# Asymptotic bounds for infinitely divisible sequences

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June 23, 2005

*In memory of Kazimierz Urbanik*

**Abstract**

We give asymptotic bounds for sample paths of discrete time infinitely divisible processes and prove the optimality of such bounds.

*Keywords and phrases. Infinitely divisible processes, boundedness of sample paths, Lévy processes, Rademacher series.*

2000 *Mathematics Subject Classification. Primary: 60G17. Secondary: 60G15*

## 1 Preliminaries and the main result

Consider a stochastic process

$$X(t) = \int_{\mathbb{R}} g(t,s)\, Z(ds) \quad t \in \mathbb{R}, \tag{1.1}$$

where $Z$ is a Lévy process and $g$ is a deterministic kernel. Suppose that we sample this process at discrete times $t_n$, which are not far apart from each other. Formally, we state such condition as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (v^2 \sup_n g(t_n,s)^2 \wedge 1)\, \Pi_Z(dv) ds < \infty, \tag{1.2}$$

where $\Pi_Z$ is the Lévy measure of $Z(1)$. Note that (1.2), without the supremum, is necessary for the integral (1.1) to exist (see, e.g., [6]). We will characterize an extremal sample behavior of $X(t_n)$ as $n \to \infty$. Roughly speaking, we show that possible heavy tails of $X(t)$ have small influence on

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the variation of the sequence $(X(t_n))$ and its behavior depends mostly on small jumps of $Z$ (see Examples 7 and 8). We quantify this dependence. We will put and solve this problem in a general framework and then apply the solution to special cases as above.

Our work extends and refines a result of Braverman [2] given for symmetric stable processes. The technique is based on series representations of infinitely divisible processes combined with precise estimates for the tail of a Rademacher series related to an infinitely divisible distribution. We further develop this technique in Section 2. Section 3 contains the proof of the main result and some applications. The proof uses series representations and methods developed in [4] as well.

A sequence $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ is said to be infinitely divisible if for every $n \in \mathbb{N}$ the random vector $(X_1, \dots, X_n)$ has an infinitely divisible distribution in $\mathbb{R}^n$. It follows from Maruyama [5] that every infinitely divisible sequence has the Lévy-Khintchine representation

$$\mathbb{E}e^{i\langle \mathbf{y},\mathbf{X}\rangle} = \exp\Big\{\langle \mathbf{y}, \mathbf{b}\rangle - \frac{1}{2}Q(\mathbf{y}) + \int_{\mathbb{R}^{\mathbb{N}}} (e^{i\langle \mathbf{y},\mathbf{x}\rangle} - 1 - i\langle \mathbf{y}, [\mathbf{x}]\rangle)\, \nu(d\mathbf{x})\Big\},$$

where $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ has only finitely many nonzero components, $\langle \mathbf{y}, \mathbf{x}\rangle := \sum_n y_n x_n$, and $[\mathbf{x}] \in \mathbb{R}^{\mathbb{N}}$, $[\mathbf{x}]_n := x_n/(|x_n| \vee 1)$. Here $\mathbf{b} \in \mathbb{R}^{\mathbb{N}}$ is a drift, $Q$ is a Gaussian covariance, and $\nu$ is a Lévy measure. I. e., $\nu$ is a Borel measure on $\mathbb{R}^{\mathbb{N}}$ such that $\nu(\{0\}) = 0$ and $\int (|x_n|^2 \wedge 1)\, \nu(d\mathbf{x}) < \infty$ for every $n \in \mathbb{N}$.

We will assume in the sequel that $\mathbf{X}$ is an infinitely divisible sequence with no Gaussian component or drift, i.e., $Q = 0$ and $\mathbf{b} = 0$. In this case the characteristic function of $\mathbf{X}$ simplifies to

$$\mathbb{E}e^{i\langle \mathbf{y},\mathbf{X}\rangle} = \exp\Big\{\int_{\mathbb{R}^{\mathbb{N}}} (e^{i\langle \mathbf{y},\mathbf{x}\rangle} - 1 - i\langle \mathbf{y}, [\mathbf{x}]\rangle)\, \nu(d\mathbf{x})\Big\}. \tag{1.3}$$

The condition corresponding to (1.2) is of the form

$$\int_{\mathbb{R}^{\mathbb{N}}_0} (\|\mathbf{x}\|_\infty^2 \wedge 1)\, \nu(dx) < \infty, \tag{1.4}$$

where $\|\mathbf{x}\|_\infty := \sup_n |x_n|$. Our main result is the following.

**Theorem 1** *Let $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ be an infinitely divisible sequence satisfying (1.3)–(1.4). Let $\theta : (0,\infty) \mapsto [0,\infty)$ be a left continuous non-increasing function such that for some $u_0 > 0$*

$$\nu(\{\mathbf{x} : \|\mathbf{x}\|_\infty \ge u\}) \le \theta(u) \quad u \in (0, u_0). \tag{1.5}$$

*Put*

$$\kappa_\theta(t) = \int_0^t (\theta^{-1}(s) \wedge 1)\, ds + \sqrt{t}\left(\int_t^\infty ((\theta^{-1}(s))^2 \wedge 1)\, ds\right)^{\frac{1}{2}} \quad t \ge 0, \tag{1.6}$$

*where $\theta^{-1}$ is the right continuous inverse of $\theta$. Then*

$$\limsup_{n\to\infty} \frac{|X_n|}{\kappa_\theta(\log n)} < \infty \quad a.s. \tag{1.7}$$

*This asymptotic bound of $\mathbf{X}$ is sharp in the following sense. For every left continuous non-increasing function $\theta : (0,\infty) \mapsto [0,\infty)$ with $\int (u^2 \wedge 1)\,(-d\theta(u)) \in (0,\infty)$ there exists an infinitely divisible* *sequence* $\mathbf{X}$ *with Lévy measure* $\nu$ *satisfying (1.3), (1.4) and (1.5) such that*

$$\limsup_{n\to\infty} \frac{|X_n|}{\kappa_\theta(\log n)} > 0 \tag{1.8}$$

*with positive probability.*

In order to use (1.7) we need information about the asymptotic behavior of $\kappa_\theta$. This is possible, to a large extend, when $\theta$ is regularly varying function at zero. We will write $f(t) \asymp g(t)$ $(t \to \infty)$ when $C^{-1}g(t) \le f(t) \le Cg(t)$ for some positive constant $C$ and all $t$ sufficiently large.

**Lemma 2** *Suppose* $\theta$ *is a regularly varying function with index* $-\alpha$ *at zero. If* $\alpha < 1$ *then* $\kappa_\theta$ *is bounded. If* $\alpha = 1$ *then*

$$\kappa_\theta(t) \asymp \int_1^t \theta^{-1}(s)\,ds. \tag{1.9}$$

*If* $1 < \alpha < 2$ *then*

$$\kappa_\theta(t) \asymp t\theta^{-1}(t). \tag{1.10}$$

*If* $\alpha = 2$ *then*

$$\kappa_\theta(t) \asymp \left( t \int_t^\infty (\theta^{-1}(s))^2\,ds \right)^{\frac{1}{2}}. \tag{1.11}$$

*If* $\alpha > 2$*, then* $\kappa_\theta(t) = \infty$.

**Example 3** If $\theta(u) = u^{-\alpha}$ then $\kappa_\theta(t) \asymp t^{1-1/\alpha}$ when $1 < \alpha < 2$ and $\kappa_\theta(t) \asymp \log t$ when $\alpha = 1$. If $\theta(u) = u^{-\alpha}\ell(u^{-\alpha})$, where $\ell$ is slowly varying at infinity, then

$$\theta^{-1}(t) \asymp t^{-\frac{1}{\alpha}}(\ell^{\#}(t))^{-\frac{1}{\alpha}}.$$

Here $\ell^{\#}$ is de Bruijn conjugate of $\ell$ defined by

$$\ell(t)\ell^{\#}(t\ell(t)) \to 1, \qquad \ell^{\#}(t)\ell(t\ell^{\#}(t)) \to 1 \quad (t \to \infty)$$

(see [1], Proposition 1.5.15). Combining these facts with Lemma 2 we can evaluate the asymptotic behavior of $\kappa_\theta$ in many cases. For example, if $\theta(u) = u^{-\alpha}\prod_{k=1}^n (\log^{(k)}(u^{-1}))^{\beta_k}$ $(u \to 0)$, where $\log^{(k)}$ is the $k$–th iterate of log and $\beta_k \in \mathbb{R}$, then for $1 < \alpha < 2$

$$\kappa_\theta(t) \asymp t^{1-\frac{1}{\alpha}} \Big( \prod_{k=1}^n (\log^{(k)} t)^{\beta_k} \Big)^{\frac{1}{\alpha}}$$

(cf. [1], Appendix 5).

## 2 Rademacher series related to infinitely divisible distributions

Our first aim is to recall precise estimates for the tail of a Rademacher series. If $\mathbf{a} = (a_i)$ is a bounded sequence of real numbers then $\mathbf{a}^* = (a_i^*)$ will stand for the nonincreasing rearrangement of $(|a_i|)$ and $\|\mathbf{a}\|_p = (\sum_{i=1}^{\infty} |a_i|^p)^{1/p}$. Let $(\epsilon_i)$ denote the Rademacher sequence, i.e. a sequence of i.i.d. symmetric random variables taking on values $\pm 1$. Assume that $\|\mathbf{a}\|_2 < \infty$ and let $S = \sum_{i=1}^{\infty} \epsilon_i a_i$ denote the corresponding Rademacher series.

For each $t > 0$ we consider the following quasi norms of $\mathbf{a} = (a_i)$

$$M_{\mathbf{a}}(t) := \left(E|S|^t\right)^{\frac{1}{t}},$$

$$K_{\mathbf{a}}(t) := \inf\left\{\|a'\|_1 + \sqrt{t}\|a''\|_2 : \; a = a' + a''\right\}, \tag{2.1}$$

and

$$H_{\mathbf{a}}(t) := \sum_{i \le t} a_i^* + \sqrt{t}\left(\sum_{i>t} a_i^{*2}\right)^{\frac{1}{2}}, \tag{2.2}$$

Then the following inequlities hold for each $t > 0$ (see [3])

$$P(|S| > eM_{\mathbf{a}}(t)) \le e^{-t} \le 2e^2 P(|S| > M_{\mathbf{a}}(t))^{\frac{1}{2}} \tag{2.3}$$

and

$$\frac{1}{4} H_{\mathbf{a}}(t) \le M_{\mathbf{a}}(t) \le K_{\mathbf{a}}(t) \le H_{\mathbf{a}}(t). \tag{2.4}$$

Finally, let

$$\psi_{\mathbf{a}}(s) := K_{\mathbf{a}}^{-1}\left(\frac{s}{2e}\right). \tag{2.5}$$

Since $t \mapsto K_{\mathbf{a}}(t)$ is a concave function (as the infinimum of concave functions) and $K_{\mathbf{a}}(0) = 0$, the function $s \mapsto \psi_{\mathbf{a}}(s)$ is convex with $\psi_{\mathbf{a}}(0) = 0$. (Note that when $K_{\mathbf{a}}(\infty) < \infty$, $\psi_{\mathbf{a}}(s) = \infty$ for large $|s|$; this occurs if and only if $\|\mathbf{a}\|_1 < \infty$). (2.3)–(2.5) yield the following estimates on the tail of $S$:

$$(4e^4)^{-1} e^{-\psi_{\mathbf{a}}(8es)} \le P(|S| > s) \le e^{-\psi_{\mathbf{a}}(2s)}, \quad s > 0. \tag{2.6}$$

Now we consider a real symmetric infinitely divisible random variable $X$ with characteristic function given by

$$Ee^{iuX} = \exp\left(-\int_0^{\infty} (1 - \cos ux)\, \tau(dx)\right)$$

where $\tau$ is a Borel measure on $\mathbb{R}_+$ such that $\int_0^{\infty} (x^2 \wedge 1)\, \tau(dx) < \infty$. Let $\theta(x) := \tau([x, \infty))$, $x > 0$ and define

$$\theta^{-1}(t) := \inf\{x > 0 : \theta(x) \le t\}, \quad t > 0. \tag{2.7}$$

$\theta^{-1}$ is the right-continuous inverse of $\theta$; $\theta^{-1}$ is a nonincreasing function. Let $\{\Gamma_i\}$ be the sequence of arrival times in a Poisson process with rate 1 independent of the Rademacher sequence $\{\epsilon_j\}$. Then we have the following representation of $X$ in distribution

$$X \stackrel{d}{=} \sum_{i=1}^{\infty} \epsilon_i \theta^{-1}(\Gamma_i) \tag{2.8}$$

(see, e.g., [7] ). By the Strong Law of Large Numbers $i^{-1}\Gamma_i \to 1$ a.s.. Therefore, (2.8) is closely related to the Rademacher series

$$S := \sum_{i=1}^{\infty} \epsilon_i \theta^{-1}(i). \tag{2.9}$$

We want to express bounds for the tail of $S$ explicitly as a function of $\theta$. To this end we introduce the function

$$\kappa_\theta(t) = \int_0^t (\theta^{-1}(s) \wedge 1)\, ds + \sqrt{t} \left( \int_t^\infty ((\theta^{-1}(s))^2 \wedge 1)\, ds \right)^{\frac{1}{2}}, \quad t > 0 \tag{2.10}$$

and $\kappa_\theta(0) = 0$, as in (1.6). Equivalently,

$$\kappa_\theta(t) = \int_{\theta^{-1}(t)}^\infty (x \wedge 1)\, \tau(dx) + \sqrt{t} \left( \int_0^{\theta^{-1}(t)} (x^2 \wedge 1)\, \tau(dx) \right)^{\frac{1}{2}}.$$

The following estimates play the key role.

**Lemma 4** *Let* $\mathbf{a} = (\theta^{-1}(i))_{i \in \mathbb{N}}$ *and let* $\psi_{\mathbf{a}}$ *be given by (2.5). Then for all* $t > 0$

$$c\kappa_\theta(t) \le \psi_{\mathbf{a}}^{-1}(t) \le C\kappa_\theta(t) \tag{2.11}$$

*where* $c = \frac{e}{8}(\theta^{-1}(1) \wedge 1)$ *and* $C = 4e(\theta^{-1}(1) \vee 1)$.

**Proof** By (2.4) it is enough to estimate $H_{\mathbf{a}}$ in (2.2). For $t \ge 1$ we have

$$\begin{aligned} H_{\mathbf{a}}(t) &= \sum_{i \le t} \theta^{-1}(i) + \sqrt{t}\Big( \sum_{i > t} (\theta^{-1}(i))^2 \Big)^{\frac{1}{2}} \\ &\le \theta^{-1}(1) + \int_1^t \theta^{-1}(s)\, ds + \sqrt{t}\Big( \int_t^\infty (\theta^{-1}(s))^2\, ds + (\theta^{-1}(t))^2 \Big)^{\frac{1}{2}} \\ &\le \theta^{-1}(1) + \int_1^t \theta^{-1}(s)\, ds + \sqrt{t}\Big( \int_t^\infty (\theta^{-1}(s))^2\, ds \Big)^{\frac{1}{2}} + \sqrt{t}\theta^{-1}(t)). \end{aligned}$$

It is easy to verify the following bounds for $t \ge 1$

$$\theta^{-1}(1) + \int_1^t \theta^{-1}(s)\, ds \le (\theta^{-1}(1) \vee 1) \int_0^t (\theta^{-1}(s) \wedge 1)\, ds,$$

$$\int_t^\infty (\theta^{-1}(s))^2\, ds \le ((\theta^{-1}(1))^2 \vee 1) \int_t^\infty ((\theta^{-1}(s))^2 \wedge 1)\, ds$$

and

$$\sqrt{t}\theta^{-1}(t) \le t\theta^{-1}(t) \le (\theta^{-1}(1) \vee 1) \int_0^t (\theta^{-1}(s) \wedge 1)\, ds.$$

Hence we get

$$H_{\mathbf{a}}(t) \le 2(\theta^{-1}(1) \vee 1)\kappa_\theta(t) \quad \text{for } t \ge 1. \tag{2.12}$$

If $t<1$ then

$$H_{\mathbf{a}}(t) \;=\; \sqrt{t}\Big(\sum_{i=1}^{\infty}(\theta^{-1}(i))^2\Big)^{1/2} \le \sqrt{t}\Big(\int_1^{\infty}(\theta^{-1}(s))^2\,ds + (\theta^{-1}(1))^2\Big)^{\frac{1}{2}}.$$

We have the following elementary bounds for $t<1$

$$\int_1^{\infty}(\theta^{-1}(s))^2\,ds \le ((\theta^{-1}(1))^2\vee 1)\int_t^{\infty}((\theta^{-1}(s))^2\wedge 1)\,ds,$$

$$(\theta^{-1}(1))^2 \le 2((\theta^{-1}(1))^2\vee 1)\int_t^{\infty}((\theta^{-1}(s))^2\wedge 1)\,ds \quad \text{if } t\in(0,1/2],$$

and

$$\sqrt{t}\theta^{-1}(1) \le \sqrt{2}t\theta^{-1}(1) \le \sqrt{2}(\theta^{-1}(1)\vee 1)\int_0^t(\theta^{-1}(s)\wedge 1)\,ds \quad \text{if } t\in(1/2,1).$$

It follows that

$$H_{\mathbf{a}}(t) \le \sqrt{3}(\theta^{-1}(1)\vee 1)\kappa_\theta(t) \quad \text{if } t<1. \tag{2.13}$$

Combining (2.12) and (2.13) we prove the upper bound in (2.11).

Now we will prove the lower bound in (2.11). For $t<1$ we have

$$\begin{aligned}
\kappa_\theta(t) &= \int_0^t(\theta^{-1}(s)\wedge 1)\,ds + \sqrt{t}\left(\int_t^{\infty}((\theta^{-1}(s))^2\wedge 1)\,ds\right)^{\frac{1}{2}} \\
&\le t+\sqrt{t}\left(1+\sum_{i=1}^{\infty}(\theta^{-1}(i))^2\right)^{\frac{1}{2}} \le 2\sqrt{t}+H_{\mathbf{a}}(t) \\
&\le \left(\frac{2}{\theta^{-1}(1)}+1\right)H_{\mathbf{a}}(t).
\end{aligned}$$

For $t\ge 1$ we obtain

$$\begin{aligned}
\kappa_\theta(t) &\le 1+\sum_{i\le t}\theta^{-1}(i)+\sqrt{t}\left((\theta^{-1}(t))^2+\sum_{i>t}(\theta^{-1}(i))^2\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{\theta^{-1}(1)}+1\right)\sum_{i\le t}\theta^{-1}(i)+\sqrt{t}\left(\sum_{i>t}(\theta^{-1}(i))^2\right)^{\frac{1}{2}}+\sqrt{t}\theta^{-1}(t).
\end{aligned}$$

Since $t\ge 1$, we get $\sum_{i\le t}\theta^{-1}(i)\ge \frac{t}{2}\theta^{-1}(t)\ge \frac{\sqrt{t}}{2}\theta^{-1}(t)$. Hence, for $t\ge 1$ we have

$$\kappa_\theta(t) \le \left(\frac{1}{\theta^{-1}(1)}+3\right)H_{\mathbf{a}}(t).$$

Using (2.4) we obtain the lower bound in (2.11). The proof of Lemma 4 is complete. □

**Lemma 5** (i). *For every $k>0$ there exists a constant $L$ such that*

$$\kappa_\theta(kt) \le L\kappa_\theta(t) \quad t\ge 0.$$

(ii). *Suppose that $\theta_1(u)\le k\theta_2(u)$ for all $u\in(0,u_0)$ and some $k,u_0>0$. Then there exists a constant $L$ such that*

$$\kappa_{\theta_1}(t) \le L\kappa_{\theta_2}(t) \quad t\ge 0.$$

**Proof** *(i).* If $k \le 1$, then by (2.11) and the fact that $\psi_{\mathbf{a}}^{-1}$ is nonincreasing, $\kappa_\theta(kt) \le c^{-1}\psi_{\mathbf{a}}^{-1}(kt) \le c^{-1}\psi_{\mathbf{a}}^{-1}(t) \le Cc^{-1}\kappa_\theta(t)$. If $k > 1$, then by the concavity of $\psi_{\mathbf{a}}^{-1}$, $\psi_{\mathbf{a}}^{-1}(0) = 0$, and (2.11) we have

$$\kappa_\theta(kt) \le c^{-1}\psi_{\mathbf{a}}^{-1}(kt) \le c^{-1}k\psi_{\mathbf{a}}^{-1}(t) \le Cc^{-1}k\kappa_\theta(t).$$

*(ii).* Put $\theta_3 = k\theta_2$. We have $\theta_1^{-1}(t) \le \theta_3^{-1}(t)$ for all $t > \theta_3(u_0)$ and $\theta_1^{-1}(t) \wedge 1 \le (\theta_3^{-1}(t) \wedge 1)/(u_0 \wedge 1)$ for all $t \le \theta_3(u_0)$. Using *(i)* and the definition of $\kappa_\theta$ we have

$$\kappa_{\theta_1}(t) \le (u_0 \wedge 1)^{-1}\kappa_{\theta_3}(t) = (u_0 \wedge 1)^{-1}k\kappa_{\theta_2}(k^{-1}t) \le L\kappa_{\theta_2}(t).$$

□

## 3 Proof of Theorem 1 and some applications

**Proof of the Theorem 1:** Let $N$ be a Poisson random measure on $\mathbb{R}^{\mathbb{N}}$ with intensity measure $\nu$. Define $f_n : \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}$ by $f_n(\mathbf{x}) = x_n$ and $f(\mathbf{x}) := \sup_n f_n(\mathbf{x}) = \|\mathbf{x}\|_\infty$. Put

$$X_n' = \int_{\mathbb{R}^{\mathbb{N}}} f_n(\mathbf{x})\,(N(d\mathbf{x}) - (|f_n(\mathbf{x})| \vee 1)^{-1}\nu(d\mathbf{x})) \quad n \in \mathbb{N}.$$

The sequence $\mathbf{X}' = (X_n')$ is well defined and its characteristic function is the same as of $\mathbf{X}$. Therefore, we may further assume that $\mathbf{X}' = \mathbf{X}$. Then we decompose $X_n$ as folows

$$X_n = a_n + \int_{f \le 1} f_n\, d(N - \nu) + \int_{f>1} f_n\, dN$$

where

$$a_n = -\int_{f>1} f_n(|f_n| \vee 1)^{-1}\, d\nu.$$

From (1.4) $\nu(f > 1) < \infty$. Hence

$$\sup_n |a_n| \le \nu(f > 1) < \infty$$

and

$$\sup_n \Big|\int_{f>1} f_n\, dN\Big| \le \int_{f>1} |f_n|\, dN \le \int_{f>1} f\, dN < \infty \quad a.s$$

because the last integral is a finite sum. Therefore, the asymptotic behavior of $X_n$ is determined by

$$Y_n := \int_{f\le 1} f_n\, d(N - \nu).$$

From now on we may and do assume that $\nu$ is concentrated on $\{f \le 1\}$, that is, all $f_n$'s are uniformly bounded by 1.

Let $Y_n' := \int f_n\, d(N' - \nu)$, where $N'$ is an independent copy of $N$. Let $\widetilde{N} = N - N'$ be the symmetrization of $N$ and let

$$\widetilde{Y}_n = Y_n - Y_n' = \int f_n\, d\widetilde{N} \tag{3.1}$$

be the corresponding symmetrization of $Y_n$. We will use series representation of $(\widetilde{Y}_n)$ given in [4], Section 5.1. Namely, let $\sum_i \delta_{s_i}$ be a Poisson point process on $S := \mathbb{R}^{\mathbb{N}}$ with intensity measure $2\nu$ independent of the Rademacher sequence $\{\epsilon_i\}$. Then $\widetilde{N} \stackrel{d}{=} \sum_i \epsilon_i \delta_{s_i}$ and $(\widetilde{Y}_n)_{n\in\mathbb{N}} \stackrel{d}{=} \left(\sum_i \epsilon_i f_n(s_i)\right)_{n\in\mathbb{N}}$. Without loss of generality, we may and do assume that

$$\widetilde{Y}_n = \sum_{i=1}^{\infty} \epsilon_i f_n(s_i).$$

Put

$$\widetilde{\theta}(u) = 2\nu(f \geq u) \quad u > 0$$

and let $\widetilde{\theta}^{-1}$ denote the right continuous inverse of $\widetilde{\theta}$ given by (2.7). Recall that $(a_i^*)_{i\in\mathbb{N}}$ stands for the nonincreasing rearrangement of any sequence $(|a_i|)_{i\in\mathbb{N}}$. Let

$$T = \sup\{i \geq 1 : f^*(s_i) > u_i\} \tag{3.2}$$

where

$$u_i := \widetilde{\theta}^{-1}\left(\frac{1}{2}i\right).$$

Write

$$\widetilde{Y}_n \stackrel{d}{=} \sum_{i\leq T} \epsilon_i f_n^*(s_i) + \sum_{i>T} \epsilon_i f_n^*(s_i) =: V_n + W_n. \tag{3.3}$$

Since $|f_n(s_i)| \leq f(s_i)$, we have $f_n^*(s_i) \leq f^*(s_i) \leq 1$ ($f \leq 1$ by our assumption). Therefore

$$|V_n| \leq T f^*(s_1) \leq T. \tag{3.4}$$

We will show that $T$ has some exponential moment finite, so that $T < \infty$ a.s. Observe that for every $k \geq 1$

$$T \geq k \iff \exists\, i \geq k : \mathrm{Card}\{j : f(s_j) > u_i\} \geq i.$$

Since $\sum_i \delta_{s_i}$ is a Poisson point process with intensity $2\nu$, $\sum_j \delta_{f(s_j)}$ is a Poisson point process on $\mathbb{R}_+$ with intensity measure $\mu = (2\nu) \circ f^{-1}$. Therefore, for each Borel set $A \subset \mathbb{R}_+$

$$\mathrm{Card}\{j : f(s_j) \in A\} \stackrel{d}{=} M(\mu(A)),$$

where $M(t)$, $t \geq 0$ is the usual Poisson process with parameter 1. Hence

$$\begin{aligned} P(T \geq k) &\leq \sum_{i=k}^{\infty} P\left(M(\mu(u_i, \infty)) \geq i\right) = \sum_{i=k}^{\infty} P\left(M(\widetilde{\theta}(u_i+)) \geq i\right) \\ &\leq \sum_{i=k}^{\infty} P\left(M\left(\frac{1}{2}i\right) \geq i\right) \end{aligned}$$

because $\widetilde{\theta}(\widetilde{\theta}^{-1}(t)+) \leq t$. Applying a large deviations estimate $P\left(|\frac{M(t)}{t} - 1| > \frac{1}{3}\right) \asymp \exp(-\alpha t)$ for certain $\alpha > 0$ as $t \to \infty$, we get that

$$Ee^{\beta T} < \infty \quad \text{for some } \beta > 0. \tag{3.5}$$

Now we consider $W_n$ in (3.3). Using the contraction principle conditionally $\{s_i\}$ we obtain

$$\begin{aligned} P_\epsilon\left(|W_n|>t\right) &\leq 2P_\epsilon\left(\Big|\sum_{i>T}\epsilon_i f^*(s_i)\Big|>t\right)\leq 4P_\epsilon\left(\Big|\sum_{i=2}^{\infty}\epsilon_i\widetilde{\theta}^{-1}(\frac{i}{2})\Big|>t\right)\\ &\leq 4P_\epsilon\left(\Big|\sum_{i=1}^{\infty}\epsilon_i\widetilde{\theta}^{-1}(i)\Big|>\frac{t}{2}\right)+4P_\epsilon\left(\Big|\sum_{i=1}^{\infty}\epsilon_i\widetilde{\theta}^{-1}(i+\frac{1}{2})\Big|>\frac{t}{2}\right)\\ &\leq 12P_\epsilon\left(\Big|\sum_{i=1}^{\infty}\epsilon_i\widetilde{\theta}^{-1}(i)\Big|>\frac{t}{2}\right). \end{aligned}$$

Hence

(3.6) $$P\left(|W_n|>t\right)\leq 12P\left(\Big|\sum_{i=1}^{\infty}\epsilon_i\widetilde{\theta}^{-1}(i)\Big|>\frac{t}{2}\right)\qquad t>0.$$

Using (3.6) and (2.6) for $\mathbf{a}=(\widetilde{\theta}^{-1}(i))_{i\in\mathbb{N}}$ we get for every $x>0$

$$\begin{aligned} P\left(\sup_{n\geq 2}\frac{|W_n|}{\kappa_{\widetilde{\theta}}(\log n)}>x\right) &\leq \sum_{n=2}^{\infty}P\left(|W_n|>x\kappa_{\widetilde{\theta}}(\log n)\right)\\ &\leq 12\sum_{n=2}^{\infty}P\left(\Big|\sum_{i=1}^{\infty}\epsilon_i\widetilde{\theta}^{-1}(i)\Big|>\frac{x}{2}\kappa_{\widetilde{\theta}}(\log n)\right)\\ &\leq 12\sum_{n=2}^{\infty}\exp\left(-\psi_{\mathbf{a}}(x\kappa_{\widetilde{\theta}}(\log n))\right). \end{aligned}$$

Applying (2.11) we get

$$\psi_{\mathbf{a}}\left(x\kappa_\theta(\log n)\right)\geq\psi_{\mathbf{a}}\left(\frac{x}{C}\psi_{\mathbf{a}}^{-1}(\log n)\right)\geq\frac{x}{C}\log n.$$

The last inequality follows from the concavity of $\psi_{\mathbf{a}}^{-1}$, provided $x\geq C$. Consequently,

$$P\left(\sup_{n\geq 2}\frac{|W_n|}{\kappa_{\widetilde{\theta}}(\log n)}>x\right)\leq\sum_{n=2}^{\infty}n^{-x/C}$$

for $x>C$. In particular, this shows that

(3.7) $$\mathbb{E}\left(\sup_{n\geq 2}\frac{|W_n|}{\kappa_{\widetilde{\theta}}(\log n)}\right)<\infty,$$

which together with (3.3), (3.4) and (3.5) yields

$$\mathbb{E}\left(\sup_{n\geq 2}\frac{|\widetilde{Y}_n|}{\kappa_{\widetilde{\theta}}(\log n)}\right)<\infty.$$

Recall (3.1), i.g., $\widetilde{Y}_n$ is the symmetrization of $Y_n$ and $\mathbb{E}Y_n=0$. Hence

$$\mathbb{E}\left(\sup_{n\geq 2}\frac{|Y_n|}{\kappa_{\widetilde{\theta}}(\log n)}\right)=\mathbb{E}\left(\sup_{n\geq 2}\frac{|Y_n-\mathbb{E}_Y(Y_n')|}{\kappa_{\widetilde{\theta}}(\log n)}\right)\leq\mathbb{E}\left(\sup_{n\geq 2}\frac{|\widetilde{Y}_n|}{\kappa_{\widetilde{\theta}}(\log n)}\right)<\infty.$$

Combining this with the first part of the proof we get

$$\sup_{n\geq 2}\frac{|X_n|}{\kappa_{\widetilde{\theta}}(\log n)}<\infty. \tag{3.8}$$

By (1.5) we have $\widetilde{\theta}(u)\leq 2\theta(u)$ for $u<u_0$. Since, by Lemma 5(ii), $\kappa_{\widetilde{\theta}}(t)\leq L\kappa_\theta(t)$, (3.8) implies (1.7).

Now we will prove the optimality of the sequence $b_n=\kappa_\theta(\log n)$. Let $\theta$ be fixed and let $\tau$ be a measure on $\mathbb{R}_+$ with $\tau([u,\infty))=\theta(u)$. Let $(\epsilon_{in})_{i,n\in\mathbb{N}}$ be an array of independent symmetric random variables taking values $\pm 1$. Consider

$$X_n:=\sum_{i=1}^{\infty}\epsilon_{in}\theta^{-1}(\Gamma_i), \tag{3.9}$$

where $(\Gamma_i)$ are arrival times in a Poisson process with rate 1 independent of $(\epsilon_{in})$. It is well known that given the condition on $\theta$, series (3.9) converges a.s., defining an infinitely divisible sequence $\mathbf{X}=(X_n)$ (see, e.g. [7]). The Lévy measure $\nu$ of $\mathbf{X}$ is the push-forward of the product of $\tau$ and $\left(\frac{1}{2}\delta_{-1}+\frac{1}{2}\delta_1\right)^{\otimes\mathbb{N}}$ by the map $(u,\mathbf{x})\mapsto u\mathbf{x}$. Therefore,

$$\nu(\mathbf{x}:\|\mathbf{x}\|_\infty\geq u)=\theta(u).$$

By the first part of Theorem 1,

$$\limsup_{n\to\infty}\frac{|X_n|}{b_n}<\infty\quad a.s. \tag{3.10}$$

Using the mentioned above large deviation estimate for a Poisson process we infer that there exists a constant $d>1$ such that the event

$$A=\{\Gamma_i\leq di,\ \forall i\in\mathbb{N}\}$$

has positive probability. Consider an i.i.d. sequence

$$S_n:=\sum_{i=1}^{\infty}\epsilon_{in}\theta^{-1}(di).$$

We will show that there exists a constant $L>0$ such that

$$\limsup_{n\to\infty}\frac{|S_n|}{b_n}\geq L\quad a.s. \tag{3.11}$$

Indeed, by (2.6) for $\mathbf{a}=\left(\theta^{-1}(di)\right)_{i\in\mathbb{N}}$

$$\sum_n P(|S_n|>Lb_n)=\sum_n P(|S_1|>Lb_n)\geq(4e^4)^{-1}\sum_n\exp\left(-\psi_{\mathbf{a}}(8eLb_n)\right).$$

By Lemmas 5 and 4

$$b_n=\kappa_\theta(\log n)\leq L_1\kappa_{d^{-1}\theta}(\log n)\leq L_1c^{-1}\psi_{\mathbf{a}}^{-1}(\log n).$$

Hence

$$\sum_n P(|S_n|>Lb_n)\geq\sum_n\exp\left(-\psi_{\mathbf{a}}(8eLL_1c^{-1}\psi_{\mathbf{a}}^{-1}(\log n))\right)=\sum_n\frac{1}{n}=\infty$$

when $L = (8eL_1)^{-1}c$. By Borel–Cantelli Lemma (3.11) holds.

Fix positive integers $m > k > 1$ and let $(\mathbf{e}_n)$ denote the standard basis in $l_\infty^m$. We have

$$\sup_{k \le n \le m} \frac{|S_n|}{b_n} = \left\| \sum_{n=k}^{m} b_n^{-1} S_n \mathbf{e}_n \right\|_\infty = \left\| \sum_{i=1}^{\infty} \theta^{-1}(di)\, \mathbf{y}_i \right\|_\infty$$

where

$$\mathbf{y}_i = \sum_{n=k}^{m} b_n^{-1} \epsilon_{in} \mathbf{e}_n.$$

$(\mathbf{y}_i)$ are independent symmetric random vectors with $\|y_i\|_\infty = b_k^{-1}$. By the contraction principle we have

$$\begin{aligned} P\left( \sup_{k \le n \le m} \frac{|S_n|}{b_n} \ge L \right) &= P\left( \left\| \sum_{i=1}^{\infty} \theta^{-1}(di)\, \mathbf{y}_i \right\|_\infty \ge L \mid A \right) \\ &\le 2P\left( \left\| \sum_{i=1}^{\infty} \theta^{-1}(\Gamma_i)\, \mathbf{y}_i \right\|_\infty \ge L \mid A \right) \\ &= 2P\left( \sup_{k \le n \le m} \frac{|X_n|}{b_n} \ge L \mid A \right). \end{aligned}$$

Letting $m \to \infty$ and then $k \to \infty$ and applying (3.11) we get

$$1 \le 2P\left( \limsup_{n \to \infty} \frac{|X_n|}{b_n} \ge L \mid A \right).$$

This proves (1.8) and completes the proof of Theorem 1. □

**Remark 6** Somewhat more explicit but equivalent form of the sequence $(X_n)$ constructed in the second part of the theorem is as follows. Let $Z(t)$, $t \ge 0$ be a symmetric Lévy process with Lévy measure

$$\Pi_Z(x : |x| \ge u) = \theta(u).$$

Let $h_n(t) = \text{sign}\,(\sin(2^n t))$ be the Rademacher functions. Then

$$X_n = \int_0^1 h_n(t)\, dZ(t).$$

If $\Pi_Z(\mathbb{R}) = \infty$, then a 0-1 law for subspaces of paths together with (1.8) imply that

$$\limsup_{n \to \infty} \frac{|X_n|}{\kappa_\theta(\log n)} > 0 \quad a.s.$$

(see, e.g., [8]). If also $\int_{|x| \le 1} |x|\, \Pi_Z(dx) = \infty$, then $\kappa_\theta(\log n) \to \infty$. In this case one can show by the Hewitt-Savage 0-1 law applied to the representation (3.9) that

$$\limsup_{n \to \infty} \frac{|X_n|}{\kappa_\theta(\log n)} = \text{Constant} > 0 \quad a.s.$$

**Proof of Lemma 2:** Notice that

$$\kappa_\theta(t) = \theta(1) + \int_{\theta(1)}^{t} \theta^{-1}(s)\,ds + \sqrt{t}\left(\int_t^\infty (\theta^{-1}(s))^2\,ds\right)^{\frac{1}{2}} \quad t \geq \theta(1). \tag{3.12}$$

The function $\theta^{-1}(t)$ is $(\frac{-1}{\alpha})$–regulary varying and $(\theta^{-1}(t))^2$ is $(\frac{-2}{\alpha})$–regulary varying at infinity, see [1], Proposition 1.5.15, page 29. By Karamata Theorem (see [1], Theorem 1.5.15 page 28) we obtain that

$$\int_1^t \theta^{-1}(s)\,ds \asymp t\theta^{-1}(t) \quad \text{if } \alpha > 1 \tag{3.13}$$

and

$$\int_t^\infty (\theta^{-1}(s))^2\,ds \asymp t(\theta^{-1}(t))^2 \quad \text{if } \alpha < 2. \tag{3.14}$$

This and (3.12) imply (1.10). If $\alpha = 1$, then (3.14) and $\int_1^t \theta^{-1}(s)\,ds \geq \frac{1}{2}t\theta^{-1}(t)$ for $t > 2$ yield (1.9). If $\alpha = 2$, then (3.13) and $\int_t^\infty (\theta^{-1}(s))^2\,ds \geq t(\theta^{-1}(2t))^2 > \frac{1}{3}t(\theta^{-1}(t))^2$ for $t$ large enough yield (1.11). The cases $\alpha < 1$ and $\alpha > 2$ are obvious. □

Now we will return to our original question concerning infinitely divisible sequences defined by a stochastic integral.

**Example 7** Let $X(t)$, $t \in \mathbb{R}$ be a stochastic process as in (1.1)–(1.2), where $Z$ is a Lévy process with no Gaussian part. For simplicity we assume that $\mathbb{E}|Z(1)| < \infty$ with $\mathbb{E}Z(1) = 0$ or $Z(1)$ is symmetric. Then we write

$$X(t_n) = b_n + X_n$$

where $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ satisfies (1.3) with $\nu$ being the push-forward of the product of $\Pi_Z$ and the Lebesgue measure on $\mathbb{R}$ by the map $(s, x) \mapsto (xg(t_n, s))_{n\in\mathbb{N}}$ and

$$b_n = -\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sign}(vg(t_n, s)) I(|vg(t_n, s)| > 1)\, \Pi_Z(dv)dt$$

or $b_0 = 0$ when $Z$ is symmetric (cf. [6]).

By (1.2) sequence $(b_n)_{n\in\mathbb{N}}$ is bounded. Define

$$\theta(u) = \int_{\mathbb{R}} \Pi_Z(\{v \in \mathbb{R} : |v| \sup_n |g(t_n, s)| \geq u\})\,ds \tag{3.15}$$

for small $u > 0$, otherwise $\theta$ can be arbitrary. By Theorem 1

$$\limsup_{n\to\infty} \frac{|X(t_n)|}{\kappa_\theta(\log n)} < \infty \quad a.s. \tag{3.16}$$

For example, if $Z$ is an $\alpha$-stable process with $\Pi_Z(dv) = (a|v|^{-\alpha-1}I(v < 0) + bv^{-\alpha-1}I(v > 0))dv$, then condition (1.2) becomes

$$c := \int_{\mathbb{R}} \sup_n |g(t_n, s)|^\alpha\,ds < \infty$$

and (3.15) gives

$$\theta(u) = \alpha^{-1}(a+b)c\,u^{-\alpha}.$$

Consider $1 \le \alpha < 2$ (assume $\mathbb{E}Z(1) = 0$ when $1 < \alpha < 2$ and $Z(1)$ symmetric when $\alpha = 1$). Then by Example 3 and (3.16)

$$\limsup_{n\to\infty} \frac{|X(t_n)|}{(\log n)^{1-\frac{1}{\alpha}}} < \infty \quad a.s. \quad \text{when } 1 < \alpha < 2 \tag{3.17}$$

and

$$\limsup_{n\to\infty} \frac{|X(t_n)|}{\log(\log n)} < \infty \quad a.s. \quad \text{when } \alpha = 1. \tag{3.18}$$

(3.17)-(3.18) were obtained by Braverman [2] for symmetric stable processes. Observe that smaller value of $\alpha$ produces lesses variability in $(X(t_n))$.

Naturally, one may consider more general infinitely divisible processes defined by stochastic integral with respect to an infinitely divisible random measure. The conclusion of Theorem 1 does not depend on a representation of an infinitely divisible process.

**Example 8** Let $M$ be an independently scattered divisible random measure on $\mathbb{R}$ with finite control measure $m$ and no Gaussian part. Assume that $M$ is symmetric and

$$\mathbb{E}e^{iuM(A)} = e^{-m(A)\psi(u)},$$

where $\psi(u) = \int_0^\infty (1-\cos uv)\,\Pi(du)$ with $\Pi$ being a Lévy measure. Consider a harmonizable process

$$X(t) = \int_{\mathbb{R}} e^{itu}\, M(du) \quad t \in \mathbb{R}.$$

Condition (1.4) of Theorem 1 holds for the real and imaginary parts of $X(t_n)$ *for any sequence* $(t_n)_{n\in\mathbb{N}}$. In this case we take

$$\theta(u) = m(\mathbb{R})\Pi([u,\infty))$$

and conclude that

$$\limsup_{n\to\infty} \frac{|X(t_n)|}{\kappa_\theta(\log n)} < \infty \quad a.s. \tag{3.19}$$

Given that $X(t_n)$ may have heavy tails and the sequence $\kappa_\theta(\log n)$ may converge to infinity very slowly (as in the case of 1-stable processes, $\kappa_\theta(\log n) = \log(\log n)$), (3.19) indicates that the sequence $(X(t_n))$ will be strongly dependent for any sequence $(t_n)_{n\in\mathbb{N}}$. This reinforces the fact that the process $(X(t))$ is never ergodic, see [5].

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