Modelling and Measuring Business Risk

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1 Introduction

As can be seen e.g. by the cutting edge section of RISK Magazine, research papers mainly focus on market risk, credit risk, and—with a little less attention—operational risk. Although these risk types are very important for financial institutions, the true landscape of risk is much more complex and far from being well explored and understood. There is a variety of "other" risks looming on the horizon, which seriously threaten a bank's profitability or which can disrupt or even destroy its business completely. Moreover, such risks often reflect an under-researched area of financial risk management, and established and ready-to-use measurement techniques are rarely available. Also banking supervisors demand that more attention is being paid to such "hard-to-measure" risks as the following Pillar-2 passages of the new international regulatory framework of Basel II [1] show:

- 731: "...Sound capital assessment include...policies and procedures designed to ensure that the bank identifies, *measures*, and reports *all material risks*."
- 742: "Although the Committee recognises that other risks [...] are not easily measurable, it expects industry to further develop techniques for managing *all aspects* of these risks."

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This view has also been confirmed by different European supervisors, confer e.g. The Committee of European Banking Supervisors [3].

Capturing all material risks of a financial institution requires a broad risk self-assessment to find out which are the most relevant risk drivers for the bank. One of the most obvious variables to be monitored in this context are earning themselves. However, none of the Pillar 1 risks take earnings volatility *directly* as a primary driver into account, instead, they usually focus on aspects of the business environment that only indirectly affect the institution' earnings by virtue of e.g. failed processes, credit defaults, drop in share prices, or interest rate changes.

For an all-encompassing risk assessment it is therefore necessary to introduce an additional kind of risk that is directly linked to the uncertainty of specific earnings components not yet associated to other risk types. Usually, such an earnings-related potential loss, which can also threaten a bank's market capitalisation, is referred to as *business risk*.

Evidence for the growing importance of business risk was recently also given in a survey undertaken by the IFRI/CRO Forum about Economic Capital practices in leading financial institutions [6] where 85 % of the participants stated to include business risk in their aggregated economic capital assessment. Yet surprisingly, there is no common agreement on a precise definition, specific risk drivers, and measurement methodology for business risk, even though its absolute size in term of economic capital is comparable to that of operational risk, see again [6]. With this regard, we also performed a benchmark exercise on a sample of 15 international banks by analysing their risk management practise as disclosed in their official financial annual reports from 2004 to 2006. Again, we found that an increasing number of institutions are trying to quantify business risk in some way, even if different definitions and assumptions are adopted. Broadly speaking, approaches for business risk quantification can be divided into two main categories; top-down and bottom-up. Top-down techniques are linked to the general trend of the business environment and benchmark analysis based on external data is used for approximating business risk. In contrast to that, bottom-up approaches try to explicitly determine the volatility of particular, bank-internal economic time series (such as volumes, earnings, revenues, and expenses) at a more granular level, which is then transformed into a measure of business risk.

Here we we propose a bottom-up approach for modelling and measuring business risk where the dynamic of the underlying earnings is described in a continuous-time model. The remainder of this chapter is structured as follows. After some preliminaries such as formulating the *discounted-cash-flow* method in continuous time, we introduce a first stochastic model for quantifying business risk in section 2.2 where we also briefly point out how such model could be implemented in practice. The results obtained are then used in section 2.3 to investigate the relation between the *Earnings-at-Risk* (EAR) measure and the so-called *Capital-at-Risk* (CAR) measure in greater detail. Finally, in section 3 we propose a possible extension of the simple business risk model.

2 Modelling Business CAR: a Simple Approach

2.1 Setting the Scene

Overlap with other risk types. Of course, the concept of "revenues" and "expenses" as we used so far is too general for measuring business risk. In particular, in order to avoid double counting and risk overlap, revenue and cost components that enter the business risk model must not directly or indirectly be used for the quantification of other risk types. To give an example, as potentially relevant revenues one may consider customer related provisions and net interest rate income, while on the cost side administrative expenses and depreciations may be included into business risk quantification. On the other hand, earnings related to trading activities would clearly cause an overlap with market risk, and should therefore not be included. Something similar holds for loan loss provisions, when they are captured within the bank's credit portfolio model.

However, the question which revenue and cost components are really relevant for modelling a particular firm's business risk, and which parts have to be excluded, is not an easy one. The answer crucially depends on the firm's definition of others risk types and its economic capital framework in general, and therefore setting up a business risk model should always be an integral part of the bank's overall risk-defining and assessment process. Moreover, one has to be aware that the availability and granularity of revenue and cost data may also depend on the firm's accounting rules, controlling standards, and IT infrastructure. As a consequence, the quality of data may differ from one legal entity to the other, and in order to achieve reliable results at aggregated level, great attention should be paid with regard to data selection and preparation. Hereafter, when we talk about earnings, we actually always mean non-credit and non-market earnings so that there is no double counting with other risk types that are already measured within a bank's economic capital model.

EAR versus CAR. Business risk can be defined as the potential loss in the company's earnings due to adverse, unexpected changes in business volume, margins, or both. Such losses can result above all from a serious deterioration of the market environment, customer shift, changes in the competitive situation, or internal restructuring. On one hand, these effects can lead to a drop in earnings in the short-term, e.g. within the next budget year, and are often measured in terms of earnings volatility are more general by EAR. On the other hand, volume or margin shrinking probably leads to a longer-lasting weakening of the earnings situation, thereby seriously diminishing the company's market capitalisation, and this risk is often referred to as CAR. As pointed out by Saita [10, 11], the recognition of such negative long-term effects on earnings and the resulting impact on the market capitalisation is particular important for the shareholder perspective on capital and should also be used in the context of risk-adjusted performance measurement, e.g. by means of RAROC, EVA, or related concepts.

A convincing analysis proving this link between earnings' related risk and a company's loss in market value is given in Morrison, Quella and Slywotzky [8]. They found out that during a period of five years, 10 % of Fortune 1,000 companies lost (at least once) 25 % of their shareholder value within a one-month period, and that nearly all of these stock drops were a result of reduced quarterly earnings or reduced expected future earnings. Moreover, the majority of these earnings-shortfalls (about 58 %) were not owing to classical financial risks or operational losses but rather to what Quella et al. refer to as strategic risk factors, such as raising costs and margin squeeze, emerging global competitors, and customer priority shift etc.

If one considers business risk as a matter of market capitalisation and therefore measures it by CAR, one has to take the uncertainty of (all) future earnings into account. As mentioned above, such earnings fluctuations, i.e. the deviations of the realised earnings from the planned earnings trajectory, my be the result of many different factors. However, for the model we suggest here, it is not necessary to explicitly link all these risk factors to future earnings. Instead we suppose that all risk factors together constitute some random "noise" effect, mixing with the expected earnings path; i.e. for $t \geq 0$ future cumulated earnings $E(t)$ can be written as

$$E(t) = f(t) + \text{"noise"}\,, \qquad t \geq 0\,,$$

where f is a nonrandom function describing the planned earnings trajectory. Consequently, in section 2.2 we model future earnings as a stochastic process $(E(t))_{t\geq 0}$.

Discounted-cash-flow method. Before we go on, however, it is worthwhile to recall some basic facts about company valuation, especially about the discounted-cash-flow method where expected future earnings are discounted to obtain the company's market value (see e.g. Goedhart, Koller and Wessels [5] or Pratt, Reilly and Schweihs [9] for a more detailed description of this subject). Denote by $\Delta E_{t_i} \in \mathbb{R}$ the company's earnings that are planned to be realized in $\Delta t_i = t_i - t_{i-1}$, i.e., between the future time periods t_{i-1} and t_i, defined for all $0 = t_0 < t_1 < \cdots < t_T$. One then usually defines the *present value* or *market value* of the company as

$$V(T) = \sum_{i=1}^{T} \frac{\Delta E_{t_i}}{(1 + r_i)^{t_i}}, \tag{2.1}$$

where $r_i \in \mathbb{R}_+$ is a risk adjusted discount rate. Expression (2.1) simply says that a company's market value is just the sum of its discounted expected future earnings. For our purposes, however, a continuous-time setting of the present value (2.1) is more feasible.

Definition 2.1 (Present value in continuous time)**.** *The* present value *of all earnings cumulated until the time horizon t is given by*

$$P(t) = P(0) + \int_0^t e^{-r(s)}\, dE(s), \qquad t \geq 0,$$

where $r(\cdot)$ is a nonrandom positive function and

$$E(t) = E(0) + \int_0^t dE(s), \qquad t \geq 0,$$

are the cumulated future earnings *as they are expected to be realised up to a time horizon t. Furthermore, we define*

$$P(\infty) := \lim_{t \to \infty} P(t),$$

provided that the limit exists.

2.2 Model Definition and First Results

We begin our analysis of business CAR with a simple model based on Brownian motion. Such a model allows for closed-form solutions for business CAR and is therefore particularly useful to understand the nature and general properties if this important risk type.

Stochastic modelling of future cash flows. We define our model in a multivariate fashion in that it takes the dependence between different *cells* into account. Each cell could simply reflect a legal entity, business division, geographical region, or a combination of them. If one explicitly splits up revenues and costs instead of considering earnings directly, revenues and costs are represented by different cells, and for each cell we can define a cash-flow process representing the stochastic evolution of revenues, costs, or earnings. In the following we treat revenues as positive and costs as negative variables.

Definition 2.2 (Brownian motion cash flow (BMC) model)**.** *Consider a d-dimensional standard Brownian motion $((W_1(t))_{t\geq 0}, \dots, (W_d(t))_{t\geq 0})$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Then, the BMC model consists of:*

(1) Cash-flow processes.
For each business risk cell, indexed by $i = 1, \dots, m$, cumulated future cash flows $X_i(t)$ for $t \geq 0$ are described by a cash-flow process*, which is the strong continuous solution to the Itô-stochastic-differential equation*

$$dX_i(t) = \alpha_i(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)\,, \qquad t \geq 0\,. \tag{2.2}$$

The bank's total aggregated cash flow is given by the aggregate cash-flow process

$$X(t) = \sum_{i=1}^{m} X_i(t) = \sum_{i=1}^{m} X_i(0) + \sum_{i=1}^{m} \int_0^t dX_i(s)\,.$$

Here, $\alpha_i(\cdot) > 0$, $i = 1, \dots, m$, and $\sigma_{ij}(\cdot)$, $i = 1, \dots, m; j = 1, \dots, d$, are nonrandom functions of time, satisfying the integrability conditions $\int_0^t |\alpha_i(s)| ds < \infty$ and $\int_0^t \sigma_{ij}^2(s) ds < \infty$.

(2) Value process.
Let $r(\cdot) > 0$ be a nonrandom function so that $\int_0^t \left(|\alpha_i(s)| e^{-r(s)\,s} + \sigma_{ij}^2(s) e^{-2r(s)\,s}\right) ds < \infty$. Then, the aggregate value process $(P(t))_{t\geq 0}$ *is defined by (setting $P(0) = 0$)*

$$\begin{aligned} P(t) &= \sum_{i=1}^{m} \int_0^t e^{-r(s)\,s}\,dX_i(s)\,, \\ &= \sum_{i=1}^{m} \int_0^t \alpha_i(s)\,e^{-r(s)\,s}\,ds + \sum_{j=1}^{d} \sum_{i=1}^{m} \int_0^t \sigma_{ij}(s)\,e^{-r(s)\,s}\,dW_j(s)\,, \quad t \geq 0\,. \end{aligned} \tag{2.3}$$

Remark 2.3. (a) For every $t \geq 0$, the BMC model describes multivariate normally distributed cumulated cash flows $X_i(\cdot),\ i = 1, \dots, m$, with expectation

$$\mathbb{E}X_i(t) = X_i(0) + \int_0^t \alpha_i(s)\, ds\,, \qquad t \geq 0\,, \tag{2.4}$$

and cross-covariance function of $X_i(\cdot)$ and $X_k(\cdot),\ i, k = 1, \dots, m$, given by

$$\operatorname{covar}(X_i(t), X_k(t)) = \int_0^t \sum_{j=1}^d \sigma_{ij}(s)\, \sigma_{kj}(s)\, ds =: \int_0^t \mathbf{\Sigma}_{ik}(s)\, ds\,, \qquad t \geq 0\,.$$

Here, $\mathbf{\Sigma} := (\mathbf{\Sigma}_{ik}(t))_{ik}$ is called instantaneous cash flow covariance matrix, which is assumed to be positive definite for all $t \geq 0$.

(b) The variance of future cash flows $X_i(\cdot),\ i = 1, \dots d$, can be written as

$$\operatorname{var}(X_i(t)) = \int_0^t \mathbf{\Sigma}_{ii}(s)\, ds =: \int_0^t \sigma_i^2(s)\, ds\,, \qquad t \geq 0\,, \tag{2.5}$$

where $\sigma_i(\cdot),\ i = 1, \dots m$, are referred to as the instantaneous cash-flow volatilities.

(c) For nonzero $\sigma_i(\cdot)$, the cross-correlation function between $X_i(\cdot)$ and $X_k(\cdot)$ is

$$\operatorname{corr}(X_i(t), X_k(t)) = \frac{\operatorname{covar}(X_i(t), X_k(t))}{\sqrt{\operatorname{var}(X_i(t))}\,\sqrt{\operatorname{var}(X_k(t))}}\,, \qquad t \geq 0\,. \tag{2.6}$$

Informally, we denote the instantaneous correlation between $dX_i(\cdot)$ and $dX_k(\cdot)$ as

$$\operatorname{corr}(dX_i(t), dX_k(t)) = \frac{\mathbf{\Sigma}_{ik}(t)}{\sigma_i(t)\, \sigma_k(t)} =: \rho_{ik}(t)\,, \qquad t \geq 0\,.$$

(d) The value of the aggregate cash-flow process $X(t)$ at time $t \geq 0$ gives the total earnings of the bank that have been realised between 0 and t (cumulated earning). Its variance is given by

$$\begin{aligned}
\operatorname{var}(X(t)) &= \int_0^t \sum_{j=1}^d \left(\sum_{i=1}^m \sigma_{ij}(s) \right)^2 ds \\
&= \int_0^t \sum_{j=1}^d \sum_{i=1}^m \sum_{k=1}^m \sigma_{ij}(s)\, \sigma_{kj}(s)\, ds \\
&= \int_0^t \sum_{i=1}^m \sum_{k=1}^m \mathbf{\Sigma}_{ik}(s)\, ds =: \int_0^t \sigma^2(s)\, ds\,, \qquad t \geq 0\,,
\end{aligned} \tag{2.7}$$

where we call $\sigma(\cdot)$ the instantaneous aggregate cash-flow volatility.

(e) Note that the number d of independent Brownian motions $W_i(\cdot)$ can be different from the number m of business risk cells. Therefore, our model also allows for such realistic scenarios where the number of risk factors (represented by different Brownian motions) is greater than the number of clusters; think e.g. of a bank with two legal entities that are exposed to three risk factors, which affect their non-credit and non-market earnings and thus business risk. $\square$

Example 2.4. [Bivariate BMC model with constant parameters]
Consider a simple bivariate BMC model with constant drift and diffusion parameters where the cash-flow processes are given by

$$\begin{aligned} dX_1(t) &= \alpha_1\,dt + \sigma_1\,dW_1(t) \\ dX_2(t) &= \alpha_2\,dt + \sigma_2\big(\rho\,dW_1(t) + \sqrt{1-\rho^2}\,dW_2(t)\big)\,, \qquad t \geq 0\,. \end{aligned} \tag{2.8}$$

From Remark 2.3 it follows that

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

implying for the variance of the i-th cumulated future cash flow

$$\mathrm{var}(X_i(t)) = \sigma_i^2 t\,, \qquad t \geq 0\,, \qquad i = 1, 2\,. \tag{2.9}$$

Moreover, the correlation between $X_1(\cdot)$ and $X_2(\cdot)$ and the instantaneous correlation are given for all $t \geq 0$ by

$$\mathrm{corr}(X_1(t), X_2(t)) = \mathrm{corr}(dX_1(t), dX_2(t)) = \rho\,.$$

Since all parameters are time-independent, this model can be calibrated quite easily. After discretisation of (2.8) by using the Euler method, σ_1 and σ_2 can be calculated directly from the standard deviations of the discrete increments $\Delta X_1(\cdot)$ and $\Delta X_2(\cdot)$. Then, according to (2.9), volatilities at a larger time-scale t can be derived by using the $\sqrt{t}$-scaling law. Finally, α_1 and α_2 can be estimated through the sample means of the discretisised incremental cash flows (2.8), or, alternatively, they can be obtained from the cumulated cash flows

$$\mathbb{E}X_i(t) = \alpha_i\,t + \mathrm{const.}, \qquad t \geq 0\,,$$

for $i = 1, 2$ by regression analysis. $\square$

Example 2.5. [Bivariate BMC model with time-dependent parameters]
We use a similar set up as in (2.8) but with time-dependent parameters for $t \geq 0$ of $\alpha_i(t) = \alpha_i t^{a_i}$ and $\sigma_i(t) = \sigma_i t^{b_i}$ for $a_i, b_i \geq 0$; $i = 1, 2$:

$$\begin{aligned} dX_1(t) &= \alpha_1 \, t^{a_1} \, dt + \sigma_1 \, t^{b_1} \, dW_1(t) \\ dX_2(t) &= \alpha_2 \, t^{a_2} \, dt + \sigma_2 \, t^{b_2} \big(\rho \, dW_1(t) + \sqrt{1-\rho^2} \, dW_2(t)\big) \,, \qquad t \geq 0 \,. \end{aligned} \tag{2.10}$$

The expectations and variances of future cash flows can be calculated from (2.4) and (2.5); for $i = 1, 2$ one obtains

$$\mathbb{E}X_i(t) = \frac{\alpha_i}{1 + a_i} \, t^{1+a_i} \,, \qquad t \geq 0 \,,$$

and

$$\operatorname{var}(X_i(t)) = \frac{\sigma_i^2}{2b_i + 1} \, t^{2b_i+1} \,, \qquad t \geq 0 \,.$$

The instantaneous correlation in this model is still ρ, however, using (2.6) we derive

$$\operatorname{corr}(X_1(t), X_2(t)) = \frac{\sqrt{2b_1 + 1} \, \sqrt{2b_2 + 1}}{1 + b_1 + b_2} \, \rho \,, \qquad t \geq 0 \,.$$

We now define the *signal-to-noise ratio*, also referred to as *Sharpe ratio*, for each cash-flow process as the ratio of its expected growths to the fluctuations, i.e. for $i = 1, 2$

$$\eta_i(t) = \frac{\mathbb{E}X_i(t) - X_i(0)}{\sqrt{\operatorname{var}(X_i(t))}} \,, \qquad t \geq 0 \,. \tag{2.11}$$

Instead of constant volatilities $\sigma_i(\cdot) = \sigma_i$ as in the Example 2.4, we are now asking for constant Sharpe ratios $\eta_i(\cdot) = \eta_i$. Obviously, the Sharpe ratios of $X_1(\cdot)$ and $X_2(\cdot)$ are here constant for $b_i = a_i + \frac{1}{2}$. A typical situation as it may occour in practice is depicted in Figure 1, which shows the monthly earnings over 5 years for two hypothetical legal entities. □

Calculating business CAR. For the purpose of CAR calculations we have to learn more about the value process $(P(t))_{t \geq 0}$. The following result is well-known and describes its distributional properties.

Proposition 2.6. *Consider the BMC model of Definition 2.2 with value process* $(P(t))_{t \geq 0}$. *Then, for every* $t > 0$, *the value* $P(t)$ *has normal distribution function* Φ_P *with expected value*

$$\mathbb{E}P(t) = \sum_{i=1}^{m} \int_0^t \alpha_i(s) \, e^{-r(s)\, s} \, ds \,, \qquad t \geq 0 \,, \tag{2.12}$$

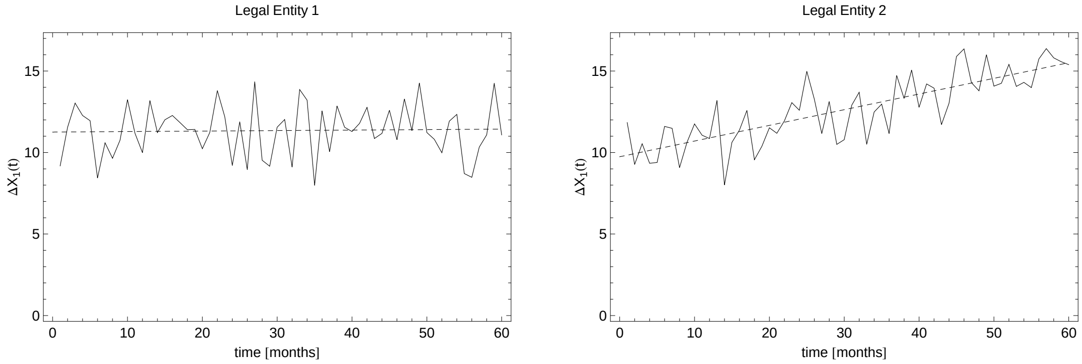

Figure 1: Illustration of monthly earnings of two different legal entities as described in Example 2.5. In contrast to legal entity 1, the monthly earnings of legal entity 2 seems to have a positive linear drift which e.g. could be determined by linear regression (showed as dashed lines). Hence, in (2.10) one could set $a_1 = 0$ and $a_2 = 1$. Regarding the volatilities one could either use $b_1 = b_2 = 0$ (constant absolute volatilities) or $b_1 = 0.5$ and $b_2 = 1.5$ (constant Sharpe ratios).

and variance

$$\mathrm{var}(P(t)) = \int_0^t \sigma^2(s)\, e^{-2\, r(s)\, s}\, ds\,, \qquad t \geq 0\,,$$

where $\sigma(\cdot)$ *is the instantaneous aggregate cash-flow volatility defined in* (2.7).

Proof. Note that on the right-hand side of equation (2.3) the integrals in the first term are standard Riemann whereas those of the second term given by $I(t) = \int_0^t \sigma_{ij}(s)\, e^{-r(s)s} dW_j(s)$ are Itô integrals with deterministic integrands. Then $I(t)$ is normally distributed with $\mathbb{E}I(t) = 0$ and $\mathrm{var}(I(t)) = \mathbb{E}\,(I(t)^2)$, see e.g. Shreve [12], Theorem 4.4.9. Using Itô's isometry we further obtain

$$\begin{aligned}
\mathrm{var}(P(t)) &= \sum_{j=1}^{d} \mathrm{var}\left(\sum_{i=1}^{m} \int_0^t \sigma_{ij}(s)\, e^{-r(s)\, s}\, dW_j(s) \right) \\
&= \sum_{j=1}^{d} \mathbb{E}\left(\sum_{i=1}^{m} \int_0^t \sigma_{ij}(s)\, e^{-r(s)\, s}\, dW_j(s) \right)^2 \\
&= \sum_{j=1}^{d} \int_0^t \left(\sum_{i=1}^{m} \sigma_{ij}(s)\, e^{-r(s)\, s} \right)^2 ds \\
&= \int_0^t \sigma^2(s)\, e^{-2\, r(s)\, s}\, ds\,, \qquad t \geq 0\,.
\end{aligned}$$

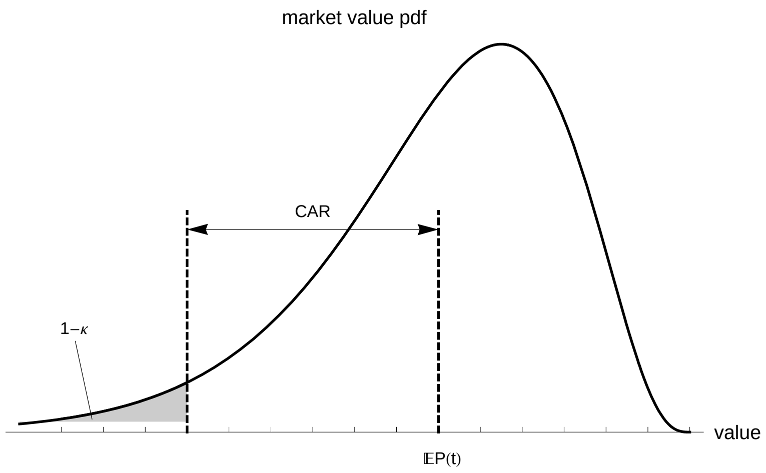

Figure 2: Business CAR is defined as the difference between the expected market value $\mathbb{E}P(t)$ of the bank's aggregate value process and a very low quantile of its market value distribution F_t.

□

Since we now know that in the BMC model the bank's total market value $P(\cdot)$ is normally distributed with distribution function Φ_P, it is straightforward to calculate business CAR. Before, however, we want to precisely define business CAR for general distributions of $P(\cdot)$.

Definition 2.7 (Business CAR). *Consider different business risk cells with cash-flow processes $X_i(\cdot)$, $i =, 1, \ldots, m$, that are not attributable to other risk types, and define their corresponding market value process $P(\cdot)$ according to* (2.3)*. For $t > 0$, suppose that F_t is the distribution function of the value $P(t)$ with mean value $\mathbb{E}P(t) < \infty$. Then,* business CAR *at time horizon t and confidence level $\kappa \in (0, 1)$ is given by*

$$CAR_\kappa(t) = \mathbb{E}P(t) - F_t^{\leftarrow}(1-\kappa)\,, \qquad t \geq 0\,, \tag{2.13}$$

where $F_t^{\leftarrow}(\kappa) = \inf\{x \in \mathbb{R} : \ F_t(x) \geq \kappa\}$, $0 < \kappa < 1$, is the generalized inverse of F_t. If F_t is strictly increasing and continuous, we may write $F_t^{\leftarrow}(\cdot) = F_t^{-1}(\cdot)$.

In the context of economic capital calculations, the confidence level κ is a number close to 1, e.g. $\kappa = 0.999$. In the case that the probability density function (pdf) of F_t exists, the definition of business CAR is illustrated in Figure 2. In general, F_t and thus business CAR cannot be calculated analytically. For the BMC model, however, a closed-form expression for business CAR is available.

Theorem 2.8 (Business CAR for the BMC model)**.** *Assume that future cash flows are described by a BMC model with instantaneous aggregate cash-flow volatility (see* *equation (2.7))*

$$\sigma(t) = \sum_{i=1}^{m} \sum_{j=1}^{d} \sum_{k=1}^{m} \sigma_{ij}(t)\, \sigma_{kj}(t)\,, \qquad t \geq 0\,, \tag{2.14}$$

and nonrandom interest rate function $r(\cdot)$. Then, business CAR at time-horizon t and confidence level $\kappa \in (0,1)$ is given by

$$CAR_\kappa(t) = \Phi^{-1}(\kappa) \sqrt{\int_0^t \sigma^2(s)\, e^{-2\,r(s)\,s}\, ds}\,, \qquad t \geq 0\,, \tag{2.15}$$

where Φ is the standard normal distribution function.

Proof. The assertion follows directly from Proposition 2.6 together with the definition of business CAR (2.13). □

This analytical expression for business CAR is mainly a consequence of using non-random interest rates. If one instead allows $r(\cdot)$ to be some continuous adapted interest rate process, the distribution function F_t, $t \geq 0$, of the aggregate value process $(P(t))_{t\geq 0}$ is in general not normal anymore, and the result for business CAR will rarely be available in closed-form.

Note that (2.15) only depends on the cash flows' covariance matrix and not on other model parameters such as drift parameters $\alpha_i(\cdot)$ or the initial values $X_i(0)$. As a consequence thereof, the BMC model can be calibrated easily, confer also Examples 2.4 and 2.5.

2.3 The Relationship Between EAR and CAR

Earnings-at-Risk. In the light of Definition 2.2 we now can qualify our notion of EAR, which we have already introduced in section 2.1. From Remark 2.3 (d) we know that $\mathrm{var}(X(t))$ is the volatility of the bank's total aggregate cash flows accumulated between time 0 and t. It is also well-known that $X(t)$, $t \geq 0$, is normally distributed, see e.g. Shreve [12], Theorem 4.4.9, so that EAR of the cumulated earnings $X(t)$ at $t \geq 0$ and confidence level $\kappa \in (0,1)$ is simply given by

$$\begin{aligned} \mathrm{EAR}_\kappa(t) &= \Phi^{-1}(\kappa)\, \sqrt{\mathrm{var}(X(t))} \\ &= \Phi^{-1}(\kappa) \sqrt{\int_0^t \sigma^2(s)\, ds}\,, \qquad t \geq 0\,, \end{aligned} \tag{2.16}$$

where Φ is the standard normal distribution function. In contrast to (2.15), we see that in (2.16) the volatility is not discounted. Moreover, it should be mentioned that according to what we have said in section 2.1, the time parameter t in (2.16) should be chosen in a way that it reflects a *short-term horizon* so that discounting effects can be neglected. Finally, we define the instantaneous EAR as

$$\text{ear}_\kappa(\cdot) \;=\; \Phi^{-1}(\kappa)\,\sigma(\cdot)\,. \tag{2.17}$$

EAR-CAR-transformation factors. An interesting question concerns the relation between EAR and CAR. Intuitively, CAR should be higher than EAR since CAR takes—in contrast to EAR—the *long-term uncertainty* of future cash flows into account. It has been suggested e.g. by Matten [7] or Saita [10, 11] that CAR is a constant multiplier of EAR, and that the multiplication factor depends only on a (risk-adjusted) interest rate (discount factor) and the time horizon t. Saita based his analysis of the EAR-CAR relationship on a discrete-time cash-flow model similar to (2.1) where EAR reflects the uncertainty of the ΔE_{t_i}, and [11], section 5.8, gives a very readable overview about this topic.

In the case of the BMC model, we see by comparing (2.15) with (2.17) that such a proportionality between EAR(t) and CAR(t) does not hold for all $t \geq 0$ because of the time dependence of the instantaneous aggregate-cash-flow volatility $\sigma(\cdot)$. However, the mean value theorem ensures that one can always chose a $\xi \in (0,t)$ so that (2.15) can be written as

$$\begin{aligned}\text{CAR}_\kappa(t) &= \Phi^{-1}(\kappa)\,\sigma(\xi)\sqrt{\int_0^t e^{-2\,r(s)\,s}\,ds}\\ &= \text{ear}_\kappa(\xi)\sqrt{\int_0^t e^{-2\,r(s)\,s}\,ds}\,, \qquad t \geq 0\,,\end{aligned}$$

and we can think of $\text{ear}_\kappa(\xi)$ as an average EAR of the time interval $[0,t]$. The following two examples illustrate the relationship between EAR and CAR, in particular showing that—even in the quite simple framework of BMC models—EAR-CAR-transformation crucially depends on the specifications of the cash-flow processes $X_i(t),\ i =, 1, \ldots, d$.

Example 2.9. [BMC model with constant diffusion parameters]
Consider a BMC model with $\sigma_{ij}(\cdot) = \sigma_{ij} =$ const. for $i = 1, \ldots m; j = 1, \ldots d$. Then also the aggregate-cash-flow volatility (2.14) is constant and we obtain

$$\text{CAR}_\kappa(t) = k_1(r,t)\,\Phi^{-1}(\kappa)\,\sigma = k_1(r,t)\,\text{ear}_\kappa\,, \qquad t \geq 0\,,$$

with EAR-CAR transformation factor

$$k_1(r,t) = \int_0^t e^{-2\,r(s)\,s}\,ds\,,$$

i.e. in this special case business CAR is proportional to the (constant) instantaneous EAR. If furthermore the interest rate is constant, $r(\cdot) = r$, we arrive at

$$k_1(r,t) = \sqrt{\frac{1-\exp(-2\,r\,t)}{2\,r}}\,, \tag{2.18}$$

which is a simple function of the time horizon t and the discount rate r. The higher r is, the smaller k_1 will be because future cash flows (and so their fluctuations) tend to have lower impact on the market value (and so on its uncertainty). Similarly, longer time horizons t lead to a growing k_1 because more uncertain future cash-flows are taken into account. In the limit $t \to \infty$ expression (2.18) simplifies to

$$\lim_{t\to\infty} k_1(r,t) = \frac{1}{\sqrt{2\,r}}\,. \tag{2.19}$$

□

Example 2.10. [BMC model with constant Sharpe ratio]
Consider a BMC model with cumulated cash-flows $X_i(\cdot)$ for $i = 1,\ldots,m$. As in Example 2.5 we consider the Sharpe ratios $\eta_i(\cdot)$ given by (2.11). Adopting a linear-growth model with constant drift parameters $\alpha_i(\cdot) = \alpha_i > 0,\ i = 1,\ldots,m$, we know from Example 2.5 that constant Sharpe ratios require square-root-of-time scalings of the instantaneous cash-flow volatilities, i.e. $\sigma_i(t) = c_i\sqrt{t}$ for some constants $c_i,\ i = 1,\ldots,m$. This implies that the aggregate-cash-flow volatility (2.14) for $t \geq 0$ can be written as $\sigma(t) = \sigma\sqrt{t}$, resulting in sample paths of $X_i(\cdot)$ that are in general more noisy than the one obtained in Example 2.9, a fact that is illustrated in the top panel of Figure 3.

Finally, using (2.15) we can calculate business CAR in the case of a constant interest rate r to

$$\mathrm{CAR}_\kappa(t) = k_2(r,t)\,\Phi^{-1}(\kappa)\,\sigma\,, \qquad t \geq 0\,,$$

with EAR-CAR transformation factor

$$k_2(r,t) = \sqrt{\int_0^t s\,e^{-2rs}\,ds} = \frac{1}{2r}\sqrt{1-(1+2rt)\exp(-2\,r\,t)} \tag{2.20}$$

and

$$\lim_{t\to\infty} k_2(r,t) = \frac{1}{2\,r}\,. \tag{2.21}$$

□

Comparing the different BMC-model specifications of Example 2.9 and 2.10, one could expect that k_2 is greater than k_1 because in the previous example the Sharpe ratio was actually increasing with $\sqrt{t}$ (implying a decrease of future cash-flow fluctuation) whereas here it is constant by construction. Hence, k_2 accumulates more future uncertainty than k_1. As we see in the bottom panel of Figure 3, this is indeed the case if t exceeds a certain threshold, i.e. $t > t_0(r)$, which for $r = 0.1$ is approximately given by $t_0(0.1) = 2.2$. Moreover, by comparing (2.19) and (2.21) it follows that for all $t \geq t_0(r)$ we have that

$$0 \leq \frac{k_2(r,t)}{k_1(r,t)} \leq \frac{1}{\sqrt{2r}} \,.$$

3 A Model with Level-Adjusted Volatility

In the BMC model the *absolute* changes of future cash flows $X_i(\cdot)$ are directly modelled by a Brownian motion, see equation (2.2), which in particular means that the uncertainty of a business cell's cash flow is independent from its absolute level. There is, however, no rational for this behaviour and intuitively one rather associates higher total earnings with higher earnings volatilities. As a possible remedy, one could describe future cash flows by a *geometric* Brownian motion as it is used e.g. for stock prices in the famous famous Black-Scholes-Merton setting. Then, for $t \geq 0$ cumulated cash-flows $X_i(\cdot)$, $i = 1, \dots, m$, would be given by

$$X_i(t) = X_i(0) \exp\left[\int_0^t \left(\alpha_i(s) - \frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2(s)\right) ds + \sum_{j=1}^d \sigma_{ij}(s)\, dW_j(s)\right],$$

implying an exponential cash-flow growth of

$$\mathbb{E}X_i(t) = X_i(0) \exp\left(\int_0^t \alpha_i(s) ds\right),$$

which, however, might be considered as too extreme and optimistic for most businesses. Alternatively, we suggest a model with still moderate growth but a kind of "cash-flow level adjusted" volatility. More precisely, we have the following definition.

Definition 3.1 (Level-adjusted BMC model). *Consider a d-dimensional standard Brownian motion $((W_1(t))_{t\geq 0}, \dots, (W_d(t))_{t\geq 0})$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Then, the level-adjusted BMC model model consists of:*

(1) Cash-flow processes.

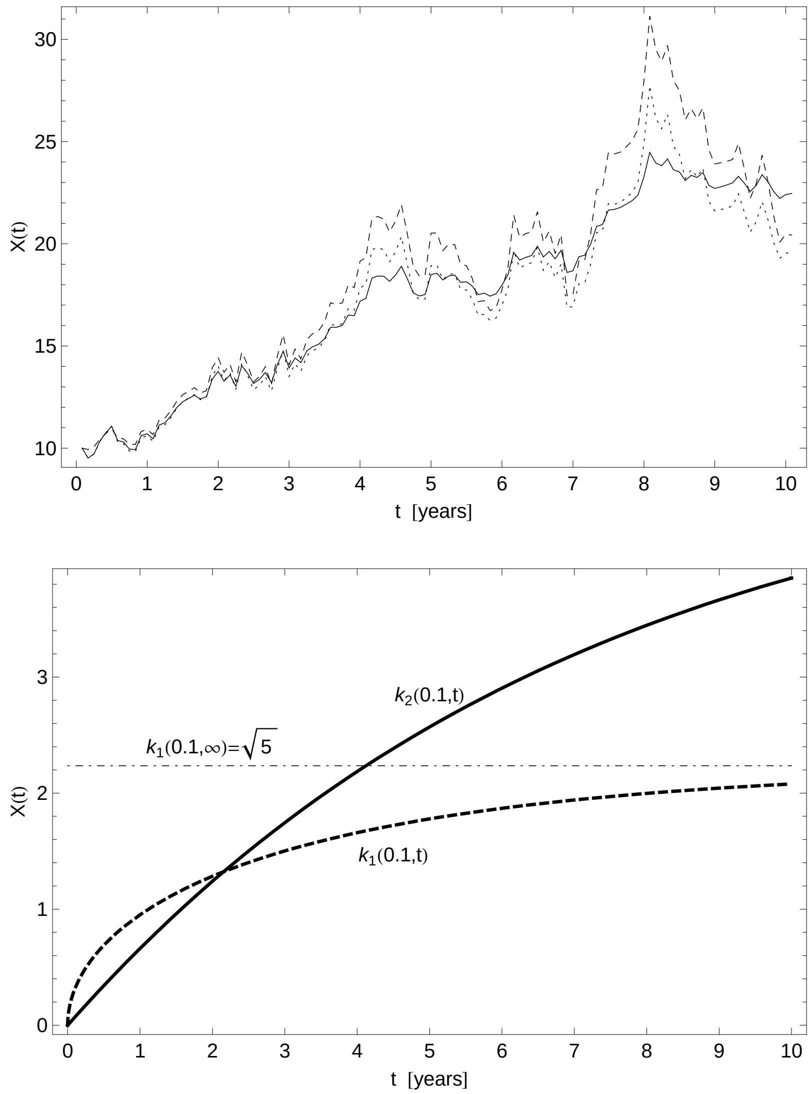

Figure 3: Top panel: Typical cash-flow paths $X_i(t)$ for BMC models with constant absolute cash-flow volatility (solid line) and constant Sharpe ratio (dashed line) referring to Examples 2.9 and 2.10, respectively. The parameters are $X_i(0) = 10$, $\alpha_i = 1.2$ and $\sigma_i = 1.4$ with a time horizon of 10 years and monthly increments. Moreover, the dotted path is obtained by the advanced model discussed in section 3, especially Example 3.2, and we set $\tilde{\sigma}_i = \sigma_i/X_i(0) = 0.14$

Bottom Panel: EAR-CAR-transformation factors $k_1(r,t)$ and $k_2(r,t)$ according to (2.18) and (2.20), respectively, as a function of time for a discount rate $r = 0.1$. The dotted-dashed line indicates the asymptote of $k_1(r,t)$ when $t \to \infty$.

For each business risk cell, indexed by $i = 1, \dots, m$, *cumulated future cash flows* $X_i(t)$ *for* $t \geq 0$ *are described by a* cash-flow process, *which is the strong continuous solution to the Itô-stochastic-differential equation*

$$dX_i(t) = \alpha_i(t)\,dt + X_i(t)\sum_{j=1}^{d} \tilde{\sigma}_{ij}(t)\,dW_j(t)\,, \qquad t \geq 0\,, \tag{3.1}$$

with aggregate cash-flow process

$$X(t) = \sum_{i=1}^{m} X_i(t) = \sum_{i=1}^{m} X_i(0) + \sum_{i=1}^{m} \int_0^t dX_i(s)\,.$$

Here, $\alpha_i(\cdot) > 0$, $i = 1, \dots, m$, *and* $\tilde{\sigma}_{ij}(\cdot)$, $i = 1, \dots, m; j = 1, \dots, d$, *are nonrandom functions of time, satisfying the integrability conditions* $\int_0^t |\alpha_i(s)| ds < \infty$ *and* $\int_0^t \tilde{\sigma}_{ij}^2(s) ds < \infty$. *The matrix* $(\tilde{\sigma}_{ij}(t))_{ij}$ *is assumed to be positive definite for all* $t \geq 0$.

(2) Value process.
Let $r(\cdot) > 0$ *be a nonrandom function so that* $\int_0^t \left(|\alpha_i(s)| e^{-r(s)\,s} + \tilde{\sigma}_{ij}^2(s) e^{-2r(s)\,s}\right) ds < \infty$. *Then, the* aggregate value process $(P(t))_{t \geq 0}$ *is defined by (setting* $P(0) = 0$*)*

$$P(t) \;=\; \sum_{i=1}^{m} \int_0^t e^{-r(s)\,s}\,dX_i(s)\,, \qquad t \geq 0\,.$$

Let us first consider the cash-flow process and compare (3.1) with (2.2) of the BMC model. For the latter, the diffusion parameters σ_{ij} play the role of an *absolute* measure of uncertainty for the increments of $X_i(\cdot)$, whereas in (3.1) the $\tilde{\sigma}_{ij}$ describe the increments' fluctuations *relative* to the level of $X_i(\cdot)$. Furthermore, instead of (3.1) we may write

$$X_i(t) = X_i(0) + \int_0^t \alpha_i(s)\,ds + \sum_{j=1}^{d} \int_0^t X_i(s)\,\tilde{\sigma}_{ij}(s)\,dW_j(s)\,, \qquad t \geq 0\,, \tag{3.2}$$

and from the martingale property of the Itô integral it immediately follows that the expectation of $X_i(\cdot)$ is given by

$$\mathbb{E}X_i(t) = X_i(0) + \int_0^t \alpha_i(s)\,ds\,, \qquad t \geq 0\,,$$

i.e. it is the same as for the BMC model, and, in particular, the model does not exhibit exponential growth as it would be the case when geometric Brownian motion is used. We close this section with an extended example that illustrates some properties of the level-adjusted volatility model.

Example 3.2. [Constant drift and diffusion parameters]
For the sake of simplicity we focus on the case of constant parameters $\alpha_i(\cdot) = \alpha_i$ and $\tilde{\sigma}_{ij}(\cdot) = \tilde{\sigma}_{ij}$. Note however, that the diffusion parameter of the process (3.1) is random and given by $X_i(\cdot)\,\tilde{\sigma}_{ij}$. In order to find a solution for (3.2) we define the function

$$F(t, W(t)) := \exp\big(-\sum_{j=1}^{d} \tilde{\sigma}_{ij} W_j(t) + \frac{1}{2}\sum_{j=1}^{d} \tilde{\sigma}_{ij}^2\, t \big)\,, \qquad t \geq 0\,.$$

Then, by using Itô's formula the differential of the product FX_i can be calculated as

$$d\big(F(t, W(t))\, X_i(t)\big) = \alpha_i\, F(t, W(t))\, dt\,,$$

which after integration finally yields (setting $W_j(0) = 0$ for $j = 1, \ldots, d$),

$$X_i(t) = F(-t, -W(t)) \left(X_i(0) + \alpha_i \int_0^t F(s, W(s))\, ds \right)\,, \qquad t \geq 0\,. \tag{3.3}$$

According to (3.3), the cumulated cash-flows $X_i(t)$ at time $t \geq 0$ are not normally distributed as they are in the BMC model. A one-dimensional example for a typical path of $X_i(\cdot)$ according to (3.3) is plotted as a dotted line in the top panel of Figure 3.

The Itô representation of the value process $(P(t))_{t\geq 0}$ is given by

$$P(t) = \sum_{i=1}^{m} \alpha_i \int_0^t e^{-r(s)s}\, ds + \sum_{j=1}^{d}\sum_{i=1}^{m} \tilde{\sigma}_{ij} \int_0^t X_i(s)\, e^{-r(s)s}\, dW_j(s)\,, \quad t \geq 0\,, \tag{3.4}$$

which cannot be calculated in closed form. Note, however, that the expectation of $P(\cdot)$ is again given by (2.12) and therefore is the same as for the BMC model. This can also be seen in Figure 4. In the top panel, we compare the distribution of the present value as obtained by (3.4) firstly to that of a normal distribution with the same mean and variance (dashed curve), and secondly to the normally distributed present value calculated from the BMC model of Example 2.9 (solid curve). One can see that (3.4) leads to a distribution that is more skewed to the right (positive skewness) and is more peaked and heavier-tailed than a normal distribution with the same variance (kurtosis larger than 3). □

EAR-CAR-transformation revisited. We have seen in section 2.2 that for the BMC model both the cumulated cash-flows $X_i(\cdot)$ and the present value $P(\cdot)$ are normally distributed. This has two important consequences. First, business CAR (and thus the EAR-CAR-transformation factor) of the BMC model is independent of the expected growth and thus of $\alpha_i(\cdot)$, $i = 1, \ldots, m$. Second, the transformation factor is

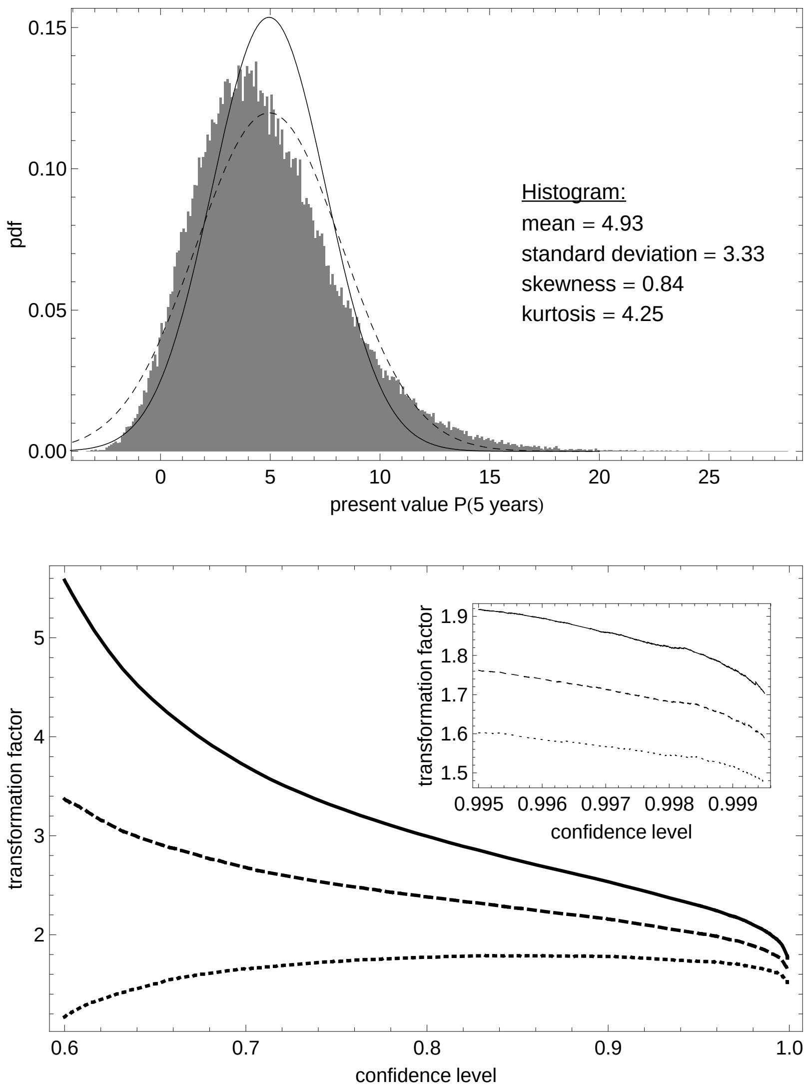

Figure 4: Top panel: The histogram shows the simulated present-value distribution for $t = 5$ years of the level-adjusted volatility model as discussed in Example 3.2 as well as the mean, standard deviation, skewness, and kurtosis parameters of the simulated data. This is compared to a normal distribution with the same mean and standard deviation as the simulated data, plotted as a dashed line. The solid curve represents the normally distributed present value as obtained from the BMC model of Example 2.9. We set $X_i(0) = 10$ and use the yearly parameters $\alpha_i = 1.2$, $\sigma_i = 1.4$, $\tilde{\sigma}_i = \sigma_i/X_i(0) = 0.14$, and a yearly interest rate of $r = 0.08$.

Bottom level: EAR-CAR-transformation factor for the level-adjusted growth model of Example 3.2 as a function of the confidence level κ and different values $\alpha_1 = 1.0$ (solid line), $\alpha_2 = 1.2$ (dashed line), and $\alpha_3 = 1.4$ (dotted line) of the growth parameter. The other parameters are the same as used above.

invariant under changes of the confidence level and equals the ratio of the present value volatility and the earnings volatility (see Examples 2.9 and 2.10). This is a consequence of the well-known property of elliptically distributed random variables, which says that their quantiles can always be expressed in terms of their standard deviation.

However, such a behaviour cannot be expected for the level-adjusted BMC model because $P(\cdot)$ is not elliptically distributed. Figure 4 serves to illustrate this for the model of Example 3.2. It shows the results of a simulation study where the transformation factor between CAR and EAR (calculated at $t = 0$ where EAR is normally distributed with standard deviation $\sigma = \sigma_i X_i(0)$) is plotted as a function of their confidence level. The growth parameter is set to be $\alpha_i = 1.0, 1.2$, and 1.4. Note that the higher the growth rate α_i is, the lower the transformation factor and therefore the ratio between CAR and EAR will be. In contrast, if we compare the ratio of the volatilities of the present value and EAR, i.e. the volatility of the simulated histogram data to the initial absolute cash-flow volatility $\sigma = \sigma_i X_i(0) = 1.4$, we obtain 2.29, 2.38, and 2.46 for $\alpha_i = 1.0, 1.2$, and 1.4, respectively. Moreover, we see that an increasing confidence level leads to a decreasing transformation factor.

Summing up we can conclude that in general the question of how a EAR can be converted into an CAR is not straightforward to answer. While for the BMC model this seems to be easier and intuitively easier to grasp (since independent of the confidence level and the expected growth rate) it becomes rather involved for more general models like the one discussed in this section.

4 Conclusion and Outlook

In this chapter we suggested a multivariate continuous-time setting for assessing business risk using stochastic models for the future cash-flows of (non-credit, non-market, etc.) earnings. In contrast to scenario-based methods for estimating business risk, our model has the advantage that it results in a time-dependent probability distribution of future earnings, which allows for an accurate definition of VAR-like risk measures at any confidence level and time horizon.

We also investigated the relationship between EAR and CAR, which for general cash-flow processes turns out to be not straightforward, and, in particular, a constant multiplier converting EAR into CAR is usually not available. However, a simple EAR-CAR-transformation factor only depending on the time horizon and the discount rate can analytically be derived for the simple BMC model. Such a result may be useful when a fast and approximative VAR estimation based on some EAR figure is needed.

Since in our model the dependence structure between different legal entities or business units is reflected by their correlation matrix, risk-reducing strategies such as known from stock portfolio analysis can be straightforwardly applied.

One could think of several enhancements of the BMC model. A particular interesting possibility would be to introduce jumps in the earnings processes representing sudden and sharp falls in a company's earnings caused by e.g. a change in the competitor environment or a customer shift. A particular important class of jump process are the *Lévy processes*, which have become quite popular also in financial modelling, see e.g. Cont & Tankov [4]. Then, in addition to the covariance structure of the multivariate Brownian motion we already discussed, jump dependence between future earnings could be modelled by so-called Lévy copulas, see e.g. Böcker & Klüppelberg [2] for an application of this technique to operational risk. However, since every model should be seen also in the light of the data needed for its calibration, it may be wise to start with a well-known and established Brownian approach.

In our opinion, the development of advanced business risk models will be an important task in quantitative risk management, despite the difficulties and complexity discussed above. Equally important is to work on a harmonised definition of this material risk type, which clearly requires a closer collaboration between practitioners and academics.

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