List Decoding for Bidirectional Broadcast Channels with Unknown Varying Channels

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***Abstract*—The concept of bidirectional relaying shows the potential to improve the performance in wireless networks such as sensor, ad-hoc, and even cellular systems. It applies to three-node networks, where a relay node establishes a bidirectional communication between two other nodes. In the first phase of a decode-and-forward protocol, the two nodes transmit their messages to a relay node, which decodes them. In the succeeding bidirectional broadcast phase, the relay broadcasts a re-encoded composition of them so that both nodes can decode the other's message using their own message as side information. We assume that the transmission is affected by unknown varying channels. Unfortunately, the unknown variation of the channel can lead to channels which completely prohibit any reliable communication. In this work, we analyze the bidirectional broadcast phase under list decoding and show that this decoding technique can improve the performance significantly in the sense that it allows to transmit reliably in scenarios where usual decoding schemes fail.**

I. Introduction

Future wireless communication systems will make great demands on coverage and throughput. This is a challenging task for cellular systems especially at the cell edges. To meet these requirements the use of relays is a promising approach, which is intensively discussed at the moment by the Third Generation Partnership Program's Long-Term Evolution Advanced (3GGP LTE-Advanced) group.

Due to practical constraints a relay cannot transmit and receive at the same time and frequency. The consequence is that orthogonal resources for transmission and reception are needed, which can be realized more efficiently, if bidirectional communication is considered [1], [2]. In this work, we consider bidirectional relaying in a three-node network, where a relay node establishes a bidirectional communication between two other nodes. This is the *bidirectional relay channel*, which is also known as the two-way relay channel.

There exist different strategies for bidirectional relaying which are usually classified by the processing at the relay node. The most common schemes are amplify-and-forward [1], decode-and-forward [1], [3], [4], and compress-and-forward [5], [6]. Here, we consider a two-phase decode-and-forward protocol, where cooperation between the encoders of nodes 1 and 2 is not allowed, cf. Figure 1. In the initial phase both nodes transmit their messages to the relay node. Since the relay node decodes both messages, we end up with the classical multiple access channel. In the succeeding phase the relay broadcasts a re-encoded message based on the network coding idea so that nodes 1 and 2 are able to decode the other's message using their own message from the previous phase as side information. Note that due to the side information at the receiving nodes this channel differs from the classical broadcast channel. To emphasize this property, this channel is called *bidirectional broadcast channel* (BBC). The optimal coding strategy and capacity region is derived in [4] under the assumption of perfect channel state information (CSI) at the transmitter and the receivers.

(a) MAC phase (b) BBC phase

Fig. 1. Multiple access (MAC) and bidirectional broadcast (BBC) phase of a bidirectional relay communication.

Due to the nature of the wireless channel the assumption that perfect CSI can be provided at all nodes is an ambitious aim. Consequently, uncertainty in the channel knowledge is an ubiquitous phenomenon in practical systems. Channel estimation is an approach to mitigate the channel uncertainty, which is quite popular nowadays. It is clear that this is only one specific approach and it seems to be natural to treat this problem from a more general point of view to gain hints and insights what is at best possible under channel uncertainty.

A well accepted model for channel uncertainty is to assume that all nodes do not know the exact channel realization, but only know that this realization belongs to a prespecified set of channels. If this channel remains fixed during the whole transmission of a codeword, this corresponds to the compound channel. Not surprisingly, this kind of uncertainty leads to a loss in capacity compared to the case of full CSI. Decode-and-forward bidirectional relaying is well understood in the compound scenario [7]. Thereby, it shows that the bidirectional character of the communication can be exploited to enhance the spectral efficiency.

The variation of the channel from symbol to symbol in an unknown and arbitrary manner is an additional effect of channel uncertainty. This is the concept of *arbitrarily varying channels* (AVC) [8], [9], [10] and it seems to be worth to include this kind of uncertainty to obtain more robust solu-

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tions. Bidirectional relaying in the arbitrarily varying scenario is analyzed in [11], [12], where it is shown that the impact of the unknown variation is much more dramatic than by compound channels. The uncertainty in the variation can lead to symmetric channels so that the receiving nodes cannot distinguish between different codewords. The consequence is that reliable communication is no longer possible.

Blinovsky et. al. [13] and Hughes [14] independently showed for the single-user AVC that the concept of list decoding might help to dissolve the ambiguity of codewords caused by symmetric channels. Then this decoding technique allows to communicate reliably in scenarios where usual decoding schemes fail. For a general survey about the concept of list decoding we refer to [15]. It seems to be natural to extend the list decoding idea to the bidirectional relaying scenario to improve the performance of the bidirectional communication. In this work, we focus on the second phase, i.e., the bidirectional broadcast channel.[1]

II. Bidirectional Broadcast Channel with Unknown Varying Channels

The transmission in the bidirectional broadcast phase is affected by channels which vary arbitrarily in an unknown manner from symbol to symbol. To model this behavior we introduce a finite state set $\mathcal{S}$. Further, let $\mathcal{X}$ and $\mathcal{Y}_k$, $k=1,2$ be finite input and output sets. Then, for a fixed state sequence $s^n \in \mathcal{S}^n$ of length n and input and output sequences $x^n \in \mathcal{X}^n$ and $y_k^n \in \mathcal{Y}_k^n$, $k=1,2$, the discrete memoryless broadcast channel is given by $W^{\otimes n}(y_1^n, y_2^n|x^n, s^n) := \prod_{i=1}^n W(y_{1,i}, y_{2,i}|x_i, s_i)$.

Definition 1: The discrete memoryless *arbitrarily varying broadcast channel* $\mathcal{W}$ is the family

$$\mathcal{W} := \left\{W^{\otimes n} : \mathcal{X}^n \times \mathcal{S}^n \to \mathcal{P}(\mathcal{Y}_1^n \times \mathcal{Y}_2^n)\right\}_{n\in\mathbb{N}, s^n\in\mathcal{S}^n}$$

where $\mathcal{P}(\cdot)$ denotes the set of all probability distributions.

Since we do not allow any cooperation between the receiving nodes, it is sufficient to consider the marginal transition probabilities $W_k^{\otimes n}(y_k^n|x^n, s^n)$, $k=1,2$, only. Further, for any probability distribution $q \in \mathcal{P}(\mathcal{S})$ we denote the averaged broadcast channel by

$$\overline{W}_q(y_1, y_2|x) := \sum_{s\in\mathcal{S}} W(y_1, y_2|x, s)q(s) \tag{1}$$

and the corresponding averaged marginal channels by $\overline{W}_{1,q}(y_1|x)$ and $\overline{W}_{2,q}(y_2|x)$.

III. Preliminaries

Before we present the main result we need to introduce some standard notation and basic concepts.

[1] *Notation:* Discrete random variables are denoted by capital letters and their corresponding realizations and ranges by lower case and calligraphic letters respectively; $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}_+$ the set of non-negative real numbers; all logarithms, exponentials, and information measures are taken to the basis 2; $\mathcal{P}(\cdot)$ denotes the set of all probability distributions; $\mathfrak{P}_L(\mathcal{M})$ is the set of all subsets of $\mathcal{M}$ with cardinality L; $\hat{\mathfrak{P}}_L(\mathcal{M})$ is the set of all subsets of $\mathcal{M}$ with cardinality at most L; lhs := rhs means the value of the right hand side (rhs) is assigned to the left hand side (lhs).

A. Definitions

We consider the standard model with a block code of arbitrary but fixed length n. Let $\mathcal{M}_k := \{1, ..., M_k^{(n)}\}$ be the message set of node k, $k=1,2$, which is also known at the relay node. Further, we make use of the abbreviation $\mathcal{M} := \mathcal{M}_1 \times \mathcal{M}_2$.

Definition 2: A deterministic $(M_1^{(n)}, M_2^{(n)}, L_1, L_2, n)$*-list code* of length n with list sizes (L_1, L_2) for the arbitrarily varying bidirectional broadcast channel (AVBBC) consists of codewords

$$x_m^n \in \mathcal{X}^n$$

one for each message $m = (m_1, m_2)$, and list decoders at nodes 1 and 2

$$\mathcal{L}^{(1)} : \mathcal{Y}_1^n \times \mathcal{M}_1 \to \hat{\mathfrak{P}}_{L_1}(\mathcal{M}_2)$$
$$\mathcal{L}^{(2)} : \mathcal{Y}_2^n \times \mathcal{M}_2 \to \hat{\mathfrak{P}}_{L_2}(\mathcal{M}_1)$$

where $\hat{\mathfrak{P}}_{L_1}(\mathcal{M}_2)$ is the set of all subsets of $\mathcal{M}_2$ with cardinality at most L_1 and similarly $\hat{\mathfrak{P}}_{L_2}(\mathcal{M}_1)$ is the set of all subsets of $\mathcal{M}_1$ with cardinality at most L_2.

When x_m^n with $m=(m_1,m_2)$ has been sent, and y_1^n and y_2^n have been received at nodes 1 and 2, the list decoder at node 1 is in error if m_2 is not in $\mathcal{L}^{(1)}(y_1^n, m_1)$. Accordingly, the list decoder at node 2 is in error if m_1 is not in $\mathcal{L}^{(2)}(y_2^n, m_2)$. This allows us to define the probabilities of error for given message $m=(m_1,m_2)$ and given state sequence $s^n \in \mathcal{S}^n$ at nodes 1 and 2 as

$$\lambda_1(m, s^n) := \sum_{y_1^n : m_2 \notin \mathcal{L}^{(1)}(y_1^n, m_1)} W_1^{\otimes n}(y_1^n|x_m^n, s^n)$$
$$\lambda_2(m, s^n) := \sum_{y_2^n : m_1 \notin \mathcal{L}^{(2)}(y_2^n, m_2)} W_2^{\otimes n}(y_2^n|x_m^n, s^n)$$

and the *average probability of error* for state sequence $s^n \in \mathcal{S}^n$ at node k, $k=1,2$, as

$$\bar{\lambda}_k^{(n)}(s^n) := \frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} \lambda_k(m, s^n).$$

Definition 3: A rate pair $(R_{\mathrm{R1}}, R_{\mathrm{R2}}) \in \mathbb{R}_+^2$ is said to be deterministically *achievable* for the AVBBC $\mathcal{W}$ if for any $\delta > 0$ there exists an $n(\delta) \in \mathbb{N}$ and a sequence of deterministic $(M_1^{(n)}, M_2^{(n)}, L_1, L_2, n)$-list codes such that for all $n \geq n(\delta)$ we have

$$\frac{1}{n}\log\left(\frac{M_1^{(n)}}{L_2}\right) \geq R_{\mathrm{R2}} - \delta \quad \text{and} \quad \frac{1}{n}\log\left(\frac{M_2^{(n)}}{L_1}\right) \geq R_{\mathrm{R1}} - \delta$$

while $\max_{s^n\in\mathcal{S}^n} \bar{\lambda}_k^{(n)}(s^n) \to 0$ as $n\to\infty$, $k=1,2$. The set of all achievable rate pairs is the *list capacity region* with list sizes (L_1, L_2) of the AVBBC $\mathcal{W}$ and is denoted by $\mathcal{R}_{\text{list}}(L_1, L_2)$.

Remark 1: The definitions above require that we have to find codes such that the average probability of error goes to zero as the block length tends to infinity for all possible state sequences simultaneously. This means the codes are universal with respect to the state sequence.

B. Types of Sequences

In this work, we need the definition of the mutual information [10, p. 21]. We denote the mutual information between the input random variable X and the output random variable Y by $I(X;Y)$. Moreover, to emphasize the dependency of the mutual information on the input distribution P_X and the channel W, we also write $I(X;Y) = I(P_X, W)$ interchangeably.

Further, we utilize the concept of types from Csiszár and Körner [10] which will be a crucial technique to prove our results. The *type* of a sequence $x^n = (x_1, ..., x_n) \in \mathcal{X}^n$ is a distribution P_{x^n} on $\mathcal{X}$ where $P_{x^n}(a)$ is the relative frequency of a in x^n. The set of all sequences of type P_{x^n} is denoted by $\tau_X = \{x^n : x^n \in \mathcal{X}^n, P_{x^n} = P_X\}$.

For η_k, $k = 1, 2$, we define a family of joint distributions P_{XSY_k} of random variables X, S, and Y_k in $\mathcal{X}$, $\mathcal{S}$, and $\mathcal{Y}_k$, respectively, by

$$\mathcal{C}_{\eta_k} := \{P_{XSY_k} : D(P_{XSY_k} \| P_X \times P_S \times W_k) \le \eta_k\}$$

where $D(\cdot\|\cdot)$ denotes the (Kullback-Leibler) information divergence [10, p. 20] and $P_X \times P_S \times W_k$ denotes a joint distribution on $\mathcal{X} \times \mathcal{S} \times \mathcal{Y}_k$ with probability mass function $P_X(x)P_S(s)W_k(y_k|x, s)$.

IV. Symmetrizability

For list decoders $\mathcal{L}^{(1)}$, $\mathcal{L}^{(2)}$ with list sizes $L_1 = 1$, $L_2 = 1$ at the receiving nodes 1 and 2, respectively, the list coding problem reduces to the usual coding problem for the AVBBC under the average error criterion which is analyzed in detail in [11] and [12]. For this purpose we define the following region

$$\begin{aligned}\mathcal{R}_{\text{ran}} := \big\{(R_{\text{R1}}, R_{\text{R2}}) : &R_{\text{R1}} \le \inf_{q \in \mathcal{P}(\mathcal{S})} I(P_{X|U}, \overline{W}_{1,q}|U),\\ &R_{\text{R2}} \le \inf_{q \in \mathcal{P}(\mathcal{S})} I(P_{X|U}, \overline{W}_{2,q}|U)\big\}\end{aligned}$$

for random variables $(U, X, Y_1, Y_2) \in \mathcal{U} \times \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$ and joint probability distributions $\{P_U(u)P_{X|U}(x|u)\overline{W}_q(y_1, y_2|x)\}_{q \in \mathcal{P}(\mathcal{S})}$. Thereby, U is an auxiliary random variable and describes a possible time-sharing operation. Note that this region corresponds to the random code capacity region of the AVBBC $\mathcal{W}$, cf. Section V-A and also [11]. Further, we define the maximum single-user random code rates as

$$R_{\text{R}k,\text{ran}} := \max_{P_X} \inf_{q \in \mathcal{P}(\mathcal{S})} I(P_X, W_{k,q}), \quad k = 1, 2.$$

In [11] we reveal the following behavior of the deterministic code capacity region $\mathcal{R}_{\text{list}}(1, 1)$ of the AVBBC $\mathcal{W}$:

$$\mathcal{R}_{\text{list}}(1, 1) = \mathcal{R}_{\text{ran}} \qquad \text{if interior}(\mathcal{R}_{\text{list}}(1, 1)) \ne \emptyset.$$

Further, it is shown that interior$(\mathcal{R}_{\text{list}}(1, 1)) \ne \emptyset$ if and only if the AVBBC $\mathcal{W}$ is non-$\mathcal{Y}_1$-symmetrizable and non-$\mathcal{Y}_2$-symmetrizable [12].

A key idea for the analysis of the list decoding technique is an extension of the concept of symmetrizability. For this purpose we follow [13], [14] and introduce a refinement of a symmetrizable channel, which distinguishes among different degrees of symmetry.

We say a channel $V(y|x_1, ..., x_t)$ with input alphabet $\mathcal{X}^t$ and output alphabet $\mathcal{Y}$ is symmetric in $x_1, ..., x_t$ if the channel is invariant over all permutations of the inputs $x_1, ..., x_t$ for all $y, x_1, ..., x_t$. This leads to the following definition.

Definition 4: For any $t_k \ge 1$, $k = 1, 2$, an AVBBC is $(\mathcal{Y}_k, t_k)$-*symmetrizable* if there is a channel $U : \mathcal{X}^{t_k} \to \mathcal{P}(\mathcal{S})$ such that

$$V_k(y_k|x, x_1, ..., x_{t_k}) := \sum_{s \in \mathcal{S}} W_k(y_k|x, s)U(s|x_1, ..., x_{t_k})$$

is symmetric in $x, x_1, ..., x_{t_k}$. For convenience, we take all AVBBC's to be $(\mathcal{Y}_k, 0)$-symmetrizable, $k = 1, 2$.

A $(\mathcal{Y}_k, t_k)$-symmetrizable channel can be interpreted as a channel where the state sequence can emulate t_k replicas of the channel input x. It is clear from the definition that if an AVBBC is $(\mathcal{Y}_k, t_k)$-symmetrizable then it is also $(\mathcal{Y}_k, t'_k)$-symmetrizable for all $0 \le t'_k \le t_k$. Consequently, a $\mathcal{Y}_k$-symmetrizable channel in the sense of [12] is $(\mathcal{Y}_k, 1)$-symmetrizable in the terminology of Definition 4. The following theorem relates the symmetrizability and the maximum single-user random code rates.

Theorem 1: If $R_{\text{R}k,\text{ran}} = 0$, $k = 1, 2$, then the AVBBC $\mathcal{W}$ is $(\mathcal{Y}_k, t_k)$-symmetrizable for all $t_k \ge 0$. If $R_{\text{R}k,\text{ran}} > 0$, then any $(\mathcal{Y}_k, t_k)$-symmetrizable AVBBC satisfy

$$t_k \le \frac{\log(\min\{|\mathcal{Y}_k|, |\mathcal{S}|\})}{R_{\text{R}k,\text{ran}}}. \tag{2}$$

The theorem is a straight forward extension of a similar result for the single-user AVC [14, Theorem 1]. For brevity the proof is omitted.

Definition 5: If interior$(\mathcal{R}_{\text{ran}}) \ne \emptyset$, the *symmetrizability* of an AVBBC is denoted by (T_1, T_2) and is defined by the largest pair of integers (t_1, t_2) such that the AVBBC is $(\mathcal{Y}_1, t_1)$-symmetrizable and $(\mathcal{Y}_2, t_2)$-symmetrizable.

It follows immediately from Theorem 1 that the symmetrizability of an AVBBC is finite whenever interior$(\mathcal{R}_{\text{ran}}) \ne \emptyset$. Moreover, it provides simple upper bounds on the symmetrizability which are given by the largest integers which satisfy the condition (2).

Lemma 1: Let (T_1, T_2) be the symmetrizability of an AVBBC. Then any list code of block length n with $M_1^{(n)} M_2^{(n)}$ messages and $L_1 \le T_1$ satisfies

$$\max_{s^n \in \mathcal{S}^n} \bar{\lambda}_1^{(n)}(s^n) \ge \left(1 - \frac{L_1}{K_1 + 1}\right)\left(\frac{M_2^{(n)} - K_1}{M_2^{(n)}}\right)$$

where $K_1 = \min\{M_2^{(n)} - 1, T_1\}$. Similarly, any list code of block length n with $M_1^{(n)} M_2^{(n)}$ messages and $L_2 \le T_2$ satisfies

$$\max_{s^n \in \mathcal{S}^n} \bar{\lambda}_2^{(n)}(s^n) \ge \left(1 - \frac{L_2}{K_2 + 1}\right)\left(\frac{M_1^{(n)} - K_2}{M_1^{(n)}}\right)$$

where $K_2 = \min\{M_1^{(n)} - 1, T_2\}$.

The lemma is an extension of [14, Lemma 4], which is a similar result for the single-user AVC. The lemma shows that if $L_k \le T_k$, $k = 1, 2$, then the list size of the decoder at node

k does not suffice to reliably restrict the transmitted message so that the probability of error is bounded from below by a positive constant.

V. List Capacity Region

Now we are in the position to state the list capacity region of the AVBBC which is the main result of this work.

Theorem 2: The list capacity region $\mathcal{R}_{\text{list}}(L_1, L_2)$ with list sizes (L_1, L_2) of the AVBBC $\mathcal{W}$ is given by

$$\mathcal{R}_{\text{list}}(L_1, L_2) = \mathcal{R}_{\text{ran}} \qquad \text{if } L_1 > T_1 \text{ and } L_2 > T_2.$$

Further, interior$(\mathcal{R}_{\text{list}}(L_1, L_2)) = \emptyset$ if and only if $L_1 \leq T_1$ or $L_2 \leq T_2$.

The theorem shows that for every AVBBC with interior$(\mathcal{R}_{\text{ran}}) \neq \emptyset$ there is a characteristic pair of minimum list sizes (T_1+1, T_2+1). This implies that reliable communication with rates $(R_{\text{R}1}, R_{\text{R}2}) \in \mathcal{R}_{\text{ran}}$ can be guaranteed provided that the list sizes at the decoders satisfy $L_k > T_k$, $k = 1, 2$, while no reliable communication is possible for smaller list sizes, i.e., $L_k \leq T_k$.

In the following subsections we present the proof of the theorem. As a first step, we characterize the general behavior of the list capacity region. Then, we present a coding strategy which achieves positive rates. As we will see, this will be sufficient to establish the capacity region. As a final task, we show that this is actually the capacity region, which means that no other rate pairs are achievable.

A. Characterization of General Behavior

First, we establish a result for the list capacity region which is similar to Ahlswede's dichotomy result for the single-user AVC [9]. This means that the list capacity region either equals the random code capacity region or else has an empty interior.

Lemma 2: The list capacity region $\mathcal{R}_{\text{list}}(L_1, L_2)$ with list sizes (L_1, L_2) displays the following behavior:

$$\mathcal{R}_{\text{list}}(L_1, L_2) = \mathcal{R}_{\text{ran}} \qquad \text{if interior}(\mathcal{R}_{\text{list}}(L_1, L_2)) \neq \emptyset.$$

Proof: From [11] we know that for the AVBBC all rate pairs $(R_{\text{R}1}, R_{\text{R}2}) \in \mathcal{R}_{\text{ran}}$ are achievable using a random coding strategy. A random code consists of a whole family of deterministic codes and a random variable. Thereby, the "random" refers to the fact that the encoder and decoders are chosen according to the common random variable whose outcome has to be known at all nodes. Unfortunately, it shows that the number of needed encoder and decoders of the random code is at least exponential.

But similarly to [11] we can apply Ahlswede's *elimination technique* [9] to construct a new random code. It shows that it suffices for the new code to consist of a relatively small number of deterministic codes, more precisely n^2 codes, and a random variable which is uniformly distributed on these deterministic codes.

If interior$(\mathcal{R}_{\text{list}}(L_1, L_2)) \neq \emptyset$, in a second step we can convert the "reduced" random code into a list code by adding short prefixes to the original codewords to inform the decoders which of the codes is actually used. Since the number of possible codes is small enough, the transmission of those additional information will not cause an essential loss in capacity. In [11] a usual deterministic prefix code is constructed. Since this is a special list code, namely with list sizes $(1, 1)$, it is evident that we can also construct a general list code with arbitrary list sizes (L_1, L_2) to inform the decoders. ■

Next, we want to analyze this behavior in more detail to connect it with the list sizes of the decoders and the symmetrizability of the channel.

Therefore, we first observe that the last part of Theorem 2, i.e., when interior$(\mathcal{R}_{\text{list}}(L_1, L_2)) = \emptyset$, already follows from Lemma 1, which shows that interior$(\mathcal{R}_{\text{list}}(L_1, L_2)) = \emptyset$ if and only if $L_1 \leq T_1$ or $L_2 \leq T_2$.

It remains to study the case when interior$(\mathcal{R}_{\text{list}}(L_1, L_2)) \neq \emptyset$, which is done in the following. Thereby, we present a strategy which achieves positive rates when $L_k > T_k$, $k = 1, 2$. Together with the characterization of the general behavior, cf. Lemma 2, the list capacity region follows immediately.

B. Coding Strategy

To achieve positive rates we need a suitable set of codewords $x^n_{m_1,m_2}$, $m_1 = 1, ..., M_1^{(n)}$, $m_2 = 1, ..., M_2^{(n)}$, with properties as stated in the following lemma.

Lemma 3: For any $L_1 \geq 1, L_2 \geq 1$, $\epsilon > 0$, $n \geq n_0(\epsilon, L_1, L_2)$, $R_{\text{R}2} = \frac{1}{n}\log(\frac{M_1^{(n)}}{L_2})$, $R_{\text{R}1} = \frac{1}{n}\log(\frac{M_2^{(n)}}{L_1})$, $M_1^{(n)} \geq L_2 2^{n\epsilon}$, $M_2^{(n)} \geq L_1 2^{n\epsilon}$, and given type P_X, there exist codewords $x^n_{m_1,m_2} \in \tau_X$, $m_1 = 1, ..., M_1^{(n)}$, $m_2 = 1, ..., M_2^{(n)}$ such that for every $x^n_{m_1,m_2} \in \tau_X$, $s^n \in \mathcal{S}^n$, and every joint type $P_{XX^{L_1}S}$ with $X^{L_1} = X_1, X_2, ..., X_{L_1}$ we have for each $m_1 \in \mathcal{M}_1$

$$|\{m_2' : (x^n_{m_1,m_2}, x^n_{m_1,m_2'}, s^n) \in \tau_{XX_kS}\}| \leq 2^{n(|R_{\text{R}1} - I(X_k;XS)|^+ + \epsilon)}$$

$$\frac{1}{M_2^{(n)}}|\{m_2 : (x^n_{m_1,m_2}, s^n) \in \tau_{XS}\}| \leq 2^{-n\frac{\epsilon}{2}} \quad \text{if } I(X;S) \geq \epsilon$$

$$\frac{1}{M_2^{(n)}}|\{m_2 : (x^n_{m_1,m_2}, x^n_{m_1,m_2'}, s^n) \in \tau_{XX_kS} \text{ for some } m_2' \neq m_2\}| \leq 2^{-n\frac{\epsilon}{2}} \quad \text{if } I(X;X_kS) - |R_{\text{R}1} - I(X_k;S)|^+ \geq \epsilon$$

for $k = 1, ..., L_1$. Moreover, if $R_{\text{R}1} < \min_k I(X_k;S)$, then $x^n_{m_1,m_2}$, $m_1 = 1, ..., M_1^{(n)}$, $m_2 = 1, ..., M_2^{(n)}$, can be selected to further satisfy

$$|\{J \in \mathfrak{P}_{L_1}(\mathcal{M}_2) : (x^n_{m_1,m_2}, x^n_{m_1,J}, s^n) \in \tau_{XX^{L_1}S}\}| \leq 2^{n\epsilon}$$

$$\frac{1}{M_2^{(n)}}|\{m_2 : (x^n_{m_1,m_2}, x^n_{m_1,J}, s^n) \in \tau_{XX^{L_1}S} \text{ for some } J \in \mathfrak{P}_{L_1}(\mathcal{M}_2 \backslash \{m_2\})\}| \leq 2^{-n\frac{\epsilon}{2}} \quad \text{if } I(X;X^{L_1}S) \geq \epsilon$$

with $J = (j_1, ..., j_{L_1}) \in \mathfrak{P}_{L_1}(\mathcal{M}_2)$ and $x^n_{m_1,J}$ denotes the ordered L_1-tuple $(x^n_{m_1,j_1}, x^n_{m_1,j_2}, ... x^n_{m_1,j_{L_1}})$ where the indices are ordered as $j_1 < j_2 < ... < j_{L_1}$. Similarly, for every $x^n_{m_1,m_2} \in \tau_X$, $s^n \in \mathcal{S}^n$, and every joint type $P_{XX^{L_2}S}$

we have for each $m_2 \in \mathcal{M}_2$

$$|\{m_1':(x^n_{m_1,m_2},x^n_{m_1',m_2},s^n)\in\tau_{XX_kS}\}|\leq 2^{n(|R_{\text{R}2}-I(X_k;XS)|^++\epsilon)}$$
$$\frac{1}{M_1^{(n)}}|\{m_1:(x^n_{m_1,m_2},s^n)\in\tau_{XS}\}|\leq 2^{-n\frac{\epsilon}{2}} \quad \text{if } I(X;S)\geq\epsilon$$
$$\frac{1}{M_1^{(n)}}|\{m_1:(x^n_{m_1,m_2},x^n_{m_1',m_2},s^n)\in\tau_{XX_kS} \text{ for some}$$
$$m_1'\neq m_1\}|\leq 2^{-n\frac{\epsilon}{2}} \quad \text{if } I(X;X_kS)-|R_{\text{R}2}-I(X_k;S)|^+\geq\epsilon$$

for $k=1,...,L_2$. Moreover, if $R_{\text{R}2}<\min_k I(X_k;S)$, then $x^n_{m_1,m_2}$, $m_1=1,...,M_1^{(n)}$, $m_2=1,...,M_2^{(n)}$, can be selected to further satisfy

$$|\{J\in\mathfrak{P}_{L_2}(\mathcal{M}_1):(x^n_{m_1,m_2},x^n_{J,m_2},s^n)\in\tau_{XX^{L_2}S}\}|\leq 2^{n\epsilon}$$
$$\frac{1}{M_1^{(n)}}|\{m_1:(x^n_{m_1,m_2},x^n_{J,m_2},s^n)\in\tau_{XX^{L_2}S} \text{ for some}$$
$$J\in\mathfrak{P}_{L_2}(\mathcal{M}_1\backslash\{m_1\})\}|\leq 2^{-n\frac{\epsilon}{2}} \quad \text{if } I(X;X^{L_2}S)\geq\epsilon.$$

The lemma shows that these good codewords are obtained by randomly selecting codewords from the set of sequences of a fixed type. All such codewords will possess the desired properties with probability arbitrarily close to 1. Thereby, the lemma is similar to [14, Lemma 1] for the single-user AVC and is only slightly modified to ensure that the desired properties of the codewords hold for both receiving nodes.

C. Decoding Strategy

A crucial part is to define suitable list decoders at the receiving nodes 1 and 2. We follow [13], [14] and use a divergence typicality decoder, which is a generalization of the one introduced in [16] for a decoder with list size one. The list decoder with list size L_1 at node 1 is defined as follows.

Definition 6: For given codewords $x^n_{m_1,m_2}\in\tau_X$, $m_1=1,...,M_1^{(n)}$, $m_2=1,...,M_2^{(n)}$, and (small) $\eta_1>0$, let $m_2\in\mathcal{L}^{(1)}(y_1^n,m_1)$ if and only if

I) there exists an $s^n\in\mathcal{S}^n$ such that $P_{x^n_{m_1,m_2},s^n,y_1^n}\in\mathcal{C}_{\eta_1}$
II) for each choice of L_1 other distinct codewords $x^n_{m_1,j_1},...,x^n_{m_1,j_{L_1}}$, where each satisfies $P_{x^n_{m_1,j_k},s^n_k,y^n_1}\in\mathcal{C}_{\eta_1}$ for some $s^n_k\in\mathcal{S}^n$, we have $I(XY_1;X^{L_1}|S)\leq\eta_1$ where $X^{L_1}=X_1,X_2,...,X_{L_1}$ and $P_{XX^{L_1}SY_1}$ is the joint type of $(x^n_{m_1,m_2},x^n_{m_1,j_1},...,x^n_{m_1,j_{L_1}},s^n,y^n_1)$.

The list decoder with list size L_2 at node 2 is defined accordingly with (small) constant $\eta_2>0$. To establish the list capacity region for $L_k>T_k$, $k=1,2$ (cf. Theorem 2), we have to ensure that the list decoders as given in the definition above are well defined, which means that they satisfy the constraint on the list size, i.e., $|\mathcal{L}^{(k)}(y^n_k,m_k)|\leq L_k$ for all $m_k\in\mathcal{M}_k$ and $y^n_k\in\mathcal{Y}^n_k$, $k=1,2$. Clearly, it suffices to show that $|\mathcal{L}^{(k)}(y^n_k,m_k)|\leq T_k+1$ for all $y^n_k\in\mathcal{Y}^n_k$ for a sufficiently small $\eta_k>0$. This is guaranteed by the following lemma.

Lemma 4: Let $\beta>0$, then for a sufficiently small η_k, $k=1,2$, no ensemble X^{T_k+2},S^{T_k+2},Y_k can simultaneously satisfy

$$\min_x P_X(x)\geq\beta$$

and

$$P_{X_i}=P,\quad P_{X_iS_iY_k}\in\mathcal{C}_{\eta_k}$$
$$I(X_iY_k;X_i^{T_k+2}|S_i)\leq\eta_k \qquad 1\leq i\leq T_k+2 \tag{3}$$

where $X_i^{T_k+2}=X_1,...,X_{i-1},X_{i+1},...,X_{T_k+2}$.

The lemma is an extension of [14, Lemma 2], which is a similar lemma for the single-user AVC. The proof is analogously and therefore omitted for brevity.

D. Positive Rates

So far we defined coding and decoding rules. Next, we show that this strategy is sufficient to achieve (small) positive rates if the list sizes are great enough, which means $L_k>T_k$, $k=1,2$.

Lemma 5: Let $L_k=T_k+1$, $k=1,2$, and $\beta>0$. For any type P_X with $\min_x P_X(x)\geq\beta$ and any δ satisfying $\frac{6}{7}\min_k\inf_{q\in\mathcal{P}(\mathcal{S})}I(P_X,\overline{W}_{k,q})\leq\delta<\min_k\inf_{q\in\mathcal{P}(\mathcal{S})}I(P_X,\overline{W}_{k,q})$, there exist positive rates $R_{\text{R}1}$ and $R_{\text{R}2}$ such that

$$\max_{s^n\in\mathcal{S}^n}\bar{\lambda}_k^{(n)}(s^n)<2^{-n\gamma_k}, \qquad k=1,2, \tag{4}$$

where $\gamma_k>0$ depends only on β, δ, and the AVBBC $\mathcal{W}$.

Outline of the Proof: First, observe that for $L_k>T_k$, $k=1,2$, we have $\inf_{q\in\mathcal{P}(\mathcal{S})}I(P_X,\overline{W}_{k,q})>0$ for any P_X with $\min_x P_X(x)>0$ so that all expressions are well defined. We choose $\epsilon=\min_k\inf_{q\in\mathcal{P}(\mathcal{S})}I(P_X,\overline{W}_{k,q})-\delta$ and set $M_1^{(n)}=M_2^{(n)}=2^{n\epsilon}>0$. Then, Lemma 3 is applicable and ensures that the all generated codewords have the desired properties. Similarly as in [14, Lemma 3], these properties are needed to show that the probability of error vanishes for increasing block length n. Further, we choose η_1,η_2 small enough so that the list decoders satisfy the constraints on the list sizes, cf. Lemma 4. With this, following [14, Lemma 3], the probability of error (4) gets arbitrary small. In this work, we concentrate on the main concepts and ideas. Since the rest of the proof is mostly technical, it is omitted for brevity.

The lemma shows that with this strategy we can achieve (small) positive rates so that interior$(\mathcal{R}_{\text{list}}(L_1,L_2))\neq\emptyset$ holds. Together with the characterization of the general behavior, cf. Lemma 2, it immediately follows that $\mathcal{R}_{\text{list}}(L_1,L_2)=\mathcal{R}_{\text{ran}}$ which completes the achievability part of the proof.

E. Optimality

It remains to show that no other rate pairs are achievable so that Theorem 2 completely characterize the list capacity region.

As a first step, it is easy to show that the average probability of error of the AVBBC $\mathcal{W}$ under list decoding equals the average probability of error of the averaged bidirectional broadcast channel under list decoding. Hence, for the AVBBC $\mathcal{W}$ under list decoding we cannot achieve higher rates as for the averaged bidirectional broadcast channel under list decoding.

For a fixed distribution $q\in\mathcal{P}(\mathcal{S})$ each marginal channel of the averaged bidirectional broadcast channel, cf. (1), corresponds to a usual discrete memoryless channel (DMC). The maximal achievable rates under list decoding of a DMC are well known [17], [18] and it shows that list decoding does not lead to higher rates. Consequently, the maximal rates with list decoding are equal to the maximal rates without list decoding, i.e., the usual decoding with list size one. Further, for such a scenario it does not matter if a deterministic coding strategy or a random coding strategy is used [10].

Since the rates have to be achievable for all possible distributions $q \in \mathcal{P}(\mathcal{S})$, the maximal rates are restricted to the infimum of these rates. Since these rates correspond to the rates given in Theorem 2, we showed that no other rate pairs are achievable.

VI. Discussion

Channel uncertainty influences the communication in wireless networks. While for compound channels communication is still possible, but at reduced rates, for unknown varying channels the impact is much more dramatic. Due to the uncertainty in the variation of the channel there are scenarios where no communication is possible, not even at very low rates. The concept of symmetrizability provides necessary and sufficient conditions to decide when reliable communication is possible and when not [16].

For bidirectional relaying it shows in [12] that symmetric channels lead to an ambiguity of the codewords which prohibit reliable communication. Unfortunately, it reveals that many practically relevant channels are symmetric and it is needless to say that one is interested in techniques which dissolve the ambiguity caused by the symmetric channels. Independently in [13] and [14] it is demonstrated for the single-user AVC that the concept of list decoding is an adequate technique to achieve this. Moreover, for practical systems the concept of list decoding will significantly improve the error-correction performance [15].

In this work, we considered list decoding for bidirectional broadcast channels with unknown varying channels and established a strategy which allows to transmit in the presence of symmetric channels. For this purpose we introduced a generalized notion of symmetrizability which distinguishes among different degrees of symmetry. This allowed us to reveal a connection between the degree of the symmetry of a channel and the needed list size at the decoder. It shows that if a list size is greater than the symmetrizability of the channel, the decoder is able to successfully dissolve a possible ambiguity of the codewords. Consequently, the concept of list decoding permit reliable communication in scenarios, where usual decoding techniques fail.

The following illustrates how bidirectional communication can benefit from the list decoding technique. While usual decoding techniques as considered in [12] fail, list decoding with sufficiently large list sizes can establish a reliable communication. Figure 2 depicts the maximal achievable rates for a given input distribution P_X and increasing list sizes (L_1, L_2) at the decoders and illustrates how the transmission collapses if the list sizes are smaller or equal than the corresponding symmetrizabilities (T_1, T_2).

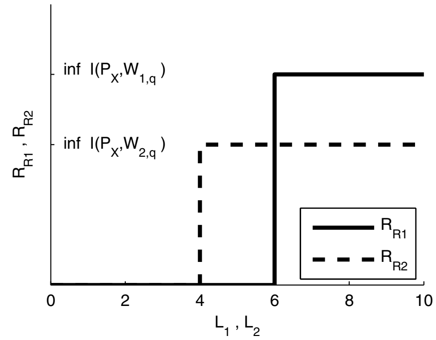

Fig. 2. Achievable rates for the AVBBC with symmetrizabilities $T_1 = 5$ and $T_2 = 3$ for a given input distribution P_X and increasing list sizes (L_1, L_2).

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