On the Relation of Circuit Theory and Signals, Systems and Communications

Josef A. Nossek and Michel T. Ivrlač
Institute for Circuit Theory and Signal Processing
Technische Universität München, D-80333 Munich, Germany
email: {nossek, ivrlac}@tum.de

***Abstract*—In signal processing, system theory, an in information theory signals are either complex valued waveforms or sequences of dimensionless complex numbers. The »power« of such signals is usually said to be the squared magnitude of these complex numbers. In an electrical circuit, power is always the product of voltage and current, both present at a port, i.e., at a pair of terminals. Here we ask the question: what is the relation between signal power and physical power? There are special cases, where the two notions of power are strictly proportional and, therefore, no conflict occurs. But in general, especially when we deal with vector signals, the commonly used squared Euclidean norm of the signal vector may not be a measure for the physical power associated with the voltages and currents, which correspond to the signal vector. In addition to the physically consistent computation of signal power, physically consistent modeling of the noise is crucial in all information processing systems. Again circuit theory provides with a methodology to model how the noise comes into the system. Modeling is one the most important tasks in engineering and, therefore, physically consistent modeling is very important for the education of electrical engineers. Circuit Theory is the essential framework to ensure physical consistency across several levels of abstraction.**

I. Scalar Signals and One-ports

Scalar signals in information processing and communication systems are usually either complex valued continuous functions of time, $x(t) \in \mathbb{C}$, or sequences $x[k] \in \mathbb{C}$ of complex numbers. The instantaneous power is in most textbooks (e.g., [1]–[3]) defined as:

$$P(t) = |x(t)|^2, \quad P[k] = |x[k]|^2. \tag{1}$$

In case that x is a random variable, it is customary to apply the expectation operation to arrive at the average power $P_{\mathrm{av}} = \mathrm{E}[P[k]]$. The physical power, which flows into a one-port is given by:

$$P_{\mathrm{phy}}(t) = u(t) \cdot i(t) = u^2(t)/R = i^2(t) \cdot R, \tag{2}$$

where we have assumed that the port is terminated by a passive linear resistor $R > 0$. In this specific case, the signal power (1) is *strictly proportional* to the physical power (2) provided that the assignment of signals to physical quantities is made in one of the following ways:

$$\left.\begin{array}{l} x(t) \sim u(t) \\ x(t) \sim i(t) \\ x(t) \sim \big(\alpha u(t) + \beta i(t)\big) \end{array}\right\} \implies P(t) \sim P_{\mathrm{phy}}(t). \tag{3}$$

Therefore, the question »what does the signal $x(t)$ mean physically?« is of secondary interest in this case. The standard textbook definition of signal power and signal energy can be applied and there is no conflict. The above reasoning is based on instantaneous values. We could have used equally well complex phasors, if the signals are mono-frequent, or at least narrow in their relative bandwidth. Note that for physical power we always have first to deal with continuous-time voltages and currents, before transforming them to discrete-time sequences consistently.

II. Vector Signals and Multi-Ports

Let us consider a complex valued vector signal $\boldsymbol{x}(t) \in \mathbb{C}^n$, or $\boldsymbol{x}[k] \in \mathbb{C}^n$, the instantaneous power of which is usually defined as:

$$P(t) = \|\boldsymbol{x}(t)\|_2^2, \quad P[k] = \|\boldsymbol{x}[k]\|_2^2. \tag{4}$$

The physical power which flows into a multi-port is equal to the sum of the powers flowing through the ports and can be computed as the dot-product of the vector containing the port voltages and the vector containing the port currents:

$$P_{\mathrm{phy}}(t) = \boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{i}(t) = \boldsymbol{i}^{\mathrm{T}}(t)\boldsymbol{u}(t). \tag{5}$$

Let us show with a simple example with only two ports, that it is not possible to have a simple assignment of signals to physical quantities, as was possible in the scalar case. The physical power which flows into the two-port in Figure 1, of course, depends on the properties of the two-port, which are reflected by any of the existing two-port matrices. Either by inspection or

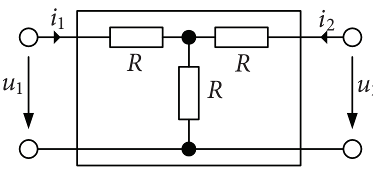

Figure 1: Resistive, passive $(R > 0)$ two-port.

formally with the aid of the impedance matrix of this twoport:

$$\boldsymbol{Z} = R\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

we obtain for the physical power:

$$\begin{aligned} P_{\mathrm{phy}}(t) &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{i} = \boldsymbol{i}^{\mathrm{T}}\boldsymbol{Z}\boldsymbol{i} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{Z}^{-1}\boldsymbol{u} \\ &= 2R\left(i_1^2 + i_2^2 + i_1 i_2\right) = \tfrac{2}{3}\left(u_1^2 + u_2^2 - u_1 u_2\right)/R. \end{aligned} \tag{6}$$

Clearly, even in this simple case, the signal power (4) is *not* proportional to the physical power (6), no matter which of the following assignments for the vector signal $\boldsymbol{x}(t)$ and the physical quantities $\boldsymbol{u}(t)$ or $\boldsymbol{i}(t)$ we choose:

$$\left.\begin{array}{l} \boldsymbol{x}(t) \sim \boldsymbol{u}(t) \\ \boldsymbol{x}(t) \sim \boldsymbol{i}(t) \\ \boldsymbol{x}(t) \sim \big(\alpha \boldsymbol{u}(t) + \beta \boldsymbol{i}(t)\big) \end{array}\right\} \implies P(t) \nsim P_{\mathrm{phy}}(t). \tag{7}$$

978-1-4244-9474-3/11/$26.00 ©2011 IEEE

Thus, in the vector case, the question »what does the signal $\boldsymbol{x}(t)$ mean physically?« is of essential importance, because the physical power is a quadratic form and not simply a squared norm. Only if the impedance matrix is a scaled identity, i.e., if the multiport is not really a multiport but rather an assembly of identical but decoupled one-ports, then the quadratic form (6) reduces to a scaled squared norm of either the current or the voltage vector. Again, the above reasoning, based on instantaneous values, can easily be extended to average power by taking expectations and to narrow-band situations by working with complex phasors.

III. How does the Noise Come into the System?

Since in almost all systems signal-to-noise-ratio (SNR) is of paramount importance, it is not only signal power which we have to model consistently. How does the noise come into the system has to be addressed with equal rigor. In most textbooks on communication systems we will find a block diagram as given in Figure 2.

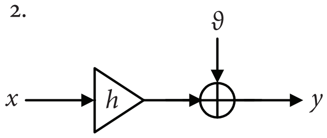

Figure 2: An additive white Gaußian noise channel with input signal x, output signal y, channel coefficient h, and additive, zero-mean, white Gaußian noise ϑ with variance σ_ϑ^2.

The SNR at the output is given by:

$$\mathrm{SNR} = \frac{\mathrm{E}\big[|y|^2 \mid \vartheta = 0\big]}{\mathrm{E}\big[|y|^2 \mid x = 0\big]} = \mathrm{E}\big[|x|^2\big] \cdot \frac{|h|^2}{\sigma_\vartheta^2}. \tag{8}$$

If the magnitude of the channel coefficient h is reduced by a factor of 2, we would expect a reduction of SNR by a factor of 4, as predicted by (8). But that would only be true, if σ_ϑ^2 is independent of h. It turns out, however, that consistent modeling of a physical system may enforce certain dependencies between h and σ_ϑ^2. Let us investigate this with a simple example system shown in Figure 3. Basic circuit analysis reveals:

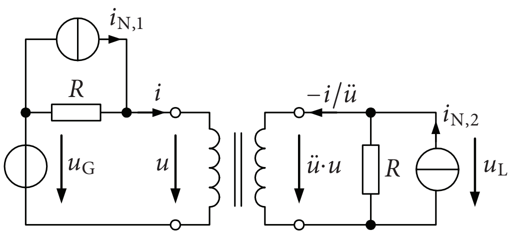

Figure 3: The channel, an ideal transformer, that is driven by a voltage source u_{G} with source impedance R, and terminated with a load impedance R, both impedances subject to thermal noise represented by the two current sources with zero mean and variance of $4\mathrm{k}TB/R$, and uncorrelated with each other.

$$u_{\mathrm{L}} = u_{\mathrm{G}} \frac{\ddot{u}}{1+\ddot{u}^2} + R\ddot{u}\frac{i_{\mathrm{N},1} + \ddot{u}\, i_{\mathrm{N},2}}{1+\ddot{u}^2}. \tag{9}$$

By assigning $x = u_{\mathrm{G}}$, and $y = u_{\mathrm{L}}$, we have established the correspondence between the abstract model from Figure 2, and the circuit from Figure 3, which leads to the implications:

$$h = \frac{\ddot{u}}{1+\ddot{u}^2} \le \frac{1}{2}, \quad \sigma_\vartheta^2 = \sigma_0^2\left(1 - \sqrt{1-4h^2}\right), \tag{10}$$

where $\sigma_0^2 = 2\mathrm{k}BTR$, and $0 < \ddot{u} \le 1$. These equations show that the channel coefficient h and the noise variance σ_ϑ^2 are *not* independent, and therefore, our conjecture, that the SNR will go down by a factor of 4 when the channel coefficient h is reduced by a factor of 2, is completely wrong. The SNR as a function of h is depicted in Figure 4 where we observe a monotonic increase of SNR as h is reduced from its maximum value of 0.5 down to smaller values. The noise comes into the system through noise sources associated with noisy circuit elements and possibly external sources of noise and interference. In general, there is not one abstract adder through which the noise is injected, but there are Kirchhoff's current and voltage laws (KCL and KVL) superimposing the noise to the signal.

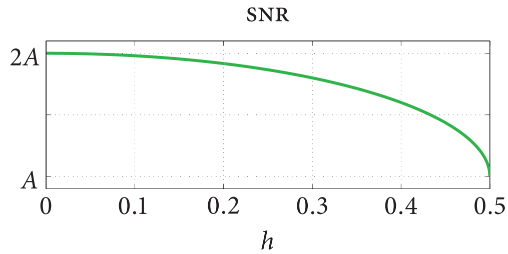

Figure 4: The SNR of the AWGN-channel from Figure 2 of the communication system from Figure 3, as function of h.

As such simple a circuit as given in Figure 3 leads to this unexpected results, we may encounter complicated dependencies between the channel matrix of multi-input multi-output (MIMO) systems and the noise covariance matrix [4].

IV. Conclusions

We have shown that common mathematical models, used deliberately in system theory and in modeling communication systems, may not always be consistent with the basic physical laws which govern the behavior of technical system implementations. Such inconsistencies can be overcome by using circuit theory as a frame work providing physical consistency along different layers of abstraction: from electromagnetics down to circuits, signal processing algorithms and information theory.

It is important that electrical engineering and information technology students are made aware of these problems. Otherwise the unreflected use of mathematical models without taking care of their physical consistency may lead to suboptimal or even erroneous designs. Therefore, circuit theory – and we stress *theory* – has to be an indispensable part of EE curricula.

References

[1] J. G. Proakis, *Digital Communications*, third ed. McGraw-Hill, 1995.
[2] A. Lapidoth, *A Foundation in Digital Communication*. Cambridge University Press, 1995.
[3] J. G. Proakis and D. G. Monolakis, *Digital Signal Processing*, 2nd ed. Macmillan Publishing Company, 1992.
[4] M. T. Ivrlač and J. A. Nossek, "Toward a Circuit Theory of Communication," *IEEE Transactions on Circuits and Systems-I: Regular Papers*, vol. 57(7), pp. 1663–1683, jul 2010.