Nullators and Norators in Circuits Education: A Benefit or an Obstacle?

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***Abstract*— The study of circuits and systems requires some basic reasoning with various equivalent interconnections of components like resistors, sources, ideal opamps, ideal transistors, short and open circuits.... Students should be able to handle a rich variety of such interconnections correctly and appropriately. Special components like nullators and norators bring in some valuable insights, flexibility and design methods based thereupon. From the practical experience of teaching nullators and norators in basic circuits courses for electrical engineering students, the authors stress a number of didactical goals and experiences. First, it is crucial that the students understand the notions of nullator and norator and are able to distinguish these from other components, such as short circuits, open circuits, resistors, and sources. Second, they should be able to reason correctly with interconnections that involve nullators and norators. Third, they should be able to find equivalents for circuits having nullators and norators. Analysis and design of circuits with nullators and norators will be presented. A particular case study of the use of nullators and norators for the implicit inversion of a 2×2 matrix will be used as an illustration of some unexpected results, which we will hardly find without nullators and norators. This will lead to an evaluation of the didactical processes of using nullators and norators.**

I. Introduction

The paper is organized as follows. First we discuss the historical context and the opportunities of nullators and norators for Circuits and Systems (CAS) education in Section II. In Section III we present a number of didactical exercises where this concept can help to clarify some notions of components and equivalent resistive circuits. In Section IV we present an application where nullators and norators elucidate the process of multiplying a vector of voltages with the inverse of a given matrix without even actually calculating this inverse. In Section V we give conclusions and some opinions on the benefits and impediments for introducing nullators and norators.

II. Historical context and opportunities

Nullators and norators have a long tradition in the CAS community [1]-[6] with famous circuit theorists like Carlin, Youla, Tellegen, Oono and Belevitch. In [1] Carlin and Youla introduce a nullator as a pathologic one port described by

$$\text{Nullator:} \qquad v = 0 \quad \text{and} \quad i = 0 \tag{1}$$

Hence it has some properties of a short circuit as well as an open circuit and it does not have an immittance or a scattering

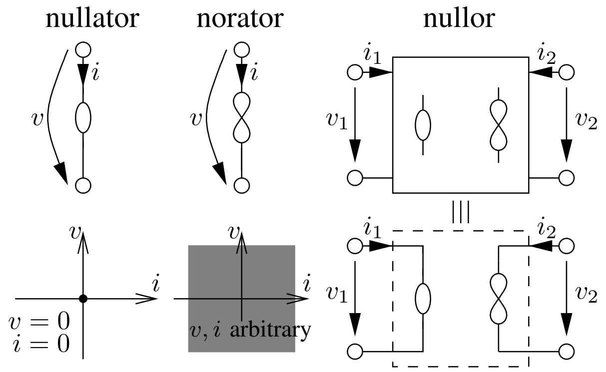

Fig. 1. Nullator, norator and nullor with their $i - v$ characteristics and equations.

representation. They define a norator in a similar way as a one port

$$\text{Norator:} \qquad \text{voltage } v \text{ and current } i \text{ arbitrary.} \tag{2}$$

So the norator exhibits also some properties of a short circuit as well as an open circuit. One can also combine these two components into a two port and define

$$\text{Nullor:} \qquad \text{port 1} = \text{nullator} \quad \& \quad \text{port 2} = \text{norator.} \tag{3}$$

Or alternatively

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} \tag{4}$$

The typical symbols for these are given in Fig. 1, where the nullator has a big zero, while the norator has an infinity symbol, and the nullor has both. The corresponding acceptable points in the $i - v$ plane are also plotted.

In fact Oono [4] and Belevitch [5] had already defined the nullor as a special anomalous two port in their circuit realization and synthesis methods. Such a nullor is typically obtained in [1][4][5] as a degenerate case of a circuit where some components go to a limit. Carlin in [2] did no longer define the nullor, nullator and norator as a limiting process but postulated these one and two ports via (1)-(4). In a more mathematical way [2] defines the nullator (resp. the norator) as the subspace of dimension 0 i.e. only the origin (resp., 2, i.e. the whole plane) in the i, v space. This is more in

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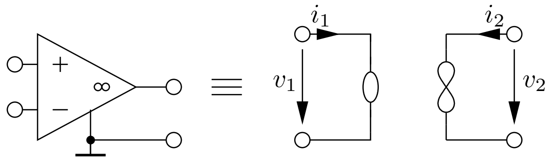
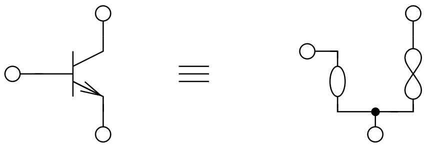
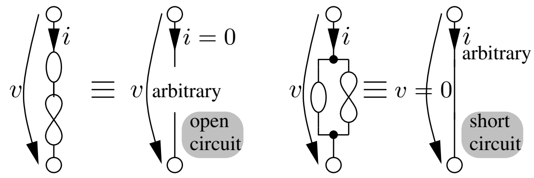


(a) An ideal OpAmp with infinite amplification and operating in the linear region can be represented by a nullator at the input and a norator at the output (i.e. a nullor)



(b) An ideal transistor and its small signal equivalent three terminal nullor

Fig. 2. Important circuit elements and their nullor equivalents



(a) (b)

Fig. 3. Interconnections including nullators, norators and nullors

line with many modern subspace methods in mathematics, circuit and system theory [6], and in particular the behavioral approach [7]. However this postulation [2] of characteristics of nullator and norator created a fierce reaction from Tellegen [3] that nullators and norators are fictitious elements that have no physical meaning while the nullor is still a physically meaningful two port. While many of the realization methods with gyrators and negative resistances have disappeared from arena of active research and have been eliminated in most EE education programs in order to make the didactical process more lean, the nullator, norator, and nullor still appear often in textbooks [6] and recent publications on design and analysis of circuits. Moreover stirred by the progress in IC technology, the operational amplifier has become an important building block, that can also be described with (1)-(3). In fact, if the opamp is ideal (with zero input current and infinite amplification and if it is working in the linear region) it can be represented by a nullor, i.e. a nullator at the input and a norator at the output (Fig. 2). Often the input is described as a virtual ground $v = 0$ and $i = 0$. Students should not confuse this with a short circuit, in other words it is not permitted to solder a conductor between the positive and negative input.

III. Correct Reasoning with Nullators and Norators Illustrated with Exercises

It is the teaching experience of the authors that the notions of nullator, norator and nullor can be introduced at an early phase in the educational process of electrical engineering and that there are interesting benefits if the students have acquired the experience to reason with these components adequately.

In the learning process there is a first important step in bringing the definitions (1)-(4) into working instruments for dealing with circuits. Next one can study the interconnections of various components including nullators, norators and nullors. Third, there are systematic ways of using these components in various designs. In fact it has been shown [8][9] that every linear network can be modeled by an equivalent network consisting only of linear resistors, capacitors and nullors and excited only by current sources. Hence the proposed approach can smoothen, strengthen and deepen the students' insights. Of course the present list of exercises is based on the authors' experience, and environment.

A. *Interconnections with nullators, norators and nullors*

When connecting a norator and a nullator in series (Fig. 3(a)), and in parallel (Fig. 3(b)) the first is equivalent with an open circuit while the second is equivalent with an short circuit. This is surprising for many students, but one should observe that in a series connection the same current is going through both the nullator and the norator. Since the nullator has current zero also the equivalent has current zero. The voltages of the nullator and the norator add up to v, and hence v is arbitrary. Any one port with arbitrary voltage and zero current is an open circuit. Similar arguments apply to the parallel connection (Fig. 3(b)) where the equivalent is a short circuit.

In fact all four types of linear controlled sources can be synthesized nicely with nullators, norators, and linear resistors (see [8], [9]). Similar to the source equivalents and v-shift and i-shift properties, students can easily derive a number of equivalences for parallel and series connections of nullators, norators and sources (see Fig. 4). Moreover it is easy to show that a tree of nullators can be replaced by any tree of nullators with the same collection of nodes. A similar equivalence property holds for a tree of norators. Various other equivalences are described also in [8].

B. *Exercises with nullators, norators and nullors*

The students should learn to find the equivalent characteristic of an interconnection of components including nullators and norators, and should be able to deal with the impact of certain modifications on it. The analog circuit simulator applet [11] is a very nice active webpage where the students can experiment and verify their insights with circuits. It would be nice if this applet is extended with nullators, norators and nullors.

In Fig. 5(a) a one port is constructed with two parallel branches. The first is a series connection of a nullator and a voltage source of 3V, and the second is a series connection of a norator and a current source of 2mA. Using the defining equations of the norator and the nullator, we can conclude that the first branch must have a voltage of 3V and the second branch must have a current of 2mA. Hence the parallel

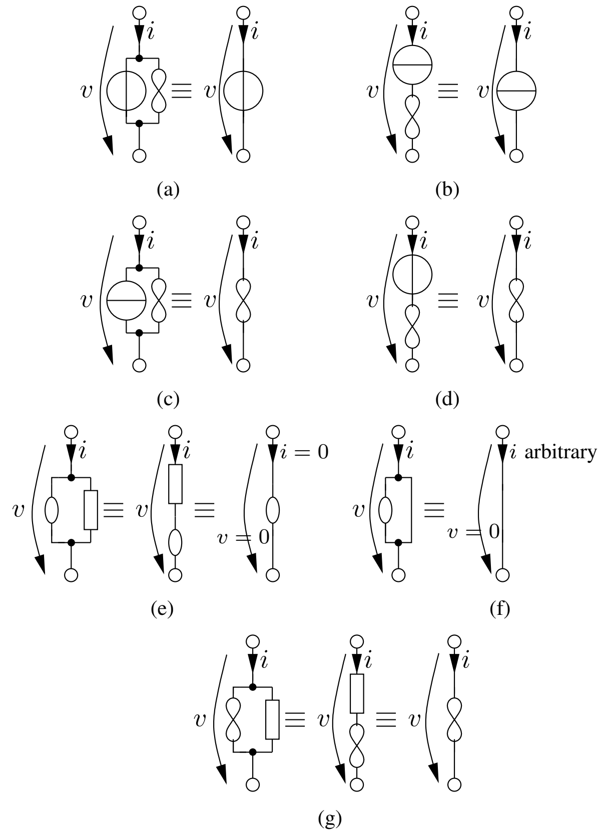


Fig. 4. Equivalences with nullators and norators

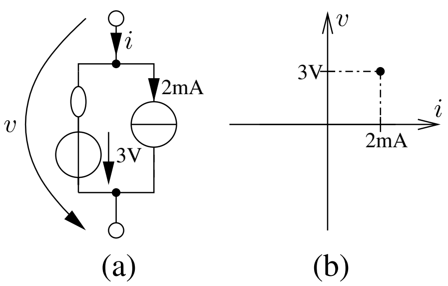


Fig. 5. Equivalent circuits comprising nullators, norators and nullors

connection should have an equivalent of a voltage $v = 3\text{V}$ and a current $i = 2\text{mA}$. More excercises for simplifying one ports involving nullators and norators are given in Fig. 6.

C. Circuit analysis and design with nullators, norators and nullors

For an implementable circuit, which with given excitation always has a unique solution for all currents and voltages, the number of nullators must always match the number of norators, such that the number of equations always matches the number of unknowns to be evaluated. This leads to the fact that nullators and norators can always be paired to form nullors, which then can be implemented by transistors or opamps. Obviously quite a variety of different opamp or transistor

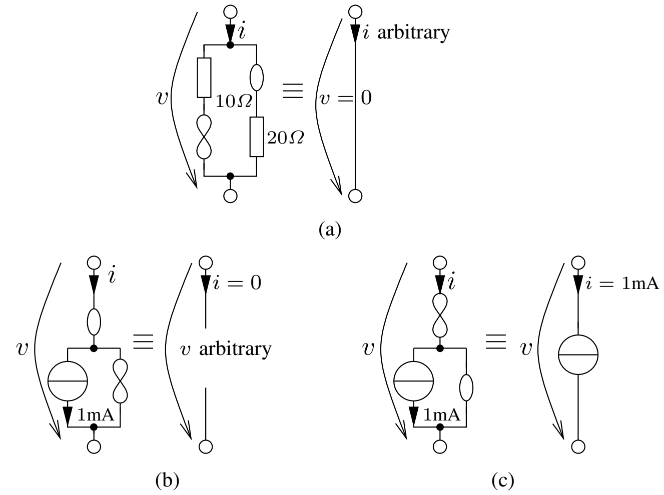


Fig. 6. Exercises simplify the one ports given at the left

circuits can be generated from the nullator-norator equivalent circuit depending on the number of nullators and norators. This opens up design choices, where the circuit designer has to find the optimal version when taking into account more detailed models *e.g.* finite gain bandwidth product of opamps. Nice illustrations of these statements are given in [8][9] and the example in the next Section.

IV. Multiplication with the inverse of a matrix with nullators and norators

Here we will show how the well known design of an inverting amplifier circuit with an opamp can be generalized to the design of a circuit performing the multiplication with the inverse of a given matrix $\boldsymbol{A}$, without actually even having to compute this inverse matrix $\boldsymbol{A}^{-1}$.

We start with a well known opamp circuit to perform the multiplication of a vector $\boldsymbol{x}$ by a matrix $\boldsymbol{A}$ delivering a vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. For the simple case of two dimensional real-valued vectors such a circuit is depicted in Fig. 7(a). The matrix

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is assumed to have negative entries, such that the conductances $-aG$, $-bG$, $-cG$ and $-dG$ are all positive. If this is not the case, inverting amplifiers can be inserted at proper places to ensure an implementation with positive conductances throughout. Let us first replace the opamps by their nullor model (see Fig. 7(b)). Since we have the nullators and norators coming pairwise, we have a unique solution for all voltages and currents in our circuit. Now, we can replace the two norators in Fig. 7(b) by two voltage sources enforcing the voltages y_1 and y_2 across the two ports, which previously have been the output and give freedom for the port voltages x_1 and x_2 by replacing the voltage sources with norators. Since the solution is unique, we will see in the circuit in Fig. 7(c) exactly the same voltages and currents as we had in Fig. 7(a) and Fig. 7(b). By construction all circuits in Fig. 7 have the same node voltages and hence they all satisfy $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ as well as

$x = A^{-1}y$. Finally we simply have to replace nullator-norator pairs by opamps to arrive at the circuit of Fig. 7(d), which now multiplies the input vector y with the inverse matrix A^{-1} to get $x = A^{-1}y$. Note that we have never actually computed that inverse, in the design of the circuit only the elements of the original matrix A have been used directly.

This design approach could be extended to higher dimensionality vectors and to complex-valued vectors and matrices. One can even design with n opamps a circuit for performing the operation $z = A^{-1}By$, where A and B are $n \times n$ matrices.

V. Conclusions and Recommendations

It is the experience of the authors that it is beneficiary to include nullators and norators in the collection of components that the students can reason with in basic circuit theory. It provides the students with a better understanding of the role of the input and output of an ideal opamp, by making an abstraction as nullator and norator branch. Moreover, once these nullator and norator characteristics are understood, the students can manipulate creatively various equivalences, which otherwise would be difficult to obtain. In a circuit with several nullator-norator pairs we have a number of different opamp/transistor implementations, which are equivalent on the level of ideal opamps/transistors, but possibly quite different from real opamps/transistors. Therefore, the designer is in a position to choose the best one for his specific application. Moreover this brings circuit theory closer to the topics in design, analysis and matrix manipulations like matrix inversion.

Acknowledgment

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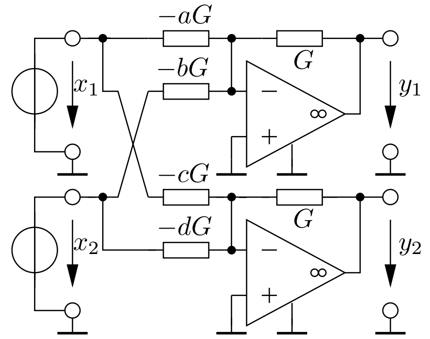
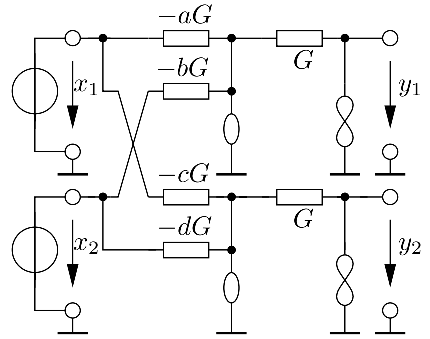
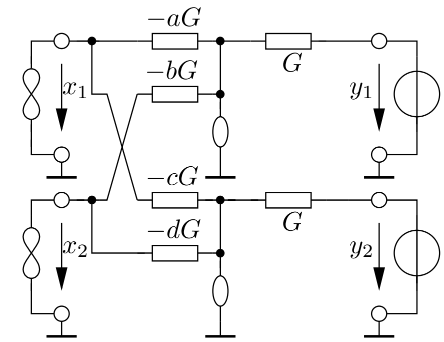


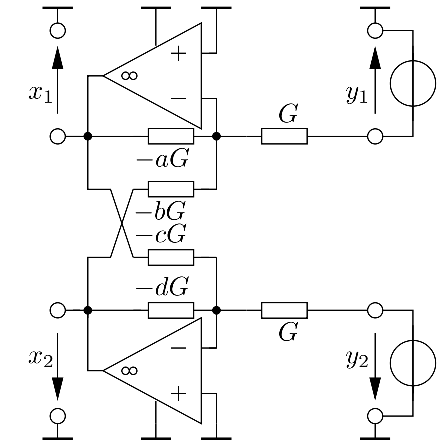
(a) Opamp circuit for implementing $y = Ax$



(b) Nullator-Norator equivalent for the opamp circuit in Fig. 7(a)



(c) Enforcing the voltage y_1 and y_2 with voltage sources and replacing the voltage sources x_1 and x_2 by Norators leads to the same voltages x_1 and x_2 across these Norators



(d) Opamp circuit for implementing $x = A^{-1}y$

Fig. 7. Development of a circuit for multiplication with A^{-1} although we know only the entries of A