A differential equation for diagonalizing complex semisimple Lie algebra elements

Uwe Helmke [a,*] Martin Kleinsteuber [**,b]

[a] *Institute of Mathematics, University of Würzburg, 97074 Würzburg, Germany*

[b] *Institute of Data Processing, Technische Universität München, 80333 München, Germany*

Abstract

In this paper, we consider a generalization of Ebenbauer's differential equation for non-symmetric matrix diagonalization to a flow on arbitrary complex semisimple Lie algebras. The flow is designed in such a way that the desired diagonalizations are precisely the equilibrium points in a given Cartan subalgebra. We characterize the set of all equilibria and establish a Morse-Bott type property of the flow. Global convergence to single equilibrium points is shown, starting from any semisimple Lie algebra element. For strongly regular initial conditions, we prove that the flow converges to an element of the Cartan subalgebra and thus achieves asymptotic diagonalization.

Key words: diagonalization, structure-preserving isospectral flow, semisimple Lie algebra

1 Introduction

The starting point for this paper has been the work by R. W. Brockett [2], who introduced the double bracket flow

$$\dot{H} = [H, [H, N]], \tag{1}$$

where $[A, B] := AB - BA$ is the matrix commutator and N a real diagonal matrix with pairwise distinct eigenvalues, as a means to diagonalize symmetric matrices H. An extension of (1) to compact Lie algebras appeared in [3,4],

* Research partially supported by DFG-SPP 1305 grant HE 1858/12-1.
Email: helmke@mathematik.uni-wuerzburg.de
**Corresponding author. Email: kleinsteuber@tum.de

together with a full phase portrait analysis. The double bracket flow has found numerous applications, such as, e.g., to linear programming, symmetric and skew-symmetric eigenvalue computations, variational problems and Hamiltonian systems; see e.g. [11] and the references therein. Taken this broad applicability of the double bracket flow into mind, one wonders, if (1) can also be used for diagonalization of non-symmetric matrices. This would be a major step forward in numerical linear algebra, as globally convergent algorithms for the non-symmetric eigenvalue problem are currently unknown. Unfortunately, this simple idea fails, as the isospectral flow (1) on non-symmetric matrices is not gradient anymore and thus the solutions will i.g. not converge to the equilibrium points.

Ebenbauer [7] had the idea of adding a suitable normalization term to (1) in order to force the flow to converge to normal matrices. Specifically, Ebenbauer considers the isospectral matrix differential equation

$$\dot{L} = [L, [L^\top + L, N]] + \rho[L, [L^\top, L]], \quad L(0) = L_0, \tag{2}$$

on arbitrary real symmetric matrices. Here, ρ is some positive constant. Of course, for $\rho = 0$ and $L = H$ symmetric this contains the double bracket flow as a special case. In [7], Ebenbauer shows, under the implicit condition that $L(0)$ is diagonalizable, that the solutions $L(t)$ of (2) converge to the set of normal matrices. Moreover, if the eigenvalues of L_0 have distinct real parts, then $L(t)$ converges to the real Jordan form of L_0, cf. [7,8].

In subsequent work, Ebenbauer and Arsie [8] introduced an extension of (2) to real semisimple Lie algebras. Thus, given any Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with Cartan involution θ and $P \in \mathfrak{p}$, they consider the isospectral flow on the semisimple Lie algebra $\mathfrak{g}$

$$\dot{X} = [X, [X - \theta X, P]] + \rho[X, [\theta X, X]], \quad X(0) = X_0. \tag{3}$$

In order to analyze the convergence properties of (3), Ebenbauer and Arsie note, that the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ on a semisimple Lie algebra defines a smooth conjugacy between the two flows (3), (2). That is, a curve $X(t)$ in $\mathfrak{g}$ solves (3) if and only if $L(t) := \mathrm{ad(X(t))}$ is a solution of (2) in $gl(\mathfrak{g})$. Here N is assumed to be equal to $\mathrm{ad(P)}$. Thus the convergence properties of the two flows are equivalent. Under the assumptions that $\mathrm{ad(P)}$ has distinct eigenvalues and the real parts of the eigenvalues of $L_0 = \mathrm{ad(X_0)}$ are distinct, the authors conclude convergence of (3) to a diagonalization of X_0. However, this extrinsic approach via the embedding of $\mathfrak{g}$ into $gl(\mathfrak{g})$ fails, as the assumptions made for convergence are far too strong and are (almost) never satisfied. In fact, except in special low dimensional cases such as e.g. $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{sp}(1)$, a semisimple Lie algebra does not contain any elements $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)$ has distinct eigenvalues. Thus one cannot apply the convergence results in [7,8] on the Ebenbauer matrix flow (2) to deduce convergence of the

Lie algebra flow (3).

In this paper, we therefore present a different, intrinsic Lie algebra approach to (3) that avoids these difficulties. Specifically, we consider the flow

$$\dot{X} = [X, [X - \theta X, \mathrm{i}N]] + \rho[X, [\theta X, X]], \quad X(0) = X_0 \tag{4}$$

on complex semisimple Lie algebras $\mathfrak{g}$. Here N is assumed to be a regular element in a torus algebra of the compact real form of $\mathfrak{g}$. Note, that we will strongly take advantage of the fact that two torus algebras in $\mathfrak{g}$ are conjugate to one another via an inner automorphism, which is not the case for general real Lie algebras. The presented results hence do not carry over straightforwardly to the real case.

Our results extend Ebenbauer's results even for the matrix flow (2), by avoiding not necessary genericity assumptions. In a first step, we generalize the classical notions of normal and diagonalizable complex matrices into a Lie algebraic setting, together with a generalization of well-known results, such as e.g. normal matrices are unitarily diagonalizable. Similarly, the notions of regular and strongly regular Lie algebra elements are introduced that extend the properties of a matrix to have pairwise distinct eigenvalues, or eigenvalues with pairwise distinct real parts, respectively. The genericity of these two classes in a semisimple Lie algebra is shown. To clarify the connection with the approach in [8] we note, that the matrix $\mathrm{ad}(X)$, associated with a regular element X in a semisimple Lie algebra, does not have pairwise distinct eigenvalues (except for few low-dimensional cases). Thus the correct regularity condition on $L_0 = \mathrm{ad}(X_0), N = \mathrm{ad}(P)$ to ensure convergence of (3) would be a suitable non-genericity condition that allows for multiple eigenvalues of $L_0 = \mathrm{ad}(X_0), N = \mathrm{ad}(P)$.

We then prove that (3) converges to the set of normal elements of the adjoint orbit of X_0 if and only if X_0 is semisimple. Moreover, answering a conjecture in [8], if X_0 is semisimple, global convergence to single equilibrium points is shown. This depends on a general convergence result, Theorem 13, for Morse-Bott type flows that is proven here and may be of independent interest; see also [12]. We characterize the critical points as normal elements in the adjoint orbit $\mathcal{O}(X_0)$ such that their symmetric part $X - \theta X$ is contained in a maximal abelian subalgebra of $\mathfrak{k}$. Under the genericity condition that X_0 is strongly regular, we finally prove pointwise convergence of the solutions to normal elements contained in a given Cartan subalgebra. This implies the desired diagonalizability property of the flow.

2 Preliminaries on complex semisimple Lie algebras

We briefly summarize some well-known facts on complex semisimple Lie algebras and introduce relevant notation; see [15] for a textbook on Lie algebras and Lie groups.

Let G denote a connected complex semisimple Lie group with maximal compact subgroup K. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the associated Lie algebras, then $\mathfrak{g}$ is a finite dimensional, complex semisimple Lie algebra with compact real form $\mathfrak{k}$, cf. [15] Thm. 6.11., satisfying

$$\mathfrak{g} = \mathfrak{k} \oplus \mathrm{i}\mathfrak{k}. \tag{5}$$

By (5), every $X \in \mathfrak{g}$ decomposes uniquely into $X = A + \mathrm{i}B$ with $A, B \in \mathfrak{k}$. Thus (5) yields the Cartan decomposition of $\mathfrak{g}$ with respect to the Cartan involution

$$\theta \colon A + \mathrm{i}B \mapsto A - \mathrm{i}B. \tag{6}$$

For every $X \in \mathfrak{g}$ we have $X - \theta X \in \mathrm{i}\mathfrak{k}$. For $Z \in \mathfrak{g}$, let

$$\mathrm{ad}_Z \colon \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto [Z, X] \tag{7}$$

be the adjoint representation of $\mathfrak{g}$ with $\mathrm{ad}(\mathfrak{g}) = \{\mathrm{ad}_Z \mid Z \in \mathfrak{g}\}$. Recall, that the *Killing form*

$$\kappa \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad (Z_1, Z_2) \mapsto \mathrm{tr}(\mathrm{ad}_{Z_1} \circ \mathrm{ad}_{Z_2}) \tag{8}$$

is ad-invariant, i.e. $\kappa(X, [Y, Z]) = -\kappa([Y, X], Z)$ holds for all $X, Y, Z \in \mathfrak{g}$. Furthermore,

$$B_\theta \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad (Z_1, Z_2) \mapsto -\mathrm{Re}\kappa(Z_1, \theta(Z_2)) \tag{9}$$

defines a positive definite symmetric $\mathbb{R}$-bilinear form on $\mathfrak{g}$. Here, $\mathrm{Re}z$ denotes the real part of a complex number z. Note, that B_θ defines a norm on $\mathfrak{g}$ which will be denoted by $\|\cdot\|$. Given any $g \in G$, let $X \mapsto g(X)$ denote the inner automorphism $Ad(g) : \mathfrak{g} \to \mathfrak{g}$ defined by g. In the special case, where $G \subset \mathbb{C}^{n\times n}$ is a connected matrix Lie group with matrix Lie algebra $\mathfrak{g}$, this inner automorphism then takes the form

$$g(X) = gXg^{-1} \quad \text{for some } g \in G. \tag{10}$$

Moreover, up to conjugation with inner automorphisms, the Cartan-involution is $\theta(X) = -X^\dagger$, where $X^\dagger$ is the conjugate transpose of X.

Given $X_0 \in \mathfrak{g}$, we define the adjoint orbit of X_0 by

$$\mathcal{O}(X_0) := \{g(X_0) \mid g \in G\}. \tag{11}$$

Being a homogeneous space, $\mathcal{O}(X_0)$ is a smooth compact manifold with tangent space at $X \in \mathcal{O}(X_0)$ given by

$$T_X\mathcal{O}(X_0) = \{[X, Z] \mid Z \in \mathfrak{g}\}. \tag{12}$$

The adjoint orbit carries a natural Riemannian metric. To this end, consider the smooth map $\pi\colon G \to \mathcal{O}(X_0), g \mapsto g(X_0)$ with

$$\mathrm{Ker} T_g\pi = T_e l_g(\mathrm{Ker}\ \mathrm{ad}_{X_0}). \tag{13}$$

Here, $e \in G$ is the identity, l_g is multiplication from the left and T stands for the tangent map. The tangent map of π at g

$$T_g\pi\colon T_gG \to T_{g(X_0)}\mathcal{O}(X_0) \tag{14}$$

maps any complement of $\mathrm{Ker}T_g\pi$ isomorphically onto the tangent space $T_{g(X_0)}\mathcal{O}(X_0)$. Since $\mathcal{O}(X_0) = \mathcal{O}(g(X_0))$ for all $g \in G$, we can construct a Riemannian metric on $\mathcal{O}(X_0)$ as follows. Let $X \in \mathcal{O}(X_0) \subset \mathfrak{g}$. Then

$$\begin{aligned} \mathrm{ad}_X\colon \mathfrak{g} &\to T_X\mathcal{O}(X_0) \\ \xi &\mapsto \mathrm{ad}_X(\xi) = [X, \xi] \end{aligned} \tag{15}$$

is surjective with kernel $\mathrm{Ker}\ \mathrm{ad}_X$. Let

$$(\mathrm{Ker}\ \mathrm{ad}_X)^\perp := \{\xi \in \mathfrak{g} \mid B_\theta(\xi, \eta) = 0 \text{ for all } \eta \in \mathrm{Ker}\ \mathrm{ad}_X\}. \tag{16}$$

A simple calculation shows

$$(\mathrm{Ker}\ \mathrm{ad}_X)^\perp = \mathrm{Im}\ \mathrm{ad}_{\theta X}. \tag{17}$$

We define the *normal Riemannian metric* on $\mathcal{O}(X_0)$ by

$$\begin{aligned} \langle \cdot, \cdot \rangle\colon T_X\mathcal{O}(X_0) \times T_X\mathcal{O}(X_0) &\to \mathbb{R} \\ (\mathrm{ad}_X(U), \mathrm{ad}_X(V)) &\mapsto B_\theta(U, V) \end{aligned} \tag{18}$$

for $U, V \in (\mathrm{Ker}\ \mathrm{ad}_X)^\perp$.

3 A flow for normalization

3.1 Normal and semisimple Lie algebra elements

We extend the notions of normal and diagonalizable complex matrices to a Lie algebraic setting and prove a generalization of the well-known fact from linear algebra, that normal matrices are unitarily diagonalizable. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathrm{i}\mathfrak{k}$

be a complex semisimple Lie algebra with compact real form $\mathfrak{k}$ and Cartan-involution θ. An element $X \in \mathfrak{g}$ is *normal*, if

$$[X, \theta X] = 0. \tag{19}$$

The following result characterizes normal elements in a Lie algebra as the critical points of the norm function on adjoint orbits.

Theorem 1 (Kempf - Ness, 1979) *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Equivalent are for $X_0 \in \mathfrak{g}$:*

(a) $\mathcal{O}(X_0)$ is Zariski-closed.
(b) $\mathcal{O}(X_0)$ is closed with respect to the standard topology.
(c) The function $V\colon \mathcal{O}(X_0) \to \mathbb{R}, X \mapsto \frac{1}{2}\|X\|^2$ possesses a critical point.

In each case, the critical points are the minima of V. The normal elements of $\mathcal{O}(X_0)$ are exactly the critical points of the norm function V. The set of critical points is a single K-orbit and thus is a smooth manifold. At each critical point X, the kernel of the Hessian of V coincides with the associated tangent space of the set of local minima of V.

Proof. Cf. [14] and [11].

The notion of a Cartan subalgebra provides the proper Lie algebraic generalization of the class of unitarily diagonalizable matrices. A *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ that is

(i) maximal Abelian;
(ii) the linear transformations $\{\mathrm{ad}_H \in \mathrm{ad}(\mathfrak{g}) \mid H \in \mathfrak{h}\}$ are simultaneously diagonalizable.

Note, that this definition differs from the standard one in the literature. However, in the semisimple case, they are equivalent, cf. [15], Prop. 2.10 and 2.13. Note also, that if $\mathfrak{t}$ is maximal Abelian in $\mathfrak{k}$, then $\mathfrak{h} := \mathfrak{t} \oplus i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and that all Cartan subalgebras are conjugate to each other via some $g \in G$.

Property (ii) implies, that the Lie algebra $\mathfrak{g}$ decomposes into the simultaneous eigenspaces

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \mathrm{ad}_H X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}, \tag{20}$$

where α is a linear functional $\alpha\colon \mathfrak{h} \to \mathbb{C}$. A nonzero functional α with $\mathfrak{g}_\alpha \neq 0$ is called a *root* and $\mathfrak{g}_\alpha$ the corresponding *root space*. By finite dimensionality of $\mathfrak{g}$, the set of roots Σ is finite. Since $\mathfrak{h}$ is maximal Abelian, $\mathfrak{g}_0 = \mathfrak{h}$ and thus any complex semsimple Lie algebra has the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Sigma - \{0\}} \mathfrak{g}_\alpha. \tag{21}$$

Let $\Sigma^+ \subset \Sigma$ denote the set of positive roots, where positivity is defined by lexicographic ordering on the set of roots. If $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$, then (21) restricts to the decomposition

$$\mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha \in \Sigma^+} (\mathfrak{g}_\alpha + \theta \mathfrak{g}_\alpha). \tag{22}$$

A fact that will be important in the sequel is that the roots are real valued functions on $\mathrm{i}\mathfrak{t}$ and purely imaginary on $\mathfrak{t}$.

We now prove the announced generalization that normal matrices are unitarily diagonalizable.

Theorem 2 (Diagonalization Theorem I) *For any normal element $X \in \mathfrak{g}$ there exists $k_0 \in K$, such that $k_0(X) \in \mathfrak{h}$.*

Proof. Let $S := X - \theta X \in \mathrm{i}\mathfrak{k}$. There exists $k_1 \in K$ such that $\Lambda := k_1(S) \in \mathrm{i}\mathfrak{t}$, cf. [15], Thm. 6.51. Let $\Psi := k_1(X + \theta X) \in \mathfrak{k}$ and let $N \in \mathfrak{t}$ be a regular element, i.e. $\{H \in \mathfrak{k} \mid [H, N] = 0\} = \mathfrak{t}$. Denote by

$$Z_K(\Lambda) := \{k \in K \mid k(\Lambda) = \Lambda\} \tag{23}$$

the centralizer of Λ in K and let $Z_K(\Lambda)^0$ be the connected component containing the identity. Note, that $Z_K(\Lambda)$, being a closed subgroup of the compact group K, is itself compact. The Lie algebra of $Z_K(\Lambda)$ is given by

$$\mathfrak{z}_\mathfrak{k}(\Lambda) := \{H \in \mathfrak{k} \mid [H, \Lambda] = 0\}. \tag{24}$$

Let κ denote the Killing form on $\mathfrak{g}$. Since $Z_K(\Lambda)$ is compact,

$$f \colon Z_K(\Lambda) \to \mathbb{R}, \quad k \mapsto \kappa(k(\Psi), N) \tag{25}$$

has a critical point, say k_2, for which

$$\kappa([H, k_2(\Psi)], N) = 0 \tag{26}$$

holds for all $H \in \mathfrak{z}_\mathfrak{k}(\Lambda)$, the Lie algebra of $Z_K(\Lambda)$. By the ad-invariance of κ, this implies that $\kappa([k_2(\Psi), N], H) = 0$ holds for all $H \in \mathfrak{z}_\mathfrak{k}(\Lambda)$, or, in other words, $[k_2(\Psi), N] \in \mathfrak{z}_\mathfrak{k}(\Lambda)^\perp$. On the other hand, using the Jacobi identity,

$$[\Lambda, [k_2(\Psi), N]] = -[k_2(\Psi), [N, \Lambda]] - [N, [\Lambda, k_2(\Psi)]] = 0. \tag{27}$$

The first summand is zero because $[N, \Lambda] = 0$. Since $[X, \theta X] = 0$ we have $[\Lambda, \Psi] = 0$. Therefore, $k_2(\Lambda) = \Lambda$ implies $[\Lambda, k_2(\Psi)] = k_2([\Lambda, \Psi] = 0$. Thus

$$[k_2(\Psi), N] \in \mathfrak{z}_{\mathfrak{k}}(\Lambda) \cap \mathfrak{z}_{\mathfrak{k}}(\Lambda)^{\perp} = 0, \tag{28}$$

and since N is regular in $\mathfrak{t}$, $k_2(\Psi) \in \mathfrak{t}$. Summarizing,

$$\begin{aligned} \mathfrak{h} \ni \Lambda + k_2(\Psi) &= k_2(\Lambda + \Psi) \\ &= k_2 k_1(X - \theta X + X + \theta X) = 2 k_2 k_1(X) \end{aligned} \tag{29}$$

and the result follows. □

An element $X \in \mathfrak{g}$ is called *semisimple*, if $\mathrm{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ is diagonalizable. By Theorem 4.1.6 in [16], X is semisimple if and only if it belongs to some Cartan subalgebra. Since all Cartan subalgebras are conjugate to each other, it follows that X is semisimple if and only if it is G-conjugate to some $H \in \mathfrak{h}$, i.e. if there exists some $g \in G$ such that $g(X) \in \mathfrak{h}$. Using the Kempf-Ness-Theorem, we now have the following result.

Lemma 3 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{t}$ any maximal abelian subalgebra of $\mathfrak{k}$ and $\mathfrak{h} = \mathfrak{t} \oplus \mathrm{i}\mathfrak{t}$ the associated Cartan subalgebra.*

(1) An element $X_0 \in \mathfrak{g}$ is semisimple if and only if $\mathcal{O}(X_0)$ contains a normal element.
(2) The set of normal elements $\mathcal{N}(X_0)$ in $\mathcal{O}(X_0)$ coincides with the set of global minima of $V(X) = \frac{1}{2}\|X\|^2$.
(3) There exists $H \in \mathcal{O}(X_0) \cap \mathfrak{h}$ with

$$\mathcal{N}(X_0) := \{k(H) \mid k \in K\}. \tag{30}$$

Proof. By Theorem 1, it remains to show that a critical point of V has to be normal. But this holds trivially, since for $\xi = \mathrm{ad}_H X \in T_X \mathcal{O}(X_0)$,

$$DV(X)\xi = -\tfrac{1}{2} D\Big(\kappa(X, \theta X)\Big)\xi = -\kappa(\mathrm{ad}_H X, \theta X) = -\kappa(H, [X, \theta X]).$$

The derivative thus vanishes if and only if $[X, \theta X] = 0$. This, together with the Kempf-Ness theorem proves the first two claims. For the last one note that the Kempf-Ness theorem asserts that $\mathcal{N}(X_0)$ is a single K-orbit. Choose any element $H_0 \in \mathcal{N}(X_0)$ and decompose it as $H_0 = \Omega + \mathrm{i}S$, $\Omega, S \in \mathfrak{k}$. Let $\mathfrak{t}_0$ denote a maximal abelian subalgebra in $\mathfrak{k}$ containing Ω, S. Since H_0 is normal, we have $[\Omega, S] = 0$ and therefore $\mathfrak{t}_0$ exists. Since all maximal abelian subalgebras in $\mathfrak{k}$ are conjugate via the adjoint action of K, there exists $k \in \mathfrak{k}$ with $H := k(H_0) \in \mathfrak{h} \cap \mathcal{N}(X_0)$. This completes the proof. □

It is now easy to construct a flow that globally converges to the set of normal elements in an adjoint orbit. Thus this flow achieves the normalization of a semisimple Lie algebra element within the adjoint orbit. In a later section, we will then discuss the more challenging problem of constructing a flow that converges to the intersection of a given Cartan subalgebra with the set of normal elements.

Theorem 4 *Let $X_0 \in \mathfrak{g}$. The solution $X(t)$ of the flow*

$$\dot{X} = [X, [\theta X, X]]. \tag{31}$$

converges to the set of isospectral normal elements, i.e. to the set $\mathcal{N}(X_0)$ given by Eq. (30), if and only if X_0 is semisimple.

Proof. Let X_0 be semisimple. We prove that $V(X) = \frac{1}{2}\|X\|^2$ is a Lyapunov function for the system (30). To see this, note that $[X, \theta X] \in \mathrm{i}\mathfrak{k}$, and $B_\theta|_{\mathrm{i}\mathfrak{k}} = \kappa|_{\mathrm{i}\mathfrak{k}}$. Hence by the ad-invariance of the Killing form,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} V(X(t)) &= B_\theta([X, [\theta X, X]], X) \\ &= \kappa([\theta X, X], [X, \theta X]) \\ &= -\|[X, \theta X]\|^2. \end{aligned}$$

Therefore, $V(X)$ is monotonically decreasing along the flow (44). By Theorem 1, $\mathcal{O}(X_0)$ is closed and therefore the sublevel sets of V are compact. Thus $X(t)$ is bounded and converges to the set of critical points, which is $\mathcal{N}(X_0)$. If X_0 is not semisimple, then by Lemma 3 (1) $\mathcal{N}(X_0)$ is empty. □

4 Diagonalization in semisimple Lie algebras

Following [15], Ch. II, Sec. 2., we introduce *regular elements* in $\mathfrak{g}$. In the special case where $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the regular elements are precisely those matrices with pairwise distinct eigenvalues. For $Z \in \mathfrak{g}$, denote by $m(Z)$ the dimension of the generalized eigenspace of ad_Z for the eigenvalue 0. Among all $Z \in \mathfrak{g}$, there is a lower bound

$$M := \min\{m(Z) \mid Z \in \mathfrak{g}\}. \tag{32}$$

An element $Z \in \mathfrak{g}$ is called *regular* if $m(Z) = M$. The set $\mathfrak{g}_r$ of regular elements in $\mathfrak{g}$ is a Zariski open subset. In particular, it is open and dense in $\mathfrak{g}$; cf. proof of Thm 2.9', Ch. II in [15].

In a Lie algebra, the natural notion of diagonalizability is as follows. An element $X \in \mathfrak{g}$ is called *diagonalizable with respect to a Cartan subalgebra* $\mathfrak{h}$, if there exists $g \in G$ such that $g(X) \in \mathfrak{h}$. Since all Cartan subalgebras in a semisimple Lie algebra are conjugate, the semisimple elements of $\mathfrak{g}$ are precisely those that are diagonalizable with respect to a given Cartan subalgebra. The following theorem gives a sufficient condition for diagonalizability. It extends the well known fact, that complex matrices with pairwise distinct eigenvalues are diagonalizable.

Theorem 5 (Diagonalization Theorem II) *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Every regular element $X \in \mathfrak{g}$ is* diagonalizable with respect to a Cartan subalgebra $\mathfrak{h}$. *In particular, regular elements are semisimple.*

Proof. By [15], Theorem 2.9., every regular element lies in some Cartan subalgebra $\mathfrak{h}'$. Theorem 2.15 in [15] proves, that any two Cartan subalgebras are conjugate to each other, i.e. that there exists some $g \in G$ with $g(\mathfrak{h}') = \mathfrak{h}$. Hence $g(X) \in \mathfrak{h}$. □

Note, that if $H \in \mathfrak{h}$ is regular, then $\mathfrak{h}$ is the centralizer of H, i.e.

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid [X, H] = 0\}. \tag{33}$$

Hence if $X_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ is a nonzero root space element, then $0 \neq [H, X_\alpha] = \alpha(H)X_\alpha$ and thus regular elements in $\mathfrak{h}$ are exactly those with

$$\mathfrak{h}_r := \{H \in \mathfrak{h} \mid \alpha(H) \neq 0 \text{ for all } \alpha \in \Sigma\}. \tag{34}$$

It follows, that the set of regular elements in $\mathfrak{g}$ is characterized by

$$\mathfrak{g}_r = \{g(\mathfrak{h}_r) \mid g \in G\}. \tag{35}$$

For our purposes, we have to specify a subset of regular elements. In the case of $\mathfrak{sl}_n(\mathbb{C})$, these matrices will be those with eigenvalues having pairwise distinct real part.

Lemma 6 *(a) The set of* strongly regular *elements*

$$\mathfrak{h}_{sr} := \{H_1 + \mathrm{i}H_2 \in \mathfrak{h} \mid H_1, H_2 \in \mathfrak{t}, H_2 \textit{ is regular}\}. \tag{36}$$

is a subset of $\mathfrak{h}_r$. It is open and dense in $\mathfrak{h}$. Moreover, let $H \in \mathfrak{h}$ and $g \in G$ with $g(H) \in \mathfrak{h}$. Then $g(H) \in \mathfrak{h}_{sr}$ if and only if $H \in \mathfrak{h}_{sr}$.

(b) The set

$$\mathfrak{g}_{sr} := \{g(\mathfrak{h}_{sr}) \mid g \in G\} \tag{37}$$

is open and dense in $\mathfrak{g}$.

Proof. (a) Let $H = H_1 + \mathrm{i}H_2 \in \mathfrak{h}_{sr}$. Then $\mathrm{Re}(\alpha(H)) = \alpha(H_2)$ and therefore $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$. Thus $H \in \mathfrak{h}_r$. The set $\mathfrak{h}_{sr}$ is obtained from $\mathfrak{h}$ by removing the finitely many hyperplanes

$$\{H \in \mathfrak{h} \mid \mathrm{Re}\alpha(H) = 0, \quad \alpha \in \Sigma\}. \tag{38}$$

It is therefore open and dense in $\mathfrak{h}$.

(b) Since the map

$$\sigma \colon G \times \mathfrak{h} \to \mathfrak{g}, \quad (g, H) \mapsto g(H) \tag{39}$$

is open, cf. [15], proof of Thm. 2.15, it follows that the set $\mathfrak{g}_{sr}$ is open and dense in $\mathfrak{g}$. □

5 A flow for diagonalization

We can now state and prove the main results of this paper. For any element $N \in \mathfrak{g}$ consider the least square distance function

$$f \colon \mathcal{O}(X_0) \to \mathbb{R}, \quad X \mapsto \tfrac{1}{2}\|X - N\|^2. \tag{40}$$

Theorem 7 *The Riemannian gradient of f is given by*

$$\mathrm{grad} f(X) = [X, [X - N, \theta X]].$$

Proof. The differential of f at a tangent vector $\xi \in T_X\mathcal{O}(X_0)$ is given by

$$Df(X)\xi = B_\theta(X - N, \xi). \tag{41}$$

Hence for $\xi = \mathrm{ad}_X(V)$ with $V \in (\mathrm{Ker}\ \mathrm{ad}_X)^\perp$, we have

$$\begin{aligned} Df(X)\mathrm{ad}_X(V) &= B_\theta(X - N, \mathrm{ad}_X(V)) \\ &= -B_\theta(\mathrm{ad}_{\theta X}(X - N), V). \end{aligned} \tag{42}$$

There exists a unique $U \in (\mathrm{Ker}\ \mathrm{ad}_X)^\perp$ such that $\mathrm{grad} f(X) = \mathrm{ad}_X(U)$. Thus for all $V \in (\mathrm{Ker}\ \mathrm{ad}_X)^\perp$

$$\begin{aligned} & - B_\theta(\mathrm{ad}_{\theta X}(X - N), V) = B_\theta(U, V) \\ \Leftrightarrow\ & B_\theta(U + \mathrm{ad}_{\theta X}(X - N), V) = 0 \\ \Leftrightarrow\ & U + \mathrm{ad}_{\theta X}(X - N) \in \mathrm{Ker}\ \mathrm{ad}_X \\ \Leftrightarrow\ & \mathrm{grad} f(X) = -\mathrm{ad}_X \mathrm{ad}_{\theta X}(X - N) \\ & = [X, [X - N, \theta X]]. \end{aligned} \tag{43}$$

□

Corollary 8 *The critical points $X \in \mathcal{O}(X_0)$ are exactly those where $[X - N, \theta X] = 0$.*

Proof. This follows by Eq. (42), since Eq. (17) yields $\mathrm{ad}_{\theta X}(X-N) \in (\mathrm{Ker}\ \mathrm{ad}_X)^{\perp}$. □

The above real analytic gradient flow has the nice feature that all solutions converge pointwise to the critical points. Unfortunately, these critical points are *not* the ones we are looking for, i.e. they are not contained in a given Cartan subalgebra. Thus, one has to modify the flow in an appropiate way. Following Ebenbauer [7,8] we propose a combination of the above gradient flow and the flow (31) for normalizing an element results in a differential equation that diagonalizes semisimple elements.

Fix a Cartan subalgebra as $\mathfrak{h} = \mathfrak{t} \oplus \mathrm{i}\mathfrak{t}$ in $\mathfrak{g}$. Let $\rho > 0$ denote any positive constant, θ the Cartan involution (6) and let $N \in \mathfrak{t}$ be an arbitrary fixed regular element. Let $X_0 \in \mathfrak{g}$. Our goal then is to construct a flow that achieves asymptotic conjugation into the Cartan subalgebra $\mathfrak{h} = \{X \in \mathfrak{g} \mid [N, X] = 0\}$ defined by N. Consider the flow on $\mathcal{O}(X_0)$ defined by

$$\dot{X} = [X, [X - \theta X, \mathrm{i}N]] + \rho[X, [\theta X, X]] \tag{44}$$

with $X(0) = X_0$. This is a Lie algebraic version of the flow considered by Ebenbauer [7]. The next lemma shows, for X_0 semisimple, that the flow converges to the subset of normal elements X whose symmetric part $X - \theta X$ is contained in $\mathrm{i}\mathfrak{t} \subset \mathfrak{h}$.

Lemma 9 *The flow (44) possesses an equilibrium point if and only if X_0 is semisimple. The set of equilibria $\mathcal{E}$ of (44) consists of all normal elements in $\mathcal{O}(X_0)$ whose projection onto $\mathfrak{k}$ commutes with N, i.e.*

$$\mathcal{E} = \mathcal{N}(X_0) \cap \{X \in \mathfrak{g} \mid [N, X - \theta X] = 0\}.$$

Proof. Since $[\mathrm{i}N, X - \theta X] \in \mathfrak{k}$ and $[X, \theta X] \in \mathrm{i}\mathfrak{k} = \mathfrak{k}^{\perp}$, we have $[\mathrm{i}N, X - \theta X] + \rho[X, \theta X] = 0$ if and only if

$$\begin{gathered} [N, X - \theta X] = 0 \text{ and} \\ [\theta X, X] = 0. \end{gathered} \tag{45}$$

The second statement follows immediately from Lemma 3. □

The above lemma implies $S := X - \theta X \in \mathrm{i}\mathfrak{t}$ for any equilibrium point X, since $[N, X - \theta X] = 0$ and N is a regular element. Thus it does not exactly achieve our main goal which is conjugation of X_0 into $\mathfrak{h}$. Before addressing this issue any further, we first show global, pointwise convergence of the flow to single equilibria. An important feature here is that the set of equilibrium points may form a continuum. Thus, we first consider the geometry of the set of equilibria in more detail.

Let W denote the Weyl group. Clearly, for any Weyl group element $w \in W$, $w(X) \in \mathcal{E}$ holds whenever $X \in \mathcal{E}$. Since the Weyl group permutes the roots, there exists an $X_{sort} \in \mathcal{E}$ such that with $S_{sort} := \frac{1}{2}(X_{sort} - \theta X_{sort})$

$$\alpha(S_{sort}) \geq 0 \text{ for all } \alpha \in \Sigma^+.$$

Let $\mathrm{stab}_W(S_{sort}) = \{w \in W \mid w(S_{sort}) = S_{sort}\}$ and $\mathrm{stab}_K(S_{sort}) = \{k \in K^0 \mid k(S_{sort}) = S_{sort}\}$.

Lemma 10 *Let $\Lambda := \{X \in \mathcal{E} \mid \frac{1}{2}(X - \theta X) = S_{sort}\}$ and $X \in \Lambda$. Then*

$$\Lambda = \{k(X) \mid k \in \mathrm{stab}_K(S_{sort})\}.$$

Moreover, the set of equilibria $\mathcal{E}$ decomposes into the finitely many connected components

$$\mathcal{E} = \bigcup_{w \in W/\mathrm{stab}_W(S_{sort})} w(\Lambda).$$

Proof. If $X \in \Lambda$ and $k \in \mathrm{stab}_K(S_{sort})$, then $k(X) \in \Lambda$ and hence $\{k(X) \mid k \in \mathrm{stab}_K(S_{sort})\} \subset \Lambda$. For the other inclusion, a similar argument as in the proof of Theorem 2 shows, that for every $X \in \Lambda$ there exists some $k \in \mathrm{stab}_K(S_{sort})$ such that $k(X) = S_{sort} + T$ with $T \in \mathfrak{t}$. The claim now follows since $S_{sort} + T \in \Lambda$.

For the second part, note that $\bigcup_{w \in W/\mathrm{stab}_W(S_{sort})} w(\Lambda) \subset \mathcal{E}$ is obvious. To see the other inclusion, let $X \in \mathcal{E}$ be given. Then X is normal and $S := \frac{1}{2}(X - \theta X) \in \mathrm{i}\mathfrak{t}$. Let $w \in W$ such that $w(S) = S_{sort}$. Then $w(X) \in \Lambda$ and the result follows. □

Lemma 11 *The restriction of the flow (44) to $\mathcal{N}(X_0)$ is the gradient vector field of the function*

$$\phi \colon \mathcal{N}(X_0) \to \mathbb{R}, \quad X \mapsto B_\theta(X - \theta X, \mathrm{i}N).$$

Proof. Let $\Omega \in \mathfrak{k}$ and let $[X, \Omega] \in T_X\mathcal{N}(X_0)$ be an arbitrary tangent element. Differentiating ϕ yields

$$\begin{aligned} D\phi(X)[X, \Omega] &= B_\theta([X, \Omega] - [\theta X, \Omega], \mathrm{i}N) \\ &= \kappa([X - \theta X, \Omega], \mathrm{i}N) \\ &= B_\theta(\Omega, [X - \theta X, \mathrm{i}N]). \end{aligned}$$

Hence $\mathrm{grad}\phi(X) = [X, [X - \theta X, \mathrm{i}N]]$. □

Lemma 12 *The critical points of ϕ are exactly the equilibria $\mathcal{E}$ of the flow*

(44) and the Hessian of ϕ at some critical point $X \in \Lambda_i$ is given by

$$\mathsf{H}_X([X,\Omega_1],[X,\Omega_2]) = \sum_{\alpha\in\Sigma^+} \alpha(S)\beta(\mathrm{i}N)\kappa(\Omega_1^{(\alpha)},\Omega_2^{(\alpha)}), \tag{46}$$

where $S := X - \theta X$ and $\Omega_i := \Omega_i^{(0)} + \sum_{\alpha\in\Sigma^+} \Omega_i^{(\alpha)}$ with respect to the root space decomposition of $\mathfrak{k}$, cf. Eq. (22). In particular, the rank of the Hessian is given by

$$\mathrm{rk}\mathsf{H}_X = \#\{\alpha \in \Sigma \mid \alpha(S) \neq 0\}.$$

Proof. We compute the Hessian of ϕ along the direction $[X,\Omega] \in T_X\mathcal{N}(X_0)$.

$$D^2\phi(X)([X,\Omega],[X,\Omega]) = \kappa([[X-\theta X,\Omega],\Omega],\mathrm{i}N) = -\kappa([X-\theta X,\Omega],[\mathrm{i}N,\Omega]).$$

Now $S := X - \theta X \in \mathrm{i}\mathfrak{t}$ and hence

$$[S,\Omega] = \sum_{\alpha\in\Sigma^+} \alpha(S)\Omega^{(\alpha)}$$

and

$$[\mathrm{i}N,\Omega] = \sum_{\alpha\in\Sigma^+} \alpha(\mathrm{i}N)\Omega^{(\alpha)}.$$

The result follows by symmetrizing and the fact that $\kappa(\Omega^{(\alpha)},\Omega^{(\beta)}) = 0$ for $\alpha \neq \beta$. □

The following theorem is essential for proving the pointwise convergence of (44). See [12] for further results in this direction.

Theorem 13 *Let $f : M \to TM$ be a smooth complete vector field on a Riemannian manifold M and $V : M \to \mathbb{R}$ a smooth Lyapunov function with compact sublevel sets. Let $A \subset M$ be a compact Riemannian submanifold of M such that the following holds:*

(i) A is the set of global minima of V and V has no other critical points.

(ii) The Hessian of V degenerates exactly on A and the bundles of positive and negative eigenspaces of the Hessian are transversal to the tangent bundle of A.

(iii) The restriction of f on A is a gradient vector field

$$f|_A = \mathrm{grad}\phi$$

of a smooth function $\phi : A \to \mathbb{R}$ and the set Λ of equilibria of f on A is nonempty and decomposes into finitely many connected components $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$.

(iv) The Hessian of $\phi : A \to \mathbb{R}$ at each critical point $a \in \Lambda_i$ has constant rank $= \mathrm{dimA} - \mathrm{dim}\Lambda_\mathrm{i}$.

If assumptions (i)-(iii) hold, then any trajectory $t \mapsto x(t)$ converges to a connected component of the set of equilibria in A. If assumptions (i)-(iv) hold, then any trajectory of f in M converges pointwise to an equilibrium point in A.

Proof. By the assumption on V, the ω–limit sets $\omega(x)$ of each $x \in M$ are non-empty, compact and connected subsets of A. Let $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$ be the disjoint decomposition of the set of critical points of ϕ into connected components. Since ϕ is constant on each Λ_i and defines a gradient flow on A, Λ is cycle-free with Λ_i isolated invariant sets. Moreover, any ω–limit set $\omega(x)$ of a point $x \in A$ is contained in Λ. Thus $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$ is a Morse decomposition and therefore, the Butler–McGehee Lemma [6] implies that any ω–limit set $\omega(x)$ of a point $x \in M$ is contained in some Λ_i. By the last condition, the gradient flow is normally hyperbolic at Λ in A and by the second assumption on V the flow f is even normally hyperbolic at Λ in M. The result follows using an argument of [1], see e.g. [11] Ch. 1, Prop. 3.8 and [12] Thm. 6. □

Theorem 14 *Let $X_0 \in \mathfrak{g}$ be semisimple. Then the flow (44) converges from any initial condition $X(0) \in \mathcal{O}(X_0)$ to a single equilibrium point $H \in \mathcal{E}$.*

Proof. We have to check the assumptions of Theorem 13. Let $X \in \mathfrak{g}$ and let $V(X) := \frac{1}{2}\|X\|^2$ and $A := \mathcal{N}(X_0)$. Then assumption (i) and (ii) are just the Kempf-Ness-Theorem 1. (iii) is Lemma 11 together with Lemma 10 and the fact, that for $X \in \Lambda$, $\phi(X) = B_\theta(X - \theta X, \mathrm{i}N) = 2B_\theta(S_{sort}, \mathrm{i}N) = const.$ For assumption (iv), by Lemma 12 the rank of the Hessian is given by $\mathrm{rk}\mathsf{H}_X = \#\{\alpha \in \Sigma \mid \alpha(S_{sort}) \neq 0\}$. Now if $S + T \in \mathrm{i}\mathfrak{t} \oplus \mathfrak{t}$ is some diagonalization of X_0, then

$$\begin{aligned}\dim A &= \dim \mathcal{N}(X_0) = \dim\{k(S+T) \mid k \in K\}\\ &= \dim\{[\Omega, S+T] \mid \Omega \in \mathfrak{k}\}\\ &= \#\{\alpha \in \Sigma \mid \alpha(S+T) \neq 0\}.\end{aligned}$$

If Λ_i is some connected component of $\mathcal{E}$, then

$$\begin{aligned}\dim \Lambda_i &= \dim \Lambda = \dim\{[\Omega, T] \mid \Omega \in \mathrm{stab}_{\mathfrak{k}}(S_{sort})\}\\ &= \#\{\alpha \in \Sigma \mid \alpha(T) \neq 0 \text{ and } \alpha(S_{sort}) \neq 0\}\\ &= \#\{\alpha \in \Sigma \mid \alpha(T) \neq 0 \text{ and } \alpha(S) \neq 0\}.\end{aligned}$$

Thus $\dim A - \dim \Lambda_i = \mathrm{rk}\mathsf{H}_X$. □

With the above result we have almost achieved our goal, in so far that pointwise convergence to single equilibrium points is shown. However, the limiting point need not be contained in a given Cartan subalgebra and thus does not really achieve diagonalization. This property will only hold if we assume a suitable genericity condition.

Corollary 15 *If $X_0 \in \mathfrak{g}$ is strongly regular, then the flow (44) converges to some diagonalization $H \in \mathcal{O}(X_0) \cap \mathfrak{h}$.*

Proof. For regular elements, $\mathcal{O}(X_0) \cap \mathfrak{h}$ is a finite set. In fact, if $H \in \mathcal{O}(X_0) \cap \mathfrak{h}$ is some diagonalization of X_0, then

$$\mathcal{O}(X_0) \cap \mathfrak{h} = \{w(H) \mid w \in \text{ Weyl group of } G\},$$

cf. [15], Ch. IV, Sec. 6. The result follows, as for a normal element $X \in \mathcal{N}(X_0)$, the condition $[X - \theta X, N] = 0$ implies $[X + \theta X, N] = 0$ and therefore $X \in \mathfrak{h}$. □

6 Numerical Examples

Since the considered differential equation (44) evolves intrinsically on the Lie algebra, it seems reasonable to exploit this rich inherent structure for numerical simulations. The development of a structure preserving integrator, however, is beyond the scope of this paper. We refer to [5],[9] and [13] for an introduction and beyond to the concepts of geometric numerical integration. The presented numerical simulations are done with the ODE solver of SCILAB 5.1. In figure 1, the initial matrix is complex and of size 15×15. It has randomly been generated with $[0, 1]$-uniformly distributed entries. As this is generic, the flow diagonalizes the initial matrix. In figure 2, X_0 is a randomly chosen matrix with spectrum $0, 1, \ldots, 5, \mathrm{i}, \ldots, 4\mathrm{i}$. Here, the initial matrix is not strongly regular. A convergence of the flow to diagonality can therefore not be expected. However, as the dotted lines illustrate, the right-hand-side of the differential equation (44) tends to 0, indicating the pointwise convergence in any case. Moreover, the flows converge to a normal matrix X_{final}, since at final time $\|[X_{final}, \theta X_{final}]\| < 10^{-15}$.

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References

[1] B. Aulbach. *Continuous and Discrete Dynamics near Manifolds of Equilibria.* Lecture Notes in Mathematics, Vol. 1058, Springer-Verlag, Heidelberg, 1985.

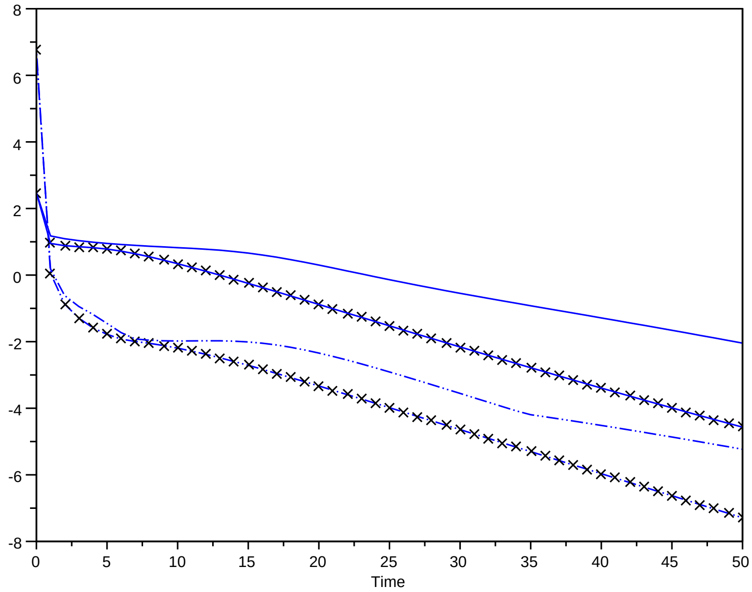

Fig. 1. Initial matrix is randomly chosen complex (15×15). The offnorm of $X(t)$ is plotted in logarithmic scale for the two parameters $\rho = 0.1$ and $\rho = 10$ (crossed lines). The dashed lines are the respective norms of $\dot{X}(t)$.

[2] R. W. Brockett. Dynamical Systems that sort lists, diagonalize matrices and solve and solve linear programming problems. *Proc. of the* 27^{th} *IEEE Conference on Decision and Control (CDC), Austin, USA, pp. 799-803, 1988.*

[3] A. Bloch, R.W. Brockett and T. Ratiu, A new formulation of the generalized Toda lattice equations and their fixed point analysis via the moment map. *Bull. Amer. Math. Soc.*, 23(2), 477–486, 1990.

[4] A. Bloch, R.W. Brockett and T. Ratiu, Completely integrable gradient flows. *Comm. Math. Phys.*, 147(1), 54–74, 1992.

[5] N. Del Buono and T. Politi (Eds). *Geometric Numerical Algorithms (Special Issue)* Future Generation Computer Systems. Vol. 19 (3), 2003.

[6] G. Butler and P. Waltman. Persistence in dynamical systems. *J. Differential Equations.* Vol. 63, 255–263, 1986.

[7] C. Ebenbauer. A Dynamical System That Computes Eigenvalues and Diagonalizes Matrices with a Real Spectrum. *Proc. of the* 46^{th} *IEEE Conference on Decision and Control (CDC), New Orleans, pp. 1704-1709, 2007.*

[8] C. Ebenbauer and A. Arsie. On an eigenflow and its Lie algebraic generaliztion. *Communications in Information and Systems*, Vol. 8 (2), 147–170, 2008.

[9] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration: Structure-Preserving Algorithms for ordinary Differential Equations. 2nd Ed.* Springer, Berlin, 2006.

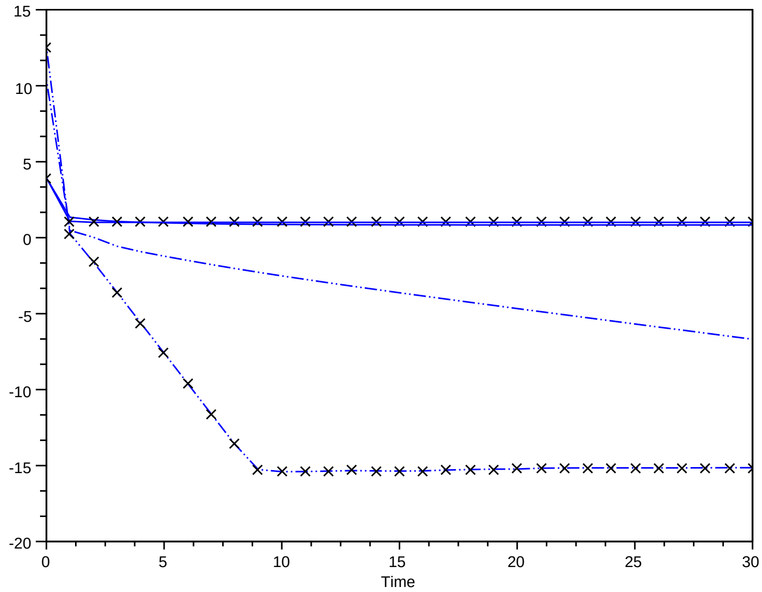

Fig. 2. Initial matrix is randomly chosen matrix with prescribed spectrum $0, 1, \ldots, 5, \mathrm{i}, \ldots, 4\mathrm{i}$. The offnorm of $X(t)$ is plotted in logarithmic scale for the two parameters $\rho = 0.1$ and $\rho = 1$ (crossed lines). The dashed lines are the respective norms of $\dot{X}(t)$.

[10] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, San Diego, 1978.

[11] U. Helmke and J.B. Moore. *Optimization and Dynamical Systems.* Springer, London, 1994.

[12] U. Helmke. Global convergence of nonlinear cascade flows with Morse–Bott zero dynamics. *Systems and Control Letters.* Vol. 58(6), 389-394, 2009.

[13] A. Iserles, H. Z. Munthe-Kaas, S. P. Nørsett, and A. Zanna. Lie-Group methods. *Acta Numerica.* Vol. 9, 215–365, 2000.

[14] G. Kempf and L. Ness. The lengths of vectors in representation spaces, in K. Lonsted (ed.), *Algebraic Geometry, Lecture Notes in Mathematics 732*, Springer: Berlin, London, New York, 233–244, 1979.

[15] A.W. Knapp. *Lie Groups Beyond an Introduction,* 2^{nd} *Ed.* Birkhäuser, Boston, 2002.

[16] V. S. Varadarajan. *Lie Groups, Lie Algebras, and Their Representations.* Springer, New York, 1984.