95	-0.04	-4.16	-0.70	4.21	-2.77
11	0.74	4.61	-2.28	3.85	3.80	8.11	-1.32	12.37	6.00	1.31
12	1.66	7.72	2.61	0.34	2.88	-8.67	3.19	2.34	0.60	-0.39
13	-2.24	1.31	-0.29	5.72	7.67	-3.21	4.01	2.84	6.51	5.83
14	3.79	3.81	0.56	2.26	9.81	4.65	4.65	-2.87	-2.36	0.91
15	2.14	-0.50	2.33	6.32	4.09	-5.80	1.06	2.06	0.72	0.92
16	2.17	2.72	1.06	6.34	11.23	8.94	3.47	6.38	3.71	-6.12

2. Prove that the following rules for expected values apply:

   a) Rule R1:

$$\mathrm{E}\{\mathbf{A}\ \mathbf{h}(\mathbf{x}) + \mathbf{b}\} = \mathbf{A}\ \mathrm{E}\{\mathbf{h}(\mathbf{x})\} + \mathbf{b}$$

   b) Rule R2:

$$\mathrm{V}\{\mathbf{A}\ \mathbf{h}(\mathbf{x}) + \mathbf{b}\} = \mathbf{A}\ \mathrm{V}\{\mathbf{h}(\mathbf{x})\}\ \mathbf{A}^T$$

   c) Rule R3:

$$\mathrm{V}\{\mathbf{h}(\mathbf{x})\} = \mathrm{E}\{\mathbf{h}(\mathbf{x}) \cdot \mathbf{h}^T(\mathbf{x})\} - \mathrm{E}\{\mathbf{h}(\mathbf{x})\} \cdot \mathrm{E}\{\mathbf{h}^T(\mathbf{x})\}$$

3. The normally distributed random variables $\mathbf{x} \sim \mathcal{N}(\mathbf{x_0}, \mathbf{C})$ have the following pdf:

$$\mathrm{pdf}(\mathbf{x}, \mathbf{x}_0, \mathbf{C}) = \frac{1}{\sqrt{2\pi}^{n_x} \cdot \sqrt{\det \mathbf{C}}}\ e^{-\frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T \cdot \mathbf{C}^{-1}(\mathbf{x}-\mathbf{x}_0)}$$

   a) Calculate the mean value of $\mathbf{x}$.

   b) Calculate the covariance matrix of $\mathbf{x}$.

   c) A performance f depends linearly on the random variables $\mathbf{x}$ as follows:

$$f = f_0 + \bigtriangledown f(\mathbf{x})^T \cdot (\mathbf{x} - \mathbf{x}_0) = f_0 + \mathbf{g}^T \cdot (\mathbf{x} - \mathbf{x}_0)$$

   Calculate the mean value m_f and variance σ_f^2 of f.

4. Estimators:

   a) Show that the expectation value estimator $\hat{\mathrm{E}}\{\mathbf{h}(\mathbf{x})\}$ is unbiased.

   b) Show that the variance value estimator $\hat{\mathrm{V}}\{\mathbf{h}(\mathbf{x})\}$ is unbiased.

5. Prove that the following rules for estimator values apply:

   a) Rule R1:

$$\hat{\mathrm{E}}\{\mathbf{A}\ \mathbf{h}(\mathbf{x}) + \mathbf{b}\} = \mathbf{A}\ \hat{\mathrm{E}}\{\mathbf{h}(\mathbf{x})\} + \mathbf{b}$$

   b) Rule R2:

$$\hat{\mathrm{V}}\{\mathbf{A}\ \mathbf{h}(\mathbf{x}) + \mathbf{b}\} = \mathbf{A}\ \hat{\mathrm{V}}\{\mathbf{h}(\mathbf{x})\}\ \mathbf{A}^T$$

   c) Rule R3:

$$\hat{\mathrm{V}}\{\mathbf{h}(\mathbf{x})\} = \frac{n_{mc}}{n_{mc} - 1}(\hat{\mathrm{E}}\{\mathbf{h}(\mathbf{x}) \cdot \mathbf{h}^T(\mathbf{x})\} - \hat{\mathrm{E}}\{\mathbf{h}(\mathbf{x})\} \cdot \hat{\mathrm{E}}\{\mathbf{h}^T(\mathbf{x})\})$$

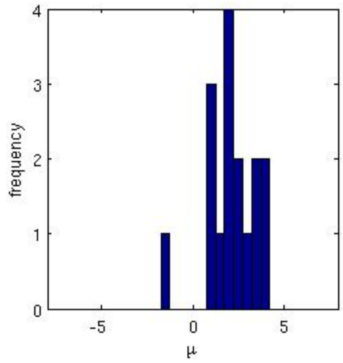
Exercise 5: Solution

1

a)

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

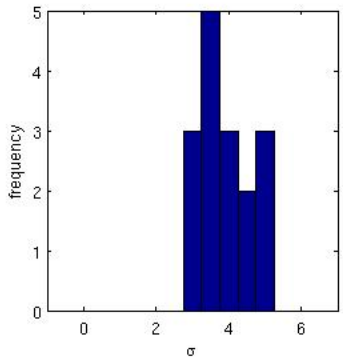
#	$\hat{\mu}$	#	$\hat{\mu}$
1	2.24	9	2.06
2	-1.64	10	1.14
3	0.89	11	3.72
4	4.12	12	1.23
5	3.72	13	2.82
6	1.76	14	2.52
7	1.76	15	1.34
8	2.33	16	3.99



b)

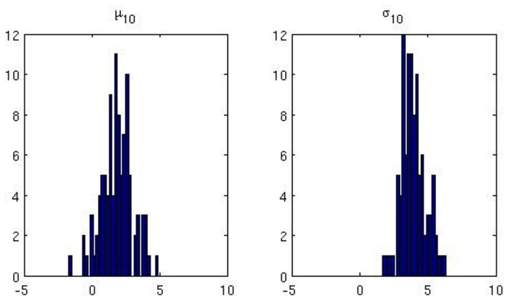
$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu})^2}$$

#	$\hat{\sigma}$	#	$\hat{\sigma}$
1	3.92	9	3.23
2	3.51	10	3.53
3	5.05	11	4.42
4	2.82	12	4.13
5	4.26	13	3.80
6	5.23	14	3.72
7	3.49	15	3.15
8	3.33	16	4.76

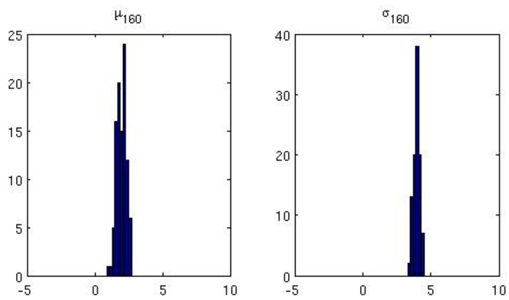


Additional experiments

Estimated mean $\hat{\mu}_{10}$ and standard deviation $\hat{\sigma}_{10}$ for 100 samples with 10 elements each.



Estimated mean $\hat{\mu}_{160}$ and standard deviation $\hat{\sigma}_{160}$ for 100 samples with 160 elements each.



Comparison

n	10	160	$\times 16$
$\hat{\mu}_{\hat{\mu_n}}$	1.8667	1.9908	
$\hat{\sigma}_{\hat{\mu_n}}$	1.124708	0.328338	$\times\frac{1}{3.425}$
$\hat{\mu}_{\hat{\sigma_n}}$	3.9392	3.9392	
$\hat{\sigma}_{\hat{\sigma_n}}$	0.877698	0.227205	$\times\frac{1}{3.863}$

True values $\mu = 2$, $\sigma = 4$.

2

a)

From definition: $E\{\boldsymbol{g}(\boldsymbol{x})\} = \int\limits_{-\infty}^{\infty}\cdots\int \boldsymbol{g}(\boldsymbol{x})\mathrm{pdf}_x(\boldsymbol{x})d\boldsymbol{x}$

$$\begin{aligned}
E\{\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x})+\boldsymbol{b}\} &= \int\limits_{-\infty}^{\infty}\cdots\int \left(\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x})+\boldsymbol{b}\right)\mathrm{pdf}_x(\boldsymbol{x})d\boldsymbol{x} \\
&= \boldsymbol{A}\underbrace{\int\limits_{-\infty}^{\infty}\cdots\int \boldsymbol{h}(\boldsymbol{x})\mathrm{pdf}_x(\boldsymbol{x})d\boldsymbol{x}}_{E\{\boldsymbol{h}(\boldsymbol{x})\}} + \boldsymbol{b}\underbrace{\int\limits_{-\infty}^{\infty}\cdots\int \mathrm{pdf}_x(\boldsymbol{x})d\boldsymbol{x}}_{1} \\
&= \boldsymbol{A}E\{\boldsymbol{h}(\boldsymbol{x})\}+\boldsymbol{b}
\end{aligned}$$

b)

From definition: $V\{\boldsymbol{g}(\boldsymbol{x})\} = E\left\{\left(\boldsymbol{g}(\boldsymbol{x})-E\{\boldsymbol{g}(\boldsymbol{x})\}\right)\left(\boldsymbol{g}(\boldsymbol{x})-E\{\boldsymbol{g}(\boldsymbol{x})\}\right)^T\right\}$

$$\begin{aligned}
\text{Let } \boldsymbol{m}_h &= E\{\boldsymbol{h}(\boldsymbol{x})\} \\
V\{\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x})+\boldsymbol{b}\} &= E\left\{(\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x})+\cancel{\boldsymbol{b}}-\boldsymbol{A}\boldsymbol{m}_h-\cancel{\boldsymbol{b}})(\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x})+\cancel{\boldsymbol{b}}-\boldsymbol{A}\boldsymbol{m}_h-\cancel{\boldsymbol{b}})^T\right\} \\
&= E\left\{\boldsymbol{A}(\boldsymbol{h}(\boldsymbol{x})-\boldsymbol{m}_h)(\boldsymbol{h}(\boldsymbol{x})-\boldsymbol{m}_h)^T\boldsymbol{A}^T\right\} \\
&= \boldsymbol{A}\underbrace{E\left\{(\boldsymbol{h}(\boldsymbol{x})-\boldsymbol{m}_h)(\boldsymbol{h}(\boldsymbol{x})-\boldsymbol{m}_h)^T\right\}}_{V\{\boldsymbol{h}(\boldsymbol{x})\}}\boldsymbol{A}^T \\
&= \boldsymbol{A}V\{\boldsymbol{h}(\boldsymbol{x})\}\boldsymbol{A}^T
\end{aligned}$$

c)

$$\begin{aligned}
V\{\boldsymbol{h}(\boldsymbol{x})\} &= E\left\{(\boldsymbol{h}(\boldsymbol{x})-\boldsymbol{m}_h)(\boldsymbol{h}(\boldsymbol{x})-\boldsymbol{m}_h)^T\right\} \\
&= E\{\boldsymbol{h}(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x})^T\} - \cancel{\boldsymbol{m}_h\underbrace{E\{\boldsymbol{h}^T(\boldsymbol{x})\}}_{\boldsymbol{m}_h^T}} - E\{\boldsymbol{h}(\boldsymbol{x})\}\underbrace{\boldsymbol{m}_h^T}_{E\{\boldsymbol{h}^T(\boldsymbol{x})\}} + \cancel{\boldsymbol{m}_h\boldsymbol{m}_h^T} \\
&= E\{\boldsymbol{h}(\boldsymbol{x})\boldsymbol{h}^T(\boldsymbol{x})\} - E\{\boldsymbol{h}(\boldsymbol{x})\}E\{\boldsymbol{h}^T(\boldsymbol{x})\}
\end{aligned}$$

3

a)

$$\mathrm{pdf}(\boldsymbol{x}, \boldsymbol{x}_0, \boldsymbol{C}) = \frac{1}{(\sqrt{2\pi})^{n_x}\sqrt{\det \boldsymbol{C}}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^T \boldsymbol{C}^{-1}(\boldsymbol{x}-\boldsymbol{x}_0)}$$

$$\boldsymbol{m}_x = E\{\boldsymbol{x}\} = \int_{-\infty}^{\infty}\cdots\int \boldsymbol{x}\,\mathrm{pdf}_x(\boldsymbol{x})\,d\boldsymbol{x}$$

Variable substitution $\boldsymbol{x} \to \boldsymbol{y}$ (according to exercise 4 problem 2c):

$$\begin{aligned}
\boldsymbol{y} &= \boldsymbol{A}^{-1}(\boldsymbol{x} - \boldsymbol{x}_0) \\
\boldsymbol{C} &= \boldsymbol{A}\boldsymbol{A}^T \\
\boldsymbol{x} &= \boldsymbol{A}\boldsymbol{y} + \boldsymbol{x}_0 \\
\mathrm{pdf}_y(\boldsymbol{y}) &= \frac{1}{(\sqrt{2\pi})^{n_x}} e^{-\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}}
\end{aligned}$$

$$\begin{aligned}
E\{\boldsymbol{y}\} &= \int_{-\infty}^{\infty}\cdots\int \boldsymbol{y}\,\mathrm{pdf}_y \boldsymbol{y}\,d\boldsymbol{y} \\
&= \int_{-\infty}^{\infty}\cdots\int \boldsymbol{y}\,\frac{1}{(\sqrt{2\pi})^{n_x}} e^{-\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}} d\boldsymbol{y} \\
&= \frac{1}{(\sqrt{2\pi})^{n_x}} \int_{-\infty}^{\infty}\cdots\int \underbrace{\boldsymbol{y}\,e^{-\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}}}_{\text{odd function}}\,d\boldsymbol{y} \\
&= \boldsymbol{0} \\
E\{\boldsymbol{x}\} &= E\{\boldsymbol{A}\boldsymbol{y} + \boldsymbol{x}_0\} \overset{R1}{=} \boldsymbol{A}E\{\boldsymbol{y}\} + \boldsymbol{x}_0 \\
&= \boldsymbol{A}\boldsymbol{0} + \boldsymbol{x}_0 = \boldsymbol{x}_0
\end{aligned}$$

b)

$$\begin{aligned}
V\{\boldsymbol{y}\} &= E\{\boldsymbol{y}\boldsymbol{y}^T\} \\
&= \int_{-\infty}^{\infty}\cdots\int \boldsymbol{y}\boldsymbol{y}^T\,\frac{1}{(\sqrt{2\pi})^{n_x}} e^{-\frac{1}{2}\boldsymbol{y}^T\boldsymbol{y}} d\boldsymbol{y} \\
&= \underbrace{\int_{-\infty}^{\infty}\cdots\int \begin{bmatrix} y_1^2 & \cdots & y_1 y_{n_x} \\ \vdots & \ddots & \vdots \\ y_{n_x} y_1 & \cdots & y_{n_x}^2 \end{bmatrix} \prod_{i=1}^{n_x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2}\,d\boldsymbol{y}}_{\boldsymbol{M}}
\end{aligned}$$

$$M_{jj} = \int_{-\infty}^{\infty}\!\!\cdots\!\int y_j^2 \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_j^2} \prod_{i \neq j} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2} d\boldsymbol{y}$$

$$= \underbrace{\int_{-\infty}^{\infty} y_j^2 \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_j^2} dy_j}_{\text{1, see below}} \prod_{i \neq j} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2} dy_i}_{\text{1, see below}} = 1$$

$$M_{jk} = \int_{-\infty}^{\infty}\!\!\cdots\!\int y_j y_k \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_j^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_k^2} \prod_{i \neq j,k} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2} d\boldsymbol{y} \qquad j \neq k$$

$$= \underbrace{\int_{-\infty}^{\infty} y_j \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_j^2} dy_j}_{\text{0, odd function}} \underbrace{\int_{-\infty}^{\infty} y_k \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_k^2} dy_k}_{\text{0, odd function}} \prod_{i \neq j,k} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2} dy_i}_{\text{1, see below}} = 0$$

$$\Rightarrow \boldsymbol{M} = \boldsymbol{I}$$

Integrals:

$$\int_{-\infty}^{\infty} y_j^2 \, \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_j^2} dy_j \overset{\text{even function}}{=} 2\frac{1}{\sqrt{2\pi}} \int_0^{\infty} y_j^2 e^{-\frac{1}{2}y_j^2} dy_j$$

$$\overset{\text{formulary}}{=} 2\frac{1}{\sqrt{2\pi}} \frac{1}{2}\sqrt{2\pi} = 1$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2} dy_i \overset{\text{even function}}{=} 2\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}y_i^2} dy_i$$

$$\overset{\text{formulary}}{=} 2\frac{1}{\sqrt{2\pi}} \frac{1}{2}\sqrt{2\pi} = 1$$

$$V\{\boldsymbol{x}\} = V\{\boldsymbol{A}\boldsymbol{y} + \boldsymbol{x}_0\} \overset{R3}{=} \boldsymbol{A} V\{\boldsymbol{y}\} \boldsymbol{A}^T$$
$$= \boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^T = \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{C}$$

c)

$$\begin{aligned} E\{\overline{f}\} &= E\{f_0 + \boldsymbol{g}^T(\boldsymbol{x} - \boldsymbol{x}_0)\} \\ &= f_0 + \boldsymbol{g}^T(E\{\boldsymbol{x}\} - \boldsymbol{x_0}) \\ &= f_0 + \boldsymbol{g}^T(\boldsymbol{x_0} - \boldsymbol{x_0}) \\ &= f_0 \end{aligned}$$

$$\begin{aligned} V\{\overline{f}\} &= V\{f_0 + \boldsymbol{g}^T(\boldsymbol{x} - \boldsymbol{x}_0)\} \\ &= \boldsymbol{g}^T V\{\boldsymbol{x}\} \boldsymbol{g} \\ &= \boldsymbol{g}^T \boldsymbol{C} \boldsymbol{g} \end{aligned}$$

4)

$$\text{estimator } \hat{f} \text{ of } f \text{ is unbiased} \Leftrightarrow E\{\hat{f}\} = f$$

a)

$$\begin{aligned}
E\big\{\hat{E}\{\boldsymbol{h}(\boldsymbol{x})\}\big\} &= E\{\hat{\boldsymbol{m}}_h\} \\
&= E\{\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{h}(\boldsymbol{x}_i)\} \\
&= \frac{1}{N}\sum_{i=1}^{N}E\{\boldsymbol{h}(\boldsymbol{x}_i)\} \\
&= \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{m}_h \\
&= \frac{N\,\boldsymbol{m}_h}{N} \\
&= \boldsymbol{m}_h
\end{aligned}$$

b)

Variance of estimated mean value.
Let $\boldsymbol{h}_i = \boldsymbol{h}(\boldsymbol{x}_i)$:

$$\sum_{i=1}^{N}(\boldsymbol{h}_i - \boldsymbol{m}_h) = \sum_{i=1}^{N}\boldsymbol{h}_i - N\boldsymbol{m}_h = N\hat{\boldsymbol{m}}_h - N\boldsymbol{m}_h = N(\hat{\boldsymbol{m}}_h - \boldsymbol{m}_h) \tag{5.1}$$

$$E\left\{\hat{V}\{\boldsymbol{h}(\boldsymbol{x})\}\right\} = \frac{1}{N-1}E\{\underbrace{\sum_{i=1}^{N}(\boldsymbol{h}_i - \hat{\boldsymbol{m}}_h)(\boldsymbol{h}_i - \hat{\boldsymbol{m}}_h)^T}_{S}\}$$

$$\begin{aligned}
S &= \sum_{i=1}^{N}(\boldsymbol{h}_i - \hat{\boldsymbol{m}}_h)(\boldsymbol{h}_i - \hat{\boldsymbol{m}}_h)^T \\
&= \sum_{i=1}^{N}\left[(\boldsymbol{h}_i - \boldsymbol{m}_h) + (\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)\right]\left[(\boldsymbol{h}_i - \boldsymbol{m}_h) + (\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)\right]^T \\
&= \sum_{i=1}^{N}(\boldsymbol{h}_i - \boldsymbol{m}_h)(\boldsymbol{h}_i - \boldsymbol{m}_h)^T + \sum_{i=1}^{N}(\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)(\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)^T + \\
&\quad + \sum_{i=1}^{N}(\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)(\boldsymbol{h}_i - \boldsymbol{m}_h)^T + \sum_{i=1}^{N}(\boldsymbol{h}_i - \boldsymbol{m}_h)(\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)^T \\
&\overset{(5.1)}{=} \sum_{i=1}^{N}(\boldsymbol{h}_i - \boldsymbol{m}_h)(\boldsymbol{h}_i - \boldsymbol{m}_h)^T + \cancel{N(\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)(\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)^T} + \\
&\quad + \cancel{N(\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)(\hat{\boldsymbol{m}}_h - \boldsymbol{m}_h)^T} + N(\hat{\boldsymbol{m}}_h - \boldsymbol{m}_h)(\boldsymbol{m}_h - \hat{\boldsymbol{m}}_h)^T \\
&= \sum_{i=1}^{N}(\boldsymbol{h}_i - \boldsymbol{m}_h)(\boldsymbol{h}_i - \boldsymbol{m}_h)^T - N(\hat{\boldsymbol{m}}_h - \boldsymbol{m}_h)(\hat{\boldsymbol{m}}_h - \boldsymbol{m}_h)^T
\end{aligned}$$

$$E\{S\} = \underbrace{E\{\sum_{i=1}^{N}(\boldsymbol{h}_i - \boldsymbol{m}_h)(\boldsymbol{h}_i - \boldsymbol{m}_h)^T\}}_{NV\{\boldsymbol{h}(\boldsymbol{x})\}} - N\,\underbrace{E\{(\hat{\boldsymbol{m}}_h - \boldsymbol{m}_h)(\hat{\boldsymbol{m}}_h - \boldsymbol{m}_h)^T\}}_{\substack{\text{Variance of expectation} \\ \text{value estimator; see} \\ \text{compendium} \\ V\{\hat{\boldsymbol{m}}_h\} = \frac{1}{N}V\{\boldsymbol{h}(\boldsymbol{x})\}}}$$

$$\begin{aligned}
E\left\{\hat{V}\{\boldsymbol{h}(\boldsymbol{x})\}\right\} &= \frac{1}{N-1}(N\,V\{\boldsymbol{h}(\boldsymbol{x})\} - V\{\boldsymbol{h}(\boldsymbol{x})\}) \\
&= V\{\boldsymbol{h}(\boldsymbol{x})\}
\end{aligned}$$

5)

Definitions:

$$\hat{E}\{\boldsymbol{g}(\boldsymbol{x})\} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{g}(\boldsymbol{x}_i)$$
$$\hat{V}\{\boldsymbol{g}(\boldsymbol{x})\} = \frac{1}{N-1}\sum_{i=1}^{N}\Big(\boldsymbol{g}(\boldsymbol{x}_i) - \hat{E}\{\boldsymbol{g}(\boldsymbol{x})\}\Big)\Big(\boldsymbol{g}(\boldsymbol{x}_i) - \hat{E}\{\boldsymbol{g}(\boldsymbol{x})\}\Big)^T$$

a)

$$\begin{aligned}\hat{E}\{\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{b}\} &= \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x}_i) + \boldsymbol{b}) \\ &= \frac{1}{N}\left(\boldsymbol{A}\sum_{i=1}^{N}\boldsymbol{h}(\boldsymbol{x}_i) + N\boldsymbol{b}\right) \\ &= \boldsymbol{A}\hat{E}\{\boldsymbol{h}(\boldsymbol{x})\} + \boldsymbol{b}\end{aligned}$$

b)

$$\begin{aligned}\text{Let } \hat{\boldsymbol{m}}_h &= \hat{E}\{\boldsymbol{h}(\boldsymbol{x})\} \\ \hat{V}\{\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{b}\} &= \frac{1}{N-1}\sum_{i=1}^{N}(\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x}) + \cancel{\boldsymbol{b}} - \boldsymbol{A}\hat{\boldsymbol{m}}_h - \cancel{\boldsymbol{b}})(\boldsymbol{A}\boldsymbol{h}(\boldsymbol{x}) + \cancel{\boldsymbol{b}} - \boldsymbol{A}\hat{\boldsymbol{m}}_h - \cancel{\boldsymbol{b}})^T \\ &= \boldsymbol{A}\underbrace{\frac{1}{N-1}\sum_{i=1}^{N}(\boldsymbol{h}(\boldsymbol{x}) - \hat{\boldsymbol{m}}_h)(\boldsymbol{h}(\boldsymbol{x}) - \hat{\boldsymbol{m}}_h)^T}_{\hat{V}\{\boldsymbol{h}(\boldsymbol{x})\}}\boldsymbol{A}^T \\ &= \boldsymbol{A}\hat{V}\{\boldsymbol{h}(\boldsymbol{x})\}\boldsymbol{A}^T\end{aligned}$$

c)

$$\begin{aligned}\hat{V}\{\boldsymbol{h}(\boldsymbol{x})\} &= \frac{1}{N-1}\sum_{i=1}^{N}(\boldsymbol{h}(\boldsymbol{x}_i) - \hat{\boldsymbol{m}}_h)(\boldsymbol{h}(\boldsymbol{x}_i) - \hat{\boldsymbol{m}}_h)^T \\ &= \frac{1}{N-1}\Big[\underbrace{\sum_{i=1}^{N}\big(\boldsymbol{h}(\boldsymbol{x}_i)\boldsymbol{h}(\boldsymbol{x}_i)^T\big)}_{N\hat{E}\{\boldsymbol{h}(\boldsymbol{x})\boldsymbol{h}(\boldsymbol{x})^T\}} - \cancel{\hat{\boldsymbol{m}}_h\underbrace{\sum_{i=1}^{N}\boldsymbol{h}^T(\boldsymbol{x}_i)}_{N\hat{\boldsymbol{m}}_h^T}} - \underbrace{\sum_{i=1}^{N}\boldsymbol{h}(\boldsymbol{x})}_{N\hat{\boldsymbol{m}}_h}\hat{\boldsymbol{m}}_h^T + \cancel{N\hat{\boldsymbol{m}}_h\hat{\boldsymbol{m}}_h^T}\Big] \\ &= \frac{N}{N-1}\Big(\hat{E}\{\boldsymbol{h}(\boldsymbol{x})\boldsymbol{h}^T(\boldsymbol{x})\} - \hat{E}\{\boldsymbol{h}(\boldsymbol{x})\}\hat{E}\{\boldsymbol{h}^T(\boldsymbol{x})\}\Big)\end{aligned}$$

Exercise 6: Yield Analysis

1. Yield estimation by Monte-Carlo analysis: The two independent statistical parameters $x_{s1} \sim U(0,1)$ and $x_{s2} \sim U(0,1)$ are uniformly distributed. A performance f depends quadratically on the statistical parameters: $f = x_{s1}^2 + x_{s2}^2$.

   Additionally, the following specification is given on the performance value: $f \leq 1$

   a) Show that the yield is equal to $\pi/4 = 0.785398163397448$.

   b) Calculate analytically the variance of the yield estimator dependent on the number of sample elements. Calculate the 95% confidence interval of the yield estimator for the sample sizes given in Table 6.1.

   c) Calculate the variance value estimator of the yield estimator for sample nr. 1 in Table 6.1. Calculate the 95% confidence interval of the yield estimator.

   d) Check if the estimated yield value for all 10 samples and all sample sizes in Table 6.1 is inside the 95% confidence interval (as computed in 1.b))

Table 6.1: Monte-Carlo analysis

	Number of sample elements						
	10	100	1,000	10,000	100,000	1,000,000	10,000,000
	Number of 'good' sample elements with $f \leq 1$						
Sample 1	9	74	755	7.873	78.394	784.957	7.853.229
Sample 2	6	87	791	7.910	78.515	785.549	7.855.640
Sample 3	8	82	793	7.874	78.400	785.741	7.851.932
Sample 4	8	78	805	7.915	78.294	785.760	7.855.647
Sample 5	7	79	790	7.833	78.503	784.826	7.852.262
Sample 6	9	80	781	7.850	78.439	785.262	7.855.425
Sample 7	9	73	788	7.824	78.703	785.885	7.853.404
Sample 8	5	74	790	7.940	78.429	785.434	7.852.424
Sample 9	10	85	773	7.870	78.629	785.580	7.853.661
Sample 10	9	78	798	7.894	78.516	785.802	7.853.915

2. Yield analysis considering operating parameters: The yield of a circuit design is analysed for one operating parameter x_r and one statistical parameter x_s. The figure below shows in each plot an aceptance region for different circuits in dependency of x_r and x_s. Inside the acceptance region the circuit meets the performance specification.

   The statistical parameter x_s is a process parameter, which is distributed uniformly $x_s \sim U(0,1)$. It is required, that the circuit meets the performance specification for any operating condition defined by a tolerance region of 0 to 100 for the operating parameter x_r.

   Mark the area in the plots, in which the circuits are located which meet the performance specification in the specified tolerance region of the operating parameter and state the yield value for each circuit.

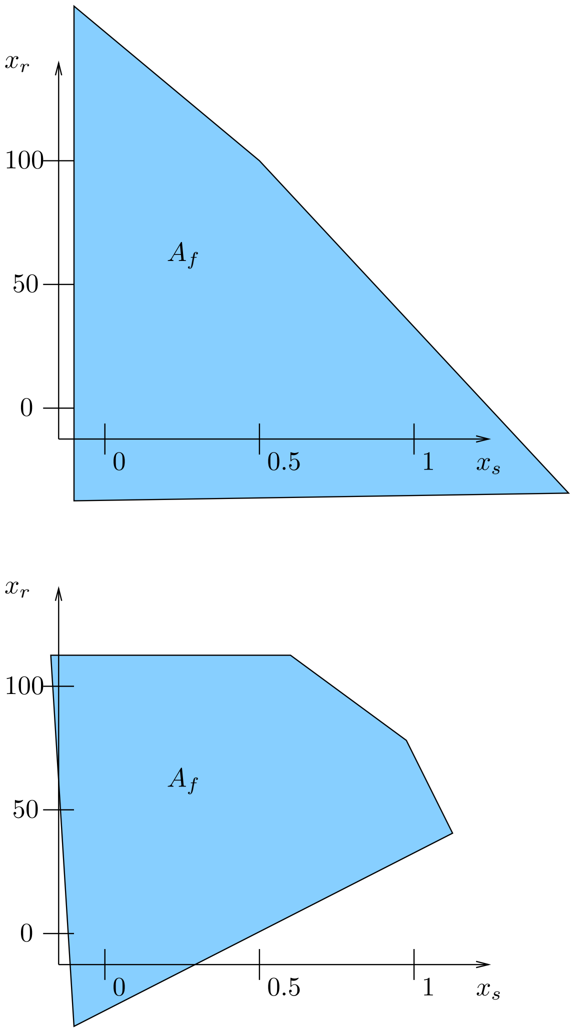


Figure 6.1: Acceptance regions

3. Geometric yield analysis: Statistical parameters $x_{s,1}$ and $x_{s,2}$ are Gaussian distributed with mean value vector $\mathbf{x}_{s,0}$ and covariance matrix $\mathbf{C}$:

$$\mathbf{x}_{s,0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 0,04 & 0 \\ 0 & 0,04 \end{bmatrix} \tag{6.1}$$

The performance function f and a specification are given by:

$$f = x_{s,1} \cdot x_{s,2}; \quad 0,5 \leq f \leq 2$$

a) Determine the yield by geometric yield analysis and a linearization of the performance at its mean value vector $\mathbf{x}_{s,0}$.

b) Determine the yield by geometric yield analysis considering the nonlinear dependency of the performance on the statistical parameters.

c) Conduct a geometric design centering on the circuit.

Exercise 6: Solution

1)

a)

$$\text{pdf}_{x_s}(\boldsymbol{x}_s) = \text{pdf}_U(x_{s,1}) \cdot \text{pdf}_U(x_{s,2}) = \begin{cases} 1 & \text{for } \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \boldsymbol{x}_s \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ 0 & \text{else} \end{cases}$$

$$Y = \iint_{A_S} \text{pdf}_{x_s}(\boldsymbol{x}_s) d\boldsymbol{x}_s = \iint_{A_S} 1 d\boldsymbol{x}_s = A_S$$

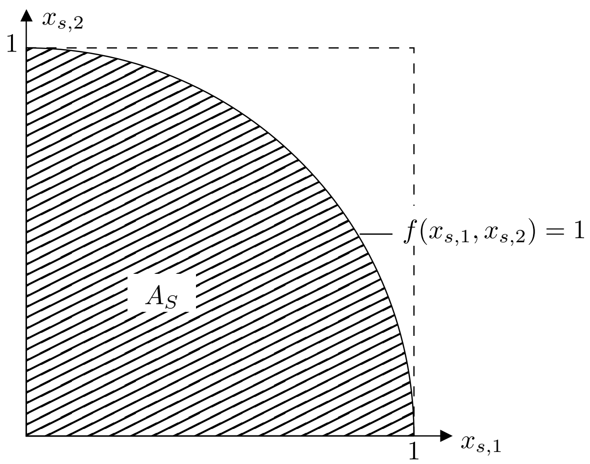


Figure 6.2: Acceptance Region

$$\Rightarrow Y = \frac{1}{4} A_{circle} = \frac{1}{4}\pi r^2 = \frac{\pi}{4} = 0.785398$$

b)

$$V\{\hat{Y}\} = \frac{1}{n_{mc}} Y(1-Y) = \frac{0.168548}{n_{mc}} = \sigma_{\hat{Y}}^2$$

Assume $\hat{Y}$ normal distributed:
Confidence Interval:

$$\hat{Y} \in [Y - \Delta Y; Y + \Delta Y]$$

Standard normal distribution table in compendium: $95\% -$ confidence interval $\Rightarrow$

$$\Delta Y = 1.96\sigma_{\hat{Y}} = 1.96\sqrt{\frac{0.168548}{n_{mc}}}$$

For $n_{mc} = 1000$: $\sigma_{\hat{Y}} = 0.01298$; $\Delta Y = 0.0254408$; confidence interval:

$$[0.785398 - 0.0254408; 0.785398 + 0.0254408] = [0.759952; 0.810844]$$

Table 6.2: Results

n_{mc}	Y	$\sigma_{\hat{y}}$	Lower bound 95% CI	Upper bound
10	0.7853981633	0.1298259944	0.5309392142	1.0398571125
100	0.7853981633	0.0410545841	0.7049311783	0.8658651484
1000	0.7853981633	0.0129825994	0.7599522684	0.8108440583
10.000	0.7853981633	0.0041054584	0.7773514648	0.7934448618
100.000	0.7853981633	0.0012982599	0.7828535739	0.7879427528
1.000.000	0.7853981633	0.0004105458	0.7845934935	0.7862028332
10.000.000	0.7853981633	0.0001298259	0.7851437044	0.7856526223

Y, $\sigma_{\hat{Y}}$ independent of sample

c)

$$\hat{V}(\hat{Y}) = \hat{\sigma}_{\hat{Y}}^2 = \frac{1}{n_{mc}-1}\hat{Y}(1-\hat{Y}) \quad \text{where } \hat{Y} = \frac{n^+}{n_{mc}}$$

confidence interval:

$$Y \in [\hat{Y} - \Delta\hat{Y}; \hat{Y} + \Delta\hat{Y}]$$

$95\% -$ confidence interval $\Rightarrow$

$$\Delta\hat{Y} = 1.96\hat{\sigma}_{\hat{Y}}$$

For $n_{mc} = 1000$: $\hat{Y} = 0.755$; $\hat{\sigma}_{\hat{Y}} = 0.0136$; $\Delta Y = 0.0267$; confidence interval:

$$[0.755 - 0.0267; 0.755 + 0.0267] = [0.728; 0.782]$$

Table 6.3: Results

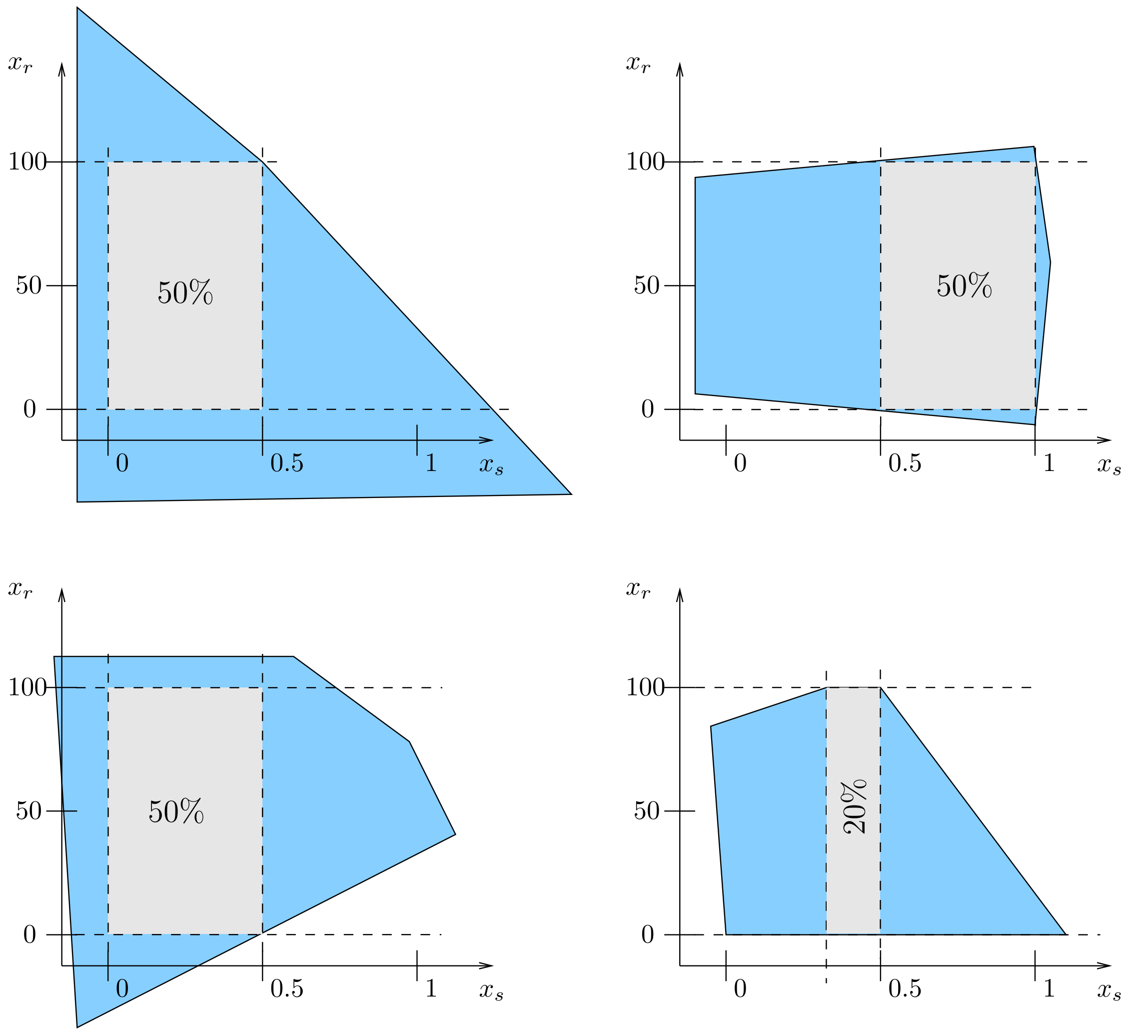
n_{mc}	$\hat{Y}$	$\hat{\sigma}_{\hat{y}}$	Lower bound 95% CI	Upper bound
10	0.9	0.1000000000	0.7040000000	1.0960000000
100	0.74	0.0440844002	0.6535945755	0.8264054244
1.000	0.755	0.0136073568	0.7283295805	0.7816704194
10.000	0.7873	0.0040923765	0.77927894191	0.7953210580
100.000	0.78394	0.0013014598	0.7813891387	0.7864908612
1.000.000	0.784957	0.0004108523	0.7841517293	0.7857622706
10.000.000	0.7853229	0.0001298425	0.7850684086	0.7855773913

$\hat{Y}$, $\hat{\sigma}_{\hat{Y}}$ dependent of sample

d)

For $n_{mc} = 1000$: sample 1 outside, all other samples inside.

2)



3)

a)

$$\begin{aligned}\overline{f} &= f(\boldsymbol{x}_{s,0}) + \underbrace{\nabla f(\boldsymbol{x}_{s,0})^T}_{[x_{s,2}\ x_{s,1}]}(\boldsymbol{x}_s - \boldsymbol{x}_{s,0}) \\ &= 1 + [1\ 1]\begin{bmatrix} x_{s,1} - 1 \\ x_{s,2} - 1 \end{bmatrix} \\ &= x_{s,1} + x_{s,2} - 1\end{aligned}$$

$$\sigma_{\overline{f}} = \sqrt{\nabla f^T \boldsymbol{C} \nabla f} = \sqrt{0.08} = 0.28$$

$\underline{f \geq f_L = 0.5}$; $f(\boldsymbol{x}_{s,0}) = 1 \geq 0.5$ $\rightarrow$ specification fulfilled at nominal point

$$\begin{aligned}\overline{\beta}_{WL} &= \frac{\overline{f}(\boldsymbol{x}_{s,0}) - f_L}{\sigma_{\overline{f}}} = 1.78 \\ \overline{Y}_L &= \int_{-\infty}^{\beta_L} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \overset{\text{table}}{=} 96.25\%\end{aligned}$$

$\underline{f \leq f_U = 2}$; $f(\boldsymbol{x}_{s,0}) = 1 \leq 2$ $\rightarrow$ specification fulfilled at nominal point

$$\begin{aligned}\overline{\beta_{WU}} &= \frac{f_U - \overline{f}(\boldsymbol{x}_{s,0})}{\sigma_{\overline{f}}} = 3.57 \\ \overline{Y}_U &\overset{\text{table}}{=} 99.99971 \\ \overline{Y}_{tot,L} &= 1 - (1 - \overline{Y}_L) - (1 - \overline{Y}_U) = 96.25\% \\ \overline{Y}_{tot,U} &= \min(96.25\%, 99.99971\%) \\ \overline{Y}_{tot} &\in [96.25\%, 96.25\%]\end{aligned}$$

b)

$$\begin{aligned}\beta^2(\boldsymbol{x}_s) &= (\boldsymbol{x}_s - \boldsymbol{x}_{s,0})^T C^{-1} (\boldsymbol{x}_s - \boldsymbol{x}_{s,0}) \\ &= 25(x_{s,1} - 1)^2 + 25(x_{s,2} - 1)^2\end{aligned}$$

$\underline{f \leq f_U = 2}$; $f(\boldsymbol{x}_{s,0}) = 1 \leq f_U \;\rightarrow$

$$\min_{\boldsymbol{x}_s} \beta^2(\boldsymbol{x}_s) \text{ s.t. } f(\boldsymbol{x}_s) \geq f_U$$

$$\Leftrightarrow\; 25(x_{s,1} - 1)^2 + 25(x_{s,2} - 1)^2 \text{ s.t. } x_{s,1}x_{s,2} \geq 2$$

$$\mathcal{L}(x_{s,1}, x_{s,2}, \lambda) = 25(x_{s,1} - 1)^2 + 25(x_{s,2} - 1)^2 - \lambda(x_{s,1}x_{s,2} - 2)$$

$$\frac{\partial \mathcal{L}}{\partial x_{s,1}} = 50(x_{s,1,WU} - 1) - \lambda_{WU} x_{s,2,WU} \overset{!}{=} 0 \tag{6.2}$$

$$\frac{\partial \mathcal{L}}{\partial x_{s,2}} = 50(x_{s,2,WU} - 1) - \lambda_{WU} x_{s,1,WU} \overset{!}{=} 0 \tag{6.3}$$

$$\text{see compendium: constraint always active } \Rightarrow\; x_{s,1,WU}x_{s,2,WU} = 2 \tag{6.4}$$

$$\begin{aligned}(6.3) - (6.2) :& 50(x_{s,2,WU} - x_{s,1,WU}) + \lambda_{WU}(-x_{s,1,WU} + x_{s,2,WU}) = 0 \\ & \underbrace{(50 + \lambda_{WU})}_{>0}(x_{s,2,WU} - x_{s,1,WU}) = 0 \\ & x_{s,1,WU} = x_{s,2,WU} \overset{(6.4)}{=} \pm\sqrt{2}\end{aligned}$$

$$\beta_{WU}(\boldsymbol{x}_s) = \sqrt{25(x_{s,1,WU} - 1)^2 + 25(x_{s,2,WU} - 1)^2} \approx \begin{cases} 2.9 & \Rightarrow \text{ value of interest} \\ 17.07 \end{cases}$$

$$\overset{\text{table}}{\Rightarrow}\; Y_U = 99.8\%$$

$\underline{f \geq f_L = 0.5}$; $f(\boldsymbol{x}_{s,0}) = 1 \geq f_L \;\rightarrow$

$$\min_{\boldsymbol{x}_s} \beta^2(\boldsymbol{x}_s) \text{ s.t. } f(\boldsymbol{x}_s) \leq f_L$$

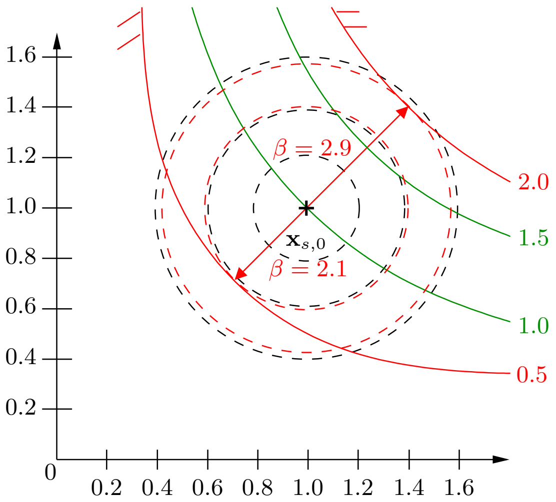
$$\vdots$$

$$x_{s,1,WL} = x_{s,2,WL} = \sqrt{0.5}$$

$$\begin{aligned}\beta_{WL}(\boldsymbol{x}_s) &= 2.1 \\ Y_L &= 98.2\%\end{aligned}$$

Total yield:

$$1 - (1 - Y_L) - (1 - Y_U) = 98\% \leq Y_{tot} \leq \min(Y_L, Y_U) = 98.2\%$$



c)

$$\min_{\boldsymbol{x}_{s,0}} e^{-\frac{1}{2}\beta_{WL}} + e^{-\frac{1}{2}\beta_{WU}}$$

$$\Rightarrow e^{-\frac{1}{2}\beta_{WL}}\frac{1}{2}\nabla\beta_{WL} + e^{-\frac{1}{2}\beta_{WU}}\frac{1}{2}\nabla\beta_{WU} = 0$$

$$\text{select} \quad \beta_{WL} = \beta_{WU} \wedge \nabla\beta_{WL} = -\nabla\beta_{WU}$$

From visual inspection using (380) (Compendium): $x^\star_{s,0,1} = x^\star_{s,0,2}$
Same distance to worst case points:

$$x^\star_{s,1} - x_{s,1,WL} = x_{s,1,WU} - x^\star_{s,1}$$

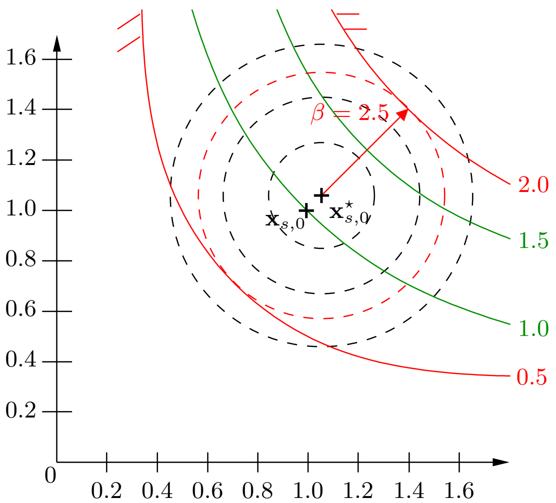
geometric design centering

$$x^\star_{s,1} = x^\star_{s,2} = \frac{x_{s,1,WU} + x_{s,1,WL}}{2} = \frac{3}{4}\sqrt{2}$$

$$\beta^\star_{WL} = \beta^\star_{WU} = 2.5$$

$$Y^\star_L = Y^\star_U = 99.4\%$$

$$\underbrace{1 - (1 - Y^\star_L) - (1 - Y^\star_U)}_{98.8\%} \leq Y^\star_{tot} \leq \underbrace{\min(Y^\star_L, Y^\star_U)}_{99.4\%}$$



Exercise 7: Line search and unconstrained optimization without derivatives

1. Univariate unconstrained optimization (line search): A performance f depends on the step size α as follows:

$$f(\alpha) = 1 - \alpha \cdot e^{-\alpha}$$

The optimum is located at α^*=1 with $f(\alpha^*)$=0.63212.

   a) The start interval is $[\mathrm{L}^0, \mathrm{R}^0] = [0, 2]$ and the first cut is at α_1=E^0=0.764. Is this a valid start interval for optimization using the golden sectioning method?

   b) What are the interval boundaries α_R,α_L for all α values which meet the Wolfe-Powell-conditions with c_1=0.1 and c_2=0.9?

   c) How many optimization steps are required for the golden sectioning method in order to obtain the optimal step size α^* with an accuracy of 0.1?

   d) Calculate two steps of the interval sectioning method by golden sectioning.

   e) Calculate two steps of the interval sectioning method by quadratic model interpolation.

2. Unconstrained optimization without derivatives: Fig. 4 shows the isolines of a function. Its minimum is located at $\mathbf{p}^*$.

   a) Carry out, graphically, several steps of the polytope method. The starting simplex is 1,2,3.

   b) What are the problem-specific parameters of the polytope method?

   c) Carry out, graphically, several steps of the coordinate search method, starting from point A.

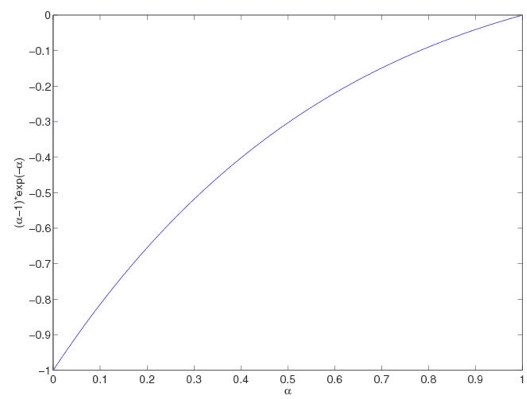


Figure 7.1: Function to line search in 1b

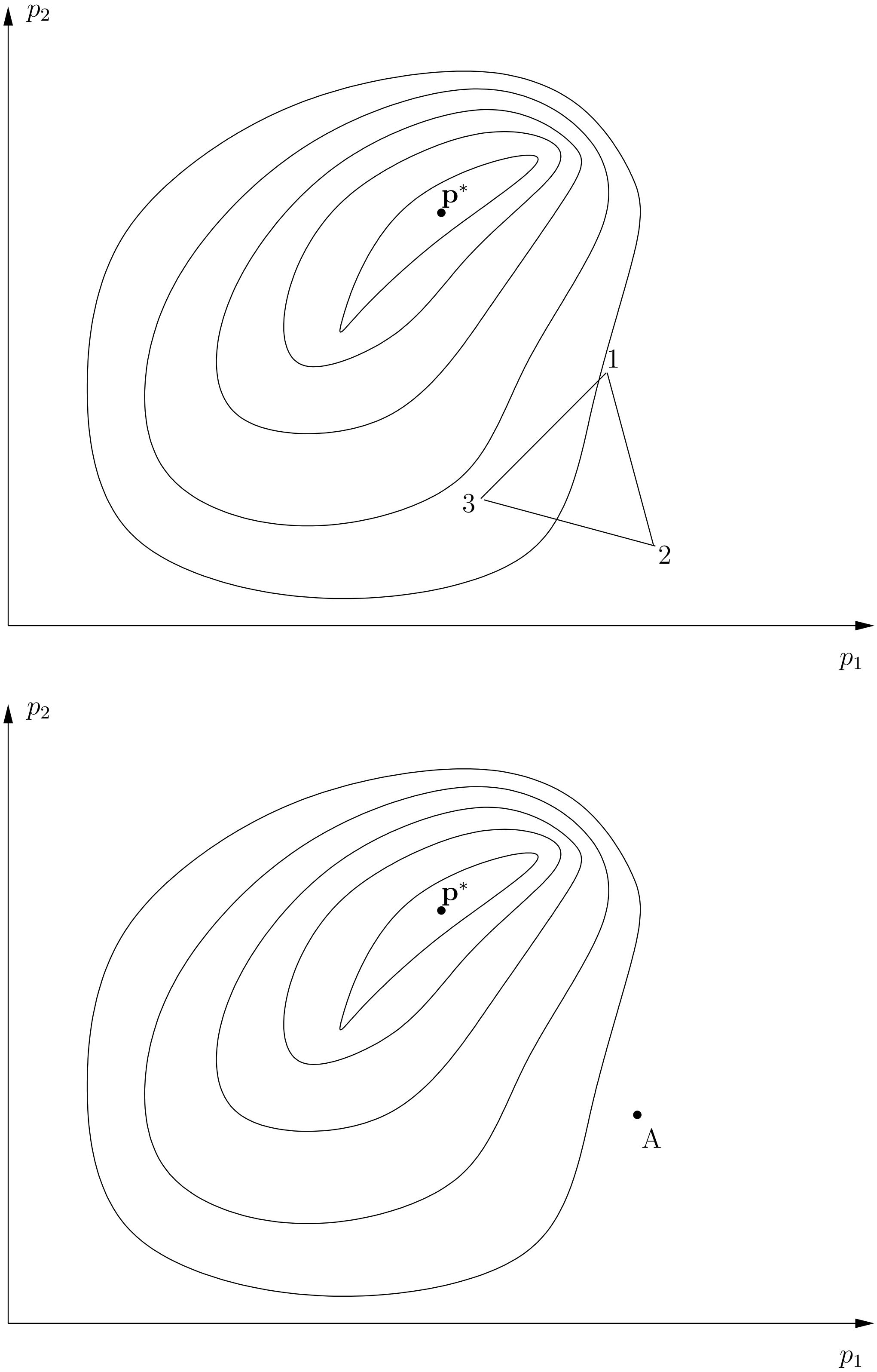


Figure 7.2: Unconstraint optimization without derivatives

Exercise 7: Solution

1

a)

$$f(L^0 = 0) = 1 \qquad f(R^0 = 2) = 0.729$$
$$f(\alpha_1 = 0.764) = 0.644$$

$$f(\alpha)\text{unimodal} \rightarrow \begin{array}{ll} f(\alpha_1) & < f(R^0) \\ f(\alpha_1) & < f(L_0) \end{array} \rightarrow \alpha^* \in [L^0, R^0] \rightarrow \text{valid start interval}$$

b)

$$\frac{df(\alpha)}{d\alpha} = -e^{-\alpha} - \alpha(-1)e^{-\alpha} = (\alpha - 1)e^{-\alpha}$$

Sufficient reduction:

$$\begin{aligned} f(\alpha_R) &\leq f(0) + \alpha_R c_1 \frac{df(0)}{d\alpha} \\ 1 - \alpha_R e^{-\alpha_R} &\leq 1 + \alpha_R \cdot 0.1 \cdot (-1) \\ -\alpha_R e^{-\alpha_R} &\leq -0.1\alpha_R \\ e^{-\alpha_R} &\geq 0.1 \\ -\alpha_R &\geq \ln 0.1 \\ \alpha_R &\leq 2.3 \end{aligned}$$

curvature condition:

$$\begin{aligned} \frac{df}{d\alpha}(\alpha_L) &\geq c_2 \frac{df}{d\alpha}(0) \\ (\alpha_L - 1)e^{-\alpha_L} &\geq 0.9 \cdot (-1) \\ (\alpha_L - 1)e^{-\alpha_L} &\geq -0.9 \end{aligned}$$

general: root finding, numerical solver
here: plot $\Rightarrow$ $\alpha_L \geq 0.07$

c)

given: $\Delta^{(0)} = R^{(0)} - L^{(0)} = 2$; to obtain: κ for which $\Delta^{(\kappa)} \leq 0.1$
from compendium:

$$\kappa = \left\lceil \frac{-1}{\log \tau} \log \frac{\Delta^{(0)}}{\Delta^{(\kappa)}} \right\rceil \approx \left\lceil 4.78 \log \frac{2}{0.1} \right\rceil = \lceil 6.22 \rceil = 7$$

d)

$$\Delta = |R - L| \qquad \alpha_1 = L + (1-\tau)\cdot\Delta \qquad \alpha_2 = L + \tau\cdot\Delta \qquad \tau = \frac{\sqrt{5}-1}{2} \approx 0.618$$

i	$L^{(i)}$	$\alpha_1^{(i)}$	$\alpha_2^{(i)}$	$R^{(i)}$	$\Delta^{(i)}$	
0	0	0.764	1.236	2	2	$f(L^{(0)}) = 1,\ f(R^{(0)}) = 0.73,$ $f(\alpha_1^{(0)}) = 0.644,\ f(\alpha_2^{(0)}) = 0.640,$ $f(\alpha_1^{(0)}) > f(\alpha_2^{(0)})$
	↙	↙		↓		
1	0.764	1.236	**1.528**	2	1.236	$f(\alpha_2^{(0)}) = f(\alpha_1^{(1)}) < f(\alpha_2^{(1)}) = 0.668$
	↓		↘	↘		
2	0.764	**1.056**	1.236	1.528	...	...

$$\alpha_2 - \alpha_1 = \frac{\Delta}{2\tau - 1}$$

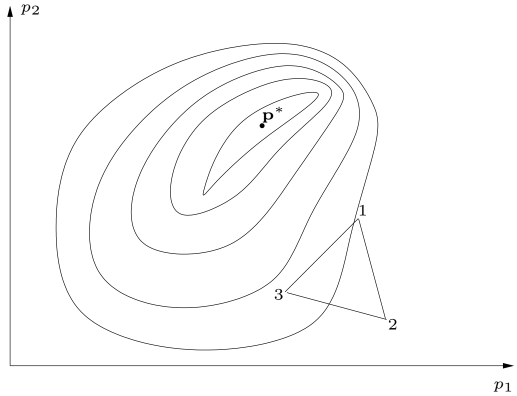
e)

See http://www.cse.uiuc.edu/iem/optimization/

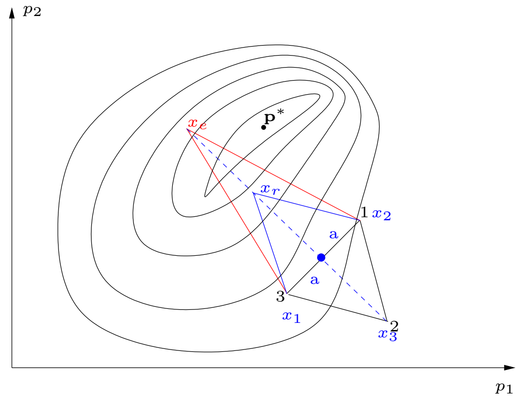
2)

a)

Start:

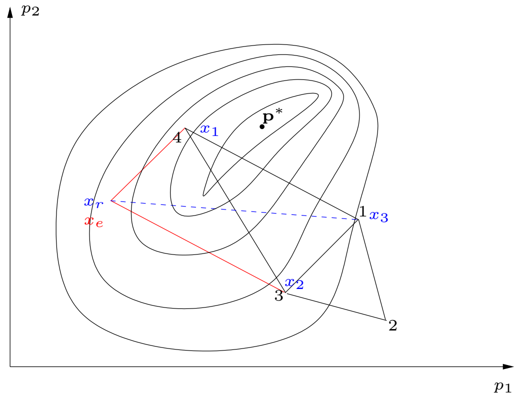


1:



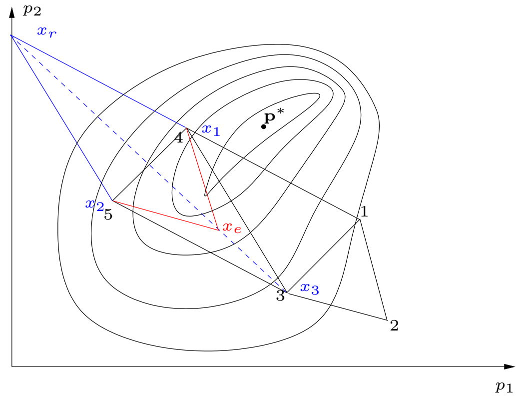
$f_r < f_1 \;\Rightarrow$ Expansion

2:



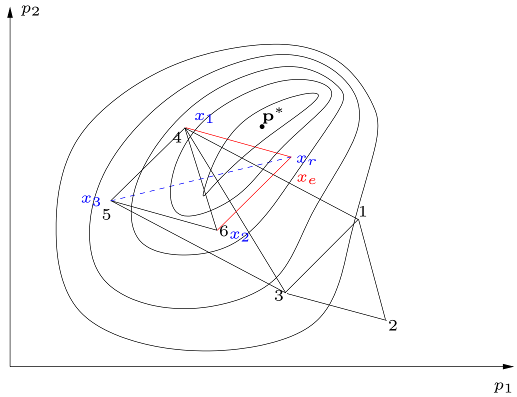
$f_1 < f_r < f_2 \;\Rightarrow$ Reorder

3:



$f_3 < f_r \;\Rightarrow$ Inner contraction

4:



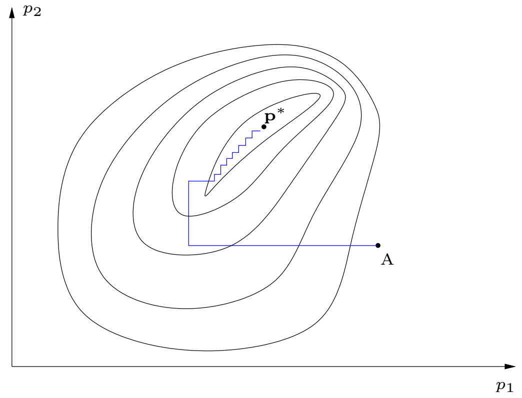
$f_1 < f_r < f_2 \;\Rightarrow$ Reorder

...

b)

- reflection coefficient
- expansion coefficient
- contraction coefficient
- reduction coefficient

c)



Exercise 8: Multivariate unconstrained optimization with derivatives

1. Positive definite Hessian matrix: The quadratic objective function f depends on two parameters x_1 and x_2:

$$f(\mathbf{x}) = f(x_1, x_2) = x_1^2 + 2x_2^2 + 4x_1 + 4x_2$$

The minimum is located at:

$$\mathbf{x}^* = \begin{bmatrix} -2 \\ -1 \end{bmatrix}; \quad f(\mathbf{x}^*) = -6$$

The start point of the optimization is given as:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad f(\mathbf{x}^{(0)}) = f^{(0)} = 0$$

a) Carry out two steps of the method of steepest descent. Use the interval sectioning method by quadratic model interpolation for the line search.

b) Carry out two steps of the Newton method. Use the interval sectioning method by quadratic model interpolation for the line search.

c) Carry out two steps of the Quasi-Newton method. Use the interval sectioning method by quadratic model interpolation for the line search and the Symmetric-Rank-1 (SR1) method to update the quadratic model.

d) Carry out two steps of the conjugent-gradient approach (CG).

2. Indefinite Hessian matrix: The objective function f depends on two parameters x_1 and x_2:

$$f(\mathbf{x}) = f(x_1, x_2) = x_1^4 + x_1 x_2 + (1 + x_2)^2$$

The minimum is located at:

$$\mathbf{x}^* = \begin{bmatrix} 0.6959 \\ -1.3479 \end{bmatrix}; \quad f(\mathbf{x}^*) = -0.5825$$

The start point of the optimization is given as:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad f(\mathbf{x}^{(0)}) = f^{(0)} = 1$$

a) Carry out one single step of the Newton method. Does this method reach an improvement of the objective function value?

b) Carry out one single step of the steepest-descent method. Does this method reach an improvement of the objective function value?

c) Carry out one single step in the Levenberg-Marquardt direction. Does this method reach an improvement of the objective function value?

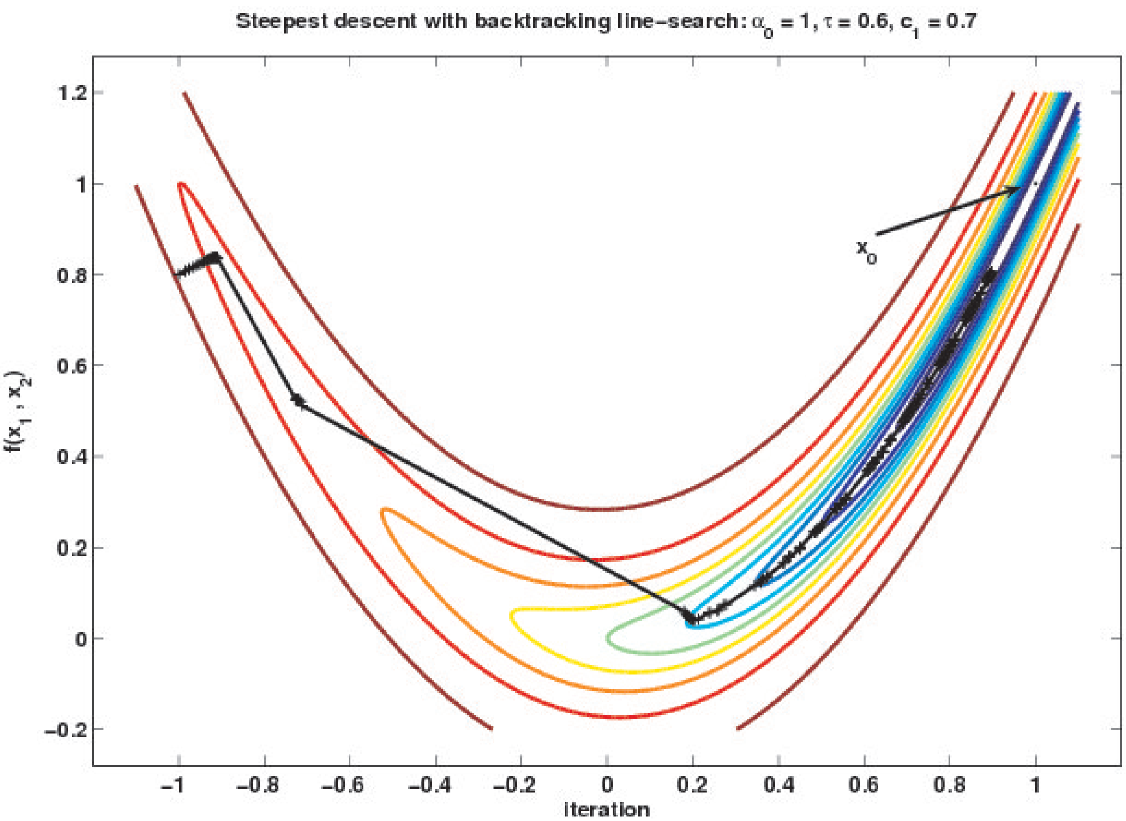


Figure 8.1: Example 1: Steepest descent method

Exercise 8: Solution

1)

1a)

Select search direction as steepest descent:

$$\boldsymbol{r}^{(\kappa)} = -\nabla f(\boldsymbol{x}^{(\kappa)}) = -\begin{bmatrix} 2x_1^{(\kappa)} + 4 \\ 4x_2^{(\kappa)} + 4 \end{bmatrix}$$

1. Search direction: $\boldsymbol{r}^{(0)} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$

   Line search: $f(\alpha) = f(\boldsymbol{x}^{(0)} + \alpha \boldsymbol{r}^{(0)})$

α	1	0.5	0
$\boldsymbol{x}^{(0)} + \alpha \boldsymbol{r}^{(0)}$	$\begin{bmatrix} -4 \\ -4 \end{bmatrix}$	$\begin{bmatrix} -2 \\ -2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
f	16	−4	0 (valid start interval)

   Determine quadratic model $a\alpha^2 + b\alpha + c \overset{!}{=} f(\alpha)$:

$$\left.\begin{array}{rcrcrcl} a & + & b & + & c & = & 16 \\ 0.25a & + & 0.5b & + & c & = & (-4) \\ & & & & c & = 0 & \end{array}\right\} \rightarrow a = 48\,;\ b = -32$$

$$m^{(0)}(\alpha) = 48\alpha^2 - 32\alpha = 16\alpha(3\alpha - 2)$$

$$\nabla m^{(0)}(\alpha^{(0)}) \overset{!}{=} 0 \Leftrightarrow \alpha^{(0)} = \frac{1}{3}$$

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \alpha^{(0)} \boldsymbol{r}^{(0)} = \begin{bmatrix} -4/3 \\ -4/3 \end{bmatrix} \qquad f^{(1)} = f(\boldsymbol{x}^{(1)}) = \frac{-16}{3} \approx -5.33$$

2. Search direction $\boldsymbol{r}^{(1)} = \begin{bmatrix} -4/3 \\ 4/3 \end{bmatrix}$

α	1	0.5	0
$\boldsymbol{x}^{(0)} + \alpha \boldsymbol{r}^{(0)}$	$\begin{bmatrix} -8/3 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -6/3 \\ -2/3 \end{bmatrix}$	$\begin{bmatrix} -4/3 \\ -4/3 \end{bmatrix}$
f	−32/9	−52/9	−48/9 (valid start interval)

   quadratic model:

$$m^{(1)}(\alpha) = \frac{48}{9}\alpha^2 - \frac{32}{9}\alpha - \frac{48}{9}$$

$$\nabla m^{(1)}(\alpha^{(1)}) \overset{!}{=} 0 \Leftrightarrow \alpha^{(1)} = \frac{1}{3}$$

$$\boldsymbol{x}^{(2)} = \boldsymbol{x}^{(1)} + \alpha^{(1)} \boldsymbol{r}^{(1)} = \begin{bmatrix} -16/9 \\ -8/9 \end{bmatrix} \qquad f^{(2)} = f(\boldsymbol{x}^{(2)}) = \frac{-160}{27} \approx -5.926$$

b)

$$\nabla^2 f(\boldsymbol{x}) = \boldsymbol{H} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{positive definite}$$

1. Search direction:

$$\boldsymbol{H}^{(0)}\boldsymbol{r}^{(0)} = -\boldsymbol{g}^{(0)}$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \boldsymbol{r}^{(0)} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \rightarrow \boldsymbol{r}^{(0)} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Line search: $f(\alpha) = f(\boldsymbol{x}^{(0)} + \alpha \boldsymbol{r}^{(0)})$

α	0	0.5	1	1.5
$\boldsymbol{x}^{(0)} + \alpha \boldsymbol{r}^{(0)}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -0.5 \end{bmatrix}$	$\begin{bmatrix} -2 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$
f	0	-4.5	-6	-4.5

start interval $[1.5; 0.5]$ valid

Quadratic model $m(\alpha) = 6\alpha^2 - 12\alpha$; $\nabla m(\alpha) = 12\alpha - 12 \overset{!}{=} 0 \Rightarrow \alpha^{(0)} = 1$

$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + 1\boldsymbol{r}^{(0)} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \qquad \nabla f(\boldsymbol{x}^{(1)}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{x}^{(1)} = \boldsymbol{x}^{\star}$$

c)

1. $\boldsymbol{B}^{(0)} = \boldsymbol{I} \;\rightarrow\; \boldsymbol{B}^{(0)}\boldsymbol{r}^{(0)} = -\boldsymbol{g}^{(0)}$

   Because $\boldsymbol{B}^{(0)} = \boldsymbol{I}$: $\boldsymbol{r}^{(0)} = -\boldsymbol{g}^{(0)}$ $\rightarrow$ similar to steepest descent $\boldsymbol{x}^{(1)} = \begin{bmatrix} -4/3 \\ -4/3 \end{bmatrix}$

   SR 1 Update:

$$\boldsymbol{s}^{(0)} = \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)} = \begin{bmatrix} -4/3 \\ -4/3 \end{bmatrix}$$

$$\boldsymbol{y}^{(0)} = \nabla f(\boldsymbol{x}^{(1)}) - \nabla f(\boldsymbol{x}^{(0)}) = \begin{bmatrix} -8/3 \\ -16/3 \end{bmatrix}$$

$$\boldsymbol{z}^{(0)} = \boldsymbol{y}^{(0)} - \boldsymbol{B}^{(0)}\boldsymbol{s}^{(0)} = \begin{bmatrix} -4/3 \\ -4 \end{bmatrix}$$

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(0)} + \frac{\boldsymbol{z}^{(0)}\boldsymbol{z}^{(0)T}}{\boldsymbol{z}^{(0)T}\boldsymbol{s}^{(0)}} = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 13/4 \end{bmatrix}$$

2.

$$\boldsymbol{B}^{(1)}\boldsymbol{r}^{(1)} = -\boldsymbol{g}^{(1)}$$

$$\begin{bmatrix} 15 & 9 \\ 9 & 39 \end{bmatrix} \boldsymbol{r}^{(1)} = \begin{bmatrix} -16 \\ 16 \end{bmatrix}$$

$$\rightarrow \boldsymbol{r}^{(1)} = \begin{bmatrix} -1.52 \\ 0.762 \end{bmatrix}$$

   Line search using quadratic model $\rightarrow \alpha^{(1)} = 0.4375$

$$\boldsymbol{x}^{(2)} = \boldsymbol{x}^{(1)} + \alpha^{(1)}\boldsymbol{r}^{(1)} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \boldsymbol{x}^{\star}$$

   Next SR1-Update: $B^{(2)} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \boldsymbol{H}$

d)

1. Search direction: $\boldsymbol{r}^{(0)} = -\nabla f(\boldsymbol{x}^{(0)}) = \boldsymbol{g}^{(0)} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$

$$\alpha^{(0)} = \frac{\|g^{(0)}\|^2}{\boldsymbol{r}^{(0)T}\boldsymbol{H}\boldsymbol{r}^{(0)}} = \frac{1}{3}$$
$$\boldsymbol{x}^{(1)} = \boldsymbol{x}^{(0)} + \alpha^{(0)}\boldsymbol{r}^{(0)} = \begin{bmatrix} -4/3 \\ -4/3 \end{bmatrix}$$

2.

$$\boldsymbol{g}^{(1)} = \begin{bmatrix} 4/3 \\ -4/3 \end{bmatrix}$$
$$\beta^{(1)} = \frac{\|\boldsymbol{g}^{(1)}\|^2}{\|\boldsymbol{g}^{(0)}\|^2} = \frac{1}{9}$$
$$\boldsymbol{r}^{(1)} = -\boldsymbol{g}^{(1)} + \beta^{(1)}\boldsymbol{r}^{(0)} = \begin{bmatrix} -4/3 \\ 4/3 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -16/9 \\ 8/9 \end{bmatrix}$$
$$\alpha^{(1)} = \frac{\|\boldsymbol{g}^{(1)}\|^2}{\boldsymbol{r}^{(1)T}\boldsymbol{H}\boldsymbol{r}^{(1)}} = \frac{3}{8}$$
$$\boldsymbol{x}^{(2)} = \boldsymbol{x}^{(1)} + \alpha^{(1)}\boldsymbol{r}^{(1)} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \boldsymbol{x}^{\star}$$
$$\boldsymbol{g}^{(2)} = 0 \;\rightarrow\; \beta^{(2)} = 0 \;\rightarrow\; \boldsymbol{r}^{(2)} = \boldsymbol{0}$$

2)

$$\boldsymbol{g} = \nabla f(\boldsymbol{x}) = \begin{bmatrix} 4x_1^3 + x_2 \\ x_1 + 2(1 + x_2) \end{bmatrix} \qquad \boldsymbol{g}^{(0)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\boldsymbol{H} = \nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} 12x_1^2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \boldsymbol{H}^{(0)} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{aligned} \det\left(\boldsymbol{H}^{(0)} - \lambda \boldsymbol{I}\right) &= \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) \\ &= (-\lambda)(2-\lambda) - 1 = \lambda^2 - 2\lambda - 1 \stackrel{!}{=} 0 \\ \lambda_{1,2} &= \frac{2 \pm \sqrt{2^2 - 4(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} \\ \lambda_1, \lambda_2 &= 1 \pm \sqrt{2} \rightarrow \text{indefinite} \end{aligned}$$

a)

Newton direction $\boldsymbol{H}^{(0)}\boldsymbol{r}^{(0)} = -\boldsymbol{g}^{(0)}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{r}^{(0)} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\boldsymbol{r}^{(0)} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Line search $f(\alpha) = (-2\alpha)^4 + 1 = 1 + 16a^4$ i.e., no reduction of f in Newton direction

b)

Steepest descent $\boldsymbol{r}^{(0)} = -\boldsymbol{g}^{(0)} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$f(\alpha) = (1 - 2\alpha)^2 \rightarrow$ reduction possible: can become smaller than $f^{(0)} = 1$

c)

$$(H^{(0)} + k\boldsymbol{I})\boldsymbol{r}^{(0)} = -\boldsymbol{g}^{(0)}$$

Select k such that $\lambda_1', \lambda_2' > 0$:

$$\begin{aligned}
\lambda_{1,2}' &= \lambda_{1,2} + k \overset{!}{>} 0 \\
k &> -\lambda_{1,2} \Rightarrow k > -(1 \pm \sqrt{2}) \\
k &> \sqrt{2} - 1 \\
\text{Select } k = 1 : \\
\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \boldsymbol{r}^{(0)} &= \begin{bmatrix} 0 \\ -2 \end{bmatrix} \\
\boldsymbol{r}^{(0)} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
\text{Line search: } f(\alpha) &= \alpha^4 - \alpha^2 + (1 - \alpha)^2 \\
&= \alpha^4 - 2\alpha + 1 \rightarrow \text{ Improvement possible}
\end{aligned}$$

Exercise 9: Quadratic optimization with constraints

1. The following quadratic optimization problem with equality constraints will be solved:

$$\min_{\mathbf{x}} x_1^2 + x_2^2 + x_3^2 \quad \text{s.t.} \quad x_1 + 2x_2 - x_3 = 4 \ \wedge \ x_1 - x_2 + x_3 = -2$$

   a) Convert the constrained optimization problem to an unconstrained optimization problem by variable substitution.

   b) Convert the constrained optimization problem to an unconstrained optimization problem by coordinate transformation.

2. The plot shows a quadratic optimization problem with inequality constraints. Conduct the steps of the active set method graphically, until the minimum is found. The active set for the starting point $\mathbf{x}^{(0)}$ is $\mathcal{A}^{(0)} = \{3, 4\}$.

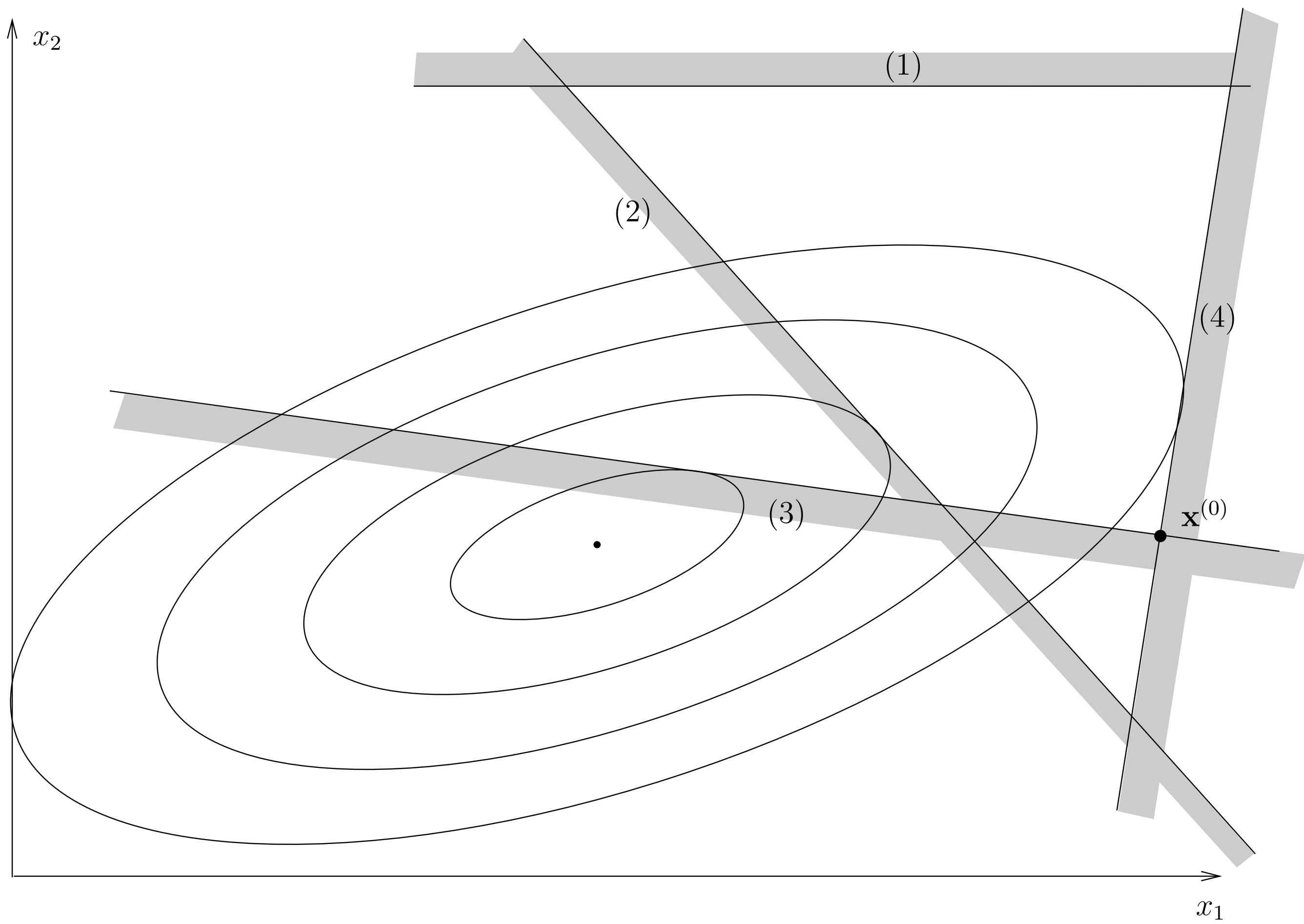


Figure 9.1: Quadratic optimization with inequality constraints

Exercise 9: Solution

1)

a)

$$
\begin{aligned}
x_1 + 2x_2 - x_3 &= 4 && (9.1)\\
x_1 - x_2 + x_3 &= -2 && (9.2)\\
(9.1) - (9.2): \quad 3x_2 - 2x_3 &= 6 \\
x_2 &= 2 + \frac{2}{3}x_3 && (9.3)\\
(9.1) + 2(9.2): \quad 3x_1 + x_3 &= 0 \\
x_1 &= -\frac{1}{3}x_3 && (9.4)
\end{aligned}
$$

(9.3), (9.4) in $f(\boldsymbol{x})$:

$$
\begin{aligned}
x_1^2 + x_2^2 + x_3^2 &= \frac{1}{9}x_3^2 + 4 + \frac{8}{3}x_3 + \frac{4}{9}x_3^2 + x_3^2 \\
&= \frac{14}{9}x_3^2 + \frac{8}{3}x_3 + 4 \\
&= \phi(x_3)
\end{aligned}
$$

$\rightarrow$

$$
\left.\begin{array}{rl}
\min & x_1^2 + x_2^2 + x_3^2 \\
\text{s.t.} & x_1 + x_2 - x_3 = 4 \\
& x_1 - x_2 + x_3 = -2
\end{array}\right\} \quad \Leftrightarrow \quad \min \phi(x_3)
$$

$$
\begin{aligned}
\nabla\phi(x_3) &\overset{!}{=} 0 \\
\frac{28}{9}x_3 + \frac{8}{3} &= 0 \\
x_3^\star &= -\frac{6}{7} \quad \rightarrow \quad x_2^\star = \frac{10}{7} \quad x_1^\star = \frac{2}{7} \\
\nabla^2\phi(x_3^\star) &= \frac{28}{9} > 0
\end{aligned}
$$

b)

Rewrite problem in matrix vector notation:

$$\min \underbrace{\begin{bmatrix}0 & 0 & 0\end{bmatrix}}_{\boldsymbol{b}^T} \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T \underbrace{\begin{bmatrix}2 & & \\ & 2 & \\ & & 2\end{bmatrix}}_{\boldsymbol{H}} \boldsymbol{x} \text{ s.t. } \underbrace{\begin{bmatrix}1 & 2 & -1 \\ 1 & -1 & 1\end{bmatrix}}_{\boldsymbol{A}} \boldsymbol{x} = \underbrace{\begin{bmatrix}4 \\ -2\end{bmatrix}}_{\boldsymbol{c}}$$

1. Q-R-decomposition of A:

$$\boldsymbol{A}^T = [\boldsymbol{Q}\boldsymbol{Q}_\perp] \begin{bmatrix}\boldsymbol{R} \\ \boldsymbol{0}\end{bmatrix}$$

$$\boldsymbol{Q} = \begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{21}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{21}}\end{bmatrix} \qquad \boldsymbol{Q}_\perp = \boldsymbol{Z} = \begin{bmatrix}\frac{1}{\sqrt{14}} \\ \frac{-2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}}\end{bmatrix} \qquad \boldsymbol{R} = \begin{bmatrix}\sqrt{6} & \frac{-\sqrt{6}}{3} \\ 0 & \frac{\sqrt{21}}{3}\end{bmatrix}$$

2. Solve $\boldsymbol{R}\boldsymbol{u} = \boldsymbol{c}$:

$$\begin{bmatrix}\sqrt{6} & 0 \\ \frac{-\sqrt{6}}{3} & \frac{\sqrt{21}}{3}\end{bmatrix} \boldsymbol{u} = \begin{bmatrix}4 \\ -2\end{bmatrix} \rightarrow \boldsymbol{u} = \begin{bmatrix}\frac{4}{\sqrt{6}} \\ \frac{-2}{\sqrt{21}}\end{bmatrix}$$

3. Calculate $\boldsymbol{Y}\boldsymbol{c} = \boldsymbol{Q}_1\boldsymbol{u}$:

$$\boldsymbol{Y}\boldsymbol{c} = \begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{21}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{21}}\end{bmatrix} \begin{bmatrix}4 \\ -2\end{bmatrix} = \begin{bmatrix}\frac{2}{7} \\ \frac{10}{7} \\ \frac{-6}{7}\end{bmatrix}$$

4. Solve $(\boldsymbol{Z}^T\boldsymbol{H}\boldsymbol{Z})\boldsymbol{y} = -\boldsymbol{Z}^T(\underbrace{\boldsymbol{b}}_{=0} + \boldsymbol{H}\boldsymbol{Y}\boldsymbol{c})$:

$$\boldsymbol{Z}^T\boldsymbol{H}\boldsymbol{Z}\boldsymbol{y} = -\boldsymbol{Z}^T\boldsymbol{H}\boldsymbol{Y}\boldsymbol{c} = \boldsymbol{0}$$
$$\boldsymbol{y} = \boldsymbol{0}$$

5. Calculate $\boldsymbol{x}^\star$:

$$\boldsymbol{x}^\star = \boldsymbol{Y}\boldsymbol{c} + \boldsymbol{Z}\boldsymbol{y} = \boldsymbol{Y}\boldsymbol{c} = \begin{bmatrix}\frac{2}{7} \\ \frac{10}{7} \\ \frac{-6}{7}\end{bmatrix}$$

2)

Step 1:

$$\begin{aligned}
\boldsymbol{\delta}^{(0)} &= \mathbf{0} \;\rightarrow\; \text{Case I} \\
\lambda_3, \lambda_4 &< 0 \\
\lambda_3 \nabla c_3 + \lambda_4 \nabla c_4 &= \nabla f \quad \text{dependent on scaling} \\
\lambda_4 &< \lambda_3 \;\rightarrow\; \text{Case Ib} \\
q &= 4 \\
\mathcal{A}^{(1)} &= \mathcal{A}^{(0)} \backslash \{4\} = \{3\} \\
\boldsymbol{x}^{(1)} &= \boldsymbol{x}^{(0)}
\end{aligned}$$

Step 2:

$$\begin{aligned}
\boldsymbol{\delta}^{(1)} &\neq \mathbf{0} \;\rightarrow\; \text{Case II} \\
\alpha^{(1)} &< 1 \;\rightarrow\; \text{Case IIb} \\
q' &= 2 \\
\mathcal{A}^{(2)} &= \mathcal{A}^{(1)} \cup \{2\} = \{2, 3\} \\
\boldsymbol{x}^{(2)} &= \boldsymbol{x}^{(1)} + \alpha^{(1)} \boldsymbol{\delta}^{(1)}
\end{aligned}$$

Step 3:

$$\begin{aligned}
\boldsymbol{\delta}^{(2)} &= \mathbf{0} \;\rightarrow\; \text{Case I} \\
\lambda_3 &< 0 \;\rightarrow\; \text{Case Ib} \\
q &= 3 \\
\mathcal{A}^{(3)} &= \mathcal{A}^{(2)} \backslash \{3\} = \{2\} \\
\boldsymbol{x}^{(3)} &= \boldsymbol{x}^{(2)}
\end{aligned}$$

Step 4:

$$\begin{aligned}
\boldsymbol{\delta}^{(3)} &\neq \mathbf{0} \;\rightarrow\; \text{Case II} \\
\alpha^{(3)} &= 1 \;\rightarrow\; \text{Case IIa} \\
\mathcal{A}^{(4)} &= \mathcal{A}^{(3)} \\
\boldsymbol{x}^{(4)} &= \boldsymbol{x}^{(3)} + \boldsymbol{\delta}^{(3)}
\end{aligned}$$

Step 5:

$$\begin{aligned}
\boldsymbol{\delta}^{(4)} &= \mathbf{0} \;\rightarrow\; \text{Case I} \\
\forall_{i \in \mathcal{A}} :& \lambda_i \geq \rightarrow\; \text{Case Ia} \\
&\rightarrow \text{first order optimality conditions fulfilled} \\
&\rightarrow \text{stop}
\end{aligned}$$

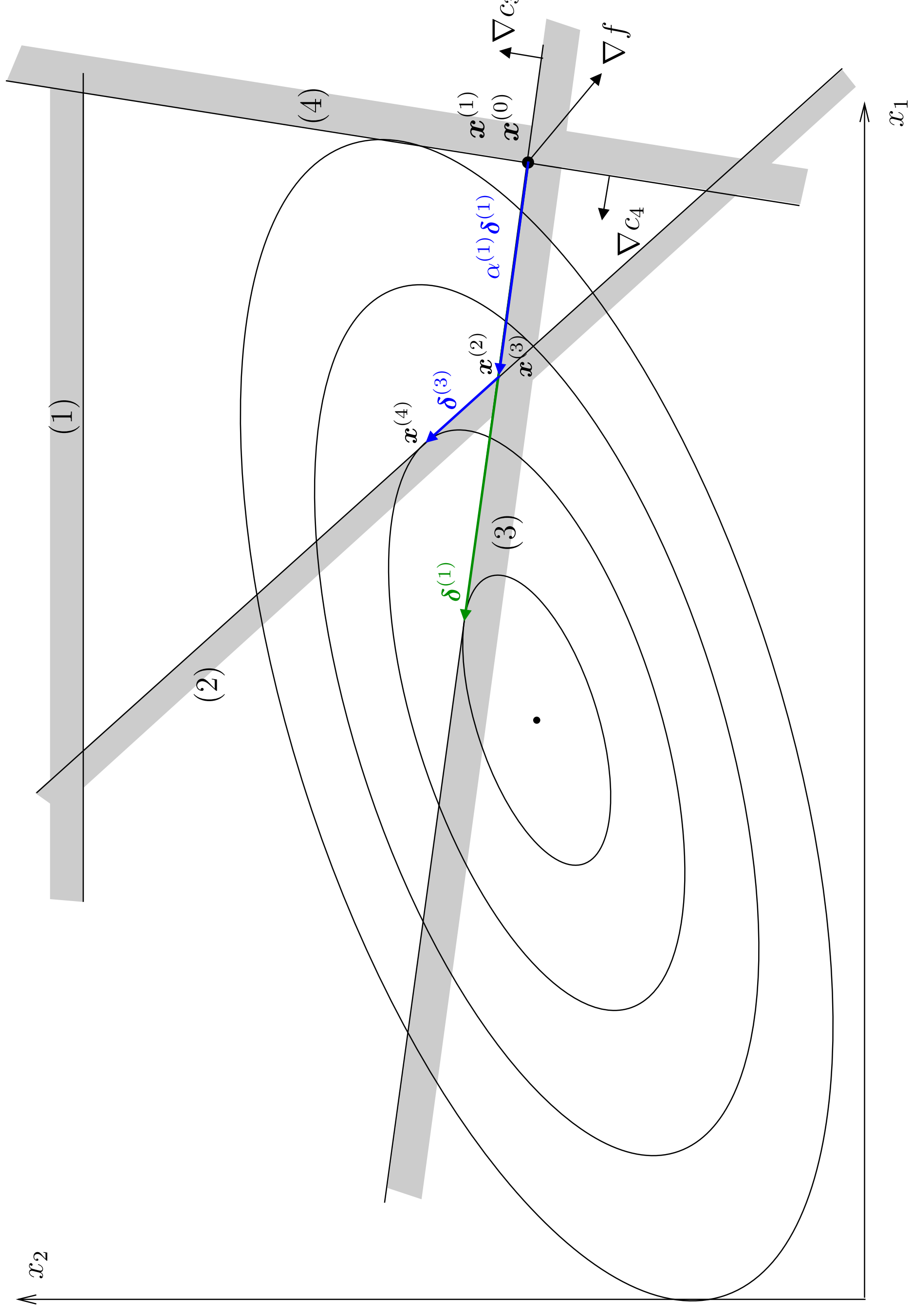


Figure 9.2: Solution of task 2

Exercise 10: Structural analysis of analog circuits

1. Fig. 8 shows the schematic of an operational amplifier.

   a) Generate the net-pin connection matrix for transistors MP1 and MP2. The supply voltage is connected to net *vdd*, the drain of MP1 to net *net1* and the drain of MP2 to net *net2*.

   b) Generate the connectivity matrix for transistors MP1 and MP2. What basic building block is implemented?

   c) Conduct a structural recognition of the basic building blocks. Only add new pairs for yet "uncoupled" transitors. Consider the building blocks in the following order:

   - Voltage reference 2 (vr2)
   - Voltage reference 1 (vr1)
   - Current load (cml)
   - Simple current mirror (scm)
   - Level shifter (ls)
   - Cascode (c)
   - Differential pair (dp)
   - Cascode current mirror (CCM)
   - 4-transistor current mirror (4TCM)
   - Wide-swing current mirror (WSCM)
   - Differential stage (DS)

   Mark the building blocks in the circuit with the given abbreviations.

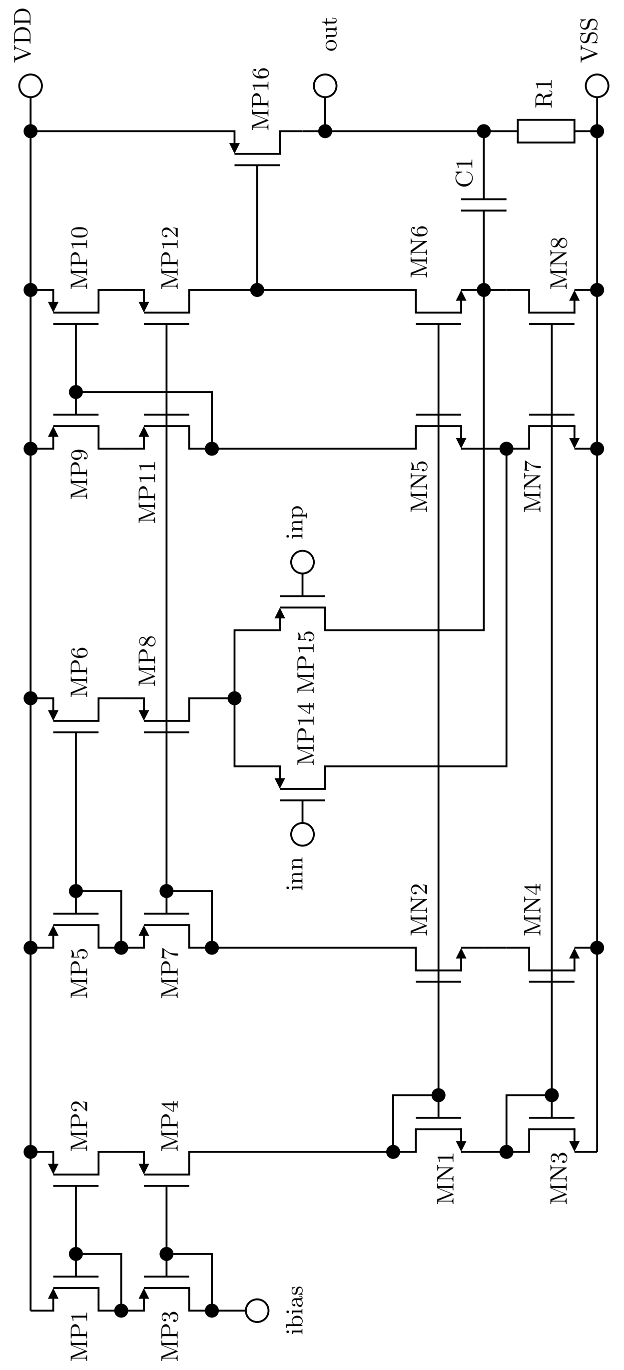


Figure 10.1: Operational amplifier

Exercise 10: Solution

1)

a)

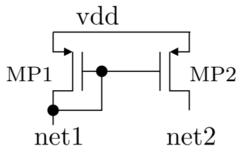


Figure 10.2: Simple Current Mirror

$$C_{MP1,MP2} = \begin{array}{c} \begin{matrix} \text{MP1.g} & \text{MP1.d} & \text{MP1.s} & \text{MP2.g} & \text{MP2.d} & \text{MP2.s} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} \text{net1} \\ \text{net2} \\ \text{vdd} \end{matrix}$$

b)

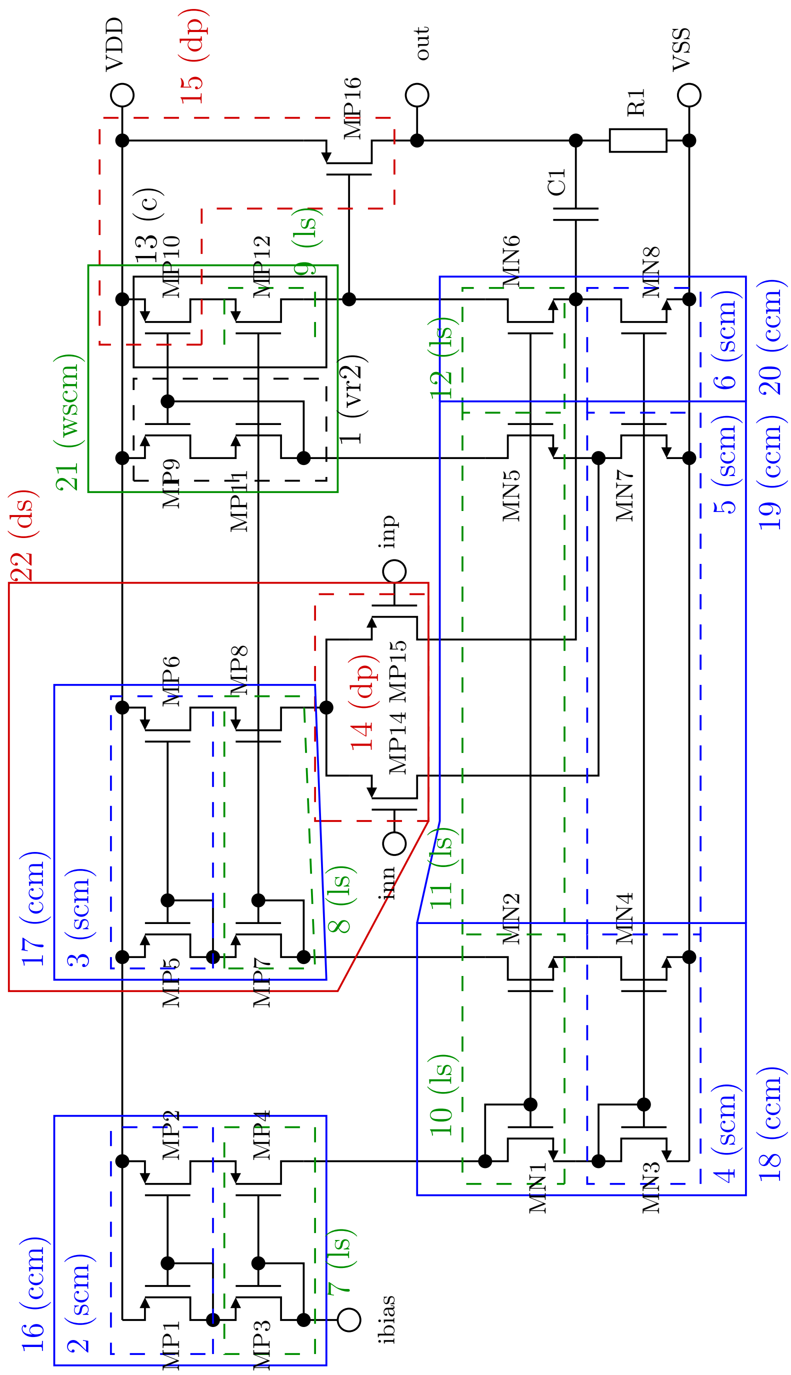
$$A = C^T_{MP1,MP2} C_{MP1,MP2}$$

$$= \begin{array}{c} \begin{matrix} \text{MP1.g} & \text{MP1.d} & \text{MP1.s} & \text{MP2.g} & \text{MP2.d} & \text{MP2.s} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} \text{MP1.g} \\ \text{MP1.d} \\ \text{MP1.s} \\ \text{MP2.g} \\ \text{MP2.d} \\ \text{MP2.s} \end{matrix}$$

1 $\rightarrow$ gates connected to one drain
1 $\rightarrow$ sources connected
$\rightarrow$ current mirror

c)

- 1 (vr2) outrules MP7, MP11 as level shifter
- 21 (wscm) outrules 9 (ls)
- 21 (wscm) outrules 15 (dp)


VDD
15 (dp)
out
VSS
MP16
R1
C1
13 (c)
MP10
MP12
9 (ls)
21 (wscm)
1 (vr2)
MP9
MP11
MN6
MN8
12 (ls)
6 (scm)
20 (ccm)
MN5
MN7
5 (scm)
19 (ccm)
22 (ds)
inp
MP14 MP15
14 (dp)
inn
MP6
MP8
17 (ccm)
3 (scm)
8 (ls)
MP5
MP7
11 (ls)
MN2
MN4
10 (ls)
MN1
MN3
4 (scm)
18 (ccm)
MP2
MP4
16 (ccm)
2 (scm)
7 (ls)
MP1
MP3
ibias