Operational VAR: meaningful means

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Abstract

Making the assumption that the distribution of operational-loss severity has finite mean, Klaus Böcker and Jacob Sprittulla suggest a refined version of the analytical operational VAR theorem derived in Böcker and Klüppelberg (2005), which significantly reduces the approximation error to operational VAR.

1 Introduction and recapitulation

Our approach to quantify operational risk belongs to the advanced measurement approaches (AMA), which allow banks to use an internal model for determining operational risk capital charges. More precisely, we work within the framework of Böcker and Klüppelberg (2005) and adopt their definition of a standard loss distribution approach (LDA), in which aggregate operational loss *S(t)* up to time *t* is described by the stochastic process

(1) $$S(t) = \sum_{i=1}^{N(t)} X_i , \quad t \geq 0$$

where $(X_i)_{i \in N}$ are positive independent and identically distributed (iid) random variables describing the magnitude of each loss event, i.e. the loss severity, and $(N(t))_{t \geq 0}$ is a counting process counting the number of loss events in the time interval *[0, t]*, also referred to as the frequency process. Furthermore, we define operational VAR at confidence level κ with $0 < \kappa < 1$ as the κ-quantile of the aggregate loss distribution $G(\cdot) = P(S \leq \cdot)$,

$$VAR(\kappa) = G^{\leftarrow}(\kappa)$$

where $G^{\leftarrow}(\kappa) = \inf\{x \in R : G(x) \geq \kappa\}$ is the generalized inverse of G.

In general, $G(\cdot)$ and thus also operational VAR cannot be analytically calculated. In Böcker and Klüppelberg (2005), however, it was shown that if the distribution function $F(\cdot)=P(X_i \le \cdot)$ of the loss severity is subexponentially distributed (and therefore belongs to the class of heavy-tail distributions), operational VAR is asymptotically given by

$$(2) \qquad VAR(\kappa)=F^{\leftarrow}\left(1-\frac{1-\kappa}{\lambda}(1+o(1))\right), \qquad \kappa \to 1$$

where $\lambda = EN(t)$ denotes frequency expectation. This result is based on the fact that for subexponential severity distributions (under weak regularity conditions) the following asymptotic relation between the tail distributions $\overline{G}(\cdot)=1-G(\cdot)$ of the aggregate loss and $\overline{F}(\cdot)=1-F(\cdot)$ of the loss severity holds:

$$(3) \qquad \overline{G}(x) \sim \lambda\, \overline{F}(x), \qquad x \to \infty.$$

Here, the symbol ~ means that the quotient of the right-hand and left-hand side tends to 1 as x tends to infinity, i.e. $\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x)=\lambda$.

Equation (3) has an interesting interpretation, which allows us to better understand the meaning of the operational VAR approximation (2). Recall that the distribution tail of aggregate loss can be written as

$$\overline{G}(x)=\sum_{n=0}^{\infty} P(S>x \mid N=n)\,P(N=n)=\sum_{n=0}^{\infty} \overline{F^{n*}}(x)\,P(N=n)$$

where $\overline{F^{n*}}(\cdot)=1-F^{n*}(\cdot)$ and F^{n*} denotes the n-fold convolution of F. Using the definition of subexponentiality (see Embrechts et al. (1997), Definition 1.3.3), $\overline{F^{n*}}(x) \sim n\overline{F}(x)$ for $x \to \infty$, we obtain

$$(4) \qquad P(S>x \mid N=n)=\overline{F^{n*}}(x) \sim n\,\overline{F}(x) \sim n\,\overline{F}(x)\,F^{n-1}(x) =: n\,p(1-p)^{n-1}, \qquad x \to \infty,$$

which can be interpreted as follows: Consider a Bernoulli random variable for which 'success', defined as a very large loss that impacts significantly the VAR figure, occurs only with a very small probability $p=P(X_i>x)=\overline{F}(x)$, and 'failure', defined as a small or negligible loss, occurs with probability

$1-p=F(x)$. From (4) we infer that for large x conditional aggregate loss $P(S>x|N=n)$ can be approximated by a binomial distribution where the number of successes that occur in n trials equals 1, which is just the 'single-loss approximation' of operational VAR.

2 Refinement by mean correction

Henceforth we assume that the severity distribution F has finite expectation $\mu=E(X_i)$ which assures that the aggregate operational loss has finite expectation given by $ES=\lambda\mu$. Note, however, that this assumption is not necessary for the definition of operational VAR, see e.g. Chavez-Demoulin et al. (2006).

Approximation (2) has to be used carefully since its estimation error can be quite large, confer e.g. Mignola and Ugoccioni (2006) in the context of operational risk. Moreover, a consequence of the single-loss interpretation is that approximation (2) usually underestimates operational VAR because in reality *all* (and not only one) loss events $X_i, i=1,...,n$ contribute to aggregate loss S and therefore to operational VAR. This is depicted in Figure 1, which also shows that the accuracy of the single-loss approximation is poor if expected loss is not negligible compared to operational VAR. Hence, especially for large frequency expectations λ and when severities are not extremely heavy-tailed, equation (2) systematically underestimates operational VAR.

These findings suggest an adjustment of the single-loss approximation by taking expected loss into account. This approach can also be motivated by the so-called large-deviation theory where one investigates asymptotic behaviour of random sums like (1) when both x and n are varying together. Then, uniform convergence can be achieved when S is replaced by the centred random variable $\widehat{S}=S-\lambda\,\mu$, see for example Proposition 8.6.4. of Embrechts et al. (1997) or Klüppelberg and Mikosch (1997).

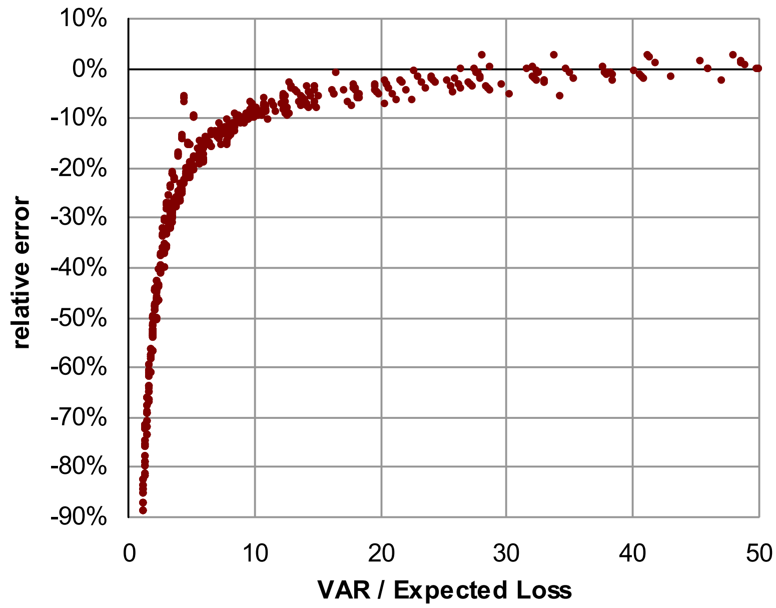

Figure 1: Relative error of the single-loss approximation (2) for the lognormal-Poisson model compared to a Monte Carlo simulation using 2 million trials. We have calculated 800 values with varying parameters within the ranges of $5 \le \lambda \le 1000$, $0.95 \le \kappa \le 0.9995$, $2 \le \mu \le 8$, and $1.5 \le \sigma \le 3$. The resulting relative error is plotted against the operational VAR-expected loss ratio.

Now, one property of subexponential distributions F is that $\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1,\ y \in R$ (see Lemma 1.3.5 of Embrechts et al. (1997)), which implies the following relationships:

$$\overline{G}(x) \sim \overline{G}(x+\lambda\mu), \qquad x \to \infty,$$
$$\overline{F}(x) \sim \overline{F}(x+\mu), \qquad x \to \infty.$$

Together with (3) we then obtain

$$\overline{G}(x+\lambda\mu) \sim \lambda\, \overline{F}(x+\mu), \qquad x \to \infty,$$

which finally yields the following refined approximation of operational VAR:

(5) $$VAR(\kappa) = F^{\leftarrow}\left(1 - \frac{1-\kappa}{\lambda}(1+o(1))\right) + (\lambda - 1)\mu\,, \qquad \kappa \to 1\,.$$

In the light of equation (5) and in contrast to the single-loss interpretation, operational VAR can be thought of as the result of two different components: first, exactly one single extreme loss at very high confidence level and, second, $(n-1)$ expected losses of expected loss size, which we refer to as *mean correction*.

Note that the mean correction term $(\lambda-1)\mu$ of equation (5) does not depend on the confidence level κ, it is just a constant. Hence, for $\kappa \to 1$ approximation (5) asymptotically equals (2). Needless to say, any other real constant C instead of $(\lambda-1)\mu$ also conserves the limit behaviour of (5), and we do not claim that our choice $C=(\lambda-1)\mu$ yields ultimate accuracy of operational VAR approximation. Instead, our choice is mainly motivated by empirical findings about the approximation error of equation (2) as well as by the fact that, as mentioned above, stronger convergence properties hold when using centred random variables. The latter is clearly demonstrated by Figures 2 - 4, which compare the mean relative errors of operational VAR for the single-loss approximation with its mean-corrected version in the case of the lognormal-Poisson model as well as for the Pareto-Poisson model.

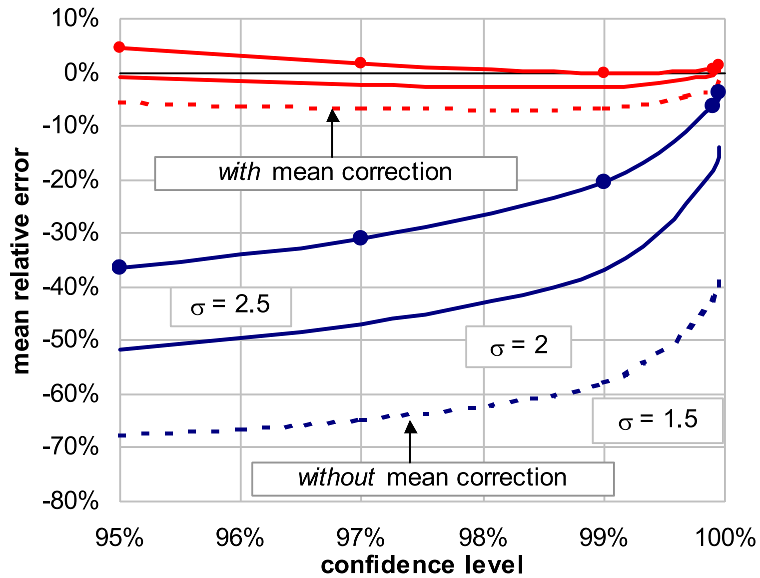

Figure 2: Comparison of the accuracy of the single-loss approximation for the lognormal-Poisson model without mean correction (blue lines) and with mean correction (red lines) for different values of σ of the lognormal distribution. Relative errors are calculated as in Figure 1. For every curve the mean of the relative errors obtained for different parameters λ and μ is plotted.

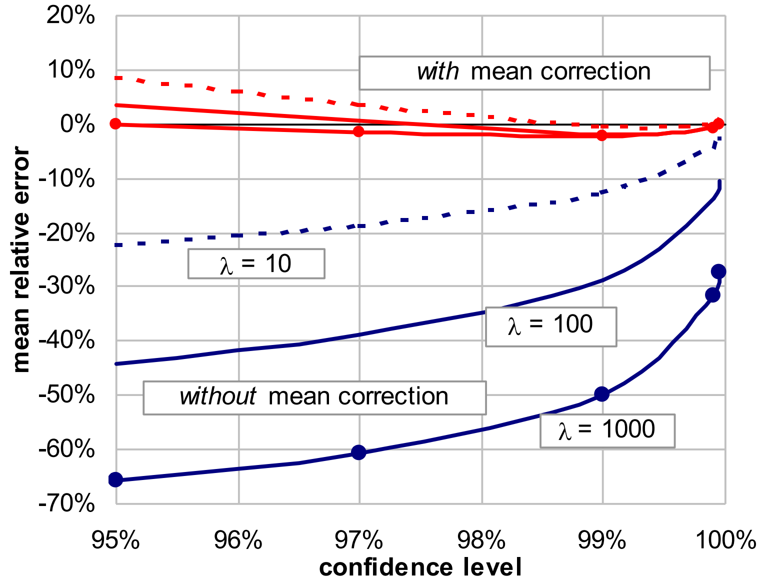

Figure 3: Comparison of the accuracy of the single-loss approximation for the lognormal-Poisson model without mean correction (blue lines) and with mean correction (red lines) for different values of intensities λ of the Poisson distribution. Relative errors are calculated as in Figure 1.

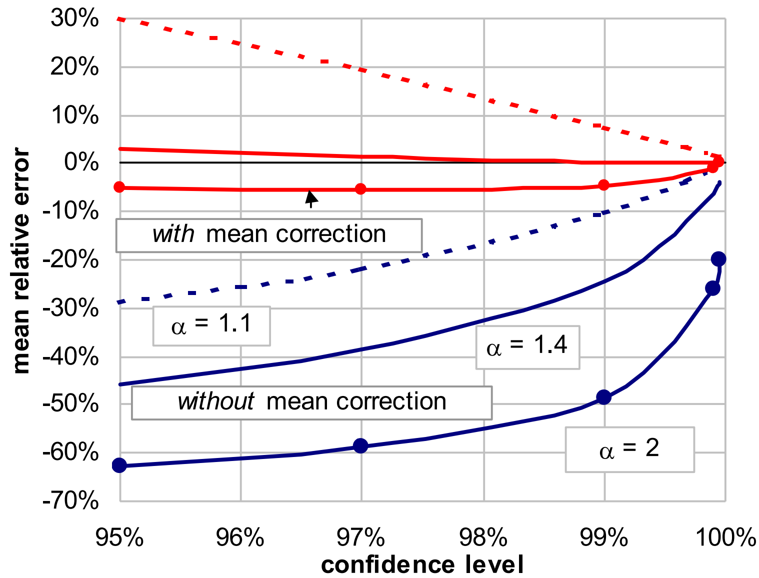

Figure 4: Comparison of the accuracy of the single-loss approximation for the Pareto-Poisson model without mean correction (blue lines) and with mean correction (red lines) for different values of the parameter $\alpha > 1$ of the Pareto distribution. Relative errors are calculated as in Figure 1. The parameter θ of the Pareto distribution varies within the range of $1 \leq \theta \leq 15$

These illustrations convincingly show that mean correction yields a significant improvement of the standard single-loss approximation, especially for lower confidence levels κ and higher values of expected loss (i.e. higher intensity parameter λ in the case of the Poisson frequency model). For instance, for the lognormal-Poisson model with a confidence level of $\kappa = 99.9$ % the maximum relative error observed was below 5 % for the refined approximation (5) compared to 72 % for the standard approximation (2). Similar simulations with other subexponential severity distributions like Weibull (with a shape parameter less than 1) and log-logistic confirm the results of the lognormal or Pareto severity model shown in figures 2-4. Keeping in mind that operational risk measurement is usually concerned with many other sources of uncertainty such as parameter estimation and data collection problems, our operational VAR approximation works remarkable well and might be a viable alternative to simulation methods, in particular in situations where complex and time-intensive calculations have to be avoided or where the sensitivity of operational VAR with respect to different model parameters shall be analysed.

3 About the Authors / Disclaimer

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