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***Abstract*—The rate balancing problem is considered in the multiple-input single-output (MIMO) broadcast channel with linear transceivers. After introducing relative per-stream rate targets as auxiliary variables, the optimization is performed by a gradient projection algorithm. In addition, the transmit and receive filters are updated and the mean square error matrices are diagonalized in each iteration. Although the algorithm can only find local optima of the considered non-convex optimization problem, numerical simulations show that it achieves good performance while it has a lower complexity than a comparable existing algorithm.**

***Index Terms*—MIMO systems, broadcast channels, linear transceivers, quality of service, gradient methods.**

I. Introduction

Many publications on multiple-input multiple-output (MIMO) broadcast channels study the problem of maximizing the (possibly weighted) sum of the rates of the individual users in the system (e.g., [1], [2], [3], [4], [5], [6]). The drawback of this optimization criterion is that some users might be served at very low rates, and there is even no guarantee that they are served at all.

Thus, we consider the problem of maximizing the throughput in a MIMO broadcast channel subject to a power constraint and to the constraint that the rates of the different users need to have certain fixed ratios. In the literature, this problem is called *rate balancing* (e.g., [7]). Interpreted in a graphical manner, the solution is the intersection of the Pareto boundary of the rate region with a straight line whose direction is determined by the relative rate constraints (e.g. [8]).

For MIMO broadcast channels with nonlinear dirty paper coding (DPC), optimal and close-to-optimum solutions to the rate balancing problem were proposed, e.g., in [7], [8], but these solutions cannot be applied to systems with linear precoding as they rely on the special properties of the interference pre-compensation achieved by DPC.

For the special case of single receive antennas, the rate balancing problem with linear precoding can be solved in a globally optimal manner, e.g., with a slightly modified version of the balancing algorithm from [9] or, in case of equal rates, using the signal-to-inferference-and-noise ratio (SINR) balancing algorithm from [10]. However, in the general case where users might have multiple receive antennas and multiple data streams, the globally optimal solution for the linear precoding case is unknown since the non-concave rate equations make the optimization problem non-convex.

A suboptimal solution was proposed in [11], [12]. However, this method has a high complexity as it is based on a geometric programming (GP) formulation. The mean-square-error-based (MSE) balancing method from [9] can be adapted such that it can be applied to the rate balancing problem in MIMO broadcast channels, but it will not be further considered in this paper since it does not achieve a satisfying performance.

In [13], the rate balancing problem was considered in a MIMO OFDM system with the restriction of exclusive assignment of subcarriers to users. Due to this restriction, the problem is not comparable to rate balancing in a MIMO broadcast channel where several users are sharing a single carrier. Another related approach that cannot be applied in our setup is the SINR balancing algorithm from [2], which balances the per-stream SINRs instead of the per-user rates.

In order to provide a good solution with reasonable complexity for the rate balancing problem in a general MIMO broadcast channel with linear precoding, we propose a method based on a gradient projection update of additionally introduced relative per-stream rate targets and on alternating filter updates in the uplink and in the downlink. After introducing the system model and stating the mathematical problem formulation in Section II, we will present the various ingredients of the algorithm in the Sections III through VI. In the derivation of the algorithm, it will become clear that certain parts are similar to the respective steps of the power minimization algorithm proposed in our companion work [14] due to the close relationship between the two considered problems. Though, other parts have to be redeveloped for the rate balancing problem considered here. Finally, the algorithm is summarized in Section VII along with some comments on the convergence and the initialization, and the paper is rounded off with some numerical results, comments on computational complexity, and concluding remarks in the Sections VIII and IX.

Notation

Vectors are typeset in boldface lowercase letters and matrices in boldface uppercase letters. We write $\bullet^{\mathrm{T}}$ for the transpose of a vector or matrix, $\bullet^{\mathrm{H}}$ for the conjugate transpose, $\mathbf{0}$ for the zero matrix or vector, and $\mathbf{I}_{\bullet}$ for the identity matrix of size $\bullet$. The vector $\mathbf{1}$ is the all-ones vector, and the vector $\boldsymbol{e}_i$ is the i-th canonical unit vector, which has a one as the i-th entry and zeros elsewhere. $[\boldsymbol{A}]_{i,j}$ is used to denote the element in the i-th row and j-th column of the matrix $\boldsymbol{A}$. $|\bullet|$ is used for the cardinality of a set, and $\delta_{i,j}$ is the Kronecker delta, which is 1

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whenever $i = j$ and 0 otherwise. The order relation $\boldsymbol{x} \geq \boldsymbol{y}$ has to be understood element-wise, and $\mathbb{R}^n_{0,+}$ is the closed positive orthant of the $\mathbb{R}^n$, i.e., $\mathbb{R}^n_{0,+} = \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{x} \geq \boldsymbol{0}\}$. We use the shorthand notation $(\bullet_k)_{k=1}^K$ for $[\bullet_1^{\mathrm{T}}, \ldots, \bullet_K^{\mathrm{T}}]^{\mathrm{T}}$.

II. System Model and Problem Formulation

We consider a downlink system with an M-antenna base station, K receivers with N_k receive antennas at the k-th receiver, and frequency flat channels $\boldsymbol{H}_k^{\mathrm{H}} \in \mathbb{C}^{N_k \times M}$, which are assumed to be perfectly known. The aim of rate balancing is to maximize the throughput of a multi-user communication system while keeping certain fixed ratios between the data rates of the individual users. Moreover, a certain transmit power limitation P has to be respected. Introducing an auxiliary variable R_0, this optimization problem can be written in an abstract form as

$$\max\ R_0 \quad \text{s.t.:} \quad r_k \geq R_0 \varrho_k\ \forall k \ \text{ and } \ \boldsymbol{r} \in \mathcal{R}(P), \tag{1}$$

where the elements of $\boldsymbol{r} = [r_1, \ldots, r_K]^{\mathrm{T}}$ are the per-user rates, and $\mathcal{R}(P)$ is the rate region for sum power P, i.e., the set of all rate vectors $\boldsymbol{r}$ that can be achieved in the considered communication system with a sum power not exceeding P. Although the rate balancing problem has a nice graphical interpretation as the intersection of the line defined by the coefficients ϱ_k and the Pareto boundary of $\mathcal{R}(P)$, the actual solution cannot be found in such an abstract manner because the rate region cannot be easily parametrized. Instead, we will perform the optimization with respect to transmit and receive filters and transmit powers.

Partitioning the information for user k into $S_k \leq \min\{M, N_k\}$ separately coded data streams, we can describe the system by

$$\hat{\boldsymbol{x}}_k = \boldsymbol{V}_k^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \sum_{k'=1}^{K} \boldsymbol{B}_{k'} \operatorname{diag}\{\boldsymbol{p}_{k'}\}^{\frac{1}{2}} \boldsymbol{x}_{k'} + \boldsymbol{V}_k^{\mathrm{H}} \boldsymbol{\eta}_k, \tag{2}$$

where $\boldsymbol{x}_k \in \mathbb{C}^{S_k}$ contains the unit-power data symbols of user k, $\boldsymbol{p}_k \in \mathbb{R}^{S_k}$ is the vector of the corresponding powers, the columns $\boldsymbol{b}_k^{(s)} \in \mathbb{C}^M$ of the matrix $\boldsymbol{B}_k \in \mathbb{C}^{M \times S_k}$ are the unit norm beamforming vectors, and the matrices $\boldsymbol{V}_k^{\mathrm{H}} \in \mathbb{C}^{S_k \times N_k}$ are the receive filters. Note that adding receive filters is not a necessity, but rather a vehicle that is used to allow a stream-wise formulation. The same is true for the fact that we describe the transmission by beamforming vectors and transmit powers instead of using the covariance matrices $\boldsymbol{C}_k = \boldsymbol{B}_k \operatorname{diag}\{\boldsymbol{p}_k\} \boldsymbol{B}_k^{\mathrm{H}}$ of the transmitted signals. As long as the receive filters provide a sufficient statistic for the received data, the formulation in (2) is without loss of generality.

The additive circularly symmetric Gaussian noise $\boldsymbol{\eta}_k \in \mathbb{C}^{N_k}$ in (2) is assumed to be white with unit variance, i.e., $\boldsymbol{\eta}_k \sim \mathcal{CN}(\boldsymbol{0}, \mathbf{I}_{N_k})$. This assumption is without loss of generality since in case of correlated noise $\check{\boldsymbol{\eta}}_k$, a pre-whitening filter $\check{\boldsymbol{V}}_k^{\mathrm{H}} = \boldsymbol{C}_{\check{\boldsymbol{\eta}}_k}^{-\frac{1}{2}}$ could be applied at the k-th receiver, where $\boldsymbol{C}_{\check{\boldsymbol{\eta}}_k}^{\frac{1}{2}}$ is a square matrix that fulfills $\boldsymbol{C}_{\check{\boldsymbol{\eta}}_k}^{\frac{1}{2}} \boldsymbol{C}_{\check{\boldsymbol{\eta}}_k}^{\frac{1}{2},\mathrm{H}} = \boldsymbol{C}_{\check{\boldsymbol{\eta}}_k}$. Considering this filter as a part of the channel, we get an equivalent system with white noise that can be described by (2) with $\boldsymbol{H}_k^{\mathrm{H}}$ being the composition of the actual channel and the pre-whitening filter in this case.

To deal with the S_k data streams of each user k, we introduce the relative per-stream rate targets $\rho_k^{(s)}$ as auxiliary variables, and we write the problem as

$$\max\ R_0 \quad \text{s.t.:}\ \boldsymbol{r}_k \geq R_0 \boldsymbol{\rho}_k\ \forall k \ \text{ and } \ (\boldsymbol{r}_k)_{k=1}^K \in \mathcal{R}'(P) \quad \text{and} \quad \sum_{s=1}^{S_k} \rho_k^{(s)} = \varrho_k\ \forall k, \tag{3}$$

where $\boldsymbol{r}_k = [r_k^{(1)}, \ldots, r_k^{(S_k)}]^{\mathrm{T}}$ is the vector of per-stream rates of user k, and $\boldsymbol{\rho}_k = [\rho_k^{(1)}, \ldots, \rho_k^{(S_k)}]^{\mathrm{T}}$ contains the per-stream rate targets. The achievable rate of each stream is given by

$$r_k^{(s)} = \log_2\left(1 + \gamma_k^{(s)}\right) \quad \text{with} \tag{4}$$

$$\gamma_k^{(s)} = \frac{p_k^{(s)} \left| \boldsymbol{v}_k^{(s),\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{b}_k^{(s)} \right|^2}{\boldsymbol{v}_k^{(s),\mathrm{H}} \boldsymbol{v}_k^{(s)} + \sum_{\substack{k'=1 \\ (k',s') \neq (k,s)}}^{K} \sum_{s'=1}^{S_{k'}} p_{k'}^{(s')} \left| \boldsymbol{v}_k^{(s),\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{b}_{k'}^{(s')} \right|^2}.$$

These per-stream rates can be considered as the rates of the $S_{\text{tot}} = \sum_{k=1}^K S_k$ virtual users of an effective multiple-input single-output (MISO) broadcast channel with channel vectors $\tilde{\boldsymbol{h}}_k^{(s),\mathrm{H}} = \boldsymbol{v}_k^{(s),\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}}$. The rate region of this effective MISO system is denoted by $\mathcal{R}'(P)$. The concept of optimizing over newly introduced relative per-stream rate targets is similar to the use of absolute per-stream rate targets in our companion work [14], where the power minimization problem was investigated.

The optimization variables in problem (3) are the receive filters $\boldsymbol{v}_k^{(s)}$, the unit norm beamforming vectors $\boldsymbol{b}_k^{(s)}$, the transmit powers $p_k^{(s)}$, the per-stream rate targets $\rho_k^{(s)}$, and the auxiliary variable R_0. The main challenge is to optimize the filters as well as the relative per-stream rate targets $\rho_k^{(s)}$ while the optimal power allocation can be found efficiently for given receive filters and rate targets. To perform the optimization of the filters, we apply an alternating optimization. Parts of this procedure will be performed in the dual uplink with uplink channel matrices $\boldsymbol{H}_k \in \mathbb{C}^{M \times N_k}$, uplink noise $\boldsymbol{\eta} \sim \mathcal{CN}(\boldsymbol{0}, \mathbf{I}_M)$, uplink beamformers $\boldsymbol{T}_k \in \mathbb{C}^{N_k \times S_k}$, and uplink receive filters $\boldsymbol{U}_k^{\mathrm{H}} \in \mathbb{C}^{S_k \times M}$. If the columns $\boldsymbol{t}_k^{(s)}$ of the beamforming matrix $\boldsymbol{T}_k$ are equal to the scaled columns of $\boldsymbol{V}_k$ and the rows $\boldsymbol{u}_k^{(s),\mathrm{H}}$ of the uplink receive filter $\boldsymbol{U}_k^{\mathrm{H}}$ are equal to the scaled rows of $\boldsymbol{B}_k^{\mathrm{H}}$, the same per-stream rates can be achieved in the uplink and in the downlink with the same sum transmit power provided that minimum mean square error (MMSE) equalization is employed and the downlink mean square error (MSE) matrix

$$\boldsymbol{E}_k^{\mathrm{DL}} = \mathrm{E}\left[(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^{\mathrm{H}}\right], \tag{5}$$

(or the uplink MSE matrix) is diagonal [15]. Fulfilling the requirement of a diagonal MSE matrix also ensures that encoding and decoding the data streams of a user separately is optimal [15]. Therefore, we can stick to a stream-wise view throughout most of the steps, and we will only switch to a user-wise perspective when absolutely necessary.

III. Solution to the Power Allocation Problem

As the first ingredient of the algorithm, we will derive how the optimal R_0 for given relative per-stream rate targets $\rho_k^{(s)}$ and given downlink receive filters $\boldsymbol{V}_k^{\mathrm{H}}$ can be computed. For fixed filters $\boldsymbol{V}_k^{\mathrm{H}}$, we can define the effective MISO channels

$$\tilde{\boldsymbol{h}}_k^{(s),\mathrm{H}} = \frac{\boldsymbol{v}_k^{(s),\mathrm{H}}}{\|\boldsymbol{v}_k^{(s)}\|_2}\boldsymbol{H}_k^{\mathrm{H}}, \tag{6}$$

and we can solve the optimization for fixed targets $\rho_k^{(s)}$ with respect to the transmit powers $q_k^{(s)}$ of the dual uplink of this effective setting:

$$\max_{R_0,(\boldsymbol{q}_k)_{k=1}^K} \quad R_0 \quad \text{s.t.: } R_k^{(s)} \geq R_0\rho_k^{(s)} \quad \forall\, k,s \tag{7}$$
$$\text{and} \quad q_k^{(s)} \geq 0 \quad \forall\, k,s \quad \text{and} \quad \sum_{k=1}^K\sum_{s=1}^{S_k} q_k^{(s)} \leq P$$

with $\boldsymbol{q}_k = [q_k^{(1)},\ldots,q_k^{(S_k)}]^{\mathrm{T}}$. Here, $R_k^{(s)}$ are the rates in the dual single-input multiple-output (SIMO) uplink given by

$$R_k^{(s)} = \log_2\left(1 + q_k^{(s)}\tilde{\boldsymbol{h}}_k^{(s),\mathrm{H}}\boldsymbol{X}_k^{(s),-1}\tilde{\boldsymbol{h}}_k^{(s)}\right) \tag{8}$$

with

$$\boldsymbol{X}_k^{(s)} = \mathbf{I}_M + \sum\nolimits_{\substack{k'=1\\(k',s')\neq(k,s)}}^K \sum\nolimits_{s'=1}^{S_{k'}} q_{k'}^{(s')}\tilde{\boldsymbol{h}}_{k'}^{(s')}\tilde{\boldsymbol{h}}_{k'}^{(s'),\mathrm{H}}, \tag{9}$$

and $q_k^{(s)}$ are the uplink transmit powers. It is known that with appropriately chosen uplink powers fulfilling $\sum_{k=1}^K\sum_{s=1}^{S_k} q_k^{(s)} = \sum_{k=1}^K\sum_{s=1}^{S_k} p_k^{(s)}$, the same rates as in the original downlink channel can be achieved, i.e., $R_k^{(s)} = r_k^{(s)}$ [16]. Therefore, the optimization (7) is equivalent to an optimization in the downlink where $q_k^{(s)}$ and $R_k^{(s)}$ are replaced by $p_k^{(s)}$ and $r_k^{(s)}$, respectively. Since (7) is formulated independently of the filters so that the only optimization variables are the uplink powers, the optimization in the uplink is much easier than in the downlink, and it can be solved efficiently as described below. Nevertheless, the downlink beamformers $\boldsymbol{b}_k^{(s)}$ are implicitly optimized together with the powers since the resulting rates $R_k^{(s)}$ from (8) are achievable in the downlink only if the downlink beamformers $\boldsymbol{b}_k^{(s)}$ are chosen to be the conjugate transpose of the optimal uplink receive filters $\boldsymbol{u}_k^{(s),\mathrm{H}}$, which will be computed in (29).

Assume that $(R_0^*,(\boldsymbol{q}_k^*)_{k=1}^K)$ is the optimizer of (7). Then, the optimizer of the problem

$$\min_{(\boldsymbol{q}_k)_{k=1}^K} \quad \sum_{k=1}^K\sum_{s=1}^{S_k} q_k^{(s)} \tag{10}$$
$$\text{s.t.: } R_k^{(s)} \geq \tilde{\rho}_k^{(s)} \quad \forall\, k,s \quad \text{and} \quad q_k^{(s)} \geq 0 \quad \forall\, k,s.$$

with $\tilde{\rho}_k^{(s)} = R_0^*\rho_k^{(s)}$ is equal to $(\boldsymbol{q}_k^*)_{k=1}^K$, and the optimum is $P_{\min} = P$. Therefore, solving (7) is equivalent to finding the value of R_0, for which the optimizer of (10) fulfills the sum power constraint with equality. This can be done by slightly extending the power minimization algorithm from [9]. In fact, a similar extension was proposed in [9] to apply the algorithm to the SINR balancing problem.

According to [9], the fixed point iteration

$$q_k^{(s)} \leftarrow \frac{2^{\tilde{\rho}_k^{(s)}} - 1}{\tilde{\boldsymbol{h}}_k^{(s),\mathrm{H}}\boldsymbol{X}_k^{(s),-1}\tilde{\boldsymbol{h}}_k^{(s)}} \tag{11}$$

converges to the globally optimal solution of (10). If we choose $\tilde{\rho}_k^{(s)} = R_0\rho_k^{(s)}$ and adapt R_0 in each iteration of the fixed point algorithm such that $\sum_{k=1}^K\sum_{s=1}^{S_k} q_k^{(s)} = P$ is fulfilled, the power constraint will be fulfilled after convergence and the final value of R_0 will be the globally optimal one. The update of R_0 is equivalent to finding the positive real root of the nonlinear function

$$f(R_0) = \sum_{k=1}^K\sum_{s=1}^{S_k} \frac{2^{R_0\rho_k^{(s)}} - 1}{\tilde{\boldsymbol{h}}_k^{(s),\mathrm{H}}\boldsymbol{X}_k^{(s),-1}\tilde{\boldsymbol{h}}_k^{(s)}} - P, \tag{12}$$

which can be efficiently done using the Newton-Raphson method (e.g., [17]).

For later reference, we define the function $\mathcal{Q}$: $\mathbb{C}^{MS_{\text{tot}}} \times \mathbb{R}_{0,+}^{S_{\text{tot}}} \mapsto \mathbb{R}_{0,+}^{1+S_{\text{tot}}}$ as

$$\mathcal{Q}\left(\left(\tilde{\boldsymbol{h}}_k^{(1)},\ldots,\tilde{\boldsymbol{h}}_k^{(S_k)},\boldsymbol{\rho}_k\right)_{k=1}^K\right) = (R_0^*,(\boldsymbol{q}_k^*)_{k=1}^K) \tag{13}$$

where $(R_0^*,(\boldsymbol{q}_k^*)_{k=1}^K)$ is the optimizer of problem (7).

IV. Update of the Per-Stream Rate Targets

Making use of the dual SIMO uplink introduced in Section III, we will now derive the gradient projection step, which can be considered as the principal step of the algorithm. We start by stating the calculation rule for the gradient in the following theorem.

Theorem 1: The partial derivatives of R_0 with respect to the relative per-stream rate targets $\rho_k^{(s)}$ are given by

$$\frac{\partial R_0}{\partial \rho_k^{(s)}} = \frac{-R_0}{\mathbf{1}^{\mathrm{T}}\boldsymbol{J}_R^{-1}\boldsymbol{\rho}_{\text{all}}}\mathbf{1}^{\mathrm{T}}\boldsymbol{J}_R^{-1}\boldsymbol{e}_k^{(s)}, \tag{14}$$

where $\boldsymbol{\rho}_{\text{all}} = (\boldsymbol{\rho}_k)_{k=1}^K$, $\boldsymbol{e}_k^{(s)} = \boldsymbol{e}_{s+\sum_{j=1}^{k-1}S_j}$, and $\boldsymbol{J}_R$ is the Jacobian matrix of the uplink rates $R_\kappa^{(\sigma)}$ with respect to the uplink powers $q_{k'}^{(s')}$, which is defined as

$$[\boldsymbol{J}_R]_{\sigma+\sum_{j=1}^{\kappa-1}S_j,\; s'+\sum_{j=1}^{k'-1}S_j} = \frac{\partial R_\kappa^{(\sigma)}}{\partial q_{k'}^{(s')}}. \tag{15}$$

Proof of Theorem 1: From [18], it follows that the minimum rate constraints and the sum power constraint are fulfilled with equality in the optimal solution of (7), i.e., $R_\kappa^{(\sigma)} = R_0\rho_\kappa^{(\sigma)}\ \forall\kappa,\sigma$ and $\sum_{k'=1}^K\sum_{s'=1}^{S_k} q_{k'}^{(s')} = P$. Taking the derivatives of these two equations, we get

$$\frac{\partial R_\kappa^{(\sigma)}}{\partial \rho_k^{(s)}} = \frac{\partial R_0}{\partial \rho_k^{(s)}}\rho_\kappa^{(\sigma)} + R_0\delta_{\kappa,k}\delta_{\sigma,s} \qquad \forall k,s,\kappa,\sigma \tag{16}$$

$$\sum_{k'=1}^K\sum_{s'=1}^{S_k} \frac{\partial q_{k'}^{(s')}}{\partial \rho_k^{(s)}} = 0 \qquad \forall k,s. \tag{17}$$

On the other hand, $\frac{\partial R_\kappa^{(\sigma)}}{\partial \rho_k^{(s)}}$ can be extended to

$$\frac{\partial R_\kappa^{(\sigma)}}{\partial \rho_k^{(s)}} = \sum_{k'=1}^{K} \sum_{s'=1}^{S_{k'}} \frac{\partial R_\kappa^{(\sigma)}}{\partial q_{k'}^{(s')}} \frac{\partial q_{k'}^{(s')}}{\partial \rho_k^{(s)}} \quad \forall k, s, \kappa, \sigma \tag{18}$$

using the chain rule. Written in matrix notation, this leads to the system of equations

$$\begin{bmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\boldsymbol{\rho}_{\text{all}} & \boldsymbol{J}_R \end{bmatrix} \begin{bmatrix} \nabla R_0 \\ \boldsymbol{J}_q \end{bmatrix} = \begin{bmatrix} \mathbf{0}^{\mathrm{T}} \\ R_0 \mathbf{I}_{S_{\text{tot}}} \end{bmatrix}, \tag{19}$$

where $\boldsymbol{J}_q$ is the Jacobian matrix of the powers $q_{k'}^{(s')}$ with respect to the rate targets $\rho_k^{(s)}$, i.e.,

$$[\boldsymbol{J}_q]_{s'+\sum_{j=1}^{k'-1} S_j,\ s+\sum_{j=1}^{k-1} S_j} = \frac{\partial q_{k'}^{(s')}}{\partial \rho_k^{(s)}}, \tag{20}$$

and the row vector ∇R_0 is the gradient of the variable R_0 with respect to the rate targets $\rho_k^{(s)}$, i.e.,

$$\frac{\partial R_0}{\partial \rho_k^{(s)}} = \nabla R_0\ \boldsymbol{e}_k^{(s)}. \tag{21}$$

The solution of (19) is given by

$$\nabla R_0 = -c^{-1} \mathbf{1}^{\mathrm{T}} \boldsymbol{J}_R^{-1} R_0 \mathbf{I}_{S_{\text{tot}}} = \frac{-R_0}{\mathbf{1}^{\mathrm{T}} \boldsymbol{J}_R^{-1} \boldsymbol{\rho}_{\text{all}}} \mathbf{1}^{\mathrm{T}} \boldsymbol{J}_R^{-1}, \tag{22}$$

where $c = \mathbf{1}^{\mathrm{T}} \boldsymbol{J}_R^{-1} \boldsymbol{\rho}_{\text{all}}$ is the Schur complement of the first matrix in (19) relative to $\boldsymbol{J}_R$ [19]. The existence of the inverse $\boldsymbol{J}_R^{-1}$ is proven in [14]. ∎

The entries of $\boldsymbol{J}_R$ can be explicitly calculated from (8) by means of standard methods of matrix calculus as

$$\frac{\partial R_\kappa^{(\sigma)}}{\partial q_\kappa^{(\sigma)}} = \frac{\tilde{\boldsymbol{h}}_\kappa^{(\sigma),\mathrm{H}} \boldsymbol{X}_\kappa^{(\sigma),-1} \tilde{\boldsymbol{h}}_\kappa^{(\sigma)}}{\ln 2 \cdot \left(1 + q_\kappa^{(\sigma)} \tilde{\boldsymbol{h}}_\kappa^{(\sigma),\mathrm{H}} \boldsymbol{X}_\kappa^{(\sigma),-1} \tilde{\boldsymbol{h}}_\kappa^{(\sigma)}\right)} \tag{23}$$

and

$$\frac{\partial R_\kappa^{(\sigma)}}{\partial q_{k'}^{(s')}} = \frac{-q_\kappa^{(\sigma)} \left|\tilde{\boldsymbol{h}}_\kappa^{(\sigma),\mathrm{H}} \boldsymbol{X}_\kappa^{(\sigma),-1} \tilde{\boldsymbol{h}}_{k'}^{(s')}\right|^2}{\ln 2 \cdot \left(1 + q_\kappa^{(\sigma)} \tilde{\boldsymbol{h}}_\kappa^{(\sigma),\mathrm{H}} \boldsymbol{X}_\kappa^{(\sigma),-1} \tilde{\boldsymbol{h}}_\kappa^{(\sigma)}\right)} \tag{24}$$

for $(k', s') \neq (\kappa, \sigma)$. In [14], it was shown that $\boldsymbol{J}_R^{-1}$ has only non-negative entries so that all partial derivatives $\frac{\partial R_0}{\partial \rho_k^{(s)}}$ are non-positive. Thus, after a gradient step

$$\tilde{\rho}_k^{(s)} \leftarrow \rho_k^{(s)} + d \frac{\partial R_0}{\partial \rho_k^{(s)}} \quad \forall k,\ \forall s \tag{25}$$

with positive step size d, the relative per-stream rate targets are decreased, and the new targets $\tilde{\rho}_k^{(s)}$ no longer satisfy the constraint $\sum_{s=1}^{S_k} \rho_k^{(s)} = \varrho_k \ \ \forall k$. Thus, a projection to the set of valid relative per-stream rate targets has to be performed by solving the optimization problem

$$\min_{\boldsymbol{\rho}_k} \ \sum_{s=1}^{S_k} \left(\rho_k^{(s)} - \tilde{\rho}_k^{(s)}\right)^2 \tag{26}$$

$$\text{s.t.: } \boldsymbol{\rho}_k \geq \mathbf{0} \ \text{ and } \ \sum_{s=1}^{S_k} \rho_k^{(s)} = \varrho_k$$

for all users k. As was derived in [14], the solution to this problem is given by the waterfilling equation

$$\rho_k^{(s)} = \max\left\{\tilde{\rho}_k^{(s)} + \mu_k,\ 0\right\}, \tag{27}$$

where the optimal water level $\mu_k \in \mathbb{R}$ is

$$\mu_k = \frac{1}{|\mathcal{S}_{k,\mathrm{a}}|} \left(\varrho_k - \sum_{s \in \mathcal{S}_{k,\mathrm{a}}} \tilde{\rho}_k^{(s)}\right) \tag{28}$$

with $\mathcal{S}_{k,\mathrm{a}}$ being the set of active streams of user k.

After the rate target update and the projection step, the new optimal value of R_0 can be calculated by means of the function $\mathcal{Q}$ defined in (13). In case that the performance after the projection step is worse than it was before the gradient step, i.e., a too large step size d has been used, the gradient-projection step has to be repeated with a reduced step size. If no increase in performance is achieved even with a very small step size (smaller than a given limit $d_{\min}$), the old rate targets are kept and the algorithm proceeds to the filter update without having changed the rate targets.

V. Update of the Filters

Apart from the special treatment of inactive streams discussed in Section VI, which has to be rederived for the rate balancing problem, the filter update works like the one used in our power minimization algorithm presented in [14]. This section is devoted to describing the various steps of the update procedure, which is an alternating optimization in the uplink and downlink and includes a diagonalization of the MSE matrices.

The uplink rates in (8) can only be achieved if the uplink receive filters provide a sufficient statistic for every data stream individually, which is the case if they are chosen optimally in the MMSE sense, i.e.,

$$\tilde{\boldsymbol{U}}_k^{\mathrm{H}} = \operatorname{diag}\{\boldsymbol{q}_k\}^{\frac{1}{2}} \tilde{\boldsymbol{H}}_k^{\mathrm{H}} \boldsymbol{X}^{-1} \quad \text{with} \quad \tilde{\boldsymbol{H}}_k = \left[\tilde{\boldsymbol{h}}_k^{(1)}, \ldots, \tilde{\boldsymbol{h}}_k^{(S_k)}\right], \tag{29}$$

where

$$\boldsymbol{X} = \mathbf{I}_M + \sum_{k=1}^{K} \sum_{s=1}^{S_k} q_k^{(s)} \tilde{\boldsymbol{h}}_k^{(s)} \tilde{\boldsymbol{h}}_k^{(s),\mathrm{H}}, \tag{30}$$

and $\tilde{\boldsymbol{h}}_k^{(s)}$ is given by (6).

After the update of the uplink receive filters, which is equivalent to updating the downlink beamformers, the system has to be transformed back to the downlink in order to update the downlink receive filters. To do so, we first diagonalize the uplink MSE matrix

$$\begin{aligned} \boldsymbol{E}_k &= \mathbf{I}_{S_k} + \tilde{\boldsymbol{U}}_k^{\mathrm{H}} \boldsymbol{X} \tilde{\boldsymbol{U}}_k - \tilde{\boldsymbol{U}}_k^{\mathrm{H}} \tilde{\boldsymbol{H}}_k \operatorname{diag}\{\boldsymbol{q}_k\}^{\frac{1}{2}} \\ &\quad - \operatorname{diag}\{\boldsymbol{q}_k\}^{\frac{1}{2}} \tilde{\boldsymbol{H}}_k^{\mathrm{H}} \tilde{\boldsymbol{U}}_k \\ &= \mathbf{I}_{S_k} + \operatorname{diag}\{\boldsymbol{q}_k\}^{\frac{1}{2}} \tilde{\boldsymbol{H}}_k^{\mathrm{H}} \boldsymbol{X}^{-1} \tilde{\boldsymbol{H}}_k \operatorname{diag}\{\boldsymbol{q}_k\}^{\frac{1}{2}} \end{aligned} \tag{31}$$

and perform a stream-wise duality transformation according to [15] afterwards. The matrix $\boldsymbol{E}_k$ is diagonal if the filters of the MIMO uplink are chosen according to

$$\boldsymbol{T}_k = \left[\frac{\boldsymbol{v}_k^{(1)}}{\|\boldsymbol{v}_k^{(1)}\|_2}, \ldots, \frac{\boldsymbol{v}_k^{(S_k)}}{\|\boldsymbol{v}_k^{(S_k)}\|_2} \right] \operatorname{diag}\left\{\boldsymbol{q}_k\right\}^{\frac{1}{2}} \boldsymbol{F}_k \tag{32}$$

$$\boldsymbol{U}_k^{\mathrm{H}} = \boldsymbol{F}_k^{\mathrm{H}} \tilde{\boldsymbol{U}}_k^{\mathrm{H}}, \tag{33}$$

where $\boldsymbol{F}_k$ is the modal matrix of the eigenvalue decomposition

$$\boldsymbol{F}_k \boldsymbol{D}_k \boldsymbol{F}_k^{\mathrm{H}} = \operatorname{diag}\left\{\boldsymbol{q}_k\right\}^{\frac{1}{2}} \tilde{\boldsymbol{H}}_k^{\mathrm{H}} \boldsymbol{X}^{-1} \tilde{\boldsymbol{H}}_k \operatorname{diag}\left\{\boldsymbol{q}_k\right\}^{\frac{1}{2}}. \tag{34}$$

Note that the uplink transmit filters $\boldsymbol{T}_k$ are not normalized, i.e., they include the uplink transmit powers $q_k^{(s)}$. Thus, the total power ist given by $\sum_k \operatorname{tr}[\boldsymbol{T}_k \boldsymbol{T}_k^{\mathrm{H}}]$. As $\boldsymbol{F}_k$ is unitary, it neither changes the transmit power spent for a user k, i.e., $\operatorname{tr}[\boldsymbol{T}_k \boldsymbol{T}_k^{\mathrm{H}}] = \mathbf{1}^{\mathrm{T}} \boldsymbol{q}_k$, nor does it influence the streams of other users. Moreover, it conserves the rate that user k could achieve with joint encoding and decoding of all streams, but it increases the rate that user k can achieve with stream-wise encoding and decoding so that the rate of the stream-wise case equals the rate of the joint case. Note that this property is another reason for diagonalizing the MSE matrices since per-stream coding is assumed in this paper, i.e., the per-user rate is assumed to be the sum of the per-stream rates. However, the mapping between the per-stream rate targets $\rho_k^{(s)}$ and the rates of the actual data streams is lost due to the diagonalization since the streams might have been resorted or even recombined by this unitary rotation of the filters. Thus, the relative per-stream rate targets $\rho_k^{(s)}$ are no longer related to the actual per-stream rates and need to be adapted to the new situation. This will be done at the end of the filter update.

The stream-wise uplink-downlink transformation is performed by choosing

$$\boldsymbol{b}_k^{(s)} \leftarrow \frac{\boldsymbol{u}_k^{(s)}}{\|\boldsymbol{u}_k^{(s)}\|_2} \quad \text{and} \quad \boldsymbol{v}_k^{(s)} \leftarrow \boldsymbol{t}_k^{(s)}, \tag{35}$$

and calculating the downlink powers $\boldsymbol{p}_k = [p_k^{(1)}, \ldots, p_k^{(S_k)}]^{\mathrm{T}}$ using

$$\boldsymbol{M} \left[\boldsymbol{p}_1^{\mathrm{T}}, \ldots, \boldsymbol{p}_K^{\mathrm{T}}\right]^{\mathrm{T}} = \boldsymbol{\tau} \tag{36}$$

with $[\boldsymbol{\tau}]_{s+\sum_{j=1}^{k-1} S_j} = \|\boldsymbol{t}_k^{(s)}\|_2^2$ and

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{1,1} & \ldots & \boldsymbol{M}_{1,K} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{K,1} & \ldots & \boldsymbol{M}_{K,K} \end{bmatrix} \in \mathbb{R}^{S_{\mathrm{tot}} \times S_{\mathrm{tot}}}, \tag{37}$$

where $[\boldsymbol{M}_{\kappa,k}]_{s,\sigma} = -|\boldsymbol{u}_k^{(s),\mathrm{H}} \boldsymbol{H}_\kappa \boldsymbol{t}_\kappa^{(\sigma)}|^2$ for $\kappa \neq k$, and $\boldsymbol{M}_{k,k} = \operatorname{diag}_{s=1}^{S_k} \left\{ \|\boldsymbol{u}_k^{(s)}\|_2^2 - \sum_{\kappa \neq k} \mathbf{1}^{\mathrm{T}} \boldsymbol{M}_{\kappa,k} \boldsymbol{e}_s \right\}$ [15].

In the downlink, optimal receive filters in the MMSE sense are given by

$$\boldsymbol{v}_k^{(s),\mathrm{H}} \leftarrow \boldsymbol{b}_k^{(s),\mathrm{H}} \boldsymbol{H}_k \left(\mathbf{I}_{N_k} + \boldsymbol{H}_k^{\mathrm{H}} \left(\sum_{j=1}^{K} \boldsymbol{B}_j \boldsymbol{B}_j^{\mathrm{H}} \right) \boldsymbol{H}_k \right)^{-1}, \tag{38}$$

for $_k^{(s)}$: $p_k^{(s)} \neq 0$. Inactive streams with $p_k^{(s)} = 0$ are excluded from this update as they would be set to zero. Note that those streams could never again be reactivated in a later step. Therefore, a different update method for the inactive streams is proposed in Section VI and Appendix A. Note that we do not diagonalize the MSE matrices in the downlink since one diagonalization per iteration suffices to guarantee that all MSE matrices are diagonal after convergence.

None of the steps performed during the filter update can decrease the per-user sum rates. Thus, by setting

$$\tilde{\rho}_k^{(s)} \leftarrow \frac{r_k^{(s)}}{R_0} \quad \forall k, s, \tag{39}$$

where $r_k^{(s)}$ are the currently achieved downlink rates, we have $\sum_{s=1}^{S_k} \tilde{\rho}_k^{(s)} \geq \varrho_k$. However, to fulfill the constraints of the optimization problem (3), equality must hold. To get a set of new relative per-stream rate targets $\rho_k^{(s)}$ that fulfills the sum constraint with equality, we can apply the projection (26) and (27) that has already been used in Section IV. Since this projection reduces the per-user sum of the relative per-stream rate targets, R_0 is increased in the next evaluation of the function $\mathcal{Q}$.

VI. Filter Update for Inactive Streams

Dealing with inactive streams is a typical problem when optimizing MIMO systems. For instance, in [12], the authors do not allow inactive streams in order to be able to apply the geometric programming framework. They argue that this is without loss of generality since streams with very small transmit powers can be treated as inactive in a practical system. However, this is not satisfying from a theoretical point of view.

In the proposed gradient-based optimization, it is possible that streams are deactivated in that the relative per-stream rate targets of certain streams are set to zero during the gradient-projection step. However, it might happen that a reactivation of certain streams becomes reasonable after the filters have been updated. As mentioned above, the filter update step has to be prevented from setting the downlink receive filters of inactive streams to zero since the effective MISO channels in the next iteration are computed using these receive filters [see (6)]. Thus, the filters should be chosen in way that they are a good complement to the spatial directions of the active streams. If this is the case, the gradient-projection step in the next iteration might activate them as additional streams, finally yielding a value of the objective function that is higher than the one that would be possible without the additional streams.

Furthermore, the filters should be chosen such that the streams do not interfer with other streams of the same user once they get activated, i.e.,

$$\boldsymbol{v}_k^{(s),\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{b}_k^{(s')} p_k^{(s'),\frac{1}{2}} = 0 \tag{40}$$

for $s' \neq s$ and for all s that correspond to inactive streams. This condition is the counterpart of the requirement of a diagonal MSE matrix, which we imposed for the active streams.

Algorithm 1 Gradient-Based Rate Balancing for MIMO BC

Require: $(\varrho_k)_{k=1}^K$, $(\boldsymbol{H}_k)_{k=1}^K$, $(\boldsymbol{V}_k)_{k=1}^K$, $(\boldsymbol{\rho}_k)_{k=1}^K$, ϵ^{Grad}, d_0, $d_{\min}$

(1) $\left(R_0, (\boldsymbol{q}_k)_{k=1}^K\right) \leftarrow \mathcal{Q}\left(\left(\tilde{\boldsymbol{h}}_k^{(1)}, \ldots, \tilde{\boldsymbol{h}}_k^{(S_k)}, \boldsymbol{\rho}_k\right)_{\forall k}\right)$

(2) **repeat**

(3) $R_{\text{last}} \leftarrow R_0$

(4) compute the uplink receivers $\tilde{\boldsymbol{U}}_k^{\text{H}}$ using (29)

(5) diagonalize the MSE matrices using (32) and (33)

(6) perform UL to DL transformation using (35) and (36)

(7) compute the downlink receivers $\boldsymbol{v}_k^{(s),\text{H}}$ of active streams using (38)

(8) compute the new per-stream rates $(\boldsymbol{r}_k)_{\forall k}$ using (4)

(9) compute the rate targets $(\boldsymbol{\rho}_k)_{\forall k}$ by applying projection rule (27) to $(\tilde{\boldsymbol{\rho}}_k)_{\forall k} \leftarrow (\boldsymbol{r}_k/R_0)_{\forall k}$

(10) $\left(R_0, (\boldsymbol{q}_k)_{k=1}^K\right) \leftarrow \mathcal{Q}\left(\left(\tilde{\boldsymbol{h}}_k^{(1)}, \ldots, \tilde{\boldsymbol{h}}_k^{(S_k)}, \boldsymbol{\rho}_k\right)_{\forall k}\right)$

(11) compute the downlink receivers $\boldsymbol{v}_k^{(s),\text{H}}$ of inactive streams by repeatedly solving (49)

(12) compute the gradient using (14)

(13) $d \leftarrow d_0$

(14) **loop**

(15) compute $(\tilde{\boldsymbol{\rho}}_k)_{\forall k}$ by performing the gradient step (25) with step size d

(16) compute the rate targets $(\boldsymbol{\rho}_{k,\text{new}})_{\forall k}$ by applying projection rule (27) to $(\tilde{\boldsymbol{\rho}}_k)_{\forall k}$

(17) $\left(R_{\text{new}}, (\boldsymbol{q}_{k,\text{new}})_{k=1}^K\right) \leftarrow \mathcal{Q}\left(\left(\tilde{\boldsymbol{h}}_k^{(1)}, \ldots, \tilde{\boldsymbol{h}}_k^{(S_k)}, \boldsymbol{\rho}_k\right)_{\forall k}\right)$

(18) **if** $R_{\text{new}} \geq R_0$ **then**

(19) $\boldsymbol{\rho}_k \leftarrow \boldsymbol{\rho}_{k,\text{new}}\ \forall k, \quad \boldsymbol{q}_k \leftarrow \boldsymbol{q}_{k,\text{new}}\ \forall k, \quad R_0 \leftarrow R_{\text{new}}$

(20) **break**

(21) **else if** $d < d_{\min}$ **then**

(22) **break**

(23) **end if**

(24) $d \leftarrow d/2$

(25) **end loop**

(26) **until** $R_0 - R_{\text{last}} \leq \epsilon^{\text{Grad}}$

A method to find vectors that comply to these two demands is described in Appendix A. It is based on solving a generalized eigenvalue problem of dimension smaller than S_k for each inactive stream.

VII. Overview, Convergence, and Initialization

The method proposed in this paper is summarized in Algorithm 1. In the main, it is an alternating optimization consisting of three steps in each iteration: the gradient-projection step for the relative per-stream rate targets, the update of the downlink transmit filters, and the update of downlink receive filters. Within these steps, the optimization from Section III, which is encapsulated in the function $\mathcal{Q}$, is called at several positions as an inner optimization to compute the currently optimal power allocation and the current value R_0 of the objective function. Convergence is guaranteed since no step can decrease the value R_0 and, on the other hand, R_0 is bounded from above by the finite optimal value.

As an initialization, we have to choose downlink receive filters $\boldsymbol{v}_k^{(s)}$ and relative per-stream rate targets $\rho_k^{(s)}$ for all k and s. However, unlike other rate balancing algorithms, such as the one proposed in [11], [12], the method does not need any initial values for the transmit powers. Instead, the initial powers depend on the initial rate targets and are computed in the first line of the algorithm. Thus, when comparing our approach with the one from [11], [12], the two optimizations are not started with the same initialization. Instead, initializing with relative per-stream rate targets instead of with per-stream powers can be considered as a specific feature of the algorithm.

In our numerical simulations, we used the initial filters

$$\boldsymbol{v}_k^{(s)} = \boldsymbol{e}_s \quad \forall\ k, s \tag{41}$$

and the initial rate targets

$$\rho_k^{(s)} = \begin{cases} \varrho_k & \text{if } s = 1 \\ 0 & \text{otherwise} \end{cases} \quad \forall\ k. \tag{42}$$

Unlike in the case of power minimization studied in [14], no feasibility considerations are needed since the rate balancing problem is always feasible. Therefore, any other choice for the initial per-stream rate targets would be applicable. However, since the algorithm is able to activate previously inactive streams, there is no need to start with more than one active stream per user. This is another difference to the algorithm in [11], [12], which does not support inactive streams and has to be initialized with all possible streams being active.

VIII. Numerical Results

To evaluate the performance of the proposed method, we have performed numerical simulations in a system with $M = 10$ transmit antennas, $K = 4$ users, $N_k = 5$ receive antennas for all users, and relative rate targets $\varrho_1 = \varrho_2 = 1$ and $\varrho_3 = \varrho_4 = 2$. All channel coefficients were drawn independently from a circularly symmetric complex Gaussian distribution with zero mean and unit variance, and the results are averaged over 1000 channel realizations.

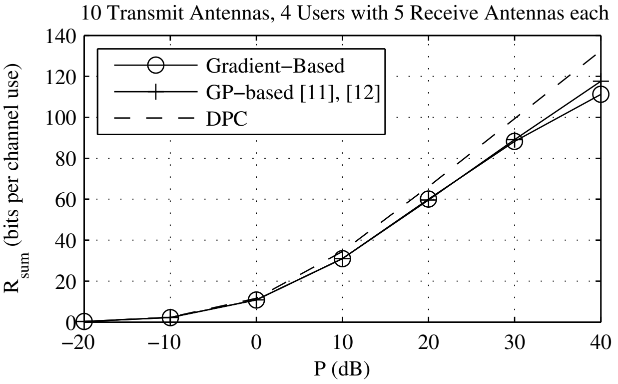

Fig. 1. Sum Rate for Relative Rate Targets $\varrho_1 = \varrho_2 = 1$, $\varrho_3 = \varrho_4 = 2$.

As can be seen in Fig. 1, which shows the sum rate $R_{\text{sum}} = \sum_{k=1}^K r_k$ versus the transmit power P, the proposed rate balancing algorithm for MIMO broadcast channels with linear precoding performs very similar to the GP-based scheme from [11], [12]. However, as will be discussed below, the computational complexity of the GP-based algorithm is significantly higher.

We have also included the curve of the globally optimal DPC solution, which can be computed as described in [8]. Although in general not achievable in systems that are constrained to use linear transceivers, this curve is an interesting benchmark as it is an upper bound for the linear precoding case. It can be seen that both the gradient-based and the GP-based scheme perform close to the optimal DPC solution, but it is not clear, which portion of the rate gap between the two schemes and the DPC optimum is a result from the suboptimality of the optimization procedures and which portion is due to the restriction to linear transceivers. Nevertheless, it can be seen that the gap is relatively small, i.e., the proposed method performs close to the globally optimal solution.

In [12], the authors acknowledge that "power allocation with GP has high complexity." Indeed, in our simulations it turned out that solving a geometric program in KN variables, as it is necessary in each iteration of the algorithm proposed in [11], [12], needs much more computation time than the execution of the operations within an iteration of the proposed gradient-projection algorithm. The computationally most complex subproblem of the gradient projection approach is the repeated evaluation of the function $\mathcal{Q}$ defined in (13). Depending on the required accuracy, this function is evaluated up to about ten times per iteration using the iterative procedure from Section III, which converges in a very small number of steps (typically about ten). As a result, this inner optimization can be performed significantly faster than the inner optimization of the GP-based method. Another computationally complex step in the gradient projection algorithm is the repeated computation of generalized eigenvalue decompositions during the update of the filters of currently inactive streams. However, firstly, as the involved matrices have moderate sizes of less than S_k, this step is also much less complex than the solution of the geometric program, and secondly, it can even be skipped if a further complexity reduction is desired. By instead leaving the filters of inactive streams as they were in the previous iteration, the resulting sum rate was decreased on average by no more than 5% in our numerical simulations.

For a lower overall complexity it is not sufficient to have a lower complexity per iteration, but it is also necessary to take the total number of iterations into account. To this end, we have included Fig. 2, where we have plotted the developing of the sum throughput over the iterations. The curves are again averaged over 1000 channel realizations. It can be seen that the number of iterations needed by the gradient-based algorithm to achieve a certain sum throughput is at least not higher than the number needed by the GP-based method. Furthermore, it can be seen that due to the difference in the initialization, which we already discussed in the previous section, the gradient projection algorithm starts at a significantly higher sum rate.

IX. Conclusion

In this work, we have derived a gradient-projection algorithm for the rate balancing problem in MIMO broadcast channels with linear transceivers. The gradient update is performed with respect to a set of auxiliary variables, the so-called relative per-stream rate targets. Further major steps of the algorithm are the power allocation in an effective MISO system, the alternating update of uplink and downlink filters of the active streams based on the MMSE rule, and the update of the filters of inactive streams based on a generalized eigenvalue problem. In numerical simulations, the proposed algorithm turned out to have a similar performance as the geometric programming based method from [11], [12] at a lower complexity. In particular, the computation time per iteration was reduced significantly, but also the number of iterations was lower.

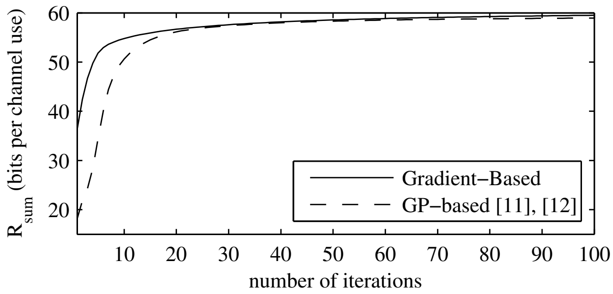

Fig. 2. Convergence of the GP-based and the Gradient-based Algorithm for $P = 20\,\mathrm{dB}$.

Appendix A
Details of the Filter Update for Inactive Streams

Any downlink receive filter that complies to the zero-interference requirement (40) can be expressed as $\boldsymbol{v}_k^{(s)} = \boldsymbol{\Phi}_k \hat{\boldsymbol{v}}_k^{(s)}$, where $\boldsymbol{\Phi}_k$ is a matrix of basis vectors of the nullspace of $\mathrm{diag}\{\boldsymbol{p}_k\}^{\frac{1}{2}} \boldsymbol{B}_k^{\mathrm{H}} \boldsymbol{H}_k$, and $\hat{\boldsymbol{v}}_k^{(s)}$ is a vector of appropriate size without any special structure. Note that currently inactive streams do not reduce the dimensionality of the nullspace since $p_k^{(s')} = 0$ for all s' that correspond to currently inactive streams.

A stream has a good chance to get activated in a gradient-projection step if the absolute value of the derivative $\frac{\partial R_0}{\partial \rho_k^{(s)}}$ is small, i.e., if the non-positive derivative is large. Therefore, our aim is now to find the vector $\boldsymbol{\Phi}_k \hat{\boldsymbol{v}}_k^{(s)}$ that maximizes this derivative.

Let us reorder the columns and rows of $\boldsymbol{J}_R$ such that the reordered version $\tilde{\boldsymbol{J}}_R$ can be partitioned into

$$\tilde{\boldsymbol{J}}_R = \begin{bmatrix} \boldsymbol{J}_{R,\mathrm{aa}} & \boldsymbol{J}_{R,\mathrm{ai}} \\ \boldsymbol{J}_{R,\mathrm{ia}} & \boldsymbol{J}_{R,\mathrm{ii}} \end{bmatrix} \tag{43}$$

where the first letter in the second subscript refers to rows belonging to active (a) or inactive (i) streams, and the second letter refers to the columns. For instance, $\boldsymbol{J}_{R,\mathrm{ia}}$ contains the derivatives $\frac{\partial R_\kappa^{(\sigma)}}{\partial q_{k'}^{(s')}}$ with ${}_\kappa^{(\sigma)}$ corresponding to inactive and ${}_{k'}^{(s')}$ corresponding to active streams. As can be easily verified,

$\frac{\partial R_{\kappa}^{(\sigma)}}{\partial q_{k'}^{(s')}} = 0$ if ${}_{\kappa}^{(\sigma)}$ is inactive and ${}_{\kappa}^{(\sigma)} \neq {}_{k'}^{(s')}$. Thus,

$$\begin{bmatrix} \boldsymbol{J}_{R,\text{aa}} & \boldsymbol{J}_{R,\text{ai}} \\ \mathbf{0} & \boldsymbol{J}_{R,\text{ii}} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{J}_{R,\text{aa}}^{-1} & -\boldsymbol{J}_{R,\text{aa}}^{-1}\boldsymbol{J}_{R,\text{ai}}\boldsymbol{J}_{R,\text{ii}}^{-1} \\ \mathbf{0} & \boldsymbol{J}_{R,\text{ii}}^{-1} \end{bmatrix}, \quad (44)$$

where the second equality is due to [19], and $\boldsymbol{J}_{R,\text{ii}}$ is the diagonal matrix

$$\boldsymbol{J}_{R,\text{ii}} = \frac{1}{\ln 2} \operatorname{diag}\left\{ \tilde{\boldsymbol{h}}_{\kappa}^{(\sigma),\text{H}} \boldsymbol{X}_{\kappa}^{(\sigma),-1} \tilde{\boldsymbol{h}}_{\kappa}^{(\sigma)} \right\}, \quad (45)$$

whose diagonal elements correspond to the inactive streams ${}_{\kappa}^{(\sigma)}$. Using $\tilde{\boldsymbol{\rho}}_{\text{all}} = [\boldsymbol{\rho}_{\text{a}}^{\text{T}}\ \boldsymbol{\rho}_{\text{i}}^{\text{T}}]^{\text{T}} = [\boldsymbol{\rho}_{\text{a}}^{\text{T}}\ \mathbf{0}^{\text{T}}]^{\text{T}}$ and $\tilde{\boldsymbol{e}}_{\kappa}^{(\sigma)}$ to denote the accordingly reordered versions of $\boldsymbol{\rho}_{\text{all}}$ and $\boldsymbol{e}_{\kappa}^{(\sigma)}$, respectively, we get for an inactive stream ${}_{\kappa}^{(\sigma)}$ [cf. (14)]:

$$\begin{aligned} \frac{\partial R_0}{\partial \rho_{\kappa}^{(\sigma)}} &= \frac{-R_0}{\mathbf{1}^{\text{T}}\boldsymbol{J}_R^{-1}\boldsymbol{\rho}_{\text{all}}} \mathbf{1}^{\text{T}}\boldsymbol{J}_R^{-1}\boldsymbol{e}_{\kappa}^{(\sigma)} = \frac{-R_0}{\mathbf{1}^{\text{T}}\tilde{\boldsymbol{J}}_R^{-1}\tilde{\boldsymbol{\rho}}_{\text{all}}} \mathbf{1}^{\text{T}}\tilde{\boldsymbol{J}}_R^{-1}\tilde{\boldsymbol{e}}_{\kappa}^{(\sigma)} \\ &= \frac{-R_0 \cdot \ln 2 \cdot \mathbf{1}^{\text{T}} \begin{bmatrix} -\boldsymbol{J}_{R,\text{aa}}^{-1}\boldsymbol{J}_{R,\text{ai}}\boldsymbol{e}_{\text{i},\kappa}^{(\sigma)} \\ \boldsymbol{e}_{\text{i},\kappa}^{(\sigma)} \end{bmatrix}}{\mathbf{1}^{\text{T}}\boldsymbol{J}_{R,\text{aa}}^{-1}\boldsymbol{\rho}_{\text{a}} \cdot \tilde{\boldsymbol{h}}_{\kappa}^{(\sigma),\text{H}}\boldsymbol{X}_{\kappa}^{(\sigma),-1}\tilde{\boldsymbol{h}}_{\kappa}^{(\sigma)}} \\ &= -\beta \cdot \frac{\boldsymbol{v}_{\kappa}^{(\sigma),\text{H}}\left(\mathbf{I}_{N_\kappa} + \boldsymbol{H}_{\kappa}^{\text{H}}\boldsymbol{D}\boldsymbol{H}_{\kappa}\right)\boldsymbol{v}_{\kappa}^{(\sigma)}}{\boldsymbol{v}_{\kappa}^{(\sigma),\text{H}}\boldsymbol{H}_{\kappa}^{\text{H}}\boldsymbol{X}^{-1}\boldsymbol{H}_{\kappa}\boldsymbol{v}_{\kappa}^{(\sigma)}}, \end{aligned} \quad (46)$$

where $\boldsymbol{e}_{\text{i},\kappa}^{(\sigma)}$ is the canonical unit vector that identifies the stream ${}_{\kappa}^{(\sigma)}$ among all inactive streams. The abbreviation

$$\beta = \frac{R_0 \cdot \ln 2}{\mathbf{1}^{\text{T}}\boldsymbol{J}_{R,\text{aa}}^{-1}\boldsymbol{\rho}_{\text{a}}} \quad (47)$$

is positive and depends solely on the active streams, and the matrix $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \sum_{{}_k^{(s)}:\ \text{active}} \frac{\mathbf{1}^{\text{T}}\boldsymbol{J}_{R,\text{aa}}^{-1}\boldsymbol{e}_{\text{a},k}^{(s)} \cdot \boldsymbol{X}_k^{(s),-1}\tilde{\boldsymbol{h}}_k^{(s)} q_k^{(s)} \tilde{\boldsymbol{h}}_k^{(s),\text{H}}\boldsymbol{X}_k^{(s),-1}}{\ln 2 \cdot \left(1 + q_k^{(s)}\tilde{\boldsymbol{h}}_k^{(s),\text{H}}\boldsymbol{X}_k^{(s),-1}\tilde{\boldsymbol{h}}_k^{(s)}\right)}, \quad (48)$$

where $\boldsymbol{e}_{\text{a},k}^{(s)}$ is the canonical unit vector that identifies the stream ${}_k^{(s)}$ among all active streams.

Thus, in order to maximize the derivative $\frac{\partial R_0}{\partial \rho_{\kappa}^{(\sigma)}}$ for the first inactive stream ${}_{\kappa}^{(\sigma)}$ of user κ, we have to solve the optimization problem

$$\boldsymbol{v}_{\kappa}^{(\sigma)} = \boldsymbol{\Phi}_{\kappa} \operatorname*{argmin}_{\hat{\boldsymbol{v}}_{\kappa}^{(\sigma)}} \frac{\hat{\boldsymbol{v}}_{\kappa}^{(\sigma),\text{H}}\boldsymbol{\Phi}_{\kappa}^{\text{H}}\left(\mathbf{I}_{N_\kappa} + \boldsymbol{H}_{\kappa}^{\text{H}}\boldsymbol{D}\boldsymbol{H}_{\kappa}\right)\boldsymbol{\Phi}_{\kappa}\hat{\boldsymbol{v}}_{\kappa}^{(\sigma)}}{\hat{\boldsymbol{v}}_{\kappa}^{(\sigma),\text{H}}\boldsymbol{\Phi}_{\kappa}^{\text{H}}\boldsymbol{H}_{\kappa}^{\text{H}}\boldsymbol{X}^{-1}\boldsymbol{H}_{\kappa}\boldsymbol{\Phi}_{\kappa}\hat{\boldsymbol{v}}_{\kappa}^{(\sigma)}}, \quad (49)$$

which is a generalized eigenvalue problem, i.e., the optimal $\hat{\boldsymbol{v}}_{\kappa}^{(\sigma)}$ is the generalized eigenvector of the matrices $\boldsymbol{\Phi}_{\kappa}^{\text{H}}\left(\mathbf{I}_{N_\kappa} + \boldsymbol{H}_{\kappa}^{\text{H}}\boldsymbol{D}\boldsymbol{H}_{\kappa}\right)\boldsymbol{\Phi}_{\kappa} \in \mathbb{C}^{S_{0,k}\times S_{0,k}}$ and $\boldsymbol{\Phi}_{\kappa}^{\text{H}}\boldsymbol{H}_{\kappa}^{\text{H}}\boldsymbol{X}^{-1}\boldsymbol{H}_{\kappa}\boldsymbol{\Phi}_{\kappa} \in \mathbb{C}^{S_{0,k}\times S_{0,k}}$ that belongs to the smallest generalized eigenvalue. Here, $S_{0,k} < S_k$ denotes the dimensionality of the nullspace of $\operatorname{diag}\{\boldsymbol{p}_k\}^{\frac{1}{2}}\boldsymbol{B}_k^{\text{H}}\boldsymbol{H}_k$. The generalized eigenvector can be scaled to unit norm without loss of generality.

The procedure is repeated for all inactive streams of a user, but after the i-th inactive filter $\boldsymbol{v}_{\kappa}^{(\sigma_i)}$ of user κ has been computed, the matrix $\boldsymbol{\Phi}_{\kappa}$ is replaced with a matrix of basis vectors of the nullspace of $[\operatorname{diag}\{\boldsymbol{p}_k\}^{\frac{1}{2}}\boldsymbol{B}_k^{\text{H}}\boldsymbol{H}_k, \boldsymbol{v}_{\kappa}^{(\sigma_1)}, \ldots, \boldsymbol{v}_{\kappa}^{(\sigma_i)}]^{\text{H}}$. This ensures that orthogonal directions are chosen for subsequently computed filters. As a result, the first computed filter has the best direction with respect to the criterion defined above, and the corresponding stream has the highest chance to get activated in the next iteration. On the other hand, the stream corresponding to the filter computed last is very unlikely to be activated.

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