Extremes of Continuous-Time Processes

Vicky Fasen[1]

Center for Mathematical Sciences, Munich University of Technology, D-85747 Garching, Germany, fasen@ma.tum.de

Abstract

In this paper we present a review on the extremal behavior of stationary continuous-time processes with emphasis on generalized Ornstein-Uhlenbeck processes. We restrict our attention to heavy-tailed models like heavy-tailed Ornstein-Uhlenbeck processes or continuous-time GARCH processes. The survey includes the tail behavior of the stationary distribution, the tail behavior of the sample maximum and the asymptotic behavior of sample maxima of our models.

1 Introduction

In this paper we study the extremal behavior of stationary continuous-time processes. The class of stationary continuous-time processes is rich, and the investigation of their extremal behavior is complex. The development of the extremal behavior of Gaussian processes, which is the origin of continuous-time extreme value theory starting with Rice [46, 47, 48], Kac [24], Kac and Slepian [25], Volkonskii and Rozanov [56, 57] and Slepian [54, 55], alone, would fill a paper. See the monograph of Leadbetter, Lindgren and Rootzén [33] or the Ph.D. thesis of Albin [2] or the paper [3] for a review of this topic. Since financial time series

- are often random with jumps,
- have heavy tails,
- exhibit clusters on high levels,

we will concentrate mainly on stationary continuous-time processes having these properties.

We will explain the basic ideas concerning extreme value theory for stationary continuous-time processes by generalized Ornstein-Uhlenbeck (GOU) processes, which are applied as stochastic volatility models in finance and as risk models in insurance. They are represented by

$$X_t = \mathrm{e}^{-\xi_t} \int_0^t \mathrm{e}^{\xi_{s-}} \, d\eta_s + \mathrm{e}^{-\xi_t} X_0, \quad t \geq 0, \tag{1}$$

where $(\xi_t, \eta_t)_{t \geq 0}$ is a bivariate Lévy process independent of the starting random variable X_0 (cf. Lindner and Maller [39] and for definitions, further details and references see also Maller, Müller and Szimayer [40] in this volume). A bivariate Lévy process is characterized by the *Lévy-Khinchine representation*

$$\mathbb{E}(\mathrm{e}^{i\langle \Theta, (\xi_t, \eta_t) \rangle}) = \exp(-t\Psi(\Theta)) \quad \text{for } \Theta \in \mathbb{R}^2,$$

where

$$\Psi(\Theta) = -i\langle \gamma, \Theta \rangle + \frac{1}{2}\langle \Theta, \Sigma\,\Theta \rangle + \int_{\mathbb{R}^2} \Big(1 - \mathrm{e}^{i\langle \Theta, (x,y) \rangle} + i\langle (x,y), \Theta \rangle\Big) \, d\Pi_{\xi,\eta}(x,y)$$

with $\gamma \in \mathbb{R}^2$, Σ a non-negative definite matrix in $\mathbb{R}^{2\times 2}$, $\langle \cdot, \cdot \rangle$ the inner product and $\Pi_{\xi,\eta}$ a measure on $\mathbb{R}^2$, called *Lévy measure*, such that $\int_{\mathbb{R}^2} \min\{\sqrt{x^2+y^2}, 1\} \, d\Pi_{\xi,\eta}(x,y) < \infty$ and $\Pi_{\xi,\eta}((0,0)) = 0$ (cf. Sato [51]).

The limit behavior of the sample maxima

$$M(T) = \sup_{0 \leq t \leq T} X_t \tag{2}$$

as $T \to \infty$ of the stationary GOU-process $(X_t)_{t \geq 0}$ will be described either when $\xi_t = \lambda t$ or when $\mathbb{E}(\mathrm{e}^{-\alpha \xi_1}) = 1$ for some $\alpha > 0$.

In Section 2, a synopsis of extreme value theory is given. Precise definitions of the GOU-models studied in this paper are presented in Section 3. We start with the investigation of the tail behavior of the sample maximum in Section 4. Section 5 on the asymptotic behavior of sample maxima $M(T)$ as $T \to \infty$ and the cluster behavior follows. Finally, Section 6 concludes with remarks on extensions of the results to more general models.

2 Extreme value theory

One method of investigating extremes of stationary continuous-time processes is to study the extremal behavior of the discrete-time skeleton

$$M_k(h) = \sup_{(k-1)h \leq s \leq kh} X_s \quad \text{for } k \in \mathbb{N} \tag{3}$$

and some fixed $h > 0$, which is again a stationary sequence. The advantage of such a skeleton is that known results for sequences can be applied, which are well investigated; see de Haan and Ferreira [15], Embrechts, Klüppelberg and Mikosch [17], Leadbetter et al. [33] and Resnick [45]. To my knowledge this idea was first applied to Gaussian processes by Leadbetter and Rootzén [34]. The monograph of Leadbetter et al. [33] and the paper of Leadbetter and Rootzén [35] contain a detailed study of extremes of discrete-time and continuous-time processes. A completely different approach to extreme value theory for continuous-time processes as presented here is given in Berman [8]. Both approaches were combined by Albin [2, 3].

2.1 Extremes of discrete-time processes

We start with an introduction into extremes of discrete-time processes. Let $(Y_n)_{n\in\mathbb{N}}$ be a stationary sequence with distribution function F and $M_n = \max\{Y_1,\ldots,Y_n\}$ for $n \in \mathbb{N}$. The simplest stationary sequence is an iid (independently and identically distributed) sequence. In this case, we find sequences of constants $a_n > 0$, $b_n \in \mathbb{R}$, such that

$$\lim_{n\to\infty} \mathbb{P}(a_n^{-1}(M_n - b_n) \leq x) = G(x) \quad \text{for } x \in \operatorname{supp}(G), \tag{4}$$

and some non-degenerate distribution function G whose support is denoted by $\operatorname{supp}(G)$, if and only if

$$\lim_{n\to\infty} n\overline{F}(a_n x + b_n) = -\log G(x) \quad \text{for } x \in \operatorname{supp}(G)\,, \tag{5}$$

where $\overline{F} = 1 - F$ denotes the *tail* of F. Then we say that F is in the *maximum domain of attraction* of G ($F \in \operatorname{MDA}(G)$). The Extremal Types Theorem (Leadbetter et al. [33], Theorem 1.4.2) says that G is either a Fréchet (Φ_α, $\alpha > 0$), a Gumbel (Λ) or a Weibull (Ψ_α, $\alpha > 0$) distribution.

For a stationary sequence $(Y_n)_{n\in\mathbb{N}}$ there exists sufficient conditions such that the extremal behavior of the stationary sequence coincides with the extremal behavior of an iid sequence with the same stationary distribution; i.e. (5) implies (4). The conditions which guarantee this conclusion are known as D and D' conditions (cf. Leadbetter et al. [33], pp. 53). The condition D is a mixing condition for the asymptotic independence of maxima, and the condition D' is an anti-clustering condition. That is, given an observation at some time n is large, the probability that any of the neighboring observations are also large is quite low.

Examples exist which do not satisfy the D' condition and which have extremal clusters on high level values. There, the *extremal index* is defined as a measure of the cluster size; i. e. if (5) holds and

$$\lim_{n\to\infty} \mathbb{P}(a_n^{-1}(M_n - b_n) \leq x) = G^\theta(x) \quad \text{for } x \in \operatorname{supp}(G),$$

then θ is called the extremal index. The parameter θ takes only values in $[0, 1]$, where $\theta = 1$ reflects no extremal clusters.

2.2 Extremes of continuous-time processes

After these basic ideas concerning extremes of stationary discrete-time processes, we continue with extremes of stationary continuous-time processes. The extremal behavior of a continuous-time process is influenced by the dependence of the process not only in large, but also in small time intervals. The dependence structure of the process in small time intervals is negated by investigating the extremal behavior of $(M_k(h))_{k\in\mathbb{N}}$ as in (3), where $\max_{k=1,\dots,n} M_k(h) = M(nh)$. The conditions D and D' on the sequence $(M_k(h))_{k\in\mathbb{N}}$ can be reformulated as conditions on the continuous-time process $(X_t)_{t\geq 0}$ known as C and C' conditions. Again, condition C is a condition on the asymptotic independence of maxima and C' on the cluster behavior of $(M_k(h))_{k\in\mathbb{N}}$. Similar to discrete-time models, an Extremal Types Theorem also holds (cf. Leadbetter et al. [33], Theorem 13.1.5). For Gaussian processes a simple condition only on the covariance function exists such that C and C' are satisfied (cf. Leadbetter et al. [33], Theorem 12.3.4).

As in discrete time, there are also continuous-time examples which do not satisfy the C' condition and have extremal clusters on high levels. In this case, the *extremal index function* $\theta : (0,\infty) \to [0,1]$ is defined as a measure for clusters, where $\theta(h)$ is the extremal index of the sequence $(M_k(h))_{k\in\mathbb{N}}$ for every $h > 0$. The function $\theta(\cdot)$ is increasing. In our context we say that a continuous-time process has *extremal clusters*, if $\lim_{h\downarrow 0} \theta(h) < 1$, and otherwise it has no clusters, i. e. $\theta(h) = 1$ for every $h > 0$, by the monotony of θ. The interpretation of an extremal cluster in continuous-time is the same as in discrete-time, i.e., a continuous-time process clusters if given a large observation at some time t, there is a positive probability that any of the neighboring observations is also large.

2.3 Extensions

At the end we also want to describe the way in which it is in mathematical terms possible to investigate the locations and heights of local maxima. One possibility is by marked point processes (cf. Daley and Vere-Jones [13], Kallenberg [26] and Resnick [45]). In our case, a marked point process counts the number of elements in the set

$$\{k : a_n^{-1}(M_k(h) - b_n) \in B_0, a_n^{-1}(X_{k+t_1} - b_n) \in B_1, \dots, a_n^{-1}(X_{k+t_l} - b_n) \in B_l\} \tag{6}$$

for any Borel sets B_j in $\operatorname{supp}(G)$, $j = 0, \dots, l$, fixed $l \in \mathbb{N}$ and $k + t_1, \dots, k + t_l \geq 0$, $n \in \mathbb{N}$. But there are slightly different ways to define them; see also Leadbetter et al. [33] and Rootzén [49]. In this way, we find the locations of high level exceedances if $M_k(h)$ is large, and we describe the behavior of the process if it is on a high level by taking the limit as $n \to \infty$ in (6). More on this idea of marked point processes for Gaussian processes can be found under the name *Slepian model* going back to Lindgren in a series of papers (cf. the survey [37]), where $a_n^{-1}(M_k(h) - b_n)$ is replaced by an upcrossing; i. e. an *upcrossing* of level u is a point t_0 for which $X_t < u$ when $t \in (t_0 - \epsilon, t_0)$ and $X_t \geq u$ when $t \in (t_0, t_0 + \epsilon)$ for some $\epsilon > 0$. These ideas have been extended to non-Gaussian models. We refer to the very readable review paper of Leadbetter and Spaniolo [36] on this topic and on the intensity of upcrossings on high levels. However, upcrossings have the disadvantage that there may be infinitely many in a finite time interval, so that the marked point processes converge to a degenerate limit as $n \to \infty$.

3 The GOU-model

Generalized Ornstein-Uhlenbeck processes are applied in various areas as, e. g., in financial and insurance mathematics or mathematical physics; we refer to Carmona et al. [11, 12] and Donati-Martin et al. [16] for an overview of applications. In the financial context, generalized Ornstein-Uhlenbeck processes are used as stochastic volatility models (cf. Barndorff-Nielsen and Shephard [6, 7], Barndorff-Nielsen, Nicolata and Shephard [5]) and as insurance risk models (cf. Paulsen [43], Klüppelberg and Kostadinova [31], Kostadinova [32]).

We assume throughout that $(X_t)_{t\geq 0}$ is a measurable, stationary càdlàg (right-continuous with left limits) version of the GOU-process as in (1) and that $\mathbb{P}(\sup_{0\leq t\leq 1} |X_t| < \infty) = 1$. For two functions, f and g, we write $f(x) \sim g(x)$ as $x \to \infty$, if $\lim_{x\to\infty} f(x)/g(x) = 1$. Two distribution functions, F and H, are called *tail-equivalent* if both have support unbounded to the right and there exists some $c > 0$ such that $\lim_{x\to\infty} \overline{F}(x)/\overline{H}(x) = c$.

3.1 The Ornstein-Uhlenbeck-process

Let $(X_t)_{t\geq 0}$ be a stationary GOU-process as in (1) with $\xi_t = \lambda t$ for some $\lambda > 0$; then the GOU-process reduces to a classical Ornstein-Uhlenbeck (OU) process

$$X_t = \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s}\, d\eta_s + \mathrm{e}^{-\lambda t} X_0, \quad t \geq 0. \tag{7}$$

A stationary version of (7) exists if and only if $\int_{\{|x|>1\}} \log(1+|x|)\, \Pi_\eta(dx) < \infty$, where Π_η is the Lévy measure of $(\eta_t)_{t\geq 0}$. This result goes back to Wolfe [59]; see also the monograph of Sato [51]. The OU-process is a popular volatility model as introduced by Barndorff-Nielsen and Shephard [7]; see also Shephard [53] in this volume.

In this paper, we study only distribution functions F of η_1 belonging to the *class of convolution equivalent distributions* denoted by $\mathcal{S}(\gamma)$ for some $\gamma \geq 0$, i. e., functions which satisfy

(i) $F(x) < 1$ for every $x \in \mathbb{R}$.

(ii) $\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = \exp(-\gamma y)$ for all $y \in \mathbb{R}$ locally uniformly.
(iii) $\lim_{x\to\infty} \overline{F*F}(x)/\overline{F}(x)$ exists and is finite.

The class $\mathcal{S}(0)$ is called the class of *subexponential distributions*. For details and further references see Embrechts et al. [17] and Watanabe [58]. An important family in $\mathcal{S}(\gamma)$ are distribution functions with tail

$$\overline{F}(x) \sim x^{-\beta} \mathrm{e}^{-\gamma x - cx^p}, \quad x \to \infty,$$

where γ, $c \geq 0$, $p < 1$, and if $c = 0$, $\beta > 1$ (cf. Klüppelberg [28], Theorem 2.1, or Pakes [42], Lemma 2.3). There are certain subclasses of generalized inverse Gaussian distributions, normal inverse Gaussian distributions, generalized hyperbolic distributions and CGMY distributions in $\mathcal{S}(\gamma)$, which are used for modelling financial time series (cf. Schoutens [52]).

We investigate two different kinds of OU-models.

(M1) OU-model with $\boldsymbol{\eta_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)}$. *Let $(X_t)_{t\geq 0}$ be a stationary OU-process as in (7). We assume that the distribution of η_1 is in $\mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ for some $\gamma \geq 0$.*

This assumption is sufficient for the existence of a stationary version of an OU-process. The following Proposition (cf. Proposition 2 and Proposition 3 of Fasen, Klüppelberg and Lindner [22]) describes the tail behavior of X_t. The proof of this result is based on the asymptotic equivalence of the tail of the distribution function and the tail of its Lévy measure for every infinitely divisible convolution equivalent distribution in $\mathcal{S}(\gamma)$, and the representation of the Lévy measure of X_t (cf. Wolfe [59], Theorem 2 and the monograph of Sato [51]) as

$$\nu_X(dx) = \frac{\nu(x,\infty)}{x}\, dx \qquad \text{for } x > 0.$$

Proposition 1. *Let $(X_t)_{t\geq 0}$ be as in (M1). Then*

$$\mathbb{P}(X_t > x) = o(\mathbb{P}(\eta_1 > x)) \qquad \text{as } x \to \infty$$

and $X_t \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$.

This result shows that the driving Lévy process and the OU-process are in the same maximum domain of attraction, but they are not tail-equivalent. The precise relationship is given in [22]. In the next model this will be different.

(M2) OU-model with $\boldsymbol{\eta_1 \in \mathcal{R}_{-\alpha}}$. *Let $(X_t)_{t\geq 0}$ be a stationary OU-process as in (7). We assume that η_1 has a regularly varying right tail distribution function, written as $\eta_1 \in \mathcal{R}_{-\alpha}$, i. e.,*

$$\mathbb{P}(\eta_1 > x) = l(x)x^{-\alpha}, \quad x \geq 0, \tag{8}$$

where $l(\cdot)$ is a slowly varying function; for more details on regular variation see Section 4 of Davis and Mikosch [14] in this volume.

Under these assumptions there exists again a stationary version of the OU-process. All distribution functions with regularly varying tails are in $\mathcal{S}(0)$ and belong to MDA(Φ_α), $\alpha > 0$. In particular this means that the distribution of η_1 is also in MDA(Φ_α). The same techniques to compute the tail behavior of X_t as in (M1), where the tail of the Lévy measure and the probability measure are compared, are also used to derive the tail behavior of X_t in (M2) (cf. [22], Proposition 3.2).

Proposition 2. *Let $(X_t)_{t\geq 0}$ be as in (M2). Then*

$$\mathbb{P}(X_t > x) \sim (\alpha\lambda)^{-1}\mathbb{P}(\eta_1 > x) \qquad \textit{as } x \to \infty.$$

This result shows that the tail of X_t is again regularly varying of index $-\alpha$, and hence, also X_t is in MDA(Φ_α).

3.2 The non-Ornstein Uhlenbeck process

The last model we investigate in this paper is again a GOU-model as in (1), but it excludes the classical OU process as in (7).

(M3) Non-OU model. *Let $(X_t)_{t\geq 0}$ be a stationary GOU-model as in (1). Let $(\eta_t)_{t\geq 0}$ be a subordinator, i. e., a Lévy process with nondecreasing sample paths, and if $(\xi_t)_{t\geq 0}$ is of finite variation, then we assume additionally that either the drift of $(\xi_t)_{t\geq 0}$ is non-zero, or that there is no $r > 0$ such that the Lévy measure of $(\xi_t)_{t\geq 0}$ is concentrated on $r\mathbb{Z}$. Furthermore, we suppose*

$$\mathbb{E}(\mathrm{e}^{-\alpha\xi_1}) = 1 \qquad \textit{for some } \alpha > 0. \tag{9}$$

We assume, finally, the moment conditions

$$\mathbb{E}|\eta_1|^{q\max\{1,d\}} < \infty \qquad \textit{and} \qquad \mathbb{E}(\mathrm{e}^{-\max\{1,d\}p\xi_1}) < \infty \tag{10}$$

for some $d > \alpha$ and $p, q > 0$ with $1/p + 1/q = 1$.

In the classical OU-model condition (9) is not satisfied. As in many studies like the GARCH-model, Lindner and Maller [39] apply the results of Kesten [27] and Goldie [23] for stochastic recurrence equations to deduce the stationarity and the heavy-tailed behavior of model (M3). In this context the stochastic recurrence equation has the form

$$X_{t+1} = A_{t+1}X_t + B_{t+1}\,, \quad t \geq 0,$$

where

$$A_t = \mathrm{e}^{-(\xi_t - \xi_{t-1})} \qquad \text{and} \qquad B_t = \mathrm{e}^{-\xi_t}\int_{t-1}^{t} \mathrm{e}^{\xi_{s-}}\, d\eta_s\,, \quad t \geq 0.$$

The result for the tail behavior as presented in Lindner and Maller [39], Theorem 4.5, is the following.

Proposition 3. *Let $(X_t)_{t\geq 0}$ be as in (M3). Then for some $C>0$,*

$$\mathbb{P}(X_t > x) \sim Cx^{-\alpha} \qquad as\ x\to\infty.$$

Typical examples which satisfy (M3) are the volatility process of the continuous-time GARCH$(1,1)$ (COGARCH$(1,1)$) model introduced by Klüppelberg et al. [29, 30] and the volatility process of Nelson's diffusion limit of a GARCH$(1,1)$-model [41].

Example 1 (COGARCH$(1,1)$ process). The right-continuous version of the volatility process of the COGARCH$(1,1)$ process is defined as GOU-process as in (1), where

$$\xi_t = ct - \sum_{0<s\leq t} \log(1+\beta \mathrm{e}^c (\Delta L_s)^2) \quad \text{and} \quad \eta_t = \lambda t \quad \text{for } t\geq 0,$$

$\lambda, c>0$, $\beta\geq 0$ are constants and $(L_t)_{t\geq 0}$ is a Lévy process (cf. Lindner [38] of this volume). The assumptions in (M3) are satisfied if and only if

$$-\alpha c + \int ((1+\beta \mathrm{e}^c y^2)^\alpha - 1)\, \Pi_L(dy) \quad \text{and} \quad \mathbb{E}|L_1|^{2\widetilde{d}} < \infty \text{ for some } \widetilde{d} > \alpha,$$

where Π_L denotes the Lévy measure of L.

Example 2 (Nelson's diffusion model). The Nelson's diffusion model, originally defined as solution of the stochastic differential equation

$$dX_t = \lambda(a - X_t)\, dt + \sigma X_t\, dB_t,$$

where $a, \lambda, \sigma > 0$ and $(B_t)_{t\geq 0}$ is a Brownian motion, is by Theorem 52 on p. 328 in Protter [44] a GOU-process with

$$\xi_t = -\sigma B_t + \left(\frac{1}{2}\sigma^2 + \lambda\right) t \quad \text{and} \quad \eta_t = \lambda a t \quad \text{for } t \geq 0.$$

Since

$$\mathbb{E}(\mathrm{e}^{-u\xi_1}) = \exp\left(\frac{1}{2}\sigma^2 u^2 - \left(\frac{1}{2}\sigma^2 + \lambda\right) u\right)$$

we have $\mathbb{E}(\mathrm{e}^{-\alpha\xi_1}) = 1$ for $\alpha = 1 + 2\lambda/\sigma^2$.

For more details on these examples we refer to Lindner [38] in this volume.

3.3 Comparison of the models

At first glance, the results presented in Propositions 1-3 are surprising. We start with a comparison of models (M1) and (M3) driven by the same Lévy process $(\eta_t)_{t\geq 0}$. In model (M1), the tail of η_1 is heavier than the tail of X_t. In contrast, in model (M3) the existence of the $q\alpha$ moment of η_1 by (10) results in the tail of η_1 being at most $-q\alpha$ regularly varying and hence, lighter tailed than X_t. Taking now, in models (M1) and (M3), the same Lévy process $(\eta_t)_{t\geq 0}$, it ensures that X_t has a different tail behavior in each model. In (M1) it is lighter tailed and in (M3) it is heavier tailed than η_1.

Next we compare the OU-models (M1) and (M2), which have the same Lévy process $(\xi_t)_{t\geq 0}$. In model (M1) we find that X_t is lighter tailed than η_1; in (M2) we find the tail-equivalence of the distribution function of X_t and η_1.

We conclude that both Lévy processes $(\xi_t)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ are contributing factors to the tail behavior of X_t.

4 Tail behavior of the sample maximum

It is the tail of the distribution of the sample maximum $M(h)$ as in (2) for some $h > 0$, rather than X_t itself, that determines the limit distribution of the normalized process $M(T)$ as $T \to \infty$. The tail behavior of $M(h)$ is affected differently in models (M1)–(M3).

For (M1) the derivation of the asymptotic behavior of the tail of $M(h)$ is much more involved than for (M2) and given in Fasen [20]. For model (M2) the following asymptotic behavior holds:

$$\begin{aligned}
&\mathbb{P}(M(h) > x) \\
&= \mathbb{P}\left(\sup_{0\leq t\leq h}\left\{\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\,d\eta_s + \mathrm{e}^{-\lambda t}X_0\right\} > x\right) \\
&\sim \mathbb{P}\left(\int_0^h \sup_{0\leq t\leq h}\left\{\mathbf{1}_{[0,t)}\,\mathrm{e}^{-\lambda(t-s)}\right\}\,d\eta_s > x\right) + \mathbb{P}\left(\sup_{0\leq t\leq h}\left\{\mathrm{e}^{-\lambda t}\right\}X_0 > x\right) \\
&= \mathbb{P}(\eta_h > x) + \mathbb{P}(X_0 > x) \qquad \text{as } x \to \infty.
\end{aligned}$$

The mathematical proof (see Fasen [18], Proposition 3.2) is based on results of Rosinski and Samorodnitsky [50] investigating the tail behavior of random variables in $\mathcal{S}(0)$, which are functionals acting on infinitely divisible processes. In model (M3) only the last summand of representation (7) influences the tail behavior, since

$$\mathbb{E}\left|\sup_{0\leq t\leq h}\mathrm{e}^{-\xi_t}\int_0^t \mathrm{e}^{\xi_{s-}}\,d\eta_s\right|^d < \infty$$

(cf. Fasen [19], Remark 2.3 *(iii)*). Hence, Klüppelberg et al. [30], Lemma 2, and Breiman [9], Proposition 3, give as $x \to \infty$,

$$\begin{aligned}\mathbb{P}(M(h) > x) &= \mathbb{P}\left(\sup_{0 \le t \le h} \left\{ \mathrm{e}^{-\xi_t} \int_0^t \mathrm{e}^{\xi_{s-}}\, d\eta_s + \mathrm{e}^{-\xi_t} X_0 \right\} > x\right) \\ &\sim \mathbb{P}\left(\sup_{0 \le t \le h} \left\{ \mathrm{e}^{-\xi_t} \right\} X_0 > x\right) \\ &\sim \mathbb{E}\left(\sup_{0 \le s \le h} \mathrm{e}^{-\alpha \xi_s}\right) \mathbb{P}(X_0 > x).\end{aligned}$$

We summarize the tail behavior of $M(h)$ for the different models.

Proposition 4.

(a) OU-model with $\eta_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ as in (M1):

$$\mathbb{P}(M(h) > x) \sim h \frac{\mathbb{E}(\mathrm{e}^{\gamma X_0})}{\mathbb{E}(\mathrm{e}^{\gamma \eta_1})} \mathbb{P}(\eta_1 > x) \qquad \text{as } x \to \infty.$$

(b) OU-model with $\eta_1 \in \mathcal{R}_{-\alpha}$ as in (M2):

$$\mathbb{P}(M(h) > x) \sim \left(h + \frac{1}{\alpha \lambda}\right) \mathbb{P}(\eta_1 > x) \qquad \text{as } x \to \infty.$$

(c) Non-OU model as in (M3):

$$\mathbb{P}(M(h) > x) \sim \mathbb{E}\left(\sup_{0 \le s \le h} \mathrm{e}^{-\alpha \xi_s}\right) \mathbb{P}(X_t > x) \qquad \text{as } x \to \infty.$$

In all three models $M(h)$ is in the same maximum domain of attraction as X_t.

5 Running sample maxima and extremal index function

The classic problem arising from studying the extremal behavior of stochastic processes is the asymptotic behavior of the sample maxima $M(T)$ as $T \to \infty$. One of the first researchers turning from the extremal behavior of Gaussian processes to stable processes was Rootzén [49]. His results already include the asymptotic behavior of sample maxima of OU-processes driven by stable Lévy motions and their marked point process behavior, where the definition of the marked point process is slightly different to Section 2.3. Generalizations of his results to regularly varying processes including model (M2) are presented in Fasen [18]. Model (M1) was investigated in Fasen et al. [22], but more details can be found in [20]. A proof of the asymptotic behavior in model (M3) is given in [19]. We denote by $x^+ = \max\{0, x\}$ for $x \in \mathbb{R}$.

Proposition 5.

(a) *OU-model with* $\eta_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ *as in (M1):*
Let $a_T > 0, b_T \in \mathbb{R}$ *be sequences of constants such that*

$$\lim_{T\to\infty} T\mathbb{P}(M(1) > a_T x + b_T) = \exp(-x) \quad \text{for } x \in \mathbb{R}.$$

Then

$$\lim_{T\to\infty} \mathbb{P}(a_T^{-1}(M(T) - b_T) \leq x) = \exp(-\mathrm{e}^{-x}) \quad \text{for } x \in \mathbb{R},$$

and

$$\theta(h) = 1 \quad \text{for } h > 0.$$

(b) *OU-model with* $\eta_1 \in \mathcal{R}_{-\alpha}$ *as in (M2):*
Let $a_T > 0$ *be a sequence of constants such that*

$$\lim_{T\to\infty} T\mathbb{P}(M(1) > a_T x) = x^{-\alpha} \quad \text{for } x > 0.$$

Then

$$\lim_{T\to\infty} \mathbb{P}(a_T^{-1} M(T) \leq x) = \exp\left(-\frac{\alpha\lambda}{\alpha\lambda + 1} x^{-\alpha}\right) \quad \text{for } x > 0,$$

and

$$\theta(h) = \frac{h\alpha\lambda}{h\alpha\lambda + 1} \quad \text{for } h > 0.$$

(c) *Non-OU model as in (M3):*
Let $a_T > 0$ *be a sequence of constants such that*

$$\lim_{T\to\infty} T\mathbb{P}(M(1) > a_T x) = x^{-\alpha} \quad \text{for } x > 0.$$

Then

$$\lim_{T\to\infty} \mathbb{P}(a_T^{-1} M(T) \leq x) = \exp\left(-\frac{\mathbb{E}\big(\sup_{0\leq s\leq 1} \mathrm{e}^{-\alpha\xi_s} - \sup_{s\geq 1} \mathrm{e}^{-\alpha\xi_s}\big)^+}{\mathbb{E}\left(\sup_{0\leq s\leq 1} \mathrm{e}^{-\alpha\xi_s}\right)} x^{-\alpha}\right)$$

for $x > 0$, *and*

$$\theta(h) = h \frac{\mathbb{E}\left(\sup_{0\leq s\leq 1} \mathrm{e}^{-\alpha\xi_s} - \sup_{s\geq 1} \mathrm{e}^{-\alpha\xi_s}\right)^+}{\mathbb{E}\left(\sup_{0\leq s\leq h} \mathrm{e}^{-\alpha\xi_s}\right)} \quad \text{for } h > 0.$$

These results reflect the fact that model (M1) has no clusters of extremes on high levels, whereas both regularly varying models (M2) and (M3) have them. In particular, models (M2) and (M3) do not satisfy the anti-cluster condition C'.

For the behavior of the marked point processes of model (M1) and (M2) we refer to [22] and of (M3) to [19].

6 Conclusion

All continuous-time models $(X_t)_{t\geq 0}$ presented in Section 3 are heavy-tailed models, which model stationary continuous-time processes with jumps. The OU-model in (M1) has no clusters of extremes. This property has been confirmed so far in all investigated OU-models in MDA(Λ), including Gaussian OU-processes (cf. Albin [1]). However, the regularly varying models (M2) and (M3) have extremal clusters on high levels.

One generalization of the OU-process is the supOU process introduced by Barndorff-Nielsen [4], where the driving Lévy process is replaced by an infinitely divisible random measure. Modelling long range dependence, in the sense that the autocovariance function decreases very slowly, is a special feature of this class of processes. All models presented in this paper have exponentially decreasing covariance functions and do not allow long range dependence. SupOU processes have an extremal behavior similar to that of OU-models, see Fasen and Klüppelberg [21]. This means that only regularly varying supOU processes have extremal clusters.

Another extension of OU-processes are continuous-time ARMA (CARMA) processes, as presented in Brockwell [10] of this volume. In such models the exponentially decreasing kernel function of an OU-process is replaced by a more general kernel function. The results of Fasen [20, 18] show that a CARMA process and an OU-process, driven by the same Lévy process, have similar extremal behavior. The regularly varying CARMA processes show extremal clusters. In the case in which the driving Lévy process of the CARMA process has marginals in $\mathcal{S}(\gamma)\cap\text{MDA}(\Lambda)$, and the kernel functions have only one maximum, there are again no extremal clusters. If they have more than one maximum, then they may also model extremal clusters.

Acknowledgement

Parts of the paper were written while the author was visiting the Department of Operations Research and Information Engineering at Cornell University. She takes pleasure in thanking the colleagues there for their hospitality. Financial support by the Deutsche Forschungsgemeinschaft through a research grant is gratefully acknowledged.

References

1. Albin, J. M. P. On extremes of infinitely divisible Ornstein-Uhlenbeck processes. Preprint, available at `http://www.math.chalmers.se/~palbin/`.
2. Albin, J. M. P. (1987). *On Extremal Theory for Nondifferentiable Stationary Processes*. Ph.D. thesis, Lund University.
3. Albin, J. M. P. (1990). On extremal theory for stationary processes. *Ann. Probab.* **18**, 92–128.

4. Barndorff-Nielsen, O. E. (2001). Superposition of Ornstein–Uhlenbeck type processes. *Theory Probab. Appl.* **45**, 175–194.
5. Barndorff-Nielsen, O. E., Nicolata, E., and Shephard, N. (2002). Some recent developments in stochastic volatility modelling. *Quantitative Finance* **2**, 11–23.
6. Barndorff-Nielsen, O. E. and Shephard, N. (2001). Modelling by Lévy processes for financial econometrics. In: O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick (Eds.), *Lévy Processes: Theory and Applications*, pp. 283–318. Birkhäuser, Boston.
7. Barndorff-Nielsen, O. E. and Shephard, N. (2001). Non-Gaussian OU based models and some of their uses in financial economics. *J. Roy. Statist. Soc. Ser. B* **63**, 167–241.
8. Berman, S. M. (1992). *Sojourns and Extremes of Stochastic Processes*. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove.
9. Breiman, L. (1965). On some limit theorems similar to the arc-sine law. *Theory Probab. Appl.* **10**, 323–331.
10. Brockwell, P. (2007). Lévy driven continuous-time ARMA processes. This volume, pp. ???
11. Carmona, P., Petit, F., and Yor, M. (1997). On the distribution and asymptotic results for exponential Lévy processes. In: M. Yor (Ed.), *Exponential Functionals and Principal Values Related to Brownian Motion*, pp. 73–130. Biblioteca de le Revista Matemtica Iberoamericana.
12. Carmona, P., Petit, F., and Yor, M. (2001). Exponential functionals of Lévy processes. In: O. E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (Eds.), *Lévy Processes, Theory and Applications*, pp. 41–55. Birkhäuser, Boston.
13. Daley, D. J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes. Vol I: Elementary Theory and Methods*. 2nd edn. Springer, New York.
14. Davis, R. and Miksoch, T. (2007). Probabilistic properties of stochastic volatility models. This volume, pp. ???
15. De Haan, L. and Ferreira, A. (2006). *Extreme Value Theory: An Introduction*. Springer.
16. Donati-Martin, C., Ghomrasni, R., and Yor, M. (2001). On certain Markov processes attached to exponential functionals of Brownian motion; application to Asian options. *Rev. Mat. Iberoamericana* **17** (1), 179–193.
17. Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.
18. Fasen, V. (2005). Extremes of regularly varying mixed moving average processes. *Adv. in Appl. Probab.* **37**, 993–1014.
19. Fasen, V. (2008). Asymptotic results for sample ACF and extremes of integrated generalized Ornstein-Uhlenbeck processes. Preprint, available at `http://www.ma.tum.de/stat/`.
20. Fasen, V. (2008). Extremes of mixed MA processes in the class of convolution equivalent distributions. Preprint, available at `http://www.ma.tum.de/stat/`.
21. Fasen, V. and Klüppelberg, C. (2007). Extremes of supOU processes. In: F. E. Benth, G. Di Nunno, T. Lindstrom, B. Oksendal, and T. Zhang (Eds.), *Stochastic Analysis and Applications: The Abel Symposium 2005*, pp. 340–359. Springer.

22. Fasen, V., Klüppelberg, C., and Lindner, A. (2006). Extremal behavior of stochastic volatility models. In: A. Shiryaev, M. d. R. Grossinho, P. Oliviera, and M. Esquivel (Eds.), *Stochastic Finance*, pp. 107–155. Springer, New York.
23. Goldie, C. M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**, 126–166.
24. Kac, M. (1943). On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.* **49**, 314–320.
25. Kac, M. and Slepian, D. (1959). Large excursions of Gaussian processes. *Ann. Math. Statist.* **30**, 1215–1228.
26. Kallenberg, O. (1997). *Foundations of Modern Probability*. Springer, New York.
27. Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207–248.
28. Klüppelberg, C. (1989). Subexponential distributions and characterizations of related classes. *Probab. Theory Relat. Fields* **82**, 259–269.
29. Klüppelberg, C., Lindner, A., and Maller, R. (2004). A continuous time GARCH process driven by a Lévy process: stationarity and second order behaviour. *J. Appl. Probab.* **41**, 601–622.
30. Klüppelberg, C., Lindner, A., and Maller, R. (2006). Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models. In: Y. Kabanov, R. Lipster, and J. Stoyanov (Eds.), *From Stochastic Calculus to Mathematical Finance. The Shiryaev Festschrift*, pp. 393–419. Springer, Berlin.
31. Klüppelberg, K. and Kostadinova, T. (2008). Integrated insurance risk models with exponential Lévy investment. *Insurance Math. Econ.* To appear.
32. Kostadinova, T. (2007). Optimal investment for insurers, when stock prices follows an exponential Lévy process. *Insurance Math. Econ.* **41**, 250–263.
33. Leadbetter, M. R., Lindgren, G., and Rootzén, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York.
34. Leadbetter, M. R. and Rootzén, H. (1982). Extreme value theory for continuous parameter stationary processes. *Z. Wahrsch. verw. Gebiete* **60**, 1–20.
35. Leadbetter, M. R. and Rootzén, H. (1988). Extremal theory for stochastic processes. *Ann. Probab.* **16**, 431–478.
36. Leadbetter, M. R. and Spaniolo, G. V. (2004). Reflections on Rice's formulae for level crossings–history, extensions and use. *Aust. N. J. Stat.* **46**, 173–180.
37. Lindgren, G. (1984). Use and structure of Slepian model processes for prediction and detection in crossing and extreme value theory. In: J. T. d. Oliveira (Ed.), *Statistical Extremes and Applications*, pp. 261–284. Reidel, Dordrecht.
38. Lindner, A. (2007). Continuous time approximations to GARCH and stochastic volatility models. This volume, pp. ???
39. Lindner, A. and Maller, R. (2005). Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes. *Stoch. Proc. Appl.* **115**, 1701–1722.
40. Maller, R., Müller, G., and Szimayer, A. (2007). Ornstein-Uhlenbeck processes and extensions. This volume, pp. ???
41. Nelson, D. B. (1990). ARCH models as diffusion approximations. *J. Econometrics* **45**, 7–38.
42. Pakes, A. G. (2004). Convolution equivalence and infinite divisibility. *J. Appl. Probab.* **41**, 407–424.

43. Paulsen, J. (1993). Risk theory in a stochastic economic environment. *Stoch. Proc. Appl.* **46**, 327–361.
44. Protter, P. E. (2004). *Stochastic Integration and Differential Equations*. 2nd edn. Springer, Berlin.
45. Resnick, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.
46. Rice, S. O. (1939). The distribution of the maximum of a random curve. *Amer. J. Math.* **61**, 409–416.
47. Rice, S. O. (1944). Mathematical analysis of random noise. *Bell. System Techn. J.* **23**, 282–332.
48. Rice, S. O. (1945). Mathematical analysis of random noise. *Bell. System Techn. J.* **24**, 46–156.
49. Rootzén, H. (1978). Extremes of moving averages of stable processes. *Ann. Probab.* **6**, 847–869.
50. Rosiński, J. and Samorodnitsky, G. (1993). Distributions of subadditive functionals of sample paths of infinitely divisible processes. *Ann. Probab.* **21**, 996–1014.
51. Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.
52. Schoutens, W. (2003). *Lévy Processes in Finance*. Wiley, Chichester.
53. Shephard, N. (2007). Stochastic volatility. This volume, pp. ???
54. Slepian, D. (1961). First passage time for particular Gaussian processes. *Ann. Math . Stat.* **32**, 610–612.
55. Slepian, D. (1962). The one-sided barrier problem for Gaussian noise. *Bell System Techn. J.* **41**, 463–501.
56. Volkonskii, V. A. and Rozanov, Y. A. (1959). Some limit theorems for random functions, I. *Theory Probab. Appl.* **4**, 178–197.
57. Volkonskii, V. A. and Rozanov, Y. A. (1961). Some limit theorems for random functions, II. *Theory Probab. Appl.* **6**, 186–198.
58. Watanabe, T. (2008). Convolution equivalence and distributions of random sums. *Probab. Theory Relat. Fields* To appear.
59. Wolfe, S. J. (1982). On a continuous analogue of the stochastic difference equation $x_n = \rho x_{n-1} + b_n$. *Stoch. Proc. Appl.* **12**, 301–312.